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Snell's laws at the interface between nonlinear dielectrics

Aceves, Alejandro Borbolla, Ph.D.

The University of Arizona, 1988

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**SNELL'S LAWS AT THE INTERFACE
BETWEEN NONLINEAR DIELECTRICS**

by

Alejandro Borbolla Aceves

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A Dissertation Submitted to the Faculty of the
COMMITTEE ON APPLIED MATHEMATICS

In Partial Fulfillment of the Requirements
For the Degree of

DOCTOR OF PHILOSHOPHY

In the Graduate College

THE UNIVERSITY OF ARIZONA

1988

THE UNIVERSITY OF ARIZONA
GRADUATE COLLEGE

As members of the Final Examination Committee, we certify that we have read
the dissertation prepared by Alejandro Borbolla Aceves
entitled SNELL'S LAWS AT THE INTERFACE BETWEEN NONLINEAR DIELECTRICS

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I hereby certify that I have read this dissertation prepared under my
direction and recommend that it be accepted as fulfilling the dissertation
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ACKNOWLEDGEMENTS

I would like to give special thanks to Alan C. Newell for being so enthusiastic and supportive throughout these years. Working with him has been a most enjoyable experience. Also, the continuous collaboration and discussions with Jerry Moloney who first introduced us to the problem, have always been very fruitful.

We will always remember our stay in Tucson. The people in the Applied Mathematics Program were always very friendly. Thank you to David McLaughlin and all the graduate students in the program who at one time or another showed their interest in my research.

The financial support I received from the Universidad de Mexico through a scholarship in the first years of my studies and the research assistanship provided from a grant by the Air Force Office of Scientific Research to the Arizona Center for Mathematical Sciences will always be greatly appreciated.

This thesis is dedicated to my family. In each and every page of this work, there is a piece of Rocio, Ale and Andy. Thank you for your love and support. I wouldn't have done it without you guys!. Also the love and support of my mother, brother and sisters made me felt not so far from them in the past years.

Finally, I want also to dedicate this work to the memory of my father who could not see it in its final version.

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ABSTRACT

A theory is presented which describes the global reflection and transmission characteristics of a self-focused channel propagating at an oblique angle of incidence to an interface separating two or more self-focusing nonlinear dielectric media. A complete characterization of the different behavior of the channel is given in the proper parameter space. In the dominant region, the nonlinear wavepacket representing the self-focused channel is represented as an equivalent particle moving in an equivalent potential.

The dynamics of the particle is described by Newton's equations of motion, with the asymptotic propagation paths of the channel being read off from the associated phase planes of the equivalent potential. The theory provides therefore, the nonlinear Snell's Laws of reflection or transmission since the particle dynamics gives the critical angle of total reflection and in the case of transmission, the corresponding angle of transmission. This theory also gives the stability characteristics of nonlinear surface waves, which had only been partially established in the past through numerical simulations.

Finally, some applications of the theory are presented such as the design of an all-optical power adjustable spatial scanning element and an all-optical switch. Extensions of the theory to waveguides with multiple interfaces are also given and possible new directions are suggested.

INTRODUCTION

The physical theory of the propagation of light in matter goes back to the previous century and it is condensed in mathematical form in Maxwell's equations. The fundamental principle involved here is that matter responds to electromagnetic waves, thus changing the field inside the medium. This distortion of the field is proportional to the polarization of the medium which in turn is proportional to the input electric field, the constant of proportionality being the susceptibility which characterizes the properties of the medium.

Until some 30 years ago the theory of optics was linear. It has been found since that some materials possess an intensity dependent index of refraction. In this case, the governing equations become nonlinear. New phenomena, which have many practical ramifications, have been observed in these materials: second harmonic generation, optical bistability, self induced transparency, and four wave mixing are some examples. It is noteworthy that new mathematical concepts originating in the last 15 years have had an immediate impact in this area, one of the most important being the concept of the soliton [1]. Similarly, new numerical techniques developed to solve nonlinear partial differential equations are now widely used to simulate nonlinear optical phenomena [2]

Two areas of nonlinear optics are of particular relevance for applications in communications, data processing, optical switches and all optical logic devices; these are fiber optics and nonlinear waveguides. In the first case of fiber optics, both the theoretical and the experimental [3] aspects have been explored in detail. In fact, fiber optics are being commercially used now. It is nonlinear waveguide theory that will be studied in this work.

A waveguide is a device composed of an array of dielectric materials. Its purpose is that indicated by its name, to guide the propagating light in a narrow beam. This is achieved by choosing appropriate geometries and refractive indices for the different media. The theory of linear waveguides, described in some detail in [4], relies on what are essentially the WKB methods of geometrical optics and is an extension of Snell's laws of reflection and transmission of electromagnetic waves at an interface between linear dielectrics [5].

It was not until about 1973 when the first publications dealing with surface waves along an interface between a linear and a nonlinear dielectric started to appear [6]. Later in 1976 the problem of incident plane waves at an interface between a linear and nonlinear dielectric was studied by A.E.Kaplan [7] and he found that the reflectivity was not only intensity dependent but exhibited a hysteretic behavior. This was demonstrated experimentally in 1979 by P.W. Smith, J.P. Hermann, W.J. Tomlinson and P.J. Maloney [8]. About that time beam propagation at an interface between a linear and a nonlinear dielectric was studied by N.N. Rozanov [9] using, as we will here, the paraxial approximation. There the author poses the problem as an initial boundary value problem in the nonlinear medium and one of the events he sees in the break up of the beam into separate channels in the nonlinear medium. We will explain with our theory why and how that happens. A particular problem that was studied is whether for beams incident at an interface, bistability in the reflectivity also occurred as in the case of a plane wave. P.W. Smith and W.J. Tomlinson showed there is no bistability [10] but later N.N. Rozanov and G.V. Khodova [11] show bistability occurs if one uses the full wave equation instead of the paraxial approximation and concluded that only if this approximation fails one could expect possible bistable behavior in the reflectivity.

It is the purpose of this work to study beam propagation at nonlinear waveguides using the paraxial approximation and what we are aiming to provide is the nonlinear analogue of Snell's laws.

As we noted before, nonlinear optics has brought into the picture areas of mathematics that have been the subject of great activity over the last years. In particular in fiber optics and nonlinear waveguides the governing equation for the field is the Nonlinear Schrödinger Equation (NLS), or some variant thereof. The NLS equation is a completely integrable equation that can be solved using inverse scattering theory [12]. It has been shown that an arbitrary initial condition, subject to certain constraints, will decompose into soliton and radiation modes. In fibers solitons are temporal pulses localized in z , the direction of propagation and time t , whereas in nonlinear waveguides one has beams or wavepackets that have a local structure in the transverse direction that could be soliton like. Although NLS solitons occur in a variety of situations, it is probably in nonlinear optics where they have made their greatest impact. Not only are they important from a theoretical point of view, but they have been observed in many experiments [13] and exploited in many devices; for example, the soliton laser [14].

In many circumstances, the NLS equation by itself is not enough to describe the physics. Extra terms which we refer to as perturbations come into play, and a natural question to ask is what do perturbations do to the solitons of the unperturbed equation. To answer that question, Kaup and Newell developed a singular perturbation theory [15] in which they computed the changes produced in the scattering data by the perturbation; it is the scattering data that determines the behavior of the solution. If in particular the initial condition consists of one or more solitons, it is found that in many cases that the perturbation will not excite new modes to

any great extent. Instead, it will principally act only on the modes contained in the initial condition. They interpreted their results by thinking of each soliton as a particle-like object whose mass, position and velocity are given by the soliton parameters. One can view the evolution of these solitons as an N particle system each moving under the action of a force consisting of an external component due to the perturbations and an internal component due to the soliton interactions. There are many examples in [15] of the use of this theory to analyze the evolution of a one soliton initial condition for various equations, and Kodama has worked on perturbations of NLS connected to fiber optics when the initial condition contains one or two soliton components [16]. In fact, the equation he analyzes, the modified nonlinear Schrödinger equation (MNLS), is completely integrable [17] as well and has been studied using inverse scattering theory [18]. Finally we will show here how by applying this idea we can construct, in a very natural way, a simple theory of optical beam propagation at interfaces.

Although there are many cases in which the perturbation will excite modes not contained in the initial condition only to small order, this is not true in general. For example, in the problem of solitary waves propagating in channels of slowly changing depth, an incoming soliton transfers an order one amount of mass to nonsoliton components as it propagates [15]. We will see subsequently that something similar may happen to beams approaching an interface between nonlinear dielectrics.

The way we present our results is as follows: In Chapter 1, we derive the governing equations for the electric field propagating in a nonlinear dielectric, and we will describe the particular geometry of the waveguide we will consider. A summary of the linear theory and particular solutions of the nonlinear case (solutions that propagate parallel to the interface with no changes in its shape) are also presented

in this chapter.

In Chapter 2 we identify the relevant parameters which will characterize the different behaviors of wavepackets incident at an interface between two nonlinear dielectrics. We will find there, by an extensive numerical analysis of the problem, that there are three different regimes of behavior. First, for broad incoming wavepackets, carrying moderate power, the reflection properties are given by the familiar linear Snell's laws. In other words, in this regime, the wavepacket splits at the interface into partially reflected and partially transmitted components and each part disperses as it propagates away from the interface. For higher powers, there is a second regime where again the wavepacket may split, but this is produced by a different mechanism, which is not linear, and we will see the amounts of reflected and transmitted powers differ from the linear predictions. In this case, we find that either the reflected or the transmitted components, and in some cases both, propagate as dispersionless solitons. Third, and the main observation coming from this Chapter is that for high power, narrow beams, most of the power remains confined in the beam when it crosses an interface, and when the mismatch of the nonlinear refractive indices is small a sharp transition between total reflection and total transmission is observed. This is the regime which dominates in parameter space and we will study it in detail in Chapters 3, 4 and 5.

In Chapter 3 we present a theory that describes the evolution of beams or wavepackets which remain entirely either in the medium at the left of the interface or in the other one. In each case, the observed fact that the beam or wavepacket remains localized as it propagates allows us to develop an equivalent particle description. Further, the propagation of each wavepacket can be inferred from the motion and phase plane portraits of an equivalent particle in a potential well. Varying the

initial angle of the beam, is equivalent to choosing different initial velocities of the particle. In Chapter 4 we extend the theory to consider wavepackets crossing the interface. There we can predict that a wavepacket crossing the interface from the medium with lower nonlinear refractive index to the medium with the higher one, can break-up into multiple channels after it has crossed the interface. We show how to compute the number and amplitudes of these channels. A similar analysis is developed for beams crossing in the other direction. If the nonlinear refractive index mismatch is small, there will be no break-up and a composite potential which describes the global dynamics of the wavepackets is constructed; beams incident at an interface from either side are either totally reflected or totally transmitted. These two Chapters give us the nonlinear Snell's laws for this regime, since now we can predict the reflected and transmitted components of the wavepacket in terms of the initial parameters of the wavepacket. Each reflected and transmitted component is an asymptote of an equivalent particle trajectory.

If there is no break-up, the nonlinear Snell's Laws are given in terms of the velocity of an equivalent particle moving under the action of an equivalent potential. It says that if the initial velocity v_0 ($v_0 = 2\beta k \sin \Psi_i$, where Ψ_i is the angle of incidence, β is the effective guide index and $k = \omega/c$) is less than the critical velocity v_c of total reflection, which is a function of both the power and the refractive indices of both media, then the final velocity of the equivalent particle is $v_f = -v_0$ which simply means that the angle of reflection is equal to the angle of incidence. If on the other hand, the initial velocity is bigger than the critical velocity, the beam will be totally transmitted and the final velocity v_f is,

$$v_f = \sqrt{v_0^2 + 2U_0 - 2U_f}$$

where U_0 and U_f are respectively the values of the equivalent potential computed at the initial and final positions of the equivalent particle and both will depend on the power of the incident beam and on the linear and nonlinear indices of refraction of the two media. Given v_f , the angle of transmission Ψ_t is obtained from the expression $v_f = 2\beta k \sin \Psi_t$.

In this Chapter we also make a connection with previous work, and in particular we will be able to infer the stability properties of the solutions obtained in Chapter 1.

Chapter 5 is devoted to show some applications of the theory. The fact that the nonlinear Snell's Laws are power dependent, brings as a direct application the design of a spatial scanner. In Figure 1, we show the superposition of different beam trajectories approaching an interface. In all cases, as we can see the angle of incidence is the same, but each case corresponds to a beam of different power. As we just said above, the power dependence of the nonlinear Snell's Laws gives as a consequence that the final location of the beam is different for each case and we can predict them from the theory described in Chapter 4.

In this Chapter we also extend the theory to a waveguide with multiple interfaces. Also, we consider the phenomena of soliton collisions in a waveguide. As an example of a specific application, we will show how to design an optical switch. We also show numerical simulations of the behavior of wavepackets that initially have two bound soliton components and we show how the nonlinear waveguide splits the wavepacket.

Figure 1. A spatial scanner. Here we show the trajectories of incident beams all at the same angle but of different powers. We see that at the end some are reflected and some are transmitted, but all at different angles. With the equivalent particle theory we will be able to compute the full trajectory for cases similar to these ones.

It is important to emphasize, that the theory presented here is tested by comparing its predictions with numerical simulations of the governing equation, since at the moment, to my knowledge, there is no experimental evidence of the results shown here.

CHAPTER 1

Linear Theory and Introduction to the Nonlinear Problem

The problem we are going to study throughout most of this work is that of the propagation of wavepackets in a planar waveguide consisting of two dielectrics which typically have nonlinear refractive indices. The plane that describes the waveguide will be the $x - z$ plane and the interface will be located at $x = 0$, then z is the direction in which the wavepacket propagates. Considering a planar waveguide effectively means that its width in the y direction is small compared to the beam width; therefore under this assumption, the electric field is y -independent.

In this geometry we define Ψ_i as the angle that an incoming beam approaching the interface from the left hand medium makes with the interface. Similarly Ψ_r and Ψ_t are respectively the angles that a reflected and a transmitted beam make with the interface (see Figure 2).

In this chapter we will first outline the derivation of the equations and will consider the case where both media have linear refractive indices. This case is of particular interest since in the next chapter a connection between linear and nonlinear theory will be made. We will conclude the chapter by solving the nonlinear problem for the case where the wavepacket is z -independent.

1.1 Derivation of the equations

The problem we will consider is that of a propagating wave in the z direction that is confined in the transverse direction x .

Figure 2. Schematic representation of the planar waveguide, where we define the angle of incidence Ψ_i .

The field will be monochromatic in time with frequency ω and in both x and z it will be almost monochromatic. The z wavenumber of the incident field will be βk , where β is sometimes called the effective guide index or the waveguide mode, and $k = \omega/c$. These waves are sometimes called nonlinear surface polaritons or simply nonlinear surface electromagnetic waves. We will neglect nonlinear optical phenomena such as second and third harmonic generation, or nonlinear wave mixing of two or more surface electromagnetic waves. Instead we assume that everything occurs at one and the same frequency. Also there are different orientations which the electric and magnetic field vectors can take as they propagate. Here we will only consider transverse electric (TE) waves where the electric vector is oriented transverse to the plane of incidence.

The wave equation that governs optical wave propagation in a medium is

$$[\nabla_{\mathbf{x}} \nabla_{\mathbf{x}} + \frac{1}{c^2} \frac{\partial^2}{\partial t^2}] \vec{E}(x, z, t) = -\frac{4\pi}{c^2} \frac{\partial^2}{\partial t^2} \vec{P}(x, z, t)$$

where $\vec{E}$, $\vec{P}$ are respectively the electric and polarization vectors, and $\nabla = \hat{x} \frac{\partial}{\partial x} + \hat{z} \frac{\partial}{\partial z}$. So, in this problem we have two similar equations, one for each side of the interface.

From the above assumptions we have that

$$\vec{E}(x, z, t) = \vec{E}(x, z) e^{-i\omega t} + cc., \quad \vec{P}(x, z, t) = \vec{P}(x, z) e^{-i\omega t} + cc.$$

In the nonlinear case, the polarizaton is typically a complicated nonlinear function of $\vec{E}$ but we will expand it in power series of $\vec{E}$ keeping only the terms to third order, and only those terms which contribute to the frequency ω . That is

$$\vec{P} = \chi^{(1)} \vec{E} + \chi^{(3)} |E|^2 \vec{E}$$

where $\chi^{(1)}$ and $\chi^{(3)}$ are respectively, the linear and third order susceptibilities which characterize the physical properties of a medium. Physically, $\chi^{(n)}$ is related to the microscopic structure of the medium and can be properly evaluated only with a full quantum-mechanical calculation. The complete set of equations that fully describes the electromagnetic field and the medium is called the Maxwell-Bloch system of equations.

