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**A COUPLED ANGULAR MOMENTUM MODEL FOR THE JOSEPHSON
JUNCTION**

The University of Arizona

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A COUPLED ANGULAR MOMENTUM MODEL
FOR THE JOSEPHSON JUNCTION

by

Andrew Lewis DiRienzo

A Dissertation Submitted to the Faculty of the

DEPARTMENT OF PHYSICS

In Partial Fulfillment of the Requirements
For the Degree of

DOCTOR OF PHILOSOPHY

In the Graduate College

THE UNIVERSITY OF ARIZONA

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THE UNIVERSITY OF ARIZONA
GRADUATE COLLEGE

As members of the Final Examination Committee, we certify that we have read
the dissertation prepared by Andrew Lewis DiRienzo
entitled A Coupled Angular Momentum Model for the Josephson Junction

and recommend that it be accepted as fulfilling the dissertation requirement
for the Degree of Doctor of Philosophy.

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SIGNED:

Andrew A. Siliezo

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In regard to others I have dealt with during my graduate career, both at Arizona and Stony Brook, the following memorable quotes can be viewed as indicative of how I was treated by them [(1) and (2)], while the last one sums up how I feel about them.

- (1) Son, let me give you the bottom line, YOU'RE WRONG!

Marlan Scully
September, 1977

- (2) Sometimes when you go out looking for diamonds you come back with golf balls, and right now you have enough golf balls to graduate!

Marlan Scully
June, 1978

- (3) Good luck with J. J. (J.J./M.S. >> 1).

Willis Lamb
June, 1979

TABLE OF CONTENTS

	Page
LIST OF ILLUSTRATIONS	v
ABSTRACT	vi
1. INTRODUCTION	1
2. PRELIMINARIES	4
A. BCS Ground State and Pseudospin	4
B. Macroscopic Angular Momentum	7
C. The Charged Superconductor	15
3. THE JOSEPHSON JUNCTION: PREVIOUS THEORIES	20
A. (n, ϕ) Approximation	23
B. Two-state Approximation	26
4. THE JOSEPHSON JUNCTION: ANGULAR MOMENTUM APPROACH	30
A. Angular Momentum Formalism	30
B. The Rotating Vector Solution	33
C. The (n, ϕ) Approximation Revisited	36
D. The Two-state Approximation Revisited	38
E. Some Numerical Values	42
5. SOLUTION TO THE EXACT EQUATIONS OF MOTION: PERTURBATION THEORY	44
6. CONCLUDING REMARKS	51
APPENDIX A: THE $\vec{k}$ -SPACE CUTOFF	52
APPENDIX B: SECOND ORDER CORRECTIONS TO THE JOSEPHSON FREQUENCY	57
REFERENCES	59

LIST OF ILLUSTRATIONS

Figure	Page
1. BCS occupation distribution	8
2. Occupation distributions in the strong-coupling limit	9
3. The expectation value of $\hat{J}$	13
4. Charge-excess state occupation distributions	17
5. Charge-depleted state occupation distributions	18
6. Josephson junction with ideal battery	21
7. Rotation of vectors $\vec{J}_{L\perp}$ and $\vec{J}_{R\perp}$	34
8. Regions in $\vec{k}$ -space	53
9. Regions associated with k_{nF}	56

ABSTRACT

A model for the Josephson junction is constructed based on two macroscopic angular momentum vectors. These vectors, which interact via a Heisenberg-like Hamiltonian, are defined using Anderson's pseudospin concept in superconductivity. Along with this, a new state vector, which affords a more complete description of the constant-charge-imbalance mode of the junction, is explicitly constructed. The resulting equations of motion lead directly to the basic Josephson results and at the same time provide a simple physical picture for the dynamical behavior of the junction. Both the Anderson (n, ϕ) and Feynman two-state models of the junction are shown to be equivalent to a restricted form of the angular momentum approach. The process of formulating the junction problem in terms of pseudo-angular-momentum together with the above identification constitutes a microscopic derivation of the Feynman method. A perturbation theory calculation is carried out within the full pseudo-angular-momentum equations of motion to determine how this approach differs from the earlier ones.

CHAPTER 1

INTRODUCTION

The theory of superconductivity, as first proposed by Bardeen, Cooper, and Schrieffer [1], represents one of the most significant achievements of theoretical physics. The BCS theory successfully accounts for many unusual and unexpected phenomena, of which the Josephson effect [2] is perhaps the most surprising. The idea that a dc current can spontaneously flow between two weakly coupled superconductors, or that a constant voltage placed across such a junction will induce an alternating current, are certainly counter to our physical intuition. For this reason a simple model for the dynamics of a Josephson junction can prove quite valuable as an aid in understanding this otherwise unusual behavior.

Among such models presently used, the two most popular are the (n, ϕ) theory first developed by Anderson [3] and the two state (two-level atom) model developed by Feynman [4]. The frequent quotation of the latter reference, in journals concerned with original research publications, serves to indicate the utility of simplified models, even for the experts.

In this dissertation, an alternative model for the junction is proposed. This new model involves: (i) an explicit construction of a state vector to represent the constant-charge-imbalance mode of the

junction, and (ii) a description of junction dynamics using the coupling of two macroscopic angular momentum vectors.

Our model is based on Anderson's reformulation of the BCS theory in terms of pseudo-angular-momentum operators [5]. Starting from a microscopic Hamiltonian, a careful use of this approach yields a description of the Josephson junction in terms of two coupled macroscopic angular momentum vectors. Thus the dynamics of a junction may be understood in terms of the classical equations of motion for two interacting angular momenta. (It should be pointed out that there is at least one other pseudo-angular-momentum model of the Josephson junction [6]. Even though both models start with Anderson's pseudo-spin concept, and are therefore by necessity similar, they are not equivalent.) As we will show, our model reproduces all the well-known Josephson results and, within certain clearly defined approximations, is equivalent to both Anderson's (n, ϕ) theory and Feynman's two-state model. (The process of making this latter identification effectively constitutes a derivation of the Feynman model from a microscopic description of the junction.)

The remainder of the dissertation is organized as follows: In Chapter 2 we give a brief description of the BCS ground state using the pseudo-angular-momentum formulation, and introduce the operators which characterize the macroscopic angular momentum variables to be used in the later calculations. This chapter also includes the construction of a state vector to represent a charged superconductor. Chapter 3 introduces the tunneling Hamiltonian model of the Josephson junction and summarizes the central results of Anderson's (n, ϕ) and

Feynman's two-state theories. In Chapter 4, we reformulate the tunneling Hamiltonian in terms of our macroscopic angular momentum variables. We obtain the equations of motion for the angular momentum variables and solve them within a particular approximation to get the Josephson current. We then go on to show that the (n, ϕ) and two-state theories are identical to this restricted form of the angular momentum approach. Chapter 5 gives a perturbation theory solution to the exact angular momentum equations of motion. Finally, Chapter 6 summarizes the results.

CHAPTER 2

PRELIMINARIES

A. BCS Ground State and Pseudospin

In 1957 Bardeen, Cooper, and Schrieffer [1] proposed that the superconducting properties of a metal may be understood by assuming that, at $T = 0$ K, the electrons in a metal are described by the highly specialized state vector

$$|\phi\rangle = \prod_{\vec{k}} [U(\vec{k}) + V(\vec{k}) e^{i\phi} C_{\vec{k}\uparrow}^{\dagger} C_{-\vec{k}\downarrow}^{\dagger}] |0\rangle, \quad (2.1)$$

where $C_{\vec{k}s}^{\dagger}$ is the fermion creation operator for an electron with wave vector $\vec{k}$ and spin s . The coefficient $U(\vec{k})$ and $V(\vec{k})$ (assumed real) satisfy the constraint $U^2(\vec{k}) + V^2(\vec{k}) = 1$. The state $|\phi\rangle$ is constructed in such a way as to emphasize the central role of pairing [7] in superconductivity and yields a groundstate energy which is independent of the phase angle ϕ .

The strong pairing correlation is nicely illustrated by using a pseudospin model to describe $|\phi\rangle$. As Anderson [5] first observed, each momentum subspace factor of $|\phi\rangle$ could be viewed as a state vector for an imaginary two-level system, i.e.,

$$|\phi\rangle = \prod_{\vec{k}} |\chi\rangle_{\vec{k}} \quad (2.2)$$

with

$$|\chi\rangle_{\vec{k}} = U(\vec{k})|0\rangle_{\vec{k}} + V(\vec{k}) e^{i\phi} C_{\vec{k}\uparrow}^{\dagger} C_{-\vec{k}\downarrow}^{\dagger} |0\rangle_{\vec{k}} \quad (2.3)$$

Thus $|\chi\rangle_{\vec{k}}$ is a linear superposition of two states: $|0\rangle_{\vec{k}}$, in which both the $\vec{k}\uparrow$ and $-\vec{k}\downarrow$ single electron states are empty, and $C_{\vec{k}\uparrow}^{\dagger} C_{-\vec{k}\downarrow}^{\dagger} |0\rangle_{\vec{k}}$, in which both of these single electron states are occupied. In Eq. (2.3) $U(\vec{k})$ and $e^{i\phi} V(\vec{k})$ are the respective amplitudes for these two basic states. The analogy to a spin- $\frac{1}{2}$ system is made obvious if we introduce the following notation:

$$\begin{aligned} \sigma_{\vec{k}}^{+} &= C_{\vec{k}\uparrow}^{\dagger} C_{-\vec{k}\downarrow}^{\dagger} && \text{-creates a pair: raises spin} \\ \sigma_{\vec{k}}^{-} &= C_{-\vec{k}\downarrow} C_{\vec{k}\uparrow} && \text{-destroys a pair: lowers spin} \\ \sigma_{\vec{k}z} &= \frac{1}{2}(n_{\vec{k}\uparrow} + n_{-\vec{k}\downarrow} - 1) && \text{-acts like a number operator for} \\ &&& \text{electron pairs or equivalently is} \\ &&& \text{the z-component of the angular} \\ &&& \text{momentum operator} \\ &&& \vec{\sigma}_{\vec{k}} \cdot \vec{n}_{\vec{k}s} = C_{\vec{k}s}^{\dagger} C_{\vec{k}s} \end{aligned} \quad (2.4)$$

As Anderson has pointed out, $(1, \sigma_{\vec{k}}^{+}, \sigma_{\vec{k}}^{-}, \sigma_{\vec{k}z})$ forms a complete set of operators for the $\vec{k}$ -subspace. (Note: The definitions given here differ slightly from those given by Anderson but are completely equivalent to them.)

