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NUMERICAL PREDICTIONS FOR UNSTEADY VISCOUS FLOW PAST AN
ARRAY OF CYLINDERS

The University of Arizona

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NUMERICAL PREDICTIONS FOR UNSTEADY VISCOUS FLOW
PAST AN ARRAY OF CYLINDERS

by

Edward Andrew Cerutti

A Dissertation Submitted to the Faculty of the
DEPARTMENT OF AEROSPACE AND MECHANICAL ENGINEERING

In Partial Fulfillment of the Requirements
For the Degree of

DOCTOR OF PHILOSOPHY
WITH A MAJOR IN MECHANICAL ENGINEERING

In the Graduate College
THE UNIVERSITY OF ARIZONA

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NOMENCLATURE

Latin

CD	drag coefficient $[\text{Drag}/(1/2 \rho U^2 D)]$
CD _f	friction drag coefficient
CD _p	pressure drag coefficient
CL	lift coefficient $[L/(1/2 \rho U^2 D)]$
CL _f	friction lift coefficient
CL _p	pressure lift coefficient
CM	moment coefficient $[M/(1/2 \rho U^2 D^2)]$
D	cylinder diameter, drag
DHL _{i,j}	length of left hand side of cell i,j
DHR _{i,j}	length of right hand side of cell i,j
DWL _{i,j}	length of lower side of cell i,j
DWU _{i,j}	length of upper side of cell i,j
$\vec{e}_\theta$	unit vector in the θ direction
$\vec{e}_r$	unit vector in the radial direction
F1,F2	weighting factors in the vorticity transport equation
F3,F4	weighting factors in the vorticity transport equation
L	lift
M	moment
p	pressure $[p/(1/2 \rho U^2)]$
$\vec{q}$	fluid velocity vector
$\vec{r}$	position vector

NOMENCLATURE--continued

$\vec{r}_{op}$	$\vec{r}_p - \vec{r}_o$
Re	Reynolds number (UD/ν)
$RB_{i,j}$	distance from node i,j to node $i,j-1$
$RL_{i,j}$	distance from node i,j to node $i-1,j$
$RR_{i,j}$	distance from node i,j to node $i+1,j$
$RU_{i,j}$	distance from node i,j to node $i,j+1$
t	time (tU/D)
$\vec{U}$	onset flow velocity
u	velocity component in the x direction
$\vec{V}_p$	velocity induced at an arbitrary point p
v	velocity component in the y direction

Greek

α	arbitrary constant used to combine particular and complementary solutions for the determination of γ , angle employed for Biot-Savart integration
γ	bound vorticity (γ/U)
δ	half the center-to-center distance between cylinders in the array
ϵ	infinitesimal distance
η	body oriented coordinate which increases in the radial direction
μ	dynamic viscosity
ν	kinematic viscosity
ξ	body oriented coordinate which increases clockwise in the tangential direction
ρ	density

NOMENCLATURE--continued

τ	shear stress [$\tau/(1/2 \rho U^2)$]
τ_0	shear stress at the surface of the cylinder
ω	free vorticity ($\omega D/U$)
ω^+	vorticity at time $t + \Delta t$

Subscripts

i	position index in the ξ direction, general index
j	position index in the η direction
n	indicates number associated with a particular cylinder in the array
o	indicates position at point o
p	indicates position at point p
q	indicates position at point q on the surface of a cylinder

ABSTRACT

The unsteady two-dimensional flow around an array of circular cylinders submerged in a uniform onset flow is analyzed. The fluid is taken to be viscous and incompressible. The array of cylinders consists of two horizontal rows extending to infinity in the upstream and downstream directions. The center-to-center distance between adjacent cylinders is a constant.

The Biot-Savart law of induced velocities is used to determine the velocity field due to the free vorticity in the surrounding fluid and the bound vorticity distributed on the surface of each cylinder. The bound vorticity is needed to enforce the no-penetration condition and to account for the production of free vorticity at the solid surfaces. It is governed by a Fredholm integral equation of the second kind. This equation is solved by numerical techniques.

