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**On the incompressible limit of the compressible Navier-Stokes
equations**

Lin, Chi-Kun, Ph.D.

The University of Arizona, 1992

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Ann Arbor, MI 48106

**ON THE INCOMPRESSIBLE LIMIT
OF
THE COMPRESSIBLE NAVIER-STOKES EQUATIONS**

by
Chi-Kun Lin

A Dissertation Submitted to the Faculty of the
DEPARTMENT OF MATHEMATICS
In Partial Fulfillment of the Requirements
For the Degree of
DOCTOR OF PHILOSOPHY
In the Graduate College
THE UNIVERSITY OF ARIZONA

1992

THE UNIVERSITY OF ARIZONA
GRADUATE COLLEGE

As members of the Final Examination Committee, we certify that we have
read the dissertation prepared by Chi-Kun Lin
entitled ON THE INCOMPRESSIBLE LIMIT OF THE NAVIER-STOKES EQUATIONS

and recommend that it be accepted as fulfilling the dissertation
requirement for the Degree of DOCTOR OF PHILOSOPHY

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To
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Abstract

Many interesting problems in classical physics involve the behavior of solutions of nonlinear hyperbolic systems as certain parameter and coefficients becomes infinite. Quite often, the limiting solution (when it exists) satisfies a completely different nonlinear partial differential equation. The incompressible limit of the compressible Navier-Stokes equations is one physical problem involving dissipation when such a singular limiting process is interesting.

In this article we study the time-discretized compressible Navier-Stokes equation and consider the incompressible limit as the Mach number tends to zero. For γ -law gas, $1 < \gamma \leq 2$, $D \leq 4$, we show that the solutions $(\rho_\epsilon, \mu_\epsilon/\epsilon)$ of the compressible Navier-Stokes system converge to the solution $(1, \bar{v})$ of the incompressible Navier-Stokes system. Furthermore we also prove that the limit also satisfies the Leray energy inequality.

I. Introduction

The Navier-Stokes equations occupy a central position in the study of non-linear partial differential equations, dynamical systems, and modern scientific computation, as well as classical fluid dynamics. Because of the complexity and variety of fluid dynamical phenomena, and the simplicity and exactitude of the governing equations, a very special depth and beauty is expected in the mathematical theory.

One of the fundamental questions concerning these equations is “How do solutions depend on the physical parameters?”. Typically, problems in this class involve **singular limits**.

Many interesting problems in classical physics involve the behavior of solutions of nonlinear systems as certain parameter and coefficients becomes infinite. Quite often, the limiting solution (when it exists) satisfies a completely different nonlinear partial differential equation. The incompressible limit of the compressible Navier-Stokes equations is one physical problem involving dissipation when such a singular limiting process is interesting.

The question of the incompressible limit in fluid dynamics has received considerable attention. The basic result, which has been proven in various contexts, is that **slightly compressible fluid flow is close to incompressible flow**, even though the equations for the latter are related to those for the former via a singular limit. This justifies the use of the incompressible flow equations for certain real fluids that are actually slightly compressible. (see [KMa1], [KMa2], [Maj1], etc.)

Another important and interesting question concerning the singular limit is “Are the Euler equations the inviscid limit of the Navier-Stokes equations?”. In the absence of boundaries the answer is yes, for short times [Kat2] or at least as long as the Euler solution is smooth [Con1]. In the presence of boundaries the answer is in general no, because of discrepancies in the boundary behavior (boundary layer).

DiPerna and Majda introduced a new concept of measure-valued solution for the 3-D incompressible Euler equations in order to incorporate the complex phenomena present in limits of approximate solutions of these equations. They

prove that a sequence of Leray-Hopf weak solutions of the Navier-Stokes equations converges in the high Reynolds number limit to a measure-valued solution of 3-D Euler equations for all positive times(see [DiPM]).

First the incompressible Navier-Stokes equations describe the evolution of the velocity field $\mathbf{u} = \mathbf{u}(x, t)$ of an idealized fluid over a given domain in $\mathbf{R}^D$.

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla P + \nu \Delta \mathbf{u}, \quad (I.1)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (I.2)$$

$$\mathbf{u}(x, 0) = \mathbf{u}_0(x), \quad x \in \mathbf{R}^D, \quad (I.3)$$

where $\nu > 0$ is the kinetic viscosity of the fluid. We call $\nu \Delta \mathbf{u}$ the **diffusion** or **dissipation** term and $(\mathbf{u} \cdot \nabla) \mathbf{u}$ the **inertia** or **convection** term. The effect of the dissipative term is one of dissipating energy and smoothing. The equations say that $\mathbf{u}$ is convected subject to pressure forces and, at the same time is dissipated. In a seminal paper of 1934, J.Leray([Ler]) proved the existence of a temporally global weak solution to these equations over the whole space $\mathbf{R}^3$ for any initial data with finite energy. The uniqueness of these solutions has not yet been either proved nor disproved; there are known sufficient conditions which guarantee uniqueness [Ser]. The regularity properties of these solutions are not sufficiently understood. It is known that the spatial BV norm is bounded in time and various temporal averages of higher derivatives are also apriori bounded(see [Con2]).

It is still unknown whether the existence of smooth solutions of the Navier-Stokes equations is a generic fact. This makes the derivation of hydrodynamics limit for weak solutions a problem of definite interest, although more complicated than for smooth solutions.

Next we consider the compressible Navier-Stokes equation

$$\partial_t \rho + \nabla \cdot \mu = 0, \quad (I.4)$$

$$\partial_t \mu + \nabla \cdot \left(\frac{\mu \otimes \mu}{\rho} \right) + \nabla P(\rho) = \nabla \cdot \left(\nu \sum \right), \quad (I.5)$$

$$\sum \equiv \left(\nabla \frac{\mu}{\rho} \right) + \left(\nabla \frac{\mu}{\rho} \right)^T - \frac{2}{D} \left(\nabla \cdot \frac{\mu}{\rho} \right) I, \quad (I.6)$$

where P is the pressure. For compressible fluid this function determines the compressibility of the fluid. To avoid complications at the boundary, we concentrate below on the case where $x \in T^D$, the D -dimensional torus.

Both the systems in (I.1) – (I.3) and the one in (I.4) – (I.6) describe fluid flow in certain regions of motion. A basic question is the following:

Q: How are the solutions of compressible and incompressible fluids related ?

Note that the momentum equation (I.1) and the equation (I.2) are of different type. This makes the existence theory more difficult. Indeed, the short-time existence theory for the compressible equations is somewhat easier than the theory for the incompressibles. However, using the idea of **initialization** for problems with different time scale, one can also treat the incompressible case as a limit of compressible ones (see [KreLN], [KreL] and [Tem1]).

Before further discussion, we describe our main motivating problem, that of the stability of compressible flow near its incompressible limit. One of the most powerful tools for examining properties of the Navier-Stokes equations is that of **scaling argument**, i.e., to scale the variables. This enables one both to simplify the equations by neglecting terms and to use one calculation or experiment to obtain results for another similar problem. In fact we can also obtain the incompressible equations by a limiting process from the equations governing compressible flows. We shall see below that the equations in (I.1) – (I.2) arise as the singular limit equations for the system in (I.11) – (I.12) as the Mach number tends to zero.

The first step in understanding this limiting process is through the **nondimensionalization** of the compressible fluid equations.

We consider (I.4) – (I.5) with initial data $\rho(x, 0) = \rho^{\text{in}}(x)$, $\mu(x, 0) = \mu^{\text{in}}(x)$ and set

$$\rho_m = \max \rho^{\text{in}}(x), \quad \mu_m = \max |\mu^{\text{in}}(x)|,$$

to represent typical values for the density and fluid momentum present in the initial data for (I.4) – (I.5). We introduce the new variables,

$$\tilde{\mu} = \frac{\mu}{\mu_m}, \quad \tilde{\rho} = \frac{\rho}{\rho_m}, \quad x' = x, \quad t' = \frac{\mu_m}{\rho_m} \epsilon t, \quad (I.7)$$

and rewrite the compressible fluid equations in (O.4) – (O.5),

$$\epsilon \partial_{t'} \tilde{\rho} + \nabla \cdot \tilde{\mu} = 0, \quad (I.8)$$

$$\epsilon \partial_{t'} \tilde{\mu} + \nabla \cdot \left(\frac{\tilde{\mu} \otimes \tilde{\mu}}{\tilde{\rho}} \right) + \lambda^2 \nabla P(\tilde{\rho}) = \frac{1}{\mu_m} \nabla \cdot \left(\nu \widetilde{\sum} \right), \quad (I.9)$$

where λ^2 is the nondimensional quantity

$$\lambda^2 = \left(\frac{dp}{d\rho}(\rho_m) / \left| \frac{\mu_m}{\rho_m} \right|^2 \right) (\gamma A)^{-1}. \quad (I.10)$$

In particular, the Mach number M is defined by the ratio of the typical fluid speed (described by $|\mathbf{v}_m| = \mu_m/\rho_m$) to the typical sound speed (described by $(dp/d\rho)^{\frac{1}{2}}$). i.e., with $c(\rho_m) = (dp/d\rho)^{1/2}$,

$$M = \frac{|\mathbf{v}_m|}{c(\rho_m)}, \quad \lambda = M^{-1} (\gamma A)^{-1/2}.$$

If the Mach number is small, say $M = O(\epsilon)$, i.e., $\lambda = (\frac{1}{\epsilon})$. Then after scaling, balancing the dissipation term and dropping the primes and the tildas, we derive the scaled compressible Navier-Stokes equations

$$\epsilon \partial_t \rho_\epsilon + \nabla \cdot \mu_\epsilon = 0, \quad (I.11)$$

$$\epsilon \partial_t \mu_\epsilon + \nabla \cdot \left(\frac{\mu_\epsilon \otimes \mu_\epsilon}{\rho_\epsilon} \right) + \nabla P(\rho_\epsilon) = \epsilon \nabla \cdot \left(\nu \sum_\epsilon \right). \quad (I.12)$$

It is obvious that $(1, 0)$ is a trivial solution of (I.11) – (I.12). We consider the perturbation near $(1, 0)$. That is looking at data near the equilibrium $(\rho, \mu) = (1, 0)$. Let

$$\mu_\epsilon = \epsilon \tilde{\mu}_\epsilon, \quad \rho_\epsilon = 1 + \epsilon^2 \tilde{\rho}_\epsilon, \quad (I.13)$$

where $\tilde{\mu}_\epsilon$, $\tilde{\rho}_\epsilon$ play the roles of the velocity fluctuation and density fluctuation respectively. Then (I.11) – (I.12) becomes

$$\epsilon^3 \partial_t \tilde{\rho}_\epsilon + \epsilon \nabla \cdot \tilde{\mu}_\epsilon = 0, \quad (I.14)$$

$$\epsilon^2 \partial_t \tilde{\mu}_\epsilon + \epsilon^2 \nabla \cdot \left(\frac{\tilde{\mu}_\epsilon \otimes \tilde{\mu}_\epsilon}{1 + \epsilon^2 \tilde{\rho}_\epsilon} \right) + \nabla P(\rho_\epsilon) = \epsilon^2 \nabla \cdot \left(\nu \widetilde{\sum}_\epsilon \right), \quad (I.15)$$

or

$$\epsilon^2 \partial_t \tilde{\rho}_\epsilon + \nabla \cdot \tilde{\mu}_\epsilon = 0, \quad (I.16)$$

$$\partial_t \tilde{\mu}_\epsilon + \nabla \cdot \left(\frac{\tilde{\mu}_\epsilon \otimes \tilde{\mu}_\epsilon}{1 + \epsilon^2 \tilde{\rho}_\epsilon} \right) + \frac{1}{\epsilon^2} \nabla P(\rho_\epsilon) = \nabla \cdot \left(\nu \widetilde{\sum}_\epsilon \right). \quad (I.17)$$

As ϵ tends to zero, the leading behavior of the fluctuations is formally consistent with the incompressible Navier-Stokes equations. In fact, formally letting $\epsilon \rightarrow 0$ in (I.16) produces the incompressibility condition $\nabla \cdot \mu = 0$. While some coefficients in (I.17) becomes infinity as $\epsilon \rightarrow 0$. So we call this is a singular limit.

Introducing the pressure fluctuation $\tilde{P}_\epsilon$ by

$$P(\rho_\epsilon) = P(1) + \epsilon^2 \tilde{P}_\epsilon, \quad (I.18)$$

then we have

$$\epsilon^2 \partial_t \tilde{\rho}_\epsilon + \nabla \cdot \tilde{\mu}_\epsilon = 0, \quad (I.19)$$

$$\partial_t \tilde{\mu}_\epsilon + \nabla \cdot \left(\frac{\tilde{\mu}_\epsilon \otimes \tilde{\mu}_\epsilon}{1 + \epsilon^2 \tilde{\rho}_\epsilon} \right) + \nabla \tilde{P}_\epsilon = \nabla \cdot \left(\nu \widetilde{\sum}_\epsilon \right). \quad (I.20)$$

Thus this last equation makes sense as an approximation to the considered compressible fluid equation around a state of identically constant density and constant momentum. If $(\tilde{\mu}_\epsilon, \tilde{P}_\epsilon)$ is compact in some topology then there exists $(\bar{\mu}, \bar{P})$ such that

$$(\tilde{\mu}_\epsilon, \tilde{P}_\epsilon) \rightarrow (\bar{\mu}, \bar{P}),$$

in this topology. Then formally letting $\epsilon \rightarrow 0$, $(\bar{\mu}, \bar{P})$ will solve the incompressible Navier-Stokes equations (I.1) – (I.2). In fact we can also see the rigorous results by asymptotic expansion of solutions of the compressible Navier-Stokes equations (I.11) – (I.12). Proceeding formally we assume the asymptotic expansion

$$\begin{aligned} \rho_\epsilon &= 1 + \epsilon^2 \rho^{(1)} + O(\epsilon^4), \\ \text{i.e., the density is almost constant,} \\ \mu_\epsilon &= \epsilon \mu^{(1)} + \epsilon^3 \mu^{(2)} + O(\epsilon^5). \end{aligned} \tag{I.21}$$

By Taylor's expansion we obtain

$$P(\rho_\epsilon) = P(1) + \epsilon^2 \frac{dp}{d\rho}(1) \rho^{(1)} + O(\epsilon^4),$$

then

$$\frac{1}{\epsilon^2} \nabla P = \nabla(\bar{c}^2 \rho^{(1)}) + O(\epsilon^2),$$

where $\bar{c}^2 = \frac{dp}{d\rho}(1)$. We substitute the expansion in (I.20) into the eqs.(I.11) – (I.12) and equate powers of ϵ to obtain

$$\begin{aligned} \nabla \cdot \mu^{(1)} &= 0, \\ \partial_t \rho^{(1)} + \nabla \cdot \mu^{(2)} &= 0, \\ \partial_t \mu^{(1)} + \nabla \cdot (\mu^{(1)} \otimes \mu^{(1)}) + \nabla P^{(1)} &= \nabla \cdot \left(\nu \sum^{(1)} \right), \end{aligned} \tag{I.22}$$

where $P^{(1)} \equiv \frac{dp}{d\rho}(1) \rho^{(1)}$ gives the pressure and density fluctuation. Take divergence and eliminate $\mu^{(1)}$ we derive the familar equation of linear **acoustics**

$$\partial_{tt} \rho^{(1)} - \bar{c}^2 \nabla \cdot (\nabla \rho^{(1)}) = 0. \tag{I.23}$$

It explains the fact that small perturbation of pressure, density and velocity propagates in a fluid in the form of the acoustics wave. As previous we can conjecture that $(\mu^{(1)}, P^{(1)})$ will solve the incompressible Navier-Stokes equations.

In fluid dynamics there appear many systems of nonlinear differential equations involving parameters such as the Mach number and the Alfvén number etc. One problem on the singular limit is to determine the limiting system which has a completely different property comparing with the original system, as such a parameter tends to some value.

When the system is hyperbolic, this problem has been studied in Kreiss-Browning ([KreB]), Klainerman - Majda ([KMa1], [KMa2]), Majda ([Maj1]), and Schobert ([Sch1]-[Sch4]). In particular, Klainerman and Majda ([KMa1]-[KMa2]) studied the convergence of classical solutions of the compressible fluid equations to their incompressible limit as the Mach number becomes small.

However, to insure that the solution is close to a solution of the incompressible equation, one has to **prepare** the initial data. The principle is the **initialization**, one choose the initial data so that a number of time derivatives at $t = 0$ remain bounded independent of the compressibility.

It is shown by Kato that unique solutions for nonviscous flow exist for all ϵ on the time interval $[0, T]$ independent of ϵ , and if the initial datum is incompressible datum, Klainerman and Majda ([KMa1]-[KMa2]) prove that the solutions converge as $\epsilon \rightarrow 0$ uniformly on $[0, T]$ to a solution of the incompressible Euler equation.

For the initial datum is not incompressible, it is solved by Ukai. But the uniform convergence breaks near $t = 0$, due to the development of initial layer.(see [Uka])

Similar results are also true for the compressible and incompressible Navier-Stokes equations but only for the periodic initial data independent of the viscosity coefficients as they tend to zero.(see [KMa1], [KMa2] and [Maj1].) For stationary motions it had been studied in [H.B. da Veiga 1-2], but only for small data(classical solutions).

In very interesting recent work, H.O.Kreiss's ([KreLN]) they have studied the slightly compressible Navier-Stokes equations, but of different form. For the Cauchy problem periodic in space, under appropriate assumptions on the initial data, the solution of the compressible equation consists —to first order— of a solution of the incompressible equations plus a function which is highly oscillating in time. The highly oscillatory part (the sound wave) can be described by wave equations, at least locally in time.

Very recently DiPerna and Lions obtained the first global existence proof of large amplitude solutions to the Boltzmann equation. Here, weak L^1 -solutions of arbitrary amplitude are constructed for initial data satisfying only the physically natural condition that the total mass, energy and entropy be finite. This fact has implications for various asymptotic problems such as the hydrodynamics limit and the large-time behavior. In many respects DiPerna-Lions theory is analogous to Leray global existence theory for the Navier-Stokes equation(see [DiPL]).

Under some mild compactness assumptions the global in time convergence of DiPerna-Lions renormalized solutions of the time-discretized Boltzmann equation to the Leray solution of the time-discretized incompressible Navier-Stokes equations is proved(see [BGL2]). Stimulated by [BGL2] we study the incompressible limit of the compressible Navier-Stokes equation for large data satisfying only the entropy inequality.

The nonstationary problems are much more difficult than the stationary problem. When we use the compactness method the main difficulty lies in the nonlinear term, for example, $(\mathbf{u}_\epsilon \cdot \nabla) \mathbf{u}_\epsilon$, in Euler equation or Navier-Stokes equation.

If $\phi(\cdot, \cdot)$ is a smooth vector field defined in $R^+ \times \Omega$ with compact support then

$$\int_0^\infty \int_\Omega (\mathbf{u}_\epsilon \cdot \nabla) \mathbf{u}_\epsilon \phi \, dx \, dt = - \int_0^\infty \int_\Omega \sum_{i,j} \mathbf{u}_\epsilon^i \mathbf{u}_\epsilon^j D_j \phi^i \, dx \, dt. \quad (I.24)$$

and under the weak convergence assumption $\mathbf{u}_\epsilon \rightharpoonup \mathbf{u}$ one may have (for example, when concentrations or oscillations occur in the approximate sequence $\mathbf{u}_\epsilon$)

$$D'(R^+ \times \Omega) - \lim_{\epsilon \rightarrow 0} \mathbf{u}_\epsilon^i \mathbf{u}_\epsilon^j \neq \mathbf{u}^i \mathbf{u}^j. \quad (I.25)$$

where D' is the class of distributions. Knowing the weak limit of a sequence does not yield the limit of its square! The difficulty turns upon the possibility of very rapid fluctuations in the functions $\mathbf{u}_\epsilon$. This is the problem of **oscillation**. Another difficulty is the problem of **concentration** we will discuss in lemma IV.5.

To ensure the equality of both sides of (I.24), one would need some pointwise convergence (at least almost everywhere) for $\mathbf{u}_\epsilon(x, t)$ in $R^+ \times \Omega$. (see **Lemma III.11**) In fact this is correspondent to the following question. Let f_n be a bounded sequence of functions in $L^p(I, B)$ where B is a Banach space and $1 \leq p \leq \infty$. When does there exist a strongly converging subsequence; that is to say when is $\{f_n\}$ relatively compact in $L^p(I, B)$?

This is the crucial point in the compactness method for existence of solutions in nonlinear partial differential equation; several examples of this method were given by J.L.Lions ([Lio]).

A first answer which allows to solve a large diversity of problems, as is shown in ([Lio], [CoF] and [Tem1]) was given by J.P.Aubin, we called it **Lions-Aubin lemma** : it suffices that the f_n are bounded in a space $L^p(I, X)$ where X is included in a space B with compact imbedding and derivatives $\partial_t f_n$ are bounded in a space $L^q(I, Y)$ where $B \subset Y$.

As we know a linear wave travels without change of profile. However, in the nonlinear wave the nonlinearity play the role to produce progressively more and more deformation in the wave profile as t increases. That's why the uniform control in time is difficult to obtain. It is a natural phenomenon for nonlinear wave.

One method to overcome the difficulty is to approximate the evolution operator by the stationary operator. The idea comes from when we discretize the time variable.