Using the fermion anticommutation rules

$$\begin{aligned} [C_{\vec{k}s}, C_{\vec{k}'s'}^{\dagger}]_{+} &= \delta_{\vec{k}\vec{k}'} \delta_{ss'} \\ [C_{\vec{k}s}^{\dagger}, C_{\vec{k}'s'}^{\dagger}]_{+} &= 0 \end{aligned}$$

one can show that the σ 's satisfy the angular momentum commutation relations

$$\begin{aligned} [\sigma_{\vec{k}}^+, \sigma_{\vec{k}}^-] &= 2 \sigma_{\vec{k}z} \delta_{\vec{k}\vec{k}}, \quad , \\ [\sigma_{\vec{k}z}, \sigma_{\vec{k}}^\pm] &= \pm \sigma_{\vec{k}}^\pm \delta_{\vec{k}\vec{k}}, \quad . \end{aligned} \quad (2.5)$$

Thus Eq. (2.3) may be written as

$$\begin{aligned} |\chi\rangle_{\vec{k}} &= U(\vec{k}) |0\rangle_{\vec{k}} + V(\vec{k}) e^{i\phi} \sigma_{\vec{k}}^+ |0\rangle_{\vec{k}} \\ &= U(\vec{k}) |\downarrow\rangle_{\vec{k}} + V(\vec{k}) e^{i\phi} |\uparrow\rangle_{\vec{k}} \quad , \end{aligned} \quad (2.6)$$

where $|\downarrow\rangle_{\vec{k}}$ and $|\uparrow\rangle_{\vec{k}}$ are the basis states for our pseudospin system. This enables us to view $|\phi\rangle$ as a product of two-level state vectors, i.e.,

$$|\phi\rangle = \prod_{\vec{k}} [U(\vec{k}) + V(\vec{k}) e^{i\phi} \sigma_{\vec{k}}^+] |0\rangle \quad , \quad (2.7a)$$

$$|\phi\rangle = \prod_{\vec{k}} [U(\vec{k}) |\downarrow\rangle_{\vec{k}} + V(\vec{k}) e^{i\phi} |\uparrow\rangle_{\vec{k}}] \quad . \quad (2.7b)$$

Consequently we have replaced the superconductor with an ensemble of imaginary spin- $\frac{1}{2}$ systems (one spin system for each $\vec{k}$ -vector), the orientation of each spin being determined by $U(\vec{k})$, $V(\vec{k})$, and ϕ .

[Note: In terms of our pseudospin picture of a superconductor, ϕ takes on the meaning of a rotation in pseudospin space. To see this consider the situation where all the spins are in the xz -plane. The state representing this is

$$\prod_{\vec{k}} [U(\vec{k}) |\uparrow\rangle_{\vec{k}} + V(\vec{k}) |\downarrow\rangle_{\vec{k}}] \quad . \quad (2.7c)$$

If each of these spins is now rotated clockwise through an angle ϕ about the z-axis, the state describing the system becomes $|\phi\rangle$ of Eq. (2.7b).

This convention is discussed more fully following Eq. (2.14).]

B. Macroscopic Angular Momentum

Now that our pseudospin system has been defined we will evaluate its total angular momentum. To do this it is necessary to give the form of $U(\vec{k})$ and $V(\vec{k})$ explicitly.

In general, for a neutral superconductor at $T = 0$ K, the amplitudes $U(\vec{k})$ and $V(\vec{k})$ have the dependence $|\vec{k}|$ shown in Fig. 1. However, for purposes of computational simplicity, we shall assume that $U(\vec{k})$ and $V(\vec{k})$ are given by the functions shown in Fig. 2. To be specific, the probability amplitudes in the BCS theory will be approximated by step functions in a region about k_F , the Fermi momentum. The energy region over which $U(\vec{k})$ and $V(\vec{k})$ deviate from their values in a normal metal is taken to be 2Δ , where Δ is the energy gap parameter. This model corresponds to the strong-coupling model of a superconductor [8].

Finally, it will be convenient for future calculations to define a maximum magnitude κ , such that the product $\prod_{\vec{k}}$ in Eq. (2.7) is understood to be over all $\vec{k}$ with $|\vec{k}| \leq \kappa$. The value of κ is chosen such that, for a neutral superconductor at $T = 0$ K, the total number of $\vec{k}$ -vectors (denoted by N_0) with energy $\epsilon_{\vec{k}} < \epsilon_F - \Delta$ is equal to the total number of $\vec{k}$ -vectors with $\epsilon_{\vec{k}} > \epsilon_F + \Delta$. Further details on this procedure are given in Appendix A.

Figure 1. BCS occupation distribution.

The BCS occupation distributions are shown for a neutral superconductor at $T = 0$ K.

Figure 2. Occupation distributions in the strong-coupling limit.

The BCS occupation distributions for a neutral superconductor in the strong coupling limit are shown. There are N_0 pseudo-spin systems ($\vec{k}$ -vectors) in regions (1) and (3), while there are m such systems in region (2).

With these conventions we now turn our attention to the macroscopic angular momentum operators defined by

$$\hat{J}^+ = \hat{J}_x + i\hat{J}_y = \sum_{\vec{k}} \sigma_{\vec{k}}^+ , \quad (2.8a)$$

$$\hat{J}^- = \hat{J}_x - i\hat{J}_y = \sum_{\vec{k}} \sigma_{\vec{k}}^- , \quad (2.8b)$$

$$\hat{J}_z = \sum_{\vec{k}} \sigma_{\vec{k}z} , \quad (2.8c)$$

where $\sum_{\vec{k}}$ means sum over all $\vec{k}$ with $|\vec{k}| \leq \kappa$. These operators clearly obey the angular momentum commutation rules and, since the number of allowed $\vec{k}$ -vectors is $\sim 10^{18}$ for 1 mm^3 of metal, represent the total angular momentum for a large spin system.

At $T = 0 \text{ K}$, the state of the superconductor is distinguished from that of a normal metal by the fact that there is a non-zero gap energy and a well-defined phase ϕ . These underlying superconducting properties of our system are reflected in the expectation value of $\hat{\vec{J}}$, i.e., once Δ and ϕ are known, then $\langle \hat{\vec{J}} \rangle$ is determined. To see this consider calculating the expectation value of $\hat{\vec{J}}$ with respect to $|\phi\rangle$. This calculation is facilitated by writing

$$\begin{aligned} \hat{\vec{J}} &= \sum_{\vec{k}}^{(1)} \vec{\sigma}_{\vec{k}} + \sum_{\vec{k}}^{(2)} \vec{\sigma}_{\vec{k}} + \sum_{\vec{k}}^{(3)} \vec{\sigma}_{\vec{k}} \\ &= \hat{\vec{J}}^{(1)} + \hat{\vec{J}}^{(2)} + \hat{\vec{J}}^{(3)} , \end{aligned} \quad (2.9)$$

where $\sum_{\vec{k}}^{(i)}$ implies the sum is only taken over the $\vec{k}$ -vectors lying in the region i ($i = 1, 2, 3$) as shown in Fig. 2. Similarly we write

$$|\phi\rangle = \prod_{\vec{k}}^{(1)} |\chi\rangle_{\vec{k}} \cdot \prod_{\vec{k}}^{(2)} |\chi\rangle_{\vec{k}} \cdot \prod_{\vec{k}}^{(3)} |\chi\rangle_{\vec{k}} \quad (2.10)$$

Referring to Fig. 2 and Eq. (2.6) we see that:

$$|\chi\rangle_{\vec{k}} = \begin{cases} e^{i\phi} |\uparrow\rangle_{\vec{k}} & (1) \\ \frac{1}{\sqrt{2}} |\downarrow\rangle_{\vec{k}} + \frac{1}{\sqrt{2}} e^{i\phi} |\uparrow\rangle_{\vec{k}} & (2) \\ |\downarrow\rangle_{\vec{k}} & (3) \end{cases} \quad \begin{matrix} (2.11a) \\ (2.11b) \\ (2.11c) \end{matrix}$$

for the respective regions of $\vec{k}$ -space. Calculating the expectation value of $\vec{\sigma}_{\vec{k}}$ in each of the three regions gives:

$$\vec{k} \langle \chi | \vec{\sigma}_{\vec{k}} | \chi \rangle_{\vec{k}} = \begin{cases} (\frac{1}{2}) \vec{e}_z & (1) \\ (\frac{1}{2}) \cos \phi \vec{e}_x - (\frac{1}{2}) \sin \phi \vec{e}_y & (2) \\ -(\frac{1}{2}) \vec{e}_z & (3) \end{cases} \quad \begin{matrix} (2.12a) \\ (2.12b) \\ (2.12c) \end{matrix}$$

where $\vec{e}_x$, $\vec{e}_y$, and $\vec{e}_z$ are unit vectors along the coordinate axes.

Remembering that there are N_0 $\vec{k}$ -vectors in regions (1) and (3), and letting m be the number of $\vec{k}$ -vectors in region (2), gives

$$\langle \phi | \hat{J}^{(1)} | \phi \rangle = \left(\frac{N_0}{2} \right) \vec{e}_z \quad , \quad (2.13a)$$

$$\langle \phi | \hat{J}^{(2)} | \phi \rangle = \left(\frac{m}{2} \right) \cos \phi \vec{e}_x - \left(\frac{m}{2} \right) \sin \phi \vec{e}_y \quad , \quad (2.13b)$$

$$\langle \phi | \hat{J}^{(3)} | \phi \rangle = - \left(\frac{N_0}{2} \right) \vec{e}_z \quad . \quad (2.13c)$$

Finally we combine these to obtain the expectation value of $\hat{\vec{J}}$:

$$\langle \phi | \hat{\vec{J}} | \phi \rangle = \left(\frac{m}{2}\right) \cos \phi \vec{e}_x - \left(\frac{m}{2}\right) \sin \phi \vec{e}_y . \quad (2.14)$$

Thus we see that, since m is directly related to Δ , the magnitude of $\langle \hat{\vec{J}} \rangle$ is determined by the gap energy, while the direction of $\langle \hat{\vec{J}} \rangle$ in the xy -plane is given by ϕ . Alternately we could say that the ground state of a superconductor is characterized by a macroscopic value ($m \sim 10^{12}$ for 1 mm^3 of metal) of the x and y components of the angular momentum with the direction of this vector determining the phase angle of the BCS ground state. It is important to note at this point that the state $|\phi\rangle$ characterizes our pseudospin system, initially described by Eq. (2.7c), after it has been rotated in pseudospin space by an angle ϕ about $\vec{e}_z$. Our sign convention is that we have called a rotation about $\vec{e}_z$ in the clockwise direction positive (Fig. 3).