The fact that we are considering TE waves propagating in an xz planar waveguide means that $\vec{E} = \hat{y}E$. That is, the electric field is simply a scalar field. Furthermore, since E is y -independent, $\nabla \cdot \vec{E} = \frac{\partial E}{\partial y} = 0$. Putting together the above assumptions and simplifications in the original equation leads us to a simpler scalar equation for E ,

$$\frac{\partial^2 E}{\partial z^2} + \frac{\partial^2 E}{\partial x^2} = -n^2 k^2 E$$

where $n^2 \equiv n_l^2 + \alpha|E|^2 = \frac{w^2}{c^2 k^2} [1 + 4\pi\chi^{(1)} + 4\pi\chi^{(3)}|E|^2]$ is the nonlinear refractive index of the medium. We will only consider the case of self focussing nonlinearities where $\chi^{(3)} > 0$. It is in these materials that one can have confined wavepackets.

The final approximation we will make, which is called the slowly varying envelope approximation (SVEA) simply states the assumption that the field is almost monochromatic in z and x . To be more precise, we will assume that $E = F(x, z)e^{i\beta kz}$, where β is of the order of the linear refractive index of either medium and F has a narrow band of z wavenumbers Δk_z of the order of the linear refractive index mismatch of the two media, and a narrow band of x wavenumbers $\Delta k_x \sim \sqrt{\Delta k_z}$. These means that the envelope F slowly changes in z compared to the wavelength. The approximation $|\frac{\partial^2 F}{\partial z^2}| \ll |\beta k \frac{\partial F}{\partial z}| \sim |\frac{\partial^2 F}{\partial x^2}| \sim |(\beta^2 - n_l^2)k^2 F| \ll |\beta k F|$ is applied and leads us to the final form of the equation

$$2i\beta k \frac{\partial F}{\partial z} + \frac{\partial^2 F}{\partial x^2} - (\beta^2 - n^2)k^2 F = 0.$$

Finally, throughout the rest of the paper we will use dimensionless spatial variables. That is we make the change of variables $x' = kx$, $z' = kz$ and then replace back x for x' and z for z' . Therefore, the final form of the equation to be studied is

$$2i\beta \frac{\partial F}{\partial z} + \frac{\partial^2 F}{\partial x^2} - (\beta^2 - n^2)F = 0. \quad (1.1)$$

In (1.1), the refractive index $n^2 = n_i^2 + \alpha_i |F|^2$, ($\alpha_i > 0$) is discontinuous at the interface. The index $i=0$ is used for the left hand medium ($x < 0$) and $i=1$ for the right hand medium ($x > 0$). Throughout this work we will consider the case where $\Delta \equiv n_0^2 - n_1^2 > 0$ and $\alpha \equiv \frac{\alpha_0}{\alpha_1} \leq 1$. If instead $\alpha > 1$, the theory presented here also works but the possible different behaviors are not as rich as the case considered here. At this point β is simply a parameter of convenience chosen to be close to the average of the linear refractive indices n_0 and n_1 .

Finally we should point out that the continuity conditions at the interface of the electric and magnetic field components tangential to the plane of incidence yz implies for the case of TE waves that both F and $\frac{dF}{dx}$ which is proportional to the tangential magnetic field H_z , are continuous at $x = 0$.

A particular case of (1.1) is when both dielectric media are linear i.e. $\alpha_0 = \alpha_1 = 0$ and this is the case we consider first.

1.II Linear theory

In this section we will look at the problem of wavepackets approaching an interface between two linear dielectrics. This problem although linear is very difficult

since the reflected and transmitted fields are given in terms of integrals and cannot be written in terms of elementary functions. Nevertheless for broad wavepackets approaching the interface at an angle of incidence close to the critical angle of total internal reflection, there is considerable literature dealing with this problem [13]. We will only outline some of the most important results, and in particular will discuss one in detail because it is not readily available in the literature and because it is relevant for comparing the linear and nonlinear theories. It is an asymptotic estimate of the amount of power carried in the reflected wavepacket in terms of the Fresnel formulae.

The standard procedure to solve the linear problem is to think of the incident, reflected and transmitted wavepackets as being a superposition of plane waves and the problem translates into finding, for each wave, the amplitudes of the reflected and transmitted components in terms of the amplitude of the incident component. Therefore we will begin by considering an incoming plane wave whose x wave number is k_x ; then the field in the left hand medium will be

$$F_{left} = e^{i(k_x x + \frac{k_z z}{2\beta})} + R(k_x) e^{-i(k_x x - \frac{k_z z}{2\beta})} \quad (1.2a)$$

where the first term represents the incoming wave and the second term is the outgoing reflected wave. Similarly, the field in the right hand medium is

$$F_{right} = T(k_x) e^{i(k'_x x - \frac{k_z z}{2\beta})} \quad (1.2b)$$

Direct substitution of (1.2a) and (1.2b) in equation (1.1) with $\alpha_0 = \alpha_1 = 0$ determines the dispersion relation for the two media

$$k_z = -k_x^2 - (\beta^2 - n_0^2) \quad (1.3a)$$

$$k_z = -k_x'^2 - (\beta^2 - n_1^2) \quad (1.3b)$$

from which one finds that $k_x'^2 = k_x^2 - (n_0^2 - n_1^2)$ which is simply the paraxial approximation of Snell's law. It is important to observe that these dispersion relations are also obtained from the dispersion relation of original equation (that is the Helmholtz equation) for the field E after expanding about the dominant wavenumber. Indeed, if x and z are in dimensionless form, $E = F e^{i\beta z}$ and the dispersion relation obtained from Helmholtz equation gives for the two media

$$(\beta + \frac{k_z}{2\beta})^2 + k_x^2 = n_0^2$$

$$(\beta + \frac{k_z}{2\beta})^2 + k_x'^2 = n_1^2.$$

Expanding these expressions and neglecting terms proportional to k_z^2 give equations (1.3a) and (1.3b).

To determine $R(k_x)$ and $T(k_x)$ one imposes the continuity conditions of F and $\frac{dF}{dx}$ valid for transverse electric (TE) fields. One obtains the Fresnel formulae

$$R(\bar{k}_x) = \frac{\bar{k}_x - \sqrt{\bar{k}_x^2 - 1}}{\bar{k}_x + \sqrt{\bar{k}_x^2 - 1}}$$

$$T(\bar{k}_x) = \frac{2\bar{k}_x}{\bar{k}_x + \sqrt{\bar{k}_x^2 - 1}}$$

where $\bar{k}_x = \frac{k_x}{\sqrt{\Delta}}$, $\Delta = n_0^2 - n_1^2 > 0$.

These results allow us to find integral expressions for the fields in both media when the incident field is a wavepacket; for this case

$$F_{left} = \int_{-\infty}^{\infty} A(k_x) e^{i(k_x x + \frac{k_x z}{2\beta})} dk_x + \int_{-\infty}^{\infty} R(k_x) A(k_x) e^{-i(k_x x - \frac{k_x z}{2\beta})} dk_x \quad (1.4)$$

and

$$F_{right} = \int_{-\infty}^{\infty} T(k_x) A(k_x) e^{i(k'_x x + \frac{k_x z}{2\beta})} dk_x \quad (1.5)$$

respectively, where $A(k_x)$ is the Fourier transform of the incoming wavepacket. Since we cannot evaluate the integrals in terms of elementary functions we will look for an asymptotic expansion for each component of the field for large values of z . We will only consider the reflected component, and there we will derive the leading order term of its asymptotic expansion for the case of an incoming broad sech-like wavepacket.

Let the incoming wavepacket be $F_{in} = E_0 \text{sech} 2\eta(x - \bar{x}) e^{i\frac{v}{2}x}$ where $\bar{x} \ll 0$ and of width $\eta^{-1} \gg 1$. Then its Fourier transform is

$$A(k_x) = \frac{E_0}{4\eta} \text{sech}\left(\frac{\pi}{4\eta}(k_x - \frac{v}{2})\right) e^{i(\frac{v}{2} - k_x)\bar{x}}$$

Observe that since F_{in} is broad, $A(k_x)$ is very narrow with its peak located at $k_x = \frac{v}{2}$. We now find the leading order term of the asymptotic expansion for large z of the second term in (1.4) using the stationary phase method. It is

$$\int_{-\infty}^{\infty} R(\bar{k}_x) A(k_x) e^{i(k_x x + \frac{k_x z}{2\beta})} dk_x \sim \frac{R(\frac{v}{2\sqrt{\Delta}})}{4\eta} \sqrt{\frac{2\beta\pi}{z}} E_0 \text{sech}\left(\frac{\pi\beta(x + \bar{x} + \frac{vz}{2\beta})}{4\eta z}\right) e^{i\phi} \quad (1.6)$$

where

$$\phi = (\beta^2 - n_0^2) \frac{z}{2\beta} - \frac{(x + \bar{x})^2}{4z}$$

In (1.6) we approximate $R(-\frac{\beta(x+\bar{x})}{\sqrt{\Delta}z}) \sim R(\frac{v}{2\sqrt{\Delta}})$ which is the first term of its Taylor expansion. Therefore if we call the reflected field given by (1.6) F_r , it follows that

$$\int_{-\infty}^{\infty} |F_r|^2 dx = |R(\frac{v}{2\sqrt{\Delta}})|^2 \int_{-\infty}^{\infty} |F_{in}|^2 dx \quad (1.7)$$

The portion of the incident power being reflected by an interface between two linear dielectrics when the incoming wavepacket is broad, is given by the Fresnel formula evaluated at $\bar{k}_x = \frac{v}{2\sqrt{\Delta}}$. One can now conclude that in this case, there is total internal reflection if $\frac{v}{2\sqrt{\Delta}} < 1$; otherwise there is partial reflection and partial transmission. Thus the critical angle Ψ_i is $\Psi_{i,crit} = \sin^{-1}(2\sqrt{\Delta})$.

The leading order of the reflected field (1.6) is called the geometrical optics component and it is easy to see why. If one describes beams of light as rays, then the incident beam is a ray given by the equation $x = \bar{x} + vz$. The reflected ray is given by the equation, $x = -\bar{x} - vz$, and the two rays will intersect at the interface. This is what geometrical optics predicts; that an interface acts like a mirror and the reflected ray is an image of the incident one.

Equation (1.6) is the only analytical result of linear theory that will be of interest for this work since in Chapter 2 we will compare it with the reflected fields for the nonlinear problem. But this is a good place to present a brief summary of the general behavior of the linear problem.

Most of the theoretical papers in this area were motivated by an experimental work done by Goos and Hänchen in 1947 [20] where they observed that the reflected

beam suffers a shift, now known as the Goos-Hänchen (GH) shift, from the location predicted by geometrical optics, and in essence what you see in these works is a correction on the reflected field F_r obtained by going to the next order term of its expansion. Here we found that the leading order term was easy to obtain, but a considerable extra effort is needed to go to the next order, since now one has contributions not only from the stationary point of the phase, but from the branch points of $R(\bar{k}_x)$ as well. Probably the most relevant work in this area is a series of papers by Horowitz and Tamir [13] (part of this work was Horowitz's Ph.D. thesis), where they first analysed an incident Gaussian beam and they not only found an explicit expression for the GH shift but they also explained its mechanism. What causes the GH shift is an interference effect between the first order contribution and one term of the second order contribution which they refer to as the near-field of the lateral wave. Meanwhile the remaining terms of the second order contribution called the intermediate and far fields, initially propagate parallel to the interface, leading eventually to weak radiance that trails the reflected beam. Finally they showed that similar results can be obtained if the beam is Cauchy type instead of Gaussian from where they infer the generality of their results for general symmetric beams, although there is no analytical result for the sech-like beam. In all of their work they considered as it is done here only broad beams and more recently Kozaki and Sakurai [21] numerically studied this problem for Gaussian beams of arbitrary width.

This phenomenon of the reflected beam been shifted was later observed by W.J. Tomlinson, J.P. Gordon, P.W. Smith and A.E. Kaplan [22], in their numerical simulations of Gaussian beams incident at an interface between a linear and a nonlinear dielectric. They called it a nonlinear Goos-Hänchen (NLGH) shift and the

explanation at that time, based on how the wavepacket evolved was that most of the power of the incident beam is transferred to a surface wave (a z independent solution of (1.1)) that propagates along the interface in the nonlinear medium; but they argued that this wave was unstable so eventually it crossed the interface propagating as a reflected wavepacket in the linear medium. The time the wavepacket spent in the nonlinear medium accounts for the shift, but no analysis was done for this problem. Later in Chapter 4 we will find an analytical expression for the NLGH shift in the case of two nonlinear dielectrics when the incident beam is narrow and the nonlinear refractive index mismatch is small.

1.III z -independent solutions

We now start considering the nonlinear problem by solving (1.1) when F is z independent. These solutions are known as nonlinear surface waves or nonlinear surface polaritons. There is an extensive literature on this topic and the kind of problems tackled include TE polaritons as well as TM polaritons; also one or multiple interfaces with linear and nonlinear dielectrics have been studied. We refer the reader to some of these works ([23], [24], [25]). In [23] one can find the general method to obtain these solutions; solutions for waveguides with self-defocusing dielectrics are given in [24], or solutions of the single interface problem with one media being linear and the other nonlinear are studied in [25]. The last case is of particular interest to us and we shall show that the solutions of that problem can be obtained as a limit of the stationary solutions of the nonlinear-nonlinear case when the nonlinearity of one medium goes to zero.

The method of finding these solutions is straightforward. One finds a particular solution in each medium; it will contain certain number of free parameters, which

together with the parameters of the nonlinear media and the location of the interfaces defines a general dispersion relation obtained from the boundary conditions at each interface. Typical z independent solutions for dielectrics with self-focusing quadratic nonlinearities are sech-like solutions and elliptic functions and we will only consider the first type of solution.

If the envelope of the wave F is independent of the direction of propagation z , then it must satisfy the equation

$$\frac{d^2 F(x)}{dx^2} - (\beta^2 - n^2(x, |F|^2))F(x) = 0 \quad (1.8)$$

Now β takes on the role of an eigenvalue; namely the existence of localized solutions of the nonlinear boundary value problem (1.8) will demand that β is a fixed function of the L_2 norm (the power) of the solution. For a given β a solution of (1.8) is given by,

$$F^+(x) = \begin{cases} \sqrt{\frac{2(\beta^2 - n_0^2)}{\alpha_0}} \text{sech}(\sqrt{\beta^2 - n_0^2}(x - x_0)) & x < 0 \\ \sqrt{\frac{2(\beta^2 - n_1^2)}{\alpha_1}} \text{sech}(\sqrt{\beta^2 - n_1^2}(x - x_1)) & x > 0 \end{cases} \quad (1.9)$$

A necessary condition for the existence of these solutions is that $\beta^2 > n_0^2$. Also note that there are two parameters to determine x_0 and x_1 . Since we are studying TE waves, both $F(x)$ and $\frac{dF}{dx}$ must be continuous at $x = 0$. That gives two equations for x_0, x_1 , namely

$$\sqrt{\frac{2(\beta^2 - n_0^2)}{\alpha_0}} \text{sech} \sqrt{\beta^2 - n_0^2} x_0 = \sqrt{\frac{2(\beta^2 - n_1^2)}{\alpha_1}} \text{sech} \sqrt{\beta^2 - n_1^2} x_1 \quad (1.10)$$

and

$$\sqrt{\beta^2 - n_0^2} \tanh \sqrt{\beta^2 - n_0^2} x_0 = \sqrt{\beta^2 - n_1^2} \tanh \sqrt{\beta^2 - n_1^2} x_1 \quad (1.11)$$

Solving for x_0, x_1 we find that

$$x_0 = \frac{1}{2\sqrt{\beta^2 - n_0^2}} \ln \frac{1 + r\sqrt{1 - \mu^2}}{1 - r\sqrt{1 - \mu^2}} \quad (1.12)$$

$$x_1 = \frac{1}{2\sqrt{\beta^2 - n_1^2}} \ln \frac{1 + \sqrt{1 - \mu^2}}{1 - \sqrt{1 - \mu^2}} \quad (1.13)$$

where $\mu^2 = \frac{n_0^2 - n_1^2}{(1 - \alpha)(\beta^2 - n_1^2)}$ must be less than one in order to have real solutions, and $r = \sqrt{\frac{\beta^2 - n_1^2}{\beta^2 - n_0^2}}$. The condition $\mu^2 < 1$ implies a condition on β , namely

$$\beta^2 > \beta_{min}^2 = n_1^2 + \frac{\Delta}{1 - \alpha} > n_0^2$$

These solutions which we refer to as $F^+(x)$, correspond to wavepackets whose envelope has a peak at $x_1 > 0$. However, observe from (1.10) and (1.11) that if x_0 and x_1 satisfy these equations, so also do, $-x_0$ and $-x_1$. Therefore there is a dual family of steady state solutions,

$$F^-(x) = \begin{cases} \sqrt{\frac{2(\beta^2 - n_0^2)}{\alpha_0}} \text{sech}(\sqrt{\beta^2 - n_0^2}(x + x_0)) & x < 0 \\ \sqrt{\frac{2(\beta^2 - n_1^2)}{\alpha_1}} \text{sech}(\sqrt{\beta^2 - n_1^2}(x + x_1)) & x > 0 \end{cases} \quad (1.14)$$

where now the peak of $F^-(x)$ is at $x = -x_0 < 0$. Typical plots of F^+ and F^- are shown in Figure 3.