The transport of free vorticity in the flow field is governed by the vorticity transport equation. This equation is discretized for a control volume and is solved numerically. Advantage is taken of spatially periodic boundary conditions in the flow direction. This reduces the computational domain to a rectangular region surrounding a single circular cylinder, but necessitates use of a non-orthogonal grid. In order to test the numerical techniques, the simpler case of unsteady flow over a single circular cylinder at various Reynolds numbers is first considered. Results compare favorably with previous experimental and numerical data.

Three cases for Reynolds numbers of 10^2 , 10^3 , and 10^4 are presented for the array of cylinders. The center-to-center distance is fixed at three diameters. The time development of constant vorticity contours as well as drag, lift, and moment coefficients are shown for each Reynolds number. The motion of stagnation and separation points with time is also given. It is found that the drag for a cylinder in the array may be as low as five percent of that for flow over a single cylinder at the same Reynolds number.

CHAPTER 1

INTRODUCTION

The advent of high-speed computers with capacious memory has revolutionized fluid flow research in the sciences and engineering. Complicated flow problems, once considered to be intractable, are now amenable to analysis through the application of numerical-solution techniques to the basic governing equations.

In the present work, an analysis is made of two-dimensional unsteady viscous incompressible flow over an array of circular cylinders arranged in two horizontal rows as shown in Figure 1.1. The two rows are assumed to extend to infinity in the positive and negative x -directions. Although the onset flow is considered to be uniform at large vertical distances above and below the array, unsteady effects arise as the flow develops around the cylinders. The onset flow is assumed to start impulsively from an initial state of rest.

This general type of flow, in which the wake from one body interferes with that of the next, has practical applications. For example, the present flow configuration is found in cross-flow heat exchangers. The determination of the forces acting on the solid bodies in this interference-type flow, and the location of separation points, is of great practical importance and requires understanding of the entire flow field around the bodies. Although this study is concerned with an infinite number of circular cylinders, the analytical and

Figure 1.1 Illustration of the flow geometry.

numerical methods employed are applicable to bodies with identical but arbitrarily shaped cross-sections and various staggered configurations which are spatially periodic.

A search through the literature for studies concerned with unsteady incompressible viscous flow over two or more bodies in the same flow field has yielded extremely limited results. The only studies found are for low Reynolds number flow past two circular cylinders. They include Yano and Kieda (1980), who used linearized versions of the Navier-Stokes equations, and Umemura (1982), who employed the method of matched asymptotic expansions.

In contrast to the above, the closely related problem of unsteady flow over a single circular cylinder has received a great deal of attention from numerous researchers. A summary of that work is necessary in order to place the present study in perspective. In addition, many of the techniques employed in this work are drawn from previous studies. An excellent survey of this specific type of work may be found in McCroskey (1977). His paper covers the development of unsteady fluid dynamics up to 1977, including linear potential theory, transonic flows, unsteady boundary layers, unsteady stall, and vortex shedding. Since the present study is strictly a numerical analysis, no experimental work is referenced.

Most methods of analyzing the unsteady flow over circular cylinders employ the vorticity transport equation and may be separated into two categories. The first determines the velocity field through the use of the stream function, while the second employs integral

relationships which are variations of the Biot-Savart law of induced velocities. Some solutions which employ the Navier-Stokes equations directly, or in boundary layer form, are also discussed.

