INFORMATION TO USERS

This was produced from a copy of a document sent to us for microfilming. While the most advanced technological means to photograph and reproduce this document have been used, the quality is heavily dependent upon the quality of the material submitted.

The following explanation of techniques is provided to help you understand markings or notations which may appear on this reproduction.

1. The sign or "target" for pages apparently lacking from the document photographed is "Missing Page(s)". If it was possible to obtain the missing page(s) or section, they are spliced into the film along with adjacent pages. This may have necessitated cutting through an image and duplicating adjacent pages to assure you of complete continuity.
2. When an image on the film is obliterated with a round black mark it is an indication that the film inspector noticed either blurred copy because of movement during exposure, or duplicate copy. Unless we meant to delete copyrighted materials that should not have been filmed, you will find a good image of the page in the adjacent frame. If copyrighted materials were deleted you will find a target note listing the pages in the adjacent frame.
3. When a map, drawing or chart, etc., is part of the material being photographed the photographer has followed a definite method in "sectioning" the material. It is customary to begin filming at the upper left hand corner of a large sheet and to continue from left to right in equal sections with small overlaps. If necessary, sectioning is continued again—beginning below the first row and continuing on until complete.
4. For any illustrations that cannot be reproduced satisfactorily by xerography, photographic prints can be purchased at additional cost and tipped into your xerographic copy. Requests can be made to our Dissertations Customer Services Department.
5. Some pages in any document may have indistinct print. In all cases we have filmed the best available copy.

University
Microfilms
International

300 N. ZEEB RD., ANN ARBOR, MI 48106

8217427

Kim, Cheol-Jung

POLYNOMIAL FIT OF INTERFEROGRAMS

The University of Arizona

PH.D. 1982

University
Microfilms
International 300 N. Zeeb Road, Ann Arbor, MI 48106

POLYNOMIAL FIT OF INTERFEROGRAMS

by

Cheol-Jung Kim

A Dissertation Submitted to the Faculty of the

COMMITTEE ON OPTICAL SCIENCES (GRADUATE)

In Partial Fulfillment of the Requirements
For the Degree of

DOCTOR OF PHILOSOPHY

In the Graduate College

THE UNIVERSITY OF ARIZONA

1 9 8 2

THE UNIVERSITY OF ARIZONA
GRADUATE COLLEGE

As members of the Final Examination Committee, we certify that we have read
the dissertation prepared by Cheol-Jung Kim
entitled Polynomial Fit of Interferograms

and recommend that it be accepted as fulfilling the dissertation requirement
for the Degree of Doctor of Philosophy .

James C. Wyant

Date 4/2/82

B. Ry Friedman

Date 4/2/82

Robert R. Shamm

4/27/82
Date

Date _____

Date _____

Final approval and acceptance of this dissertation is contingent upon the candidate's submission of the final copy of the dissertation to the Graduate College.

I hereby certify that I have read this dissertation prepared under my direction and recommend that it be accepted as fulfilling the dissertation requirement.

James C. Wyant
Dissertation Director

Date 4/2/82

STATEMENT BY AUTHOR

This dissertation has been submitted in partial fulfillment of requirements for an advanced degree at The University of Arizona and is deposited in the University Library to be made available to borrowers under rules of the Library.

Brief quotations from this dissertation are allowable without special permission, provided that accurate acknowledgment of source is made. Requests for permission for extended quotation from or reproduction of this manuscript in whole or in part may be granted by the head of the major department or the Dean of the Graduate College when in his judgment the proposed use of the material is in the interests of scholarship. In all other instances, however, permission must be obtained from the author.

SIGNED: Cheol Jang Kim

ACKNOWLEDGMENTS

I wish to thank the following people for their contributions to this work: Dr. James C. Wyant who, as my dissertation advisor, supported this research and provided continuous encouragement and refreshing suggestions, Professor Robert R. Shannon, Robert E. Parks, Dr. Chris L. Koliopoulos and Ker-Li Shu for their many helpful discussions. I also wish to thank Kathleen Seeley and Sherrie Cornett who carefully and patiently corrected and typed the illegible manuscript. Finally, I express my appreciation to Soyoung, my wife, for her patience and encouragement and my parents for their guidance.

TABLE OF CONTENTS

	Page
LIST OF ILLUSTRATIONS	vi
LIST OF TABLES	x
ABSTRACT	xi
1. INTRODUCTION	1
2. CONVENTIONAL CIRCULAR INTERFEROGRAM ANALYSIS	4
Least Squares Criterion	4
Best Reference and Relative Reference	12
Advantages of Orthonormal Polynomials	15
Digitization Error Projection	21
Comparison of Zernike Polynomials and X-Y Polynomials	29
Error Introduced by Use of Finite Number of Digitized Points	35
Fit Error Projection	43
Average of Several Interferograms	46
Edge Effect	47
3. NON-CIRCULAR APERTURE INTERFEROGRAM AND AXICON INTERFEROGRAM ANALYSIS	49
Projection of Roundoff Error	50
Reduction of Roundoff Error	55
Non-Circular Aperture Interferogram	59
Axicon Interferogram Analysis	70
4. SUBAPERTURE INTERFEROGRAM ANALYSIS	84
Criterion of Subaperture Interferogram Analysis	84
Error Magnification	89
Projection of Random Error in the Polynomial Space on Correlated Information	90
Projection of the Fit Error	99
Test of Large Flat or Parabola	100
Test of Fast Aperture Optics	122
Test of Aspheric Surface	128
Local Connection Method	132

TABLE OF CONTENTS--Continued

	Page
5. APPLICATIONS OF SUBAPERTURE INTERFEROGRAM ANALYSIS	145
Average of Interferograms	145
Lateral Shearing Interferogram Analysis	153
6. CONCLUSIONS	164
APPENDIX A: BASIC MATHEMATICS	166
REFERENCES	177

LIST OF ILLUSTRATIONS

Figure	Page
2.1. A conventional circular interferogram	5
2.2. Least squares in vector	8
2.3. Projection of digitization error	9
2.4. Orthogonalization	10
2.5. Best reference and relative reference	14
2.6. Why orthonormal polynomials?	16
2.7. Comparison of condition number	22
2.8. Reduction of the projection of the digitization error . . .	25
2.9. Comparison between different orders of fit	27
2.10. Orthogonal contribution of a polynomial	28
2.11. Comparison of error with and without digitization error . .	37
2.12. Magnification factors of σ_c/σ_d	40
2.13. Comparison between uniform digitization and non-uniform digitizations	44
2.14. Edge effect	48
3.1. $ A-A^* $ and $ Q-PA^* $	52
3.2. Dependence of roundoff error on wavefront	54
3.3. Correlated polynomial	56
3.4. Distribution of condition numbers	61
3.5. $ P /P_{11}$ from different apertures	65
3.6. Results from roundoff error simulation	67

LIST OF ILLUSTRATIONS--Continued

Figure	Page
3.7. Finite digitization error from several different radius apertures	69
3.8. Contour map from oversized Zernike circle	71
3.9. Analysis of axicon interferogram	73
3.10. Fit error of cone and decenter	74
3.11. Difference in fit error	77
3.12. Contour map of axicon	78
3.13. Radial profile	80
3.14. Adjustment of cone	82
4.1. Correlated information	85
4.2. Fit error magnification	91
4.3. Effect of random error	91
4.4. Best reference of the entire aperture	94
4.5. Subapertures of 1/2, 1/3, and 1/4 radius	101
4.6. Magnification factors of reference and surface error to the projected random error	104
4.7. Sensitivity of surface coefficients to the projected random error	105
4.8. Fit error projection	106
4.9. Magnification factors of reference and surface error to the fit error	107
4.10. Sensitivity of surface coefficients to the fit error . . .	108
4.11. Dependence on parameters	110
4.12. Magnification from random digitization	111

LIST OF ILLUSTRATIONS--Continued

Figure	Page
4.13. Magnification factors of reference and surface error to the projected error without overlap	112
4.14. Magnification factors of reference and surface error to the fit error without overlap	113
4.15. Nine 2/5 radius subaperture	116
4.16. Magnification factors of nine 2/5 radius subaperture . . .	117
4.17. Interferograms without fit error	118
4.18. Interferograms with fit error	119
4.19. Difference	120
4.20. Magnification of random error with 4 reference polynomials	124
4.21. Sensitivity of surface coefficients to random error with 4 reference polynomials	125
4.22. Magnification of fit error with 4 reference polynomials . .	126
4.23. Sensitivity of surface coefficients to fit error with 4 reference polynomials	127
4.24. Four annular subapertures of the same area	130
4.25. Magnification of random error and sensitivity	133
4.26. Magnification of fit error and sensitivity	134
4.27. Error magnification and projection of surface on the reference	135
4.28. Magnification of random projected error in the relative reference	138
4.29. Reduction of error by constraints	141
4.30. Magnification of 6 circular local connections with and without constraints	143

LIST OF ILLUSTRATIONS--Continued

Figure	Page
4.31. Magnification of 9 linear local connection without constraint	144
5.1. The ratio of $(\Delta S)_c$ to the projected random error	147
5.2. Projection of surface into best reference	150
5.3. Ratio of $(\Delta S)_f$ to the projected random error	152
5.4. Magnification factor from the digitized error vs number of interferograms	159
5.5. Magnification factors vs shear distance	161
5.6. Magnification factors of surface error to the fit error . .	162
5.7. Sensitivity of surface coefficients	163

LIST OF TABLES

Table	Page
2.1. Normalized Zernike polynomials	17
2.2. Useless polynomials and overfit	28
2.3. X-Y polynomials	30
2.4. Comparison of condition number (Zernike and X-Y)	31
2.5. Error in the coefficients	32
2.6. Comparison of roundoff error	34
2.7. Comparison of error with random digitization error	36
2.8. Comparison of condition number (uniform and non-uniform)	45
2.9. Sampled rms value of spherical aberration of wave rms at best focus	45
3.1. Condition number for reference	62
3.2. Correlated polynomials and condition number reduction	64
3.3. Condition number of oversized Zernike circle	68
3.4. Orthonormalized axicon aberrations	81
3.5. Reduction of condition number by orthonormalization	81
4.1. Condition number (subaperture with 3 references)	102
4.2. Number of digitized points in each subaperture	114
4.3. Condition number (subaperture with 4 references)	123
4.4. Condition number vs grid size	131
5.1. Rms surface wavefront error introduced by uniform random error of .577 rms	154

ABSTRACT

The conventional Zernike polynomial fit of circular aperture interferograms is reviewed and a more quantitative and statistical analysis is added. Some conventional questions such as the required number of polynomials, sampling requirements, and how to determine the optimum reference surface are answered. Then, the analysis is applied to the polynomial fit of noncircular aperture interferograms and axicon interferograms. The problems and limitations of using Zernike polynomials are presented. A method of obtaining the surface figure error information from several smaller subaperture interferograms is analyzed. The limitations of the analysis for testing a large flat, a large parabola, or an aspheric surface are presented. The analysis is compared with the local connection method using overlapped wavefront information. Finally, the subaperture interferogram analysis is used to average several interferograms and to analyze lateral shearing interferograms.

CHAPTER 1

INTRODUCTION

During the past decade, it has become common to perform computer analysis of interferograms (Rimmer, King and Fox, 1972; Loomis, 1976). The most important information obtained from interferograms has been the contour map of the surface figure error. However, phase values that are necessary for determining the surface figure error can be measured only over a limited number of usually non-uniformly distributed points. Consequently, available contour plotting programs which require phase values distributed over a square grid of points cannot readily be used unless local interpolation or polynomial fitting is used to obtain the phase value at the required grid points. Unfortunately, the use of local interpolation is limited almost entirely to contour maps because the information in a contour map cannot be separated into different categories of aberration and continuous information is impossible. On the other hand, the polynomial fit can provide continuous and separable phase information, i.e., the coefficients of polynomials. In addition, the polynomial fit is smoother and faster than local interpolation. This dissertation is concerned primarily with the polynomial fit.

The information derived from the coefficients of polynomials has many applications, as listed below:

1. Deleting test optics error (Parks, 1978)

2. Lateral shearing interferogram analysis (Rimmer and Wyant, 1975)
3. Ritchey-Common test (Shu, Parks and Shannon, 1981)
4. Slope sensing interferogram analysis (Davis and Fritz, 1979)
5. Separation of symmetric and asymmetric wavefront error in axicon test
6. Add, delete, or average of interferograms
7. Alignment error analysis.

The famous least squares criterion has been used in the polynomial fit, but its limitations and requirements have not been examined well. Furthermore, the statistical nature of the polynomial fit has been neglected.

In the next chapter, the conventional polynomial fit of a circular aperture interferogram with Zernike polynomials is reviewed and an error analysis is added. Several traditional questions about the polynomial fit are discussed from a statistical viewpoint. Then, the analysis is applied to solve more difficult problems in Chapters 3 through 5. The use of Zernike polynomials in the analysis of noncircular aperture interferograms and axicon interferograms is discussed in Chapter 3. In Chapter 4, surface figure error information is obtained from several subaperture interferograms. The previous error analysis is extended to find the limitations of this approach. This subaperture interferogram analysis is applied to testing a large flat, a parabola, or aspheric surfaces. The interpretation of lateral shearing interferograms and the average of several interferograms from the viewpoint of subaperture interferogram

analysis are given in Chapter 5. A more stable averaging algorithm is developed and general lateral shearing interferograms are analyzed simply and accurately. The basic mathematics necessary for understanding this dissertation are summarized in Appendix A.

In conclusion, this dissertation provides more mathematical background for understanding the use of a polynomial fit in interferometric optical testing. This background is applied to the analysis of more complicated interferograms like subaperture interferograms and lateral shearing interferograms, and can be applied to testing large optical components or aspheric surfaces.

CHAPTER 2

CONVENTIONAL CIRCULAR INTERFEROGRAM ANALYSIS

A conventional circular interferogram is shown in Fig. 2.1. Usually a large amount of tilt is introduced and an order number is assigned to each fringe. Then the interferogram is digitized along each fringe and the phase value at each point is obtained in units of fringe order number. Each fringe corresponds to an optical path difference of an integer number of waves. The distribution of the digitized points depends on the fringe pattern and usually is nonuniform over the aperture. However, we want continuous phase information over the aperture. In other words, the polynomial fit is a kind of estimation, but the quantitative error analysis has been neglected. In this chapter, a statistical analysis is added to the previous work of others and the error is analyzed more quantitatively.

Least Squares Criterion

The least squares criterion has been used in the polynomial fit. Mathematically, it can be represented by

$$\sum_{i=1}^N \left[W(x_i, y_i) - \sum_{j=1}^M A_j Z_j(x_i, y_i) \right]^2 = \min. \quad (2.1)$$

Fig. 2.1. A conventional circular interferogram.

Suppose we have N phase values W digitized over N points (x_i, y_i) . We want to find the M coefficients of the polynomials Z_j such that the quantity in Eq. (2.1) is minimized. Before solving Eq. (2.1), let us introduce the vector representation of the least squares criterion. Let us assign an orthonormal base vector to each digitized point. Then, the N digitized points form an N -dimensional space and the digitized data can be represented with a vector $\vec{W}$

$$\vec{W} = \frac{1}{\sqrt{N}} (W(x_1, y_1)\hat{a}_1 + W(x_2, y_2)\hat{a}_2 + \dots + W(x_N, y_N)\hat{a}_N) . \quad (2.2)$$

The vector is normalized such that the size of the vector corresponds to the square root of the second moment of phase over the digitized points

$$\begin{aligned} |\vec{W}| &= (\vec{W} \cdot \vec{W})^{\frac{1}{2}} \\ &= \left[\frac{1}{N} \sum_{i=1}^N W(x_i, y_i)^2 \right]^{\frac{1}{2}} . \end{aligned} \quad (2.3)$$

Similarly, each polynomial can be represented with a vector

$$\vec{Z}_j = \frac{1}{\sqrt{N}} (Z_j(x_1, y_1)\hat{a}_1 + Z_j(x_2, y_2)\hat{a}_2 + \dots + Z_j(x_N, y_N)\hat{a}_N) \quad (2.4)$$

$$\begin{aligned} |\vec{Z}_j| &= (\vec{Z}_j \cdot \vec{Z}_j)^{\frac{1}{2}} \\ &= \left[\frac{1}{N} \sum_{i=1}^N Z_j(x_i, y_i)^2 \right]^{\frac{1}{2}} . \end{aligned} \quad (2.5)$$

The size of each polynomial vector corresponds to the square root of the second moment of the polynomial over the digitized points. These M polynomial vectors form an M -dimensional subspace S_z in Fig. 2.2. Now, the least squares criterion can be represented with the vector equation

$$\begin{aligned}
 |\vec{W} - \vec{Z}|^2 &= \left| \vec{W} - \sum_{j=1}^M A_j \vec{Z}_j \right|^2 \\
 &= |\vec{F}|^2 \\
 &= \min.
 \end{aligned} \tag{2.6}$$

Equation (2.6) in vector form is a more general criterion than the least squares criterion. If a set of non-normal but orthogonal base vectors is used for each point, it becomes a non-uniformly weighted least squares criterion. Then the size of the vector is the square root of the second moment but with nonuniform weight. On the other hand, if a set of non-orthogonal base vectors is used, it will include the correlated terms between the different points because of the non-orthogonality. Then the size of the vector is not related to the square root of the second moment. This approach is much more difficult to solve. However, if the surface can be represented with a set of polynomials and we can neglect the digitization error and the calculation error, we can restore the correct coefficients with any kind of base vectors because Eq. (2.6) becomes zero with correct coefficients (and this is the minimum). However, random digitization error is inevitable. Usually it can be estimated only with the root-mean-square value, and its

Fig. 2.2. Least squares in vector.

expectation value at each point is the same. However, if a non-orthonormal base vector is used, the projection of the random digitization error in the vector representation is not uniform as shown in Fig. 2.3. Therefore, if we can digitize uniformly over enough points, it is not unreasonable to assign an orthonormal base vector to each point. The question of "How many points are enough?" will be considered later. Furthermore, this approach is easy to solve.