Using the idea of Langbein, Lederer and Ponath [32], if we plot the solutions in a $\frac{dF}{dx}$ vs. F plane one can see that (1.9) and (1.14) are, for a given β , the only solutions satisfying the boundary conditions at the interface and at $x = \infty$ and $x = -\infty$.

The powers $P^\pm = \int_{-\infty}^{\infty} |F^\pm(x)|^2 dx$ corresponding to these solutions are,

Figure 3. Plots of F^+ and F^- for values of $\alpha_0 = 0.5$, $\alpha_1 = 2.$, $n_0^2 = 0.11$, $n_1^2 = 0.01$ and $\beta = 0.7$.

$$P^{\pm} = 2 \left[\frac{\sqrt{\beta^2 - n_0^2}}{\alpha_0} + \frac{\sqrt{\beta^2 - n_1^2}}{\alpha_1} \pm \left(\frac{1}{\alpha_1} - \frac{1}{\alpha_0} \right) \sqrt{\beta^2 - n_1^2 - \frac{\Delta}{1-\alpha}} \right]. \quad (1.15)$$

In Figure 4, we plot β vs P for the case $\alpha = 0.25$ where the branch **ABCD** corresponds to P^+ and the branch **DE** to P^- . We note that β_{min} corresponds to the value of β where $F^+(x) = F^-(x)$, or where $x_0 = x_1 = 0$, and that there is a critical power P_C below which no steady state solutions exists. Also plotted in Figure 4 is the corresponding $\bar{x}$ vs P graph, where $\bar{x}$ is the position of the peak of the steady state solution; that is $\bar{x} = x_1$ for $F^+(x)$ and $\bar{x} = x_0$ for $F^-(x)$. We label the branches the same way as in the β vs. P graph.

Given the steady state solutions a natural question to ask is whether they are stable or unstable. The general question of the stability of surface waves on nonlinear waveguides has only been studied numerically [25], [26] and for general symmetric or asymmetric thin film waveguides, Jones and Moloney [27] showed analytically the linear stability properties of the corresponding surface waves. There has been no proof of the global stability properties of these solutions and with the equivalent particle theory derived in Chapter 3 we will be able to answer these question for $F^+(x)$ and $F^-(x)$. We will show that the branch **ABC** is stable and the branch **CDE** is unstable even though, in this case, $\frac{\partial P}{\partial \beta} > 0$ along **DE**. This means that the stability criterion suggested in [25] that P increasing (decreasing) with respect to β infers stability (instability) is not true in general. It is however true in the particular situation of the linear-nonlinear problem. Also, we notice that the stable solutions occur only when the peak is in the medium with the higher nonlinear refractive index α_1 .

We conclude this chapter by showing that one can recover the steady state solutions of the linear-nonlinear case given in [25].

Figure 4. Plots of $\bar{x}$ vs. P and β vs. P for $\alpha = 0.25$. Branch ABCD corresponds to F^+ solutions, and branch DE to F^- . Branch ABC is stable and branch CDE is unstable.

If one takes the limit $\alpha = 0$ in $F^+(x)$ one has for $x > 0$,

$$\lim_{\alpha_0 \rightarrow 0} F^+(x) = \sqrt{\frac{2(\beta^2 - n_1^2)}{\alpha_1}} \text{sech}(\sqrt{\beta^2 - n_1^2}(x - x_1)) \quad (1.15)$$

with

$$x_1 = \frac{1}{2\sqrt{\beta^2 - n_1^2}} \ln \frac{\sqrt{\beta^2 - n_1^2} + \sqrt{\beta^2 - n_0^2}}{\sqrt{\beta^2 - n_1^2} - \sqrt{\beta^2 - n_0^2}} \quad (1.16)$$

For $x < 0$

$$\lim_{\alpha_0 \rightarrow 0} F^+(x) = \sqrt{\frac{2(n_0^2 - n_1^2)}{\alpha_1}} e^{\sqrt{\beta^2 - n_0^2}x} \quad (1.17)$$

which is found from the limit of equation (1.12) as $\alpha_0 \rightarrow 0$ using the fact that

$$\lim_{\alpha_0 \rightarrow 0} \frac{1}{\sqrt{\alpha_0}} e^{-\sqrt{\beta^2 - n_0^2}x_0} = \sqrt{\frac{\Delta}{4\alpha_1(\beta^2 - n_0^2)}}.$$

Equations (1.15), (1.16) and (1.17) are the solutions of the linear-nonlinear case [25]. For this case $\lim_{\alpha_0 \rightarrow 0} P_D = \infty$, in agreement with the fact that all steady state solutions of the linear-nonlinear case have their peak at positive x . This fact is obvious once we observe that since equation (1.1) in the linear case is purely dispersive, an initial wavepacket propagating in such medium will always spread losing its original shape.

Figures 5a and 5b show, respectively, the β vs P and $\bar{x}$ vs P plots for the case $\alpha = 0$. The stability analysis of these solutions will also be discussed in Chapter 3.

Figure 5. Plots of β vs. P and $\bar{x}$ vs. P for $\alpha = 0$. The $A_0B_0C_0$ branch is stable and the $C_0D_0E_0$ branch is unstable.

CHAPTER 2

Parameter Space Representation

The focus of this Chapter is to characterize numerically in parameter space the different behaviors of wavepackets approaching an interface. As in the linear case, we will restrict the analysis to incoming wavepackets that have sech-like shapes. Since the equation in the left hand media is nonlinear Schrödinger, there is a large class of initial wavepackets that if propagated towards the interface, will split into sech-like shapes before reaching it.

The characterization given here is based on results obtained by numerical integrations of (1.1) and an outline of the numerical techniques used is given in Appendix I. In the first section we identify all the parameters of the problem and we will define our parameter space. Then we conclude this chapter by establishing three different regimes in parameter space and we will describe the main features of the different phenomena observed in each regime.

2.I Identification of relevant parameters

The parameters of this problem are, the linear and nonlinear refractive indices of the two media (n_i^2 , α_i , $i = 0, 1$), and on the other the parameters of the incident wavepacket, its power ($\int_{-\infty}^{\infty} |F|^2 dx$) and angle of incidence. In total we have six parameters and the purpose of this section is to find appropriate relations between them that can give us a full characterization of the problem with as few parameters as possible. We will obtain the reduced parameter space from a proper choice of rescalings and transformations in the field and the equation.

Consider the following set of transformations

$$F(x, z) = \sqrt{\frac{2\Delta}{\alpha_0}} F'(x', z') e^{i(\beta^2 - n_0^2)z/(2\beta)}$$

where

$$x' = \sqrt{\Delta}x, \quad , \quad z' = \frac{\Delta z}{2\beta}$$

then $F'(x', z')$ satisfies

$$i\frac{\partial F'}{\partial z'} + \frac{\partial^2 F'}{\partial x'^2} + 2|F'|^2 F' = \begin{cases} 0 & x < 0 \\ F' - 2(\alpha^{-1} - 1)|F'|^2 F' & x > 0 \end{cases} .$$

That is, the equation for F' has only one parameter α . Now, if the incident beam $F(x, 0)$ has the form of an exact soliton travelling in the left hand medium,

$$F(x, 0) = \sqrt{\frac{2}{\alpha_0}} 2\eta_0 \operatorname{sech} 2\eta_0(x - \bar{x}_0) e^{i\frac{v_0 x}{2}}$$

then

$$F'(x', 0) = \frac{\eta_0}{\sqrt{\Delta}} \operatorname{sech} \frac{2\eta_0(x' - \bar{x}')}{\sqrt{\Delta}} e^{i\frac{v_0 x'}{2\sqrt{\Delta}}} .$$

What this means is that for a fixed value of α all initial conditions with the same values of $\frac{\eta_0}{\sqrt{\Delta}}$ and $\frac{v_0}{\sqrt{\Delta}}$ will show the same behavior except for a multiplicative constant $\sqrt{\frac{2}{\alpha_0}}$ of their respective amplitudes. Since we will characterize the different behaviors in parameter space in terms of the ratio of reflected to incident power this constant is not relevant to the analysis.

Now that we have been able to reduce the number of parameters to three (see Table 1), in the next section we will show, in this parameter space, the regimes of

different behavior of an incident sech-like wavepacket at an interface between two nonlinear dielectrics.

nondimensional power	$\frac{\eta_0}{\sqrt{\Delta}}$
nondimensional angle	$\frac{v_0}{2\sqrt{\Delta}}$
ratio of nonlinear index	α

Table 1. Table of the three relevant parameters of the problem.

2.II Parameter space representation

In order to obtain a detailed picture in parameter space we performed extensive numerical simulations covering a wide range of parameters of a single wavepacket approaching the interface from the left. In each case studied, the amount of power reflected was measured from the numerical simulation and the asymptotic state of the field was established.

We have observed three different regimes in each $\alpha = \text{const.}$ plane that we shall define as the linear, the intermediate nonlinear and the strongly nonlinear regimes. The linear regime is found for small values of $\frac{\eta_0}{\sqrt{\Delta}}$ (less than 0.05 in all cases), that is, when the incident wavepacket is broad. Here the evolution of the wavepacket as well as the amount of power being reflected was insensitive to α . Furthermore, the way the wavepacket is distorted at the interface before it splits into a reflected and a transmitted component is very reminiscent of the linear case, that is, we see an interference pattern at the interface that creates two dispersive waves, one is transmitted and the other is reflected (see Figure 6a).

Figure 6. Evolution of the wavepacket for three different values of $\frac{\eta_0}{\Delta}$; (a) 0.05, (b) 0.3, (c) and (d) 0.8. The angle of incidence in c(d) is below (above) the critical angle of total reflection. In all cases $\alpha = 1$.

The reflection coefficient follows very closely that of the linear case obtained in the previous chapter as we can observe in Figure 7a where we compare $|R(\frac{v_0}{\sqrt{\Delta}})|^2$ given by the Fresnel formula with the numerical measurements obtained for $\frac{\eta_0}{\sqrt{\Delta}} = 0.05$ and $\alpha = 1$. As we just said above, if we take a different α the results are essentially the same.

Once we start increasing the parameter $\frac{\eta_0}{\sqrt{\Delta}}$, the behavior is no longer linear, but we still see two different behaviors which we define as the intermediate nonlinear and the strongly nonlinear regimes. It is the strongly nonlinear regime where we developed a full theory which is presented in the next two chapters and we want now to specify both numerically and analytically the boundaries separating this regime from the intermediate one. To do that, we numerically solve (1.1) for a given initial wavepacket whose width η_0^{-1} is small enough so that it lies outside the linear regime. If the evolution of the field is as predicted by the theory developed in the next chapters, then such case would be in the nonlinear regime, otherwise it would correspond to an intermediate case. In order to understand what characterizes the nonlinear regimes for the different values of the parameters it is necessary to present here some of the predictions that will be proved and explained in detail in Chapters 3 and 4.

We begin by considering the case $\alpha > \frac{4}{9}$, since for these values of α the equivalent particle theory predicts that in the strongly nonlinear regime a single wavepacket will remain a coherent wavepacket when it crosses the interface with very small losses in radiation (of the order of $(1 - \sqrt{\alpha})^2$).

Figure 7. Plot of (a) the reflection coefficient $|R|^2$ versus scaled incident angle $\frac{v_0}{\sqrt{\Delta}}$ for $\alpha = 1$. Three cases, corresponding to different $\frac{\eta_0}{\sqrt{\Delta}}$ are shown and dashed is the curve of the Fresnel formula given in Chapter 1. And (b) $(\frac{\eta_0}{\sqrt{\Delta}}, \frac{v_0}{\sqrt{\Delta}})$ parameter space representation of the different regimes of behavior as a function of α . The picture shows boundaries for $\alpha = 1.0$, 0.9 and 0.5 .

Also, we find that a particular combination of the parameters, $S_0 = \frac{1}{4(\frac{\eta_0}{\sqrt{\Delta}})^2(1-\alpha)}$ plays an important role in the global dynamics and in particular, it gives an explicit relation, for a fixed α , between $\frac{\eta_0}{\sqrt{\Delta}}$ and $\frac{v_0}{\sqrt{\Delta}}$ that separates the cases of total reflection from those of total transmission, whenever $S_0 > \text{Max}(\frac{1}{\alpha}, \frac{2}{3\alpha^2})$. The expression is

$$\frac{4(1-\alpha^2)(\frac{\eta_0}{\sqrt{\Delta}})^2}{3\alpha^2} + \frac{(\frac{v_0}{\sqrt{\Delta}})^2}{4} = 1. \quad (2.1)$$

This represents an ellipse in the $(\frac{\eta_0}{\sqrt{\Delta}}, \frac{v_0}{\sqrt{\Delta}})$ plane, and we will now see that a section of it defines one of the boundaries of separation between the intermediate and the strongly nonlinear regimes.

From our numerical analysis we were able to conclude that if $\alpha < 1$, a beam that crosses the interface from left to right always behaves as predicted by the strongly nonlinear theory. It reshapes (becomes narrower) to conform to the new medium and gives off a very small amount of radiation. This is intuitively clear since the wavepacket is now in a medium with a higher nonlinearity so the focusing effect is more prominent. If the wavepacket was reflected and crossed the interface going again to the left hand medium, we observed in some, but not all cases, a dramatic change in the original wavepacket. There we saw bigger losses of power into radiation or even soliton break-up, both being contrary to the predictions given by the strongly nonlinear theory.

All these observations mean, first, that for $\frac{4}{9} < \alpha < 1$, and $\frac{\eta_0}{\sqrt{\Delta}} > 0.05$, if $\frac{v}{\sqrt{\Delta}}$ is such that the wavepacket is transmitted, then such case will lie in the strongly nonlinear regime. This means that one of the boundaries that separates the intermediate from the strongly nonlinear regimes has an analytical expression given, for each α , by equation (2.1). On the other hand if the angle of incidence is small enough

so that the peak of the wavepacket does not cross the interface the equivalent particle dynamics prevails. Therefore we expect a second boundary to the left of the section of the ellipse, solution of (2.1). For this boundary we found no analytical expression, since as we just said above, whenever $\frac{v}{\sqrt{\Delta}}$ is below the critical value of total transmission, the behavior of the wavepacket was sensitive to α . Instead we inferred its form from the numerical simulations. It is important to mention that although these boundaries as well as the boundary separating the linear from the nonlinear regimes were constructed from the numerical simulations, they are, as well as equation (2.1), the leading order terms of asymptotic expressions at the transition regions. Nevertheless, what we observed in all cases was that transitions, whether from linear to nonlinear or from the intermediate to the strongly nonlinear regimes were very sharp so that in fact, one has narrow bands separating the different regimes. Still we expect to find in the future, analytical arguments similar to equation (2.1) that define these other boundaries. Finally we observed that except for the case $\alpha = 1$, both boundaries separating the intermediate from the strongly nonlinear regimes intersect at a finite value of $\frac{\eta_0}{\sqrt{\Delta}}$. That is, for every α bigger than $\frac{4}{9}$, there is a critical $\frac{\eta_0}{\sqrt{\Delta}}$, that depends on α , above which the strongly nonlinear regime prevails for all angles of incidence.