More precisely (see [Lio]) we can approximate the operator $\frac{d}{dt}$ by the difference quotient

$$\frac{I - \tau_h}{h}, \quad (I.26)$$

where

$$\tau_h f(x) = f(x - h),$$

i.e.,

$$\tau_h = \text{translation in } t \text{ of distance } h.$$

Then we can approximate the equation

$$\frac{d\mathbf{u}}{dt} + A(\mathbf{u}) = f, \quad (I.27)$$

by

$$\frac{I - \tau_h}{h} \mathbf{u}_h + A(\mathbf{u}_h) = f. \quad (I.28)$$

Usually the existence, uniqueness and the regularity results for (I.28) are easy to derive. We hope in some proper topology, by passing to subsequence we have

$$\mathbf{u}_{h'} \rightharpoonup \mathbf{u}, \quad \text{as } h' \rightarrow 0.$$

and $\mathbf{u}$ satisfies (I.27) in this topology. However, we have the same difficulty as ([BGL2]); lacking a compactness result in the time variable. Therefore for the incompressible limit we only deal with the time discretized case. To conclude this introduction let us briefly discuss the contents of this article.

Chapter II contains the basic background and description for the compressible and incompressible fluid flow. The transport theorem provides the basic tool to get the energy estimates.

In chapter III, we introduce the idea of entropy, relative entropy which give us the apriori bounds. The Legendre transformation and the generalized Young's

inequality are important when we prove the weak L^1 compactness of the nonlinear term $\{\frac{1}{\rho_\epsilon}(\frac{\mu_\epsilon}{\epsilon} \otimes \frac{\mu_\epsilon}{\epsilon})\}$.

In chapter IV, we obtain the entropy inequality for the scaled compressible Navier-Stokes equation. We also introduce the function space $C(I, X)$, $L^p(I, X)$ and Orlicz space L_Ψ which are natural function spaces for the evolution problems. Here we also state the Arzela-Ascoli theorem and the well known Dunford-Pettis theorem. The generalized Lebesgue theorem is important when we want to get the strong L^1 -compactness for the sequence of the density $\{\rho_\epsilon\}$.

Chapter V is mainly devoted to the study of the scaled compressible Navier-Stokes equation. We obtain some compactness results which are straight implication of the entropy inequality.

Chapter VI is the heart of this article. We prove the incompressible limit of the solutions of the time-discretized compressible Navier-Stokes equation (for the γ -law gas) as the Mach number become small.

In order to be sure that the discussion above make sense, we prove the existence, uniqueness and the regularity theorem for the parabolic regularization system by the semigroup theory in Appendix A.

In appendix B we state the Helmholtz decomposition theorem in $\mathbf{R}^D$ and T^D . It guarantees the existence of the limit of the pressure term.

II. Preliminaries

In this chapter we state the basic equations of fluid mechanics. These equations are derived from the conservation laws of mass, momentum and energy. The foundation of the study of fluid motion lies in **kinematics**, the analysis of motion and deformation without reference to the forces that are brought into play.

Fluid mechanics concerns itself with the study of the motion of fluids (gases and liquids). Gas-dynamics, or compressible fluid flow, is the study of those motions for which changes in fluid density play an essential role. Although there is some density change in every physical flow, it is often possible to neglect such changes and treat the flow according to the idealization that the fluid is incompressible. This approximation may be applicable to gases, for example, in low-speed flow around an airplane or flow through a vacuum cleaner, as well as to liquids.

An incompressible flow of a homogeneous fluid in all of space $\mathbf{R}^D$; is determined by the system of equations

$$\frac{D\mathbf{u}}{Dt} = -\nabla P + \nu \Delta \mathbf{u}, \quad (II.1)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (II.2)$$

$$\mathbf{u}(x, 0) = \mathbf{u}_0(x); \quad x \in \mathbf{R}^D. \quad (II.3)$$

In the above

$\mathbf{u}(x, t) \equiv (u^1, u^2, \dots, u^D)$ is the fluid velocity,

$P(x, t)$ is the scalar pressure,

$\frac{D\mathbf{u}}{Dt}$ is the convective derivative,

(i.e. the derivative along particle trajectories.)

$$\frac{D}{Dt} \equiv \partial_t + \sum_{j=1}^D u^j \partial_{x_j} . \quad (II.4)$$

It takes into account the fact that the fluid is moving and that the positions of fluid particles change with time. Indeed, if $f(x(t), t)$ is any function of position and time (scalar or vector) , then by the chain rule ,

$$\frac{d}{dt}(f(x(t), t)) = \partial_t f + \mathbf{u} \cdot \nabla f = \frac{Df}{Dt}(x(t), t) .$$

The divergence of a vector field is given by

$$\nabla \cdot \mathbf{u} = \sum_{j=1}^D \partial_{x_j} u^j . \quad (II.5)$$

A given constant viscosity coefficient $\nu \geq 0$ can be viewed as the reciprocal of Reynolds number Re . The Reynolds number is a dimensionless number and it is evidently a measure of the importance of the terms due to viscosity. For $\nu > 0$ the equation (II.1) is called the **Navier-Stokes equation**, for $\nu = 0$ it reduces to the **Euler equation**. In most applications, ν is an extremely small quantity with $\nu \ll 1$ and typically ν has sizes from 10^{-3} to 10^{-6} in turbulent flows. Thus, one can anticipate that the behavior of inviscid solutions of the Euler equations with $\nu = 0$ is quite important in describing solutions of the Navier-Stokes equations when ν is small; there are rigorous theorems supporting this fact (see [Con1],[Kat1]). The rigorous theorem (see [Con1]) supports the fact that solutions of the Navier-Stokes equations at sufficiently high Reynolds numbers remain regular on any time interval where the Euler equations remain smooth. These equations follow from the conservation of momentum for a continuum. The equation in (II.2) expresses the incompressibility of the fluid. We will explain this more specific later.

The initial value problem (II.1) – (II.3) is unusual because it contains the time derivative of only D out of $D + 1$ unknown functions.

For the compressible fluid flow

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (II.6)$$

$$\rho \left(\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) + \nabla P = \nabla \cdot \left(\nu \sum \right), \quad (II.7)$$

where

$$\sum \equiv (\nabla \mathbf{u}) + (\nabla \mathbf{u})^T - \frac{2}{D} (\nabla \cdot \mathbf{u}) I. \quad (II.8)$$

Sometimes it is easier to study $\rho \mathbf{u}$ rather than ρ and $\mathbf{u}$ separately. Let

$$\boldsymbol{\mu} \equiv \rho \mathbf{u} \text{ (momentum);} \quad (II.9)$$

with the variables $(\rho, \boldsymbol{\mu})$, the compressible equations (I.6) – (I.8) become

$$\partial_t \rho + \nabla \cdot \boldsymbol{\mu} = 0, \quad (II.10)$$

$$\partial_t \boldsymbol{\mu} + \nabla \cdot \left(\frac{\boldsymbol{\mu} \otimes \boldsymbol{\mu}}{\rho} \right) + \nabla P(\rho) = \nabla \cdot \left(\nu \sum \right), \quad (II.11)$$

$$\sum \equiv \left(\nabla \frac{\boldsymbol{\mu}}{\rho} \right) + \left(\nabla \frac{\boldsymbol{\mu}}{\rho} \right)^T - \frac{2}{D} \left(\nabla \cdot \frac{\boldsymbol{\mu}}{\rho} \right) I. \quad (II.12)$$

Similar to the incompressible fluid, for $\nu > 0$, (II.6) – (II.8) is called the compressible Navier-Stokes equation, for $\nu = 0$, the compressible Euler equation.

For compressible motions the fluid pressure depends only on the density ρ are said to be **barotropic**. This leads to a simplification in the equations of motion almost as great as if the density itself were taken constant. We simply write

$$P = P(\rho). \quad (II.13)$$

Since pressure cannot decrease as density increase, so we require the physical condition that

$$P'(\rho) \text{ is nonnegative.}$$

Thus we can define a function c by

$$c = (P'(\rho))^{\frac{1}{2}}. \quad (II.14)$$

This c is the sound speed of the fluid, and like ρ , it depends on x and t . It is emphasized that the sound speed is not a constant for a particular fluid; it is a thermodynamic variable and depends on the state of the fluid. In this article we will study the fluid flows in the **periodic** case, i.e., we require that

$$\mathbf{u}(x + e_i, t) = \mathbf{u}(x, t); i = 1, 2, \dots, D, \quad (II.15)$$

for all x and $t \geq 0$, where e_i are the standard basis vector in $\mathbf{R}^D$,

$$e_i = (0, \dots, 0, 1, 0, \dots, 0).$$

The advantage of the boundary condition (II.15) is that it leads to a simpler functional setting, while many of the mathematical difficulties remain unchanged (except of course those related to the boundary layer difficulties).

Periodic flows provide prototypical example for fluid flows in bounded domain $\Omega \subset \mathbf{R}^D$. In this case the bounded domain Ω is the D -dimensional torus $\mathbf{T}^D$. Although such flows (spatially periodic flows) do not have the boundary layers caused by viscosity effects in a neighborhood of boundaries, they are significantly different than flows in all of spaces.

One of the elementary concepts is the particle trajectories mapping

$$\mathbf{X}(\cdot, t) : \mathbf{R}^D \ni \alpha \mapsto \mathbf{X}(\alpha, t) \in \mathbf{R}^D. \quad (II.16)$$

Given a fluid flow with velocity field $\mathbf{u}(x, t)$, a **streamline** at a fixed time is an integral curve of $\mathbf{u}$; i.e., if $\mathbf{X}(\alpha, s)$ is a streamline parametrized by s at the instant t , then $\mathbf{X}(\alpha, s)$ satisfies

$$\frac{d\mathbf{X}}{ds}(\alpha, s) = \mathbf{u}(\mathbf{X}(\alpha, s), t); \quad t \text{ fixed.} \quad (II.17)$$

We define a **trajectory** to be the curve traced out by a particle as time progresses; i.e.,

$$\mathbf{X}(\alpha, t) = (x_1, x_2, \dots, x_D),$$

is the location at time t of a fluid particle initially placed at the point $\alpha = (\alpha_1, \dots, \alpha_D)$ at time $t = 0$ and α is a Lagrangian particle marker. This particle trajectory mapping is determined by the nonlinear *O.D.E.*

$$\frac{d\mathbf{X}}{dt}(\alpha, t) = \mathbf{u}(\mathbf{X}(\alpha, t), t), \quad (II.18)$$

$$\mathbf{X}(\alpha, t)|_{t=0} = \alpha. \quad (II.19)$$

If $\mathbf{u}$ is independent of t (i.e., $\partial_t \mathbf{u} = 0$), streamlines and trajectories coincide. In this case, the flow is called **stationary**. This condition means that the “shape” of the fluid flow is not changing. Even if each particle is moving under the flow, the global configuration of the fluid does not change.

The particle trajectory mapping $\mathbf{X}$ has a useful interpretation : an initial domain $\Omega \subset \mathbb{R}^D$ in a fluid evolves in time to

$$\mathbf{X}(\Omega, t) = \{\mathbf{X}(\alpha, t); \quad \alpha \in \Omega\},$$

with the space-time vector $(\mathbf{u}, 1)$ tangent to the particle trajectories. This is known as **Lagrangian** description for the fluid flows.

The Lagrangian description of the fluid flow is not used too often because the Navier-Stokes equations in Lagrangian coordinates are **strongly nonlinear**. It plays an important role, however, in two cases at least : it is used in the numerical computation of a flow with a free boundary, and, in the mathematical theory of

the Navier-Stokes equations, it is the starting point of the geometrical approach developed by V. I. Arnold, D. Ebin-J. Marsden among others. (see [Arn], [Ebi1-2] and [EbM])

It is well known that the Lagrangian coordinates are very useful in studying 1-D piston motions, in gas-dynamics and 1-D combustion theory. It also play an important role in the geometric theory of shock formation for scalar conservation law in several space variables. (see [Maj1])

Next, we recall some elementary properties of $\mathbf{X}(\cdot, t)$. Define the Jacobian of this transformation by

$$J(\alpha, t) = \det \left(\nabla_{\alpha} \mathbf{X}(\alpha, t) \right). \quad (II.20)$$

The quantity $J(\alpha, t)$ measures the infinitesimal ratio of volumes in the image to volumes initially along the curve $X(\alpha, t)$. It is called the **dilatation**. The following formula is well known as the so-called Euler expansion formula.

Proposition II.1. Let $\mathbf{u} \in \mathbf{R}^D$ be a smooth velocity field, and $\mathbf{X}(\cdot, t)$ the particle trajectory mapping. Then we have

$$\partial_t J(\alpha, t) = (\nabla \cdot \mathbf{u})|_{(X(\alpha, t), t)} J(\alpha, t), \quad (II.21)$$

or

$$d(\ln J(\alpha, t))/dt = \nabla \cdot \mathbf{u}.$$

We thus have an important physical meaning for the divergence of the velocity field. It is the relative rate of change of the dilatation following a particle path. We will also frequently need an important kinematical theorem to determine the rate of change of a given function $f(x, t)$ in a domain $\mathbf{X}(\Omega, t)$ moving with the fluid. This calculus formula, called the **transport formula**, can be derived from eq.(II.21). It is due to Reynolds and concerns the rate of change not of an infinitesimal element of volume but of any volume integral.

Theorem II.2(Transport Theorem). Let $\Omega \in \mathbb{R}^D$ —an open, bounded domain with a smooth boundary, and $\mathbf{X}$ be a given smooth particle trajectory mapping of a velocity field $\mathbf{u}$. Then for any smooth function $f(x, t)$ the following holds

$$\frac{d}{dt} \int_{\mathbf{X}(\Omega, t)} f \, dx = \int_{\mathbf{X}(\Omega, t)} \left(\partial_t f + \nabla_x \cdot (f \mathbf{u}) \right) dx, \quad (II.22)$$

where

$$\nabla \cdot (f \mathbf{u}) = \mathbf{u} \cdot \nabla f + f \nabla \cdot \mathbf{u}. \quad (II.23)$$

As an immediate application of **Proposition I.1** and **Theorem I.2**, we interpret the equations $J(\alpha, t) = 1$ and $\nabla \cdot \mathbf{u} = 0$ as the incompressibility condition. Our intuition tells us that a flow is incompressible if for all subregions Ω with smooth boundary and $t \geq 0$ it is volume preserving, i.e.,

$$\text{Vol}(\mathbf{X}(\Omega, t)) = \text{Vol}(\Omega).$$

Thus applying the transport formula (II.22) for $f = 1$, we get $\nabla \cdot \mathbf{u} = 0$. However, then (II.21) yields

$$J(\alpha, t) = J(\alpha, 0) = 1,$$

so we have

Proposition II.3. For the incompressible flow the following three conditions are equivalent:

- (i) $\text{Vol}(\mathbf{X}(\Omega, t)) = \text{Vol}(\Omega),$
- (ii) $\nabla \cdot \mathbf{u} = 0,$
- (iii) $J(\alpha, t) = 1.$

From the equation of continuity

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0,$$

i.e.,

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0,$$

and the fact that $\rho > 0$, we see that a fluid is incompressible if and only if

$$\frac{D\rho}{Dt} = 0,$$

i.e., the mass density is constant following the fluid. If the fluid is **homogeneous**, i.e., $\rho = \text{constant}$ in space, it follows that it is incompressible if and only if ρ is constant in time as well.

Problems involving inhomogeneous incompressible flow occur, for example, in oceanography. In this case one speaks of **stratified flow**.

III. Entropy and Convexity

Conservation laws are a common feature of mathematical physics, where they describe the conservation of fundamental physical quantities. The purpose of this chapter is simply to discuss the mathematical structure of hyperbolic conservation law. The concept of entropy is very important for nonlinear partial differential equations especially for conservation law.

Let Ω be a bounded open set of $\mathbf{R}^D$ with smooth boundary $\Gamma = \partial\Omega$ and outward normal $\vec{n}$. Consider

$$U : \Omega \times \mathbf{R}^+ \mapsto \mathbf{R}^m$$

the system of hyperbolic conservation law is given by

$$\partial_t U + \sum_{i=1}^D \partial_{x_i} (F_i(U)) = 0, \quad (III.1a)$$

or

$$\partial_t U + \sum_{i=1}^D A_i(U) \partial_{x_i} U = 0, \quad (III.1b)$$

where $x = (x_1, x_2, \dots, x_D) \in \mathbf{R}^D$, $U = (u_1, u_2, \dots, u_m)^T$, and $F_i(U)$ are smooth nonlinear mappings defined on an open subset of $\mathbf{R}^D$ and mapping back into $\mathbf{R}^m$ with

$$A_i(U) = \frac{\partial F_i}{\partial U}; \quad i = 1, 2, \dots, D,$$

are the corresponding $m \times m$ Jacobian matrix.

Following Friedrich's theory, (III.1b) is called the **symmetrizable** if there exists a smoothly varying, positive definite, symmetric matrix, A_0 with the property that

$$A_0 \cdot A_i \text{ is symmetric, } 1 \leq i \leq D. \quad (III.2)$$

The structure condition (III.2) plays an important role for the well-posed problem of the linearized equations. Important physical applications where the systems in (III.1) arises and (III.2) is satisfied include gas dynamics, shallow water theory, and magnetofluid dynamics where, of course, $D = 1, 2$, or 3 .

Remark. Let $\omega \in S^{D-1}$ and $\lambda_k(U, \omega)$; $1 \leq k \leq m$, be the eigenvalue of the matrix $A(U, \omega) = \sum_{i=1}^D A_i(U) \omega_i$. Since $A_0(U)$ is invertible, we have

$$\begin{aligned} 0 &= \det \left(\lambda I - \sum_{i=1}^D A_i(U) \omega_i \right) \det A_0(U) \\ &= \det \left(\lambda A_0(U) - \sum_{i=1}^D A_0(U) A_i(U) \omega_i \right). \end{aligned} \tag{III.3}$$

We say that the system of *P.D.E.'s* (III.1a) or (III.1b), is **hyperbolic** (strictly hyperbolic) if $A(U, \omega)$ has real (real and distinct) eigenvalues and is diagonalizable.

Despite the abundance of practical applications of the above systems of conservation law, remarkably little is known rigorously regarding the solutions of these physical systems in two or three space dimensions. In [Kat2], Kato proved the local existence in time of general smooth solutions of (III.1) provided that (III.2) is satisfied and the initial data $U_0(x)$ is a sufficiently smooth function – Kato required $U_0(x) \in H^s(\mathbb{R}^D)$ for $s > \frac{D}{2} + 1$. When $D = 1$, the theory of discontinuous solutions (even globally in time) of (III.1) is well-understood as a consequence of the work of Glimm in [Gli]. In that work, one of the main ideas involved using the explicit structure of the nonlinear progressing wave solutions of (III.1) in a single space variable given by the solution of the Riemann problem and constructed by Lax in [Lax1,2]. Lax also discuss, for weak waves, the fashion in which the nonlinear shock waves and rarefaction waves generalize the progressing wave solutions of the linearized version of (III.1) in a single space variable.

First we shall recall the definition of entropy as formulated by Lax and discuss the role it plays in hyperbolic conservation law. We start with the system

of conservation laws (III.1a). Let H be some function of U . When does H satisfy a conservation law, i.e., a law of the form

$$\partial_t H + \sum_{i=1}^D \partial_{x_i} (G_i(U)) = 0, \quad (III.4)$$

where G_i is some function of U ? Carrying out the differentiation in (III.4) we obtain

$$\partial_U H \partial_t U + \sum_{i=1}^D \partial_U (G_i(U)) \partial_{x_i} U = 0. \quad (III.5)$$

To deduce this from (III.1b),

$$\partial_t U + \sum_{i=1}^D A_i(U) \partial_{x_i} U = 0, \quad (III.1b)$$

we multiply (III.1b) on the left side with $\partial_U H(U)$; (III.4) results if and only if the relation

$$\sum_{i=1}^D \partial_U H \cdot A_i = \sum_{i=1}^D \partial_U G_i$$

holds. Therefore we have the following

Definition(Entropy). For $F \in C^1$, $H(U)$ is called the entropy of (III.1) if associated with $F_i(U)$ there exists $G_i(U)$ such that

$$\partial_U H(U) \cdot \partial_U F(U) = \partial_U G(U). \quad (III.6)$$

The vector field $G(U) = (G_i(U))_i$ is called the **entropy flux** and (H, G) the **entropy pair** of (III.1).

Proposition III.1. For $F = (F_i(U)) \in C^2$, suppose (III.1) admits an entropy function $H(U)$ then

$$\partial_{UU} H(U) \cdot \partial_U F_i(U) \text{ is symmetric.} \quad (III.7)$$

Proof: Let $F_i^k(U)$ be the k^{th} component of $F_i(U)$. The relation (III.6) gives us

$$\frac{\partial G_i}{\partial u_m} = \sum_r \frac{\partial H}{\partial u_r} \frac{\partial F_i^r}{\partial u_m}.$$

Since G_i is regular, we have

$$\frac{\partial^2 G_i}{\partial u_m \partial u_j} = \frac{\partial^2 G_i}{\partial u_j \partial u_m},$$

i.e.

$$\frac{\partial}{\partial u_j} \left(\sum \frac{\partial H}{\partial u_r} \frac{\partial F_i^r}{\partial u_m} \right) = \frac{\partial}{\partial u_m} \left(\sum \frac{\partial H}{\partial u_r} \frac{\partial F_i^r}{\partial u_j} \right). \quad (III.8)$$

But

$$\frac{\partial}{\partial u_j} \left(\frac{\partial F_i^r}{\partial u_m} \right) = \frac{\partial}{\partial u_m} \left(\frac{\partial F_i^r}{\partial u_j} \right).$$

Expanding (III.8) we obtain

$$\sum_r \frac{\partial^2 H}{\partial u_r \partial u_j} \frac{\partial F_i^r}{\partial u_m} = \sum_r \frac{\partial^2 H}{\partial u_r \partial u_m} \frac{\partial F_i^r}{\partial u_j},$$

i.e.,

$$\partial_{UU} H(U) \cdot \partial_U F_i(U) = (\partial F_i(U))^* \cdot \partial_{UU} H(U). \quad (III.9)$$

Hence the result.