One last note on this subject: From Eq. (2.14) we see that $\langle \hat{J}_z \rangle = 0$, for a neutral superconductor at $T = 0 \text{ K}$. This is directly related to our choice of κ and the relationship of $\hat{J}_z$ to the total pair number operator. To see this, consider the following.

The pair number operator for the $\vec{k}$ -th pair is

$$\hat{N}_{\vec{k}} = \frac{1}{2} (n_{\vec{k}\uparrow} + n_{-\vec{k}\downarrow}) . \quad (2.15)$$

From Eq. (2.4) we see that

$$\hat{N}_{\vec{k}} = \sigma_{\vec{k}z} + \frac{1}{2} . \quad (2.16)$$

Figure 3. The expectation value of $\hat{J}$.

The expectation value of $\hat{J}$ with respect to $|\phi\rangle$ is shown. The clockwise rotation, shown above, corresponds to ϕ positive.

Consequently the total pair number operator for our superconductor is

$$\begin{aligned}\hat{N} &= \sum_{\vec{k}} \hat{N}_{\vec{k}} \\ &= \sum_{\vec{k}} \left(\frac{1}{2}\right) (n_{\vec{k}\uparrow} + n_{-\vec{k}\downarrow})\end{aligned}\quad (2.17a)$$

$$= \sum_{\vec{k}} (\sigma_{\vec{k}z} + \frac{1}{2}) \quad . \quad (2.17b)$$

Technically the range of $\vec{k}$ in the above sums is unbounded. But since we will only be dealing with superconductors at $T = 0$ K, we can cut off the sums at any convenient wave vector (as long as there are no filled states above it) and not affect $\langle \hat{N} \rangle$. In order to make contact with our pseudospin operators we choose κ as the cutoff. Then Eqs. (2.8c) and (2.17b) give

$$\begin{aligned}\hat{N} &= \hat{J}_z + \sum_{\vec{k}} \left(\frac{1}{2}\right) \vec{k} \\ &= \hat{J}_z + \frac{1}{2} (2N_0 + m) \quad ,\end{aligned}\quad (2.18)$$

where the last equality follows from Fig. 2.

Now a straightforward calculation of $\langle \hat{N} \rangle$ using $|\phi\rangle$ and Eq. (2.17a) yields $\langle \hat{N} \rangle = N_0 + (m/2)$. Comparing this to the value of $\langle \hat{N} \rangle$ obtained by using Eq. (2.18) we see that our choice of κ has fixed the value of $\langle \hat{J}_z \rangle$, for a neutral superconductor at $T = 0$ K, at zero. Note finally, that once κ has been chosen for a particular superconductor it remains fixed regardless of how the physical state of the

system changes. In other words, κ defines the pseudospin system we are dealing with.

C. The Charged Superconductor

As was shown above, a neutral superconductor may be characterized by a macroscopic angular momentum vector, $\langle \phi | \hat{\vec{J}} | \phi \rangle$, lying in the xy-plane. The fact that $\langle \phi | \hat{J}_z | \phi \rangle = 0$ is due to our particular choice of cut off κ . Since an analysis of the Josephson junction will, of necessity, include the possibility of excess charge on a superconductor, we must incorporate a description of the charge imbalance state into our angular momentum formalism. The natural way to do this is to require the expectation value of $\hat{J}_z$ to be nonzero for a charged superconductor. In particular, we shall characterize a superconductor with n excess electron pairs by the state $|\phi\rangle_n$ (to be described below) such that

$${}_n \langle \phi | \hat{J}_z | \phi \rangle_n = n \quad (2.19)$$

and

$${}_n \langle \phi | \hat{\vec{J}} | \phi \rangle_n = \left(\frac{m}{2}\right) \cos \phi \vec{e}_x - \left(\frac{m}{2}\right) \sin \phi \vec{e}_y + n \vec{e}_z \quad . \quad (2.20)$$

The macroscopic angular momentum is thereby able to simultaneously describe the superconducting properties (through m and ϕ) and the amount of charge imbalance (through n).

From our discussion at the end of the last section, it is easy to see that if a state $|\phi\rangle_n$ exists, then $\hat{J}_z$ will have the expectation value given in Eq. (2.19) since

$${}_n \langle \phi | \hat{N} | \phi \rangle_n = n + (N_0 + \frac{m}{2}) \quad . \quad (2.21)$$

Therefore our goal in this section is to obtain such a state.

To motivate Eq. (2.20) and provide a concrete model for $|\phi\rangle_n$ we first consider a neutral, nearly-free-electron metal, at $T = 0$ K. If n extra electron pairs are deposited on this metal one can, in principle, solve for the self-consistent one electron eigenstates. Filling these states with electron up to a Fermi energy ϵ_{nF} (Fermi momentum k_{nF}) would then produce the electronic charge density inside the metal. The resulting charge density would correspond to having the excess charge $-2|e|n$ localized near the surface. This excess charge density need not, however, be associated with single electrons localized near the surface. Rather, it can be thought of as being due to a large number of itinerant bulk electrons whose weakly perturbed self-consistent wave functions result in excess charge density near the surface.

Now imagine turning on the electron-phonon interaction in this charged metal. Since all electron states are delocalized we may take as one model for $|\phi\rangle_n$ the expression [9]

$$|\phi\rangle_n = \prod_{\vec{k}} [U_n(\vec{k}) + V_n(\vec{k}) e^{i\phi} \sigma_{\vec{k}}^+] |0\rangle \quad , \quad (2.22)$$

where $U_n(\vec{k})$ and $V_n(\vec{k})$ have the same step function shape as in the neutral metal but are now centered about the new Fermi momentum k_{nF} (Fig. 4). This model treats all electrons as being in Cooper pairs. (Note: If n pairs have been removed from a neutral superconductor, Fig. 5, then the state $|\phi\rangle_{-n}$ is obtained in a similar fashion to $|\phi\rangle_n$).

Figure 4. Charge-excess state occupation distributions.

Figure 5. Charge-depleted state occupation distributions.

This shift in Fermi momentum to k_{nF} results in an increase by n in the number of $\vec{k}$ -states in region (1), a decrease by n in region (3), and the number of states in region (2) remains constant. Repeating the analysis followed in Eqs. (2.9)-(2.13) we find that

$${}_n \langle \phi | \hat{J}^{(1)} | \phi \rangle_n = {}_n \langle \phi | \hat{J}_z^{(1)} | \phi \rangle_n \vec{e}_z = \left(\frac{N_0 + n}{2} \right) \vec{e}_z, \quad (2.23a)$$

$${}_n \langle \phi | \hat{J}^{(3)} | \phi \rangle_n = {}_n \langle \phi | \hat{J}_z^{(3)} | \phi \rangle_n \vec{e}_z = - \left(\frac{N_0 - n}{2} \right) \vec{e}_z, \quad (2.23b)$$

while ${}_n \langle \phi | \hat{J}^{(2)} | \phi \rangle_n$ has the same value as for the uncharged metal. Thus ${}_n \langle \phi | \hat{J} | \phi \rangle_n$ is given by Eq. (2.20). It is this equation together with Eq. (2.8) that allows us to represent a superconductor by an angular momentum operator whose expectation value reflects both the superconducting properties and the charge imbalance.

CHAPTER 3

THE JOSEPHSON JUNCTION: PREVIOUS THEORIES

We now turn our attention to the description of two weakly coupled superconductors to which we apply a constant voltage. This arrangement is shown in Fig. 6. In our discussion of this problem we shall assume the battery to be ideal so that the potential V is independent of the current flowing in the circuit (i.e., the battery maintains a constant charge imbalance between the sides of the junction). In addition we shall assume that the two superconductors forming the junction are identical in size and material. This considerably simplifies the algebra without sacrificing the essential physics.

To describe the coupling of the two superconductors, due to the possibility that electrons may tunnel through the thin insulating barrier, we use the tunneling Hamiltonian [10]

$$\hat{H}_g = \sum_{\vec{k}, \vec{q}} g_{\vec{k}\vec{q}} (C_{\vec{k}s}^{\dagger} C_{\vec{q}s} + C_{\vec{q}s}^{\dagger} C_{\vec{k}s}) \quad , \quad (3.1)$$

where the $\vec{k}(\vec{q})$ subscript refers to the left(right) superconductor. Since $\hat{H}_g$ contains only single electron tunneling terms the tunneling of paired electrons will occur in second order in $\hat{H}_g$. As shown by Anderson [3] and Wallace and Stavn [11], the effective second order tunneling Hamiltonian, which gives rise to the Josephson effect, may be written as

Figure 6. Josephson junction with ideal battery.

$$\hat{H}_T'' = \sum_{\vec{k}, \vec{q}} T_{\vec{k}, \vec{q}} (\sigma_{\vec{k}}^+ \sigma_{\vec{q}}^- + \sigma_{\vec{k}}^- \sigma_{\vec{q}}^+) \quad , \quad (3.2)$$

where $T_{\vec{k}\vec{q}}$ is the effective second order matrix element for pair tunneling. To further simplify the problem we may set $T_{\vec{k}\vec{q}} = T$ (a constant). This may be justified [9] by noting that for state vectors of the form $|\phi\rangle_n$ the important terms in $\hat{H}_T''$ are for those values of $\vec{k}$ and $\vec{q}$ which lie in the 2Δ -region (Fig. 4), tunneling into other states being prohibited by the exclusion principle. We may thus set $T_{\vec{k}\vec{q}}$ equal to its value at ϵ_F , and obtain

$$\hat{H}_T = T \sum_{\vec{k}, \vec{q}} (\sigma_{\vec{k}}^+ \sigma_{\vec{q}}^- + \sigma_{\vec{k}}^- \sigma_{\vec{q}}^+) \quad . \quad (3.3)$$

To include the electrostatic effects of the battery potential V , we use the Hamiltonian

$$\hat{H}_V = \frac{qV}{2} (\hat{N}_L - \hat{N}_R) \quad , \quad (3.4)$$

where q is the charge of a pair of electrons ($q = -2|e|$) and $\hat{N}_L - \hat{N}_R$ is the difference in the number of pairs on the left and right superconductors. Note that if one pair moves from right to left, N_R decreases by one, N_L increases by one, and the total change in energy is $2(qV/2)$ as expected. The total Hamiltonian for the junction is then given by

$$\hat{H} = \hat{H}_T + \hat{H}_V \quad . \quad (3.5)$$

A. (n, ϕ) Approximation

Anderson's approach [3] to the solution of a quantum mechanical system defined by Eqs. (3.3)-(3.5) proceeds by first noting that a pair correlated superconducting state $|N\rangle$, with definite pair number N , may be obtained from the BCS type state $|\phi\rangle$, Eq. (2.1), by the transformation

$$|N\rangle = \frac{1}{2\pi} \int_0^{2\pi} e^{-iN\phi} |\phi\rangle \quad . \quad (3.6)$$

Equation (3.6) suggests that N and ϕ may be taken to be canonically conjugate variables with

$$[\hat{N}, \hat{\phi}] = -i \quad (3.7a)$$

or

$$\hat{N} = -i \frac{\partial}{\partial \phi} \quad . \quad (3.7b)$$

Since $|\phi\rangle$ in Eq. (2.1) is a linear superposition of fixed particle number states, and every state with N' pairs is multiplied by $\exp(iN'\phi)$, the representation $\hat{N} = -i \partial/\partial\phi$ is indeed valid.