Figure 7b shows the three regimes for three values of α , 1., 0.9 and 0.5 where we see that as α decreases, the nonlinear regime becomes more dominant. It is interesting to observe that the value of S at the intersection point of the two boundaries was found to be always close to 4 in all the different α 's that were studied. This suggests the possibility of defining the corners by the condition

$$\frac{1}{4(\frac{\eta_0}{\sqrt{\Delta}})^2(1-\alpha)} = 4.$$

This defines the maximum value of $\frac{\eta_0}{\sqrt{\Delta}}$ for which the intermediate regime can exist; it is

$$\left(\frac{\eta_0}{\sqrt{\Delta}}\right)_{max}^2 = \frac{1}{16(1-\alpha)}.$$

Note that $\left(\frac{\eta_0}{\sqrt{\Delta}}\right)_{max}^2 \rightarrow \infty$ as $\alpha \rightarrow 1$ consistent with the numerically observed fact that for the $\alpha = 1$ case, the intermediate regime extend to all possible values of $\left(\frac{\eta_0}{\sqrt{\Delta}}\right)^2$. An explanation of these phenomena could be that for this case, the incoming wavepacket can lose power into radiation or even break-up at the first crossing of the interface contrary to the $\alpha < 1$ case. So even for transmitted wavepackets, losses in power not predicted by the nonlinear theory were always observed (see Figure 6b). But as $\frac{\eta_0}{\sqrt{\Delta}}$ increases, the nonlinear regime dominates, and there we see for the $\alpha = 1$ case, either total reflection (Figure 6c) or total transmission (Figure 6d).

If $\alpha < \frac{4}{9}$, we will see in Chapter 4 how the description of the nonlinear regime becomes slightly more complicated because for these values of α , the transmitted wavepacket will split into several soliton components in the right hand medium. Each soliton will then evolve separately and, in some cases one of the new wavepackets may be reflected back to the left hand medium. In this case the asymptotic state will consist of several soliton components some transmitted, some reflected and radiation. Here, we have not found explicit boundaries separating the different regimes, although we can say that the behavior described here and explained in more detail in Chapter 4, is a good approximation of the true behavior for a wide range of parameters (see for example Table 2)

In conclusion, in this Chapter we provided the nonlinear analogue of the Snell's laws. Three regimes characterize the reflection and transmission laws for two nonlinear media separated by an interface. The linear regime follows the results of Chapter

1 and of the listed references. The nonlinear regime will be fully described in the next two Chapters although we already discussed some relevant results. So far we have characterized the intermediate regime by simply saying that in this regime the behavior is neither linear nor could it be described by an equivalent particle description. A full analysis of this regime should include equations of evolution for the continuous spectrum of the linear eigenvalue problem associated to NLS (see equation A.1) since what we observe is that at the interface, the perturbation excites other nonlinear modes instead of only perturbing the parameters of the incoming soliton.

CHAPTER 3

Equivalent Particle Theory

In this Chapter the dynamics of a single wavepacket near an interface for the case where the peak of the wavepacket stays in one side of the interface for all time will be described. In these situations, the interface only interacts with the tail of the beam and therefore the discontinuity in refractive index at the interface acts as a uniformly small perturbation. This interaction in turn, perturbs the propagation of the wavepacket which is described by following the location of its peak.

Throughout this work, we will refer to this analysis as the equivalent particle representation and by that we mean that the motion of the beam will be represented by the motion of an equivalent particle whose position at all times coincides with the position of the peak of the beam inside the dielectric.

Comparison of theoretical trajectories with those obtained by numerically integrating the full equation (1.1) will be shown, and we will indicate how to infer the stability properties of the z -independent solutions found in Chapter 1. The theory here developed also indicates under which conditions a wavepacket approaching the interface will cross it.

3.1 The wavepacket stays in the left hand medium

Suppose that a single wavepacket evolves in z such that its peak always remains in the left hand medium. It will soon be clear what this assumption means in terms of the shape of the envelope F of the TE field.

Let

$$F(x, z) = \sqrt{\frac{2}{\alpha_0}} A(x, \tau) e^{i(\beta^2 - n_0^2)z/(2\beta)}$$

where $\tau = \frac{z}{2\beta}$, whence $A(x, \tau)$ satisfies

$$i\frac{\partial A}{\partial \tau} + \frac{\partial^2 A}{\partial x^2} + 2|A|^2 A = VA. \quad (3.1)$$

The perturbation potential V is

$$V = \begin{cases} 0 & x < 0 \\ \Delta - 2(\alpha^{-1} - 1)|A|^2 & x > 0. \end{cases} \quad (3.2)$$

where $\Delta = n_0^2 - n_1^2$. We now see that by defining F and τ as it is done here the $A(x, \tau)$ satisfies the NLS equation with an extra term on the right hand side. In the absence of an interface, $\alpha = 1$, $\Delta = 0$ and therefore $V \equiv 0$. In that case it is well known that the general solution will consists of soliton and nonsoliton components. The amount of each can be determined from $A(x, 0)$ using inverse scattering theory. Given (3.1), it is easy to write the following exact expressions for the rate of change of the normalized dimensionless power

$$p = \int_{-\infty}^{\infty} AA^* dx$$

(the total power $P = \int_{-\infty}^{\infty} FF^* dx = \frac{2p}{\alpha_0}$), the average position

$$\bar{x} = p^{-1} \int_{-\infty}^{\infty} xAA^* dx$$

and the average velocity

$$v = ip^{-1} \int_{-\infty}^{\infty} (A \frac{\partial A^*}{\partial x} - A^* \frac{\partial A}{\partial x}) dx$$

of the normalized field $A(x, \tau)$.

We obtain,

$$\frac{dp}{d\tau} = 0 \quad (3.3a)$$

$$\frac{d\bar{x}}{d\tau} = v \quad (3.3b)$$

$$\frac{dv}{d\tau} = -2p^{-1} \int_{-\infty}^{\infty} \frac{\partial V}{\partial x} AA^* dx. \quad (3.3c)$$

These expressions are exact and involve no approximation. The equivalent particle description is obtained by making the assumption that the field moves collectively, in the sense that its local intensity AA^* is a function only of $x - \bar{x}(\tau)$. The motivation for this approximation is as follows. An arbitrary wavepacket, initially a long way from the interface, will break up into soliton, multisoliton and radiation components and the interaction of these different components can be treated separately. For simplicity, we are always going to assume that the incoming wavepacket is a single soliton for the medium in which it initially travels, which for the moment is the left hand medium,

$$A(x, \tau) = 2\eta_0 \operatorname{sech} 2\eta_0(x - \bar{x}) e^{i(\frac{v\bar{x}}{2} + 2\sigma)} \quad (3.4)$$

with $\frac{d\bar{x}}{d\tau} = v$, and $\frac{d\sigma}{d\tau} = -\frac{v^2}{8} + 2\eta_0^2$.

In Chapter 5, we will indicate what happens if the initial wavepacket consists of several such solitons bound together. Because we assume in this first section that the wavepacket will always reside in the left medium, the perturbation term on the right hand side of (3.1) is small, even for reasonably large nonlinear refractive index mismatches. We can capture the effects of the interface by allowing the soliton parameters, its velocity v and amplitude η_0 to vary slowly with τ . Before going to

the details of the theory for this problem it is important to say that the approach here presented relies on the fact that the right hand side of (3.1) is small. In this context it should be applicable in many physical situations where the NLS equation describes the evolution of some physical quantity and V represents perturbations such as material defects, higher order dispersion or nonlinear terms that could exist in the original equation but are neglected to first approximation. Indeed, it has proven to be succesful for example when V is proportional to x where, the solution found using this approach agrees with the exact solution which is obtained after a transformation that reduces the equation to an exact NLS [15]. This particular case will be discussed in more detail in Chapter 5 where an x dependent refractive index material is included in the design of an optical switch.

Another example of the usefulness of this approach is in the problem of evolution of pulses in optical fibers where a typical V is of the form $(i\beta_1 \frac{\partial^3}{\partial x^3} + \beta_2 \frac{\partial^4}{\partial x^4}) + |\frac{\partial A}{\partial x}|^2$ [28] and where the theory predicts that soliton pulses under this type of perturbations will decelerate at a rate proportional to the pulse width [16].

This approach has also been taken for situations in which V contains terms that accounts for gains or losses in which p is no longer a constant. To see some examples the reader is referred to [15]. Finally there is also the case of the dynamics of solitons under random perturbations where V would be a random function of x and/or τ [29].

Going back to the problem considered here, from the conservation of power, if $A(x, \tau)$ is given by (3.4), then the soliton width η_0 is constant but from (3.3b,c)

$$\frac{dv}{d\tau} = \frac{d^2 \bar{x}}{d\tau^2} = -2p^{-1} \int_{-\infty}^{\infty} \frac{\partial V}{\partial x} AA^* dx. \quad (3.5)$$

Since AA^* is simply a function of $x - \bar{x}(\tau)$, the force on the right hand side is a

function only of $\bar{x}$ and can be written as the gradient of a potential,

$$\frac{d^2 \bar{x}}{d\tau^2} = -\frac{\partial U_L(\bar{x})}{\partial \bar{x}} \quad (3.6)$$

where $U_L(\bar{x})$ is the integral in $\bar{x}$ of the right hand side of (3.5).

To determine the explicit form of U_L one has, from (3.2) that

$$\frac{\partial V}{\partial x} = \Delta \delta(x) + 2(1 - \alpha^{-1})\delta(x)|A|^2 + 2(1 - \alpha^{-1})H(x)\frac{\partial |A|^2}{\partial x}$$

where $\delta(x)$ and $H(x)$ are the Dirac delta function and the Heaviside function respectively; then

$$2p^{-1} \int_{-\infty}^{\infty} \frac{\partial V}{\partial x} AA^* dx = 2p^{-1}(\Delta |A(\bar{x})|^2 + (1 - \alpha^{-1})|A(\bar{x})|^4).$$

Therefore if one defines

$$M(|A|^2) = 2\Delta |A|^2 + 2(1 - \alpha^{-1})|A|^4$$

equation (3.5) becomes

$$\frac{d^2 \bar{x}}{d\tau^2} = -p^{-1}M(|A(\bar{x})|^2).$$

Finally by comparing (3.5) and (3.6) one has that

$$U_L(\bar{x}) = p^{-1} \int^{\bar{x}} M(|A(s)|^2) ds.$$

Using the soliton expression for $A(x, \tau)$, the equivalent potential is

$$U_L(\bar{x}) = \Delta(1 - (\alpha S_0)^{-1})\tanh(2\eta_0 \bar{x}) + (\Delta(3\alpha S_0)^{-1})\tanh^3(2\eta_0 \bar{x}) \quad (3.7)$$

where S_0 , the mismatch ration is given by $S_0 = \frac{1}{4(\frac{\eta_0}{\sqrt{\Delta}})^2(1-\alpha)}$. This parameter S_0 which depends on the power of the packet and on the nonlinear refractive index mismatch, characterizes the shape of U_L which together with the initial kinetic energy $\frac{v_0^2}{2}$, defines in a unique way the full evolution of the equivalent particle. The first result one can obtain from taking the first derivative of U_L ,

$$\frac{dU_L}{d\bar{x}} = 2\eta_0 \Delta \operatorname{sech}^2(2\eta_0 \bar{x}) [1 - (\alpha S_0)^{-1} \operatorname{sech}^2(2\eta_0 \bar{x})]$$

that a critical point of U_L ($\frac{dU_L}{d\bar{x}} = 0$) exists only if $\alpha S_0 < 1$ and that it is always a maximum. If on the other hand $\alpha S_0 \geq 1$, U_L is an increasing function of $\bar{x}$.

Equation (3.6) is simply Newton's second law of motion and we know that for a given set of initial conditions (position and velocity), there is a unique trajectory that the particle will follow. The set of all possible trajectories is presented in a phase plane ($\bar{x}$ vs v) plot, since they have an analytical expression,

$$v^2(\bar{x}) - v_0^2 = 2(U_L(\bar{x}_0) - U_L(\bar{x}))$$

where $(\bar{x}_0, v_0)$ are the initial position and velocity of the particle. Given this one can say that the motion of the particle and thereby the dynamics of the surface wave can be read off directly from the graphs of $U_L(\bar{x})$. Three graphs of $U_L(\bar{x})$ are drawn in Figure 8, the first two for cases where $\alpha S_0 < 1$ show a maximum for U_L whereas for the case $\alpha S_0 > 1$ the potential is always increasing. In all cases U_L is only defined for $\bar{x} \leq 0$ since the theory was developed for the left hand medium.

In order to start interpreting the results, let us review what the motion of the equivalent particle says about the evolution of the beam in the dielectric.

Figure 8. Graph of $U_L(\bar{x})$ for $\alpha S_0 < 1$ and $\alpha S_0 \geq 1$ respectively. In the first case, U_L has a maximum for some $\bar{x} < 0$ and in the second case U_L is an increasing function of $\bar{x}$.

First, let us remember that the position of the particle represents the location of the beam in the medium and if the equivalent particle approaches the position $\bar{x} = 0$ with velocity v_i this means that the beam is approaching the interface with an angle of incidence Ψ_i which is proportional to $\sin^{-1}v_i$. Similarly, when the particle stops and bounces back toward $\bar{x} = -\infty$ with speed $|v_r|$, this means that the beam is reflected at an angle Ψ_r proportional to $\sin^{-1}|v_r|$. Finally, if the beam crosses the interface, and this case will be discussed in the next Chapter, and departs from it at an angle Ψ_t , the equivalent particle would be in a trajectory going $+\infty$ with velocity proportional to $\sin\Psi_t$. This correspondence of an equivalent particle dynamics and the evolution of a single wavepacket will be carried out throughout this work whenever such wavepacket is not interacting with other beams. If this situation occurs some extra considerations need to be taken and we will do so in Chapter 5.

Let us start now by observing that particles initially far away from the interface but with initial velocity v_0 such that its total energy $\frac{v_0^2}{2} + U_L(x_0)$ is bigger than $\text{Max}(U_L)$ will cross the interface. We will deal with that case in section III of this chapter.

We can already find two global results from this part of the analysis. First, the fact that the theory shows that if $\alpha S_0 \geq 1$, there are no critical points in $U_L(\bar{x})$, that is, there are no corresponding steady state solutions in the left hand media, on the other hand if $\alpha S_0 < 1$, there is a maximum in $U_L(\bar{x})$ corresponding to the existence of an unstable steady state surface wave centered in the left medium and corresponding to the exact solution $F^-(x)$ in (2.7), which we now see is unstable. Also at the critical value $\alpha S_0 = 1$ the maximum is at $\bar{x} = 0$ corresponding to point D in Figures 4a and 4b.

We compare the trajectories of the equivalent particle given by solving (3.6) with the trajectory of the peak of the wavepacket found by numerically integrating the full equation (1.1). In Figure 9 we show in a phase plane diagram two trajectories of reflected wavepackets corresponding to $\alpha = 0.75$ and $\alpha = 0.25$. We observe that the agreement between theory and the numerical simulation is remarkable, even for larger nonlinear refractive index mismatch. Both trajectories exhibit a large phase shift, a nonlinear analogue of the Goos-Hänchen shift. Since, as we will see, the nonlinear Goos-Hänchen shift also exists for reflected beams that cross the interface, we will discuss this phenomenon only after considering all possible situations in the next Chapter.

3.II The wavepacket stays in the right hand medium

An exactly parallel analysis obtains when the wavepacket stays in the right hand medium except that, in this case, we will find that the critical point of the equivalent potential is a local maximum and therefore stable.

Let

$$F(x, z) = \sqrt{\frac{2}{\alpha_1}} A(x, \tau) e^{\frac{i(\beta^2 - n_1^2)z}{2\beta}}$$

then $A(x, \tau)$ satisfies (3.1) with

$$V(\bar{x}) = \begin{cases} -\Delta - 2(\alpha^{-1} - 1)|A|^2 & x < 0 \\ 0 & x > 0. \end{cases}$$

Figure 9. Phase plane of the equivalent particle (continuous curves) corresponding to wavepackets travelling in the left hand medium at all times. Here the interface is at $\bar{x} = 0$. Dotted and dashed trajectories were obtained by integrating equation (1.1) for $\alpha = 0.25$ and 0.75 respectively. In the second case, the agreement is so good that it is difficult to distinguish the real trajectory from the theoretical one.

If $A(x, \tau) = 2\eta_1 \operatorname{sech} 2\eta_1(x - \bar{x})e^{i(\frac{v\bar{x}}{2} + 2\sigma)}$, where $\frac{d\sigma}{d\tau} = -\frac{v^2}{8} + 2\eta_1^2$, the corresponding equations to (3.3a,b,c) are

$$\frac{dp}{d\tau} = 0$$

$$\frac{d\bar{x}}{d\tau} = v$$

$$\frac{dv}{d\tau} = -\frac{\partial U_R}{\partial \bar{x}}$$

where the equivalent potential defined for $\bar{x} \geq 0$ is

$$U_R(\bar{x}) = \Delta(1 - S_1^{-1})\tanh(2\eta_1\bar{x}) + \frac{\Delta}{3S_1}\tanh^3(2\eta_1\bar{x}) \quad (3.8)$$

where $S_1 = \frac{1}{4(\frac{\eta_1}{\sqrt{\Delta}})^2(1-\alpha)}$.