Remark 1: In particular, the existence of strictly convex entropy will imply that the system (III.1) is symmetrizable.

Proof: Let H be the strict convex entropy of (III.1) then $\partial_{UU} H$ is positive definite and symmetric. Set $A_0 = \partial_{UU} H$ then by (III.7) we have the result.

Remark 2: The entropy is not unique. For example we can look at the following example.

$$\partial_t U + \partial_x (\partial_U S(U)) = 0$$

Multiply the equation by U and $\partial_{UU} S$ respectively then

$$\partial_t S(U) + \partial_x \left(\frac{1}{2} \partial_U S(U) \cdot \partial_U S(U) \right) = 0$$

$$\partial_t \left(\frac{1}{2} |U|^2 \right) + \partial_x \left(U \partial_U S(U) - S(U) \right) = 0$$

Thus S and $\frac{1}{2}|U|^2$ are both entropies.

Remark 3: In the case of systems of more than two equations, the compatibility equation (III.6) is formally overdetermined and the existence of an entropy pair is a rare event. Nevertheless, the systems of mechanics are naturally endowed with an entropy pair. In addition, the natural pair has the property that H is strictly convex. Henceforth, we shall restrict our attention to strictly hyperbolic systems (III.1) which possess a strictly convex entropy pair.

In the following we give two concrete examples to see the entropy more clearly.

Example 1(P - System).

$$\begin{aligned} \partial_t u + \partial_x P(v) &= 0, \\ \partial_t v - \partial_x u &= 0. \end{aligned} \tag{III.10}$$

Multiply the first equation by u and the second by $-P(v)$, we obtain the entropy

$$H(u, v) = \frac{1}{2} |u|^2 + \int_{v_0}^v P(s) ds$$

and the entropy flux

$$G(u, v) = P(v)u.$$

If $P'(s)$ is negative, i.e., (III.10) is a hyperbolic system, then H is a strict convex entropy of the P-system.

Example 2(Inviscid Compresible Fluid).

$$\begin{aligned} \partial_t \rho + \nabla \cdot (\rho v) &= 0, \\ \partial_t (\rho v) + \nabla \cdot (\rho v \otimes v) + \nabla P(\rho) &= 0. \end{aligned} \tag{III.11}$$

Let $\Phi(\rho)$ be defined by

$$P(\rho) = \rho\Phi'(\rho) - \Phi(\rho), \quad (III.12)$$

then by standard energy method we have the entropy

$$H(\rho, v) = \frac{1}{2}\rho|v|^2 + \Phi(\rho).$$

In fact the standard candidate for the entropy function is the **energy function**.

For the compressible Navier-Stokes equation in the barotropic case,

$$\partial_t \rho + \nabla \cdot \mu = 0, \quad (III.13)$$

$$\partial_t \mu + \nabla \cdot \left(\frac{\mu \otimes \mu}{\rho} \right) + \nabla P(\rho) = \nabla \cdot \left(\nu \sum \right), \quad (III.14)$$

$$\sum \equiv \left(\nabla \frac{\mu}{\rho} \right) + \left(\nabla \frac{\mu}{\rho} \right)^T - \frac{2}{D} \left(\nabla \cdot \frac{\mu}{\rho} \right) I, \quad (III.15)$$

where ν is the viscosity. We assume

$$P'(\rho) = \rho\Phi''(\rho) > 0,$$

i.e., no phase transition, then we have

Lemma III.2. $H(\rho, \mu) = \frac{1}{2}|\mu|^2/\rho + \Phi(\rho)$ is a strict convex entropy of (III.13) – (III.15).

Proof : By direct calculation we have

$$\partial_t H + \nabla \cdot \left[\left(\frac{1}{2} \frac{|\mu|^2}{\rho} + \rho\Phi'(\rho) \right) \frac{\mu}{\rho} - \nu \sum : \frac{\mu}{\rho} \right] = 0,$$

where

$$A : B = \text{tr}(A \cdot B).$$

The Hessian matrix of H is

$$\begin{pmatrix} H_{\rho\rho} & H_{\rho\mu} \\ H_{\mu\rho} & H_{\mu\mu} \end{pmatrix} = \begin{pmatrix} \Phi_{\rho\rho} + \frac{|\mu|^2}{\rho^3} & (-\frac{\mu}{\rho^2})^T \\ -\frac{\mu}{\rho^2} & \frac{1}{\rho} I \end{pmatrix}.$$

Now for all (α, β) ,

$$\begin{aligned} & (\alpha, \beta)^T \begin{pmatrix} \Phi_{\rho\rho} + \frac{|\mu|^2}{\rho^3} & (-\frac{\mu}{\rho^2})^T \\ -\frac{\mu}{\rho^2} & \frac{1}{\rho} I \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ &= \alpha^2 \Phi_{\rho\rho} + \alpha^2 \frac{|\mu|^2}{\rho^3} - 2\alpha \frac{\beta \cdot \mu}{\rho^2} + \frac{|\beta|^2}{\rho} \\ &= \alpha \Phi_{\rho\rho} + \frac{1}{\rho} \left| \alpha \frac{\mu}{\rho} - \beta \right|^2. \end{aligned}$$

Thus H is convex if and only if Φ is convex.

More on the entropy function $H(\rho, \mu)$.

Since H is strict convex, it is natural to consider the Legendre transformation of H . Before we go further, let us recall some properties of Legendre transformation.

The Legendre transformation is a very useful mathematical tool: it transforms functions on a vector space to functions on the dual space. For example, by means of Legendre transformation, a Lagrangian system of second order differential equations is converted into a remarkable symmetrical system of $2D$, D is the space dimension, first order equations called a Hamiltonian system of equations.

Definition. X is a vector space, suppose $\Phi \in C^1$ and $\Phi : X \mapsto R$ is convex, i.e.

$$\Phi(\alpha x_1 + (1 - \alpha)x_2) \leq \alpha \Phi(x_1) + (1 - \alpha)\Phi(x_2),$$

for $0 \leq \alpha \leq 1$. Then let $y = \Phi_x(x)$, where

$$\Phi_x : X \mapsto X^*,$$

we define the Legendre transformation of Φ as

$$\Phi^*(y) = x\Phi_x(x) - \Phi(x).$$

with domain $X^* \equiv \{y \in R; \Phi^*(y) < \infty\}$. This transformation is suggest by the geometric interpretation of the differential equation if we represent the integral surface by its tangent plane coordinates instead of by point coordinates.

Example 3. For $\Phi(x) = \frac{1}{p}|x|^p$, the Legendre transformation is

$$\begin{aligned}\Phi^*(y) &= x\Phi_x(x) - \Phi(x) = |x|^p - \frac{1}{p}|x|^p \\ &= (1 - \frac{1}{p})|x|^p = \frac{1}{p^*}|y|^{p^*},\end{aligned}$$

where $y = |x|^{p-2}x$.

The Legendre transformation has several important and beautiful properties. For convenience we assume that Φ is at least C^2 .

Lemma III.3. If Φ is convex and $\Phi'' > 0$ then Φ^* is also convex.

Lemma III.4. $\Phi^{**} = \Phi$ iff $X = X^*$, i.e., the Legendre transformation is involutive.

By the definition of the Legendre transformation, we have the following important inequality.

Lemma III.5 (Generalized Young's Inequality).

$$y \cdot x \leq \Phi(x) + \Phi^*(y); \quad \forall x \in X, y \in X^*.$$

We return to the compressible Navier-Stokes equation (III.13) – (III.15).

Let

$$\begin{aligned}\sigma &= \frac{\partial H}{\partial \rho} = -\frac{|\mu|^2}{2\rho^2} + \Phi'(\rho), \\ \eta_i &= \frac{\partial H}{\partial \mu_i} = \frac{\mu_i}{\rho}, \quad i = 1, \dots, D,\end{aligned}\tag{III.16}$$

we obtain

$$\Phi'(\rho) = \sigma + \frac{|\mu|^2}{2\rho^2} = \sigma + \frac{1}{2}|\eta|^2.$$

Then the Legendre transformation is

$$\begin{aligned}
 H^*(\sigma, \eta) &= \rho \frac{\partial H}{\partial \rho} + \sum_{i=1}^D \mu_i \frac{\partial H}{\partial \mu_i} - H \\
 &= \rho \Phi'(\rho) - \Phi(\rho) \\
 &= P(\rho) = \Phi^*(\sigma + \frac{1}{2}|\eta|^2).
 \end{aligned} \tag{III.17}$$

For convenience we assume that Φ is of the particular form

$$\Phi(\rho) = \frac{1}{\gamma} \rho^\gamma, \tag{III.18}$$

then

$$\Phi^*(\sigma) = \frac{1}{\gamma^*} \sigma^{\gamma^*}, \tag{III.19}$$

where

$$\frac{1}{\gamma} + \frac{1}{\gamma^*} = 1.$$

In this case

$$P(\rho) = (1 - \frac{1}{\gamma}) \rho^\gamma. \tag{III.20}$$

Evidently we have the Young's inequality,

$$(\sigma, \eta) \cdot (\rho, \mu) = \sigma \rho + \eta \cdot \mu \leq H^*(\sigma, \eta) + H(\rho, \mu). \tag{III.21}$$

In the sequel we will use (III.21) to show some weak compactness results.

But our main concern is the scaled compressible Navier-Stokes equation

$$\epsilon \partial_t \rho_\epsilon + \nabla \cdot \mu_\epsilon = 0, \tag{III.22a}$$

$$\epsilon \partial_t \mu_\epsilon + \nabla \cdot \left(\frac{\mu_\epsilon \otimes \mu_\epsilon}{\rho_\epsilon} \right) + \nabla P(\rho_\epsilon) = \epsilon \nabla \cdot \left(\nu \sum_\epsilon \right). \tag{III.22b}$$

By the same technique of (III.13) – (III.21), we have the following result.

$$\partial_t \left(\frac{|\mu_\epsilon|^2}{2\rho_\epsilon \epsilon^2} + \frac{1}{\epsilon^2} \Phi(\rho_\epsilon) \right) + \nabla \cdot \left[\left(\frac{|\mu_\epsilon|^2}{2\rho_\epsilon \epsilon^2} + \frac{\rho_\epsilon}{\epsilon^2} \Phi'(\rho_\epsilon) \right) \frac{\mu_\epsilon}{\epsilon \rho_\epsilon} - \frac{\nu}{\epsilon^2} \sum_\epsilon : \frac{\mu_\epsilon}{\rho_\epsilon} \right] = 0. \tag{III.23}$$

Lemma III.6. For strict convex function $\Phi(\rho)$, then

$$H(\rho, \mu) = \frac{1}{2} \frac{|\mu|^2}{\rho} + \Phi(\rho),$$

is also a strict convex entropy of (III.22).

Since we consider the perturbation near the constant state $(1, 0)$. Hence the entropy is given by

$$\begin{aligned} \tilde{H}(\rho, \mu) &= H(\rho, \mu) - H(1, 0) - \frac{\partial H}{\partial \rho}(1, 0)(\rho - 1) - \sum_{i=1}^D \mu_i \frac{\partial H}{\partial \mu_i}(1, 0) \\ &= \frac{1}{2} \frac{|\mu|^2}{\rho} + \Phi(\rho) - \Phi(1) - \Phi'(1)(\rho - 1) \\ &= \frac{1}{2} \frac{|\mu|^2}{\rho} + \Psi(\rho), \end{aligned} \quad (III.24)$$

where $\Psi(\rho) \equiv \Phi(\rho) - \Phi(1) - \Phi'(1)(\rho - 1)$. This is the so called **relative entropy** of $H(\cdot, \cdot)$ with respect to the absolute equilibrium $(\rho, \mu) = (1, 0)$, it provides a natural measure of proximity of H to that equilibrium.

Later on we will see that the entropy and the entropy dissipation provide the weak compactness statements regarding (ρ, μ) , $\frac{\mu}{\rho}$ respectively. In this case we have the conservative equation

$$\partial_t \tilde{H} + \nabla \cdot \left[\left(\frac{1}{2} \frac{|\mu|^2}{\rho} + \rho \Phi'(\rho) \right) \frac{\mu}{\rho} - \Phi'(1) \mu - \nu \sum : \frac{\mu}{\rho} \right] = 0. \quad (III.25)$$

Let

$$\begin{aligned} \sigma &= \frac{\partial \tilde{H}}{\partial \rho} = -\frac{1}{2} \frac{|\mu|^2}{\rho^2} + \Phi'(\rho) - \Phi'(1) = -\frac{1}{2} \frac{|\mu|^2}{\rho^2} + \Psi'(\rho), \\ \eta &= \frac{\partial \tilde{H}}{\partial \mu} = \frac{\mu}{\rho}. \end{aligned} \quad (III.26)$$

So

$$\Psi'(\rho) = \sigma + \frac{1}{2} \frac{|\mu|^2}{\rho^2} = \sigma + \frac{1}{2} |\eta|^2,$$

then the dual Legendre transformation is given by

$$\begin{aligned}
\tilde{H}^*(\sigma, \eta) &= \rho \frac{\partial \tilde{H}}{\partial \rho} + \sum_{i=1}^D \mu_i \frac{\partial \tilde{H}}{\partial \mu_i} - \tilde{H} \\
&= \rho \Psi'(\rho) - \Psi(\rho) \\
&= \Psi^*\left(\sigma + \frac{1}{2} |\eta|^2\right).
\end{aligned}
\tag{III.27}$$

IV. Apriori Bounds and Weak L^1 Compactness

In this chapter we derive some (formal) a priori estimate of the barotropic compressible Navier-Stokes equation (III.13) – (III.15) by energy method. The **energy method** is a basic technique in the modern theory of partial differential operators. We also collect some weakly compact theorems that have been found important for nonlinear partial differential equations by the technique of weak convergence.

Taking the inner product of (III.14) and $\frac{\mu}{\rho}$, then integrating over Ω employing (III.13), and **Theorem II.2 (Transport Theorem)** (II.22) we obtain

$$\begin{aligned} & \int_{\Omega} \partial_t \left(\frac{1}{2} \frac{|\mu|^2}{\rho} \right) + \frac{|\mu|^2}{2\rho^2} \left(-\nabla \cdot \mu \right) + \left(\nabla \cdot \left(\frac{\mu \otimes \mu}{\rho} \right) \right) \cdot \frac{\mu}{\rho} dx \\ &= \int_{\Omega} \left(\nabla \cdot (\nu \sum) - \nabla P \right) \cdot \frac{\mu}{\rho} dx. \end{aligned} \quad (IV.1)$$

But for any vector $\mathbf{v} = (v_1, v_2, \dots, v_D)$ we have

$$\nabla \cdot (\mathbf{v} \otimes \mathbf{v}) = \sum_{i=1}^D \partial_{x_i} (v_i \mathbf{v}).$$

Therefore by (II.23) we have

$$\nabla \cdot \left(\frac{\mu \otimes \mu}{\rho} \right) = (\mu \cdot \nabla) \frac{\mu}{\rho} + \frac{\mu}{\rho^2} (\mu \cdot \nabla) \rho + \mu \left(\nabla \cdot \frac{\mu}{\rho} \right). \quad (IV.2)$$

Take the inner product of (IV.2) and $\frac{\mu}{\rho}$, then

$$\left(\nabla \cdot \left(\frac{\mu \otimes \mu}{\rho} \right) \right) \cdot \frac{\mu}{\rho} = \frac{|\mu|^2}{\rho^2} \nabla \cdot \mu + \frac{\mu}{\rho^2} \cdot \nabla \left(\frac{|\mu|^2}{2} \right) - \frac{|\mu|^2}{\rho^3} (\mu \cdot \nabla \rho). \quad (IV.3)$$

On the other hand

$$\begin{aligned} \nabla \cdot \left(\frac{1}{2} \frac{|\mu|^2}{\rho} \frac{\mu}{\rho} \right) &= \frac{|\mu|^2}{2\rho^2} \nabla \cdot \mu - \frac{|\mu|^2}{2\rho^3} (\mu \cdot \nabla \rho) + \frac{\mu}{\rho} \cdot \nabla \left(\frac{1}{2} \frac{|\mu|^2}{\rho} \right) \\ &= \frac{|\mu|^2}{2\rho^2} \nabla \cdot \mu + \frac{\mu}{\rho^2} \cdot \nabla \left(\frac{|\mu|^2}{2} \right) - \frac{|\mu|^2}{\rho^3} (\mu \cdot \nabla \rho). \end{aligned}$$

thus

$$\begin{aligned} \text{L.H.S. of (IV.1)} &= \int_{\Omega} \partial_t \left(\frac{1}{2} \frac{|\mu|^2}{\rho} \right) + \nabla \cdot \left(\frac{1}{2} \frac{|\mu|^2}{\rho} \frac{\mu}{\rho} \right) dx \\ &= \frac{d}{dt} \int_{\Omega} \frac{1}{2} \frac{|\mu|^2}{\rho} dx. \quad (\text{Transport Theorem}) \end{aligned}$$

Integrating by parts we have

$$\frac{d}{dt} \int_{\Omega} \frac{1}{2} \frac{|\mu|^2}{\rho} dx + D(t) = \int_{\Omega} P \nabla \cdot \frac{\mu}{\rho} dx, \quad (\text{IV.4})$$

where

$$D(t) = \nu \int_{\Omega} \left| \sum \right|^2 dx.$$

On the other hand, by transport theorem and (III, 11a),

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \Phi(\rho) dx &= \int_{\Omega} \Phi'(\rho) \partial_t \rho + \frac{\mu}{\rho} \cdot \nabla \Phi(\rho) + \Phi(\rho) \nabla \cdot \frac{\mu}{\rho} dx \\ &= \int_{\Omega} \left(\Phi(\rho) - \rho \Phi'(\rho) \right) \nabla \cdot \frac{\mu}{\rho} dx. \end{aligned} \quad (\text{IV.5})$$

But

$$P(\rho) = \rho \Phi'(\rho) - \Phi(\rho),$$

hence (IV.4) + (IV.5) we derive

$$\frac{d}{dt} \int_{\Omega} H(\rho, \mu) dx + D(t) = 0. \quad (\text{IV.6})$$

This formula explains why the viscosity coefficient ν must be positive if we want to have a dissipation of energy in the Navier-Stokes equation. It will also have theoretical applications in studying the mathematical properties of solutions of Navier-Stokes equation.

In fact dissipation plays an important role in balancing the steepening due to the nonlinearity, so that when these effects are present a steep but smooth wave may be formed which then propagate for all time.

A typical example of dissipative nonlinear waves is the shock wave produced by a supersonic object. In that case the gas across the shock front is heated by viscosity, and the dissipation so caused then balances the steepening of the wave produced by the nonlinearity in the governing equations.

On the other hand, another very interesting typical example of a nonlinear wave propagating in a **dispersive** medium, in which the phase velocity varies with the wavelength, is the **solitary wave**. This is a pulse formed as a result of balancing occurring between the steepening effect due to the nonlinearity and the smoothing effect of the dispersion. But it does not belong to the scope of this article.

Similar, for the scaled system (III.22) we have the following results.

Lemma IV.1 Let $(\rho_\epsilon, \mu_\epsilon)$ be a weak solution of (III.22) then we have the energy inequality

$$\int_{\Omega} \tilde{H}(U_\epsilon(T)) dx + \int_0^T \int_{\Omega} \sum_{\epsilon} : \sum_{\epsilon} dx dt \leq \int_{\Omega} \tilde{H}(U_\epsilon(0)) dx.$$

where

$$U_\epsilon(t) = \left(\rho_\epsilon(t), \mu_\epsilon(t) \right)^T. \quad (IV.7)$$

Most of the partial differential equations that arise in continuum mechanics and physics are nonlinear. The linearized equations are now well understood but methods for the nonlinear ones still require improvement.

Twenty years ago there were mainly two methods for attacking nonlinear equations (J.L. Lions, [Lio]): one being by a **compactness argument** while the other, by a **monotonicity argument**, which usually relied on some kind of convexity result.

Weak convergence is one of the most important techniques for studying nonlinear partial differential equations. The ideal is following; Suppose we wish to solve some nonlinear PDE, which we write symbolically as

$$A[u] = f, \quad (IV.8)$$

$A[\cdot]$ denoting a given nonlinear operator, f a given function, and u the unknown. To establish the existence of a solution u of (III.8), an obvious idea is first to invent an appropriate collection of nicer, approximating problems, which we can in fact solve.

These we write abstractly as

$$A_n[u_n] = f_n; \quad (n = 1, 2, \dots), \quad (IV.9)$$

where $A_n[\cdot]$ represents a nonlinear operator which is somehow closed to $A[\cdot]$ for large n , f_n is closed to f , and u_n is a solution. The hope now is that the functions $\{u_n\}_{n=1}^{\infty}$ will converge to a solution u of (IV.8)

In practice, the operator $A_n[\cdot]$ may represent finite-dimensional projections, singular regularizations, discretizations, gradients of approximate energy functionals, etc.