Now, let us solve the least squares problem. As shown in Fig. 2.2, the fit error vector $\vec{F}$ is minimized if the fit error vector is perpendicular to the subspace S_z formed by the polynomials. Mathematically, this can be stated as

$$\begin{aligned}
 (\vec{W} - \vec{Z}) \cdot \vec{Z}_1 &= 0 \\
 (\vec{W} - \vec{Z}) \cdot \vec{Z}_2 &= 0 \\
 &\vdots \\
 (\vec{W} - \vec{Z}) \cdot \vec{Z}_M &= 0
 \end{aligned} \tag{2.7}$$

Fig. 2.3. Projection of digitization error.

where

$$\vec{Z} = \sum_{i=1}^M A_i \vec{Z}_i \quad .$$

Therefore, we can obtain the following linear equation

$$\begin{pmatrix} \vec{Z}_1 \cdot \vec{Z}_1 & \vec{Z}_1 \cdot \vec{Z}_2 & \dots & \vec{Z}_1 \cdot \vec{Z}_M \\ \vec{Z}_2 \cdot \vec{Z}_1 & \vec{Z}_2 \cdot \vec{Z}_2 & \dots & \vec{Z}_2 \cdot \vec{Z}_M \\ \vdots & \vdots & \ddots & \vdots \\ \vec{Z}_M \cdot \vec{Z}_1 & \vec{Z}_M \cdot \vec{Z}_2 & \dots & \vec{Z}_M \cdot \vec{Z}_M \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_M \end{pmatrix} = \begin{pmatrix} \vec{W} \cdot \vec{Z}_1 \\ \vec{W} \cdot \vec{Z}_2 \\ \vdots \\ \vec{W} \cdot \vec{Z}_M \end{pmatrix} \quad (2.8)$$

This equation is termed the normal equation and can be solved by triangulizing the matrix and back substitution (Stewart, 1973, pp. 105-148), that is, it can be done by orthogonalizing the polynomials. Therefore, let us form a set of orthogonal polynomials from the set of non-orthogonal polynomials as shown in Fig. 2.4

Fig. 2.4. Orthogonalization.

$$\vec{z}'_1 = \vec{z}_1 \quad (2.9)$$

$$\vec{z}'_2 = \vec{z}_2 - \frac{(\vec{z}_2 \cdot \vec{z}_1)}{(\vec{z}_1 \cdot \vec{z}_1)} \vec{z}_1 \quad (2.10)$$

$$\vec{z}'_2 \cdot \vec{z}_1 = 0 \quad (2.11)$$

and

$$\vec{z}'_2 \cdot \vec{z}'_2 = \vec{z}_2 \cdot \vec{z}_2 \quad (2.12)$$

Similarly, we know that

$$\vec{z}'_k \cdot \vec{z}_i = 0 \quad \text{for } i < k \quad (2.13)$$

and

$$\vec{z}'_k \cdot \vec{z}'_k = \vec{z}_k \cdot \vec{z}_k \quad (2.14)$$

Then the fit error is orthogonal to the set of these orthogonal polynomials. The solution can also be represented with orthogonal polynomials

$$\vec{z} = \sum_{i=1}^M A_i' \vec{z}'_i \quad (2.15)$$

Therefore, the vector product between the fit error vector and the set of orthogonalized polynomials is zero.

$$\begin{aligned}
 (\vec{W} - \vec{Z}) \cdot \vec{Z}_1' &= 0 \\
 (\vec{W} - \vec{Z}) \cdot \vec{Z}_2' &= 0 \\
 &\vdots \\
 (\vec{W} - \vec{Z}) \cdot \vec{Z}_M' &= 0
 \end{aligned} \tag{2.16}$$

If $\vec{Z}$ is represented with the orthogonal polynomials, then

$$\begin{pmatrix} \vec{Z}_1' \cdot \vec{Z}_1' & 0 & \dots & 0 \\ 0 & \vec{Z}_2' \cdot \vec{Z}_2' & \dots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \dots & \vec{Z}_M' \cdot \vec{Z}_M' \end{pmatrix} \begin{pmatrix} A_1' \\ A_2' \\ \vdots \\ A_M' \end{pmatrix} = \begin{pmatrix} \vec{W} \cdot \vec{Z}_1' \\ \vec{W} \cdot \vec{Z}_2' \\ \vdots \\ \vec{W} \cdot \vec{Z}_M' \end{pmatrix}. \tag{2.17}$$

Now, if Z is represented with the original non-orthogonal polynomials,

$$\text{then } \begin{pmatrix} \vec{Z}_1' \cdot \vec{Z}_1 & \vec{Z}_1' \cdot \vec{Z}_2 & \dots & \vec{Z}_1' \cdot \vec{Z}_M \\ \vec{Z}_2' \cdot \vec{Z}_1 & \vec{Z}_2' \cdot \vec{Z}_2 & \dots & \vec{Z}_2' \cdot \vec{Z}_M \\ \vdots & \vdots & & \vdots \\ \vec{Z}_M' \cdot \vec{Z}_1 & \vec{Z}_M' \cdot \vec{Z}_2 & \dots & \vec{Z}_M' \cdot \vec{Z}_M \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_M \end{pmatrix} = \begin{pmatrix} \vec{W} \cdot \vec{Z}_1' \\ \vec{W} \cdot \vec{Z}_2' \\ \vdots \\ \vec{W} \cdot \vec{Z}_M' \end{pmatrix}. \tag{2.18}$$

Because of Eq. (2.14), the orthogonal contribution (O.C.) of each polynomial can be found from Eq. (2.18)

$$\begin{aligned}
 OC_i &= A_i' |\vec{Z}_i'| \\
 &= \frac{(\vec{W} \cdot \vec{Z}_i')}{(\vec{Z}_i' \cdot \vec{Z}_i')^{1/2}} = \frac{(\vec{W} \cdot \vec{Z}_i')}{(\vec{Z}_i' \cdot \vec{Z}_i')^{1/2}}
 \end{aligned} \tag{2.19}$$

This orthogonal contribution can be applied to find the useless polynomials in the polynomial fit. Also, the lower half of the matrix in Eq. (2.18) is zero because of the orthogonality (see Eq. 2.13). Equation (2.18) is the desired triangular matrix. This triangular matrix can be obtained from the normal matrix by several different methods (Stewart, 1973, pp. 113-148). Because the normal matrix is a positive definite matrix (Stewart, 1973, p. 139), the roundoff error is considerably smaller compared to that of a usual matrix (Stewart, 1973, p. 153). This approach has been used in interferogram analysis programs because it is fast, needs less computer memory, and is accurate enough for analyzing circular interferograms with Zernike polynomials (Wang and Silva, 1980). However, this normal equation approach should be distinguished from the Gram-Schmidt method (Stewart, 1973, pp. 244-245; Lawson and Hanson, 1974, pp. 121-133). Both methods give the same triangular matrix, but the roundoff error is large for the normal equation approach. This will be explained in the next chapter.

Best Reference and Relative Reference

In optical testing, the wavefront information obtained from the interferogram is never the same as the surface figure error we want to correct. Usually, a large tilt is introduced for easy digitization, and some alignment error is inevitable. The best reference wavefront $\vec{R}_b$ is one that minimizes the rms of the surface wavefront after the reference wavefront is adjusted. However, sometimes we cannot minimize the surface wavefront. Consequently, the reference wavefront $\vec{R}_r$ is used as the

relative reference as shown in Fig. 2.5. S and R indicate the surface polynomial space and the reference polynomial space.

Mathematically, the best reference is defined by

$$\sum_{i=1}^N [W(x_i, y_i) - \sum_{j=1}^K B_j Z_j(x_i, y_i)]^2 = \min, \quad (2.20)$$

where K is the number of reference polynomials. Usually, the reference polynomial includes constant, x-y tilt, and defocus. The reference determined by this equation is the best reference, which minimizes the rms value of the surface wavefront error as shown in Fig. 2.5. However, if the polynomials are not orthogonal to each other, the coefficients of the reference polynomials change as the number of polynomials increases. The reference determined by the M polynomial fit is the Mth relative reference. Mathematically, we define this as

$$\sum_{i=1}^N [W(x_i, y_i) - \sum_{j=1}^M A_j Z_j(x_i, y_i)]^2 = \min. \quad (2.21)$$

The best surface wavefront that has minimum rms value over the digitized points can be obtained by

$$\vec{S}_b = \sum_{i=1}^K (A_i - B_i) \vec{Z}_i + \sum_{j=K+1}^M A_j \vec{Z}_j. \quad (2.22)$$

The relative surface wavefront is

$$\vec{S}_r = \sum_{j=K+1}^M A_j \vec{Z}_j. \quad (2.23)$$

Fig. 2.5. Best reference and relative reference.

Then, which reference should be deleted in the polynomial fit of a circular interferogram with Zernike polynomials? Zernike polynomials are orthogonal over a circular aperture, but they are not orthogonal over the digitized points. Therefore, the two references are usually different. Deleting the best reference minimizes the surface wavefront over the digitized points, but deleting the relative reference minimizes the surface wavefront over the continuous aperture because of the orthogonality of the Zernike polynomials. The difference originates from the insufficient number of digitized points. We cannot determine which approach is better, but during the period of polishing and correcting, it would be better to minimize the surface wavefront over the continuous aperture. Then, overcorrection is less likely. However, if the polynomials are not orthogonal over the continuous aperture, the best reference should be deleted if enough points are digitized. This will be shown in the analysis of the non-circular aperture interferograms in the next chapter.

Advantages of Orthonormal Polynomials

If orthonormal polynomials are used, the normal equation is well-conditioned and the roundoff error is negligible. This roundoff error analysis will be given in the next chapter. Furthermore, the solution vector has some random error from the digitization error as will be shown in the next section. If non-orthonormal polynomials are used, as shown in Fig. 2.6, the error in the coefficient of each polynomial is magnified and can be larger than the error in the total polynomial fit. If we are interested in each coefficient separately, this magnification of error should be considered. However, the best reference wavefront and the best surface wavefront are orthogonal to each other, so the error magnification can be neglected if the best surface contour map is wanted. This advantage will be used in the next chapter for non-circular aperture interferogram analysis.

Usually, the Zernike polynomial is used in the polynomial fit of circular interferograms. However, the non-normal version of the Zernike polynomials has been used because of its familiarity. In this dissertation, the normalized version is used (see Table 2.1). Then, each coefficient is weighted uniformly to represent the rms error over the continuous aperture.

Now let us examine more carefully the non-orthogonal polynomials. Non-orthogonality means that the polynomials are correlated; in other words, one polynomial can be approximated with other polynomials. Therefore, this kind of correlated information is useless except for sub-aperture interferogram analysis. Suppose Zernike polynomials are used

Fig. 2.6. Why orthonormal polynomials?

Table 2.1. Normalized Zernike polynomials.

#	Polynomial	Wave Aberration Order
1	1	
2	$2R\cos\theta$	
3	$2R\sin\theta$	
4	$\sqrt{3} (2R^2-1)$	
5	$\sqrt{6} R^2\cos 2\theta$	
6	$\sqrt{6} R^2\sin 2\theta$	
7	$\sqrt{8} (3R^2-2)R\cos\theta$	
8	$\sqrt{8} (3R^2-2)R\sin\theta$	
9	$\sqrt{5} (6R^4-6R^2+1)$	
10	$\sqrt{8} R^3\cos 3\theta$	
11	$\sqrt{8} R^3\sin 3\theta$	
12	$\sqrt{10} (4R^2-3)R^2\cos 2\theta$	
13	$\sqrt{10} (4R^2-3)R^2\sin 2\theta$	
14	$\sqrt{12} (10R^4-12R^2+3)R\cos\theta$	
15	$\sqrt{12} (10R^4-12R^2+3)R\sin\theta$	
16	$\sqrt{7} (20R^6-30R^4+12R^2-1)$	
17	$\sqrt{10} R^4\cos 4\theta$	
18	$\sqrt{10} R^4\sin 4\theta$	
19	$\sqrt{12} (5R^2-4)R^3\cos 3\theta$	
20	$\sqrt{12} (5R^2-4)R^3\sin 3\theta$	
21	$\sqrt{14} (15R^4-20R^2+6)R^2\cos 2\theta$	
22	$\sqrt{14} (15R^4-20R^2+6)R^2\sin 2\theta$	
23	$4 (35R^6-60R^4+30R^2-4)R\cos\theta$	
24	$4 (35R^6-60R^4+30R^2-4)R\sin\theta$	
25	$3 (70R^8-140R^6+90R^4-20R^2+1)$	
26	$\sqrt{12} R^5\cos 5\theta$	
27	$\sqrt{12} R^5\sin 5\theta$	
28	$\sqrt{14} (6R^2-5)R^4\cos 4\theta$	
29	$\sqrt{14} (6R^2-5)R^4\sin 4\theta$	

Table 2.1.--Continued

#	Polynomial
30	$4 (21R^4 - 30R^2 + 10)R^3 \cos 3\theta$
31	$4 (21R^4 - 30R^2 + 10)R^3 \sin 3\theta$
32	$\sqrt{18} (56R^6 - 105R^4 + 60R^2 - 10)R^2 \cos 2\theta$
33	$\sqrt{18} (56R^6 - 105R^4 + 60R^2 - 10)R^2 \sin 2\theta$
34	$\sqrt{20} (126R^8 - 280R^6 + 210R^4 - 60R^2 + 5)R \cos \theta$
35	$\sqrt{20} (126R^8 - 280R^6 + 210R^4 - 60R^2 + 5)R \sin \theta$
36	$\sqrt{11} (252R^{10} - 630R^8 + 560R^6 - 210R^4 + 30R^2 - 1)$
37	$\sqrt{14} R^6 \cos 6\theta$
38	$\sqrt{14} R^6 \sin 6\theta$
39	$4 (7R^2 - 6)R^5 \cos 5\theta$
40	$4 (7R^2 - 6)R^5 \sin 4\theta$
41	$\sqrt{18} (28R^4 - 42R^2 + 15)R^4 \cos 4\theta$
42	$\sqrt{18} (28R^4 - 42R^2 + 15)R^4 \sin 4\theta$
43	$\sqrt{20} (84R^6 - 168R^4 + 105R^2 - 20)R^3 \cos 3\theta$
44	$\sqrt{20} (84R^6 - 168R^4 + 105R^2 - 20)R^3 \sin 3\theta$
45	$\sqrt{22} (210R^8 - 504R^6 + 420R^4 - 140R^2 + 15)R^2 \cos 2\theta$
46	$\sqrt{22} (210R^8 - 504R^6 + 420R^4 - 140R^2 + 15)R^2 \sin 2\theta$
47	$\sqrt{24} (462R^{10} - 1260R^8 + 1260R^6 - 560R^4 + 105R^2 - 6)R \cos \theta$
48	$\sqrt{24} (462R^{10} - 1260R^8 + 1260R^6 - 560R^4 + 105R^2 - 6)R \sin \theta$
49	$\sqrt{13} (924R^{12} - 2772R^{10} + 3150R^8 - 1680R^6 + 420R^4 - 42R^2 + 1)$

X-Y Power

1	1
2	$2R \cos \theta$
3	$2R \sin \theta$
4	$\sqrt{6} R^2 \cos 2\theta$
5	$\sqrt{6} R^2 \sin 2\theta$
6	$\sqrt{3} (2R^2 - 1)$
7	$\sqrt{8} (3R^2 - 2)R \cos \theta$
8	$\sqrt{8} (3R^2 - 2)R \sin \theta$

Table 2.1.--Continued

#	Polynomial
9	$\sqrt{8} R^3 \cos 3\theta$
10	$\sqrt{8} R^3 \sin 3\theta$
11	$\sqrt{10} (4R^2-3)R^2 \sin 2\theta$
12	$\sqrt{10} (4R^2-3)R^2 \sin 2\theta$
13	$\sqrt{10} R^4 \cos 4\theta$
14	$\sqrt{10} R^4 \sin 4\theta$
15	$\sqrt{5} (6R^4-6R^2+1)$
16	$\sqrt{12} (10R^4-12R^2+3)R \cos \theta$
17	$\sqrt{12} (10R^4-12R^2+3)R \sin \theta$
18	$\sqrt{12} (5R^2-4)R^3 \cos 3\theta$
19	$\sqrt{12} (5R^2-4)R^3 \sin 3\theta$
20	$\sqrt{12} R^5 \cos 5\theta$
21	$\sqrt{12} R^5 \sin 5\theta$
22	$\sqrt{14} (15R^4-20R^2+6)R^2 \cos 2\theta$
23	$\sqrt{14} (15R^4-20R^2+6)R^2 \sin 2\theta$
24	$\sqrt{14} (6R^2-5)R^4 \cos 4\theta$
25	$\sqrt{14} (6R^2-5)R^4 \sin 4\theta$
26	$\sqrt{14} R^6 \cos 6\theta$
27	$\sqrt{14} R^6 \sin 6\theta$
28	$\sqrt{7} (20R^6-30R^4+12R^2-1)$
29	$4 (35R^6-60R^4+30R^2-4)R \cos \theta$
30	$4 (35R^6-60R^4+30R^2-4)R \sin \theta$
31	$4 (21R^4-30R^2+10)R^3 \cos 3\theta$
32	$4 (21R^4-30R^2+10)R^3 \sin 3\theta$
33	$4 (7R^2-6)R^5 \cos 5\theta$
34	$4 (7R^2-6)R^5 \sin 5\theta$
35	$4 R^7 \cos 7\theta$
36	$4 R^7 \sin 7\theta$
37	$\sqrt{18} (56R^6-105R^4+60R^2-10)R^2 \cos 2\theta$
38	$\sqrt{18} (56R^6-105R^4+60R^2-10)R^2 \sin 2\theta$
39	$\sqrt{18} (28R^4-42R^2+15)R^4 \cos 4\theta$

Table 2.1.--Continued

#	Polynomial
40	$\sqrt{18} (28R^4 - 42R^2 + 15)R^4 \sin 4\theta$
41	$\sqrt{18} (8R^2 - 7)R^6 \cos 6\theta$
42	$\sqrt{18} (8R^2 - 7)R^6 \sin 6\theta$
43	$\sqrt{18} R^8 \cos 8\theta$
44	$\sqrt{18} R^8 \sin 8\theta$
45	$3 (70R^8 - 140R^6 + 90R^4 - 20R^2 + 1)$

in the polynomial fit and it is asked, "How much third order spherical aberration is in this interferogram?" The aberration coefficient changes as the number of Zernike polynomials used in the fit increases because third order spherical aberration is correlated with higher order spherical aberrations. If the number of polynomials is not specified, the coefficient cannot be determined and is useless. In other words, as far as testing is concerned, it would be better to represent the surface with orthogonal polynomials. This correlation of polynomials will be discussed later in determining the correlated polynomials in Chapter 3.

The order of Zernike polynomials is presented differently for different computer programs. In the computer program FRINGE written by Loomis (1976) the order of wavefront aberrations is determined by the program. This order includes more higher order terms of radially symmetric aberrations, whereas the x-y power order includes more higher order terms of azimuthally symmetric aberration. Empirically, the order

of wavefront aberration fits the optical surface well. The same order is chosen in this dissertation.

The condition number of a matrix represents the singularity of the matrix (Stewart, 1973, p. 190; Forsythe, Malcolm and Moler, 1977, pp. 41-47). A large condition number means large roundoff error (see Appendix A). Also, the condition number of the normal matrix represents the orthonormality of the polynomials. The condition numbers of the normal matrix from the aberration order fit and the x-y power order fit are compared for different number of polynomials in Fig. 2.7. Digitization over a 21x21 (317 points) and 41x41 (1257 points) grid was used. As shown in Fig. 2.7, more digitized points and fewer polynomials give a small condition number. The two different orders have almost the same condition number, so the orthogonality of the polynomials over the digitized points is about the same and the roundoff error is expected to be the same.

Digitization Error Projection

Suppose we have a random uncorrelated digitization error. Then the error vector can be defined from the N digitization error $E(x_i, y_i)$

$$\vec{E} = \frac{1}{\sqrt{N}} \{E(x_1, y_1)\hat{a}_1 + E(x_2, y_2)\hat{a}_2 + \dots + E(x_N, y_N)\hat{a}_N\} \quad (2.24)$$

and the size of the vector is the rms digitization error

$$|\vec{E}| = \left[\frac{1}{N} \sum_{i=1}^N E(x_i, y_i)^2 \right]^{1/2}. \quad (2.25)$$

Fig. 2.7. Comparison of condition number.

Let us find the expectation value of the digitization error from several measurements

$$\langle |\vec{E}|^2 \rangle = \left\langle \frac{1}{N} \sum_{i=1}^N E(x_i, y_i)^2 \right\rangle . \quad (2.26)$$

But, the digitization error is uncorrelated, so

$$\langle |\vec{E}|^2 \rangle = \frac{1}{N} \sum_{i=1}^N \langle E(x_i, y_i)^2 \rangle . \quad (2.27)$$

However, the error is random, so

$$\langle E(x_i, y_i)^2 \rangle = \sigma_E^2 \quad \text{for } i=1 \text{ to } N, \quad (2.28)$$

and therefore,

$$\langle |\vec{E}|^2 \rangle = \sigma_E^2 . \quad (2.29)$$

In other words, the digitization error can be represented with the N-dimensional sphere of radius σ_E .

Now let us consider the projection of the digitization error into the M-dimensional subspace S_z formed by the polynomials. In other words, let us find the contribution of the digitization error into the polynomial fit. We represent the N-dimensional data space with M orthonormal vectors that span the polynomial subspace S_z and N-M orthonormal vectors that are orthogonal to the polynomial subspace S_z . Then the digitization error vector can be divided into the digitization error vector fitted into the polynomials $\vec{E}_z$ and the vector orthogonal to it $\vec{E}_o$

$$\vec{E} = \vec{E}_z + \vec{E}_O \quad (\vec{E}_z \cdot \vec{E}_O = 0) \quad . \quad (2.30)$$

Then the expectation value of the fitted term is

$$\langle |\vec{E}_z|^2 \rangle = \left\langle \frac{1}{N} \sum_{i=1}^M e_i^2 \right\rangle \quad (2.31)$$

where e_i is the projection of a digitization error on the i th orthonormal polynomial in the polynomial space. Because of the orthogonality

$$\langle |\vec{E}_z|^2 \rangle = \frac{1}{N} \sum_{i=1}^M \langle e_i^2 \rangle \quad (2.32)$$

and because of the symmetry

$$\langle e_i^2 \rangle = \sigma_E^2 \quad , \quad (2.33)$$

therefore

$$\langle |\vec{E}_z|^2 \rangle = \frac{M}{N} \sigma_E^2 \quad . \quad (2.34)$$

The projected error is reduced by the factor of $\sqrt{M/N}$ as shown in Fig.

2.8. If the digitization error moves along the circle, the projection moves along a straight line AB. Therefore, the expectation value of the rms is decreased.

Let us consider what number of polynomials is enough. One criterion can be obtained if the noise level is known. Basically, we do not want to fit the digitization error more than permitted in the above estimation. Therefore, we can increase the number of polynomials until the rms of the fit error is equal to $\{(N-M)/N\}^{1/2}$ times the rms of the

Fig. 2.8. Reduction of the projection of the digitization error.

digitization error. An erroneous effect was reported in the polynomial fit with a larger number of polynomials by Underwood and Woodfin (1980).