We show the graph of $U_R(\bar{x})$ in Figure 10 for three typical cases; the first one corresponds to the case $S_1 > 1$ and $U_R(\bar{x})$ is an increasing function of $\bar{x}$. Both 10b and 10c correspond to cases where $S_1 < 1$ and they show a minimum for $U_R(\bar{x})$ but in 8c $U_R(0) > U_R(\infty)$ and the opposite is true in 8b. As we can see the theory predicts that above a critical power which occurs when $S_1 < 1$, there is a minimum for $U_R(\bar{x})$ and corresponds to the existence of a stable steady state solution in the right hand medium. This solution is $F^+(\bar{x})$ given in (1.9) for the branch **ABC**.

So far these equivalent potentials have given us the stability properties of branches **DE** from the potential U_L obtained in the previous section, and now of **ABC** of Figures 4a, 4b. What we see is that at this level of approximation, this theory does not predict the existence of the solutions $F^+(x)$ on branch **CD**.

Figure 10. Graph of $U_R(\bar{x})$ for three cases. The first one is for $S_1 \geq 1$ and the second and third are for $S_1 < 1$, where there is always a minimum for some $\bar{x} > 0$. In (b), $U(0) < U(\infty)$ and the opposite is true in (c).

This is because in order to find an explicit form of the potential, we approximated the solution by an exact soliton; therefore one can expect that corrections on this approximation would produce slight changes in the potentials. Since as you already saw in the previous section, and you will see throughout this work, in terms of the global dynamics this approximation is good enough to describe and predict the behavior for a wide range of initial conditions on the wavepacket and parameters of the dielectrics, we will not attempt to construct a next order term for the equivalent potentials. Instead we will simply argue that based on the shape of the potentials and on numerical evidence, for powers of the wavepacket corresponding to the branch **CD** of Figure 4 the correction on U_R would be such that it would have a maximum between $\bar{x} = 0$ and the position of the minimum; and that would correspond to the $F^+(x)$ solutions of that branch which therefore would be unstable.

We will now study the global dynamics. As in the previous case, we tested theoretical trajectories with trajectories obtained from the numerical solution of equation (1.1) and in Figure 11 we show in a phase plane two trajectories; (i) with parameters such that the equivalent potential is that of Figure 10a in which case the particle approaches the interface from the right but with initial energy less than $U_R(0)$ and therefore circles the center and returns to $+\infty$ without crossing the interface and (ii) a particle oscillating about the minimum of $U_R(\bar{x})$. We observe that the agreement is excellent even in case (ii) where the wavepacket stays close to the interface for all times. The numerical and theoretical trajectories remained close even after five cycles. We note however that a particle with enough initial energy will cross the interface. Therefore it is clear now that we need to extend the theory if we want to consider the situation in which a wavepacket crosses the interface either from left to right or vice versa, and that will be done in the next Chapter.

Figure 11. Phase plane of the equivalent particle (continuous curves) corresponding to wavepackets travelling in the right medium at all times. Again the interface is at $\bar{x} = 0$ and $\alpha = 0.75$ for both trajectories.

CHAPTER 4

General Description of the Fully

Nonlinear Regime

In this Chapter we will address the question of the decomposition and subsequent dynamics of a wavepacket as it crosses the interfaces, both from left to right and from right to left.

If we are in the nonlinear regime shown in Figure 7b, then whenever the initial angle of incidence is big enough to guarantee that the wavepacket will cross the interface, it will do it over an effective propagation distance of $v^{-1}\eta_0$. In most of the cases this distance is small and under these conditions the assumption we take is that the wavepacket does not significantly change shape during the crossing. We will find that this approximation turns out to be excellent in almost all cases. However, once it has crossed it will experience the new medium with new parameters and therefore the wavepacket will reshape. We will extend the theory to explain the subsequent behavior. We start the Chapter by discussing this mode decomposition. We will see that if the nonlinear refractive index ratio α is greater than $\frac{4}{9}$, then a single wavepacket crossing from left to right evolves into a single wavepacket with a small amount of radiation left over. If the power of the incident beam is $P_I = \frac{8\eta_0}{\alpha_0}$, then the power in the transmitted beam is $P_T = \frac{16\eta_0}{\alpha_1}(\frac{1}{\sqrt{\alpha}} - \frac{1}{2})$ and the power in radiation which from the conservation of total power is equal to $P_I - P_T = \frac{8\eta_0}{\alpha_0}(1 - 2\sqrt{\alpha} + \alpha)$. Therefore $P_{Rad} = \frac{8\eta_0}{\alpha_0}(1 - \sqrt{\alpha})^2$. For this case, a global dynamics will be constructed in section 2, which is an extension of the theory developed in the previous Chapter and we will make a connection with the numerical results obtained in Chapter 2.

We will also make contact with previous work, where phenomena similar to that predicted in this work has been observed and left unexplained. We conclude the Chapter by defining a Nonlinear Goos-Hänchen Shift (NGS) which has a natural interpretation in the equivalent particle description.

4.I The wavepacket crosses the interface

We address the question of the decomposition and subsequent dynamics of a wavepacket as it crosses the interface. We observe in Chapter 3 that in the nonlinear regime generically the potentials $U_L(\bar{x})$ and $U_R(\bar{x})$ have finite slopes at $\bar{x} = 0$ and therefore the velocity v of the equivalent particle at the interface is finite and the crossing occurs quickly in a time of v^{-1} times the soliton inverse width η_0 . In Figure 7b, it is only in the regions inside the boundaries separating the nonlinear from the intermediate regimes that the crossing of the beam is slow enough that the assumptions made in this part do not apply. But when the crossing time is indeed short, the wavepacket does not significantly change shape during the crossing. However, once it has crossed, it will no longer be a perfect soliton of the new medium and it will consequently decompose into the normal modes associated with the new medium. We present the details of the analysis on how to determine this decomposition in Appendix B. Here we note the results. If the wavepacket with modulus

$$|F(x, z)| = \sqrt{\frac{2}{\alpha_0}} 2\eta_0 \operatorname{sech} 2\eta_0(x - \bar{x})$$

crosses the interface from left to right and if the speed of the equivalent particle at the time it crosses the interface is v , it will decompose into a sequence of N solitons with

$$\frac{1}{\sqrt{\alpha}} - \frac{1}{2} < N \leq \frac{1}{\sqrt{\alpha}} + \frac{1}{2} \quad (4.1)$$

and radiation. Observe that the number of new soliton wavepackets increases as the nonlinear refractive index of the right hand medium, α_1 , increases. The initial velocity of each of the new solitons is also v but the amplitudes are given by

$$\eta_{1r} = 2\eta_0\left(\frac{1}{\sqrt{\alpha}} - r + \frac{1}{2}\right), \quad r = 1, \dots, N \quad (4.2)$$

The amount of radiation can be calculated using the trace formula (see Appendix B),

$$\int_{-\infty}^{\infty} AA^* dx = 4 \sum_{r=1}^N \eta_{1r} - \frac{2}{\pi} \int_0^{\infty} \ln(1 - |R|^2) d\varsigma \quad (4.3)$$

where $R(\varsigma)$ is the reflection coefficient obtained from the scattering data of the linear eigenvalue problem (B.1). It measures the amount of radiation in the mode ς . In the limit of small $|R(\varsigma)|$, $-\ln(1 - |R|^2) \sim |R|^2$, and in the absence of soliton components, (4.3) is simply Parseval's identity.

Since the total power $P = \int_{-\infty}^{\infty} FF^* dx = \frac{8\eta_0}{\alpha_0}$ is conserved, we have that

$$\frac{8\eta_0}{\alpha_0} = \frac{8}{\alpha_1} \sum_{r=1}^N \eta_{1r} - \frac{4}{\alpha_1 \pi} \int_0^{\infty} \ln(1 - |R|^2) d\varsigma.$$

The difference between $\frac{8\eta_0}{\alpha_0}$ and $\frac{8}{\alpha_1} \sum_{k=1}^N \eta_{1r}$, is the amount converted into radiation. That is, from (4.2), $P_{Rad} = 8\frac{\eta_0}{\alpha_0}(1 - N\sqrt{\alpha})^2$. Observe that, as α tends to one, the amount of radiation produced is negligible. In fact, if $\alpha > \frac{4}{9}$, then $N = 1$ and $\eta_1 = 2\eta_0(\frac{1}{\sqrt{\alpha}} - \frac{1}{2})$. Now the amount of power $P_T = \frac{8\eta_1}{\alpha_1}$ carried by the new wavepacket in the right medium is $(2\sqrt{\alpha} - \alpha)P_I$ where $P_I = \frac{8\eta_0}{\alpha_0}$ is the power of the

original wavepacket. This means that the amount of power converted into radiation is

$$\frac{4}{\alpha_1 \pi} \int_0^\infty \ln(1 - |R|^2) d\zeta = (1 - \sqrt{\alpha})^2 \frac{8\eta_0}{\alpha_0}.$$

For the worst possible case $\alpha = \frac{4}{9}$, $(1 - \sqrt{\alpha})^2 \sim 0.11$, therefore in all cases where $N = 1$ the amount of power lost into radiation is less than 11 percent and goes to zero as α tends to one.

In the new medium, a single wavepacket will propagate as an equivalent particle in a potential defined by the material properties and its own power. However, if a bound state of N solitons is produced, the problem becomes one of N interacting particles in a composite potential. Eventually the bound state is broken into N individual packets each travelling in their own effective potentials. When does the break up occurs depends on the velocity of the equivalent particle at the interface. The smaller the velocity, the sooner the break up will occur, and in most cases studies here with the initial velocity always of the order of $\sqrt{\Delta}$, break up always occur immediately after crossing the interface. In particular, the packets with the smaller amplitudes will have higher S values and this may very well mean that they will reflect back into the left medium. The N -particle theory will not be carried out in this thesis.

At this point, however, we present some data which supports the accuracy of our decomposition computation. Table 2 compares the predicted and observed values of η_{1r} . The agreement is very good. Figure 12 shows two cases of a single wavepacket decomposing into multisoliton components in the right hand medium. In the first one two sizeable wavepackets are transmitted to, and remain in, the right hand medium and one very small one is being reflected.

α	Normalized Power of initial wavepacket	Theoretical η_i 's of transmitted wavapackets	Numerical transmitted three	η_i 's wavepackets different velocities	of for initial
0.125	2	$\eta_1 = 0.292$ $\eta_2 = 0.166$ $\eta_3 = 0.038$	Initial velocity		
			0.40	0.20	0.160
	4	$\eta_1 = 0.584$ $\eta_2 = 0.332$ $\eta_3 = 0.076$	0.33	0.313	
			0.50	0.46	
0.25	2	$\eta_1 = 0.375$ $\eta_2 = 0.125$	0.38	0.372	0.373
	4	$\eta_1 = 0.750$ $\eta_2 = 0.250$	0.65 0.26	0.630	
0.375	2	$\eta_1 = 0.425$ $\eta_2 = 0.050$	0.442	0.435	0.410
	4	$\eta_1 = 0.850$ $\eta_2 = 0.100$	0.790	0.770	
0.5	2	$\eta_1 = 0.457$	0.460	0.464	0.46
	4	$\eta_1 = 0.914$	0.860	0.870	
0.625	2	$\eta_1 = 0.482$	0.483	0.483	0.496
	4	$\eta_1 = 0.964$	0.93	0.93	

Table 2. Comparison between the predicted values of the amplitudes (η_i) of the solitons that are created after break-up in the right hand media, with the obtained values computed from the numerical integration of (1.1), for different initial velocities (angles of incidence). At the blank spaces, we were no able to numerically determine the values of the amplitudes of those solitons.

Figure 12. Soliton break-up in the right hand medium. In(a), an incoming packet breaks into three as it crosses the interface, with the smallest one being reflected back to the left medium where eventually disperses into radiation. In (b) the angle of incidence is bigger so that no wavepacket is reflected. In both cases, the amplitudes of each soliton are very close to the theoretical predictions. Here $P = 4$, $v_0 = 0.4$ in (a) and 1.0 in (b), $\Delta = 0.1$ and $\alpha = 0.125$.

Since the latter does not have enough power to create a soliton wavepacket in the left medium when it recrosses the interface, it eventually disperses as radiation and decays. In the second case, the angle of incidence of the incoming wavepacket is big enough to guarantee that all three transmitted wavepackets remain in the right hand medium.

When a wavepacket which has a perfect soliton shape for the right hand medium crosses to the left, the number of wavepackets N lies between $\sqrt{\alpha} - \frac{1}{2}$ and $\sqrt{\alpha} + \frac{1}{2}$. Since $\alpha \leq 1$, this means that a wavepacket entering from the right will decompose into at most one soliton whose initial velocity is v and whose amplitude η_0 is $2\eta_1(\sqrt{\alpha} - \frac{1}{2})$. Again note that as $\alpha \rightarrow 1$, $\eta_0 \rightarrow \eta_1$. The trace formula now gives

$$\frac{8\eta_1}{\alpha_1} = H(\sqrt{\alpha} - \frac{1}{2}) \frac{8\eta_0}{\alpha_0} - \frac{4}{\alpha_0 \pi} \int_0^\infty \ln(1 - |R|^2) d\zeta$$

where H is the Heaviside function. For $\alpha > \frac{1}{4}$, the amount of radiation created is proportional to $\frac{(1-\sqrt{\alpha})^2}{\alpha}$ which is again small for α close to unity. In Figure 13, we show a situation in which $\alpha = \frac{1}{4}$ which as we said above, is too small to create a soliton packet in the left medium. The packet of radiation which is created disperses and decays.

4.II Multiple crossings

Although the dynamics is more complicated when $0 < \alpha < \frac{4}{9}$, one can carry out the decomposition calculation of each packet as it crosses the interface and follow the dynamics of the different components. However, care has to be taken because the different new components can interact with each other and produce phase shifts. Nevertheless, in principle, the total dynamics can be reconstructed by piecing together the trajectories of each wavepacket.

Figure 13. Evolution of a wavepacket that initially is in the right hand medium and once it crosses the interface, it loses its soliton shape as predicted by the theory. Here $\tau = \frac{z}{2\beta}$. The parameters are the same of figure 13 except that the wavepacket is initially in the right hand medium and $v_0 = -1.0$

However, if the nonlinear refractive index mismatch α is greater than $\frac{4}{9}$, a soliton wavepacket in one medium gives rise to a single soliton wavepacket in the other and very little radiation is produced at each crossing (see for example Figure 14). Therefore, in this case we can construct a composite theory which describes the global dynamics of a single soliton wavepacket. To do this we simply assume no radiation losses and piece together the left and the right potentials $U_L(\bar{x})$ and $U_R(\bar{x})$ by defining the composite potential

$$U(\bar{x}) = H(\bar{x})U_R(\bar{x}) + (1 - H(\bar{x}))U_L(\bar{x}) \quad (4.4)$$

where H is the Heaviside function and $U_L(\bar{x})$ and $U_R(\bar{x})$ are given by the expressions (3.7) and (3.8) derived in the previous Chapter. We emphasize that if P is the initial power of the wavepacket, neglecting radiation losses effectively means that in (4.4), $\eta_0 = \frac{\alpha_0 P}{8}$ in U_L and $\eta_1 = \frac{\alpha_1 P}{8}$ in U_R , or $\eta_1 = \frac{\eta_0}{\alpha}$. From this we see that the parameter S evaluated in the right hand medium is equal to α^2 times that value evaluated in the left hand medium; i.e. $S_1 = \alpha^2 S_0$. This is a better approximation to the true values of η than keeping the same η for both U_L and U_R . To see this, we calculate the difference between $\eta_{1exact} = 2\eta_0(\frac{1}{\sqrt{\alpha}} - \frac{1}{2})$ and the two possible approximations $\eta_1 = \eta_0$ and $\eta_1 = \frac{\eta_0}{\alpha}$. In the former, this difference is proportional to $1 - \sqrt{\alpha}$; in the latter it is proportional to $(1 - \sqrt{\alpha})^2$. In (4.4), $U(\bar{x})$ is continuous but its derivative, the equivalent force, has a small discontinuity at $\bar{x} = 0$. The motion of the particle and therefore the dynamics of the surface wave along any part of the waveguide can be described directly from the graph of $U(\bar{x})$ which is drawn, for all regimes in the S vs. α plane, $\alpha > \frac{4}{9}$ and S corresponding to the right medium in Figure 15. In cases 15c, and d we have a corner at $\bar{x} = 0$.