Indeed, given a nonlinear PDE like (IV.8), it is usually not particularly difficult to dream up some reasonable and solvable approximation: the trick is to demonstrate that solutions of (IV.9) really do converge to a solution of (IV.8).

The overall impediment is of course the nonlinearity. Whereas it is often the case that certain uniform estimates can be had for the family $\{u_n\}_{n=1}^{\infty}$, it is equally often also true that these best available bounds are none too strong. With such relatively poor estimates in hand, we can consequently usually show only that the functions $\{u_n\}_{n=1}^{\infty}$ (or a subsequence) **converge weakly** in some function space to a limit u :

$$u_n \xrightarrow{w} u, \quad \text{as } n \rightarrow \infty. \quad (IV.10)$$

Given the strong nonlinearities we are trying to cope with, this rather feeble mode of convergence is almost always a definite problem. Granted that we have somehow constructed approximate operator $A_n[\cdot]$ which tend to $A[\cdot]$ in some sense, it is in applications by no means certain that the weak convergence (IV.8) implies

$$A_n[u_n] \rightarrow A[u], \quad \text{as } n \rightarrow \infty. \quad (IV.11)$$

in any way whatsoever. The obstruction is that weak convergence is terribly behaved with respect to nonlinearities as discussed in the introduction. Hence we need some strong compactness result, i.e., u_n solves the appropriately designed approximation (IV.9) provides enough extra control, to justify passing to limits as we have discussed in (I.13) – (I.14).

For convenience we state some lemmas that we will need in the following sections. We only give the definitions and main theorems. Most of the theorems are standard and their proofs as well as a deeper analysis are available in several classical textbooks. The general references are [DiU], [DuS], [Kuf], and [Yos].

Definition If X is any Banach space, we set $X = [X]^D$, X^* will denote the dual space. For $1 \leq p < \infty$ and X a Banach space with norm $\cdot_X$, we denote by $L^p(I, X)$ the set of all mappings $f : I = (0, T) \mapsto X$ which are strongly measurable and such that

$$\int_0^T \|f\|_X^p dx < \infty. \quad (IV.12)$$

Denote $C(I, X)$ the space of continuous functions from I into X with the uniform convergence norm. At last let $1_{[t_1, t_2]} f \in L^p(I, X)$, $\forall [t_1, t_2] \subset I$ equipped with the semi-norm

$$\|f\|_{L^p((t_1, t_2), X)} = \left(\int_{t_1}^{t_2} \|f\|_X^p dt \right)^{\frac{1}{p}}.$$

We shall use the notation $w - X$ to indicate the space X equipped with its weak topology, that is the coarsest topology on X for which each of the linear forms

$$u \mapsto \langle w; u \rangle \quad \text{for } w \in X^*,$$

is continuous. Here $\langle \cdot; \cdot \rangle$ is the natural bilinear form relating X^* and X .

We also denote by $C(I, w - X)$ the space of all continuous functions from I into $w - X$, that is the set of functions u for which

$$x \mapsto \langle w; u(x) \rangle \text{ is in } C(I) \text{ for each } w \in X^*.$$

We shall use the fact that an Arzela-Ascoli theorem holds for such spaces.

Definition. For any topological linear space X , a subset K in $C(I, X)$ is called **equicontinuous** at $t_0 \in [a, b]$ if given a neighborhood V of the origin in X there exists $\delta(V) > 0$ such that $f(t) - f(t_0) \in V$ for each $t \in I, |t - t_0| \leq \delta(V)$, and uniformly for $f \in K$. The subset K is equicontinuous on I if it is equicontinuous at each $t \in I$.

Theorem IV.2 If X is a separable space and $1 < p < \infty$ then

$$(L^p(I, X))^* = L^q(I, X^*), \quad (IV.13)$$

where

$$\frac{1}{p} + \frac{1}{q} = 1.$$

We first define abstractly the notion of weak convergence and apply then these results to L^p spaces. We start with the definitions.

Definition. Let X be a Banach space, X^* its dual and $\langle \cdot; \cdot \rangle$ the bilinear canonical pairing over $X \times X^*$

(i) We say that, $x_n, x \in X$, x_n converges weakly to x and we denote

$$x_n \xrightarrow{w} x \text{ in } X,$$

if

$$\langle x_n; x^* \rangle \rightarrow \langle x; x^* \rangle,$$

for every $x^* \in X^*$.

(ii) We say that, $x_n^*, x^* \in X^*$, x_n^* converges weak star to x^* and we denote

$$x_n^* \xrightarrow{*} x^* \text{ in } X^*,$$

if

$$\langle x; x_n^* \rangle \rightarrow \langle x; x^* \rangle,$$

for every $x \in X$.

We then have the following theorem.

Theorem IV.3.

(A) Let X be a Banach space.

(1) Let $x_n \xrightarrow{w} x$, then there exists $K > 0$ such that

$$\|x_n\| \leq K,$$

furthermore

$$\|x\| \leq \liminf_{n \rightarrow \infty} \|x_n\|.$$

(2) Let $x_n^* \xrightarrow{*} x^*$, then there exists $K > 0$ such that

$$\|x^*\|_{X^*} \leq K,$$

furthermore

$$\|x^*\|_{X^*} \leq \liminf_{n \rightarrow \infty} \|x_n^*\|_{X^*}.$$

(3) If $x_n \xrightarrow{s} x$ (strongly), then $x_n \xrightarrow{w} x$ (weakly).

(4) If $x_n^* \xrightarrow{s} x^*$ (strongly), then $x_n^* \xrightarrow{*} x^*$ (weakly star).

(B) Let X be a reflexive Banach space, let $K > 0$ and let

$$\|x_n\| \leq K,$$

then there exists $x \in X$ and a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$x_{n_j} \xrightarrow{w} x \text{ in } X.$$

(C) Let X be a separable Banach space, let $K > 0$ and let

$$\|x^*\|_{X^*} \leq K,$$

then there exists $x^* \in X^*$ and a subsequence $\{x_{n_j}^*\}$ of $\{x_n^*\}$ such that

$$x_{n_j}^* \xrightarrow{*} x^* \text{ in } X^*.$$

We now see how the above results can be applied to L^p spaces. Here and hereafter we will denote by Ω an open, bounded, smooth subset of $\mathbb{R}^D$ ($D \geq 2$).

For $1 \leq p < \infty$ and $q = \frac{p}{p-1}$ we have the following

Definition. A sequence $\{f_n\}_{n=1}^\infty \subset L^p(\Omega)$ converges weakly to f in $L^p(\Omega)$, written

$$f_n \xrightarrow{w} f \text{ in } L^p(\Omega),$$

if

$$\lim_{n \rightarrow \infty} \int_{\Omega} f_n g \, dx = \int_{\Omega} f g \, dx \quad (IV.14)$$

for each $g \in L^q(\Omega)$.

Theorem IV.4 (Boundedness of weakly convergent sequences).

Assume $f_n \xrightarrow{w} f$ in $L^p(\Omega)$. Then

$$(i) \{f_n\}_{n=1}^\infty \text{ is bounded in } L^p(\Omega),$$

and

$$(ii) \|f\|_{L^p(\Omega)} \leq \liminf_{n \rightarrow \infty} \|f_n\|_{L^p(\Omega)}.$$

In view of (i) we see that if $f_n \xrightarrow{w} f$ in $L^p(\Omega)$ and $g_n \xrightarrow{s} g$ in $L^q(\Omega)$, then

$$\int_{\Omega} f_n g_n \, dx \rightarrow \int_{\Omega} f g \, dx. \quad (IV.15)$$

A refinement of (ii) holds: if $1 < p < \infty$, $f_n \xrightarrow{w} f$ in $L^p(\Omega)$ and

$$\|f\|_{L^p(\Omega)} = \liminf_{n \rightarrow \infty} \|f_n\|_{L^p(\Omega)},$$

then

$$f_n \xrightarrow{s} f, \quad \text{in } s\text{-}L^p(\Omega).$$

Theorem IV.5 (Weak compactness). Assume $1 < p < \infty$ and the sequence $\{f_n\}_{n=1}^\infty$ is bounded in $L^p(\Omega)$. Then there exists a subsequence $\{f_{n_j}\}_{j=1}^\infty \subset \{f_n\}_{n=1}^\infty$ and a function $f \in L^p(\Omega)$ with

$$f_{n_j} \xrightarrow{w} f \quad \text{in } L^p(\Omega).$$

In the case $p = \infty$ the terminology is slightly different. We say that $\{f_n\}_{n=1}^\infty \subset L^\infty(\Omega)$ converges **weakly star** to $f \in L^\infty(\Omega)$, written

$$f_n \xrightarrow{*} f \quad \text{in } L^\infty(\Omega),$$

provided (IV.14) holds for all g in $L^1(\Omega)$. The analogues of **Theorem IV.3** and **IV.4** are valid.

The weak compactness theorem is definitely false if $p = 1$, as simple examples illustrate. This failure of compactness is to a certain extent offset by regarding $L^1(\Omega)$ in a nature way as a subset of $M(\Omega)$, the space of signed Radon measures on Ω with finite mass.

Let $C_c(\Omega)$ denote the space of continuous, real-valued functions on Ω , with compact support.

Definition. A sequence $\{\mu_n\}_{n=1}^\infty \subset M(\Omega)$ converges **weakly** to $\mu \in M(\Omega)$, written

$$\mu_n \xrightarrow{w} \mu, \quad \text{in } M(\Omega),$$

provided

$$\int_{\Omega} g d\mu_n \rightarrow \int_{\Omega} g d\mu; \quad \text{as } n \rightarrow \infty.$$

for each $g \in C_c(\Omega)$.

Now we turn to the concept of weak continuity.

Definition. $f \in C([0, \infty), w - L^1(dx))$, iff

$$\int_{\Omega} f(t, x) g(x) dx \in C([0, \infty), \mathbf{R}); \quad \forall g \in L^\infty(\Omega),$$

i.e., for all $\epsilon > 0$ there exists $\delta > 0$ such that $|t_2 - t_1| < \delta$, one has

$$\int_{\Omega} |f(t_2, x) - f(t_1, x)| g(x) dx < \epsilon.$$

Definition. A subset K of $L^1([a, b], X)$ is called **equiintegrable** if, given $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that

$$\int_E \|f(t)\|_X dt < \epsilon, \quad (IV.16)$$

for each measurable subset E of $[a, b]$ whose Lebesgue measure is less than $\delta(\epsilon) > 0$ and uniformly for all $f \in K$.

Remark: For compact interval I , it readily follows that each equiintegrable subset in $L^1(I, X)$ is bounded in the norm of the latter.

Remark: If K is a bounded subset in $L^p(I, X)$ with $1 < p \leq \infty$, then K is equiintegrable.

Indeed, from Hölder inequality we have

$$\int_E \|f(t)\|_X dt \leq \left(\int_E \|f(t)\|^p dt \right)^{\frac{1}{p}} \left(\int_E dt \right)^{\frac{1}{q}}, \quad 1/p + 1/q = 1.$$

for each measurable subset E in I and for each $f \in K$, thereby completing the proof.

The following is a deeper result and is known as the **Dunford-Pettis theorem**.

Theorem IV.6 (Dunford-Pettis). Let $\{f_n\} \subset L^1(\Omega)$ then $\{f_n\}$ is relatively weakly compact iff

- (i) $\{f_n\}$ is equibounded in $L^1(\Omega)$,
- (ii) $\{f_n\}$ is equiintegrable.

As a corollary of Theorem IV.2, we have

Corollary IV.7. Let X be reflexive and $1 < p < \infty$. A subset in $L^p(I, X)$ is weakly relatively compact if and only if it is bounded in the norm of the latter.

Theorem IV.8 (Arzela-Ascoli Theorem). A subset K of $C([0, \infty), w - X)$ is relatively compact iff

- (i) K is equicontinuous in $C([0, \infty), w - X)$,
- (ii) $K(t) = \{f(t); f \in K\}$ is relatively compact in $w - X$.

The following Theorem is a very powerful tool for proving weakly compactness in $L^1(\Omega)$. So we state and prove it.

Theorem IV.9. Let $|\Omega| < \infty$ and S is a strictly convex function. We have

$$\int_{\Omega} S(f_{\alpha}) dx \leq M < \infty,$$

for some M , for all f_{α} . Then f_{α} is relatively compact in $w - L^1(\Omega)$

Proof: Since S is convex and

$$\frac{S(x)}{x} \rightarrow \infty; \text{ as } x \rightarrow \infty.$$

Hence given $\epsilon > 0$, let $K_{\epsilon} = M/\epsilon$ and choose λ_n such that $x > \lambda$ implies

$$(S(x)/x) > K_{\epsilon}.$$

Hence

$$\int_{\{|f_{\alpha}| > \lambda_n\}} |f_{\alpha}| dx \leq \frac{1}{K_{\epsilon}} \int_{\{|f_{\alpha}| \geq \lambda_n\}} S(f_{\alpha}) dx \leq M/K_{\epsilon} = \epsilon.$$

Since this is uniform in α , and $\epsilon > 0$ is arbitrary. Therefore

$$\lim_{\lambda \rightarrow \infty} \int_{\{|f_{\alpha}| > \lambda\}} |f_{\alpha}| dx = 0, \quad (IV.17)$$

uniformly in α . Now we have

$$\begin{aligned} \int_{\Omega} |f_{\alpha}| dx &= \int_{\{|f_{\alpha}| \leq \lambda\}} |f_{\alpha}| dx + \int_{\{|f_{\alpha}| > \lambda\}} |f_{\alpha}| dx \\ &\leq \lambda |\Omega| + 1 \leq \lambda M + 1, \end{aligned} \quad (IV.18)$$

where λ is chosen such that the second integral is almost 1 by (IV.17), i.e., $f_\alpha \in L^1(\Omega)$ and f_α is uniformly bounded in $L^1(\Omega)$. Let $A \subset \Omega$, by (IV.18), we have

$$\begin{aligned} \int_A |f_\alpha| dx &= \int_{\{A \cap \{|f_\alpha| > \lambda\}\}} |f_\alpha| dx + \int_{\{A \cap \{|f_\alpha| \leq \lambda\}\}} |f_\alpha| dx \\ &\leq \int_{\{|f_\alpha| > \lambda\}} |f_\alpha| dx + \lambda |A|. \end{aligned}$$

So for all $\epsilon > 0$, by (IV.17), we can choose $\lambda_\epsilon > 0$ such that

$$\int_{\{|f_\alpha| > \lambda_\epsilon\}} |f_\alpha| dx < \epsilon/2,$$

and then for this $\lambda_\epsilon > 0$, choose $|A| < \delta_\epsilon = \epsilon/2\lambda_\epsilon$. This shows that

$$|A| < \delta_\epsilon \Rightarrow \int_A |f_\alpha| dx < \epsilon,$$

uniformly in α . That is $\{f_\alpha\}$ are equiintegrable. Therefore by Theorem IV.6 (Dunford-Pettis Theorem), we conclude that

$$\{f_\alpha\} \text{ are relatively compact in } w - L^1(\Omega).$$

Remark: S is convex, so $S(x) \geq ax + b$ for some a and b , i.e., the curve is above the line which support it. Then

$$\int_\Omega (a|f_\alpha| + b) dx \leq \int_\Omega S(|f_\alpha|) < C,$$

for all α , where C is a constant. Thus

$$\int_\Omega |f_\alpha| dx < C,$$

uniformly in α .

Theorem IV.9 is a very deep result. As an application of the Theorem we have the following corollary, which is a generalization of Lebesgues' Theorem which will be extremely useful.

Corollary IV.10 (Generalized Lebesgue Theorem).

Let $\{f_n\}$ be an equiintegrable sequence of $L^1(\Omega)$, such that

$$f_n(x) \rightarrow f(x), \quad \text{a.e. in } \Omega.$$

Then f is integrable and

$$f_n \xrightarrow{s} f \quad \text{in } L^1(\Omega).$$

The following lemma is important when we use the weak convergence for the nonlinear problems, which means the almost everywhere convergence guarantees the weak convergence to the limit.

Lemma IV.11. Let Ω be a bounded open subset of $\mathbf{R}^D$, f_n and f are functions of $L^p(\Omega)$, $1 < p < \infty$ such that

$$\|f_n\|_{L^p(\Omega)} \leq C, \quad f_n \rightarrow f \quad \text{a.e. in } \Omega.$$

Then

$$f_n \xrightarrow{w} f \quad \text{in } L^p(\Omega) \text{ weakly.}$$

V. Implications of the Entropy Inequality

In this chapter we obtain some results which are direct consequence of the entropy inequality. In fact most of the problems of partial differential equations are reduced to the problem of obtaining the apriori estimate. An apriori estimates for solutions of a partial differential equation is simply an inequality which is valid for all solutions whose data and coefficients obey certain restrictions. The entropy bound will gives us the apriori estimate. To prove the weak L^1 compactness we also need some restriction for the equation of state.

A specific family of fluids which will be frequently referred to is given by the varying equations of state

$$P_\epsilon(\rho) = \frac{1}{\epsilon^2} P(\rho), \quad 0 < \epsilon \ll 1. \quad (V.1)$$

as shown in (I, 11) – (I, 12), with fixed $P(\rho)$, and $\partial p / \partial \rho > 0$. Such family arise in a natural fashion from a fixed fluid with an equation of state $P(\rho) = A\rho^\gamma$, $\gamma > 1$.

Thus we consider a sequence of weak solutions $(\rho_\epsilon, \mu_\epsilon)$ to the scaled compressible Navier-Stokes equation

$$\epsilon \partial_t \rho_\epsilon + \nabla \cdot \mu_\epsilon = 0, \quad (V.2a)$$

$$\epsilon \partial_t \mu_\epsilon + \nabla \cdot \left(\frac{\mu_\epsilon \otimes \mu_\epsilon}{\rho_\epsilon} \right) + \nabla P(\rho_\epsilon) = \epsilon \nabla \cdot (\nu \sum_\epsilon). \quad (V.2b)$$

Due to (V.2a), equation of continuity, and the total mass of fluid is given, without loss of generality we may impose the condition

$$\frac{1}{|\Omega|} \int_\Omega \rho_\epsilon(x, t) dx = \frac{1}{|\Omega|} \int_\Omega \rho_\epsilon(x, 0) dx = 1. \quad \forall \epsilon \quad (V.3)$$

In fact we will consider the solutions $(\rho_\epsilon, \mu_\epsilon)$ near the equilibrium (1,0). Similar to lemma IV.1, for weak solutions $(\rho_\epsilon, \mu_\epsilon)$ we have the following energy inequality.

Theorem V.1. Let $(\rho_\epsilon, \mu_\epsilon)$ be a weak solution of (V.2), then it satisfies the following global entropy inequality,

$$\int_{\Omega} \tilde{H}(\rho_\epsilon(t), \mu_\epsilon(t)) dx + \nu \int_0^t \int_{\Omega} \sum_{\epsilon} : \sum_{\epsilon} dx ds \leq \int_{\Omega} \tilde{H}(\rho_\epsilon(0), \mu_\epsilon(0)) dx. \quad (V.4)$$

It is interesting to notice that the use of the Orlicz space is naturally suggested by the constitutive equations of the energy for the fluid.

Let M be an Young's function, by $\tilde{L}_M$ we denote the Orlicz class corresponding to M , i.e., the set of all Lebesgue measurable functions u in Ω such that

$$\int_{\Omega} M(u) dx < \infty. \quad (V.5)$$

The Orlicz space $L_M(\Omega)$ is defined to be the linear hull of the Orlicz class $\tilde{L}_M(\Omega)$ with the Luxemburg norm

$$\|u\|_M = \inf \{ h > 0 ; \int M(u(x))/h dx \leq 1 \}. \quad (V.6)$$

For $\|u\|_M > 1$ we have

$$\int_{\Omega} M(u(x))/\|u\|_M dx \leq 1. \quad (V.7)$$

From (III.23) it is nature to consider a sequence of solutions U_ϵ indexed by a vanishing positive sequence ϵ such that for some constant $C > 0$, the initial data $U_\epsilon^{in} = (\rho_\epsilon(0), \mu_\epsilon(0))^T$ satisfies the entropy bound

$$\int_{\Omega} \tilde{H}(U_\epsilon^{in}) dx \leq C\epsilon^2, \quad (V.8)$$

where $0 < \epsilon \ll 1$.

This then implies bounds on the sequence U_ϵ through the entropy inequality (V.4). This section contains results that follow directly from the convexity of the integrands in the entropy inequality (V.4) and the entropy bound (V.8). The following lemma is straightforward from (V.4) and (V.8).