We may now write the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} |\Psi\rangle = \hat{H} |\Psi\rangle \quad (3.8)$$

for the junction using the set of product states

$$\{|\phi_L, \phi_R\rangle_n = |\phi_L\rangle_n |\phi_R\rangle_{-n}\} \quad (3.9)$$

as our restricted basis set, where n indicates the number of excess pairs on the left side of the junction. Equation (3.8) then becomes

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \sum_n \langle \phi_L, \phi_R | \Psi \rangle &= \\ &= \sum_{n'} \int d\phi'_L d\phi'_R \sum_n \langle \phi_L, \phi_R | \hat{H} | \phi'_L, \phi'_R \rangle_{n'} \sum_n \langle \phi'_L, \phi'_R | \Psi \rangle \end{aligned} \quad (3.10)$$

If we now assume that only the diagonal matrix elements of the Hamiltonian are important, we obtain an effective Hamiltonian of the form

$$\begin{aligned} \hat{H}' &= \sum_n \langle \phi_L, \phi_R | \hat{H} | \phi'_L, \phi'_R \rangle_n \\ &= \left[\left(\frac{qV}{2} \right) (\hat{N}_L - \hat{N}_R) + \lambda \cos(\phi_L - \phi_R) \right] \delta(\phi_R - \phi'_R) \delta(\phi_L - \phi'_L) \end{aligned} \quad (3.11)$$

where $\hat{N}_L = -i \partial / \partial \phi_L$, $\hat{N}_R = -i \partial / \partial \phi_R$, λ is a constant, and n is fixed by the voltage of the battery and the capacitance of the junction through the relation

$$n = \frac{CV}{2|e|} \quad (3.12)$$

(Here, and throughout this dissertation, when an ideal battery is placed across a junction, the pair difference between the sides is taken to be a constant, i.e., neither voltage fluctuations nor charge build up are being considered.)

We may now define the operators

$$\hat{n} = \left(\frac{1}{2} \right) (\hat{N}_L - \hat{N}_R) \quad (3.13a)$$

and

$$\hat{\Phi} = \hat{\Phi}_L - \hat{\Phi}_R \quad . \quad (3.13b)$$

Using these operators we obtain the (n, ϕ) Hamiltonian for the junction in the form

$$H_{n, \phi} = q V \hat{n} + \lambda \cos \hat{\Phi} \quad (3.14)$$

with

$$[\hat{n}, \hat{\Phi}] = -i \quad . \quad (3.15)$$

The Heisenberg equations of motion for $\hat{n}$ and $\hat{\Phi}$ may be determined from Eq. (3.14) with the result

$$\frac{d\hat{n}}{dt} = \left(\frac{-i}{\hbar} \right) [\hat{n}, H_{n, \phi}] = \left(\frac{\lambda}{\hbar} \right) \sin \hat{\Phi} \quad (3.16)$$

$$\frac{d\hat{\Phi}}{dt} = \left(\frac{-i}{\hbar} \right) [\hat{\Phi}, H_{n, \phi}] = \frac{qV}{\hbar} \quad .$$

Using the fact that the current operator is given by

$$\hat{I} = -q \frac{d\hat{n}}{dt} \quad (3.17)$$

(current is taken as positive when positive charge moves from left to right), Eq. (3.16) yields the well-known Josephson relation

$$\begin{aligned} I &= -q \left(\frac{\lambda}{\hbar} \right) \sin \left[\left(\frac{qV}{\hbar} \right) t + \phi_L(0) - \phi_R(0) \right] \\ &= q \left(\frac{\lambda}{\hbar} \right) \sin [\omega t - \phi_L(0) + \phi_R(0)] \end{aligned} \quad (3.18)$$

with $\omega = (2|e|V)/\hbar$. It should be noted that Eq. (3.16) determines only the change in n due to the tunneling Hamiltonian; our calculation assumes that in actual fact $dn/dt = 0$ with the ideal battery providing the necessary current to keep the total charge on each side of the junction a constant. Thus, the current given by Eq. (3.18) is the current which flows in the circuit.

B. Two-state Approximation

The procedure developed by Feynman [4] to solve the Schrödinger equation for the Josephson junction involves much more intuitive arguments than the (n, ϕ) approach. Note that

$$[\sigma_{\vec{k}}^+, \sigma_{\vec{k}'}^+] = 0$$

and referring to Eq. (2.5), with $\vec{k} \neq \vec{k}'$, we see that the pair creation and annihilation operators obey Bose-Einstein commutation relations. Thus we can consider the electron pairs to be bosons with the result that at $T = 0$ K all pairs will be in the ground state ψ . The probability of finding a pair at a given point in the superconductor is $\psi^*\psi$ and we expect this product to be proportional to the density of pairs. Thus Feynman assumes that each superconductor may be described by a macroscopic wavefunction of the form

$$\begin{aligned}\psi_L &= \sqrt{\rho_L} e^{i\theta_L}, \\ \psi_R &= \sqrt{\rho_R} e^{i\theta_R},\end{aligned}\tag{3.19}$$

where $\rho_{L(R)}$, the charge density, is assumed to be spatially uniform. The diagonal matrix representing $\hat{H}_V$, Eq. (3.4), in the two-state basis defined by Eq. (3.19) is given by

$$\hat{H}_V = \begin{pmatrix} qV/2 & 0 \\ 0 & -qV/2 \end{pmatrix}, \quad (3.20)$$

since

$$\begin{aligned} qN_L &= \int_{\Omega} \psi_L^* \psi_L d\tau \\ qN_R &= \int_{\Omega} \psi_R^* \psi_R d\tau \end{aligned} \quad (3.21)$$

and Ω is the volume of each of the identical superconductors. The Hamiltonian $\hat{H}_T$, Eq. (3.3), allows pairs to tunnel and may thus be represented by the off diagonal matrix

$$\hat{H}_T = \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix}. \quad (3.22)$$

The time dependent Schrödinger equation assumes the form

$$\begin{aligned} i\hbar \frac{\partial \psi_L}{\partial t} &= \left(\frac{qV}{2} \right) \psi_L + K \psi_R, \\ i\hbar \frac{\partial \psi_R}{\partial t} &= -\left(\frac{qV}{2} \right) \psi_R + K \psi_L \end{aligned} \quad (3.23)$$

which, using Eq. (3.19), may be written as

$$\frac{d\rho_L}{dt} = -\frac{d\rho_R}{dt} = \left(\frac{2}{\hbar}\right) K\sqrt{\rho_L\rho_R} \sin(\theta_R - \theta_L) \quad , \quad (3.24a)$$

$$\frac{d\theta_L}{dt} = \left(\frac{K}{\hbar}\right) \sqrt{\frac{\rho_R}{\rho_L}} \cos(\theta_R - \theta_L) - \left(\frac{qV}{2\hbar}\right) \quad , \quad (3.24b)$$

$$\frac{d\theta_R}{dt} = \left(\frac{K}{\hbar}\right) \sqrt{\frac{\rho_L}{\rho_R}} \cos(\theta_R - \theta_L) + \left(\frac{qV}{2\hbar}\right) \quad . \quad (3.24c)$$

These equations may be integrated to yield

$$\theta_R - \theta_L = \left(\frac{qV}{\hbar}\right)t + \theta_R(0) - \theta_L(0) \quad . \quad (3.25)$$

With the current I defined as $-\Omega(d\rho_L/dt)$, Eqs. (3.24a) and (3.25) can be used to obtain

$$\begin{aligned} I &= -\left(\frac{2\Omega K}{\hbar}\right) \sqrt{\rho_L\rho_R} \sin \left[\left(\frac{qV}{\hbar}\right)t + \theta_R(0) - \theta_L(0) \right] \\ &= \left(\frac{2\Omega K}{\hbar}\right) \sqrt{\rho_L\rho_R} \sin [\omega t + \theta_L(0) - \theta_R(0)] \quad . \end{aligned} \quad (3.26)$$

Here again we have assumed that, due to the presence of the perfect battery, ρ_L and ρ_R may be considered constant on the right hand side of Eq. (3.24). Equation (3.26) has the same structure as Eq. (3.18) and thus gives rise to the Josephson effect. The two equations are in fact identical if we make the identifications

$$\begin{aligned} \theta_L &= -\phi_L \\ \theta_R &= -\phi_R \\ q\lambda &= 2\Omega K \sqrt{\rho_L\rho_R} \quad . \end{aligned} \quad (3.27)$$

In the following chapter we shall, in the course of solving the junction problem using the angular momentum formalism, obtain an alternative derivation of Eq. (3.26).

CHAPTER 4

THE JOSEPHSON JUNCTION: ANGULAR MOMENTUM APPROACH

We begin this chapter by reformulating the junction problem in terms of our macroscopic angular momentum operators [Eq. (2.8), with one angular momentum vector for each side, $\hat{\vec{J}}_{L(R)}$]. The equations of motion for $\hat{\vec{J}}_{L(R)}$ now determine the dynamics of the junction, and their solution leads directly to the Josephson current. Finally we show that under a specific approximation the angular momentum formalism reduces to the (n, ϕ) and two-state methods.