The vorticity-stream function approach to the solution of unsteady flow over a circular cylinder is the conventional route taken by many researchers. Wang (1967) employed the method of matched asymptotic expansions, in conjunction with this approach, to model the flow in the initial stages of its development for high Reynolds numbers (1000 to ∞). Son and Hanratty (1969) solved the vorticity and stream function equations numerically for Reynolds numbers of 40, 200, and 500. Thoman and Szewczyk (1969) later extended this approach to a Reynolds number of 3×10^5 . They employed a hybrid cell structure composed of cylindrical cells around the cylinder surface and variable-width rectangular cells in the remainder of the flow region. Jordan and Fromm (1972) employed a correction to the free stream values for the stream function at the outer edge of the computational domain. Their calculations were run for Reynolds numbers of 10, 40, 100, 400, and 1000. A variation of this approach was presented by Collins and Dennis (1973). They employed a stream-function formulation and Fourier series expansion to reduce the unsteady equations of motion to a series of ordinary nonlinear differential equations of second order. The Reynolds number of their results varies from 5 to infinity. However, above a Reynolds number of 100, the implicit technique which they employed to advance the flow in time experienced convergence problems. Bar-Lev and Yang (1975) once again employed the method of

matched asymptotic expansions, coupled with the vorticity-stream function approach, to analyze the flow field in its initial stages of development.

The alternative approach taken by researchers in the calculation of the velocity field has been to employ an integral relation which can be derived from the Biot-Savart law of induced velocities. One of the earliest studies of unsteady viscous flow past a circular cylinder, which also employed this novel technique, was performed by Payne (1958). He investigated flow at Reynolds numbers of 40 and 100. The solid boundary, in his work, was duplicated through the use of image vortices, while the no-slip condition was enforced by vorticity generation on the surface of the cylinder. Unfortunately, the concept of image vortices is extremely difficult to apply to geometries other than circular cylinders and to multiple bodies. Wu and Thompson (1971 and 1973) employed a similar approach for Reynolds numbers up to 300. In their case, the vorticity at the wall was obtained from the curl of the velocity, with knowledge of the velocity boundary conditions. A rectangular coordinate system made it difficult to apply boundary conditions at the surface, and to apply velocity derivatives. Nevertheless, satisfactory results were obtained for the force coefficients. An extremely interesting and novel variation to this technique was presented by Chorin (1973). In his method, vorticity production at the surface of the body and diffusion were handled through a random-number numerical simulation. He was thus able to avoid the use of a computational grid. The Reynolds numbers in this study extended

to 10,000. Panniker and Lavan (1975) investigated the unsteady flow over circular and elliptic cylinders. Although the stream function appears in their formulation, an integral relation is employed to determine the velocity field. They employed Green's functions to relax the far field boundary conditions for the stream function, and obtained results at a Reynolds number of 500 for flow over a cylinder.

Some researchers have investigated the unsteady flow over circular cylinders by solving the Navier-Stokes equations directly. Deffenbaugh and Marshall (1976) interfaced an inner boundary-layer solution with an outer potential discrete-point-vortex wake solution. Their work was concentrated in the high laminar region ($10^4 < Re < 10^5$). Daube and Loc (1978) numerically solved the Navier-Stokes equations for this type of flow. They employed a combination of second-order and fourth-order accurate finite-difference schemes. Loc (1980) analyzed the growth of primary and secondary vortices with time for Reynolds numbers of 300, 500 and 1000. He employed a combination of second-order and fourth-order compact finite-difference schemes to analyze the unsteady nature of this flow. Cowley (1983) extended the Blasius expansion in order to analyze the initial stages of flow development.

The technique employed in the present study employs the vorticity transport equation in conjunction with the Biot-Savart law of induced velocities. Unlike other formulations, it completely avoids calculation of the vorticity at the body surface. A relationship for the direct evaluation of the flux of vorticity at the wall makes this

step unnecessary. The methods utilized in the present work are the logical extension of previous studies which, although not necessarily concerned with the flow over a cylinder, are worthy of mention.