Lemma V.2. Let $(\rho_\epsilon, \mu_\epsilon)$ satisfy the same hypothesis of Lemma IV.1 and the initial entropy satisfies (V.8) then

$$(i) \quad \{\rho_\epsilon\}_\epsilon \text{ remains bounded in } L^\infty(I, L_\Psi(\Omega)), \quad (V.9)$$

$$(ii) \quad \left\{ \frac{1}{\rho_\epsilon} \left| \frac{\mu_\epsilon}{\epsilon} \right|^2 \right\}_\epsilon \text{ remains bounded in } L^\infty(I, L^1(\Omega)), \quad (V.10)$$

$$(iii) \quad \left\{ \frac{1}{\rho_\epsilon} \frac{\mu_\epsilon}{\epsilon} \right\}_\epsilon \text{ remains bounded in } L^2(I, H^1(\Omega)). \quad (V.11)$$

Moreover, even for unbounded domain Ω , from (IV.3) we have

$$(iv) \quad \{\rho_\epsilon\}_\epsilon \text{ remains bounded in } L^\infty(I, L_\Psi(\Omega)) \cap L^\infty(I, L^1(\Omega)). \quad (V.12)$$

Since H is strictly convex, of course $\tilde{H}$ is also strictly convex, then by Theorem IV.9 we have

Lemma V.3. For $(\rho_\epsilon, \mu_\epsilon)$ satisfying the same hypothesis as Theorem V.1 then

$$\left(\rho_\epsilon(t), \frac{\mu_\epsilon}{\epsilon}(t) \right) \text{ is compact in } w - L^1(\Omega). \quad (V.13)$$

for all t fixed but arbitrary.

From the strict convexity of Ψ and the entropy bound (V.8) it follows directly that

Lemma V.4. $\left\| \frac{\rho_\epsilon(t) - 1}{\epsilon^2} \right\|_{L^1(\Omega)}$ is uniformly bounded.

Remark: From the uniform bound

$$\left\| \frac{\rho_\epsilon(t) - 1}{\epsilon^2} \right\|_{L^1(\Omega)} < M,$$

it follows directly that

$$\rho_\epsilon \rightarrow 1 \quad \text{in } s - L^1(\Omega).$$

By passing to subsequence we also have

$$\rho_\epsilon \rightarrow 1 \quad \text{a.e. in } \Omega.$$

The following lemma is a refinement of Fatou's lemma which is due to Brezis and Lieb [BrL]. We will need it in the proof of Theorem V.6.

Lemma V.5. Suppose $f_n \rightarrow f$ a.e. in Ω and $\|f_n\|_{L^p(\Omega)} \leq C < \infty$ for all n and for some $0 < p < \infty$ then

$$\lim_{n \rightarrow \infty} \left(\|f_n\|_{L^p(\Omega)}^p - \|f_n - f\|_{L^p(\Omega)}^p \right) = \|f\|_{L^p(\Omega)}^p. \quad (V.14)$$

The point of the lemma is that f_n decouples in the limit as measured in the L^p norm into $f_n - f$ and f . Observe that even if we somehow know $f_n \rightarrow f$ a.e. in Ω , so that wild oscillations are excluded, we still cannot legitimately deduce strong convergence in $L^p(\Omega)$. The obstruction is that the mass of $|f_n - f|^p$ may somehow coalesce onto a set of zero Lebesgue measure. This is the problem of **concentration**.

Now we come back to

Theorem V.6. For all $t \in (0, \infty)$ fixed but arbitrary one has

$$\left\{ \frac{\rho_\epsilon(t) - 1}{\epsilon^2} \right\}_\epsilon \text{ is compact in } w - L^1(\Omega). \quad (V.15)$$

Proof: Because of lemma V.4, we only need to show that

$$\frac{\rho_\epsilon - 1}{\epsilon^2} \text{ is equiintegrable.}$$

We start with the generalized Young's inequality

$$(\sigma, \eta) \cdot (\rho, \mu) \leq \tilde{H}^*(\sigma, \eta) + \tilde{H}(\rho, \mu).$$

Take $\sigma = \frac{(\rho_\epsilon - 1)}{\alpha|\rho_\epsilon - 1|}$, $\eta = 0$, $\rho = \frac{\rho_\epsilon + 1}{\epsilon^2}$, and $|\mu|^2 = \frac{1}{\epsilon^4} \frac{\rho_\epsilon + 1}{\rho_\epsilon} |\mu_\epsilon|^2$ then the generalized Young's inequality becomes

$$\begin{aligned} \frac{\rho_\epsilon - 1}{\epsilon^2} &\leq \frac{(\rho_\epsilon - 1)(\rho_\epsilon + 1)}{\epsilon^2 |\rho_\epsilon - 1|} \leq \alpha \tilde{H}^*\left(\frac{\rho_\epsilon - 1}{\alpha|\rho_\epsilon - 1|}, 0\right) + \alpha \tilde{H}\left(\frac{\rho_\epsilon + 1}{\epsilon^2}, \mu\right) \\ &= \alpha \Psi^*\left(\frac{\rho_\epsilon - 1}{\alpha|\rho_\epsilon - 1|}\right) + \alpha \tilde{H}\left(\frac{\rho_\epsilon + 1}{\epsilon^2}, \mu\right). \end{aligned} \quad (V.16)$$

But for $1 < \gamma \leq 2$, i.e., $\gamma^* > 2$, Ψ^* has the superquadratic homogeneity for $\sigma > 0$; this means

$$\frac{d}{d\lambda} \left(\frac{1}{\lambda^2} \Psi^*(\lambda\sigma) \right) \geq 0, \quad (V.17)$$

for any $\sigma > 0$ and $0 \leq \lambda \leq 1$. In particular

$$\frac{1}{\lambda^2} \Psi^*(\lambda\sigma) \leq \Psi^*(\sigma),$$

so (V.16) becomes

$$\frac{\rho_\epsilon - 1}{\epsilon^2} \leq \frac{1}{\alpha} \Psi^* \left(\frac{\rho_\epsilon - 1}{|\rho_\epsilon - 1|} \right) + \alpha \tilde{H} \left(\frac{\rho_\epsilon + 1}{\epsilon^2}, \mu \right). \quad (V.18)$$

On the other hand $\tilde{H}$ is the relative entropy with respect to $(1, 0)$, from the explicit form (III.24) and the fact that the density is always nonnegative we obtain

$$\tilde{H} \left(\frac{\rho_\epsilon + 1}{\epsilon^2}, \mu \right) = \frac{1}{\epsilon^2} \frac{|\mu_\epsilon|^2}{2\rho_\epsilon} + \Psi \left(\frac{\rho_\epsilon + 1}{\epsilon^2} \right). \quad (V.19)$$

But for γ -law gas, $1 < \gamma \leq 2$, the structure of Ψ gives us

$$\Psi \left(\frac{\rho_\epsilon + 1}{\epsilon^2} \right) \leq \frac{1}{\epsilon^2} \Psi(\rho_\epsilon + 1),$$

therefore

$$\tilde{H} \left(\frac{\rho_\epsilon + 1}{\epsilon^2}, \mu \right) \leq \frac{1}{\epsilon^2} \left[\frac{|\mu_\epsilon|^2}{2\rho_\epsilon} + \Psi(\rho_\epsilon + 1) \right]. \quad (V.20)$$

The convexity of Ψ gives

$$\Psi(\rho_\epsilon + 1) \leq \Psi(\rho_\epsilon) + \Psi'(\rho_\epsilon + 1)$$

Therefore (V.18) becomes

$$\frac{\rho_\epsilon - 1}{\epsilon^2} \leq \frac{1}{\alpha} \Psi^* \left(\frac{\rho_\epsilon - 1}{|\rho_\epsilon - 1|} \right) + \frac{\alpha}{\epsilon^2} \tilde{H}(\rho_\epsilon, \mu) + \frac{\alpha}{\epsilon^2} \Psi'(\rho_\epsilon + 1). \quad (V.21)$$

Set

$$M_\alpha = \sup_{|w|=1} \Psi^*(w).$$

Since Ω is a periodic domain by Hölder inequality and the structure of the γ -law gas we derive the inequality

$$\frac{1}{\epsilon^2} \int_{\Omega} \Psi'(\rho_{\epsilon} + 1) dx \leq C_1 \left(\frac{1}{\epsilon^2} \int_{\Omega} \Psi(\rho_{\epsilon} + 1) dx \right)^{\frac{\gamma-1}{\gamma}},$$

where the constant C_1 depends on γ and Ω . From the structure of Ψ and $\Phi'(1) \geq 0$ we have

$$\begin{aligned} \Psi(\rho_{\epsilon} + 1) - \Psi(\rho_{\epsilon}) &= \Phi(\rho_{\epsilon} + 1) - \Phi(\rho_{\epsilon}) - \Phi'(1) \\ &\leq \Phi(\rho_{\epsilon} + 1) - \Phi(\rho_{\epsilon}). \end{aligned}$$

Now we apply lemma V.5 to Φ with $f = 1$, $f_n = \rho_{\epsilon} + 1$, then by Fatou's lemma we obtain

$$\lim_{\epsilon \rightarrow 0} \int_{\Omega} (\Phi(\rho_{\epsilon} + 1) - \Phi(\rho_{\epsilon})) dx = \int_{\Omega} \Phi(1) dx \leq \liminf_{\epsilon \rightarrow 0} \int_{\Omega} \Phi(\rho_{\epsilon}) dx.$$

Integrating (V.21) over any measurable $E \subset \Omega$ and using (V.19), (V.20), entropy bound, the convexity of Ψ , and $\rho_{\epsilon} \geq 0$ gives

$$\int_E \frac{\rho_{\epsilon} - 1}{\epsilon^2} dx \leq \frac{1}{\alpha} \int_E \Psi^* \left(\frac{\rho_{\epsilon} - 1}{|\rho_{\epsilon} - 1|} \right) dx + \alpha(C + C_1). \quad (V.22)$$

Now for all $\epsilon' > 0$, first choose $\alpha = \frac{\epsilon'}{2(C+C_1)}$ then pick $\delta > 0$ such that

$$|E| < \delta \quad \text{implies} \quad \frac{1}{\alpha} M_{\alpha} \delta < \frac{\epsilon'}{2}.$$

Then for $|E| < \delta$ we have

$$\int_E \frac{\rho_{\epsilon} - 1}{\epsilon^2} dx < \epsilon'.$$

That is

$$\frac{\rho_{\epsilon} - 1}{\epsilon^2} \quad \text{is equiintegrable.}$$

Therefore by Dunford-Pettis Theorem we prove that

$$\left\{ \frac{\rho_{\epsilon}(t) - 1}{\epsilon^2} \right\}_{\epsilon} \quad \text{is compact in} \quad w - L^1(\Omega).$$

Furthermore,

Theorem V.7. $(\rho_\epsilon, \frac{\mu_\epsilon}{\epsilon})$ is compact in $C([0, \infty), w - L^1(\Omega))$.

Proof: By Theorem IV.8 (Arzela-Ascoli Theorem) we only need to show that

$$(\rho_\epsilon, \frac{\mu_\epsilon}{\epsilon}) \text{ is weakly equicontinuous.}$$

From the weak formulation of (V.2a),

$$\epsilon \int_{\Omega} (\rho_\epsilon(t_2) - \rho_\epsilon(t_1)) \phi \, dx - \int_{t_1}^{t_2} \int_{\Omega} \nabla \phi \cdot \mu_\epsilon \, dx \, dt = 0,$$

for all $\phi \in C_c^\infty(\Omega; \mathbf{R})$. Then by Cauchy-Schwartz inequality we have

$$\begin{aligned} \epsilon \int_{\Omega} |\rho_\epsilon(t_2) - \rho_\epsilon(t_1)| \phi \, dx &\leq \|\nabla \phi\|_{L^\infty(\Omega)} \int_{t_1}^{t_2} \|\rho_\epsilon\|_{L^1(\Omega)}^{\frac{1}{2}} \left\| \frac{|\mu_\epsilon|^2}{\rho_\epsilon} \right\|_{L^1(\Omega)}^{\frac{1}{2}} dt \\ &\leq \epsilon C |t_2 - t_1|. \end{aligned} \quad (V.23)$$

The constant C is independent of ϵ . Thus $\{\rho_\epsilon\}$ is weakly equicontinuous. Similarly from (V.2b) we deduce

$$\begin{aligned} \epsilon \int_{\Omega} (\mu_\epsilon(t_2) - \mu_\epsilon(t_1)) \cdot \psi \, dx &= \int_{t_1}^{t_2} \int_{\Omega} \nabla \psi : \frac{\mu_\epsilon \otimes \mu_\epsilon}{\rho_\epsilon} \, dx \, dt \\ &+ \int_{t_1}^{t_2} \int_{\Omega} \nabla \cdot \psi P(\rho_\epsilon) \, dx \, dt - \epsilon \nu \int_{t_1}^{t_2} \int_{\Omega} \nabla \psi : \sum_{\epsilon} \, dx \, dt, \end{aligned} \quad (V.24)$$

for all $\psi \in C_c^\infty(\Omega; \mathbf{R}^D)$. But by (V.8) and the Cauchy-Schwartz inequality, we have

$$\left| \int_{t_1}^{t_2} \int_{\Omega} \nabla \psi : \frac{\mu_\epsilon \otimes \mu_\epsilon}{\rho_\epsilon} \, dx \, dt \right| \leq \|\nabla \psi\|_{L^\infty(\Omega)} \int_{t_1}^{t_2} \int_{\Omega} \frac{|\mu_\epsilon|^2}{\rho_\epsilon} \, dx \, dt \leq C_1 \epsilon^2 |t_2 - t_1|,$$

$$\left| \int_{t_1}^{t_2} \int_{\Omega} \nabla \cdot \psi P(\rho_\epsilon) \, dx \, dt \right| \leq \|\nabla \psi\|_{L^\infty(\Omega)} \int_{t_1}^{t_2} \int_{\Omega} P(\rho_\epsilon) \, dx \, dt \leq C_2 \epsilon^2 |t_2 - t_1|,$$

$$\begin{aligned} \left| \epsilon \nu \int_{t_1}^{t_2} \int_{\Omega} \nabla \psi : \sum_{\epsilon} \, dx \, dt \right| \\ \leq \epsilon \nu \|\nabla \psi\|_{L^\infty(\Omega)} |t_2 - t_1|^{\frac{1}{2}} \left(\int_{t_1}^{t_2} \int_{\Omega} \sum_{\epsilon} : \sum_{\epsilon} \, dx \, dt \right)^{\frac{1}{2}}. \end{aligned}$$

Thus

$$\left| \int_{\Omega} \left(\frac{\mu_{\epsilon}}{\epsilon}(t_2) - \frac{\mu_{\epsilon}}{\epsilon}(t_1) \right) \cdot \psi \, dx \right| \leq M_1 |t_2 - t_1| + M_2 |t_2 - t_1|^{\frac{1}{2}}.$$

That is, $\frac{\mu_{\epsilon}}{\epsilon}$ is also weakly equicontinuous. Hence the result.

Theorem V.6 gives the weak compactness of $\{\frac{\rho_{\epsilon}-1}{\epsilon^2}\}$ and also $\rho_{\epsilon} \rightarrow 1$ in $s - L^1(\Omega)$ and a.e. in Ω . In fact from the structure of Ψ we have more strong result.

Lemma V.8. For γ -law gas we have

$$\rho_{\epsilon} \rightarrow 1 \quad \text{in} \quad w - L^{\gamma}(\Omega). \quad (V.25)$$

Remark: Furthermore by interpolation theorem we know

$$\rho_{\epsilon} \rightarrow 1 \quad \text{in} \quad s - L^{\theta} \quad 1 \leq \theta < \gamma.$$

VI. Incompressible Limit for The Time Discretized Compressible Navier-Stokes Equations

The goal of this chapter is to prove the incompressible limit of the time-discretized compressible Navier-Stokes equations. For γ -law gas $1 < \gamma \leq 2$ and the space dimension $D \leq 4$, we show that $(\rho_\epsilon, \frac{\mu_\epsilon}{\epsilon})$ converge to $(1, \bar{\nu})$ of the incompressible Navier-Stokes equations. The restriction of the space dimension $D \leq 4$ is because we use the Sobolev imbedding theorem. The main mathematical tool that we shall use in our study is the **weak compactness**. The generalized Young's inequality provides the fundamental technique to show the **equiintegrability**.

The scaled time discretized compressible Navier-Stokes equations are

$$\epsilon \frac{\rho_\epsilon - \rho_\epsilon^{in}}{\Delta t} + \nabla \cdot \mu_\epsilon = 0, \quad (VI.1a)$$

$$\epsilon \frac{\mu_\epsilon - \mu_\epsilon^{in}}{\Delta t} + \nabla \cdot \left(\frac{\mu_\epsilon \otimes \mu_\epsilon}{\rho_\epsilon} \right) + \nabla P(\rho_\epsilon) = \epsilon \nabla \cdot (\nu \sum_\epsilon); \quad (VI.1b)$$

it is an implicit time discretization of the scaled compressible Navier-Stokes equations (I.7) – (I.8). Throughout this article, we shall always set the time step $\Delta t = 1$. The theory in chapter IV can be transposed to this new problem without any significant change. The form of the the entropy inequality is however somewhat different.

$$\int_\Omega \tilde{H}(U_\epsilon) dx + J(U_\epsilon^{in}, U_\epsilon) + \nu \int_\Omega \sum_\epsilon : \sum_\epsilon dx \leq \int_\Omega \tilde{H}(U_\epsilon^{in}) dx, \quad (VI.2)$$

where $J(U_\epsilon^{in}, U_\epsilon)$ is the **relative entropy** of U_ϵ^{in} with respect to U_ϵ which is given by

$$\begin{aligned} J(U_\epsilon^{in}, U_\epsilon) = \int_\Omega & \left[\frac{1}{2} \frac{|\mu_\epsilon^{in}|^2}{\rho_\epsilon^{in}} + \Psi(\rho_\epsilon^{in}) - \left(\frac{1}{2} \frac{|\mu_\epsilon|^2}{\rho_\epsilon} + \Psi(\rho_\epsilon) \right) \right. \\ & \left. - \left(-\frac{1}{2} \frac{|\mu_\epsilon|^2}{\rho_\epsilon} + \Psi'(\rho_\epsilon) \right) (\rho_\epsilon^{in} - \rho_\epsilon) - \left(\frac{\mu_\epsilon}{\rho_\epsilon} \right) \cdot (\mu_\epsilon^{in} - \mu_\epsilon) \right] dx, \end{aligned} \quad (VI.3)$$

where

$$\Psi(\rho) = \Phi(\rho) - \Phi(1) - \Phi'(1)(\rho - 1).$$

Estimate on $\mathbf{v}_\epsilon = \mu_\epsilon / \rho_\epsilon$

First from the dissipation inequality (V.2) we have the following uniform bound,

$$\nu \int_{\Omega} \left| \nabla \frac{\mu_\epsilon}{\rho_\epsilon} \right|^2 dx \leq C \epsilon^2,$$

i.e.,

$$\left\| \frac{\mathbf{v}_\epsilon}{\epsilon} \right\|_{H^1(\Omega)} \leq C. \quad (VI.4)$$

By theorem IV.5. the weakly compact theorem there exists $\bar{\mathbf{v}}$ in $H^1(\Omega)$ such that (after passing to subsequence),

$$\frac{\mathbf{v}_\epsilon}{\epsilon} \rightharpoonup \bar{\mathbf{v}} \quad \text{in } w - H^1(\Omega).$$

If $\mathbf{v}_\epsilon$ is of mean zero then by Rellich lemma,

$$H^1(\Omega) \hookrightarrow L^q(\Omega)$$

is a compact imbedding for $1 \leq q < \frac{2D}{D-2}$,. By passing to subsequence again we have

$$\frac{\mathbf{v}_\epsilon}{\epsilon} \rightarrow \bar{\mathbf{v}} \quad \text{in } s - L^q(\Omega) \quad \text{a.e. in } \Omega.$$

Thus we prove the zero mean part. However, even for the general $\mathbf{v}_\epsilon$ we still have

Lemma VI.1. For periodic domain $\Omega = T^D$ and $1 \leq 2\gamma^* < 2D/(D-2)$, we have

$$\left\{ \frac{\mathbf{v}_\epsilon}{\epsilon} \right\} \quad \text{is compact in } s - L^{2\gamma^*}(\Omega). \quad (V.5)$$

Proof: We decompose $\mathbf{v}_\epsilon$ into

$$\mathbf{v}_\epsilon = \bar{\mathbf{v}}_\epsilon + \tilde{\mathbf{v}}_\epsilon, \quad \int_{\Omega} \tilde{\mathbf{v}}_\epsilon dx = 0. \quad (V.6)$$

We have proven that

$$\left\{ \frac{\tilde{\mathbf{v}}_\epsilon}{\epsilon} \right\} \text{ is compact in } s - L^{2\gamma^*}(\Omega).$$

Integrating the equation of the conservation of momentum over Ω we get

$$\begin{aligned} 0 &= \int_{\Omega} \frac{\mu_\epsilon}{\epsilon} dx = \int_{\Omega} \rho_\epsilon \frac{\mathbf{v}_\epsilon}{\epsilon} dx = \int_{\Omega} (1 + \epsilon \tilde{\rho}_\epsilon) \left(\frac{\bar{\mathbf{v}}_\epsilon}{\epsilon} + \frac{\tilde{\mathbf{v}}_\epsilon}{\epsilon} \right) dx \\ &= \frac{\bar{\mathbf{v}}_\epsilon}{\epsilon} \int_{\Omega} (1 + \epsilon \tilde{\rho}_\epsilon) dx + \int_{\Omega} \tilde{\rho}_\epsilon \tilde{\mathbf{v}}_\epsilon dx = \frac{\bar{\mathbf{v}}_\epsilon}{\epsilon} \int_{\Omega} \rho_\epsilon dx + \int_{\Omega} \tilde{\rho}_\epsilon \tilde{\mathbf{v}}_\epsilon dx \\ &= C \frac{\bar{\mathbf{v}}_\epsilon}{\epsilon} + \int_{\Omega} \tilde{\rho}_\epsilon \tilde{\mathbf{v}}_\epsilon dx. \end{aligned}$$

But

$$\int_{\Omega} \left| \frac{\mu_\epsilon}{\epsilon} \right| dx \leq C < \infty.$$

Thus $\{\frac{\bar{\mathbf{v}}_\epsilon}{\epsilon}\}$ is uniformly bounded in $\mathbf{R}$. By Bolzano-Weierstrass theorem we deduce

$$\left\{ \frac{\bar{\mathbf{v}}_\epsilon}{\epsilon} \right\} \text{ is compact in } \mathbf{R}.$$

Therefore

$$\left\{ \frac{\mathbf{v}_\epsilon}{\epsilon} \right\} = \left\{ \frac{\bar{\mathbf{v}}_\epsilon}{\epsilon} + \frac{\tilde{\mathbf{v}}_\epsilon}{\epsilon} \right\} \text{ is compact in } s - L^{2\gamma^*}(\Omega).$$

or

$$\left\{ \frac{\mathbf{v}_\epsilon}{\epsilon} \otimes \frac{\mathbf{v}_\epsilon}{\epsilon} \right\} \text{ is compact in } s - L^{\gamma^*}(\Omega).$$

So the lemma is proved.