Figure 14. Full evolution of a transmitted wavepacket for $\alpha = 0.75$. The initial and final wavepackets show how it reshapes and that little radiation is produced.

Figure 15. Graphs of $U(\bar{x})$ for regions of different behavior in (S_1, α) plane where $\alpha > \frac{4}{9}$. Observe that there are not stable critical points for negative $\bar{x}$. In c and d, the maximum is always a corner at $\bar{x} = 0$. In e, f and g, there is a maximum for $\bar{x} < 0$. In d, e and f an equivalent particle which enters from the left with enough energy to overcome the maximum will travel to $\bar{x} = \infty$ and not return. On the other hand in b, c and g the particle (or the wavepacket) may enter the right medium, remain there for awhile, and eventually return to the left medium.

These regions in parameter space correspond to the interval (P_C, P_D) of Figures 4a, 4b, and as we argued in 3.II, at the next order of the perturbation theory, one will find that the corner at $\bar{x} = 0$ both smooths and moves slightly to the right.

We again tested our theory by comparing it with real trajectories obtained by numerically integrating the full equation (1.1). This comparison was comprehensive. Several comparisons of the true trajectory (formed by numerically integrating the full equation (1.1) and graphing the position of the wavepacket peak versus its velocity) and the equivalent particle trajectory were made at each of the different equivalent potential shapes. In Figures 16 and 17, we show two potentials and phase planes, one corresponding to values of S_0 and α so that we are in region (15e) of parameter space and another corresponding to region (15g). The two cases are important because as we show in Figure 16b for the region (15e) and Figure 17b for the region (15g), they contain all trajectory types: (1) a packet entering from the left and failing to make it over the unstable maximum therefore ending up at $\bar{x} = -\infty$, (2) a packet entering from the left with enough energy to overcome U_{max} and end up at $\bar{x} = \infty$, (3) a packet entering from the right with enough energy to overcome U_{max} ending up at $\bar{x} = -\infty$, (4) of Figure 17a where a packet entering from $\bar{x} = \infty$ circles the center of the phase plane and returns to $\bar{x} = \infty$. For this last trajectory, according to the theory, the packet can cross the interface twice (first from right to left, then from left to right). In the case of a potential like that of region (15g), one has instead of this last trajectory, one that starts at $-\infty$ circles the center of the phase plane and returns to $-\infty$ (trajectory 4 of Figure 17b). And (5) a packet oscillating about the stable center. For the other potentials of Figure 15, every beam trajectory will be similar to one shown in these two cases. In Figure 18 we show the evolution of the beam corresponding to trajectory (4) on 17b.

Figure 16. Potential and phase plane for values of the parameters $\alpha = 0.75$, $\Delta = 0.1$ and $P = 2$. We show all typical trajectories. Observe that the agreement is better when the wavepacket does not cross the interface, which in this case are the oscillatory trajectory in the right medium and the trajectory that comes from $-\infty$ and is reflected before crossing the interface. In all other cases the agreement is very good.

Figure 17. Potential and phase plane for values of the parameters $\alpha = 0.75$, $\Delta = 0.1$ and $P = 1.52$. The difference with the previous case of Figure 16 is that now we can have a wavepacket coming from $-\infty$, crosses the interface and is reflected ending up at $-\infty$ (trajectory 2), which is the opposite of trajectory 4 of Figure 16. Again the agreement is good in all cases but better when the wavepacket does not cross the interface.

Figure 18. Evolution of the wavepacket corresponding to trajectory 2 of Figure 17. We can see that every time it crosses the interface, the wavepacket reshapes and no radiation can be detected.

We can see that each time it crosses the interface it reshapes. Since radiation losses are very small, therefore we can expect good agreement between theory and the numerical simulations and indeed from Figures 16b and 17b, we find that the agreement is excellent.

We are now in a position to explain the analytic expression that defines one of the boundaries separating the nonlinear from the intermediate regimes (equation (2.1)). It was argued that the numerical simulations gave a value of S_0 bigger than $Max(\frac{1}{\alpha}, \frac{2}{3\alpha^2})$ for all cases where the transition from intermediate nonlinear to strongly nonlinear regimes happened. What this means is that at transition, in the strongly nonlinear regime the equivalent potential is like that of Figures 15b or 15c where one has, in both cases, that there is no local maximum (although there is a corner at $\bar{x} = 0$ in 15c) and that the global maximum of U is $U(\infty)$. For those potentials of Figures 15b and 15c, the critical initial velocity v_0 of the equivalent particle (the critical angle of incidence of the beam) which gives the trajectory that separates trajectories that end up at ∞ from those that are reflected is

$$\frac{v_0^2}{2} = U(\infty) - U(-\infty).$$

This condition gives for a fixed α a relation between $\frac{v_0}{2\sqrt{\Delta}}$ and η_0 which based on the numerical simulations, defines the right boundaries of Figure 7 separating the intermediate nonlinear from the strongly nonlinear regimes. After substitution of $U(\bar{x})$ in this expression one obtains equation (2.1).

4.III Relation with previous work

We have shown that the equivalent particle theory models the numerical simulations extremely well in the cases where either (a) the wavepacket stays in one

medium or (b) the nonlinear mismatch α is greater than $\frac{4}{9}$. However, as we shall now see, the theory still produces a useful qualitative picture even when the medium on the left is linear i.e. $\alpha = 0$. This is an important case since it has been studied previously, and in particular we now discuss the numerical results found in [22] in which Tomlinson, et.al. investigate the dynamics of a series of Gaussian wavepackets all carrying the same power, which are propagated from the linear medium towards the interface at the same angle of incidence, for a range of values of α_1 .

To make contact with our theory, one can now recognize that once the packet crosses the interface in the nonlinear medium it will decompose into soliton and radiation components just as in the nonlinear-nonlinear case. The difficulty, which is only technical, is to determine what these components are. Instead we will make the rather crude assumption that the Gaussian becomes a single soliton in the right hand media with no radiation losses. Once we have a soliton in the nonlinear medium, we know we can determine its further evolution in that medium using the equivalent particle dynamics. To do that one needs to determine for each case studied by Tomlinson et.al., the equivalent potential in the right hand medium U_R and the local angle of incidence at the interface (or the velocity of the equivalent particle at $\bar{x} = 0$). From the power of the incident Gaussian beam P_g and the nonlinear refractive index coefficient of the right hand medium α_1 one can compute the soliton parameter η_1 ; $\eta_1 = \frac{\alpha_1 P_g}{8}$. This determines S_1 , thus U_R in the right hand medium. Finally, we determine the velocity of the equivalent particle at the interface from the angle of incidence given in [22]. Under the above assumptions, each beam trajectory shown by Tomlinson et. al. corresponds, in terms of our theory, to a trajectory of an equivalent particle. Each having the same initial velocity but moving on a different potential U_R due to the change in α_1 from case to case. Looking

at Figure 10, whether the beam is transmitted or eventually reflected back to the linear medium, simply depends respectively on whether the initial kinetic energy of the equivalent particle $K = v_0^2/2$, which is the same in all trajectories, is bigger or smaller than $U_R(\infty) = \Delta(1 - \frac{2}{3S_1})$.

For the linear/nonlinear ($\alpha = 0$) case

$$S_1^{-1} = 4 \frac{\eta_1^2}{\Delta} = \frac{\alpha_1^2 P_g^2}{16\Delta}.$$

When we substitute this expression in $U(\infty)$ we have that

$$U(\infty) = \Delta(1 - \frac{\alpha_1^2 P_g^2}{24\Delta})$$

which shows that $U(\infty)$ is a decreasing function of α_1 and $U(\infty) \rightarrow -\infty$ as $\alpha_1 \rightarrow \infty$. Therefore, there is a value of α_1 , α_{1c} , for which $K = U(\infty)$, and in terms of α_1 , the packets will be reflected for values of $\alpha_1 > \alpha_{1c}$ ($K < U(\infty)$); for values for which $\alpha_1 < \alpha_{1c}$ they will be transmitted. This is in complete agreement with the behavior of the trajectories in [22]. Furthermore, using the data of that reference, we estimated the value of α_{1c} and find it to be 0.016. This is an upper bound to the value of 0.0112 found by Tomlinson et al. All but three curves in [22] (curves C, D, E), indeed correspond to the particle trajectories whose potentials are like that of Figures 10a or 10b. Notice that if α_1 is close to α_{1c} , the particle (the self-focused beam), stays a long time in the neighborhood of the interface and may interact strongly with it. In such circumstances, the cumulative radiation loss leads to a continual decrease of the soliton wavepacket η_1 and this in turn leads to higher values of S_1 . However, as S_1 increases, the potential $U_R(\bar{x})$ becomes more like (10a) or (10b) for which cases we know the packet is either reflected or in very special

cases trapped. This scenario is in principle consistent with the three trajectories in [22] mentioned above, which the authors refer to as unexplained behavior.

We can also interpret the results of Akhmediev et al. [25]. They did a numerical investigation of the stability properties of the stationary solutions in a linear/nonlinear case. In particular they found a sensitive behavior when they perturbed the unstable branch (C_0, B_0) of the F^+ solution close to C_0 of Figure 5. If they choose an initial perturbation such that the power P was slightly greater than that of F^+ , they found the packet ended up at the position of a stable stationary solution. On the other hand, if the perturbed F had slightly less power than F^+ , the packet was ejected into the linear medium. This sensitive behavior is a consequence of the fact that for P near C_0 in Figure 5, the corresponding value of S_1 in the soliton picture is close to unity, the dividing parameter between a and b in Figure 10. Therefore, if we chose a particular F whose power is slightly greater (less) than P_0 , S_1 is less (greater) than one. Figure 6a (6b) obtains and the packet will move towards the stable center (will asymptote to $-\infty$).

4.IV The Nonlinear Goos-Hänchen Shift

As it was said in Chapter 1, for linear dielectrics with an interface, it was experimentally observed that a reflected wavepacket suffered a shift from the position predicted by the geometrical optics approximation. For the nonlinear case, and with the equivalent particle description there is a natural analog of the Goos-Hänchen shift. It is simply the difference in the asymptotic value of the z location of the equivalent reflected particle as $\bar{x} \rightarrow -\infty$ and the location of a free particle that bounces at the interface. From this definition the analytical expression for the nonlinear Goos-Hänchen shift δz is,

$$\delta z = 2\beta \left[\int_{-\infty}^{x_f} (v^{-1}(x) - v_0^{-1}) dx + \frac{x_f}{v_0} \right] \quad (4.5)$$

where

$$v^2(x) = v_0^2 - 2U(\bar{x}) + 2U(-\infty)$$

and x_f is the penetration distance, namely the value of $\bar{x}$ at which $v_0^2 = 2(U(\bar{x}) - U(-\infty)) = 2K$. The graph of δz vs. K is shown in Figures 19a and 19b; in the first case, which is typical for potentials with $U_{max} < U(\infty)$, we have two singularities; a logarithmic singularity for $K_1 = U_{max} - U(-\infty)$ which corresponds to the case of a beam that stays trapped close to the unstable nonlinear surface wave. In a true experiment one of course would expect a large shift for this case instead of an infinite one. The second singularity is algebraic for $K_2 = U(\infty) - U(-\infty)$ which corresponds to the critical trajectory that separates transmission from total reflection. For the second type of potential where $U(\infty) < U_{max}$ only the logarithmic singularity exists for $K = U_{max} - U(-\infty)$.

For the particular case where $\alpha = 1$, $U(\bar{x}) = \Delta \tanh(2\eta\bar{x})$ and one can solve for δz exactly; the result is

$$\delta z = \frac{\beta}{\eta v_0} \left(\ln\left(\frac{v_0}{\sqrt{\Delta}}\right) + \frac{\frac{v_0}{\sqrt{\Delta}}}{\sqrt{4 - \left(\frac{v_0}{\sqrt{\Delta}}\right)^2}} \tan^{-1}\left(\frac{\frac{v_0}{\sqrt{\Delta}}}{\sqrt{4 - \left(\frac{v_0}{\sqrt{\Delta}}\right)^2}}\right) \right)$$

and in Figure 20 we show a comparison between this expression and the resulting shift of the numerical reflected trajectories for this choice of α .

Figure 19. Plot of $\Delta\tau = \frac{\delta z}{2\beta}$ the nonlinear Goos-Hänchen shift as a function of the total initial energy $E = \frac{v_0^2}{2} + U(-\infty)$, of the equivalent particle. Two typical cases occur; case (a) when U is like that of Figures 15c, or g, and (b) when U is like 15b, d, e, or f.

Figure 20. Comparison between the theoretical Goos-Hänchen shift (solid line) and the numerical ones (x) for reflected wavepackets as a function of the total initial energy E of the equivalent particle. Here $\alpha_0 = \alpha_1 = 2$, $\Delta = 0.1$ and $\eta = 0.5$. Again $\Delta\tau = \frac{\delta z}{2\beta}$.

CHAPTER 5

Multiple Interfaces and Applications

Our goal in this Chapter is to show how the theory may be used as a building block in the design of devices. We have already shown in the introduction how the single interface model can be used as a spatial scanner (Figure 1). We can now understand why incident beams of different powers $P = \frac{8\eta_0}{\alpha_0}$ end up at different places at the end of the waveguide, since from equation (4.5), for the reflected beams the shift of the beam is η dependent whereas for the transmitted beams, the angle of transmission given in terms of the final velocity of the equivalent particle $v_f = \sqrt{v_0^2 + 2U(-\infty) - 2U(\bar{x})}$ is also η dependent. We will show here two other possible applications although there are many others. The first is to destabilize stable surface waves propagating parallel to an interface (like F^+ given in Chapter 1) through a collision with an incident auxiliary packet. We will show how to predict its further evolution after the collision. The second application goes even further and discusses means of trapping the displaced beam at a neighboring interface. Before discussing these ideas in detail, we must first extend the theory slightly in order to cover (I) multiple interfaces and (II) wavepacket collisions and bending. In the first section the case of a symmetric waveguide with two interfaces will be studied. There, we present the different potentials as function of the ratio of the wavepacket width to distance between interfaces. We conclude the Chapter by showing numerical simulations of the behavior of a two soliton initial condition in the single interface problem. We observed in all cases splitting of the original bound wavepacket, and we argue why this happens based on the equivalent particle theory. We also suggest the approach to take to fully understand the global dynamics for this case.

5.I Multiple Interfaces

Consider a geometry for a waveguide where three different media are separated by two interfaces at $x = x_1$ and $x = x_2$. The refractive indices in the various media are given by

$$n^2(x, |F|^2) = n_j^2 + \alpha_j |F|^2, \quad j = 1, 2, 3$$

where we only consider the case in which the ratio of smaller to bigger α_j on two neighbouring materials is always bigger than $\frac{4}{9}$ so that a single wavepacket prevails in all media.

To construct the equivalent potential we proceed as in the previous Chapters by assuming first that a wavepacket is propagating in one medium, say medium two, and does not cross any interface. We define the field F to be of the form

$$F(x, z) = \sqrt{\frac{2}{\alpha_2}} A(x, \tau) e^{i(\beta^2 - n_2^2)z}$$

then $A(x, \tau)$ satisfies equation (3.1) where now V is

$$V(x) = \begin{cases} \Delta_2^{(1)} - 2(\alpha_{12} - 1)|A|^2 & x_0 < x < x_1 \\ 0 & x_1 < x < x_2 \\ \Delta_2^{(3)} - 2(\alpha_{32} - 1)|A|^2 & x_2 < x < x_4 \end{cases}$$

where $\Delta_2^{(j)} = n_2^2 - n_j^2$, $\alpha_{j2} = \frac{\alpha_j}{\alpha_2}$, $x_0 = -\infty$, and $x_4 = \infty$.

In (3.1) we use for A a soliton form, following the steps of Chapter 3, the new effective potential for and equivalent particle dynamics of a wavepacket travelling in medium two is

$$\begin{aligned}
U_2(\bar{x}) = & -\Delta_2^{(1)}\left(1 - \frac{1}{S_2^{(1)}}\right)\tanh 2\eta_2(\bar{x} - x_1) - \frac{\Delta_2^{(1)}}{3S_2^{(1)}}\tanh^3 2\eta_2(\bar{x} - x_1) \\
& + \Delta_2^{(3)}\left(1 - \frac{1}{S_2^{(3)}}\right)\tanh 2\eta_2(\bar{x} - x_2) + \frac{\Delta_2^{(3)}}{3S_2^{(3)}}\tanh^3 2\eta_2(\bar{x} - x_2) + C_2
\end{aligned} \quad (5.1)$$

where $S_2^{(j)} = \frac{\Delta_2^{(j)}\alpha_2}{4(\alpha_j - \alpha_2)\eta_2^2}$, $\eta_2 = \frac{\alpha_2 P^2}{8}$, and P the total power.