A. Angular Momentum Formalism

Using the definitions of $\hat{J}^+$ and $\hat{J}^-$ given by Eq. (2.8), the tunneling Hamiltonian, Eq. (3.3), may be written in the simple form

$$\begin{aligned} H_T &= T(\hat{J}_L^+ \hat{J}_R^- + \hat{J}_L^- \hat{J}_R^+) \\ &= 2T \hat{\vec{J}}_{L\perp} \cdot \hat{\vec{J}}_{R\perp} , \end{aligned} \tag{4.1}$$

where

$$\hat{\vec{J}}_{\perp} = \hat{J}_x \vec{e}_x + \hat{J}_y \vec{e}_y \tag{4.2}$$

is the perpendicular (to the z-axis) component of the angular momentum operator. $\hat{H}_T$ thus assumes the form identified with the commonly used xy-model of magnetism. To rewrite $\hat{H}_V$ in terms of our angular momentum operators we need only substitute the appropriate form of Eq. (2.18)

for $\hat{N}_L$ and $\hat{N}_R$ (remembering that the superconductors have been assumed identical), into Eq. (3.4). This gives

$$\hat{H}_V = (qV/2) (\hat{J}_{Lz} - \hat{J}_{Rz}) \quad (4.3)$$

and the total Hamiltonian then becomes

$$\hat{H} = (qV/2) (\hat{J}_{Lz} - \hat{J}_{Rz}) + 2T \hat{\vec{J}}_{L\perp} \cdot \hat{\vec{J}}_{R\perp} \quad (4.4)$$

Using the standard angular momentum commutation rules, the Heisenberg equations of motion yield

$$\begin{aligned} \frac{d\hat{\vec{J}}_L}{dt} &= \left(\frac{-i}{\hbar} \right) [\hat{\vec{J}}_L, \hat{H}] = \left(\frac{-1}{\hbar} \right) \hat{\vec{J}}_L \times \hat{\vec{B}}_R, \\ \frac{d\hat{\vec{J}}_R}{dt} &= \left(\frac{-i}{\hbar} \right) [\hat{\vec{J}}_R, \hat{H}] = \left(\frac{-1}{\hbar} \right) \hat{\vec{J}}_R \times \hat{\vec{B}}_L, \end{aligned} \quad (4.5)$$

where

$$\begin{aligned} \hat{\vec{B}}_L &= 2T \hat{\vec{J}}_{L\perp} - b\vec{e}_z, \\ \hat{\vec{B}}_R &= 2T \hat{\vec{J}}_{R\perp} + b\vec{e}_z, \end{aligned} \quad (4.6)$$

and $b = qV/2$. When written as above, the dynamics of a Josephson junction may be viewed in terms of two coupled angular momenta which have equal but opposite, external "magnetic" fields applied along the z-axis ($\hat{\vec{B}}_L \cdot \vec{e}_z = -\hat{\vec{B}}_R \cdot \vec{e}_z$).

By using $\hat{\vec{J}}_{L(R)\perp}$ the equations of motion may be written in a more transparent fashion as

$$\frac{d\hat{J}_{Lz}}{dt} = \left(\frac{-2T}{\hbar} \right) [\vec{e}_z \cdot (\hat{\vec{J}}_{L\perp} \times \hat{\vec{J}}_{R\perp})] , \quad (4.7a)$$

$$\frac{d\hat{J}_{Rz}}{dt} = \left(\frac{-2T}{\hbar} \right) [\vec{e}_z \cdot (\hat{\vec{J}}_{R\perp} \times \hat{\vec{J}}_{L\perp})] , \quad (4.7b)$$

$$\frac{d\hat{\vec{J}}_{L\perp}}{dt} = \left(\frac{-b}{\hbar} \right) (\hat{\vec{J}}_{L\perp} \times \vec{e}_z) + \left(\frac{2T}{\hbar} \right) \hat{J}_{Lz} (\hat{\vec{J}}_{R\perp} \times \vec{e}_z) , \quad (4.7c)$$

$$\frac{d\hat{\vec{J}}_{R\perp}}{dt} = \left(\frac{b}{\hbar} \right) (\hat{\vec{J}}_{R\perp} \times \vec{e}_z) + \left(\frac{2T}{\hbar} \right) \hat{J}_{Rz} (\hat{\vec{J}}_{L\perp} \times \vec{e}_z) . \quad (4.7d)$$

We shall now proceed to obtain solutions to the above set of equations. It is important to note that Eqs. (4.7) are operator equations. We can, however, take the expectation value of both sides of these equations and thereby obtain equations which determine the time evolution of the expectation value of the operators. In that case the expectation values will be taken with respect to the state $|\alpha_L, \alpha_R\rangle_n$, i.e., $|\alpha_L, \alpha_R\rangle_n = |\alpha_L\rangle_n |\alpha_R\rangle_{-n}$ where $|\alpha_L\rangle_n$ is given by Eq. (2.22) with the phase angle now labeled as α_L .

Our notation in what follows will be

$$\vec{J}_{L(R)} = \langle \hat{\vec{J}}_{L(R)} \rangle$$

$$J_{L(R)} = |\vec{J}_{L(R)}| = |\langle \hat{\vec{J}}_{L(R)} \rangle|$$

with $\vec{J}_{L(R)}$ and $J_{L(R)}$ considered as classical objects.

B. The Rotating Vector Solution

As will be shown later, the term containing T in Eqs. (4.7c) and (4.7d) is in general much smaller than the term proportional to b . If we drop the former term and take expectation values, the equations of motion become

$$\frac{dJ_{Lz}}{dt} = -\frac{dJ_{Rz}}{dt} = \left(\frac{-2T}{\hbar} \right) [\vec{e}_z \cdot (\vec{J}_{L\perp} \cdot \vec{J}_{R\perp})] , \quad (4.8a)$$

$$\frac{d\vec{J}_{L\perp}}{dt} = \left(\frac{-b}{\hbar} \right) (\vec{J}_{L\perp} \times \vec{e}_z) , \quad (4.8b)$$

$$\frac{d\vec{J}_{R\perp}}{dt} = \left(\frac{b}{\hbar} \right) (\vec{J}_{R\perp} \times \vec{e}_z) . \quad (4.8c)$$

To understand these equations we first note that the expectation values $(\vec{J}_{L\perp}$ and $\vec{J}_{R\perp})$ of our Heisenberg operators, at $t = 0$, can be obtained by repeating the steps leading to Eq. (2.14) but now using the state $|\alpha_L, \alpha_R\rangle_n$. Doing so gives

$$\vec{J}_{L\perp}(0) = \left(\frac{m}{2} \right) \cos \alpha_L(0) \vec{e}_x - \left(\frac{m}{2} \right) \sin \alpha_L(0) \vec{e}_y ,$$

$$\vec{J}_{R\perp}(0) = \left(\frac{m}{2} \right) \cos \alpha_R(0) \vec{e}_x - \left(\frac{m}{2} \right) \sin \alpha_R(0) \vec{e}_y ,$$

with the angles α_L and α_R defined in Fig. 7. (It is important to remember that $\alpha_{L(R)}$ are defined so that a clockwise rotation corresponds to increasing $\alpha_{L(R)}$.) Thus $\vec{J}_{L\perp}(0)$ and $\vec{J}_{R\perp}(0)$ are two vectors lying in the xy -plane. The form of Eqs. (4.8b) and (4.8c) then immediately shows that $J_{L\perp}$ and $J_{R\perp}$ are constant and that $\vec{J}_{L\perp}$ rotates in a

Figure 7. Rotation of vectors $\vec{J}_{L1}$ and $\vec{J}_{R1}$.

clockwise direction while $\vec{J}_{R\perp}$ rotates in a counterclockwise sense, that is

$$\begin{aligned}\alpha_L(t) &= \left[\frac{-b}{\hbar} \right] t + \alpha_L(0) \quad , \\ \alpha_R(t) &= \left[\frac{b}{\hbar} \right] t + \alpha_R(0) \quad .\end{aligned}\tag{4.9}$$

This allows us to write

$$\begin{aligned}\vec{J}_{L\perp}(t) &= \left[\frac{m}{2} \right] \cos \alpha_L(t) \vec{e}_x - \left[\frac{m}{2} \right] \sin \alpha_L(t) \vec{e}_y \quad \dots \\ \vec{J}_{R\perp}(t) &= \left[\frac{m}{2} \right] \cos \alpha_R(t) \vec{e}_x - \left[\frac{m}{2} \right] \sin \alpha_R(t) \vec{e}_y\end{aligned}$$

which we can substitute into Eq. (4.8a) to obtain

$$\begin{aligned}\frac{dJ_{Lz}}{dt} &= \left[\frac{-2T}{\hbar} \right] \left[\frac{m}{2} \right]^2 \sin (\alpha_L - \alpha_R) \\ &= \left[\frac{-2T}{\hbar} \right] \left[\frac{m}{2} \right]^2 \sin \left[\left[\frac{-2b}{\hbar} \right] t + \alpha_L(0) - \alpha_R(0) \right] \quad .\end{aligned}\tag{4.10}$$

The current operator is defined as

$$\begin{aligned}\hat{I} &= -q \frac{d\hat{N}_L}{dt} \\ &= -q \frac{d\hat{J}_{Lz}}{dt} \quad ,\end{aligned}\tag{4.11}$$

where the last equality follows from Eq. (2.18). Noting that $b = qV/2$ and $J_{L\perp} = J_{R\perp} = m/2$ gives

$$\begin{aligned}
I &= -q \left[\frac{2T}{\hbar} \left(\frac{m}{2} \right)^2 \right] \sin \left[\left[\frac{qV}{\hbar} \right] t + \alpha_R(0) - \alpha_L(0) \right] \\
&= q \left[\frac{2T}{\hbar} \left(\frac{m}{2} \right)^2 \right] \sin [\omega t + \alpha_L(0) - \alpha_R(0)]
\end{aligned}
\tag{4.12}$$

by substituting Eq. (4.10) into (4.11). Thus, by using our macroscopic angular momenta we have been able to derive the Josephson current relationship.