For γ -law gas, from the entropy inequality we have $\{\rho_\epsilon\}$ is uniformly bounded in $L^\gamma(\Omega)$. But $L^\gamma(\Omega)$ is reflexive for $\gamma > 1$. By theorem IV.5 (Eberlein-Shmul'yan theorem) we obtain $\{\rho_\epsilon\}$ is compact in $w - L^\gamma(\Omega)$.

Now $\frac{1}{\gamma} + \frac{1}{\gamma^*} = 1$ by (IV.12) we derive

Lemma VI.2.

$$\left\{ \rho_\epsilon \left(\frac{\mathbf{v}_\epsilon}{\epsilon} \otimes \frac{\mathbf{v}_\epsilon}{\epsilon} \right) \right\} \text{ is compact in } w - L^1(\Omega). \quad (VI.7)$$

From the entropy inequality we know that $\{\frac{\mu_\epsilon}{\epsilon}\}$ is compact in $w - L^1(\Omega)$, i.e., there exists $\bar{\mu} \in L^1(\Omega)$ such that

$$\frac{\mu_\epsilon}{\epsilon} \rightharpoonup \bar{\mu}. \quad (VI.8)$$

But $\rho_\epsilon \rightarrow 1$ in $s - L^\theta(\Omega)$ for all $1 \leq \theta < \gamma$. Therefore if we require

$$\frac{2D}{D+2} = (2^*)' < \gamma, \quad (VI.9)$$

then

$$\rho_\epsilon \rightarrow 1 \quad \text{in} \quad s - L^{\frac{2D}{D+2}}(\Omega).$$

Now we can conclude

Lemma VI.3. $\bar{\mu} = \bar{v}$ a.e. in Ω .

Proof: By triangle and Hölder inequalities

$$\begin{aligned} \left\| \frac{\mu_\epsilon}{\epsilon} - \bar{v} \right\|_{L^1(\Omega)} &= \left\| \frac{\mu_\epsilon}{\epsilon} - \frac{v_\epsilon}{\epsilon} + \frac{v_\epsilon}{\epsilon} - \bar{v} \right\|_{L^1(\Omega)} \\ &\leq \|\rho_\epsilon - 1\|_{L^{\frac{2D}{D+2}}} \left\| \frac{v_\epsilon}{\epsilon} \right\|_{L^{2^*}} + \left\| \frac{v_\epsilon}{\epsilon} - \bar{v} \right\|_{L^1(\Omega)} \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

By passing to subsequence

$$\frac{\mu_\epsilon}{\epsilon} \rightarrow \bar{v} \quad \text{a.e. in } \Omega.$$

By (VI.8) we obtain

$$\bar{\mu} = \bar{v} \quad \text{a.e. in } \Omega.$$

From the entropy inequality we already have

$$\left\{ \frac{1}{\rho_\epsilon} \left| \frac{\mu_\epsilon}{\epsilon} \right|^2 \right\} \quad \text{is uniformly bounded in } L^1(\Omega).$$

In order to show that

$$\left\{ \frac{1}{\rho_\epsilon} \left(\frac{\mu_\epsilon}{\epsilon} \otimes \frac{\mu_\epsilon}{\epsilon} \right) \right\} \quad \text{is compact in } w - L^1(\Omega), \quad (VI.10)$$

by Dunford-Pettis theorem we only need to prove the **equiintegrability**. To this end we will employ the Young's inequality (III.19) and the Rellich lemma. We start from the Young's inequality

$$(\sigma, \eta) \cdot (\rho, \mu) = \sigma \rho + \eta \cdot \mu \leq H^*(\sigma, \eta) + H(\rho, \mu),$$

$\forall \alpha > 0$, $\mu_\epsilon = (\mu_\epsilon^1, \mu_\epsilon^2, \dots, \mu_\epsilon^D)$, choose

$$\rho = \frac{\rho_\epsilon}{\epsilon} \quad \mu = \frac{\mu_\epsilon^i}{\epsilon} \mathbf{e}_i \quad \sigma = 0 \quad \eta = \frac{1}{\alpha \rho_\epsilon} \left(\frac{\mu_\epsilon^j}{\epsilon} \right) \mathbf{e}_i,$$

where $\mathbf{e}_i$, $i = 1, 2, \dots, D$ is the standard basis. Then the Young's inequality becomes

$$\begin{aligned} \frac{1}{\rho_\epsilon} \left(\frac{\mu_\epsilon^i \mu_\epsilon^j}{\epsilon^2} \right) &\leq \alpha H\left(\frac{\rho_\epsilon}{\epsilon}, \frac{\mu_\epsilon^i}{\epsilon} \mathbf{e}_i\right) + \alpha H^*\left(0, \frac{1}{\alpha \rho_\epsilon} \left(\frac{\mu_\epsilon^j}{\epsilon} \right) \mathbf{e}_i\right) \\ &= \alpha H\left(\frac{\rho_\epsilon}{\epsilon}, \frac{\mu_\epsilon^i}{\epsilon} \mathbf{e}_i\right) + \alpha \Psi^*\left(\frac{1}{2} \left| \frac{\mu_\epsilon^j}{\alpha \rho_\epsilon \epsilon} \mathbf{e}_i \right|^2\right) \\ &\leq \alpha H\left(\frac{\rho_\epsilon}{\epsilon}, \frac{\mu_\epsilon}{\epsilon}\right) + \alpha \Psi^*\left(\frac{1}{2} \left| \frac{\mu_\epsilon}{\alpha \rho_\epsilon \epsilon} \right|^2\right) \\ &= \alpha H\left(\frac{\rho_\epsilon}{\epsilon}, \frac{\mu_\epsilon}{\epsilon}\right) + C \left| \frac{\mathbf{v}_\epsilon}{\epsilon} \right|^{2\gamma^*} \end{aligned}$$

where C is a constant depends on α and γ^* , i.e., it depends on α and also the space dimension D . If we require

$$1 \leq 2\gamma^* < \frac{2D}{D-2} \quad \text{i.e., } \gamma > \frac{D}{2} \quad (\text{VI.11})$$

then applying the Rellich lemma we obtain

$$\frac{\mathbf{v}_\epsilon}{\epsilon} = \frac{\mu_\epsilon}{\epsilon \rho_\epsilon} \rightarrow \bar{\mathbf{v}} \quad \text{in } s - L^{2\gamma^*}(\Omega).$$

By Vitali convergent theorem we know that

$$\left| \frac{\mathbf{v}_\epsilon}{\epsilon} \right|^{2\gamma^*} = \left| \frac{\mu_\epsilon}{\epsilon \rho_\epsilon} \right|^{2\gamma^*} \quad \text{is equiintegrable.} \quad (\text{VI.12})$$

Hence $\forall \epsilon' > 0$ we can choose E a measurable subset of Ω with $|E| < \delta$ such that

$$C \int_E \left| \frac{\mathbf{v}_\epsilon}{\epsilon} \right|^{2\gamma^*} dx < \frac{\epsilon'}{2}.$$

Also α is arbitrary, so let $\alpha \rightarrow 0$ then for all $\epsilon' > 0$ we have

$$\alpha \int_E H\left(\frac{\rho_\epsilon}{\epsilon}, \frac{\mu_\epsilon}{\epsilon}\right) dx < \alpha \frac{1}{\epsilon^2} \int_\Omega H(\rho_\epsilon, \mu_\epsilon) dx < \alpha C < \epsilon'.$$

Therefore

$$\int_E \frac{1}{\rho_\epsilon} \left(\frac{\mu_\epsilon^i \mu_\epsilon^j}{\epsilon^2} \right) < \epsilon'$$

i.e., $\left\{ \frac{1}{\rho_\epsilon} \left(\frac{\mu_\epsilon^i \mu_\epsilon^j}{\epsilon^2} \right) \right\}$ is equiintegrable. Hence by Dunford-Pettis theorem we can conclude

$$\left\{ \frac{1}{\rho_\epsilon} \left(\frac{\mu_\epsilon^i \mu_\epsilon^j}{\epsilon^2} \right) \right\} \text{ is ccompact in } w - L^1(\Omega).$$

Thus we have the following result.

Theorem VI.4. For $\gamma > \frac{D}{2}$, the following is true

$$\left\{ \frac{1}{\rho_\epsilon} \left(\frac{\mu_\epsilon}{\epsilon} \otimes \frac{\mu_\epsilon}{\epsilon} \right) \right\} \text{ is compact in } w - L^1(\Omega). \quad (VI.13)$$

From the equation of continuity (VI.1a)

$$\epsilon \left(\rho_\epsilon - \rho_\epsilon^{in} \right) + \nabla \cdot \mu_\epsilon = 0,$$

and the entropy inequality,

$$\left\| \frac{\rho_\epsilon - 1}{\epsilon^2} \right\|_{L^1(\Omega)} \leq C < \infty.$$

Thus $\forall \phi \in C_c^\infty(\Omega; \mathbf{R})$ we have

$$\begin{aligned}
|\langle \nabla \cdot \bar{\mathbf{v}}, \phi \rangle| &= |\langle \bar{\mathbf{v}}, \nabla \phi \rangle| \\
&\leq |\langle \bar{\mathbf{v}} - \frac{\mu_\epsilon}{\epsilon}, \nabla \phi \rangle| + |\langle \frac{\mu_\epsilon}{\epsilon}, \nabla \phi \rangle| \\
&\leq \left\| \bar{\mathbf{v}} - \frac{\mu_\epsilon}{\epsilon} \right\|_{L^1(\Omega)} \sup_x |\nabla \phi| + \left| \langle \frac{\mu_\epsilon}{\epsilon}, \nabla \phi \rangle \right| \\
&\leq \left\| \bar{\mathbf{v}} - \frac{\mu_\epsilon}{\epsilon} \right\|_{L^1(\Omega)} \sup_x |\nabla \phi| + \epsilon \left\| \frac{\rho_\epsilon - \rho_\epsilon^{in}}{\epsilon^2} \right\|_{L^1(\Omega)} \sup_x |\nabla \phi| \\
&\rightarrow 0 \quad \text{as } \epsilon \rightarrow 0,
\end{aligned} \tag{VI.14}$$

where

$$\langle f, g \rangle \equiv \int_{\Omega} f g \, dx.$$

Therefore we prove

Lemma VI.5. $\bar{\mathbf{v}}$ is weakly divergence free, i.e., $\nabla \cdot \bar{\mathbf{v}} = 0$ in the sense of distribution.

The weak formulation of (VI.1b) is given by

$$\begin{aligned}
\epsilon \int_{\Omega} (\mu_\epsilon - \mu_\epsilon^{in}) \cdot \psi \, dx &= \int_{\Omega} \nabla \psi : \frac{\mu_\epsilon \otimes \mu_\epsilon}{\rho_\epsilon} \, dx \\
&+ \int_{\Omega} \nabla \cdot \psi P(\rho_\epsilon) \, dx - \epsilon \nu \int_{\Omega} \nabla \psi : \sum_\epsilon \, dx,
\end{aligned} \tag{VI.15}$$

for all $\psi \in C_c^\infty(\Omega; \mathbf{R}^D)$.

If we choose the test function ψ in the space of divergence free, i.e.,

$$V \equiv \{ \psi \in C_c^\infty(\Omega; \mathbf{R}^D), \quad \nabla \cdot \psi = 0 \}$$

then the third term of (VI.15) will eliminate. In fact this idea is equivalent to apply the orthogonal projection of L^p space onto the subspace that characterized by divergence free condition. Since

$$\frac{\mu_\epsilon}{\epsilon} \rightharpoonup \bar{\mathbf{v}} \quad \text{in } w - L^1(\Omega). \tag{VI.16}$$

It is clear that

$$\frac{1}{\epsilon} \int_{\Omega} \mu_{\epsilon} \cdot \psi \, dx \longrightarrow \int_{\Omega} \bar{\mathbf{v}} \cdot \psi \, dx, \quad (VI.17)$$

$$\frac{1}{\epsilon} \int_{\Omega} \mu_{\epsilon}^{in} \cdot \psi \, dx \longrightarrow \int_{\Omega} \bar{\mathbf{v}}^{in} \cdot \psi \, dx, \quad (VI.18)$$

$$-\frac{\nu}{\epsilon} \int_{\Omega} \nabla \psi : \sum_{\epsilon} dx \longrightarrow \nu \int_{\Omega} \Delta \psi \cdot \bar{\mathbf{v}} \, dx. \quad (V.19)$$

Since

$$\begin{aligned} \rho_{\epsilon} &\rightarrow 1 \quad \text{a.e. in } \Omega, \\ \frac{\mu_{\epsilon}}{\epsilon} &\rightarrow \bar{\mathbf{v}} \quad \text{a.e. in } \Omega. \end{aligned}$$

So we have

$$\frac{1}{\rho_{\epsilon}} \left(\frac{\mu_{\epsilon}}{\epsilon} \otimes \frac{\mu_{\epsilon}}{\epsilon} \right) \rightarrow \bar{\mathbf{v}} \otimes \bar{\mathbf{v}} \quad \text{a.e. in } \Omega. \quad (VI.20)$$

But $\left\{ \frac{1}{\rho_{\epsilon}} \left(\frac{\mu_{\epsilon}}{\epsilon} \otimes \frac{\mu_{\epsilon}}{\epsilon} \right) \right\}$ is compact in $w - L^1(\Omega)$. Thus

$$\frac{1}{\rho_{\epsilon}} \left(\frac{\mu_{\epsilon}}{\epsilon} \otimes \frac{\mu_{\epsilon}}{\epsilon} \right) \rightarrow \bar{\mathbf{v}} \otimes \bar{\mathbf{v}} \quad \text{in } w - L^1(\Omega), \quad (VI.21)$$

therefore we derive

$$\int_{\Omega} \nabla \psi : \frac{1}{\rho_{\epsilon}} \left(\frac{\mu_{\epsilon}}{\epsilon} \otimes \frac{\mu_{\epsilon}}{\epsilon} \right) dx \rightarrow \int_{\Omega} \nabla \psi : \bar{\mathbf{v}} \otimes \bar{\mathbf{v}} \, dx. \quad (VI.22)$$

Take all the convergence results into account we obtain

$$\int_{\Omega} \psi \cdot \bar{\mathbf{v}} \, dx - \int_{\Omega} \psi \cdot \bar{\mathbf{v}}^{in} \, dx - \int_{\Omega} (\nabla \psi)^T : (\bar{\mathbf{v}} \otimes \bar{\mathbf{v}}) \, dx = \nu \int_{\Omega} \bar{\mathbf{v}} \cdot \Delta \psi \, dx. \quad (VI.23)$$

Note that the vector field ψ is of divergence free. Hence by (VI.23) we have

$$\begin{aligned}\mathcal{P}(\bar{\mathbf{v}} - \bar{\mathbf{v}}^{in} + \bar{\mathbf{v}} \cdot \nabla \bar{\mathbf{v}} - \nu \Delta \bar{\mathbf{v}}) &= 0 \quad \text{in } \Omega, \\ \nabla \cdot \bar{\mathbf{v}} &= 0 \quad \text{in } \Omega,\end{aligned}\tag{VI.24}$$

where $\mathcal{P}$ is the projection (see Appendix B). Next we discuss the convergence of the pressure term. From Theorem B.1 (Helmholtz Decomposition) it follows that there exists Π such that

$$\bar{\mathbf{v}} - \bar{\mathbf{v}}^{in} + (\bar{\mathbf{v}} \cdot \nabla) \bar{\mathbf{v}} + \nabla \Pi = \nu \Delta \bar{\mathbf{v}},\tag{VI.25}$$

and

$$\nabla \cdot \bar{\mathbf{v}} = \nabla \cdot \bar{\mathbf{v}}^{in} = 0.\tag{VI.26}$$

If we apply the divergence operator to (VI.25) and employing (VI.26) then Π satisfies the Poisson equation. (see Appendix B)

$$\Delta \Pi + \text{tr}(\nabla \bar{\mathbf{v}} : \nabla \bar{\mathbf{v}}) = 0.\tag{VI.27}$$

The regularity theory of the elliptic partial differential equations tell us that we will “gain” two derivatives for Π . However Π is not uniquely determined by the incompressible equations. Now

$$\begin{aligned}\frac{1}{\epsilon}(\mu_\epsilon - \mu_\epsilon^{in}) + \frac{1}{\epsilon^2} \nabla \cdot \left(\frac{\mu_\epsilon \otimes \mu_\epsilon}{\rho_\epsilon} \right) - \frac{1}{\epsilon} \nabla \cdot (\nu \sum_\epsilon) \\ \xrightarrow{w} \bar{\mathbf{v}} - \bar{\mathbf{v}}^{in} + (\bar{\mathbf{v}} \cdot \nabla) \bar{\mathbf{v}} - \nu \Delta \bar{\mathbf{v}} \quad \text{in } w - L^1(\Omega).\end{aligned}\tag{VI.28}$$

From (VI.28), we deduce immediately that

$$\frac{1}{\epsilon^2} \nabla P(\rho_\epsilon) \xrightarrow{w} \nabla \Pi \quad \text{in } w - L^1(\Omega).\tag{VI.29}$$

From the convexity of $P(\rho_\epsilon)$ we have

$$\bar{C}^2(\rho_\epsilon - 1) \leq P(\rho_\epsilon) - P(1) \leq (\bar{C}^2 + \delta(\epsilon))(\rho_\epsilon - 1),\tag{VI.30}$$

where $\bar{C}^2 = \frac{dp}{d\rho}(1)$, and $\delta(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. Then it follows immediately by Theorem V.4 that

$$\left\{ \frac{P(\rho_\epsilon) - P(1)}{\epsilon^2} \right\} \quad \text{is compact in } w - L^1(\Omega).\tag{VI.31}$$

After modification of the constant, for the pressure fluctuation we have

$$\tilde{P}(\rho_\epsilon) = \frac{P(\rho_\epsilon) - P(1)}{\epsilon^2} \rightharpoonup \Pi \quad \text{in } w - L^1(\Omega). \quad (VI.32)$$

or

$$\bar{C}^2 \frac{\rho_\epsilon - 1}{\epsilon^2} \rightharpoonup \Pi \quad \text{in } w - L^1(\Omega). \quad (VI.33)$$

Proof of the Leray energy inequality

We start from the entropy inequality

$$\int_{\Omega} \tilde{H}(U_\epsilon) dx + J(U_\epsilon^{in}, U_\epsilon) + \nu \int_{\Omega} \sum_{\epsilon} : \sum_{\epsilon} dx \leq \int_{\Omega} \tilde{H}(U_\epsilon^{in}) dx. \quad (VI.2)$$

The convexity for the integrands of all but the middle term on the left side of inequality (VI.2) is as it was for the continuous time problem. The middle term is just the entropy of U_ϵ^{in} ; as such, its integrand is easily to understood to be a nonnegative convex function of U_ϵ^{in} . It is one of the beautiful properties of the classical entropy that this integrand is a joint convex function of both its argument, U_ϵ^{in} and U_ϵ . The convexity of J gives the following inequality

$$\begin{aligned} J(U^{in}, U) + \partial_x J(U^{in}, U) (U_\epsilon^{in} - U^{in}) + \partial_y J(U^{in}, U) (U_\epsilon - U) \\ \leq J(U_\epsilon^{in}, U). \end{aligned} \quad (VI.34)$$

Taking the limits we have

$$\frac{1}{2} \int_{\Omega} |\bar{\mathbf{v}} - \bar{\mathbf{v}}^{in}|^2 dx \leq \liminf_{\epsilon \rightarrow 0} \frac{1}{\epsilon^2} J(U_\epsilon^{in}, U_\epsilon). \quad (VI.35)$$

Also by lower semicontinuity

$$\nu \int_{\Omega} |\nabla \bar{\mathbf{v}}|^2 dx \leq \liminf_{\epsilon \rightarrow 0} \frac{\nu}{\epsilon^2} \int_{\Omega} \sum_{\epsilon} : \sum_{\epsilon} dx, \quad (VI.36)$$

$$\frac{1}{2} \int_{\Omega} |\bar{\mathbf{v}}|^2 dx \leq \liminf_{\epsilon \rightarrow 0} \frac{1}{\epsilon^2} \int_{\Omega} \tilde{H}(U_\epsilon). \quad (VI.37)$$

But

$$\frac{1}{\epsilon^2} \int_{\Omega} \tilde{H}(U_{\epsilon}^{in}) dx \rightarrow \frac{1}{2} \int_{\Omega} |\bar{\mathbf{v}}^{in}|^2 dx \quad (VI.38)$$

Combine (VI.34) – (VI.38), we deduce the Leray energy inequality

$$\int_{\Omega} |\bar{\mathbf{v}}|^2 dx + \nu \int_{\Omega} |\nabla \bar{\mathbf{v}}|^2 dx \leq \int_{\Omega} \bar{\mathbf{v}} \cdot \bar{\mathbf{v}}^{in} dx. \quad (VI.39)$$

Appendix A

Global Existence and Uniqueness for The Parabolic Regularization Systems of The Compressible Navier-Stokes Equations

The purpose of this appendix is to deduce the existence theorem through a Hilbert space approach, making use of the theory of fractional powers of operators and the theory of semi-groups of operator. We shall reduce the initial value problem of (P.R.) to an abstract initial value problem for an operator differential equation in a Hilbert space.