If all the nonlinear mismatches are close to one, we neglect radiation as we did in the case of the single interface, and write a composite potential to describe the global dynamics of a wavepacket propagating in such a waveguide,

$$U(\bar{x}) = (1 - H(\bar{x} - x_1))U_1(\bar{x}) + (H(\bar{x} - x_1) - H(\bar{x} - x_2))U_2(\bar{x}) + H(\bar{x} - x_2)U_3(\bar{x})$$

where H is the Heaviside function,

$$\begin{aligned}
U_j(\bar{x}) = & \sum_{l=1, l \neq j}^3 [\Delta_j^{(l)}\left(1 - \frac{1}{S_j^{(l)}}\right)(\tanh 2\eta_j(\bar{x} - x_{l-1}) - \tanh 2\eta_j(\bar{x} - x_l)) \\
& + \frac{\Delta_j^{(l)}}{3S_j^{(l)}}(\tanh^3 2\eta_j(\bar{x} - x_{l-1}) - \tanh^3 2\eta_j(\bar{x} - x_l)) + C_j]
\end{aligned} \quad (5.2)$$

and $\Delta_j^{(l)} = n_j^2 - n_l^2$, $\eta_j = \frac{\alpha_j P}{8}$, $S_j^{(l)} = \frac{\Delta_j^{(l)}\alpha_j}{4(\alpha_j - \alpha_l)\eta_j^2}$. We choose the constant C_j in (5.2) so that $U(\bar{x})$ is continuous at each interface.

In general, if we have N interfaces located at x_j , $j = 0, 1, 2, \dots, N+1$ with $x_0 = -\infty$, and $x_{N+1} = \infty$, the composite potential is

$$U(\bar{x}) = \sum_{j=1}^{N+1} (H(\bar{x} - x_{j-1}) - H(\bar{x} - x_j))U_j(\bar{x}) \quad (5.3)$$

where $U_j(\bar{x})$ is as in (5.2) where now l runs from 1 to $N + 1$, omitting $l = j$.

We now consider two cases involving multiple interfaces. First, we have that if the wavepacket width is small compared to the distance between interfaces, then if the wavepacket is close to one interface, say x_j , the equivalent potential is of the form,

$$\begin{aligned}
 U(\bar{x}) \sim & (1 - H(\bar{x} - x_j)) \left[\Delta_j^{(j+1)} \left(1 - \frac{1}{S_j^{(j+1)}} \right) \tanh 2\eta_j(\bar{x} - x_j) + \frac{\Delta_j^{(j+1)}}{3S_j^{(j+1)}} \tanh^3 2\eta_j(\bar{x} - x_j) \right] \\
 & + H(-x_j) \left[-\Delta_{j+1}^{(j)} \left(1 - \frac{1}{S_{j+1}^{(j)}} \right) \tanh 2\eta_{j+1}(\bar{x} - x_j) - \frac{\Delta_{j+1}^{(j)}}{3S_{j+1}^{(j)}} \tanh^3 2\eta_{j+1}(\bar{x} - x_j) \right] + G(\bar{x})
 \end{aligned} \tag{5.4}$$

where $G(\bar{x})$ accounts for the contributions from (5.2) not included in (5.3) all of which are slowly varying functions of $\bar{x}$ or in other words $|\frac{dG}{d\bar{x}}| \ll 1$. This is true because when the wavepacket is narrow and propagating in the neighborhood of a particular interface, other interfaces in effect, do not perturb its evolution. In terms of the equivalent particle, what we are saying is that the local potential in the neighborhood of an interface is like that of the single interface case, thus the composite potential can be constructed for this particular case by simply 'patching' potentials like that of Figure 15.

In Figures 21 and 22, we examine two situations, each with three interfaces, and in both cases there are three minima in $U(\bar{x})$ corresponding to three stable stationary solutions. We confirmed the existence of these solutions by numerically integrating the full equation, with wavepackets centered at the location of the local minima used as initial conditions. Figure 23 shows one of these solutions when the potential is shown in Figure 21.

Figure 21. Potential and phase plane for a nonlinear waveguide with three interfaces located at $x = -12$, $x = 0$ and $x = 12$. In (b) we show three typical trajectories of the equivalent particle corresponding to the evolution of the bigger wavepacket shown in figures 26 a, b and c where an incident second wavepacket collides with him. For the case shown in figure 26a, the corresponding trajectory is **ABCD** where the particle oscillates between the first and second interface. In the second one **AB'C'D'**, the particle ends up at ∞ (see figure 26b), and in the third one, **AB''C''D''B''**, the particle oscillates about the center to the right and near the second interface (figure 26c).

Figure 22. Potential and phase plane for case study two, where now the local maximum of U near the third interface is bigger than the local maximum near the second interface. In the phase plane we show the trajectory of the equivalent particle of a packet initially propagating parallel to the second interface, after a second wavepacket collides with it and shifts its position to B'' (see figure 27). After that, the particle starts moving to the right until it has another collision at C'' , now in opposite directions, with the second particle ending up at $-\infty$.

Figure 23. Evolution of a wavepacket of power $P = 2$ propagating parallel to the second interface, where the equivalent potential has a minimum (see figure 21). As we can see this wavepacket is stable as predicted by the theory.

The exact form of the z independent solutions for this case are also sech-like solutions in each medium with the particular parameters obtained from the continuity conditions.

One can also expect that in this approximation if we put two equivalent particles, each in a different well of the composite potential, they will not interact with each other and remain in their original positions. Figure 24 shows one such case where two narrow wavepackets propagate undisturbed, each near a minimum of the equivalent potential. Although we did not consider other cases with three or more packets the same should be true. In fact, we have found new multipeak stable stationary wavepackets in a multiple interface waveguide by simply locating each peak at the minimum of a composite potential.

The approximation used above fails when the distance between interfaces becomes of the same order of the width of the wavepacket. We present here as an example, a symmetric waveguide with two interfaces one at $x = -x_1$ and the other at $x = x_1$. For $x < -x_1$ and $x > x_1$, the nonlinear index of refraction is $n^2 = n_1^2 + \alpha_1^2|F|^2$, whereas for the dielectric between the two interfaces n^2 is $n_0^2 + \alpha_0|F|^2$, and we will only consider the case $n_0^2 > n_1^2$, $\alpha_0 < \alpha_1$, since any other combination could be studied in a similar way. Also we only consider narrow wavepackets and $\frac{\alpha_0}{\alpha_1} > \frac{4}{9}$ so that again we neglect radiation. This problem is similar to the thin film waveguide where $\alpha_0 = 0$ which has been studied before [26]. To characterize the different behavior of the composite potential a new parameter will be introduced; it is $\eta_0 d$, where $d = 2x_1$ is the distance between the two interfaces. We will now show how $U(\bar{x})$ behaves as a function of this parameters and how the stability properties of the stationary surface waves change as $\eta_0 d$ changes. The fact that we are considering a symmetric waveguide simplifies considerably the analysis.

Figure 24. Two narrow wavepackets propagating each near and parallel to an interface. In the equivalent potential (see figure 21), a minimum of U is located at the peak of each wavepacket. The power of the field $P = 4.0$ is twice the power of the single peak packet of the previous figure.

First, as one can expect, the composite potential is symmetric, therefore we only need to look at its behavior for $\bar{x} \geq 0$ and it has a simpler form than the general case;

$$U_s(\bar{x}) = (1 - H(\bar{x} + x_1) + H(\bar{x} - x_1))U_{s1}(\bar{x}) + (H(\bar{x} + x_1) - H(\bar{x} - x_1))U_{s0}(\bar{x})$$

where,

$$U_{si} = \Delta_s \left(1 - \frac{1}{S^{(i)}}\right) [\tanh 2\eta_i(\bar{x} + x_1) - \tanh 2\eta_i(\bar{x} - x_1)] \\ + \frac{\Delta_s}{3S^{(i)}} [\tanh^3 2\eta_i(\bar{x} + x_1) - \tanh^3 2\eta_i(\bar{x} - x_1)] + C_i, \quad i = 0, 1$$

$$\eta_i = \frac{\alpha_i P}{8}, \quad S^{(i)} = \frac{\Delta_s \alpha_i}{4(\alpha_0 - \alpha_1)\eta_i^2} \quad \text{and} \quad \Delta_s = n_1^2 - n_0^2.$$

We observe first that from its symmetry properties, U_s must have a critical point at $\bar{x} = 0$. Whether it is a maximum or a minimum depends on $\frac{d^2 U_s}{d\bar{x}^2}$. We have that,

$$\frac{d^2 U_s}{d\bar{x}^2}(\bar{x} = 0) = -8\eta_0 \Delta_s \tanh(\eta_0 d) \operatorname{sech}^2(\eta_0 d) \left[1 - \frac{2}{S^{(0)}} \operatorname{sech}^2(\eta_0 d)\right]$$

therefore $\bar{x} = 0$ is a minimum (maximum) if $S^{(0)} > (<) 2 \operatorname{sech}^2 \eta_0 d$.

For $0 < \bar{x} < x_1$, there is at most one critical point of U_s (this result was confirmed numerically since we have no proof of it). Instead of explicitly finding it, when it exists, it is easier only to proof its existence by looking at the behavior of $\frac{dU_s}{d\bar{x}}$ at the interface x_1^- . After simplifications,

$$\frac{dU_s}{d\bar{x}} = -2\eta_0 \Delta_s (1 - \operatorname{sech}^2 2\eta_0 d) \left(1 - \frac{1 + \operatorname{sech}^2 2\eta_0 d}{S^{(0)}}\right),$$

therefore $\frac{dU_s}{d\bar{x}} \leq (>)0$ if $S^{(0)} \geq (<)(1 + \text{sech}^2 2\eta_0 d)$. This result combined with the behavior of U_s at $\bar{x} = 0$ gives as necessary conditions for having two symmetrical critical points inside the interfaces that $S^{(0)} > 1 + \text{sech}^2 2\eta_0 d$ and $S^{(0)} < 2\text{sech}^2 \eta_0 d$, then the critical points are minima; or that $S^{(0)} < 1 + \text{sech}^2 2\eta_0 d$ and $S^{(0)} > 2\text{sech}^2 \eta_0 d$ with the critical points being maxima.

The behavior of U_s in the outer media can be analysed similarly. We have observed from numerical evaluations of U_s for different parameter values that again there is at most one critical value of U_s for $\bar{x} > x_1$. Since U_s is an increasing function of $\bar{x}$ as $\bar{x} \rightarrow \infty$, will infer the existence of the critical point from the behavior of U_s at x_1^+ . We have that,

$$\frac{dU_s}{d\bar{x}}(x_1^+) = -2\eta_1 \Delta_s (1 - \text{sech}^2 2\eta_1 d) \left(1 - \frac{1 + \text{sech}^2 2\eta_1 d}{S^{(1)}}\right).$$

Using the fact that $\eta_1 = \frac{\alpha_1}{\alpha_0} \eta_0$ and $S^{(1)} = \frac{\alpha_1}{\alpha_0} S^{(0)}$, the condition for existence of critical points in the outer media is $\frac{dU_s}{d\bar{x}}(x_1^+) < 0$ or $S^{(0)} < \frac{\alpha_0}{\alpha_1} (1 + \text{sech}^2 2\frac{\alpha_1}{\alpha_0} \eta_0 d)$, and they are always minima.

We show in Figure 25 in a $(S^{(0)}, \eta_0 d)$ plane all the regions of different potentials for fixed values of Δ_s and $\frac{\alpha_0}{\alpha_1}$. From here we can extract to first approximation the existence and stability properties of surface waves in such a configuration. Consistent with the single interface case, we expect that corrections of these potentials should smooth the corners into new local minima. We also observed that when $\eta_0 d \gg 1$, U_s indeed looks like 'patched' potentials of the single interface problem. We have corroborates the existence of all regions but (II) of Figure 25 through extensive numerical simulations of the governing equation, although we have not established the errors on the boundaries separating the different regimes.

Figure 25. $(S_0, \eta_0 d)$ parameter representation of the different potentials in a symmetric waveguide. Here $n_0^2 - n_1^2 = 0.1$ and $\alpha = 0.5$. Six different regimes are obtained and each is characterized by the critical points of the equivalent potential. In our notation, $C_{i(o)}$ means that there is a minimum between (outside) the interfaces, S means that there is a maximum at $\bar{x} = 0$ and $\bar{C}$ means there is a corner at an interface.

5.II Soliton Switch

In this section we are going to show two applications of the theory; a spatial scanner and a switch. In both cases, what we will do in effect is destabilize stable stationary wavepackets propagating close to one interface, by making it collide with a second wavepacket.

It is well known that in NLS, solitons have the remarkable property that when two of them with different amplitudes η_i and velocities v_i , $i = 1, 2$ collide they remain unchanged and each will continue propagating with the same velocity v_i . The net effect of the collision is that the center of mass $\bar{x}_i$ of each is displaced by an amount $\delta\bar{x}_i$, $i = 1, 2$ where

$$\delta\bar{x}_1 = \frac{v_1 - v_2}{|v_1 - v_2|} \frac{1}{4\eta_1} \ln \left[\frac{(\eta_1 + \eta_2)^2 + \frac{(v_1 - v_2)^2}{4}}{(\eta_1 - \eta_2)^2 + \frac{(v_1 - v_2)^2}{4}} \right] \quad (5.5)$$

and $\delta\bar{x}_2 = -\frac{\eta_1}{\eta_2} \delta\bar{x}_1$. Observe that the sign of $\delta\bar{x}_1$ is the same as that of the relative velocities. Since we will consider collisions between wavepackets with relative velocities such that the collision time is small we will approximate its effect by ignoring the perturbation terms due to the interfaces, thus the wavepackets will suffer a displacement that will be given by (5.5). Therefore if a small moving wavepacket collides with a stationary large one, i.e., $\eta_2 < \eta_1$, $v_1 - v_2 = -v_2 < 0$, the phase shift experienced by the larger packet is negative, and the amount of this shift depends on v_2 . These assumptions were tested using waveguides whose parameters are that of Figures 21 and 22. In all cases studied, the stationary wavepacket before the collision propagates parallel and to the right of the second interface (see Figure 23). Its evolution after the collision is obtained by first calculating, using (5.5), its phase shift due to the collision, the equivalent particle is thus displaced to the left

by the computed amount. By ignoring other effects during and after the collision, the subsequent dynamics of the equivalent particle will be determined from its new location. Consider for example the case corresponding to Figure 21 where there are three possibilities of subsequent behavior. First, the displacement is sufficiently large so that the equivalent particle originally located at the bottom of the well to the right of the second interface, is displaced to the left of the nearby maximum and its subsequent dynamics consists of oscillations in the deep potential well between interfaces one and two. The whole trajectory follows **ABCD** in Figure 21. A numerical calculation of (1.1) shows the corresponding evolution of the wavepacket in Figure 26. The other two possibilities occur if the displacement leaves the particle between the maximum and its original position. In these cases, either the new potential energy of the particle is sufficient to carry it past the new maximum to its right (trajectories **AB'C'D'** in Figure 21b and **AB''C''C'''D'''E'''** in Figure 22b), or if it is not large, it will simply oscillate about its original equilibrium position (**AB''C''D''B''** in Figure 21b). As in the previous case, the corresponding evolution of the wavepackets was obtained from direct integration of the governing equation (1.1), and we show the results in Figures 26 and 27. Note that in some cases, there can be more than one collision between the two wavepackets and each collision changes the trajectory according to (5.5). Although our main focus has been the evolution of the originally stationary wavepacket, one can also describe the small incident wavepacket. Figure 28 shows the potential and phase portrait of its equivalent particle, where the theoretical trajectory **abcdef** contains all the features of the actual evolution seen on Figure 27. We can think of this effect as that of a scanner where the parameters of the second wavepacket determines the asymptotic location of the first one.

Figure 26. Evolution of two wavepackets in a nonlinear waveguide. The equivalent potential for the bigger one is that of Figure 21. The three cases correspond, respectively to the three trajectories presented in Figure 21b.

Figure 27. Evolution of two wavepackets in a nonlinear waveguide. The equivalent potential for the bigger wavepacket is that of Figure 22. Here, the packet originally propagating parallel to the second interface, evolves according to the theoretical trajectory shown in Figure 22b.

Figure 28. Potential and phase plane of the equivalent particle corresponding to the smaller wavepacket of Figure 27. Again, in the phase plane we show the theoretical trajectory of the equivalent particle.