The above method of solution provides us with the following concrete picture of the dynamics of the Josephson junction: The Josephson current, Eq. (4.12), is the result of taking the cross product of two, constant magnitude, macroscopic pseudo-angular-momentum vectors which lie in the xy-plane. These vectors in turn precess about the z-axis in opposite directions each with frequency $|e|V/\hbar$ (Fig. 7).

Finally, a comparison of Eq. (4.12) to Eqs. (3.18) and (3.26) allows the following identifications:

$$\alpha_L = -\phi_L = \theta_L, \tag{4.13a}$$

$$\alpha_R = -\phi_R = \theta_R, \tag{4.13b}$$

$$q \left[\frac{2T}{\hbar} \left(\frac{m}{2} \right)^2 \right] = q \left(\frac{\lambda}{\hbar} \right) = \left(\frac{2\Omega K}{\hbar} \right) \sqrt{\rho_L \rho_R}. \tag{4.13c}$$

C. The (n, ϕ) Approximation Revisited

In this section we will obtain the (n, ϕ) theory from the angular momentum approach. To do this we will only need the approximated form of the angular momentum equations of motion given in section B. One

result of this approximation is that the magnitudes of $\vec{J}_{L\perp}$ and $\vec{J}_{R\perp}$ are constant.

We start by writing our Hilbert space Hamiltonian, Eq. (4.4), in the $\{|\alpha_L, \alpha_R\rangle\}$ basis. Following the same procedures as in Chapter 3, section A gives for $\hat{H}_T$

$$\begin{aligned}\hat{H}_T &= 2T J_{L\perp} J_{R\perp} \cos(\alpha_L - \alpha_R) \\ &= 2T \left(\frac{m}{2}\right)^2 \cos(\alpha_L - \alpha_R) \quad .\end{aligned}\tag{4.14}$$

Using Eqs. (3.13) and (2.18) we may also write

$$\hat{n} = \frac{1}{2} (\hat{N}_L - \hat{N}_R) = \frac{1}{2} (\hat{J}_{Lz} - \hat{J}_{Rz}) \tag{4.15}$$

so that our total Hamiltonian assumes the form

$$\hat{H} = qV\hat{n} + 2T \left(\frac{m}{2}\right)^2 \cos(\alpha_L - \alpha_R) \quad . \tag{4.16}$$

To determine the relationship between $\hat{n}$ and $(\alpha_L - \alpha_R)$ we require that the equations of motion for $\langle \hat{n} \rangle$ and $\langle \frac{1}{2}(\hat{J}_{Lz} - \hat{J}_{Rz}) \rangle$ be identical. This implies, by referring to Eq. (4.8a), that

$$\begin{aligned}\left\langle \frac{d\hat{n}}{dt} \right\rangle &= \frac{1}{i\hbar} \langle [\hat{n}, \hat{H}] \rangle \\ &= \frac{-2T}{\hbar} \left(\frac{m}{2}\right)^2 \sin(\alpha_L - \alpha_R) \quad .\end{aligned}\tag{4.17}$$

Defining $\alpha = \alpha_L - \alpha_R$ and inserting $\hat{H}$ into (4.7) we find that

$$-i [\hat{n}, \cos \alpha] = -\sin \alpha .$$

Consequently

$$\hat{n} = i \frac{\partial}{\partial \alpha} \quad (4.18)$$

is an operator of $\hat{n}$ in terms of α , and

$$[\hat{n}, \alpha] = i . \quad (4.19)$$

Equation (4.19) is identical to the commutator $[\hat{n}, \Phi] = -i$ used in the (n, Φ) theory since Eq. (4.13) gives $\alpha = -\Phi$.

Thus, the (n, Φ) theory is equivalent to the approximated form of the pseudo-angular-momentum theory given in section B, i.e., the pseudo-angular-momentum theory with

$$J_{L\perp} = |\langle \hat{J}_{L\perp} \rangle|$$

$$J_{R\perp} = |\langle \hat{J}_{R\perp} \rangle|$$

taken as time dependent.

D. The Two-state Approximation Revisited

In this section we show that Feynman's two-state method of handling the Josephson problem can be derived from the restricted form of the pseudo-angular-momentum theory of section B. Our demonstration starts by taking the Feynman variables ρ_L and ρ_R , and identifying them with their angular momentum counterparts. These then are used to

rewrite the angular momentum equations of motion, Eq. (4.8), in terms of the Feynman variables and thereby obtain the two-state equations of motion. To carry out this program it is convenient to go to a frame of reference which rotates about the z-axis with angular velocity $v(t)$ (i.e., the angle of rotation is $\int v(t) dt$). In this rotating frame Eqs. (4.8) become

$$\frac{dJ_{Lz}}{dt} = - \frac{dJ_{Rz}}{dt} = \left(\frac{2V}{\hbar} \right) J_{L\perp} J_{R\perp} \sin (\alpha'_R - \alpha'_L) , \quad (4.20a)$$

$$\frac{d\alpha'_L}{dt} = v(t) - \left(\frac{qV}{2\hbar} \right) , \quad (4.20b)$$

$$\frac{d\alpha'_R}{dt} = v(t) + \left(\frac{qV}{2\hbar} \right) . \quad (4.20c)$$

According to Feynman, ρ_L is the electron density in the metal. Consequently $\Omega\rho_L$ is the total electronic charge. This quantity can be written in terms of our angular momentum variables by using the expectation value of Eq. (2.18)

$$\Omega\rho_L = qN_L = q(J_{Lz} + N_0 + J_{L\perp}) , \quad (4.21)$$

$$\Omega\rho_R = qN_R = q(J_{Rz} + N_0 + J_{R\perp}) ,$$

where $J_L = J_R = m/2$. As a consequence of these we have

$$\begin{aligned} \Omega \frac{d\rho_L}{dt} &= q \frac{dN_L}{dt} = q \frac{dJ_{Lz}}{dt} , \\ \Omega \frac{d\rho_R}{dt} &= q \frac{dN_R}{dt} = q \frac{dJ_{Rz}}{dt} . \end{aligned} \quad (4.22)$$

At this point we place the same conditions on ρ_L and ρ_R that Feynman does. To begin with, the time derivatives in Eq. (4.22) only refer to the changes due to tunneling. These densities are in fact constant due to the action of the ideal battery. Furthermore, the densities are approximated as being equal, $\rho_L = \rho_R$. This is equivalent to saying that

$$\begin{aligned}\Omega\rho_L &= q(N_o + J_{L\perp}) \quad , \\ \Omega\rho_R &= q(N_o + J_{R\perp}) \quad .\end{aligned}\tag{4.23}$$

Substituting Eqs. (4.22) and (4.23) into (4.20a) gives

$$\begin{aligned}\frac{d\rho_L}{dt} &= \frac{q}{\Omega} \left(\frac{2T}{\hbar} \right) \left[\left(\frac{\Omega\rho_L}{q} - N_o \right) \left(\frac{\Omega\rho_R}{q} - N_o \right) \right] \sin(\alpha'_R - \alpha'_L) \\ &= \left(\frac{2}{\hbar} \right) K \sqrt{\rho_L \rho_R} \sin(\alpha'_R - \alpha'_L) \quad ,\end{aligned}\tag{4.24a}$$

if we define K as

$$K = T \left[\left(\frac{J_{L\perp}}{\sqrt{N_o + J_{L\perp}}} \right) \left(\frac{J_{R\perp}}{\sqrt{N_o + J_{R\perp}}} \right) \right] .$$

Continuing, we define our rotating coordinate system such that the angular velocity is

$$\begin{aligned}v(t) &= \left(\frac{K}{\hbar} \right) \left[\frac{(N_o + J_L)}{(N_o + J_R)} \right]^{\frac{1}{2}} \cos(\alpha'_R - \alpha'_L) \\ &= \left(\frac{K}{\hbar} \right) \sqrt{\frac{\rho_L}{\rho_R}} \cos(\alpha'_R - \alpha'_L) \quad .\end{aligned}$$

Substituting this into Eqs. (4.20b) and (4.20c) gives

$$\frac{d\alpha'_L}{dt} = \left(\frac{K}{\hbar}\right) \sqrt{\frac{\rho_R}{\rho_L}} \cos(\alpha'_R - \alpha'_L) - \left(\frac{qV}{2\hbar}\right) , \quad (4.24b)$$

$$\frac{d\alpha'_R}{dt} = \left(\frac{K}{\hbar}\right) \sqrt{\frac{\rho_L}{\rho_R}} \cos(\alpha'_R - \alpha'_L) + \left(\frac{qV}{2\hbar}\right) , \quad (4.24c)$$

where we have used the fact that $\rho_L = \rho_R$. Noting that

$$\alpha'_R - \alpha'_L = \alpha_R - \alpha_L = \theta_R - \theta_L$$

we see that Eqs. (4.24) are identical to Eqs. (3.24). Consequently, we have converted the angular momentum equations of motion into the equations for Feynman's coupled two-state system.

Finally, the above derivation allows us to write Feynman's macroscopic wavefunctions, Eq. (3.19), as

$$\begin{aligned} \psi_L &= \left[\frac{q(J_{Lz} + N_O + J_{L\perp})}{\Omega} \right]^{\frac{1}{2}} e^{i\alpha'_L} , \\ \psi_R &= \left[\frac{q(J_{Rz} + N_O + J_{R\perp})}{\Omega} \right]^{\frac{1}{2}} e^{i\alpha'_R} . \end{aligned} \quad (4.25)$$

Thus, the pseudo-angular-momentum formulation (sections A and B), along with the work of this section, effectively constitute a microscopic derivation of Feynman's two-state model for the Josephson junction, and provide a microscopic interpretation of ψ_L , ψ_R , and K .

E. Some Numerical Values

In order to maintain the train of thought, we have, in the previous sections, often made approximations which at the time were not completely justified. In this section we shall provide some order of magnitude numbers which may be used to confirm our approximations.