An interferogram is made such that coma, astigmatism, and spherical aberration each introduce 1 rms wave error at best tilt and focus. Then, uniform random error between 0.2 and -0.2 waves is introduced and digitized over an 11x11 grid. The ratio of the digitization error to the surface figure in rms is

$$\frac{\sigma_E}{S_{\text{rms}}} = \frac{\sqrt{\frac{(0.2 \times 2)^2}{12}}}{\sqrt{1^2 + 1^2 + 1^2}}$$

$$= \frac{1}{15} \quad .$$

Eighty-one points are digitized. This interferogram is fitted with 9 (up to 4th order aberrations), 16 (up to 6th order aberrations), 25 (up to 8th order aberrations), and 37 (up to 10th order aberrations and 12th order spherical aberrations) polynomials. The contour maps of the surface figure error and those obtained from the difference between the correct

wavefront and the wavefront from each different order of fit are shown in Fig. 2.9.

If one polynomial contributes to the wavefront less than the order of the projected digitization error, we can delete this polynomial and add another polynomial without increasing the number of polynomials. As shown before the orthogonal contribution is

$$OC = A_i \sqrt{|\vec{Z}_i|^2}$$

and is explained in Fig. 2.10. In Table 2.2., from the above interferograms, the useless polynomials are determined by

$$\text{Orthogonal Contribution} < \sqrt{\frac{M}{N}} \sigma_E$$

and the overfit is determined by

$$\text{RMS Fit Error} < \sqrt{\frac{N-M}{N}} \sigma_E ,$$

where N is the number of digitized points, M is the number of polynomials, and σ_E is the digitization error in rms. This is a somewhat arbitrary criterion, and a safety constant factor can be multiplied.

Only the 9th and 16th order fits are not overfitted. We can find which polynomials should be deleted for the overfitted cases from the size of the orthogonal contribution of each polynomial. Because we want to estimate the continuous surface from the finite digitized data, the erroneous effect is more serious for undersampled interferograms. This will be discussed later.

Figure 2.9. Comparison between different orders of fit.

- (a) Surface figure error.
- (b) Differences between the surface figure error and several different order fits.

Fig. 2.10. Orthogonal contribution of a polynomial.

Table 2.2. Useless polynomials and overfit.

Order of Fit	Useless Polynomials	Overfit
9	None	No
16	z_{12}	No
25	z_{12}, z_{18}	Yes
37	$z_{12}, z_{18}, z_{20}, z_{29}, z_{37}$	Yes

Comparison of Zernike Polynomials
and X-Y Polynomials

Now, let us compare Zernike polynomials and X-Y polynomials in analyzing circular interferograms. The effects of roundoff and digitization error are emphasized and a 41x41 grid and 1500 random points are used to neglect the non-orthogonality of Zernike polynomials over the digitized points. In other words, we compare a set of orthonormal polynomials and a set of non-orthonormal polynomials. To compare them, the Zernike polynomials are arranged in x-y power order rather than wave aberration order in this section. Forty-five x-y polynomials up to the eighth order are used in the fit as shown in Table 2.3. In Table 2.4, the random digitization is worse than the uniform grid digitization and the x-y polynomial fit is seriously ill-conditioned. The same interferogram composed with coma, astigmatism, and spherical aberration is used in Table 2.5, where only the error from the 41x41 grid digitization is compared. Uniform random error between 0.2 and -0.2 waves is added for the digitization error simulation. An Eclipse A/130 minicomputer was used for simulations in this dissertation and its calculation accuracy is 6 hexadecimal digits (7.2 decimal digits). As shown in Table 2.5, the error in the coefficients of the x-y polynomials is large, even without digitization error, but it represents the wavefront almost correctly as shown in Table 2.6. The error comes from the correlation of x-y polynomials. The roundoff error increases if a less non-orthogonal polynomial is used. The condition number represents the non-orthogonality. However, this degree of ill-conditioning of the polynomials introduces only small errors in the contour map even if the coefficients are completely

Table 2.3. X-Y polynomials.

#		#	
1	1	24	X^4Y^2
2	X	25	X^3Y^3
3	Y	26	X^2Y^4
4	X^2	27	XY^5
5	XY	28	Y^6
6	Y^2	29	X^7
7	X^3	30	X^6Y
8	X^2Y	31	X^5Y^2
9	XY^2	32	X^4Y^3
10	Y^3	33	X^3Y^4
11	X^4	34	X^2Y^5
12	X^3Y	35	XY^6
13	X^2Y^2	36	Y^7
14	XY^3	37	X^8
15	Y^4	38	X^7Y
16	X^5	39	X^6Y^2
17	X^4Y	40	X^5Y^3
18	X^3Y^2	41	X^4Y^4
19	X^2Y^3	42	X^3Y^5
20	XY^4	43	X^2Y^6
21	Y^5	44	XY^7
22	X^6	45	Y^8
23	X^5Y		

Table 2.4. Comparison of condition number (Zernike and X-Y).

Digitization	Zernike	X-Y
41 by 41 grid	1.78	6.1×10^6
1500 random	3.19	1.1×10^7
1500 random	3.42	9.7×10^6

Table 2.5. Error in the coefficients.

#	Zernike Polynomial			X-Y Polynomial		
	Input	Output Without Digiti- zation Error	Output With Digiti- zation Error	Input	Output Without Digiti- zation Error	Output With Digiti- zation Error
1	0.0	-.000	.001	2.236	2.176	2.173
2	0.0	-.000	.003	-4.0	-4.198	-4.063
3	0.0	.000	.005	-4.0	-4.122	-4.078
4	0.0	.000	-.003	-11.685	-10.358	-10.016
5	.707	.707	.701	3.464	3.061	2.944
6	.707	.707	.706	-15.149	-13.953	-13.962
7	.707	.707	.707	6.0	7.126	6.256
8	.707	.707	.706	6.0	7.057	6.995
9	0.0	-.000	.000	6.0	8.487	7.993
10	0.0	-.000	-.003	6.0	6.782	6.524
11	1.0	1.000	1.001	13.417	7.051	5.299
12	0.0	.000	.001	0.0	1.779	1.736
13	0.0	.000	.003	26.833	15.339	14.116
14	0.0	.000	.006	0.0	3.191	3.665
15	0.0	.000	.001	13.417	8.034	8.177
16	0.0	.000	.004	0.0	-1.882	-.115
17	0.0	.000	.001	0.0	-2.162	-2.690
18	0.0	.000	-.004	0.0	-7.954	-6.731
19	0.0	.000	-.004	0.0	-3.799	-1.806
20	0.0	.000	.009	0.0	-5.773	-5.285
21	0.0	.000	.004	0.0	-1.413	-1.109
22	0.0	.000	.002	0.0	10.363	13.209
23	0.0	.000	.004	0.0	-2.390	-2.229
24	0.0	.000	-.002	0.0	27.399	30.128
25	0.0	.000	-.005	0.0	-9.254	-9.372
26	0.0	.000	.005	0.0	26.982	27.522
27	0.0	.000	.003	0.0	-6.093	-6.210
28	0.0	.000	.001	0.0	8.339	8.382
29	0.0	.000	-.003	0.0	.969	-.072
30	0.0	.000	-.001	0.0	1.259	1.819
31	0.0	.000	-.004	0.0	5.868	4.668
32	0.0	.000	.000	0.0	3.522	2.557
33	0.0	.000	-.001	0.0	8.459	7.992
34	0.0	.000	.006	0.0	3.032	.486
35	0.0	.000	-.002	0.0	3.603	3.727
36	0.0	.000	-.004	0.0	.770	.763
37	0.0	.000	-.003	0.0	-5.370	-6.828
38	0.0	.000	-.001	0.0	1.002	1.010
39	0.0	.000	.000	0.0	-18.563	-20.194

Table 2.5.--Continued

#	Zernike Polynomial			X-Y Polynomial		
	Input	Output Without Digiti- zation Error	Output With Digiti- zation Error	Input	Output Without Digiti- zation Error	Output With Digiti- zation Error
40	0.0	.000	-.001	0.0	6.275	7.015
41	0.0	.000	.001	0.0	-27.293	-28.261
42	0.0	.000	-.000	0.0	8.509	7.305
43	0.0	.000	-.000	0.0	-17.970	-17.459
44	0.0	.000	-.003	0.0	3.368	3.165
45	0.0	.000	-.003	0.0	-4.154	-4.359

Table 2.6. Comparison of roundoff error.

	PTS	AV	RMS	MAX	MIN	SPAN
over 41 by 41 grid	1257	0.0017	1.735	7.182	-2.800	9.982
roundoff error (Zernike)	1257	0.0000	0.0001	0.0003	-0.0002	0.0004
roundoff error (X-Y)	1257	0.0002	0.0288	0.1192	-0.0581	0.1773
over 1500 random points	1500	0.0004	1.685	6.888	-2.794	9.682
roundoff error (Zernike)	1500	0.0000	0.0001	0.0003	-0.0002	0.0005
roundoff error (X-Y)	1500	0.0003	0.0740	0.3111	-0.1704	0.4815
over 1500 random points	1500	0.0052	1.705	7.024	-2.803	9.827
roundoff error (Zernike)	1500	0.0000	0.0001	0.0003	-0.0002	0.0004
roundoff error (X-Y)	1500	0.0004	0.0596	0.2995	-0.1366	0.4361

different. A more detailed analysis will be given in Chapter 3. The random digitization is nearly as accurate as the grid digitization. However, digitization error is inevitable in the polynomial fit. The statistics of errors in the contour maps after adding a uniform random error between 0.2 and -0.2 waves are compared in Table 2.7. Now the difference is negligible. Contour maps of errors from a 41x41 grid digitization are compared in Fig. 2.11.

A large reference aberration like tilt, when introduced into the reference wavefront, increases the roundoff error because it increases the rms of the total wavefront. But by deleting the best reference first, this effect can be neglected. The best reference can be obtained accurately even with non-orthogonal polynomials. The projection of the digitization error on the polynomial space does not depend on the condition number. Therefore, a large condition number can be allowed if the roundoff error projection is smaller than the projection of the digitization error. In the above example, the roundoff error from an x-y polynomial fit is greater than the projection of the digitization error. The projection of the roundoff error will be estimated in the next chapter.

Error Introduced by Use of Finite Number of Digitized Points

To estimate the continuous surface wavefront correctly, the sampled wavefront should be similar to the continuous wavefront. What kind of criterion can be used? The statistics of good digitization should not be affected by more digitized points. So, the rms of the

Table 2.7. Comparison of error with random digitization error.

	PTS	AV	RMS	MAX	MIN	SPAN
over 41 by 41 grid	1257	0.0026	1.736	7.067	-2.882	9.949
error (Zernike)	1257	-0.0009	0.0218	0.1169	-0.1017	0.2187
error (X-Y)	1257	-0.0007	0.0366	0.1938	-0.0825	0.2763

Fig. 2.11. Comparison of error with and without digitization error.

sampled wavefront should be the same as the rms of the continuous wavefront at least. Let us estimate the difference. Suppose a continuous surface is represented with M polynomials excluding the reference polynomials. Also, let us assume the polynomials are orthonormal over the continuous surface. Then, the surface wavefront is

$$\vec{S} = \sum_{i=1}^M A_i \vec{Z}_i \quad (2.35)$$

and the rms of the surface wavefront over the continuous surface is

$$\sigma_c^2 = \sum_{i=1}^M A_i^2 \quad (2.36)$$

The rms of the surface wavefront over the digitized points can be obtained with the orthonormal polynomials over the digitized points. From Eqs. (2.17) and (2.18), the coefficients A_i' of the orthogonal polynomial can be found, but they are not normalized. Therefore, the coefficients A_i'' of the orthonormalized polynomials over the digitized points are

$$\begin{pmatrix} A_1'' \\ A_2'' \\ \vdots \\ A_M'' \end{pmatrix} = \begin{pmatrix} |\vec{Z}_1'| & & 0 \\ & |\vec{Z}_2'| & \\ & & \ddots \\ 0 & & & |\vec{Z}_M'| \end{pmatrix} \begin{pmatrix} A_1' \\ A_2' \\ \vdots \\ A_M' \end{pmatrix} \quad (2.37)$$

Then, the matrix equation between A_i and A_i'' is

$$\begin{pmatrix} A_1'' \\ A_2'' \\ \vdots \\ A_M'' \end{pmatrix} = H \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_M \end{pmatrix} . \quad (2.38)$$

The rms of the surface wavefront over the digitized points is

$$\sigma_d^2 = \sum_{i=1}^M A_i''^2 . \quad (2.39)$$

Now we can estimate the magnification of σ_c/σ_d with the singular value analysis as given in Appendix A.

Results are shown in Fig. 2.12. Three reference polynomials, a constant, and x-y tilt, are deleted in estimating the magnification. The singular values are plotted in increasing order. The average magnification factors are shown on each figure. In Fig. 2.12a, the dependence on the size of the grid (number of points) is shown. The results from three random digitizations of 149 points are shown in Fig. 2.12b. Compare the results with those from a 15x15 grid digitization over the same 149 points in Fig. 2.12a. The dependence on the number of polynomials is shown in Fig. 2.12c.

If we can digitize more closely along the fringe, we can increase the number of digitized points, and thereby reduce the contribution of digitization error and provide more surface figure information. However, this kind of non-uniform digitization was considered to degrade the orthogonality of the Zernike polynomials. Therefore we avoided this kind of digitization. However, as shown above, the non-orthogonality is not the real problem in the polynomial fit, but rather, the sampled data

(a)

Fig. 2.12. Magnification factors of σ_c/σ_d .

- (a) Dependence on the grid size.
 34 surface polynomials from Z_4 to Z_{37}
 were used.

(b)

Fig. 2.12.--Continued

(b) Random digitization.
 34 surface polynomials from Z_4 to Z_{37} were used.

(c)

Fig. 2.12.--Continued

(c) Dependence on the number of polynomials.
 A 31 by 31 grid digitization was used. The
 number of polynomials includes 3 reference
 polynomials.

should be similar to the continuous wavefront in rms. Some non-uniform digitizations are compared to uniform digitization in Fig. 2.13 and Table 2.8. As shown in Table 2.8, the 41x11 grid digitization is better conditioned than the 11x11 grid digitization. In other words, it is more orthogonal. For the third order spherical aberration of 1 wave rms at best focus, the sampled rms is given in Table 2.9. In conclusion, if we have about 10 fringes, we can digitize at an interval equal to about $\frac{1}{4}$ or smaller the average fringe spacing.

Fit Error Projection

The continuous wavefront can be represented with an infinite number of Zernike polynomials. However, only M polynomials are used in the polynomial fit

$$W_c = \sum_{i=1}^M A_i Z_i + \sum_{i=M+1}^{\infty} A_i Z_i \quad (2.40)$$

Ideally, the fit error term includes an infinite number of Zernike polynomials. However, the measurement error is inevitable and we are interested only within this error limit. Therefore, if enough points are digitized, the sampled wavefront can be estimated by

$$W_s = \sum_{i=1}^M A_i Z_i + \sum_{i=M+1}^K A_i Z_i + E \quad (2.41)$$

where E is still almost random digitization error and K is usually much less than the number of points if the surface is smooth. The digitization error was considered above, so, let us neglect this term.

Fig. 2.13. Comparison between uniform digitization and non-uniform digitizations.

34 surface polynomials from Z_4 to Z_{37} were used.

Table 2.8. Comparison of condition number (uniform and non-uniform).

Grid	PTS	Condition Number
11 by 11	81	2.08x10
41 by 11	311	5.69
101 by 11	763	6.55

Table 2.9. Sampled rms value of spherical aberration of 1 wave rms at best focus.

Grid	rms (waves)
11 by 11	1.179
21 by 11	1.086
41 by 11	1.025
101 by 11	0.985

The second term is the effective fit error. However, the effective fit error is not orthogonal to the polynomial space over the digitized points, even if it is orthogonal over the continuous aperture. Therefore, its projection into the polynomial space introduces some error in the coefficients

$$W_s = \sum_{i=1}^M (A_i + \Delta A_i) Z_i + F, \quad (2.42)$$

where F represents the fit error over the digitized points. However, if we are interested in the orthogonal components of the polynomial fit, such as the best reference wavefront and the best surface wavefront, the errors introduced on them are usually much less than the effective fit error because the effective fit error has a small projection to the polynomial space and the projection of fit error is not magnified. However, if we are interested in correlated information, the projection of fit error can be magnified considerably. This will be explained in the sub-aperture interferogram analysis in Chapter 4.

Average of Several Interferograms

The averaging of several interferograms has been used to decrease the contribution from the digitized error and to obtain more accurate surface wavefront information. However, as shown in the previous section, if not enough points are digitized, the surface wavefront information obtained from the polynomial fit does not represent the real continuous surface wavefront. In this case, the average of the coefficient is not the average of similar terms. Therefore, if we want to decrease the contribution of the digitization error by an average, many points should be

digitized and then the coefficients can be averaged. However, we cannot increase the number of digitized points for a conventional digitization technique like the graphics tablet because a large tilt should be introduced to increase the digitized points which would degrade the measurement accuracy. The average of the under-sampled interferograms will be given in Chapter 5.

Edge Effect

As shown previously, the contour maps obtained from a Zernike polynomial fit have high frequency fluctuation at the edge and appear to be incorrect. The reason is that the higher order polynomials have a fast slope at the edge. Therefore, even if the fit error is not large at the edge for uniform digitization, the contour map may deviate significantly at the edge. More dense digitization at the edge has been used at the Optical Sciences Center to control the edge effect. The results are compared in Fig. 2.14. The interferogram in Fig. 2.1 is digitized once including the edge and then excluding the points at the edge. The digitization is compared in Fig. 2.14a. The contour maps from the 37 polynomial fit is compared in Fig. 2.14b. The other approach is to adjust the Zernike circle larger than the circular aperture of the interferogram. This technique will be discussed for non-circular aperture analysis in Chapter 3.

(a)

(b)

Fig. 2.14. Edge effect.

- (a) Verification of digitized points.
- (b) Contour maps from Zernike polynomial fit.

CHAPTER 3

NON-CIRCULAR APERTURE INTERFEROGRAM AND AXICON INTERFEROGRAM ANALYSIS

The Zernike polynomial has been used for the analysis of circular aperture interferograms because of its orthogonality over a circular aperture. But what kind of polynomials should be used for the analysis of non-circular aperture interferograms. Do we have to use a set of orthogonal polynomials? This approach was used by Swantner and Lowrey (1980) for the analysis of annular aperture interferograms. However, each set of orthogonal polynomials is different for different types of apertures. Also, it is not easy to interpret the polynomial.

In Chapter 2 it was shown that a set of x-y polynomials can be used for the analysis of circular aperture interferograms. Therefore, the same Zernike polynomials can be used to analyze non-circular aperture interferograms. However, roundoff error from an ill-conditioned matrix limits the use of Zernike polynomials. Therefore, in this chapter we estimate this error for some aperture shapes and introduce methods that can reduce the roundoff error. We also solve the problem of the ill-conditioned matrix for the analysis of axicon interferograms and use a set of orthogonal polynomials for the axicon test.