The present section is concerned with the initial value problem for the parabolic regularization system of (III.11) - (III.13).

$$\partial_t \rho_\epsilon + \nabla \cdot \mu_\epsilon = \epsilon \Delta \rho_\epsilon, \quad (P.R.a)$$

$$\partial_t \mu_\epsilon + \nabla \cdot \left(\frac{\mu_\epsilon \otimes \mu_\epsilon}{\rho_\epsilon + \epsilon} \right) + \nabla P(\rho_\epsilon) = \nabla \cdot \left(\nu \sum_\epsilon \right) - \epsilon \Delta^{2s} \mu_\epsilon, \quad (P.R.b)$$

$$\sum_\epsilon \equiv \left(\nabla \frac{\mu_\epsilon}{\rho_\epsilon + \epsilon} \right) + \left(\nabla \frac{\mu_\epsilon}{\rho_\epsilon + \epsilon} \right)^T - \frac{2}{D} \left(\nabla \cdot \frac{\mu_\epsilon}{\rho_\epsilon + \epsilon} \right) I.$$

The Laplacian term, $\epsilon \Delta \rho_\epsilon$, is introduced due to the physical consideration. For we only interested in the situation when the density is nonnegative. Since the solution of the heat equation is positivity preserving. In this way, formally we make (P.R.a) to be a nonhomogeneous heat equation of ρ_ϵ . The higher Laplacian term, $-\epsilon \Delta^{2s}$ is standard. Also the term, $(\rho_\epsilon + \epsilon)$, is introduced to overcome the case when vacuum occurs.

Before we go further, let us recall some important estimates that we will need very soon. For strongly elliptic differential operator the following fundamental a-priori estimates have been established.(see [GiT] and [Paz])

Theorem A.1(Basic Estimate for Strongly Elliptic Operator).

Let L be a strongly elliptic operator of order $2m$ on a bounded domain Ω with smooth boundary $\partial\Omega$ in $\mathbf{R}^D$ and let $1 < p < \infty$. There exists a constant C such that

$$\|u\|_{W^{2m,p}(\Omega)} \leq C(\|Lu\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}), \quad (A.1)$$

for all $u \in W^{2m,p}(\Omega) \cap W_0^{m,p}(\Omega)$.

Theorem A.2(Gagliardo-Nirenberg's Inequality).

Let Ω be a bounded domain in $\mathbf{R}^D$ with boundary $\partial\Omega$ of class C^m and let $u \in W^{m,r}(\Omega) \cap L^q(\Omega)$, where $1 \leq r, q \leq \infty$. For any integer j , $0 \leq j \leq m$ and $j/m \leq \theta \leq 1$ we have

$$\|\nabla^j u\|_{L^p(\Omega)} \leq C \|u\|_{W^{m,r}(\Omega)}^\theta \|u\|_{L^q(\Omega)}^{1-\theta}, \quad (A.2)$$

provided that

$$\frac{1}{p} - \frac{j}{D} = \theta \left(\frac{1}{r} - \frac{m}{D} \right) + (1 - \theta) \frac{1}{q}, \quad (A.3)$$

and $m - j - \frac{D}{r}$ is not a nonnegative integer. If $m - j - \frac{D}{r}$ is a nonnegative integer, then the inequality (A.2) holds with $\theta = j/m$.

Theorem A.3. Let A be the generator of a bounded analytic semigroup of angle θ , $\theta > 0$, defined in X (X is a Banach space). Then for each $m > 0$, and each $\phi \in X$, $\exp(-tA) \cdot \phi \in D(A^m)$ and

$$\|A^m \exp(-tA) \cdot \phi\| \leq C \|\phi\|/|t|^m, \quad (A.4)$$

for all $t > 0$. C depends on A and m but not on ϕ .

In particular when $A = \Delta$, Δ^{2s} we have the result as (A.4).

Theorem A.4. Suppose $\exp(tA)$ is an analytic semigroup on each L^p space. Also, suppose there exists a positive integer m such that for each p , $D_p(A)$, with its graph

norm is continuously embedded in $W^{m,p}(\Omega)$. Assume Ω is sufficiently regular so that the Gagliardo-Nirenberg's inequality (A.3) holds with this m . Then

$$\exp(tA) : L^p \mapsto L^q \quad (A.5)$$

is a bounded map, whenever $1 < p < \infty$ and $t > 0$. Furthermore for any $T > 0$, there is a constant C (depending on p and q) such that

$$\|\exp(tA)\phi\|_{L^q(\Omega)} \leq Ct^{-\frac{p}{mr}} \|\phi\|_{L^p(\Omega)}, \quad (A.6)$$

where $t \in (0, T]$ and $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$.

Proof : Using in turn, Theorem A.1, Theorem A.2, and Theorem A.3 we obtain for all $t > 0$

$$\begin{aligned} \|\exp(tA)\|_{L^q(\Omega)} &\leq C \|\exp(tA)\phi\|_{W^{2,p}(\Omega)}^\theta \|\exp(tA)\phi\|_{L^p(\Omega)}^{1-\theta} \\ &\leq C \left(t^{-1} \|\phi\|_{L^p(\Omega)} + \|\exp(tA)\phi\|_{L^p(\Omega)} \right)^\theta \|\exp(tA)\phi\|_{L^p(\Omega)}^{1-\theta} \\ &\leq C \left(t^{-1} \|\phi\|_{L^p(\Omega)} + \|\phi\|_{L^p(\Omega)} \right)^\theta \|\phi\|_{L^p(\Omega)}^{1-\theta} \\ &\leq C(1 + t^{-1})^\theta \|\phi\|_{L^p(\Omega)}. \end{aligned}$$

Hence the result.

If $-A$ is a positive-definite, self-adjoint operator then the operator $\exp(tA)$, $t \geq 0$, and $(-A)^\alpha$ ($0 < \alpha \leq 1$) can be defined by the usual procedure. In fact we have

Proposition A.5. $\exp(tA)$ and $A^\alpha \exp(tA)$, $t > 0$, $0 < \alpha \leq 1$ are bounded linear operators. More precisely

$$\|\exp(tA)\| \leq 1 \quad \text{and} \quad \|(-A)^\alpha \exp(tA)\| \leq t^{-\alpha}. \quad (A.7)$$

Let $U_\epsilon = (\rho_\epsilon, \mu_\epsilon)$, then the equations (P.R.) become

$$\frac{dU_\epsilon}{dt} - \epsilon \begin{bmatrix} \Delta & 0 \\ 0 & -\Delta^{2s} \end{bmatrix} U_\epsilon + \left[\nabla \cdot \left(\frac{\mu_\epsilon \otimes \mu_\epsilon}{\rho_\epsilon + \epsilon} \right) + \nabla P_\epsilon - \nabla \cdot (\nu \Sigma_\epsilon) \right] = 0. \quad (A.8)$$

For convenience, let

$$A = \begin{bmatrix} \Delta & 0 \\ 0 & -\Delta^{2s} \end{bmatrix}, \quad (A.9)$$

$$N_\epsilon = \nabla \cdot \left(\frac{\mu_\epsilon \otimes \mu_\epsilon}{\rho_\epsilon + \epsilon} \right) + \nabla P_\epsilon - \nabla \cdot (\nu \sum_\epsilon), \quad (A.10)$$

and

$$F_\epsilon = \begin{bmatrix} \nabla \cdot \mu_\epsilon \\ N_\epsilon \end{bmatrix}. \quad (A.11)$$

We can rewrite (A.8) as

$$\frac{dU_\epsilon}{dt} - \epsilon A U_\epsilon + F_\epsilon = 0, \quad (A.12)$$

with the initial data

$$U_\epsilon(x, 0) = U_\epsilon^{in}(x) = \left(\rho_\epsilon^{in}(x), \mu_\epsilon^{in}(x) \right)^T.$$

Let $X = L^2(\Omega) \times L^2(\Omega)$ then the domain of A is

$$D(A) = [H^2(\Omega) \cap H_0^1(\Omega)] \times [H^{4s}(\Omega) \cap H_0^{2s}(\Omega)]. \quad (A.13)$$

Since $C_c^\infty(\Omega) \times C_c^\infty(\Omega)$ is dense in $D(A)$. Therefore A is densely defined in X .

Let X be a Banach space and let X^* be its dual. We denote the value of $x^* \in X^*$ at $x \in X$ by $\langle x^*, x \rangle$ or $\langle x, x^* \rangle$. For every $x \in X$ we define the duality set $F(x) \subset X^*$ by

$$F(x) = \{x^* : x^* \in X^* \text{ and } \langle x^*, x \rangle = \|x\|^2 = \|x^*\|^2\}.$$

From the Hahn-Banach theorem it follows that $F(x) \neq \emptyset$ for every $x \in X$.

Definition. A linear operator A is **dissipative** if for every $x \in D(A)$ there is a $x^* \in F(x)$ such that $\operatorname{Re} \langle Ax, x^* \rangle \leq 0$.

A useful characterization of dissipative operators is given next.

Theorem A.6. A linear operator A is dissipative if and only if

$$\|(\lambda I - A)x\| \geq \lambda \|x\| \quad \text{for all } x \in D(A) \text{ and } \lambda > 0.$$

It is clear that A is symmetric, furthermore we have

Lemma A.7. Let A be defined as (A.9), then A is dissipative hence ϵA .

Proof : For all $\lambda > 0$ we have

$$\begin{aligned} \|(I - \lambda A)U\|_X &= \|[\rho - \lambda \Delta \rho, \mu + \lambda \Delta^{2s} \mu]^T\|_X \\ &= \|(1 - \lambda \Delta)\rho\|_{L^2} + \|(1 + \lambda \Delta^{2s})\mu\|_{L^2}. \end{aligned}$$

But $1 - \lambda \Delta$ and $1 + \lambda \Delta^{2s}$ are positive definite. Hence by Poincare inequality and Garding inequality, we conclude A is dissipative. According to the previous results we obtain

Theorem A.8. A satisfies the same hypothesis as Lemma A.6 then $-A$ (or $-\epsilon A$) is the infinitesimal generator of an analytic semigroup of operator on $L^2(\Omega) \times L^2(\Omega)$.

Since $-A$ generates an analytic semigroup, we can formally rewrite the system (A.12) as

$$U_\epsilon(t) = \exp(t\epsilon A) U_\epsilon^{in} + \int_0^t \exp((t - \tau)\epsilon A) F_\epsilon d\tau, \quad 0 \leq t \leq T, \quad (A.14)$$

i.e.,

$$\rho_\epsilon(t) = \exp(t\epsilon \Delta) \rho_\epsilon^{in} + \int_0^t \exp(\epsilon(t - \tau)\Delta) (\nabla \cdot \mu_\epsilon) d\tau, \quad (A.15)$$

$$\mu_\epsilon(t) = \exp(-t\epsilon \Delta^{2s}) \mu_\epsilon^{in} + \int_0^t \exp(-\epsilon(t - \tau)\Delta^{2s}) N_\epsilon d\tau. \quad (A.16)$$

We shall refer to a continuous solution of (A.14) as a **mild solution** of (A.8). More precisely let $U_\epsilon^{in} \in X$ and $F_\epsilon \in L^1([0, T], X)$. The function $U_\epsilon \in C([0, T], X) \equiv Y$ given by (A.14) is the mild solution of the initial value problem (A.12) on $[0, T]$.

Due to integration by part, we may replace (A.15) - (A.16) by

$$\rho_\epsilon(t) = \exp(t\epsilon\Delta)\rho_\epsilon^{in} - \int_0^t \nabla \exp(\epsilon(t-\tau)\Delta)\mu_\epsilon d\tau, \quad (A.17)$$

$$\begin{aligned} \mu_\epsilon(t) = & \exp(-t\epsilon\Delta^{2s})\mu_\epsilon^{in} \\ & - \int_0^t \nabla \exp(-\epsilon(t-\tau)\Delta^{2s}) \left(\frac{\mu_\epsilon \otimes \mu_\epsilon}{\rho_\epsilon + \epsilon} - P_\epsilon - \nu \sum_\epsilon \right) d\tau. \end{aligned} \quad (A.18)$$

Denote the projection of X onto its first and second coordinates by Π_1 and Π_2 . Let

$$\Phi \equiv \Pi_1 \Phi + \Pi_2 \Phi, \quad (A.19)$$

where

$$\Pi_1 \Phi U_\epsilon(t) \equiv \exp(t\epsilon\Delta)\rho_\epsilon^{in} - \int_0^t \nabla \exp(\epsilon(t-\tau)\Delta)\mu_\epsilon d\tau, \quad (A.20)$$

$$\begin{aligned} \Pi_2 \Phi U_\epsilon(t) \equiv & \exp(-t\epsilon\Delta^{2s})\mu_\epsilon^{in} \\ & - \int_0^t \nabla \exp(-\epsilon(t-\tau)\Delta^{2s}) \left(\frac{\mu_\epsilon \otimes \mu_\epsilon}{\rho_\epsilon + \epsilon} - P_\epsilon - \nu \sum_\epsilon \right) d\tau. \end{aligned} \quad (A.21)$$

Theorem A.9. For Φ defined by (A.19), then it is a contraction in $Y \equiv C([0, T], X)$.

In order to show that Φ is a contraction, we need to estimate

$$\|\Phi U_\epsilon^1 - \Phi U_\epsilon^2\|_Y = \sup_{0 \leq t \leq T} \|\Phi U_\epsilon^1 - \Phi U_\epsilon^2\|_X. \quad (A.22)$$

We treat the first and second coordinates separately.

(i) Estimate of $\|\Pi_1 \Phi U_\epsilon^1 - \Pi_1 \Phi U_\epsilon^2\|_{L^2(\Omega)}$

Before estimating, we need some results about the analytic semigroup $e\Delta$.

As a corollary of Theorem A.4 we have

Proposition A.10 Let $1 \leq p < q \leq \infty$ then for any $t > 0$,

$$\exp(\epsilon t \Delta) : L^p(\Omega) \mapsto L^q(\Omega) \quad (A.23)$$

is a bounded map. Moreover, for each $T > 0$ there is a constant C such that

$$\|\exp(\epsilon t \Delta) f\|_{L^q(\Omega)} = \|G_t * f\|_{L^q(\Omega)} \leq C t^{-\frac{D}{2r}} \|f\|_{L^p(\Omega)}, \quad (A.24)$$

for all $f \in L^p(\Omega)$, $t \in (0, T]$ and $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$. Where

$$G_t = (4\pi\epsilon t)^{-\frac{D}{2}} \exp\left(-\frac{|x|^2}{4\epsilon t}\right) \quad (A.25)$$

is the Gaussian kernel of

$$\partial_t u - \epsilon \Delta u = 0.$$

Remark : By the maximum principle for the heat equation, it follows that

$$\exp(\epsilon t \Delta) \phi \leq G_t * \phi \quad (A.26)$$

on Ω , for all $\phi \geq 0$. That is, $\exp(\epsilon t \Delta)$ is a positivity preserving, contraction analytic C_0 semigroup on $L^p(\Omega)$. Due to the Gaussian kernel we know

$$\exp(\epsilon t \Delta) V = G_t * V.$$

Integrating by part, we deduce

$$-\nabla \exp(\epsilon t \Delta) V = K_t * V, \quad (A.27)$$

where

$$K_t = (4\pi\epsilon t)^{-\frac{D}{2}} \frac{x}{2\epsilon t} \exp\left(-\frac{|x|^2}{4\epsilon t}\right).$$

Now we are in a position to estimate.

$$\begin{aligned} \|\Pi_1 \Phi U_\epsilon^1(t) - \Pi_1 \Phi U_\epsilon^2(t)\|_{L^2(\Omega)} &= \left\| \int_0^t K_{t-\tau} * (\mu_\epsilon^1 - \mu_\epsilon^2) d\tau \right\|_{L^2(\Omega)} \\ &\leq \int_0^t \|K_{t-\tau} * (\mu_\epsilon^1 - \mu_\epsilon^2)\|_{L^2(\Omega)} d\tau \\ &\leq \int_0^t \|K_{t-\tau}\|_{L^1(\Omega)} \|\mu_\epsilon^1 - \mu_\epsilon^2\|_{L^2(\Omega)} d\tau. \end{aligned} \quad (A.28)$$

But

$$\|K_t\|_{L^1(\Omega)} = C/t^{\frac{D-1}{2}},$$

hence

$$\int_0^t \|K_t\|_{L^1(\Omega)} d\tau = C/t^{\frac{D-3}{2}}.$$

Thus we have

$$\|\Pi_1 \Phi U_\epsilon^1 - \Pi_1 \Phi U_\epsilon^2\|_Y \leq C \|U_\epsilon^1 - U_\epsilon^2\|_Y. \quad (A.29)$$

(ii) Estimate of $\|\Pi_2 \Phi U_\epsilon^1 - \Pi_2 \Phi U_\epsilon^2\|_Y$

Let W_t be the fundamental solution of

$$\partial_t \mu_\epsilon + \epsilon \Delta^{2s} \mu_\epsilon = 0$$

then W_t has the following properties.

Lemma A.11. (see [Fri1])

$$(i) \|D_x^m W_t\| \leq \frac{C_1}{(t-\tau)^{\frac{D+|m|}{4s}}} \exp\left(-C_2 \left(\frac{|x-\xi|^{4s}}{t-\tau}\right)^{\frac{1}{4s-1}}\right),$$

where C_1 and C_2 are positive constants and $0 \leq m < 4s$.

$$(ii) \|D_x^{m+a} D_\xi^b W_t\| \leq \frac{C_1}{(t-\tau)^{\frac{D+|m|+|a|+|b|}{4s}}} \exp\left(-C_2 \left(\frac{|x-\xi|^{4s}}{t-\tau}\right)^{\frac{1}{4s-1}}\right),$$

$$(iii) \|D_x^m D_\xi^b W(\xi+x, t, \xi, \tau)\| \leq \frac{C_1}{(t-\tau)^{\frac{D+|m|}{4s}}} \exp\left(-C_2 \left(\frac{|x|^{4s}}{t-\tau}\right)^{\frac{1}{4s-1}}\right),$$

where $0 \leq |a| + |b| \leq r$ and $0 \leq |m| < 4s$.

For convenience we rewrite $(\rho_\epsilon^1, \mu_\epsilon^1)^T$, $(\rho_\epsilon^2, \mu_\epsilon^2)^T$ as (ρ_1, μ_1) and (ρ_2, μ_2) .

From (A.21) we have

$$\begin{aligned}
& \|\Pi_2 \Phi U_\epsilon^1 - \Pi_2 \Phi U_\epsilon^2\|_{L^2(\Omega)} \\
& \leq \int_0^t \left\| \tilde{W}_{t-\tau} * \left(\frac{\mu_1 \otimes \mu_1}{\rho_1 + \epsilon} - \frac{\mu_2 \otimes \mu_2}{\rho_2 + \epsilon} \right) \right\|_{L^2(\Omega)} d\tau \\
& + \int_0^t \|\tilde{W}_{t-\tau} * (P(\rho_1) - P(\rho_2))\|_{L^2(\Omega)} d\tau \\
& + \int_0^t \nu \left\| \tilde{W}_{t-\tau} * (\sum_1 - \sum_2) \right\|_{L^2(\Omega)} d\tau \\
& \equiv I_1 + I_2 + I_3.
\end{aligned} \tag{A.30}$$

We will treat the integrals I_1 , I_2 and I_3 separately. Before estimating the nonlinear terms, we state some results about the product formula of Sobolev space.

Proposition A.12.

- (i) $H^s H^t \hookrightarrow H^r$, where $r = \min\{s, t, s + t - [D/2] - 1\} \geq 0$,
- (ii) H^s is a Banach algebra when $s > D/2$.

For the general Sobolev space $W^{m,p}(\Omega)$, $p \geq 1$ if Ω is a domain in $\mathbb{R}^D$ having the cone property then we have the extension of Proposition A.11.

Proposition A.13. (see [Ada])

- (i) If $sp > D$ and $f \in W^{s,p}(\Omega)$, $g \in W^{s_1,p}(\Omega)$ with $s \geq s_1 > 0$ then

$$fg \in W^{s_1,p}(\Omega),$$

and

$$\|fg\|_{W^{s_1,p}} \leq C \|f\|_{W^{s,p}} \|g\|_{W^{s_1,p}}. \tag{A.31}$$

- (ii) $W^{s,p}(\Omega)$ is a Banach algebra if $sp > D$.