Looking back to the case shown in Figure 27, where the wavepacket that originally was propagating near the second interface ends up in the far right, if in the medium next to the third interface, we add a suitable x dependence on the linear refractive index to design a switch.

It is known that if there are no interfaces but instead the refractive index of the dielectric n^2 has the form $n_1^2 + \alpha_1 |F|^2 - \gamma x$, then $A(x, \tau) = \sqrt{\frac{\alpha_1}{2}} F(x, z) e^{i(\beta^2 - n_1^2)z/(2\beta)}$ will satisfy

$$\frac{\partial A}{\partial \tau} - i \frac{\partial^2 A}{\partial x^2} - 2i|A|^2 A = -2i\gamma x A.$$

Using perturbation theory as we did in Chapter 3, or by calculating the solution exactly [15], one finds that A has the soliton shape where now

$$\frac{d\bar{x}}{d\tau} = v, \quad \frac{dv}{d\tau} = -4\gamma$$

that is, the equivalent particle experiences a deceleration of 4γ .

Having introduced this new behavior, the designing of the switch becomes very simple. We now assume for the case shown in Figure 27 that the dielectric at the right of the third interface has a refractive index that starts decreasing by γx after some x' to the right of the location of the minima in the equivalent potential. Then, the equivalent particle, once it crosses x' , will start decelerating and with the proper choice of parameters, the idea is to eventually have the particle oscillating about the well next to the third interface. We achieved this by choosing x' to be one unit distance from the location of the minima and $\gamma = 0.05$ (see Figure 29). The whole experiment describes a switch of a wavepacket initially propagating parallel to the second interface, to a new trajectory parallel to the third interface. The switching mechanism was triggered by a second wavepacket.

Figure 29. Trapping of the wavepacket originally propagating parallel to the second interface located at $x = 0$. For $x > 13.8$, the linear refractive index coefficient decreases linearly in x .

5.III Bound Solitons

For the most part of this work, we have considered initial wavepackets that have the sech-like shape typical of the one soliton solution of NLS, but interesting phenomena can occur for different initial conditions. as we said before, if initially the wavepacket is of arbitrary shape and far away from an interface, it will decompose into its soliton and radiation components, and if each soliton has different velocity, they will eventually separate before reaching the interface and evolve as individual wavepackets. After that, the dynamics of each component is determined by its own potential as we showed in the previous chapters. Possible collisions between wavepackets can now be understood in the context of the equivalent particle dynamics. But if instead our initial wavepacket is a bound N soliton state, that is N soliton components each with the same parameter v , the dynamics is more complicated since the nonlinear interaction between solitons so far ignored in the analysis is essential for this case. We present here some numerical simulations for the simplest case of a bound two soliton initial condition [12], as it approaches an interface.

In the absence of an interface the soliton components interact in a periodic fashion as we can see in Figure 30. There the two solitons, each with different η , oscillate about the center of mass which moves with velocity v . We now put in the interface. Since each soliton component has a different η , we can expect that the effect of the interface is different for each one (one has to remember that in the single soliton dynamics the equivalent potential was η dependent). The question is whether this different effect is enough to cause the solitons to split. Two cases are shown in Figures 31 and 32; both cases have the same parameters except for the velocity of the center of mass which is bigger in Figure 32.

Figure 30. Evolution of an initial bound two soliton wavepacket in a nonlinear media with no interface. The soliton parameters are $\eta_1 = 0.35$, $\eta_2 = 0.25$, and the center of mass velocity is $v_0 = -0.2$

Figure 31. Evolution of an initial bound two soliton wavepacket with the parameters of figure 30 approaching an interface. Splitting is observed, with one soliton staying in the right hand medium. Here $\Delta = 0.1$ and $\alpha = 0.75$.

Figure 32. Evolution of an initial bound two soliton wavepacket approaching an interface. Here the velocity of the center of mass is now -0.3 bigger than in the previous figure. Even though splitting still occurs, both solitons cross the interface.

Indeed we see splitting occuring at the interface in both cases, and in the second case we see that one of the components seems to be trapped near the corresponding minimum of its equivalent potential.

To understand the dynamics with an equivalent two particle description, we follow the same approach used in Chapter 3 and using the conservation laws obtain a nonlinear system of equations for the soliton parameters v_i, η_i . The first is $\frac{d}{d\tau}(\eta_1 + \eta_2) = 0$. Alternatively one can use the perturbation theory on the scattering data of the linear problem associated to NLS (A.1) [1]. A set of four equations, one for each parameter obtains. The first two are, $\frac{d\eta_i}{d\tau} = 0, \quad i = 1, 2$ and the last two

$$\frac{dv_i}{d\tau} = - \int_{-\infty}^{\infty} V(A\psi_{2i}^2 - A^*\psi_{1i}^2)dx, \quad i = 1, 2$$

where $\psi_{ni}, \quad n = 1, 2$ are the eigenfunctions of (A.1) when $A(x, 0)$ is the exact two soliton solution of NLS [12].

In contrast with the one soliton case, we cannot solve these equations analytically, one would have to do it numerically. This approach was used by Kodama et. al in [16], to describe soliton splitting in an optical fiber. We are not going to do it for this case here, instead, we will only say that once they are separated the propagation of each component can be described from its equivalent potential.

APPENDIX A

Numerical Techniques

The method that will be used in all numerical simulations of (1.10) is the split-step method that was proven to be the most efficient for the case of the exact NLS [30]. The same method, or minor variations of it has been extensively used in problems arising from other nonlinear optical phenomena [2]. We never compared this method with any other used in these type of equations because it is not the aim of this work, but we tested our code by solving the exact NLS when the initial conditions are a one or a two soliton state. Although we only compared our solutions with the analytical ones by simply overlapping them, we made quantitative comparisons of the first three conserved quantities of NLS, mass, momentum and energy. Both the qualitative and quantitative comparisons showed the expected accuracy and stability of the method. For the case of equation (1.1) only the first integral $\int_{-\infty}^{\infty} FF^* dx$ is conserved, and in all simulations the error on this quantity was within one percent until some component of the field reached the numerical boundary.

We now outline the method. If we write the equation for the field $F(x, z)$ in the form

$$\frac{\partial F}{\partial z} = i(L + N)F(x, z) \quad (A.1)$$

where $L = \frac{\partial^2}{\partial x^2}$ and N are operators that represent, respectively, the linear and the remaining part of the equation. Since both L and N do not explicitly depend in z we can formally write the solution at $z + \delta z$ in terms of F at z as

$$F(x, z + \delta z) = e^{i(L+N)\delta z} F(x, z) \quad (A.2)$$

where, the integrating operator is, from its Taylor expansion

$$e^{i(L+N)z} = I + i(L+N)\delta z + \dots + \frac{i^n(L+N)^n(\delta z)^n}{n!} + \dots \quad (A.3)$$

We approximate the operator

$$e^{i(L+N)\delta z} \sim e^{\frac{iL\delta z}{2}} e^{iN\delta z} e^{\frac{iL\delta z}{2}} \quad (A.4)$$

where the error is $O((\delta z)^3)$ which is obtained after Taylor expanding each operator of the right hand side of (A.4) multiply them and compare the result with (A.3).

What this means in terms of the error on the solution is that if $\bar{F}(x, z)$ is given by

$$\bar{F}(x, z + \delta z) = T\bar{F}(x, z)$$

where $T = e^{\frac{iL\delta z}{2}} e^{iN\delta z} e^{\frac{iL\delta z}{2}}$ then,

$$\begin{aligned} |F(x, z + \delta z) - \bar{F}(x, z + \delta z)| &= |e^{i(L+N)\delta z} F(x, z) - T\bar{F}(x, z)| = \\ |(e^{i(L+N)\delta z} - T)F(x, z) + T(F(x, z) - \bar{F}(x, z))| &\leq |(e^{i(L+N)} - T)F(x, z)| + \\ ||T|| |F(x, z) - \bar{F}(x, z)| &= ||T|| |F(x, z) - \bar{F}(x, z)| + O((\delta z)^3). \end{aligned}$$

Since $||T|| < \infty$ by a simple induction argument the error is

$$|F(x, z) - \bar{F}(x, z)| = O((n+1)(\delta z)^3)$$

where $z = n\delta z$. One should note that the splitting of the integrating operator here presented is different than that of [30]. Effectively what this does is increase the accuracy by one order in δz .

To find $\bar{F}(x, z + \delta z)$ under the action of the operator T is equivalent to solving first

$$\frac{\partial F_1}{\partial z} = iL F_1 \quad (\text{A.5})$$

in an interval $\frac{\delta z}{2}$ with initial condition $\bar{F}(x, z)$. Then solve

$$\frac{\partial F_2}{\partial z} = iN F_2 \quad (\text{A.6})$$

in an interval δz with initial condition F_1 solution of (A.5). Finally one solves (A.5) again for an interval $\frac{\delta z}{2}$ but with initial condition F_2 solution of (A.5); this will be $\bar{F}(x, z + \delta z)$. The advantage of this method is that both (A.4) and (A.5) are easy to solve. The solution of (A.4) is found by using the Fourier transform and is given by

$$F_1 = \mathbf{F}^{-1}(e^{ik^2 \frac{\delta z}{2}} \mathbf{F}(F_{1in}))$$

where F_{1in} is the initial condition and $\mathbf{F}$ is the Fourier transform. Although this is an exact solution of (A.4) in the numerical scheme we use a discrete Fourier transform that introduces errors in the solution of order $O((\delta x)^n)$ where δx is the distance between points in our spatial discretization, and n is the number of modes one considers in the discrete finite Fourier transform.

The solution of (A.5) is simply $F_2 = F_{2in} e^{iN(|F_{2in}|)\delta z}$ where one uses the fact that N being dependent only on $|F|$ which is z independent is in terms of z a

constant factor in (A.5).

Finally, from the solutions we will be monitoring quantities such as power, center of mass, momentum, etc. which are given in terms of spatial integrals of F and its first and second derivatives. We approximate the derivatives using forward Euler approximations and for each integral that we compute we use the trapezoidal rule.

APPENDIX B

The Linear Eigenvalue Problem

It is well known that for the NLS equation there is an associated linear eigenvalue problem, first obtained by Zakharov and Shabat [12], which determines from the scattering data, the soliton and radiation components of a given initial condition $A(x, 0)$. The associated linear problem is

$$i \frac{\partial \psi_1}{\partial x} + A(x, 0) \psi_2 = \varsigma \psi_1$$

(B.1)

$$-i \frac{\partial \psi_2}{\partial x} + A^*(x, 0) \psi_1 = \varsigma \psi_2$$

and Satsuma and Yajima [31] obtained the scattering data of (B.1) for the class of potentials

$$A(x, 0) = A \operatorname{sech} x e^{i \frac{vx}{2}}.$$

They found that there are N discrete eigenvalues of (B.1) $\varsigma_n = -\frac{v}{4} + i(A - n + \frac{1}{2})$ $n = 1, \dots, N$ where $A - \frac{1}{2} < N \leq A + \frac{1}{2}$ and there is also some continuous spectrum. For NLS this means that the outcome of this initial condition is a bound N -soliton component plus a radiation component. The bound state propagates with velocity v and in the evolution it shows periodic behavior as in Figure 30. We now relate this result to our problem. The question that was left open in the Chapter 3, was what happens to a wavepacket when it crosses the interface. We already said in Chapter 2 that in the nonlinear regime, which is the case of interest here, the wavepacket suffers

no initial changes as it crosses the interface. However, once it has crossed, it will no longer be a perfect soliton of the new medium and it will consequently decompose into the normal modes associated with the new medium. The way we will find these new modes is by solving the corresponding linear eigenvalue problem equivalent to (B.1) for the nonlinear Schrödinger like equation in the right hand medium, where the potential in the linear problem will be the incoming wavepacket. The result will be, after a proper scaling, similar to the one shown above.

Suppose that one has as an initial condition for the field in the right hand medium where the nonlinear refractive index is α_1 a 'natural' soliton of the left hand medium given by

$$F(x) = \sqrt{\frac{2}{\alpha_0}} 2\eta_0 \operatorname{sech} 2\eta_0(x - \bar{x}) e^{i(\frac{v\bar{x}}{2} + 2\sigma)}. \quad (B.2)$$

If, as we did in Chapter 3, we define the field to be

$$F(x, z) = \sqrt{\frac{2}{\alpha_1}} A(x, \tau) e^{i(\beta^2 - n_1^2)\tau}$$

then $A(x, \tau)$ satisfies for the right hand medium the exact NLS, and from (B.2) its initial condition would be

$$\frac{1}{\sqrt{\alpha}} 2\eta_0 \operatorname{sech} 2\eta_0(x - \bar{x}) e^{i(\frac{v\bar{x}}{2} + 2\sigma_0)}. \quad (B.3)$$

Now the question of finding the normal mode decomposition of a 'natural' soliton of the left hand medium in the right hand medium, translates into finding the scattering data of (B.1) with equation (B.3) as potential. But this is now a simple problem since if we make the following transformations,

$$x' = 2\eta_0(x - \bar{x}), \quad \psi'_1 = \psi_1 e^{-i(v\frac{x}{2} + 2\sigma_0)}, \quad \psi'_2 = \psi_2 e^{i(v\frac{x}{2} + 2\sigma_0)}$$

then (B.1) becomes

$$i \frac{\partial \psi'_1}{\partial x'} + A'(x') \psi'_2 = \varsigma' \psi'_1 \tag{B.4}$$

$$-i \frac{\partial \psi'_2}{\partial x'} + A'^*(x') \psi'_1 = \varsigma' \psi'_2$$

where now

$A'(x') = \frac{1}{\sqrt{\alpha}} \text{sech} x' e^{i\frac{v x'}{2\eta_0}}$ and $\varsigma' = \frac{\varsigma}{2\eta_0}$. This is exactly the case mentioned at the begining of this section, and if we translate the result shown above to this particular problem one has that (4.3) will decompose into N solitons and radiation where now $\frac{1}{\sqrt{\alpha}} - \frac{1}{2} < N \leq \frac{1}{\sqrt{\alpha}}$. Each soliton has parameters $\varsigma_n = -\frac{v_n}{4} + i\eta_n = -\frac{v}{4} + i(\frac{1}{\sqrt{\alpha}} - n + \frac{1}{2})2\eta_0$, $n = 1, \dots, N$. Observe that since $\alpha = \frac{\alpha_0}{\alpha_1} \leq 1$, $N \geq 1$.

We can also quantify the amount of power transfered into radiation modes using the trace formulae which are the nonlinear analogue of the Parseval's relations [15]. These formulae relates the conseved quantities of NLS, C_n , with the scattering data of (B.1) and they are

$$C_n = 2i \sum_{k=1}^N \frac{\varsigma_k^n - \varsigma_k^{*2}}{n} + \frac{1}{\pi} \int_{-\infty}^{\infty} \varsigma^{n-1} \ln(1 - |R|^2) d\varsigma$$

where ς_k are the discrete eigenvalues of (B.1) and R is the reflection coefficient given in terms of the contiuous spectrum of (B.1). For $n=1$ one has $C_1 = -\int_{-\infty}^{\infty} A A^* dx$, therefore

$$-C_1 = 4 \sum_{k=1}^N \eta_k - \frac{1}{\pi} \int_{-\infty}^{\infty} \ln(1 - |R|^2) d\zeta. \quad (B.5)$$

If $A(x, 0)$ is an almost reflectionless potential, which as we will see, is the case in our problem, $|R| \ll 1$ and in this case the amount of power in the radiation modes given by the integral of $|R|^2$ simply is

$$\int_{-\infty}^{\infty} |R|^2 d\zeta = \pi \int_{-\infty}^{\infty} AA^* dx - 4\pi \sum_{k=1}^N \eta_k \quad (B.6)$$

If instead one has the case of a soliton initially in the right hand medium and crossing the interface, to determine the decomposition of this soliton in the new medium, one can proceed as before to find that for this case, the number of soliton components is $\sqrt{\alpha} - \frac{1}{2} < N \leq \sqrt{\alpha} + \frac{1}{2}$. Therefore, if $0 < \alpha < 0.25$ the field in the left hand medium will contain no soliton components and (B.6) is for this case Parseval's identity. If $0.25 \leq \alpha \leq 1$ a single soliton will prevail but with new parameters $\varsigma_1 = -\frac{\nu}{4} + (\alpha - \frac{1}{2})2\eta_1$ and radiation, with the powers carried by each part given by (B.6).

The interpretation of these results in terms of the original wavepackets and the subsequent dynamics of each new component is explained in Chapter 4.

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