Projection of Roundoff Error

The normal equation approach involves solving a matrix equation

$$P A = Q \quad . \quad (3.1)$$

P is an M by M matrix calculated from matrix Z by

$$P = Z^t Z$$

where

$$Z = \frac{1}{\sqrt{N}} \begin{pmatrix} Z_1(x_1, y_1) & Z_2(x_1, y_1) & \dots & Z_M(x_1, y_1) \\ Z_1(x_2, y_2) & Z_2(x_2, y_2) & \dots & Z_M(x_2, y_2) \\ \vdots & & & \\ Z_1(x_N, y_N) & Z_2(x_N, y_N) & \dots & Z_M(x_N, y_N) \end{pmatrix}$$

and Q is an M by 1 matrix calculated from matrix Z and W

$$Q = Z^t W$$

where

$$W = \frac{1}{\sqrt{N}} \begin{pmatrix} W(x_1, y_1) \\ W(x_2, y_2) \\ \vdots \\ W(x_N, y_N) \end{pmatrix} .$$

M is the number of polynomials, and N is the number of digitized points. The calculated solution A^* is not the same as the real solution owing to the roundoff error. This error was estimated as shown in Appendix A where

$$\frac{||Q-PA^*||}{||P|| ||A^*||} < \gamma \beta^{-t} \quad (3.2)$$

$$\frac{||A-A^*||}{||A^*||} < \text{COND}(P) \gamma \beta^{-t} \quad (3.3)$$

The factor β is the base, t is the number of digits and γ is a number on the order of β . The term $||A-A^*||$ estimates only the error in the coefficient and is the square root of the sum of squares of the errors in the coefficients. The term $||Q-PA^*||$ is the square root of the sum of squares of the projections of the error to each polynomial because

$$\begin{aligned} ||Q-PA^*|| &= \left(\sum_{i=1}^M [(\vec{W} - \sum_{j=1}^M A_j^* \vec{Z}_j) \cdot \vec{Z}_i]^2 \right)^{\frac{1}{2}} \\ &= \left[\sum_{i=1}^M (\Delta \vec{Z} \cdot \vec{Z}_i)^2 \right]^{\frac{1}{2}} \\ &= \left[\sum_{i=1}^M (|\Delta \vec{Z}| \cos \theta_i |\vec{Z}_i|)^2 \right]^{\frac{1}{2}}, \end{aligned} \quad (3.4)$$

where $\Delta \vec{Z}$ is the error in the polynomial representation from the roundoff error. This is shown in Fig. 3.1. However, what is important is the ratio of the rms error of the total wavefront to the rms of the total wavefront. Let us consider $|\vec{Z}|^2$

Fig. 3.1. $||A-A^*||$ and $||Q-PA^*||$.

$$\begin{aligned}
|\vec{Z}|^2 &= \left(\sum_{j=1}^M A_j \vec{Z}_j \right) \cdot \left(\sum_{k=1}^M A_k \vec{Z}_k \right) \\
&= \sum_{j=1}^M A_j \left(\sum_{k=1}^M A_k \vec{Z}_j \cdot \vec{Z}_k \right) \\
&= \sum_{j=1}^M A_j \left(\sum_{k=1}^M A_k \left[\frac{1}{N} \sum_{i=1}^N Z_j(x_i, y_i) Z_k(x_i, y_i) \right] \right) \\
&= \sum_{j=1}^M A_j \left(\sum_{k=1}^M A_k [(Z^t Z)_{jk}] \right) \\
&= \sum_{j=1}^M A_j \left(\sum_{k=1}^M A_k (Z^t Z)_{jk} \right) \\
&= \sum_{j=1}^M A_j (Z^t Z A)_j \\
&= A^t Z^t Z A \\
&= A^t P A \\
&= ||A^t P A|| \quad (\text{constant}) \quad .
\end{aligned} \tag{3.5}$$

In the same way,

$$\begin{aligned}
|\Delta \vec{Z}|^2 &= ||\Delta A^t P \Delta A|| \\
&< ||\Delta A^t|| ||P \Delta A|| \quad (\text{see Appendix A}) \\
&< \text{cond}(P) ||P|| ||A||^2 (\gamma \beta^{-t})^2 \quad (\text{from Eqs. 3.2 and 3.3})
\end{aligned} \tag{3.6}$$

where $\Delta A = A - A^*$. On the other hand, as shown in Appendix A

$$\frac{||A||^2}{||P^{-1}||} < ||A^t PA|| < ||P|| ||A||^2 \quad (3.7)$$

Therefore, from Eqs. (3.6) and (3.7) and the definition of the condition number

$$\frac{|\vec{\Delta Z}|}{|\vec{Z}|} < \text{COND}(P) \gamma \beta^{-t} \quad (3.8)$$

However, Eq. (3.8) is usually overestimated and the rms of the digitized data $|\vec{Z}|$ is usually calculated in the analysis program. Therefore, $|\vec{Z}|$ can be compared with $||A||$, which is the square root of the sum of the squares of coefficients. The resulting ratio depends on the wavefront. If this ratio is large, the roundoff error can be small because from Eq. (3.6)

$$\frac{|\vec{\Delta Z}|}{|\vec{Z}|} < \frac{(\text{COND}(P) ||P||)^{\frac{1}{2}} \gamma \beta^{-t}}{|\vec{Z}| / ||A||} \quad (3.9)$$

This relationship is explained graphically in Fig. 3.2.

Fig. 3.2. Dependence of roundoff error on wavefront.

As shown in Chapter 2, if we are interested in uncorrelated information like the best reference and best surface wavefront, we can use a non-orthogonal polynomial set if the roundoff error is not large. Because digitization error is inevitable in optical testing, a non-orthogonal polynomial set can be used in the polynomial fit if the roundoff error is not large compared to the projection of the digitization error. However, a large reference tilt introduced in the test increases the roundoff error because, from Eq. (3.6)

$$|\Delta \vec{Z}| < (\text{Cond}(P) ||P||)^{\frac{1}{2}} ||A|| \gamma \beta^{-t}, \quad (3.10)$$

and $||A||$ becomes large owing to the large reference tilt. As will be shown later, the condition number of the matrix for the best reference is small. Therefore, the best reference can be calculated accurately even with non-orthonormal polynomials. Consequently, we can delete this best reference without any loss of accuracy. Therefore, a large reference tilt does not degrade accuracy.

Reduction of Roundoff Error

The main source of the ill-conditioned matrix is the fact that the higher order polynomials can be represented with lower order polynomials; in other words, they are correlated. These correlated polynomials can be deleted from the fit without any loss of accuracy. As shown in Fig. 3.3, if the orthogonal projection of a polynomial is, say, 1/20, we can delete this polynomial with no more than a 5% loss of accuracy. Also, we can delete polynomials whose orthogonal contribution

Fig. 3.3. Correlated polynomial.

is small compared to the projection from the digitization error as shown in Chapter 2. In any case, the normal equation approach is not a stable method. The reason is that the normal matrix P is more ill-conditioned than the matrix Z , as shown in Appendix A

$$\text{COND}(P) = \text{COND}^2(Z) \quad . \quad (3.11)$$

Therefore, the stable algorithm does not use the normal matrix P but Z . The stable algorithm finds the solution A that minimizes the residual of $ZA \cong W$ by orthogonal decomposition. The N by M matrix Z can be orthogonally decomposed by (Lawson and Hanson, 1974, pp. 5-11; Stewart, 1973, pp. 214-215)

$$Z = QR \quad (3.12)$$

where Q is an N by N orthogonal matrix (see Appendix A) and R is an N by M upper triangular matrix

$$R = \begin{pmatrix} R_{11} & R_{12} & \cdot & \cdot & \cdot & R_{1M} \\ 0 & R_{22} & \cdot & \cdot & \cdot & R_{2M} \\ 0 & 0 & R_{33} & \cdot & \cdot & R_{3M} \\ \vdots & \vdots & & & & \vdots \\ 0 & 0 & \cdot & \cdot & \cdot & R_{MM} \\ \vdots & \vdots & & & & \vdots \\ \vdots & \vdots & & & & \vdots \\ 0 & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix}$$

Then, the residual becomes

$$\begin{aligned} \gamma &= ||ZA-W|| \\ &= ||QRA-W|| \\ &= ||RA-Q^t W|| \quad (QQ^t = I) \quad . \end{aligned} \quad (3.13)$$

However, only the first M coefficients of RA are nonzero. Therefore, the solution A that minimizes the residual can be obtained by solving this upper M by M triangular matrix

$$(RA)_M = (Q^t W)_M \quad , \quad (3.14)$$

where $()_M$ represents the matrix of upper M rows. Then, the minimized residual is

$$\gamma_{\min} = \left(\sum_{i=M+1}^N (Q^t W)_i^2 \right)^{\frac{1}{2}} \quad . \quad (3.15)$$

The orthogonal decomposition of Z can be done by several different methods; the modified Gram-Schmidt orthogonalization, the Householder transformation, etc. (Stewart, 1973, pp. 230-242; Lawson and Hanson, 1974, pp. 9-22). Calculating the orthogonal polynomials one by one directly from the matrix Z is more stable than the normal equation approach which calculates the orthogonal polynomials indirectly from the normal matrix. Therefore, the stable method is slower even if more accurate (Lawson and Hanson, 1974, p. 122). The orthogonal polynomials obtained from the stable methods are more orthogonal for an ill-conditioned matrix Z . However, to calculate the i th orthogonal polynomial we have to know the orthogonal polynomials from the 1st to the $i-1$ th polynomial. Therefore, the stable method requires an N by M computer memory for the matrix Z and will be overwritten by the first M columns of orthogonal matrix Q . These M columns are enough to obtain the solution A , as shown in Eq. (3.14). However, for a typical interferogram analysis, the number of digitized points is a few hundred and the number of polynomials is about 40. This requires too much memory. Without using large memory, we can calculate the orthogonal polynomials each time they are required by using the matrix R and the digitized data, but this method requires more calculation and needs digitized data each time in calculating another orthogonal polynomial. Another approach is to process an order of M digitized data at a time sequentially (Lawson and Hanson, 1974, pp. 207-212). This approach requires about 2 times more calculation, but does not require large memory.

We can also reduce the roundoff error of the normal equation approach by using double precision calculations. Another approach is to use a set of orthogonal polynomials, which allows us to reduce the condition number. However, a different set of orthogonal polynomials has to be found for each different application and it is not easy to interpret the sets. Also, the orthogonal polynomials over the continuous aperture do not guarantee a small condition number for the under-sampled or seriously nonuniformly digitized interferograms.

Non-Circular Aperture Interferogram

We compared the condition numbers of several different aperture shapes. Two annular shapes with obstruction ratios of 0.25 and 0.5, an equilateral triangle, and a square were compared with a circle. Previously, a set of orthogonal polynomials for an annular shape was used by Swantner and Lowrey (1980) in the polynomial fit. However, we used the same Zernike polynomials in the polynomial fit because it can be done with minor modification.

We arranged the Zernike circle to circumscribe the non-circular aperture. However, this is not the optimum position and the resulting condition number can be decreased less. In each case, 37 polynomials were used. As shown here, they included up to 10th order aberrations and 12th order spherical aberration. If fewer polynomials are used, the condition number can be decreased. Each aperture was digitized over 11 by 11, 21 by 21, 31 by 31, 41 by 41 square grids. The Zernike circle, which inscribed the square grid, was digitized over 500 random points

five times. The distribution of the condition numbers is shown in Fig. 3.4. The average value is used for random digitization. The fluctuation of the condition number for random digitization is shown in Fig. 3.4b.

The condition number for calculating the best reference is shown in Table 3.1. It is apparent that the roundoff error is negligible when the best reference is calculated. The serious ill-conditioning of the annular shape with an obstruction ratio of 0.5 is due to the correlation of the higher order spherical aberrations. If the higher order spherical aberration terms are deleted in the fit, the condition number can be decreased. For the triangular shape, the source is not evident. But, the orthogonal projection of each polynomial is calculated and is used in determining the correlated polynomials in the fit. The orthogonal projection is calculated by

$$\begin{aligned}
 OP_i &= \frac{|\vec{Z}_i'|}{|\vec{Z}_i|} \\
 &= \frac{(\vec{Z}_i' \cdot \vec{Z}_i')^{1/2}}{(\vec{Z}_i \cdot \vec{Z}_i)^{1/2}} = \frac{(\vec{Z}_i' \cdot \vec{Z}_i)^{1/2}}{(\vec{Z}_i \cdot \vec{Z}_i)^{1/2}} \quad . \quad (3.16)
 \end{aligned}$$

The correlated polynomials can be determined without phase information, in contrast to the determination of the useless polynomials by the orthogonal contribution in Chapter 2. These correlated polynomials may contribute much more than the projection of digitization error. However, the error from deleting a correlated polynomial in the fit is smaller

(a)

Vertical bar at 500 random digitization represents the rms of the fluctuation in 3.4b.

(b)

Fig. 3.4. Distribution of condition numbers.

- (a) Condition numbers of different apertures.
- (b) Fluctuation of condition numbers.

Table 3.1. Condition number for reference.

Aperture Shape	Condition Number
circular	1.01
annular (0.25)	1.31
annular (0.5)	3.15
square	1.01
triangle	1.92

A 31 by 31 grid digitization and constant, x-y tilt reference were used.

than the ratio of the orthogonal projection as explained before. Also, correlated polynomials can be calculated easily during the normal equation approach. Correlated polynomials are found from the above criterion, and the effect of deleting them in the fit is shown in Table 3.2. As shown in the table, the condition number can be decreased considerably by deleting only one or two polynomials for a severely ill-conditioned case. The small decrease for the triangle is due to other correlated polynomials. Because these polynomials are correlated, they are usually useless. In conclusion, we can use Zernike polynomials for non-circular aperture interferogram analysis.

As explained above, Eq. (3.8) is overestimated, but, usually Eq. (3.6) is acceptable. However, P_{11} is equal to one because

$$\begin{aligned} P_{11} &= \frac{1}{N} \sum_{i=1}^N Z_1(x_i, y_i)^2 \\ &= 1 \quad (\text{because } Z_1=1) \end{aligned} \quad (3.17)$$

Also, the norm of the matrix P is almost of the order of P_{11} as shown in Fig. 3.5. The term $\frac{||P||}{P_{11}}$ is shown in Fig. 3.5. It does not deviate from 1 even if the condition number is as large as 10^7 . Let us define R as

$$R = \frac{||A||^2}{|\vec{Z}|^2} \quad (3.18)$$

and use the approximation of $||P|| \cong 1$. Then, from Eq. (3.9) we have

Table 3.2. Correlated polynomials and condition number reduction.

<u>Orthogonal Projection</u>		
<u>Aperture Shape</u>	<u>Order of Polynomial</u>	<u>Orthogonal Projection</u>
annular (0.5)	36	0.251
	37	0.174
triangle	29	0.098
	31	0.100

<u>Reduction of Condition Number</u>		
<u>Aperture Shape</u>	<u>Orders of Deleted Polynomials</u>	<u>Condition Number</u>
annular (0.5)	None	3.29×10^4
	37	3.65×10^3
	36 and 37	5.12×10^2
triangle	None	1.31×10^5
	29	7.74×10^4
	29 and 31	7.46×10^4

A 41 by 41 grid digitization was used.

Fig. 3.5. $\frac{\|P\|}{P_{11}}$ from different apertures.

$$\frac{|\Delta \vec{Z}|}{|\vec{Z}|} < (\text{COND}(P)R)^{\frac{1}{2}} \gamma \beta^{-t} \quad . \quad (3.19)$$

Some results from the roundoff error simulation are shown in Fig. 3.6a. Figure 3.6a is redrawn on this basis including the R factor in Fig. 3.6b. As shown in the figure, this factor is better for estimating the roundoff error.

To reduce the edge effect, the oversized Zernike circle can be used for circular interferograms. Its condition number is shown in Table 3.3. The edge effect is due to under sampling at the edge. Uniform digitization of a circular interferogram is not enough at the edge because of the fast slope of the higher order Zernike polynomials. Therefore, by deleting the fast slope information of the Zernike polynomials at the edge, we can decrease the error from the finite digitization. Now, the continuous wavefront over the aperture is more similar to the finite wavefront over the digitized data (see Fig. 3.7). The radius of the clear aperture is adjusted to 1.0, 0.95, and 0.90 of the Zernike radius and a 21x21 grid digitization over the Zernike circle is assumed. The continuous wavefronts of the 0.95 and 0.90 radius clear aperture were approximated by a 151x151 grid digitization over the Zernike circle. Thirty-seven polynomials including three reference polynomials, a constant, and x-y tilt, are used in the fit. As the radius decreases, the error from the finite digitization decreases, but the roundoff error increases owing to the large condition number. However, the above condition number is acceptable, so we can use about a 0.9

The ratio is the rms surface error from the roundoff error to the rms surface wavefront.

Fig. 3.6. Results from roundoff error simulation.

- (a) $\Delta S/S$ vs. condition number.
- (b) $\Delta S/S$ vs. $(\text{condition number} \times R)^{\frac{1}{2}}$.

Table 3.3. Condition number of oversized Zernike circle.

Radius	Number of Points	Condition Number
.9	1009	3.04×10^3
.95	1129	8.71×10
1.00	1257	1.78

A 41 by 41 grid digitization over the Zernike circle was used.

Fig. 3.7. Finite digitization error from several different radius apertures.

Thirty-four subspace polynomials from Z_4 to Z_{37} were used.

clear aperture for circular aperture interferograms with a few hundred digitized points.

To compare the effect, the same digitized data including the edge from Fig. 2.14a are used, but the Zernike circle is adjusted such that the 0.9 radius of the Zernike circle covers the aperture of the interferogram. The contour map is shown in Fig. 3.8. Only a 0.9 radius of the circle covers the aperture of the interferogram. Compare it with those in Fig. 2.14b. The edge is somewhat smoother.

Because we are not interested in correlated information, we do not need a stable approach for non-circular aperture interferogram analysis. Also, for usual null interferograms, the required ratio of $\Delta S/S$ is small. In conclusion, the normal equation approach can be used for a condition number up to 10^4 even with a 7.2 decimal digit accuracy (Wang and Silva, 1980). If an R factor is used, we can allow up to 10^8 of the condition number times R factor.

Axicon Interferogram Analysis

It has been known that cone and decenter aberrations are introduced by transverse and longitudinal shifts in the axicon test. These aberrations are used in correcting the surface figure error (Loomis, 1979). The best reference is found from the constant, x-y tilt, x-y decenter, and cone polynomial fit, and it is deleted to minimize the rms wavefront over the digitized points. Then, the minimized wavefront is fitted with Zernike polynomials. However, this approach generates a large fit error at the center. Therefore, the contour map from the Zernike fit is not correct at the center. However, the Zernike

Fig. 3.8. Contour map from oversized Zernike circle.

polynomial fit is important in the axicon test. It can separate the wavefront into an azimuthally symmetric part and an asymmetric part. The symmetric part can be corrected by correcting the inner cone and the asymmetric part by correcting the outer cone, thereby reducing the polishing time.

This analysis is explained in Fig. 3.9. Only the symmetric part is shown for simplicity. The interferometric setup is adjusted to make the fringes as parallel as possible. Then, the best reference is deleted and the best surface wavefront $\vec{S}_b$ is found. This best surface wavefront $\vec{S}_b$ is fitted with Zernike polynomials only and the wavefront $\vec{W}_z$ is obtained as the figure error to be polished. However, $\vec{W}_z$ deviates by the fit error $\vec{E}$ from the best surface wavefront that minimizes the rms surface error. This fit error $\vec{E}$ is generated from the Zernike fit of the cone aberration $\vec{C}$, which is used to reduce the rms surface error. This fit error is shown in Fig. 3.10. Only the first orders of cone and decenter aberrations are used because the higher order terms depend on the shape and size of the axicon. As shown in Fig. 3.10, the fit error is large at the center. The new approach is to fit the best surface wavefront $\vec{S}_b$ with cone and x-y decenter and Zernike polynomials altogether. Then, we can correct the large fit error at the center.

The fit errors obtained from a real axicon interferogram by both methods are compared in Fig. 3.11. The fit error at the center is reduced much. Contour maps made by three different methods are compared in Fig. 3.12. The contour map from the linear interpolation is of poor quality at the edge. The sharp figure error at the center is generated

Fig. 3.9. Analysis of axicon interferogram.

(a)

Fig. 3.10. Fit error of cone and decenter.

(a) Fit error of cone.

(b)

Fig. 3.10.--Continued

(b) Fit error of X decenter.

37 Zernike polynomials were used.

(c)

Fig. 3.10.--Continued

(c) rms error of 1 wave cone and X-Y decenter.

DIGITIZED POINTS WITH FIT ERROR GREATER THAN .100 WAVES

OLD

DIGITIZED POINTS WITH FIT ERROR GREATER THAN .100 WAVES

NEW

Fig. 3.11. Difference in fit error.

LINEAR INT.

CONTOUR STEP .200

OLD

CONTOUR STEP .200

NEW

CONTOUR STEP .200

Fig. 3.12. Contour map of axicon.

by using the cone aberration to reduce the rms surface error, but it is not shown in the old method. Radial profiles of this real axicon interferogram are compared in Fig. 3.13. The difference is just the same as the fit error of the cone. Linear interpolation cannot provide this information.

If the axicon aberrations are not orthonormalized, the matrix is ill-conditioned because of the correlation between cone and spherical aberrations. Let us estimate the ratio of the square root of the sum of squares of the coefficient errors to the rms of the wavefront

$$\frac{||\Delta A||}{|\vec{Z}|} < \frac{\text{COND}(P) ||A|| \gamma \beta^t}{|\vec{Z}|} . \quad (3.20)$$

This is more meaningful than $||dA||/||A||$ given above. Also, $|\vec{Z}| \cong ||A||/\sqrt{R}$ as shown before, therefore

$$\frac{||\Delta A||}{|\vec{Z}|} < \text{cond}(P) \sqrt{R} \gamma \beta^{-t} . \quad (3.21)$$

This relationship can be much worse than the estimation Eq. (3.3). Consequently, the axicon aberrations are orthonormalized to decrease the calculation error (see Table 3.4). The effect is shown in Table 3.5.