To estimate the value of m , ($J_{L1} = J_{R1} = m/2$), we set

$$m = \mathfrak{D}(\epsilon_F) \cdot 2\Delta, \quad (4.26)$$

where $\mathfrak{D}(\epsilon_F)$ is the density of states per unit energy at the Fermi energy and Δ is the BCS energy gap parameter at $T = 0$ K. For a metal with a volume $\approx 10^{-6}$ cm³ and $\epsilon_F \approx 5 \times 10^4$ K, we have $\mathfrak{D}(\epsilon_F) \approx 6 \times 10^{11}$ states/°K. Taking $\Delta \approx 5$ K, this implies that

$$J_{L1} \approx 1.5 \times 10^{12}. \quad (4.27)$$

To estimate $|J_{Lz}| = n$ (n = number of excess pairs), we may write

$$|J_{Lz}| = n = \left| \frac{CV}{q} \right|. \quad (4.28)$$

Taking $C \approx 10^{-6}$ F, $V \approx 10^{-4}$ Volts, and $q = -2|e|$ yields

$$|J_{Lz}| \approx 3 \times 10^8$$

and

$$\frac{|J_{Lz}|}{|J_{L1}|} \approx 10^{-4}. \quad (4.29)$$

Equation (4.29) justifies the approximation

$$J_{Lz} + N_o + J_{L1} \approx N_o + J_{L1}$$

used in deriving the Feynman model of the junction.

Finally, in all the solutions discussed so far we have assumed that

$$|2T J_{Lz}| \ll |b| \quad (4.30)$$

so that Eqs. (4.7) are to a good approximation, equivalent to Eqs. (4.8). To determine the validity of (4.30) we use Eq. (4.12) to write

$$|T| = \frac{2\hbar j_1}{|q| m^2} \quad (4.31)$$

where j_1 is the amplitude of the ac Josephson current. Using Eq. (4.28) we may write

$$\left| \frac{2T J_{Lz}}{b} \right| = \frac{4j_1 \hbar}{q m^2} \left(\frac{2C}{q^2} \right) .$$

Then taking $j_1 \approx 10^{-3}$ amperes, together with our previous results, gives

$$\left| \frac{2T J_{Lz}}{b} \right| \approx 10^{-11} . \quad (4.32)$$

Equations (4.8) thus represent an excellent approximation to the junction problem.

For use in the next section we define the variable a as

$$a = \frac{2T J_{Lz}}{\hbar \omega} = \frac{-T J_{Lz}}{b} \quad (4.33)$$

$$|a| \approx 0.5 \times 10^{-11} .$$

CHAPTER 5

SOLUTION TO THE EXACT EQUATIONS OF MOTION: PERTURBATION THEORY

In spite of the small value obtained in Eq. (4.32), it is worthwhile to explore a more exact solution of Eqs. (4.7). Taking expectation values, these equations become

$$\frac{dJ_{Lz}}{dt} = \left(\frac{-2T}{\hbar} \right) [\vec{e}_z \cdot (\vec{J}_{L\perp} \times \vec{J}_{R\perp})] \quad , \quad (5.1a)$$

$$\frac{d\vec{J}_{L\perp}}{dt} = \left(\frac{-b}{\hbar} \right) (\vec{J}_{L\perp} \times \vec{e}_z) + \left(\frac{2T}{\hbar} \right) J_{Lz} (\vec{J}_{R\perp} \times \vec{e}_z) \quad , \quad (5.1b)$$

$$\frac{d\vec{J}_{R\perp}}{dt} = \left(\frac{b}{\hbar} \right) (\vec{J}_{R\perp} \times \vec{e}_z) + \left(\frac{2T}{\hbar} \right) J_{Rz} (\vec{J}_{L\perp} \times \vec{e}_z) \quad . \quad (5.1c)$$

Starting with Eqs. (5.1b) and (5.1c) we see that the simple analysis of Chapter 4, section B, is no longer possible because of terms containing T . That is, even though $\vec{J}_{L\perp}$ and $\vec{J}_{R\perp}$ are still in the xy -plane, their magnitudes are (in general) not constant, while $\alpha_L(t)$ and $\alpha_R(t)$ are no longer given by Eq. (4.9). Consequently we write the solution for $\vec{J}_{L\perp}(t)$ and $\vec{J}_{R\perp}(t)$ as

$$\begin{aligned} \vec{J}_L(t) &= J_{L\perp}(t) [\cos \alpha_L(t) \vec{e}_x - \sin \alpha_L(t) \vec{e}_y] \quad , \\ \vec{J}_R(t) &= J_{R\perp}(t) [\cos \alpha_R(t) \vec{e}_x - \sin \alpha_R(t) \vec{e}_y] \quad , \end{aligned} \quad (5.2)$$

with $J_{L\perp}(t)$, $J_{R\perp}(t)$, $\alpha_L(t)$, and $\alpha_R(t)$ to be determined.

With the above information on $\vec{J}_{L\perp}$ and $\vec{J}_{R\perp}$, and for the purpose of simplifying our differential equations, we make the following identifications

$$\begin{aligned} J_L^- &\equiv \langle \hat{J}_L^- \rangle \\ &= J_{Lx} - iJ_{Ly} = J_{L\perp}(t) e^{i\alpha_L(t)} , \end{aligned} \quad (5.3)$$

$$\begin{aligned} J_R^- &\equiv \langle \hat{J}_R^- \rangle \\ &= J_{Rx} - iJ_{Ry} = J_{R\perp}(t) e^{i\alpha_R(t)} . \end{aligned}$$

Using Eqs. (5.2) and (5.3) we can rewrite the equations of motion as

$$\frac{dJ_{Lz}}{dt} = \left(\frac{2T}{\hbar} \right) J_{L\perp}(t) J_{R\perp}(t) \sin [\alpha_R(t) - \alpha_L(t)] , \quad (5.4a)$$

$$\frac{dJ_L^-}{dt} = -i \left(\frac{b}{\hbar} \right) J_L^- + i \left(\frac{2T}{\hbar} \right) J_{Lz} J_R^- , \quad (5.4b)$$

$$\frac{dJ_R^-}{dt} = i \left(\frac{b}{\hbar} \right) J_R^- + i \left(\frac{2T}{\hbar} \right) J_{Rz} J_L^- . \quad (5.4c)$$

Equations (5.4b) and (5.4c) may be further simplified by writing

$$\begin{aligned} J_L^- &= \mathfrak{J}_L(t) e^{-ibt/\hbar} , \\ J_R^- &= \mathfrak{J}_R(t) e^{+ibt/\hbar} , \end{aligned} \quad (5.5)$$

to obtain

$$\begin{aligned}\dot{\mathfrak{J}}_L(t) &= i \left(\frac{2T}{\hbar} \right) J_{Lz} e^{-i\omega t} \mathfrak{J}_R(t) , \\ \dot{\mathfrak{J}}_R(t) &= i \left(\frac{2T}{\hbar} \right) J_{Rz} e^{+i\omega t} \mathfrak{J}_L(t) ,\end{aligned}\tag{5.6}$$

with $\omega = -2b/\hbar = 2|e|V/\hbar$. Since we have assumed an ideal battery to be placed across the junction, we may take $J_{Lz} = -J_{Rz} = \text{constant}$ in Eqs. (5.6). [Remember that Eq. (5.4a) describes only the change in J_{Lz} due to tunneling.] In this case Eqs. (5.6) may be solved exactly. We shall, however, use a perturbation theory approach due to the size of our expansion parameter, which we choose as a of Eq. (4.33).

To solve Eqs. (5.6) we first assume that $\mathfrak{J}_{L(R)}$ can be written in terms of a power series in a

$$\mathfrak{J}_{L(R)} = \sum_{s=0}^{\infty} \mathfrak{J}_{L(R)}^{(s)} ,\tag{5.7}$$

where the superscript denotes the power to which a appears. Because a is so small, we restrict ourselves to first order corrections. Substituting into Eq. (5.6) gives

$$\begin{aligned}\dot{\mathfrak{J}}_L^{(1)}(t) &= i \left(\frac{2TJ_{Lz}}{\hbar} \right) e^{-i\omega t} \mathfrak{J}_R^{(0)}(t) , \\ \dot{\mathfrak{J}}_R^{(1)}(t) &= i \left(\frac{2TJ_{Rz}}{\hbar} \right) e^{+i\omega t} \mathfrak{J}_L^{(0)}(t) .\end{aligned}\tag{5.8}$$

By replacing T with

$$\lim_{\epsilon \rightarrow 0} T e^{\epsilon t} ,$$

so that the perturbation is turned on adiabatically, we can define the zeroth order term as the solution to our differential equations when the perturbation is turned off [Eqs. (5.1b) and (5.1c) with $T = 0$].

These were given in Chapter 4, section B, and result in

$$\begin{aligned} \mathfrak{J}_L^{(0)}(t) &= \left(\frac{m}{2}\right) e^{i\alpha_L^{(0)}(0)} , \\ \mathfrak{J}_R^{(0)}(t) &= \left(\frac{m}{2}\right) e^{i\alpha_R^{(0)}(0)} . \end{aligned}$$

Finally, integrating Eq. (5.8) and taking $\lim \epsilon \rightarrow 0$, gives for the first order corrections,

$$\begin{aligned} \mathfrak{J}_L^{(1)}(t) &= - \left(\frac{2TJ_{Lz}}{\hbar\omega} \right) e^{-i\omega t} \mathfrak{J}_R^{(0)} , \\ \mathfrak{J}_R^{(1)}(t) &= - \left(\frac{2TJ_{Lz}}{\hbar\omega} \right) e^{+i\omega t} \mathfrak{J}_L^{(0)} . \end{aligned} \tag{5.9}$$

Consequently, the solutions to Eqs. (5.6) are, to first order,

$$\begin{aligned} \mathfrak{J}_L(t) &\approx \mathfrak{J}_L^{(0)} [1 - a e^{-i(\omega t + \delta)}] , \\ \mathfrak{J}_R(t) &\approx \mathfrak{J}_R^{(0)} [1 - a e^{+i(\omega t + \delta)}] , \end{aligned} \tag{5.10}$$

where $\delta = \alpha_L^{(0)}(0) - \alpha_R^{(0)}(0)$.