Kinney and Paolino (1974) pioneered the present method in the study of unsteady flow near the leading edge of a semi-infinite flat plate at zero angle of attack. Schmall and Kinney (1974) then extended the technique to unsteady viscous flow past a flat plate at different angles of attack. Kinney and Cielak (1977) [see also Cielak and Kinney (1978)] investigated the unsteady viscous flow past a two-dimensional airfoil in finite thickness. They obtained results at angles of attack of zero and 30 degrees for a Reynolds number of 400. Recently, Taslim (1981) [see also Taslim, Kinney and Paolino (1984)] applied the technique to analyze the unsteady two-dimensional flow over elliptical bodies in arbitrary unsteady motion. The results include flow over circular cylinders with constant and varying angular velocities as well as flow over elliptical cylinders of varying aspect ratios undergoing translation and pitching oscillation.

The objective of the present work is to apply the above-mentioned analytical and numerical techniques to analyze the unsteady flow past multiple cylinders in a regular array. The flow is assumed to be spatially periodic in the flow direction. In addition, there is top-bottom alternating symmetry. Such a pattern of flow is amenable to analysis using currently available computers. Specific items to be investigated include grid design and automatic generation, as well as methods for enforcing the spatial periodicity of the flow. The use of

the Biot-Savart law for the velocity calculation precludes the enforcement of velocity boundary conditions on the periphery of the computational grid.

From the predicted results, it will be possible to determine the drag on any cylinder in the array. Also of interest will be the time-periodic nature of the flow. The location and movement of flow-stagnation and separation points will aid in understanding the dynamics of the wake development and the effect of cylinder-to-cylinder interference on the mechanism of vortex shedding.

CHAPTER 2

THEORETICAL ANALYSIS

Unsteady two-dimensional flow problems are generally treated through the use of the scalar vorticity transport equation,

$$\frac{D\omega}{Dt} = \nu \nabla^2 \omega \quad (2.1)$$

which is obtained by taking the curl of the incompressible form of the two-dimensional Navier-Stokes equations. Conventional methods then introduce the stream function Ψ , which is related to velocity through the relationships

$$u = \frac{\partial \Psi}{\partial y} \quad (2.2)$$

$$v = - \frac{\partial \Psi}{\partial x} \quad (2.3)$$

Since the vorticity is defined as the curl of the velocity, the relationship

$$\nabla^2 \Psi = -\omega \quad (2.4)$$

is obtained.

In order to obtain the vorticity distribution, Equation 2.1 is solved. Once the vorticity distribution is known, the stream function is obtained through the use of Equation 2.4. The velocity distribution is then obtained from the stream function using Equations 2.2 and 2.3.

This method suffers from the fact that numerical difficulties can be encountered when solving the system of coupled partial differential equations, especially when iterative methods are employed. Furthermore, the method is only applicable to two-dimensional flows since a stream function cannot be defined in three dimensions. The alternative approach of directly solving the Navier-Stokes equations requires that the pressure distribution over the fluid region be obtained. The pressure satisfies a Poisson-type equation, analogous to that satisfied by the stream function. Therefore, it must be calculated over the entire region of the fluid where it differs from the pressure in the undisturbed external flow.

In order to bypass those limitations, the method utilized in this work is based on an alternative approach which was originally suggested by Lighthill (Rosenhead, 1963, pp. 57-60). This particular approach decomposes the analysis into purely kinematical and dynamical portions. The kinematical part is governed by the Biot-Savart law of induced velocities, while the dynamical part, which deals with the production and transport of vorticity, is governed by the vorticity transport equation.

In order to apply the Biot-Savart law, the flow field must be simply connected. To satisfy this requirement, bodies are replaced by fluid with the same density as the undisturbed surroundings. In the particular case treated here, where the various bodies remain motionless with respect to an onset flow, this is achieved by distributing over the cross-section of all bodies, fluid which is not in motion.

On the surface of the bodies is placed a distribution of concentrated vorticity in the form of a vortex sheet. This vortex distribution will henceforth be called the "bound vorticity" since its location always remains fixed to the body surface. This bound vorticity (referred to by the symbol γ) is taken to be the amount of total vorticity per unit surface area which is produced at a given instant in time and which diffuses into the fluid over a given time interval. It has units of length/time or velocity.