The relative cone obtained from the new approach is useful in several ways. First, only the Zernike part $\vec{S}_Z$ in Fig. 3.9 can be separated and second, the cone that minimizes the polishing volume or the rms wavefront error can be found as shown in Fig. 3.14. If correlated information like the Zernike part or the relative cone is required, the orthogonalization of the axicon aberration is essential. In this

Fig. 3.13. Radial profile.

Table 3.4. Orthonormalized axicon aberrations.

$$R' = 247 \left(R - \frac{2}{3} - \frac{2}{5} Z_4 + \frac{2}{21} Z_9 - \frac{2}{45} Z_{16} + \frac{2}{77} Z_{25} - \frac{2}{117} Z_{36} + \frac{2}{165} Z_{37} \right)$$

$$\left(\frac{x}{R} \right)' = 16.7 \left(\frac{x}{R} - \frac{4}{3} Z_2 + \frac{8}{15} Z_7 - \frac{12}{35} Z_{14} + \frac{16}{63} Z_{23} - \frac{20}{99} Z_{34} \right)$$

$$\left(\frac{y}{R} \right)' = 16.7 \left(\frac{y}{R} - \frac{4}{3} Z_3 + \frac{8}{15} Z_8 - \frac{12}{35} Z_{15} + \frac{16}{63} Z_{24} - \frac{20}{99} Z_{35} \right)$$

Z_i is i th unnormalized Zernike polynomial.

Table 3.5. Reduction of condition number by orthonormalization.

Digitization	Condition Number	Orthonormalization
31 by 31 grid	4.27×10^4	No
11 by 11 grid	8.79×10	Yes
21 by 21 grid	4.00	Yes
31 by 31 grid	2.14	Yes
100 random	8.64×10^2	Yes
100 random	1.78×10^2	Yes
500 random	4.96	Yes
500 random	3.00	Yes

case, the projection of the digitization error should also be reduced by digitizing more points or by measuring more accurately.

The axicon interferogram is seriously non-uniform. Therefore, minimizing the rms wavefront over the digitized points is not the best technique because an average is required. However, the average of the coefficients is not accurate because of non-uniformity. A stable method of averaging is described in Chapter 5.

A heterodyne interferometer can solve many of the previously mentioned problems. Because the higher order Zernike polynomials are more orthogonal to the cone, the relative cone obtained in the 37 polynomials fit is accurate even if the Zernike part has a large fit error. It can also be used with information from the heterodyne interferometer.

CHAPTER 4

SUBAPERTURE INTERFEROGRAM ANALYSIS

Suppose we want to test a large parabola. Then, a large flat is required for the null test. However, to build a large flat, a larger spherical mirror is necessary for the Ritchey-Common test. This is not cost effective and an accurate analysis of the Ritchey-Common interferogram is difficult (Shu, Parks and Shannon, 1981). In this chapter, a method using several smaller subaperture interferograms is analyzed. The limitations of its use are examined, and the subaperture interferogram analysis technique is extended for a fast aspheric surface test. Also, a method using overlapped wavefront data is compared to the subaperture technique.

Criterion of Subaperture Interferogram Analysis

If the slope of the surface figure error is continuous at each subaperture, then the surface figure error can be obtained from several subaperture interferograms by fitting the interferograms with polynomials and then deleting the relative reference from each interferogram. However, we cannot delete the best reference of each subaperture as shown in Fig. 4.1. Consequently, the best surface wavefront $\vec{S}_b$ is useless. As mentioned previously, the relative surface wavefront $\vec{S}_r$ is correlated and causes considerable error magnification. However, this correlation does not depend on the orthogonality of the polynomials but comes from the

Fig. 4.1. Correlated information.

nature of subaperture interferogram analysis. Also, it limits the use of smaller subapertures. However, a set of orthonormal polynomials can reduce the roundoff error.

Let us look at the criterion of subaperture interferogram analysis. Mathematically, we want to minimize the quantity

$$\begin{aligned} & \sum_1 \left(W_1 - \sum_{i=1}^{M_r} B_{1i} Z_{1i} - \sum_{j=M_r+1}^M A_j Z_j \right)^2 \\ & + \sum_2 \left(W_2 - \sum_{i=1}^{M_r} B_{2i} Z_{2i} - \sum_{j=M_r+1}^M A_j Z_j \right)^2 + \dots \\ & + \sum_K \left(W_K - \sum_{i=1}^{M_r} B_{Ki} Z_{Ki} - \sum_{j=M_r+1}^M A_j Z_j \right)^2 = \min, \quad (4.1) \end{aligned}$$

where K is the number of subapertures, M_r is the number of reference polynomials, and M is the order of fit. The term $M_r K + (M - M_r)$ is the total number of polynomials used in the polynomial fit.

As shown in the equation, each datum is weighted the same. If we want to weight each subaperture the same, then the summation over each subaperture has to be divided by N_i , where N_i is the number of digitized points in the i th subaperture. However, some subapertures can be smaller than others, so the uniform weight of each subaperture is unreasonable. Therefore, the uniform weight for each digitized point is used in this dissertation.

The main idea is to fit digitized data with different sets of reference polynomials for each subaperture and with one common set of surface polynomials. Therefore, in the i th subaperture, we use both a

set of surface polynomials and an i th set of reference polynomials in the polynomial fit. The surface wavefront information can be obtained from the relative surface wavefront or by deleting the relative reference from the digitized data. Both methods will be compared later.

Let us express the polynomials as vectors in the N -dimensional space formed by N digitized data points. The j th reference polynomial in the i th subaperture can be represented by

$$\begin{aligned} \vec{z}_{ij} = \frac{1}{\sqrt{N}} (0, \dots, 0, Z_j(x_{i1}, y_{i1}), Z_j(x_{i2}, y_{i2}), \dots, \\ Z_j(x_{iN_i}, y_{iN_i}), 0, \dots, 0) \end{aligned} \quad (4.2)$$

where N_i is the number of digitized points in the i th subaperture, and

$$\begin{aligned} |\vec{z}_{ij}| &= \left[\frac{1}{N} \sum_{k=1}^N Z_{ij}(x_k, y_k)^2 \right]^{1/2} \\ &= \left[\frac{1}{N} \sum_{k=1}^{N_i} Z_j(x_{ik}, y_{ik})^2 \right]^{1/2} \\ &= \left[\frac{N_i}{N} \left(\frac{1}{N_i} \sum_{k=1}^{N_i} Z_j(x_{ik}, y_{ik})^2 \right) \right]^{1/2}. \end{aligned} \quad (4.3)$$

The size of the reference polynomial is the same as the square root of the second moment of the reference over the entire aperture, but it is proportional to the square root of the second moment of the reference in that subaperture. Also, from the definition of $\vec{z}_{ij}$ we find that

$$\vec{z}_{ij} \cdot \vec{z}_{mn} = 0 \quad \text{for } i \neq m. \quad (4.4)$$

In other words, each subaperture reference is considered to be uncorrelated. However, the set of reference polynomials for each subaperture is not generally orthogonal. Therefore, to obtain a smaller condition number it is better to make the reference polynomials in each subaperture orthonormal to others. In the same way, the surface polynomial $\vec{z}_k$ can be represented by

$$\vec{z}_k = \frac{1}{N} (z_k(x_1, y_1) \dots z_k(x_N, y_N)) \quad (4.5)$$

and

$$|\vec{z}_k| = \left[\frac{1}{N} \sum_{i=1}^N z_k(x_i, y_i)^2 \right]^{\frac{1}{2}} \quad (4.6)$$

The size of the surface polynomial is the square root of the second moment of the polynomial over the digitized points. But, this polynomial represents a continuous surface. Therefore, if there is overlap between the subapertures, the result is similar to non-uniform digitization. Consequently, a hexagonal subaperture is better than a circular subaperture. However, overlap does not have much affect in determining the relative reference for each subaperture, if each subaperture is digitized uniformly over enough points. In this sense, deleting the relative reference is a better approach than the relative surface wavefront method for determining surface figure. Also, we can use the data over only one subaperture for the overlap. In addition, the condition number can be reduced by using an orthonormal set of surface polynomials. However, the non-orthogonality between the reference polynomials and the surface polynomials is inevitable because we want correlated information

as mentioned above. This non-orthogonality is the main source of ill-conditioning and is worse for smaller subapertures.

Error Magnification

As mentioned before, the roundoff error can be reduced by deleting the best reference of each subaperture first. However, we want correlated information, so the problem of roundoff error is more serious for subaperture analysis than for the analysis of a non-circular aperture. Therefore, the normal equation approach may not be accurate enough for 7.2 decimal digit calculations and a more stable method may be required. However, the projection of digitization error does not depend on an algorithm and a more stable method is not helpful. Furthermore, large numbers of polynomials are used in the fit, so the projection of the digitization error is large. Therefore, we have to increase the number of digitized points. The effect of random error in the polynomial space on the accuracy of correlated information will be analyzed statistically in the next section.

As mentioned above, the surface figure error we want to find is correlated with the relative reference, as is the fit error. It is orthogonal only to the relative surface wavefront. Therefore, the fit error has some projection onto the total polynomial space. Usually, this projection is small, but the error in the relative reference and the relative surface can be much larger than this projection. The reason is that the projection is also orthogonal to the relative surface, but the relative surface and the relative reference are usually almost parallel

because of the correlation. Consequently, the fit error projection can be magnified considerably as shown in Fig. 4.2.

Projection of Random Error in the Polynomial Space on Correlated Information

Let us consider mathematically the effect of random error in the polynomial space on the error of the relative surface and the relative reference (see Fig. 4.3). We want to estimate the statistics of error in the relative surface or relative reference as the random error spans the polynomial space. This can be estimated by using a singular values analysis. However, each space should be represented with a set of orthonormal polynomials to estimate the magnification by a singular values analysis.

As shown in Chapter 2, the coefficients of the orthonormalized polynomials can be related to the coefficients of the polynomials used in the fit. From Eq. (2.38) we have

$$C = H A \quad (4.7)$$

where C represents the coefficients of the orthonormalized polynomials and A represents the coefficients of the polynomials used in the fit. Now, if the projected random error from the digitization error or the roundoff error into the total polynomial space is represented with orthonormalized polynomials, the rms of the projected random error is

$$\sigma_{N_z} = \sum_{i=1}^M C_i^2 \quad (4.8)$$

$$|\vec{F}_z| \ll |\vec{F}|, \text{ but } |\Delta \vec{S}_F| \gg |\vec{F}_z| \text{ and } |\Delta \vec{R}_F| \gg |\vec{F}_z|$$

Fig. 4.2. Fit error magnification.

Fig. 4.3. Effect of random error.

However, what we want to find is the magnification from the error in the coefficient C to the error in the coefficient A . Therefore, let us invert the matrix H , which is easy to do because the matrix H is triangular

$$A = H^{-1} C . \quad (4.9)$$

If there are K subapertures and if M polynomials (including the M_r reference polynomials) are used in the fit, the dimension of the square matrix is $M_r K + (M - M_r)$. The term $M_r K$ is the total number of reference polynomials and $(M - M_r)$ is the total number of surface polynomials. So, the matrix can be divided into two parts, one for the relative reference and the other for the relative surface

$$A = \begin{pmatrix} H_r^{-1} \\ H_s^{-1} \end{pmatrix} C . \quad (4.10)$$

We can find the magnification of the error in the relative reference or the relative surface by using the corresponding part of the matrix. Let us estimate the magnification error in the relative reference. The magnification for the relative surface can be calculated more easily, so we will use the upper part

$$A_r = H_r^{-1} C , \quad (4.11)$$

where M_r is an $M_r K$ by $M_r K + (M - M_r)$ matrix. Let us divide the matrix into two matrices S and T , where S is an $M_r K$ by $M_r K$ square matrix and T is an $M_r K$ by $(M - M_r)$ matrix. Then, we have

$$A_r = (S T) C \quad . \quad (4.12)$$

However, the reference polynomials are not orthonormalized, so we will use the orthonormalized polynomials to represent the reference. This is the same as multiplying A_r by the inverse of the matrix S . Then, the final orthonormal coefficients of the relative reference are

$$\begin{aligned} A_r' &= S^{-1} A_r \\ &= S^{-1} (S T) C \\ &= (I S^{-1} T) C \\ &= (I Q) C \quad , \end{aligned} \quad (4.13)$$

where I is the identity matrix. Now, C is the set of coefficients of the orthonormal polynomials that represents the projected random error in the total polynomial space and A_r' is the set of coefficients of the orthonormal polynomials that represents the projected random error in the reference space. The rms of the projected random error in the reference space is

$$\sigma_r^2 = \sum_{i=1}^{M_r K} A_{r_i}'^2 \quad . \quad (4.14)$$

We are ready to use the singular values analysis, but we have to delete the best reference of the entire aperture $\vec{R}_{be}$ because this is not the real error (see Fig. 4.4). $\vec{R}'$ is the real projected error. Therefore, we have to find and delete the best reference of the entire aperture from the reference of each subaperture. In other words, we have to

Fig. 4.4. Best reference of the entire aperture.

find the best reference of the input wavefront composed of the reference polynomials of each subaperture

$$\vec{W} = A_{11}\vec{Z}_{11} + A_{12}\vec{Z}_{12} + A_{13}\vec{Z}_{13} + A_{21}\vec{Z}_{21} + A_{22}\vec{Z}_{22} + \dots + A_{K3}\vec{Z}_{K3}$$

Three reference polynomials of constant and x-y tilt are used.

Suppose $\vec{T}_1$, $\vec{T}_2$ and $\vec{T}_3$ represent the best reference polynomials of the entire aperture. Then the coefficients of the best reference t_i are

$$\begin{pmatrix} \vec{T}_1 \cdot \vec{T}_1 & \vec{T}_1 \cdot \vec{T}_2 & \vec{T}_1 \cdot \vec{T}_3 \\ \vec{T}_2 \cdot \vec{T}_1 & \vec{T}_2 \cdot \vec{T}_2 & \vec{T}_2 \cdot \vec{T}_3 \\ \vec{T}_3 \cdot \vec{T}_1 & \vec{T}_3 \cdot \vec{T}_2 & \vec{T}_3 \cdot \vec{T}_3 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = \begin{pmatrix} \vec{W} \cdot \vec{T}_1 \\ \vec{W} \cdot \vec{T}_2 \\ \vec{W} \cdot \vec{T}_3 \end{pmatrix} \quad (4.15)$$

However, Eq. (4.15) becomes

$$\begin{pmatrix} \vec{W} \cdot \vec{T}_1 \\ \vec{W} \cdot \vec{T}_2 \\ \vec{W} \cdot \vec{T}_3 \end{pmatrix} = \begin{pmatrix} \vec{T}_1 \cdot \vec{Z}_{11} & \vec{T}_1 \cdot \vec{Z}_{12} & \dots & \vec{T}_1 \cdot \vec{Z}_{K3} \\ \vec{T}_2 \cdot \vec{Z}_{11} & \vec{T}_2 \cdot \vec{Z}_{12} & \dots & \vec{T}_2 \cdot \vec{Z}_{K3} \\ \vec{T}_3 \cdot \vec{Z}_{11} & \vec{T}_3 \cdot \vec{Z}_{12} & \dots & \vec{T}_3 \cdot \vec{Z}_{K3} \end{pmatrix} \begin{pmatrix} A_{11} \\ A_{12} \\ \vdots \\ A_{K3} \end{pmatrix}. \quad (4.16)$$

Therefore from Eqs. (4.15) and (4.16) we derive

$$Tt = T_z A_r \quad \text{and} \quad t = T^{-1} T_z A_r \quad (4.17)$$

where T and T_z are the corresponding matrices in Eqs. (4.15) and (4.16). To delete this entire best reference, we have to transform it to the coefficients of the subaperture reference polynomials. Now we have to follow the inverse step. The input wavefront is composed of the set of entire reference polynomials

$$\vec{W} = t_1 \vec{T}_1 + t_2 \vec{T}_2 + t_3 \vec{T}_3 \quad .$$

The coefficients of the subaperture reference polynomials are obtained by

$$\begin{pmatrix} \vec{Z}_{11} \cdot \vec{Z}_{11} & \vec{Z}_{11} \cdot \vec{Z}_{12} & \dots & \vec{Z}_{11} \cdot \vec{Z}_{K3} \\ \vec{Z}_{12} \cdot \vec{Z}_{11} & \vec{Z}_{12} \cdot \vec{Z}_{12} & \dots & \vec{Z}_{12} \cdot \vec{Z}_{K3} \\ \vdots & \vdots & \ddots & \vdots \\ \vec{Z}_{K3} \cdot \vec{Z}_{11} & \vec{Z}_{K3} \cdot \vec{Z}_{12} & \dots & \vec{Z}_{K3} \cdot \vec{Z}_{K3} \end{pmatrix} \begin{pmatrix} A_{11} \\ A_{12} \\ \vdots \\ A_{K3} \end{pmatrix} = \begin{pmatrix} \vec{W} \cdot \vec{Z}_{11} \\ \vec{W} \cdot \vec{Z}_{12} \\ \vdots \\ \vec{W} \cdot \vec{Z}_{K3} \end{pmatrix} \quad (4.18)$$

and

$$\begin{pmatrix} \vec{W} \cdot \vec{Z}_{11} \\ \vec{W} \cdot \vec{Z}_{12} \\ \vdots \\ \vec{W} \cdot \vec{Z}_{K3} \end{pmatrix} = \begin{pmatrix} \vec{Z}_{11} \cdot \vec{T}_1 & \vec{Z}_{11} \cdot \vec{T}_2 & \vec{Z}_{11} \cdot \vec{T}_3 \\ \vec{Z}_{12} \cdot \vec{T}_1 & \vec{Z}_{12} \cdot \vec{T}_2 & \vec{Z}_{12} \cdot \vec{T}_3 \\ \vdots & \vdots & \vdots \\ \vec{Z}_{K3} \cdot \vec{T}_1 & \vec{Z}_{K3} \cdot \vec{T}_2 & \vec{Z}_{K3} \cdot \vec{T}_3 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} . \quad (4.19)$$

Therefore from Eqs. (4.18) and (4.19)

$$Z A_{rb} = Z_T t \quad \text{and} \quad A_{rb} = Z^{-1} Z_T t , \quad (4.20)$$

where Z and Z_T are the corresponding matrices in Eqs. (4.18) and (4.19) and A_{rb} is the best reference of the entire aperture represented with the coefficients of the subaperture reference polynomials. Then from Eq. (4.17) we have

$$A_{rb} = Z^{-1} Z_T T^{-1} T_Z A_r . \quad (4.21)$$

Therefore, the final error in the coefficients of the subaperture reference polynomials is

$$\Delta A_r = A_{rb} - A_r . \quad (4.22)$$

We have to transform it to the orthonormal polynomial representation to use the singular values analysis. In orthonormal polynomial representation, it is

$$\Delta A_r' = S^{-1} \Delta A_r , \quad (4.23)$$

where S is the matrix defined above to orthonormalize the subaperture reference. Finally, the error is

$$\Delta A_r' = S^{-1}(Z^{-1}Z_T T^{-1} T_Z - I)S(IQ)C, \quad (4.24)$$

where I is a $3K$ by $3K$ identity matrix, and

$$\Delta A_r' = PC. \quad (4.25)$$

Now, we can run the singular values analysis to find the $3K + M - 3$ singular values. These singular values describe the magnification of error in each orthonormal polynomial of the total polynomial space. The square root of the second moment of the singular values represents the average magnification of the projection of random error from the total polynomial space to the relative reference space, as explained in Appendix A.

The previous analysis is an estimation of the rms error in a relative reference or a relative surface. Then, how can we estimate the error in a coefficient. From Eq. (4.10), the coefficients of the surface polynomials from a random error can be obtained by

$$\Delta A_s = H_s^{-1} C \quad (4.26)$$

Then, the error in the k th surface polynomial is

$$\Delta A_{sk} = (H_s^{-1})_{k1} C_1 + (H_s^{-1})_{k2} C_2 \dots + (H_s^{-1})_{kN} C_N \quad (4.27)$$

where N is the number of total polynomials used in the fit $3K + M - 3$.