By Young's convolution Theorem, we arrive

$$I_1 \leq \int_0^t \|\tilde{W}_{t-\tau}\|_{L^2(\Omega)} \left\| \frac{\mu_1^2}{\rho_1 + \epsilon} - \frac{\mu_2^2}{\rho_2 + \epsilon} \right\|_{L^1(\Omega)} d\tau,$$

but the Cauchy-Schwartz's inequality gives us

$$\begin{aligned}
\left\| \frac{\mu_1^2}{\rho_1 + \epsilon} - \frac{\mu_2^2}{\rho_2 + \epsilon} \right\|_{L^1(\Omega)} &\leq \left\| \frac{\mu_1^2 - \mu_2^2}{\rho_1 + \epsilon} \right\|_{L^1(\Omega)} + \left\| \frac{\mu_2^2(\rho_1 - \rho_2)}{(\rho_1 + \epsilon)(\rho_2 + \epsilon)} \right\|_{L^1(\Omega)} \\
&\leq \|\mu_1^2 - \mu_2^2\|_{L^1(\Omega)} + \|\mu_2^2(\rho_1 - \rho_2)\|_{L^1(\Omega)} \\
&\leq (\|\mu_1\|_{L^2(\Omega)} + \|\mu_2\|_{L^2(\Omega)}) \|\mu_1 - \mu_2\|_{L^2(\Omega)} \\
&\quad + \|\mu\|_{L^4(\Omega)}^2 \|\rho_1 - \rho_2\|_{L^2(\Omega)}.
\end{aligned} \tag{A.32}$$

Now applying Theorem A.2(Gagliardo-Nirenberg's inequality) to μ_2 we obtain

$$\|\mu_2\|_{L^4} \leq C \|\nabla \mu_2\|_{L^2}^\theta \|\mu_2\|_{L^2}^{1-\theta}, \tag{A.33}$$

where $\theta = \frac{D}{4} < 1$, i.e., (A.33) is true for $D < 4$. Especially we are interested in the case when space dimension $D = 2$ or 3 . If we require

$$\|\mu_i\|_{L^2} \leq R, \quad \|\nabla \mu_i\|_{L^2} \leq R, \quad i = 1, 2.$$

i.e., $\|\mu_i\|_{H^1} \leq R$ then (A.32) becomes

$$\begin{aligned}
\left\| \frac{\mu_1^2}{\rho_1 + \epsilon} - \frac{\mu_2^2}{\rho_2 + \epsilon} \right\|_{L^1(\Omega)} &\leq 2R \|\mu_1 - \mu_2\|_{L^2(\Omega)} + C_1 R^2 \|\rho_1 - \rho_2\|_{L^2} \\
&\leq M \|U_1 - U_2\|_X,
\end{aligned} \tag{A.34}$$

where $M = \max\{2R, CR^2\}$. Hence we can conclude

$$I_1 \leq M \|U_1 - U_2\|_Y \int_0^t \|\tilde{W}_{t-\tau}\|_{L^2} d\tau \equiv \tilde{M} \|U_1 - U_2\|_Y. \tag{A.35}$$

For the γ -law gas $P(\rho) = c\rho^\gamma$, if $1 < \gamma \leq 2$ and also require

$$\|\rho_i\|_{L^\gamma} \leq K, \tag{A.36}$$

then

$$\begin{aligned}
\|P(\rho_1) - P(\rho_2)\|_{L^1(\Omega)} &= C \|\rho_1^\gamma - \rho_2^\gamma\|_{L^1(\Omega)} \\
&\leq CK^{\gamma-1} \|\rho_1 - \rho_2\|_{L^\gamma(\Omega)} \\
&\leq CK^{\gamma-1} \|\rho_1 - \rho_2\|_{L^2(\Omega)}.
\end{aligned}$$

Hence we can estimate integral I_2 ,

$$\begin{aligned}
I_2 &\leq \int_0^t \|\tilde{W}_{t-\tau}\|_{L^2} \|P(\rho_1) - P(\rho_2)\|_{L^1} d\tau \\
&\leq CK^{\gamma-1} \sup_t \|\rho_1(t) - \rho_2(t)\|_{L^2} \\
&\leq M_2 \|U_1 - U_2\|_Y.
\end{aligned} \tag{A.37}$$

For integral I_3 , recall

$$\sum_i \equiv \left(\nabla \frac{\mu_i}{\rho_i + \epsilon} \right) + \left(\nabla \frac{\mu_i}{\rho_i + \epsilon} \right)^T - \frac{2}{D} \left(\nabla \cdot \frac{\mu_i}{\rho_i + \epsilon} \right) I.$$

Therefore in order to estimate $\|\sum_1 - \sum_2\|_{L^1}$, it suffices to estimate

$$\left\| \frac{\mu_1}{\rho_1 + \epsilon} - \frac{\mu_2}{\rho_2 + \epsilon} \right\|_{L^2}.$$

By Cauchy-Schwartz inequality, (A.34) and $L^2(\Omega) \subset L^1(\Omega)$ (for bounded domain Ω) we have

$$\begin{aligned}
\left\| \frac{\mu_1}{\rho_1 + \epsilon} - \frac{\mu_2}{\rho_2 + \epsilon} \right\|_{L^2} &\leq \left\| \frac{\mu_1 - \mu_2}{\rho_1 + \epsilon} \right\|_{L^1} + \left\| \frac{\mu_2(\rho_1 - \rho_2)}{(\rho_1 + \epsilon)(\rho_2 + \epsilon)} \right\|_{L^1} \\
&\leq \|\mu_1 - \mu_2\|_{L^2} + \|\mu_2(\rho_1 - \rho_2)\|_{L^1} \\
&\leq \|\mu_1 - \mu_2\|_{L^2} + \|\mu_2\|_{L^2} \|\rho_1 - \rho_2\|_{L^2} \\
&\leq R \|U_1 - U_2\|_X.
\end{aligned} \tag{A.38}$$

Therefore by Young's convolution inequality,

$$\begin{aligned}
I_3 &= \nu \int_0^t \left\| \tilde{W}_{t-\tau} * (\sum_1 - \sum_2) \right\|_{L^2} d\tau \\
&\leq \nu \int_0^t \|\tilde{W}_{t-\tau}\|_{L^2} \|\sum_1 - \sum_2\|_{L^1} d\tau \\
&\leq C \sup_t \|U_1 - U_2\|_X \int_0^t \|\tilde{W}_{t-\tau}\|_{L^2} d\tau \\
&\leq M_3 \|U_1 - U_2\|_Y.
\end{aligned} \tag{A.39}$$

Now we can combine (i) and (ii) to deduce the existence of solution for the initial value problem of the parabolic regularization system. It is clear that Φ maps X to X . Let $M \subset X = L^2(\Omega) \times L^2(\Omega)$ such that

- (1) $U = (\rho, \mu)^T \in M$ is strongly measurable,
- (2) $\rho(t) \geq 0$, for all $t \geq 0$,
- (3) $\|\rho(t)\|_{L^2} \leq R, \|\mu(t)\|_{L^2} \leq R$, and $\|\nabla \mu(t)\|_{L^2} \leq R$

with the metric $d(U, V) = \sup_{0 \leq t \leq T} \|U(t) - V(t)\|_X$, where

$$\|U(t)\|_X = \|\rho(t)\|_{L^2} + \|\mu(t)\|_{L^2}$$

Then M is a nonempty complete metric space. By (i) and (ii), we see that for sufficient small T

$$\Phi : M \mapsto M$$

is a strict contraction. Thus Theorem A.8 is proved. Therefore applying the Picard-Banach fixed point theorem we obtain the unique mild solution U_ϵ for the initial value problem of the parabolic regularization system, (P.R.), with

$$U_\epsilon \in M \subset C([0, T], X) = Y$$

Sum up we have the following local existence theorem.

Theorem A.14. For $D < 4$, $1 < \gamma \leq 2$, $(\rho_\epsilon, \mu_\epsilon) \in D(A)$ satisfy (A.34) and (A.36) then there exist a unique mild solution on $[0, T]$ of (A.17) – (A.18).

Furthermore we have the global existence result.

Theorem A.15. Under the hypothesis of Theorem A.13 then there exists a unique continuous mild solution on $[0, \infty)$ of (A.17) – (A.18).

Proof: By Theorem A.13, we have

$$\|\Phi(U(t)) - \Phi(V(t))\|_Y \leq C(t) \|U - V\|_Y,$$

for all $t > 0$ and $U, V \in M \subset C([0, T], X) = Y$. We may assume that $t \mapsto C(t)$ is monotone nondecreasing. By induction we have

$$\|\Phi^n(U(t)) - \Phi^n(V(t))\|_Y \leq (C(t))^n / n! \|U - V\|_Y,$$

for all $n \in \mathbb{N}$. Now let $\tau > 0$ be fixed but otherwise arbitrary. Choose n so large that

$$\alpha = (C(\tau))^n / n! < 1.$$

Then

$$\|\Phi^n U - \Phi^n V\|_Y \leq \alpha \|U - V\|_Y,$$

for all $U, V \in C([0, \tau], X) = Y$.

Hence Φ has a unique fixed point in Y . So (A.17) – (A.18) has a unique continuous mild solution on $[0, \tau]$. Let τ tend to ∞ , we have the global solution.

We also concern about the regularity of the mild solution for the analytic semigroup.

Theorem A.16. Let $U_\epsilon = (\rho_\epsilon, \mu_\epsilon) \in Y = C([0, \infty))$ solve (A.17) – (A.18) then

$$\begin{aligned} (i) \quad & \rho_\epsilon \in C([0, \infty), H^2 \cap H_0^1), \\ (ii) \quad & \mu_\epsilon \in C([0, \infty), H^{4s} \cap H_0^{2s}). \end{aligned}$$

Proof: (i): Since $\mu_\epsilon \in L^2$ and $\nabla \mu_\epsilon \in L^2$, therefore by the regularity theory for the heat equation we have

$$\begin{aligned} \partial_t \rho_\epsilon &\in L^2(\Omega \times [0, T]), \\ \rho_\epsilon &\in L^2(\Omega \times [0, T]). \end{aligned} \tag{A.40}$$

Furthermore from (A.17), the explicit expression of ρ_ϵ we take ∇ on both sides then

$$\begin{aligned} \|\nabla \rho_\epsilon(t)\|_{L^2} &\leq \|K_t * \rho_\epsilon^{in}\|_{L^2} + \int_0^t \|\nabla K_{t-\tau} * \mu_\epsilon\|_{L^2} d\tau \\ &\leq \|K_t\|_{L^1} \|\rho_\epsilon^{in}\|_{L^2} + \int_0^t \|\nabla K_{t-\tau}\|_{L^1} \|\mu_\epsilon\|_{L^2} d\tau < \infty. \end{aligned} \tag{A.41}$$

Thus

$$\rho_\epsilon \in C([0, \infty), L^2(\Omega)).$$

By induction we obtain

$$\rho_\epsilon \in C([0, \infty), H^s(\Omega)), \forall s.$$

(ii): Applying ∇ to the both sides of (A.18) then

$$\begin{aligned} \|\nabla \mu_\epsilon(t)\|_{L^2} &\leq \|\nabla W_t * \mu_\epsilon^{in}\|_{L^2} + \int_0^t \left\| \nabla \tilde{W}_{t-\tau} * \frac{\mu_\epsilon}{\rho_\epsilon + \epsilon} \right\|_{L^2} d\tau \\ &\quad + \int_0^t \|\nabla \tilde{W}_{t-\tau} * P(\rho_\epsilon)\|_{L^2} d\tau + \nu \int_0^t \left\| \nabla \tilde{W}_{t-\tau} * \sum_\epsilon \right\|_{L^2} d\tau \\ &\leq \|\nabla W_t\|_{L^1} \|\mu_\epsilon^{in}\|_{L^2} + \sup_t \|\mu_\epsilon(t)\|_{L^2} \int_0^t \|\nabla \tilde{W}_{t-\tau}\|_{L^1} d\tau \quad (A.42) \\ &\quad + C_1 \sup_t \|\rho_\epsilon(t)\|_{L^2} \int_0^t \|\nabla \tilde{W}_{t-\tau}\|_{L^2} d\tau \\ &\quad + C_2 \sup_t \|\mu_\epsilon(t)\|_{L^2}^2 \int_0^t \|\nabla \tilde{W}_{t-\tau}\|_{L^2} d\tau. \end{aligned}$$

Thus

$$\mu_\epsilon \in C([0, \infty), H^1).$$

By induction

$$\mu_\epsilon \in C([0, \infty), H^s).$$

It is a direct consequence of Sobolev imbedding theorem that

Corollary A.17. The mild solution $U_\epsilon = (\rho_\epsilon, \mu_\epsilon)$ is a classical solution when $s > D/2$ and the initial data satisfies

$$U_\epsilon^{in} = (\rho_\epsilon^{in}, \mu_\epsilon^{in}) \in D(A) = [H^2(\Omega) \cap H_0^1(\Omega)] \times [H^{4s}(\Omega) \cap H_0^{2s}(\Omega)].$$

Appendix B

The Helmholtz Decomposition of Vector Fields

The derivation of Leray's formulation for the incompressible Navier-Stokes equation was projecting the Navier-Stokes equation on the space of divergence-free vector field to eliminate ∇P . Then we recovered the pressure P from the gradient part of $(\mathbf{u} \cdot \nabla)\mathbf{u}$. This result suggests a general concept of decomposing vector field into the divergence-free and gradient part. We state the Helmholtz theorem for bounded domain and also the periodic domain in this chapter.

It is well known (see [Tem]) that the Hilbert space $L^2(\Omega)$ ($\partial\Omega$ is smooth) admits the following orthogonal decomposition, the Helmholtz decomposition :

$$L^2(\Omega) = X_2 \oplus G_2. \quad (B.1)$$

Here X_2 is the closure in $L^2(\Omega)$ of the space

$$(C_0^\infty(\Omega))_\sigma = \{\mathbf{u} \in C_0^\infty(\Omega); \quad \nabla \cdot \mathbf{u} = 0, \quad \text{in } \Omega\}, \quad (B.2)$$

and

$$G_2 = \{\nabla P \in L^2(\Omega); \quad P \in L_{loc}^2(\bar{\Omega})\}, \quad (B.3)$$

with $\bar{\Omega}$, the closure of Ω . It means that the field of velocity $\mathbf{u}$ is expressible as the sum of an irrotational(potential) field with the divergent-free field.

$$\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2; \quad \nabla \wedge \mathbf{u}_1 = 0, \quad \nabla \cdot \mathbf{u}_2 = 0. \quad (B.4)$$

Let $W^{s,p}(\Omega)$ ($s \in \mathbb{R}, 1 < p < \infty$) be the Sobolev space of order s such that $W^{0,p}(\Omega) = L^p(\Omega)$, and let $\|f\|_{s,p}$ be the norm of f in $W^{s,p}(\Omega)$. Set

$$\begin{aligned} X_p &= \text{closure in } L^p(\Omega) \text{ of } \{\mathbf{u} \in C_0^\infty(\Omega); \nabla \cdot \mathbf{u} = 0\}, \\ G_p &= \{\nabla f; f \in W^{1,p}(\Omega)\}, \\ \mathcal{P} = \mathcal{P}_p : L^p(\Omega) &\mapsto X_p(\Omega) \quad \text{is the projection along } G_p(\Omega). \end{aligned} \quad (B.5)$$

We then have the following generalization due to [FuM] on the direct sum decomposition of $L^p(\Omega)$, $1 < p < \infty$.

Theorem B.1 (Helmholtz Decomposition).

$$\begin{aligned}
(i) \quad & L^p(\Omega) = X_p \oplus G_p(\Omega), \quad (\text{direct sum}) \\
(ii) \quad & \mathcal{P}_{p^*} = \mathcal{P}_q, \quad q = p/p - 1, \quad (\mathcal{P}_p^* \text{ is the dual operator}) \\
(iii) \quad & X_p(\Omega)^\perp = G_q(\Omega), \quad (X_p(\Omega)^\perp \text{ denotes the annihilator of } X_p(\Omega)) \\
(iv) \quad & X_p(\Omega)^* = X_q(\Omega).
\end{aligned} \tag{B.6}$$

Note : $G_p(\Omega)$ is also a closed subspace of $L^p(\Omega)$.

The following fact is very useful for studying incompressible flows.

Theorem B.2. Every vector field $\mathbf{u} \in L^2(\mathbf{R}^D) \cap C^\infty(\mathbf{R}^D)$ has the unique orthogonal decomposition,

$$\mathbf{u} = \mathbf{w} + \nabla g, \quad \nabla \cdot \mathbf{w} = 0, \tag{B.7}$$

with the following properties,

$$\begin{aligned}
(i) \quad & \mathbf{w}, \nabla g \in L^2(\mathbf{R}^D) \cap C^\infty(\mathbf{R}^D), \\
(ii) \quad & \mathbf{w} \perp \nabla g \quad \text{in } L^2; \quad \text{i.e., } (\mathbf{w}, \nabla g)_0 = 0, \\
(iii) \quad & \|\mathbf{u}\|_0^2 = \|\mathbf{w}\|_0^2 + \|\nabla g\|_0^2, \\
(iv) \quad & \|D^\alpha \mathbf{u}\|_0^2 = \|D^\alpha \mathbf{w}\|_0^2 + \|\nabla D^\alpha g\|_0^2,
\end{aligned} \tag{B.8}$$

for any multi-index of the derivative D^α .

where $\|\cdot\|_0$ denotes the L^2 -norm.

Next we will formulate the decomposition for periodic vector field $\mathbf{u}(x) = \mathbf{u}(x + e_i)$ which we identify the functions on a torus T^D . The natural tools for studying periodic problems are the Fourier series. Here we recall their definitions and basic properties.

Proposition B.3. Let $\mathbf{u}$ be a smooth 1-periodic function in $\mathbb{R}^D$. Define the Fourier series of $\mathbf{u}$ by

$$\mathbf{u}(x) = \sum_{k \in \mathbb{Z}^D} \exp(2\pi i x \cdot k) \hat{\mathbf{u}}(k), \quad (B.9)$$

where $\hat{\mathbf{u}}(k)$ are the Fourier coefficients,

$$\hat{\mathbf{u}}(k) = \int_{T^D} \exp(-2\pi i x \cdot k) \mathbf{u}(x) dx, \quad (B.10)$$

then

$$\begin{aligned} (i) \quad \|\mathbf{u}\|_0^2 &= \sum_{k \in \mathbb{Z}^D} |\hat{\mathbf{u}}(k)|^2, \\ (ii) \quad D^\alpha \mathbf{u}(x) &= \sum_{k \in \mathbb{Z}^D} \exp(2\pi i x \cdot k) (2\pi i k)^\alpha \hat{\mathbf{u}}(k), \end{aligned} \quad (B.11)$$

for any multi-index α .

From Theorem B.2 we know that if we have a decomposition

$$\mathbf{u} = \mathbf{w} + \nabla g, \quad \nabla \cdot \mathbf{w} = 0, \quad (B.12)$$

then g is determined by the Poisson's equation

$$\Delta g = \nabla \cdot \mathbf{u}.$$

But the Poisson's equation on torus T^D does not necessary have a solution, for we have

Proposition B.4. Let $f \in C^\infty(T^D)$, and consider the Poisson's equation on a torus T^D

$$-\Delta \mathbf{u} = f \quad x \in T^D. \quad (B.13)$$

There exists a solution $\mathbf{u} \in C^\infty(T^D)$ if and only if

$$\int_{T^D} f(x) dx = 0. \quad (B.14)$$

Moreover, the explicit solution can be obtained by Fourier methods as

$$\mathbf{u}(x) = \sum_{k \neq 0} \exp(2\pi i k x) (4\pi |k|^2)^{-1} \hat{f}(k), \quad (B.15)$$

and

$$\|\mathbf{u}\|_{H^2} \leq C\|f\|. \quad (B.16)$$

Thus we “gain” two derivatives.

In our case $f = \nabla \cdot \mathbf{u}$, the compatibility condition (B.12) is automatically satisfied, and we can formulate.

Proposition B.5(Decomposition on T^D). Every vector field $\mathbf{u} \in C^\infty(T^D)$ on a torus T^D has the unique orthogonal decomposition

$$\mathbf{u} = \bar{\mathbf{u}} + \mathbf{w} + \nabla g, \quad \bar{\mathbf{u}} = \int_{T^D} \mathbf{u} \, dx, \quad \nabla \cdot \mathbf{w} = 0, \quad (B.17)$$

where

$$\nabla g = \sum_{|k| \neq 0} \exp(2\pi kx) \frac{k \otimes k}{|k|^2} \hat{\mathbf{u}}(k), \quad (B.18)$$

and $k \otimes k \equiv (k_i k_j)$. This decomposition has the following properties.

$$\begin{aligned} (i) \quad & \mathbf{w}, \nabla g \in C^\infty(T^D), \\ (ii) \quad & \mathbf{w} \perp \nabla g \text{ in } L^2(T^D), \quad \text{i.e.,} \quad \int_{T^D} \mathbf{w} \cdot \nabla g \, dx = 0, \\ (iii) \quad & \|\mathbf{u} - \bar{\mathbf{u}}\|_0^2 = \|\mathbf{w}\|_0^2 + \|\nabla g\|_0^2, \\ (iv) \quad & \|D^\alpha \mathbf{u}\|_0^2 = \|D^\alpha \mathbf{w}\|_0^2 + \|\nabla D^\alpha g\|_0^2. \end{aligned} \quad (B.19)$$

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