The above solutions can now be employed to evaluate dJ_{LZ}/dt and therefore the Josephson current. We start by noting that, for a complex number f , with $|f| \ll 1$,

$$1 + f \approx (1 + f_1) e^{if_2} ,$$

where $f = f_1 + i f_2$. Defining f as

$$f = -a e^{-i(\omega t + \delta)} ,$$

Eqs. (5.10) become

$$\mathfrak{J}_L(t) \approx \mathfrak{J}_L^{(0)} [(1+f_1) e^{if_2}] ,$$

$$\mathfrak{J}_R(t) \approx \mathfrak{J}_R^{(0)} [(1+f_1) e^{-if_2}] .$$

Substituting these into Eqs. (5.5) gives

$$J_L^-(t) = \left(\frac{m}{2}\right) (1+f_1) \exp \left\{ i \left[f_2 + \alpha_L^{(0)}(0) - \frac{bt}{\hbar} \right] \right\} ,$$

$$J_R^-(t) = \left(\frac{m}{2}\right) (1+f_1) \exp \left\{ i \left[-f_2 + \alpha_R^{(0)}(0) + \frac{bt}{\hbar} \right] \right\} ,$$

which allows us to identify $J_{L\perp}(t)$, $J_{R\perp}(t)$, $\alpha_L(t)$, and $\alpha_R(t)$ as

$$\begin{aligned} J_{L\perp}(t) &= J_{R\perp}(t) = \left(\frac{m}{2}\right) (1+f_1) , \\ \alpha_L(t) &= -\frac{bt}{\hbar} + \alpha_L^{(0)}(0) + f_2 , \\ \alpha_R(t) &= \frac{bt}{\hbar} + \alpha_R^{(0)}(0) - f_2 , \end{aligned} \tag{5.12}$$

by using Eq. (5.3). As mentioned above the magnitudes of our angular momentum vectors are now time dependent and the rotation angles are no longer given by Eq. (4.9). Continuing, we substitute from (5.12) into (5.4a) obtaining

$$\frac{dJ_{Lz}}{dt} = - \left(\frac{2T}{\hbar} \right) \left(\frac{m}{2} \right)^2 (1+f_1)^2 \sin (\omega t + \delta + 2f_2) \quad . \quad (5.13)$$

Since f_1 and f_2 are both proportional to a , we can write

$$\begin{aligned} \frac{dJ_{Lz}}{dt} \approx & - \left(\frac{2T}{\hbar} \right) \left(\frac{m}{2} \right)^2 \\ & \times [\sin (\omega t + \delta) + 2f_1 \sin (\omega t + \delta) + 2f_2 \cos (\omega t + \delta)] \end{aligned}$$

to first order in our expansion parameter. It is now obvious from the definitions of f_1 and f_2 that dJ_{Lz}/dt reduces to

$$\frac{dJ_{Lz}}{dt} = - \left(\frac{2T}{\hbar} \right) \left(\frac{m}{2} \right)^2 \sin (\omega t + \delta) \quad , \quad (5.14)$$

and the resulting Josephson current is identical to that given in Eq. (4.12). Thus we see that, to first order in a , the exact angular momentum equations of motion, Eqs. (5.1), have led to the same results as the approximated equations of motion, Eqs. (4.8).

By using a slightly different perturbation technique (Appendix B) we have been able to determine higher order corrections. The lowest nonvanishing correction to the Josephson result occurs in second order in a . In particular the Josephson frequency ω is replaced by

$$\omega' = \omega (1 - 2 a^2) \quad . \quad (5.15)$$

Since $a \approx 10^{-12}$, this correction is entirely negligible!

CHAPTER 6

CONCLUDING REMARKS

This dissertation describes the dynamics of the Josephson junction in terms of the coupling of two macroscopic angular momentum vectors. We started by demonstrating how the BCS ground state could be viewed as a state describing a large spin system. We then showed that, for the purposes of Josephson tunneling at $T = 0$ K, each superconductor could be represented by a single macroscopic angular momentum vector. These macroscopic angular momenta were then used to reformulate the Hamiltonian for the junction. The resulting equations of motion were seen to lead directly to the Josephson current, while at the same time providing us with an extremely graphic picture of the dynamical behavior of a junction. We then went on to show how our formalism, under a particular approximation, reduces to the standard (n, ϕ) theory. More significantly though, by writing our equations of motion in a rotating reference frame, we have been able to provide, starting with a microscopic Hamiltonian, a clear derivation of Feynman's two-state model. Finally, by solving the exact equations of motion in a perturbation theory we were able to say the following: Any anomalies which arise from using the angular momentum approach in conjunction with the basic Josephson problem are entirely negligible. In light of the above we feel that the angular momentum model of the Josephson junction provides us with a viable alternative to the Anderson and Feynman methods.

APPENDIX A

THE $\vec{k}$ -SPACE CUTOFF

In this appendix we consider, in somewhat more detail, the role of κ in our pseudospin system and in the definition of the charge-imbalance state.

The number of pseudospin systems in a given region of $\vec{k}$ -space is needed in many of our calculations. By definition this is equal to the number of $\vec{k}$ -vectors in the given region. [From Eq. (2.7) we see that the $\vec{k}$ subscript is used to distinguish one spin- $\frac{1}{2}$ subsystem from another.] Thus by choosing κ and considering only those $\vec{k}$'s such that $|\vec{k}| \leq \kappa$, we are specifying how many spin- $\frac{1}{2}$ systems we have and which ones they are. Consequently once κ has been chosen, it must remain fixed regardless of how the physical state of the superconductor may change. (If this is not done there is the possibility of inadvertently allowing the total spin system we are dealing with to change.)

We begin by considering a charge neutral superconductor at $T = 0$ K. Within the strong-coupling model, the state of this system is described by the BCS ground state, Eq. (2.7), with $U^2(\vec{k})$ and $V^2(\vec{k})$ as shown in Fig. 8. To be specific, the occupation distributions of the BCS theory have been approximated as step functions in the region about k_F . The region, in energy, over which this step occurs is of length 2Δ , where Δ is the energy-gap parameter. The boundaries of this region in $\vec{k}$ -space are given by

Figure 8. Regions in $\vec{k}$ -space.

$$\begin{aligned}
k_U &= \left[\left(\frac{2m_e}{\hbar^2} \right) (\epsilon_F + \Delta) \right]^{\frac{1}{2}} \\
&\approx k_F + k_\Delta \\
k_L &= \left[\left(\frac{2m_e}{\hbar^2} \right) (\epsilon_F - \Delta) \right]^{\frac{1}{2}} \\
&\approx k_F - k_\Delta,
\end{aligned}
\tag{A.1}$$

with

$$\begin{aligned}
k_\Delta &= \frac{1}{2} \left(\frac{2m_e}{\hbar^2} \right)^{\frac{1}{2}} \left(\frac{\Delta}{\epsilon_F^{\frac{1}{2}}} \right) \\
&= \frac{1}{2} \left(\frac{\Delta}{\epsilon_F} \right) k_F.
\end{aligned}
\tag{A.2}$$

In the BCS ground state the allowed $\vec{k}$ -vectors technically run out to infinity. But since we will be doing all our calculations at $T = 0$ K, we can restrict the range of $\vec{k}$ to be below some specific value, $|\vec{k}| \leq \kappa$. This procedure is valid as long as there are no filled states above κ .

In regard to Fig. 8, k_L , k_U , and κ define three regions in $\vec{k}$ -space. N_1 , N_2 , and N_3 are the number of pseudospin systems ($\vec{k}$ -vectors) in the respective regions. The following ansatz is now made concerning these three regions and κ :

(i) For a given charge-neutral superconductor at $T = 0$ K, even though κ can be chosen almost arbitrarily, choose it such that $N_1 = N_3$ and relabel them as N_0 . Then, once κ has been defined, it remains fixed regardless of how the physical state of the superconductor changes.

(ii) The three regions in $\vec{k}$ -space are always defined in the same way, that is,

$$\begin{aligned}
 (1) \quad & 0 \leq k \leq k_F - k_\Delta \\
 (2) \quad & k_F - k_\Delta \leq k \leq k_F + k_\Delta \\
 (3) \quad & k_F + k_\Delta < k \leq \kappa
 \end{aligned} \tag{A.3}$$

where it is understood that k_F may change depending on the state of the system (i.e., from Chapter 2, part C, if charge is added then k_F is replaced by k_{nF} and Fig. 8 is replaced by Fig. 9). In terms of our spin- $\frac{1}{2}$ systems, all those whose $\vec{k}$'s are in region (1) are pointing in the (+z)-direction, while those in region (3) are in the (-z)-direction. The ones in region (2) are in the xy-plane.

If we now relabel N_2 as m , the consequences of the above are the following: Regardless of how the physical state of the system may change, the total number of $\vec{k}$ -vectors from 0 to κ is fixed at $2N_0 + m$.

Figure 9. Regions associated with k_{nF} .

APPENDIX B

SECOND ORDER CORRECTIONS TO THE JOSEPHSON FREQUENCY

To obtain the exact solution to Eqs. (5.6) we differentiate both with respect to time to get

$$\begin{aligned}\ddot{\mathfrak{J}}_L + i\omega \dot{\mathfrak{J}}_L - \omega^2 a^2 \mathfrak{J}_L &= 0 \quad , \\ \ddot{\mathfrak{J}}_R - i\omega \dot{\mathfrak{J}}_R - \omega^2 a^2 \mathfrak{J}_R &= 0 \quad .\end{aligned}\tag{B.1}$$

These equations imply that we may write

$$\begin{aligned}\mathfrak{J}_L &= I_L^+ e^{i\beta^+ t} + I_L^- e^{i\beta^- t} \quad , \\ \mathfrak{J}_R &= I_R^+ e^{i\gamma^+ t} + I_R^- e^{i\gamma^- t} \quad ,\end{aligned}\tag{B.2}$$

with, to lowest order in a^2 ,

$$\begin{aligned}\beta^+ &= -\gamma^- = -\omega (1-a^2) \quad , \\ \beta^- &= -\gamma^+ = -\omega a^2 \quad .\end{aligned}\tag{B.3}$$

Inserting Eqs. (B.3) and (B.2) into (5.6) gives us

$$\begin{aligned}
 \beta^- I_L^- &= \left(\frac{2TJ_{Lz}}{\hbar} \right) I_R^- , \\
 \beta^+ I_L^+ &= \left(\frac{2TJ_{Lz}}{\hbar} \right) I_R^+ .
 \end{aligned}
 \tag{B.4}$$

Finally, by choosing

$$\begin{aligned}
 I_L^- &= \left(\frac{m}{2} \right) e^{i\alpha_L(0)} , \\
 I_R^+ &= \left(\frac{m}{2} \right) e^{i\alpha_R(0)} ,
 \end{aligned}$$

we find that the Josephson current is given by

$$I = j_1 \sin (\omega' t - \delta) , \tag{B.5}$$

with ω' defined in Eq. (5.15).

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