Therefore, the ratio of the maximum error in the k th coefficient to the

random error in rms is the largest coefficient in absolute value in the kth row of H_s^{-1}

$$\frac{\text{Maximum error in the kth surface polynomial}}{\text{Random error in rms}} = \max_{1 \leq i \leq N} (|(H_s^{-1})_{ki}|). \quad (4.28)$$

Also, C_i is the coefficient of the orthonormal polynomials, so the square root of the second moment of the kth row represents the average sensitivity

$$\begin{aligned} \langle (\Delta A_{sk})^2 \rangle &= \langle [(H_s^{-1})_{k1} C_1 + \dots + (H_s^{-1})_{kN} C_N]^2 \rangle \\ &= \sum_{i=1}^N (H_s^{-1})_{ki}^2 \langle C_i^2 \rangle \quad (\text{orthogonality}) \\ &= \sum_{i=1}^N (H_s^{-1})_{ki}^2 \cdot \frac{\sigma_{N_z}^2}{N} \quad (\text{random}) \\ &= \sigma_{N_z}^2 \cdot \left[\frac{1}{N} \sum_{i=1}^N (H_s^{-1})_{ki}^2 \right] \\ &= \sigma_{N_z}^2 m^2, \end{aligned} \quad (4.29)$$

where σ_{N_z} is the rms error in the total polynomial space, and m is the average magnification factor. Therefore, we find that

$$\frac{\sqrt{\langle (\Delta A_{sk})^2 \rangle}}{\sigma_{N_z}} = \left[\frac{1}{N} \sum_{i=1}^N (H_s^{-1})_{ki}^2 \right]^{1/2}. \quad (4.30)$$

This information is useful for determining the accuracy of the coefficient. Usually the coefficients of lower order polynomials are more sensitive because they have a higher correlation to the relative reference.

Projection of the Fit Error

As mentioned in Chapter 2, the fit error includes up to some higher order such that the polynomial representation is the same as the real surface within the digitized error. Even if the digitization error can be averaged out, the fit error can be reduced only by increasing the number of polynomials. Also, the fit error can introduce a large error on the relative reference or the relative surface for a seriously correlated case. Therefore, let us estimate the effect of the fit error. This can be done in almost the same way as before. The coefficient of the projection of fit error A can be found from the normal equation

$$Z^t Z A = Z^t W \quad . \quad (4.31)$$

But, now we see that

$$\vec{W} = C_1 \vec{Z}_{M+1} + C_2 \vec{Z}_{M+2} + \dots + C_K \vec{Z}_{M+K} \quad . \quad (4.32)$$

Then this leads to

$$Z^t W = \begin{pmatrix} \vec{Z}_1 \cdot \vec{Z}_{M+1} & \vec{Z}_1 \cdot \vec{Z}_{M+2} & \dots & \vec{Z}_1 \cdot \vec{Z}_{M+K} \\ \vec{Z}_2 \cdot \vec{Z}_{M+1} & \vec{Z}_2 \cdot \vec{Z}_{M+2} & \dots & \vec{Z}_2 \cdot \vec{Z}_{M+K} \\ \vdots & \vdots & \ddots & \vdots \\ \vec{Z}_M \cdot \vec{Z}_{M+1} & \vec{Z}_M \cdot \vec{Z}_{M+2} & \dots & \vec{Z}_M \cdot \vec{Z}_{M+K} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_K \end{pmatrix} , \quad (4.33)$$

where M is the number of polynomials used in the fit and K polynomials represent the fit error terms. Therefore, it can be represented as

$$A = X C \quad . \quad (4.34)$$

Now, we have to orthonormalize C and A as before. Then, we can estimate the rms magnification of the projected fit error on the relative reference or the relative surface. Also, we can find the sensitivity of each coefficient to the fit error in the same way as before.

Test of Large Flat or Parabola

Figure 4.5 shows three subapertures that are used for simulation. Their radius is 1/2, 1/3, and 1/4 that of the full aperture, respectively. Thirty-seven polynomials including three reference polynomials are used in the fit. A square grid over the entire aperture is used for grid digitization. A random number generator is used for random digitization. The condition number of each normal matrix is shown in Table 4.1. To reduce the condition number, the reference polynomials are orthonormalized in each subaperture. The set of orthonormalized reference polynomials is

$$\begin{aligned}
 R_{i1} &= \frac{1}{\text{RAD}_i} \\
 R_{i2} &= 2 \left[\frac{x - C_{xi}}{\text{RAD}_x} \right] \frac{1}{\text{RAD}_i} \\
 R_{i3} &= 2 \left[\frac{y - C_{yi}}{\text{RAD}_i} \right] \frac{1}{\text{RAD}_i}
 \end{aligned} \tag{4.35}$$

where C_{xi} , C_{yi} are the coordinates of the center of the i th subaperture and RAD_i is the radius of it. The purpose of dividing by the radius of the subaperture is to compensate for the smaller aperture of the reference polynomials. The resulting condition number is small enough to use the normal equation approach.

Fig. 4.5. Subapertures of $1/2$, $1/3$, and $1/4$ radius.

Table 4.1. Condition number (subaperture with 3 references).

Radius of Subaperture	Number of Subapertures	Condition Number
$\frac{1}{2}$	6	1.10×10^2
$\frac{1}{3}$	9	7.28×10^2
$\frac{1}{4}$	19	1.71×10^3

37 polynomial fit and a 31 by 31 grid digitization were used.

The magnification factors of the relative surface error and the relative reference error are shown in Fig. 4.6 and the average sensitivity of the surface coefficients is shown in Fig. 4.7. The surface or reference error magnification represents the ratio of the rms error projected into the relative surface or the relative reference to the rms error projected into the total polynomial space. The sensitivity of each coefficient is drawn in increasing order from the 4th to the 37th polynomial. These magnification factors provide information about how many averages are enough. Suppose the rms digitized error is σ_E and it is averaged K times. Then the expected rms surface error is

$$\sigma_s = \sigma_E \left(\frac{M_t}{N_t} \right)^{\frac{1}{2}} M_s \frac{1}{\sqrt{K}} \quad , \quad (4.36)$$

where M_t is the total number of polynomials used in the fit, N_t is the total number of digitized data, and M_s is the average magnification of the surface error from the random error. As shown in Fig. 4.7, lower order coefficients are sensitive to random error because lower order polynomials are seriously correlated with reference polynomials.

The fit error magnification is simulated with 12 10th order aberrations except spherical aberrations. Also, a 51x51 grid is used to reduce the finite digitization error. As shown in Fig. 4.8, the error on the reference is larger than the error on the surface. This is the advantage of using surface polynomials for surface figure error information.

Fig. 4.6. Magnification factors of reference and surface error to the projected random error.

37 polynomial fit and a 31 by 31 grid digitization were used.

Fig. 4.7. Sensitivity of surface coefficients to the projected random error.

37 polynomial fit including 3 reference polynomials and a 31 by 31 grid digitization were used.

Fig. 4.8. Fit error projection.

The magnification factors are shown in Fig. 4.9 and the average sensitivity of coefficients is shown in Fig. 4.10. The reference or surface error magnification represents the ratio of the rms error projected into the relative reference or into the relative surface from the fit error to the rms fit error. The sensitivity is drawn in increasing order from the 4th to the 37th polynomial. The small magnification of the reference for 1/2 radius is due to subtraction of the best reference over the entire aperture. Lower order coefficients are sensitive to fit error because projection of the fit error is magnified substantially owing to the correlation between the reference polynomials and the lower order surface polynomials.

Fig. 4.9. Magnification factors of reference and surface error to the fit error.

37 polynomials were used in the fit and 12 higher order polynomials were used in the simulation of the fit error. A 51 by 51 grid digitization was used.

Fig. 4.10. Sensitivity of surface coefficients to the fit error.

37 polynomials were used in the fit and 12 higher order polynomials were used in the simulation of the fit error. A 51 by 51 grid digitization was used.

Now let us look at the dependence of the magnification factors on the parameters. First, as shown in Fig. 4.11a, the magnification factor does not depend on the order of fit because a large portion of the magnification comes from the correlation between the reference polynomials and the lower order surface polynomials. In contrast, the average magnification factor decreases because higher order polynomials are less correlated. Second, as shown in Fig. 4.11b, the magnification factor does not depend on the size of the grid. It depends only on the geometry of the subaperture if enough points are digitized. Even random digitization gives the same magnification as shown in Fig. 4.12. Therefore, from now on, only grid digitizations are used for simulation.

The subapertures are overlapped in the above simulation, but, as mentioned before, the overlap degrades the orthonormality of the surface polynomials; in other words, it causes non-uniform digitization for the surface polynomials. Therefore, only one subaperture is used in the simulation. The magnification factors of the surface and reference from the random error are shown in Fig. 4.13. Compare this figure with Fig. 4.6. The magnification factors of the surface and the reference from the fit error are shown in Fig. 4.14. Compare it with Fig. 4.9. The magnification factors increase because the size of the subaperture is reduced effectively. The effect is serious for the $1/2$ radius subaperture because the size of the last subaperture is reduced considerably. The digitized points are compared in Table 4.2. Because the most important factor of magnification is the size of the subaperture, a small overlap is better than no overlap as shown in the above figures.

Fig. 4.11. Dependence on parameters.

- (a) Surface error magnification from different order fits.
 (b) Surface error magnification from different grid sizes.

Fig. 4.12. Magnification from random digitization.

Nine $1/3$ radius subapertures were used. 37 polynomial fit was used. 648 points were randomly digitized to compare with a 31 by 31 grid digitization.

Fig. 4.13. Magnification factors of reference and surface error to the projected error without overlap.

37 polynomial fit and a 31 by 31 grid digitization were used.

Fig. 4.14. Magnification factors of reference and surface error to the fit error without overlap.

37 polynomials were used in the fit and 12 higher order polynomials were used in the simulation of fit error. A 51 by 51 grid digitization was used.

Table 4.2. Number of digitized points in each subaperture.

Subaperture	Six $1/2$ Radius		Nine $1/3$ Radius		Nineteen $1/4$ Radius	
	With Overlap	Without Overlap	With Overlap	Without Overlap	With Overlap	Without Overlap
1	494	494	221	221	121	121
2	493	300	217	217	121	115
3	493	297	220	192	121	109
4	494	301	217	189	121	115
5	493	272	220	192	121	109
6	493	133	217	189	121	103
7			220	192	121	103
8			217	189	127	111
9			220	164	127	111
10					100	83
11					100	83
12					100	83
13					100	83
14					120	101
15					120	101
16					120	101
17					120	101
18					100	78
19					100	78

A 51 by 51 grid digitization was used. Subaperture number is shown in Fig. 4.5.

The nine $2/5$ radius subapertures shown in Fig. 4.15 are used for the real simulation. The magnification factors of the relative surface error from the projected random error and the fit error are shown in Fig. 4.16. A set of interferograms with the same surface figure error but with a different reference is generated. The surface figure error is composed of

- 2 waves rms of defocus
- 0.5 waves rms of x and y third order astigmatism
- 0.5 waves rms of x and y third order coma
- 1 wave rms of third order spherical aberration
- 2.45 waves rms surface figure error.

Another set of interferograms is generated by adding some higher order aberrations to test the fit error magnification. The twelve 10th order aberrations except 10th order spherical aberration are used. Each term contributes $1/20$ wave rms to the surface figure error. The total fit error is 0.173 wave rms. The interferograms are shown in Figs. 4.17 and 4.18. The fringes continue over the full width of the aperture but are too fine to be drawn. However, by adjusting the tilt, the subaperture interferograms are smooth. The diameter of the entire aperture is about 6 inches. A graphics tablet was used for digitization and about 2000 points were digitized over the entire aperture. The 37-polynomial fit was used and the difference was obtained from comparing only 37 terms (see Fig. 4.19). The difference without fit error was 0.053 wave rms. If the average magnification factor is obtained from the simulation of a 51×51 grid,

Fig. 4.15. Nine $2/5$ radius subaperture.

Fig. 4.16. Magnification factors of nine $2/5$ radius subaperture.

37 polynomials were used in the fit and 12 higher order polynomials were used in the simulation of fit error. A 51 by 51 grid digitization was used.

Fig. 4.17. Interferograms without fit error.

Fig. 4.18. Interferograms with fit error.

DIFFERENCE FROM 37 POL. FIT WITHOUT FIT ERROR

RMS ERROR .053

CONTOUR STEP .100
-.05

DIFFERENCE FROM 37 POL. FIT WITH .173 RMS FIT ERROR

RMS ERROR .28

CONTOUR STEP .400
-.20

Fig. 4.19. Difference.

then the average magnification from the digitization error is

$$\begin{aligned}
 M &= (\text{average magnification from projected digitization error}) \\
 &\quad \cdot (\text{projection ratio}) \\
 &= 2.28 \left(\frac{61}{2000} \right)^{\frac{1}{2}} \approx .4 \quad , \quad (4.37)
 \end{aligned}$$

where 61 is the total number of polynomials used in the fit and 2000 is the approximate number of digitized points. Because the digitization error is expected to be a little less than 0.1 wave rms, the digitization error magnification is a little larger than average. However, the estimation is good, and if several results are averaged, the accuracy will be much better.

Now, let us compare the error to the surface figure

$$\begin{aligned}
 R &= \frac{(\text{Error})_{\text{rms}}}{(\text{Surface})_{\text{rms}}} \\
 &= \frac{0.053}{2.45} \approx \frac{1}{46} \quad . \quad (4.38)
 \end{aligned}$$

This kind of accuracy is difficult to obtain without using a subaperture. We could reduce the surface figure error over each subaperture by adjusting the reference. The fit error magnification is almost the same as the average magnification

$$\begin{aligned}
 (\text{relative surface error})_{\text{rms}} &= (\text{avg. magnification}) \cdot (\text{rms fit error}) \\
 &= 1.58 \times 0.173 \\
 &\cong 0.27 \quad . \quad (4.39)
 \end{aligned}$$

This error can be reduced only by increasing the order of fit. If 49 orders of fit are used, the error is about 1/20 wave rms again.

The subaperture test of large flats or parabolas can be used to align a large telescope by analyzing the entire wavefront from the subaperture interferograms.

Test of Fast Aperture Optics

Now let us use one more reference polynomial for the different defocus of each subaperture. Then, we can test fast optics without a fast diverging lens. The defocus of the subaperture is orthonormalized to reduce the condition number

$$R_{i4} = \sqrt{3} \left(\frac{2 \left[(x-Cx_i)^2 + (y-Cy_i)^2 \right]}{RAD_i^2} - 1 \right) / RAD_i \quad . \quad (4.40)$$

Six 1/2 radius subapertures and nine 1/3 radius subapertures shown in Fig. 4.5 are used for simulation. The condition number is shown in Table 4.3. The condition number is small enough to use the normal equation approach for up to a 1/3 radius subaperture. The magnification of random error is shown in Fig. 4.20, and the sensitivity of surface coefficients is shown in Fig. 4.21. The magnification of fit error and the sensitivity are shown in Figs. 4.22 and 4.23. The fit error magnification is large because the projection of the fit error is increased because of one more reference polynomial. The coefficients of third order astigmatism are very sensitive to random error and fit error because they are correlated with the reference polynomials. As shown

Table 4.3. Condition number (subaperture with 4 references).

Radius of Subaperture	Number of Subapertures	Condition Number
$\frac{1}{2}$	6	1.60×10^2
$\frac{1}{3}$	9	1.52×10^3

37 polynomial fit and a 31 by 31 grid digitization were used.

Fig. 4.20. Magnification of random error with 4 reference polynomials.
 37 polynomial fit and a 31 by 31 grid digitization were used.

Fig. 4.21. Sensitivity of surface coefficients to random error with 4 reference polynomials.

37 polynomial fit and a 31 by 31 grid digitization were used.

Fig. 4.22. Magnification of fit error with 4 reference polynomials.

37 polynomials were used in the fit and 12 higher order polynomials were used in the simulation of fit error. A 51 by 51 grid digitization was used.

Fig. 4.23. Sensitivity of surface coefficients to fit error with 4 reference polynomials.

37 polynomials were used in the fit and 12 higher order polynomials were used in the simulation of fit error. A 51 by 51 grid digitization was used.

above, it is not difficult to use up to a 1/3 radius subaperture. It can be used to test fast optics without a fast diverging lens or to test large surface figure error by reducing the rms wavefront of each subaperture by adjusting x-y tilt and defocus.

Test of Aspheric Surface

Suppose we want to test a surface with spherical aberration. Then the defocus is adjusted to reduce the rms wavefront and the maximum slope. The rms value can be reduced by a factor of 4 and the maximum slope by a factor of 2. Therefore, if the aperture is divided into several subapertures and the defocus is adjusted in each subaperture, the rms value and the maximum slope can be reduced more. We can test large spherical aberration with an annular subaperture. First, let us orthonormalize reference polynomials over the annular subaperture. They are

$$\begin{aligned}
 R_{i1} &= \frac{1}{\sqrt{1-e^2}} \frac{1}{\text{RAD}_i} \\
 R_{i2} &= \frac{1}{\sqrt{1-e^4}} \left[\frac{2x}{\text{RAD}_i} \right] \frac{1}{\text{RAD}_i} \\
 R_{i3} &= \frac{1}{\sqrt{1-e^4}} \left[\frac{2y}{\text{RAD}_i} \right] \frac{1}{\text{RAD}_i} \\
 R_{i4} &= \frac{1}{(\sqrt{1-e^2})^3} \sqrt{3} \left(\frac{2(x^2+y^2)}{\text{RAD}_i^2} - 1 - e^2 \right) \frac{1}{\text{RAD}_i} , \quad (4.41)
 \end{aligned}$$

where e is the obstruction ratio of the i th subaperture and RAD_i is its radius.

The orthogonalized spherical aberration over the annular aperture is

$$S_p' = \left[R^4 - (1+e^2) \left(R^2 - \frac{(1+e^2)}{2} \right) - \frac{(1+e^2+e^4)}{3} \right] . \quad (4.42)$$

Let us look at the slope of spherical aberration at the edge. Without defocus adjustment, it is

$$(\text{Slope})_{ed} = 4 \text{ waves/rad} \quad (4.43)$$

and with defocus adjustment, it is

$$(\text{Slope})_{ed} = 4 - 2(1+e^2) \text{ waves/rad} . \quad (4.44)$$

If the obstruction ratio is large, it can be reduced considerably. So, let us use the four annular subapertures shown in Fig. 4.24. Each subaperture has the same area. The obstruction ratio of the largest annular subaperture is 0.866 and the slope at the edge can be reduced by a factor of 8. Also, the rms value of spherical aberration can be reduced by a factor of about 63 from simulation.

The condition number is reduced by using orthonormal reference polynomials (see Table 4.4). The change of the condition number can be explained as follows. First, the reference polynomials are not orthogonal because the constant term and the defocus term are correlated from under sampling. Therefore, the condition number is large for a 31x31 grid. Then, the reference polynomials become orthogonal and the condition number decreases. However, higher order spherical aberrations are correlated with the defocus reference. Consequently, the surface

Fig. 4.24. Four annular subapertures of the same area.

Table 4.4. Condition number vs grid size.

Grid	Number of Points	Condition Number
31 by 31	709	2.67×10^3
51 by 51	1,961	1.65×10^3
75 by 75	4,293	1.61×10^3
101 by 101	7,845	2.63×10^3
121 by 121	11,289	2.66×10^3

polynomials and the reference polynomials are correlated and the condition number increases. At about a 101×101 grid, it can be approximated to be continuous. Therefore, we have to digitize many points to neglect the finite digitization error. The 101×101 grid is used in the simulation. The magnification of random error is shown in Fig. 4.25a. Even if the magnification factor is large, the large number of digitized points reduces the effect of random error. The sensitivity of surface coefficients is shown in Fig. 4.25b. Spherical aberration terms are very sensitive because they are correlated with the reference polynomials. The fit error magnification and the sensitivity are shown in Fig. 4.26.

Local Connection Method

Let us look at another approach that does not depend on fit error. Suppose W_{12} is the digitized data from interferogram 1 over the overlapped area and W_{21} is from interferogram 2 (see Fig. 4.27a). If we digitize over the same points over the overlap, then the surface figure from both data should be the same and only the best reference can be different (see Fig. 4.27b). Therefore, we can connect two interferograms by deleting the difference of the best reference. Now, there is no magnification of random error and fit error because correlated information is no longer wanted. Furthermore, projection of random error onto the polynomial space is small because of the small number of polynomials, only three for constant and x-y tilt. The projection of the digitization error is

$$\sigma_{N_z} = \sqrt{\frac{3}{N}} \times \sigma_E, \quad (4.45)$$

Fig. 4.25. Magnification of random error and sensitivity.

37 polynomial fit and a 101 by 101 grid digitization were used.

- (a) Random error magnification.
- (b) Sensitivity of coefficients to random error.

Fig. 4.26. Magnification of fit error and sensitivity.

37 polynomial fit and a 101 by 101 grid digitization were used.

(a) Fit error magnification.

(b) Sensitivity of coefficients to fit error.

Fig. 4.27. Error magnification and projection of surface on the reference.

- (a) Error magnification.
- (b) Same digitization.
- (c) Different digitization.

where σ_E is the rms digitization error and N is the number of digitized points for each interferogram. Because we are interested in the difference and we can assume that the error is uncorrelated, the expected error in the reference is

$$\begin{aligned}\sigma_{N_z}' &= \sqrt{2} \sigma_{N_z} \\ &= \sqrt{\frac{6}{N}} \sigma_E.\end{aligned}\quad (4.46)$$

However, this error is interpolated over the entire aperture (see Fig. 4.27a). Therefore, the rms error over the entire aperture is magnified from the rms error over the overlapped area. This magnification will be estimated later. However, if the digitized points are different, the

projection of the surface wavefront to the best reference is different (see Fig. 4.27c). Therefore, we have to find the relative reference and delete the difference of the relative reference. The relative reference has no projection from the surface wavefront because all the reference and surface polynomials are used together in the fit, so it is not affected by the different digitization. However, it is affected by the digitization error and the fit error as shown above in the subaperture interferogram analysis. But the subaperture size is the same as the entire aperture, so there is no serious magnification of the error from the correlation between the relative surface space and the relative reference space if the polynomials are orthogonal. But it is not easy to find the orthogonal polynomials for each differently shaped overlap. Therefore, let us generate a circular aperture within the overlap. Then the Zernike polynomials can be used for connecting the interferograms. Even if the number of digitized points decreases, the large magnification factor can be avoided by using orthogonal polynomials. Now, let us estimate the magnification. As shown above, coefficients of the reference can be represented with coefficients of the polynomials, which are orthonormal over the digitized points. From Eq. (4.11) we have

$$A_r = H_r^{-1} C, \quad (4.47)$$

where the dimension of A_r is $2M_r$ and that of C is $M+M_r$. However, the difference of reference is the real projected relative reference error. So, the real projected relative reference error is

$$\begin{aligned}
A_{r1}' &= A_{r1} - A_{r \ M_r+1} \\
A_{r2}' &= A_{r2} - A_{r \ M_r+2} \\
&\vdots \\
A_{rM_r}' &= A_{rM_r} - A_{rM_r+M_r}
\end{aligned} \tag{4.48}$$

Therefore we have

$$A_r' = X C, \tag{4.49}$$

where X can be obtained from H_r^{-1} easily.

Because Zernike polynomials are orthonormal over a circular aperture, we can use the singular value analysis to estimate the magnification of random error. Fifty points are randomly digitized over the circular portion of overlap from each interferogram. A 37-polynomial fit is used. The results obtained from three different simulations are shown in Fig. 4.28. Thirty-four zero singular values correspond to 34 surface polynomials that have no projection to the relative reference. Three more zero singular values correspond to three out of six orthogonal reference polynomials that have same projection to both subapertures. As shown above, about 50 digitizations from each interferogram are enough. The expected rms error in the relative reference is

$$\begin{aligned}
\sigma_{N_z} &= \text{projected error in the relative reference} \\
&= (\text{digitization error}) \cdot (\text{projection error}) \cdot (\text{average magnification}) \\
&= \sigma_E \left(\frac{M+M_r}{2N} \right)^{\frac{1}{2}} M_s.
\end{aligned} \tag{4.50}$$

Fig. 4.28. Magnification of random projected error in the relative reference.

For three references and 50 points digitization, we have

$$\begin{aligned}\sigma_{N_z} &= \sigma_E \left(\frac{3+37}{2 \times 50} \right)^{\frac{1}{2}} \times 0.75 \\ &\approx 0.475 \sigma_E .\end{aligned}\tag{4.51}$$

On the other hand, the projected error from the best reference approach is

$$\begin{aligned}\sigma_{N_z} &= \left(\frac{6}{50} \right)^{\frac{1}{2}} \sigma_E \\ &\approx 0.35 \sigma_E .\end{aligned}\tag{4.52}$$

The result is comparable.

Now, let us look at the magnification from the fit error. Because the fit error is not random, the projection to each interferogram is almost the same. Therefore, the difference that is of interest is not large. Furthermore, the surface and the reference are almost orthogonal, so the magnification from the correlation is small. In addition, the 37-polynomial fit over the small circular portion of the overlapped area is usually accurate enough. So, the effect of fit error is negligible. Therefore, the different digitizations can be used to connect the interferograms.

Now, suppose we want to connect the $1/2 \times 6$ subaperture shown in Fig. 4.5. Because of the reference error, after one turn the final reference is not the same as the original one. One simple solution is to subtract the average of the difference. Suppose a common axis is used

for each overlapped reference and the reference difference in each overlap is ΔR_{12} , ΔR_{23} , ΔR_{N1} for N overlap such that

$$\frac{1}{N} \sum_{i=1}^N \Delta R_{ii+1} = \Delta R \neq 0 \quad (N+1=1) \quad . \quad (4.53)$$

Then, using the reference difference of

$$\begin{aligned} \Delta R_{12}' &= \Delta R_{12} - \Delta R \\ \Delta R_{23}' &= \Delta R_{23} - \Delta R \\ &\vdots \\ \Delta R_{N1}' &= \Delta R_{N1} - \Delta R \end{aligned} \quad (4.54)$$

gives us

$$\sum_{i=1}^N \Delta R_{ii+1}' = 0 \quad (4.55)$$

and there is no contradiction. However, this approach does not consider the least squares criterion. As shown in Fig. 4.29, this solution does not have a minimum fit error ($|E'| < |E''|$). The other approach is to use the constraints and the least squares criterion at the same time. The result has a minimum fit error among the solutions that satisfy the constraints (see Fig. 4.29). From each constraint, we can reduce one polynomial in the fit, so the projection of random error on the polynomial space is reduced as shown in Fig. 4.28 ($|N_Z'| < |N_Z|$). This approach is practically impossible to use for the relative reference because it requires a large array. If we have N overlap and M

Fig. 4.29. Reduction of error by constraints.

polynomials including M_r reference polynomials for fit, we need an $(M+M_r)N$ by $(M+M_r)N$ array. However, for the best reference approach, we need an $M_r \cdot N$ by $M_r \cdot N$ array. We can use the following least squares criterion for the best reference approach

$$\sum_1 (W-A_1R)^2 + \sum_2 (W-A_2R)^2 + \dots + \sum_N (W-A_NR)^2 = \min.$$

$$\begin{pmatrix} A_{M+1} \\ A_{M+2} \\ A_{M+3} \\ \vdots \\ A_N \end{pmatrix} = C \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_M \end{pmatrix}, \quad (4.56)$$

where each A_i represents the set of coefficients of the i th subaperture reference polynomials, N is the number of overlapped areas, and $(N-M)$ is the number of constraints. It is assumed that a common axis is used to represent the reference of each subaperture. Then the equations of the constraints can be found easily.

Finally, let us estimate the magnification. We assume a random reference error over the overlap even if it is unreasonable at times. Then, the magnification of the rms error over the entire aperture from the rms error over the overlap can be estimated by the singular value analysis. The result is shown in Fig. 4.30. Six $1/2$ radius subapertures and three reference polynomials are assumed. The three zero-magnification factors in Fig. 4.30a are unreasonable because they correspond to the case where all the random error is in the last overlap, which is not used in connecting interferograms. However, we do obtain some information about how serious the magnification can be. In Fig. 4.30b, three constraints of zero sum of three references are used to reduce the projection of digitized error. Furthermore, the magnification factors are smaller. In effect, three zero factors were used to reduce the projection, but these three zero factors do not correspond to unreasonable cases. The constraint is very effective for reducing error. In Fig. 4.31, the magnification from the nine linear local interpolations, which are overlapped over a half-diameter, is shown. It can be used to test a 5×1 rectangular aperture. Because some random reference error over the overlap is unreasonable, these magnifications are overestimated,

Fig. 4.30. Magnification of 6 circular local connections with and without constraints.

- (a) Without constraint.
- (b) With constraint.

but they can provide information about how many digitized points are required.

CHAPTER 5

APPLICATIONS OF SUBAPERTURE INTERFEROGRAM ANALYSIS

As shown in the previous chapter, subaperture interferogram analysis is useful for analyzing several interferograms at one time. It can be applied to the average of under-sampled interferograms. This averaging approach is more reliable than the average of coefficients technique. Also, lateral shearing interferograms can be analyzed by the subaperture approach because we need several lateral shearing interferograms to obtain surface figure information.

Average of Interferograms

Suppose a wavefront is represented with a set of orthonormal polynomials over the continuous aperture

$$W = A_1 \vec{Z}_1 + A_2 \vec{Z}_2 + \dots + A_{M_r} \vec{Z}_{M_r} + A_{M_{r+1}} \vec{Z}_{M_{r+1}} + \dots + A_M \vec{Z}_M, \quad (5.1)$$

where $\vec{Z}_i$ is orthonormal over the continuous aperture, M_r is the number of reference polynomials, and M is the total number of polynomials. If more than M points are digitized, then the M coefficients A_i can be obtained by the least squares approach if there is no digitization error and the roundoff error can be neglected. However, these errors introduce some error ΔA_i to each coefficient, and this error can be large if insufficient points are digitized. The reason is that the polynomials are not orthonormal over the digitized points any more.

Now the rms error in the surface wavefront over the continuous aperture is

$$(\Delta S)_c = \sum_{i=M_r+1}^M \Delta A_i^2 . \quad (5.2)$$

However, from previous analysis, the error in each coefficient introduced by random error can be represented with a set of orthonormal polynomials over the digitized points. From Eq. (4.10)

$$\Delta A_s = \begin{pmatrix} \Delta A_{M_{r+1}} \\ \Delta A_{M_{r+2}} \\ \vdots \\ \Delta A_M \end{pmatrix} = H_s^{-1} C , \quad (5.3)$$

where H_s^{-1} is an $M-M_r$ by M matrix. Therefore, the statistics of $(\Delta S)_c$ can be obtained by running the singular values analysis to the matrix H_s^{-1} . Then, we can derive the average magnification factor m_s

$$\sqrt{\langle (\Delta S)_c^2 \rangle} = \sigma_E \sqrt{\frac{M}{N}} m_s , \quad (5.4)$$

where σ_E is the rms digitized error and N is the number of digitized points. If N is not much larger than M , the average magnification factor m_s can be very large as shown in Fig. 5.1. In this case, obtaining the average of these coefficients is not a reliable method because, by averaging K interferograms, the error can be decreased only by a factor of $\sqrt{1/K}$ if we assume each error is uncorrelated. However,

Fig. 5.1. The ratio of $(\Delta S)_c$ to the projected random error.

the large magnification factor m_s makes it useless, as can be seen from Eq. (5.5)

$$\begin{aligned}\sqrt{\langle (\Delta S)_c^2 \rangle} &= \sigma_E \sqrt{\frac{M}{N}} m_s \sqrt{\frac{1}{K}} \\ &= \sigma_E \sqrt{\frac{M}{N_t}} m_s, \end{aligned} \quad (5.5)$$

where N_t is the total number of digitized points $N \times K$.

This analysis is similar to the analysis of the error resulting from using a finite number of digitized points as described in Chapter 2. The above magnification can be considered in two steps. First, the projected digitization error introduces error on the finite surface wavefront over the digitized points. But, the error on the finite surface wavefront can be magnified because the finite surface wavefront is not orthogonal to the finite reference wavefront. Next, the continuous surface wavefront is magnified from the finite surface wavefront. This second magnification is the error from the finite digitization. In Fig. 5.1, 37 polynomials are used in the fit and 4 reference polynomials, constant, x-y tilt, and defocus, are assumed. Typical magnification from 50, 100, and 200 random distributions is shown. Also, the fit error from the continuous surface wavefront is not orthogonal to the finite surface wavefront and the finite reference wavefront. So, its projection onto the total polynomial space can introduce a magnified error on the surface wavefront, too.

As mentioned above, projection of random error into the best reference is small even if ill-conditioned. The reason is that even

if the surface polynomials are not orthogonal to each other, the reference polynomials are almost orthogonal to each other. So, there is no magnification from the correlation.

Therefore, let us delete the best reference from each interferogram and fit the total digitized data at one time with surface polynomials. Now, the total digitized data have random digitization error plus some random reference error

$$\begin{aligned}\sigma_E'^2 &= \sigma_E^2 + \sigma_E^2 \left(\frac{M_r}{N} \right) m_r^2 \\ &\approx \sigma_E^2\end{aligned}\tag{5.6}$$

where m_r is the average magnification factor of approximately 1. The number of reference polynomials M_r is much less than the number of digitized points N . Therefore, the projected error in the surface polynomial fit is

$$\begin{aligned}(\Delta S)_f &\approx \sigma_E \sqrt{\frac{M-M_r}{NK}} \\ &\approx \sigma_E \sqrt{\frac{M-M_r}{N_t}}\end{aligned}\tag{5.7}$$

This error is the rms surface wavefront error over the finite digitized points and is almost the same as the rms error in the continuous surface wavefront because $N \cdot K$ large numbers of points are considered. However, the projection of the surface wavefront into the best reference is different for each different interferogram as shown in Fig. 4.27c. This error of the best reference is added uncorrelatedly because the best

reference from each interferogram is orthogonal as shown in Fig. 5.2a. Then, the total sum of the best reference errors introduces error on the finite surface wavefront as shown in Fig. 5.2b. So, this method is not reliable if the surface figure error is large.

If a subaperture interferogram is used, we can find surface information from all the digitized data from several interferograms. Also, the relative reference of each interferogram is found from the digitized data of each interferogram. The subaperture size is the same as the entire aperture size, so there is no correlation between reference and surface space. Therefore, the magnification is small.

Fig. 5.2. Projection of surface into best reference.

- (a) Total sum of the best reference errors.
- (b) Surface error from the total sum of the best reference errors.

Let us estimate the projected surface error in rms over the digitized points. This projected error is almost the same as the rms surface error over the continuous aperture because all the digitized data are combined

$$(\Delta S)_f = \sigma_E \sqrt{\frac{M_r K + M - M_r}{NK}} m_s' , \quad (5.8)$$

where σ_E is the random digitized error in rms and m_s' is the average magnification of the projected random error. But m_s' is much smaller than m_s when coefficients are averaged (see Fig. 5.3). In Fig. 5.3, the same 37 polynomials are used. Five interferograms each with 50 random digitized points are combined for each simulation and five different simulations are used.

The m_s' from the K interferogram average, with N digitized points in each interferogram, is almost the same as m_s with $N*K$ digitized points. Therefore, as shown in Fig. 5.1, if more than 100 points are digitized in each interferogram, the average magnification does not decrease much. But the maximum magnification factor can still be reduced. So this method is more reliable than averaging of coefficients.

This subaperture analysis is also affected by the surface figure error, but only from the fit error. Therefore, the smooth surface does not degrade the accuracy. Furthermore, projection of the fit error is small and is not magnified because of the orthogonality between the surface space and the reference space.

Fig. 5.3. Ratio of $(\Delta S)_f$ to the projected random error.

The following simulation is exaggerated to show the difference between the three approaches. Five interferograms are used as an average and each interferogram is digitized randomly over 50 points. Thirty-seven polynomials are used in the fit and 4 reference polynomials are assumed. The results are shown in Table 5.1.

S_1 has flat surface figure error and S_2 has 5.74 rms surface figure error that is generated by $Z_4=1$, $Z_5=1$, ..., $Z_{37}=1$. The error is almost the same as the estimation. The approach of deleting the best reference is not reliable if the surface figure error is large, but the subaperture approach is reliable in any case.

Lateral Shearing Interferogram Analysis

The lateral shearing interferometer has been used frequently because it is self-referenced and can reduce the slope of the wavefront. However, it is not easy to analyze the interferograms. The old analysis methods are

1. Uniform grid digitization (Rimmer, 1974; Saunders, 1970).
2. Coefficient transformation (Rimmer and Wyant, 1975).
3. Fourier transformation (Gruenzel, 1976).

The lateral shearing interferometer is the same as other interferometers. It uses the interference of the reference and the test wavefront, but the reference is not as well defined as it is in an ordinary interferometer. This is due to the nature of the self-reference. However, if a polynomial representation is used for the reference wavefront, the reference wavefront can be approximated. Because total wavefront

Table 5.1. Rms surface wavefront error introduced by uniform random error of .577 rms.

Method	Surface	
	S_1	S_2
Averaging Coefficients	2.16	2.16
Deleting Best Reference	0.27	1.44
Subaperture Analysis	0.31	0.31

information is obtained from several shearing interferograms, the subaperture interferogram analysis can be applied. This subaperture interferogram analysis approach is similar to the coefficient transformation approach, but it is more general, simple, and accurate.

Let us review the coefficient transformation approach. The following material is extracted from Rimmer and Wyant (1975). From two x-sheared and y-sheared interferograms, two sets of polynomial coefficients A and B are obtained from the polynomial fit. Assume a set of coefficients C for the unknown wavefront. Find the two transformation matrices U and V such that the coefficients of the sheared wavefront can be obtained from the unknown wavefront. Calculate the two matrices G_1 and G_2 to obtain the variance of the two sheared wavefronts from the sets of coefficients. Then, minimize the variance of the wavefront obtained from the difference between the measured interferograms and those obtained from the unknown wavefront. Mathematically this becomes

$$W = (A-UC)^t G_1 (A-UC) + (B-VC)^t G_2 (B-VC) = \min. \quad (5.9)$$

Then the solution can be obtained by

$$C = Q^{-1} D \quad (5.10)$$

where

$$Q = U^t G_1 U + V^t G_2 V$$

$$D = U^t G_1 A + V^t G_2 B \quad .$$

For circular aperture interferograms, Zernike polynomials are used and, if the x-y shear is small, the matrices G_1 and G_2 are almost the same and diagonal. So, this approach can be easily applicable for circular interferograms. However, it is not easy to find the matrices G_1 and G_2 for the general aperture. Also, this approach cannot handle several undersampled shearing interferograms because the coefficients are not reliable as shown in the previous section.

Now, let us look at the subaperture approach. The relationship between the measured phase information S_i from the i th shearing interferogram and the i th wavefront W_i is

$$S_i(x,y) = W_i(x,y) - W_i(x+S_{xi}, y+S_{yi}) + N_i(x,y) \quad (5.11)$$

where N_i is the digitized error. Also, we can represent the i th wavefront W_i by

$$W_i(x,y) = \sum_{j=1}^{M_r} A_{ij} Z_j(x,y) + \sum_{j=M_r+1}^M A_j Z_j(x,y) + E(x,y) , \quad (5.12)$$

where M_r is the number of reference polynomials and E is the fit error term. Therefore, each wavefront may have different coefficients for the reference polynomials but should have the same coefficients for the surface polynomials. These coefficients of the surface polynomials are what we want to find. Therefore, we have

$$\begin{aligned}
 S_i(x,y) = & \sum_{j=1}^{M_r} A_{ij} (Z_j(x,y) - Z_j(x+S_{xi}, y+S_{yi})) \\
 & + \sum_{j=M_r+1}^M A_j (Z_j(x,y) - Z_j(x+S_{xi}, y+S_{yi})) \\
 & + E(x,y) - E(x+S_{xi}, y+S_{yi}) + N_i(x,y) \quad . \quad (5.13)
 \end{aligned}$$

Now, let us define a set of functions G_{ij} by

$$G_{ij}(x,y) = Z_j(x,y) - Z_j(x+S_{xi}, y+S_{yi}) \quad . \quad (5.14)$$

Then, it follows that

$$\begin{aligned}
 S_i(x,y) = & \sum_{j=1}^{M_r} A_{ij} G_{ij}(x,y) + \sum_{j=M_r+1}^M A_j G_{ij}(x,y) \\
 & + E'(x,y) + N_i(x,y) \quad . \quad (5.15)
 \end{aligned}$$

This equation is the same as that of the subaperture interferogram analysis. Let us define the surface polynomial by

$$G_j'(x,y) = \sum_{i=1}^K G_{ij}(x,y) \quad \text{for } j=M_r+1 \text{ to } M \quad (5.16)$$

where K is the number of different shearing interferograms. Now the least squares criterion is used

$$\begin{aligned} \sum_1 (S_1 - \sum_{j=1}^{M_r} A_{1j} G_{1j} - \sum_{j=M_r+1}^M A_j G_j)^2 + \dots \\ + \sum_K (S_K - \sum_{j=1}^{M_r} A_{Kj} G_{Kj} - \sum_{j=M_r+1}^M A_j G_j)^2 = \min. \end{aligned} \quad (5.17)$$

Then, we can restore the coefficients of the surface polynomials. However, even if the fit error E is orthogonal to the polynomial space formed by Z_i , the fit error E' is not orthogonal to the space formed by G_i' . Therefore, the fit error introduces some error in the coefficients of the surface polynomials just as in the subaperture interferogram analysis. The digitized error also introduces some error. This error magnification will be discussed later.

First, let us look at the sheared reference polynomials

$$\begin{aligned} \text{constant } (Z_1=1) &\rightarrow 0 & (G_{i1}=0) \\ \text{x-tilt } (Z_2=x) &\rightarrow \text{constant} & (G_{i2}=S_{xi}) \\ \text{y-tilt } (Z_3=y) &\rightarrow \text{constant} & (G_{i3}=S_{yi}) \end{aligned}$$

The i th sheared reference is

$$R_i = A_{i1} G_{i1} + A_{i2} G_{i2} + A_{i3} G_{i3} \quad . \quad (5.18)$$

We cannot find each reference coefficient because of the singularity. However, we do not want coefficients of the reference, but rather we

want to find each relative sheared reference R_i . This can be represented by one constant polynomial

$$R_i = A_{i1} G_{i1} \quad (G_{i1} = \text{constant}) \quad . \quad (5.19)$$

Therefore, each shearing interferogram is fitted with one different constant and with common sheared surface polynomials for the coherent defocus case. If each shearing interferogram has different defocus, each shearing interferogram should be fitted with a constant and a sheared defocus polynomial and common sheared surface polynomials. However, if N reference polynomials are used, at least $N+1$ different shearing interferograms are needed. If the total surface polynomial is made from $N+1$ subaperture surface polynomials, any total surface polynomial cannot be parallel to all the N reference polynomials of any shearing interferogram.

Generally, surface space is not orthogonal to reference space. So, the same analysis of the subaperture interferogram approach can be applied for error magnification. The error magnification analysis is the same as the subaperture interferogram analysis, but the magnification factor is not serious because of the large subaperture size. Also, the sensitivity of each coefficient can be obtained.

Let us look at some simulations. A typical magnification factor of projected random error on the surface wavefront is shown in Fig. 5.4. The magnification is not large if enough different shearing interferograms (at least three for incoherent defocus) are used. Also, increasing the number of shearing interferograms does not decrease the magnification

Fig. 5.4. Magnification factor from the digitized error vs number of interferograms.

37 polynomial fit and a 31 by 31 grid digitization were used.

- (a) Coherent defocus.
- (b) Incoherent defocus.

factor. However, large numbers of points can be combined by using several interferograms, so several shearing interferograms can be used for undersampled interferograms. The dashed line represents the average magnification factor. The magnification factor is inversely proportional to the shear distance for the coherent and incoherent defocus as shown in Fig. 5.5. The magnification factor of the fit error on the surface wavefront is shown in Fig. 5.6. It is much smaller compared with the magnification factor of the projected random error. Typical sensitivity of the coefficients is shown in Fig. 5.7. The sensitivity is not uniform and coefficients of the lower order polynomials are sensitive to random error. The sensitivity is drawn in increasing order of the polynomials from M_r+1 to M . The sensitivity of Z_4 is not drawn for the incoherent case.

Fig. 5.5. Magnification factors vs shear distance.

37 polynomial fit and a 31 by 31 grid digitization were used.

(a) Two shears with coherent defocus.

(b) Three shears with incoherent defocus.

Fig. 5.6. Magnification factors of surface error to the fit error.

3rd polynomials were used in the fit error and 12 higher order polynomials were used in the simulation of fit error. A 31 by 31 grid digitization was used.

- (a) 0.1 radius shear with coherent defocus.
- (b) 0.1 radius shear with incoherent defocus.

Fig. 5.7. Sensitivity of surface coefficients.

37 polynomials were used in the fit and 12 higher order polynomials were used in the simulation of fit error. A 31 by 31 grid digitization was used.

- (a) Sensitivity of surface coefficients to random error.
- (b) Sensitivity of surface coefficients to fit error.

CHAPTER 6

CONCLUSIONS

The polynomial fit of circular aperture interferograms with Zernike polynomials is reviewed and the advantages of using a set of orthonormal polynomials are presented. However, the polynomial is orthogonal only over a continuous aperture and not over digitized points. The effect of non-orthogonality is quantitatively and statistically analyzed, and it is shown that non-orthogonality does not degrade accuracy much if only uncorrelated information like the best reference wavefront or the best surface wavefront is wanted. This suggests the possible use of non-orthogonal polynomials.

The results from both Zernike polynomials and x-y polynomials are compared and indicate it is possible to use non-orthogonal polynomials. Non-circular aperture interferograms are analyzed with non-orthogonal Zernike polynomials because the Zernike polynomial is familiar and is used for the analysis of circular aperture interferograms. Its limitations are presented, but it is shown that it can be used for the analysis of several kinds of non-circular aperture interferograms with only minor modification like deleting correlated and useless polynomials.

On the other hand, if correlated information like the relative surface wavefront or the relative reference wavefront is desired for a subaperture interferogram analysis, the error is magnified. Therefore,

error magnification limits the use of a smaller subaperture. However, from statistical analysis, it is shown that up to a $1/4$ subaperture reference can be used to test a large flat or a large parabola with coherent defocus. Fast optics can be tested with up to a $1/3$ subaperture reference with incoherent defocus. The maximum slope of the aspheric surface can be reduced by a factor of about 8 with four annular subapertures.

Subaperture analysis using overlapped wavefront information is compared with the polynomial fit approach. This technique requires digitizing the same data points over the overlapped area, but the possibility of using different digitized data points is presented. It is shown that if about 50 points are digitized over each circular overlapped area, the surface wavefront can be connected by deleting the difference of the relative reference wavefront.

Finally a subaperture interferogram analysis is applied to the averaging of several interferograms and to the analysis of lateral shearing interferograms. Both must be done with several interferograms, so a subaperture interferogram analysis can be easily used. It can average undersampled interferograms and analyze the lateral shearing interferograms more accurately.

APPENDIX A

BASIC MATHEMATICS

The following algebra is essential to understanding the material in this dissertation. The explanation is brief, but more detailed information can be found in Forsythe and Moler (1967), Lawson and Hanson (1974), Stewart (1973), and Forsythe, Malcolm and Moler (1977).

Norm of Vector and Matrix

Let us consider a vector in an n-dimensional space. The norm or Euclidean length of a vector is defined by

$$\begin{aligned} ||x|| &= (x^t x)^{\frac{1}{2}} \\ &= \left(\sum_{i=1}^n x_i^2 \right)^{\frac{1}{2}} \end{aligned} \quad (A.1)$$

where

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad .$$

Now, let us define the norm of an N by N matrix A by

$$\begin{aligned} ||A|| &= \text{Max.}_{x \neq 0} \left(\frac{||Ax||}{||x||} \right) \\ &= \text{Max.}_{||x||=1} (||Ax||) \end{aligned} \quad (A.2)$$

The matrix A transforms the unit sphere to an ellipsoid, as shown in Fig. A.1. Then, the norm of the matrix A is just the maximum radius of the ellipsoid. Generally, the norm of the matrix satisfies

$$||AB|| < ||A|| ||B|| \quad (\text{A.3})$$

Singular Values Analysis

An M by N matrix equation can be considered as a transformation of an N -dimensional vector x to an M -dimensional vector y

$$Ax = y \quad (\text{A.4})$$

where

$$A = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1N} \\ A_{21} & A_{22} & \dots & A_{2N} \\ \vdots & & & \\ A_{M1} & A_{M2} & & A_{MN} \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} \quad \text{and } y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix}$$

Usually, the norm of a vector is changed under transformation. However, an orthogonal matrix whose transpose matrix is its inverse matrix does not change the norm of vectors. Suppose

$$Ux = y \quad (\text{A.5})$$

where

$$UU^t = U^tU = I \quad , \quad (\text{A.6})$$

Fig. A.1. Norm and condition number of a matrix A.

then

$$\begin{aligned}
 \|y\| &= (y^t y)^{\frac{1}{2}} \\
 &= [(Ux)^t (Ux)]^{\frac{1}{2}} \\
 &= [x^t U^t U x]^{\frac{1}{2}} \\
 &= (x^t x)^{\frac{1}{2}} \quad (\text{from Eq. A.6}) \\
 &= \|x\| \quad .
 \end{aligned} \tag{A.7}$$

As shown in Eq. (A.6), each column vector is orthogonal to the other column vectors and it is the same for the row vector

$$\begin{aligned}
 \sum_{j=1}^N U_{ij} U_{kj} &= \delta_{ik} \\
 \sum_{i=1}^N U_{ij} U_{ik} &= \delta_{jk} \quad .
 \end{aligned} \tag{A.8}$$

Generally, an M by N matrix A can be decomposed with two orthogonal matrices U and V and a diagonal matrix S. This is called the singular values decomposition. A FORTRAN program of the singular values decomposition is available (Forsythe, Malcolm and Moler, 1977, pp. 229-235).

Therefore,

$$A = U S V \quad (\text{A.9})$$

where A is an M by N matrix, U is an M by M orthogonal matrix, S is an M by N diagonal matrix and V is an N by N orthogonal matrix. Now, let us consider M by N linear equations

$$\begin{aligned} y &= Ax \\ &= USVx \end{aligned} \quad (\text{A.10})$$

Let us define an M-dimensional vector U_i and an N-dimensional vector V_i^t by

$$U_i = \begin{pmatrix} U_{1i} \\ U_{2i} \\ \vdots \\ U_{Mi} \end{pmatrix} \quad V_i^t = \begin{pmatrix} V_{i1} \\ V_{i2} \\ \vdots \\ V_{iN} \end{pmatrix} \quad (\text{A.11})$$

Then, any N-dimensional vector x can be decomposed with N orthonormal vectors V_i^t and any M-dimensional vector y can be decomposed with M orthonormal vectors U_i

$$\begin{aligned}
 x &= x_1 V_1^t + x_2 V_2^t + \dots + x_N V_N^t \\
 y &= y_1 U_1 + y_2 U_2 + \dots + y_M U_M .
 \end{aligned}
 \tag{A.12}$$

Furthermore, from the orthonormality of U_i and V_i^t ,

$$\begin{aligned}
 A V_i^t &= U S V V_i^t \\
 &= U S \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (\text{only } i\text{th row is } 1) \\
 &= S_{ii} U_i \quad (\text{matrix } S \text{ is diagonal}) \\
 &\quad \text{or } 0 \quad \text{if } i > M
 \end{aligned}
 \tag{A.13}$$

Therefore, the norms of vector x and y can be obtained by

$$||x|| = \left(\sum_{i=1}^N x_i^2 \right)^{\frac{1}{2}}
 \tag{A.14}$$

$$\begin{aligned}
 ||y|| &= ||Ax|| = \left(\sum_{i=1}^M y_i^2 \right)^{\frac{1}{2}} \\
 &= \left(\sum_{i=1}^M S_{ii}^2 x_i^2 \right)^{\frac{1}{2}}
 \end{aligned}
 \tag{A.15}$$

In conclusion, the singular values decomposition provides M orthonormal base vectors to represent M -dimensional vector y and N orthonormal base vectors to represent N -dimensional vector x . Furthermore, the matrix

transformation from an N-dimensional space to an M-dimensional space can be represented with the singular values S_{ii} . If $M < N$, we can imagine that $S_{ii} = 0$ for $i > M$ although S is an M by N diagonal matrix. Then, we can find N parameters that represent the transformation of N base vectors.

Suppose we are interested in the magnification of the norm of vector y from the norm of vector x by the matrix A when the vector x moves, along the unit sphere

$$\begin{aligned} ||y|| &= ||Ax|| \\ &= \left(\sum_{i=1}^N S_{ii}^2 x_i^2 \right)^{1/2} \quad \left(\text{for } \sum_{i=1}^N x_i^2 = 1 \right) \end{aligned} \quad (\text{A.16})$$

Let us consider the average of (expectation value of) the magnification. Because of symmetry

$$\begin{aligned} \int \dots \int_{\sum_{i=1}^N x_i^2=1} x_i^2 dx_1 dx_2 \dots dx_N &= \int \dots \int_{\sum_{i=1}^N x_i^2=1} x_j^2 dx_1 dx_2 \dots dx_N \\ &= \frac{1}{N} \int \dots \int_{\sum_{i=1}^N x_i^2=1} (x_1^2 + x_2^2 + \dots + x_N^2) dx_1 dx_2 \dots dx_N \\ &= \frac{1}{N} \int \dots \int_{\sum_{i=1}^N x_i^2=1} dx_1 dx_2 \dots dx_N \quad (\text{because } x_1^2 + x_2^2 + \dots + x_N^2 = 1) \end{aligned} \quad (\text{A.17})$$

Let us assume

$$\iint \dots \int \text{ is } \iint \dots \int_{\sum_{i=1}^N x_i^2 = 1} .$$

Then

$$\begin{aligned} \left\langle \frac{||y||^2}{||x||^2} \right\rangle &= \left\langle ||y||^2 \right\rangle \quad (\text{because } ||x|| = 1) \\ &= \frac{\iint \dots \int \left(\sum_{i=1}^N S_{ii}^2 x_i^2 \right) dx_1 dx_2 \dots dx_N}{\iint \dots \int dx_1 dx_2 \dots dx_N} \\ &= \frac{\sum_{i=1}^N S_{ii}^2 \iint \dots \int x_i^2 dx_1 dx_2 \dots dx_N}{\iint \dots \int dx_1 dx_2 \dots dx_N} \\ &= \frac{1}{N} \sum_{i=1}^N S_{ii}^2 \quad (\text{from Eq. A.17}) \\ &= \text{Avg. } (S_{ii}^2) \end{aligned} \tag{A.18}$$

Condition Number and Error Estimation

The condition number of a non-singular matrix A is defined by

$$\text{COND}(A) = ||A|| \ ||A^{-1}|| \tag{A.19}$$

From the above singular value analysis, we can find the norm of a matrix A

$$||A|| = \text{maximum of singular values.}$$

Also, we can find the singular values decomposition of A^{-1}

$$\begin{aligned} A^{-1} &= (U S V)^{-1} \\ &= V^{-1} S^{-1} U^{-1} \\ &= V^t S^t U^t \end{aligned} \tag{A.20}$$

Therefore, the singular values of A^{-1} are the inverses of the singular values of A . So, the norm of A^{-1} is

$$||A^{-1}|| = \text{inverse of the minimum of the singular values of } A.$$

Finally, the condition number of a matrix A can be defined by

$$\text{COND}(A) = \frac{\text{Maximum singular value of } A}{\text{Minimum singular value of } A} . \tag{A.21}$$

The meaning of the condition number is explained graphically in Fig. A.1.

Suppose there is some error in the measurement of the vector y .

Then the linear equations become .

$$A(x + \Delta x) = (y + \Delta y) , \tag{A.22}$$

where Δy is the measurement error of y and Δx is the resulting error introduced in x . Then, it follows that

$$\begin{aligned} ||y|| &= ||Ax|| \\ &< ||A|| ||x|| \end{aligned} \tag{A.23}$$

$$\begin{aligned}
||\Delta y|| &= ||A\Delta x|| \\
&> ||\Delta x|| / ||A^{-1}|| \quad (\text{because } A^{-1}\Delta y = \Delta x) \quad . \quad (\text{A.24})
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{||\Delta x||}{||x||} &< ||A|| \, ||A^{-1}|| \, \frac{||\Delta y||}{||y||} \\
&< \text{COND}(A) \, \frac{||\Delta y||}{||y||} \quad (\text{A.25})
\end{aligned}$$

Consequently, the relative error in x introduced by the measurement error of y can be magnified by the factor of condition number from the relative error of the measurement.

Now, let us estimate the roundoff error of the linear equation $Ax=y$. Suppose the base of the floating point number is β and the number of digits is t , then the calculated solution x^* satisfies

$$(A+E)x^* = y \quad . \quad (\text{A.26})$$

Furthermore, we know that (Forsythe, Malcolm and Moler, 1977, p. 46)

$$\frac{||E||}{||A||} = \gamma \beta^{-t} \quad (\text{A.27})$$

where γ will rarely be bigger than β . Therefore, it follows that

$$\begin{aligned}
||y-Ax^*|| &= ||Ex^*|| \\
&< ||E|| \, ||x^*|| \quad (\text{A.28})
\end{aligned}$$

and

$$\frac{||y - Ax^*||}{||A|| ||x^*||} < \gamma\beta^{-t} \quad (\text{from Eq. A.27}) \quad (\text{A.29})$$

Also, the error of the solution is

$$x - x^* = A^{-1}(y - Ax^*) \quad (\text{A.30})$$

Therefore, the norm of error satisfies

$$\begin{aligned} ||x - x^*|| &< ||A^{-1}|| ||E|| ||x^*|| \\ &< ||A^{-1}|| ||A|| ||x^*|| \gamma\beta^{-t} \end{aligned} \quad (\text{A.31})$$

Finally

$$\frac{||x - x^*||}{||x^*||} < \text{COND}(A) \gamma\beta^{-t} \quad (\text{A.32})$$

$$||A^t P A||$$

Because P is symmetric

$$P = V^t S V \quad (\text{A.33})$$

So,

$$||A^t P A|| = ||(VA)^t S (VA)|| \quad (\text{A.34})$$

Therefore

$$(S_{ii})_{\min} ||VA||^2 < ||A^t P A|| < (S_{ii})_{\max} ||VA||^2 \quad (\text{A.35})$$

Furthermore

$$||VA|| = ||A|| \quad (\text{because V is orthogonal})$$

Therefore

$$\frac{||A||^2}{||P^{-1}||} < ||A^t P A|| < ||P|| ||A||^2 \quad . \quad (A.36)$$

$$\underline{\text{COND}(Z^t Z)}$$

Suppose Z is decomposed by singular values analysis, then

$$Z = U S V \quad (A.37)$$

and

$$\text{COND}(Z) = \frac{(S_{ii})_{\max}}{(S_{ii})_{\min}} \quad . \quad (A.38)$$

On the other hand

$$\begin{aligned} Z^t Z &= V^t S^t U^t U S V \\ &= V^t S^t S V \\ &= V^t S^2 V \quad (S = S^t) \quad . \end{aligned} \quad (A.39)$$

Now

$$\begin{aligned} \text{COND}(Z^t Z) &= \frac{(S_{ii})_{\max}^2}{(S_{ii})_{\min}^2} \\ &= \text{COND}^2(Z) \quad . \end{aligned} \quad (A.40)$$

REFERENCES

- Davis, Nancy, and T. A. Fritz, "Application of Zernike Polynomials to Reduction of Wavefront Slope Data," Optical Fabrication and Testing Workshop, OSA Technical Notebook, pp. 39-42, Tucson, AZ (1979).
- Forsythe, G. E., M. A. Malcolm, and C. B. Moler, Computer Methods for Mathematical Computations, Prentice Hall, Inc., Englewood Cliffs, N.J. (1977).
- Forsythe, G. E., and C. B. Moler, Computer Solution of Linear Algebraic Systems, Prentice Hall, Inc., Englewood Cliffs, N.J. (1967).
- Gruenzel, R. R., "Application of Lateral Shearing Interferometry to Stochastic Inputs," JOSA, 66, 1341 (1976).
- Lawson, C. L., and R. J. Hanson, Solving Least Squares Problems, Prentice Hall, Inc., Englewood Cliffs, N.J. (1974).
- Loomis, J. S., FRINGE User's Manual, Optical Sciences Center, University of Arizona (1976).
- Loomis, J. S., "Analysis of Interferograms from Waxicons," Optical Components: Manufacture and Evaluation Proc., SPIE, 171, 64 (1979).
- Parks, R. E., "Removal of Test Optics Error," Advances in Optical Metrology Proc., SPIE, 153, 56 (1978).
- Rimmer, M. P., "Method for Evaluating Lateral Shearing Interferograms," Appl. Opt., 13, 623 (1974).
- Rimmer, M. P., C. M. King, and D. G. Fox, "Computer Program for the Analysis of Interferometric Test Data," Appl. Opt., 11, 2790 (1972).
- Rimmer, M. P., and J. C. Wyant, "Evaluation of Large Aberrations Using a Lateral-Shear Interferometer Having Variable Shear," Appl. Opt., 14, 142 (1975).
- Saunders, J. B., "A Simple Interferometric Method for Workshop Testing of Optics," Appl. Opt., 9, 1623 (1970).

- Shu, Ker-Li, R. E. Parks, and R. R. Shannon, "Exact Ray Trace Analysis of the Ritchey-Common Test," JOSA, 71, 1587 (1981).
- Stewart, G. W., Introduction to Matrix Computations, Academic Press, New York (1973).
- Swantner, W. M., and W. H. Lowrey, "Zernike-Taitan Polynomials for Interferogram Reduction," Appl. Opt., 19, 161 (1980).
- Underwood, K. L., and G. Woodfin, "Use of Zernike Polynomials in Evaluating Interferograms," JOSA, 70, 1591 (1980).
- Wang, J. Y., and D. E. Silva, "Wave-front Interpretation with Zernike Polynomials," Appl. Opt., 19, 1510 (1980).