The diffusion of bound vorticity gives rise to "free vorticity", ω , in the surrounding fluid. The distribution of this free vorticity is governed by the vorticity transport equation (2.1).

For any given distribution of ω and prescribed motion of the surrounding fluid, a distribution of γ can always be found such that the internal fluid of the various bodies has zero velocity (Hess and Smith, 1967). This bound vorticity is governed by a Fredholm integral equation of the second kind. The distribution of γ is made unique by enforcing a global vorticity conservation principle. In the case of this work, since the bodies in the flow field are not undergoing rotation, the vorticity produced over the surface of each body cannot produce any change in circulation around that specific body with time.

The relationship between the bound and free vorticities is determined by the enforcement of the zero-slip condition at all points on the surface of every body. At any instant in time, the fluid may have an "apparent" tangential velocity on the surface of any of the bodies. This slip velocity is numerically equal to $-\gamma$ at each point

on the surface of the body. Lighthill (Rosenhead, 1963, pp. 57-60) explains that this slip velocity must be reduced to zero by the generation of a precise amount of free vorticity at each point on the body. This vorticity production thus relates the free and bound vorticities. As the flow develops, this vorticity produced on the surface of every body diffuses into the flow and becomes free vorticity. The vorticity transport equation then governs the subsequent distribution of this free vorticity.

This chapter presents the theoretical aspects of the analysis. All bodies are assumed to be circular in cross-section and the fluid is assumed to have constant properties. The first part deals with the kinematic relationships between the velocity and vorticity fields. Here the Biot-Savart law of induced velocities is employed. In addition, the development of the integral equation necessary in the determination of the bound vorticity is shown. The second part covers the dynamic aspects of this problem. Here the vorticity transport equation is derived, and the interrelationship between the kinematical and dynamic portions of the work are discussed.

The Biot-Savart Law of Induced Velocities

Knowledge of the velocity field yields the vorticity field in a straightforward manner. From the definition of vorticity, (i.e., $\vec{\omega} = \text{curl } \vec{V}$), $\vec{\omega}$ can be obtained through relatively simple differentiation. The reverse procedure, however, involves the difficulty of an integration, and in many cases, it must be performed numerically.

In the present work, the velocity field is determined from the integral relationship which gives the Biot-Savart Law of Induced Velocities (Milne-Thomson, 1966; Karamcheti, 1966). This law requires two conditions to be met. The first is that the flow field must be simply connected and unbounded. The second requirement is that at a great distance from the vortex element, the speed of the fluid must be at least of order $\frac{1}{|\vec{r}_0 - \vec{r}|^2}$. Please refer to Figure 2.1. This last condition ensures that the fluid is at rest at infinity. Of course, a uniform onset flow can always be superimposed without affecting ω . In such a case, the vorticity is said to induce a perturbation velocity field to the uniform onset flow.

In three dimensions, the Biot-Savart law is expressed as,

$$\vec{V}_p(r,t) = - \frac{1}{4\pi} \iiint \frac{\vec{\omega}_0 \times (\vec{r}_0 - \vec{r})}{|\vec{r}_0 - \vec{r}|^3} dV \quad (2.5)$$

where the vector quantities $\vec{\omega}_0$, $\vec{r}$ and $\vec{r}_0$ are shown in Figure 2.1. $\vec{V}_p$ is the velocity of the fluid at point P relative to the cylinder which is motionless. In two dimensions, the direction of $\vec{\omega}_0$ is perpendicular to the plane of the paper. When integrated along this direction between the limits of positive and negative infinity, Equation 2.5 reduces to

$$\vec{V}_p(r,t) = - \frac{1}{2\pi} \iint \frac{\vec{\omega}_0 \times (\vec{r}_0 - \vec{r})}{|\vec{r}_0 - \vec{r}|^2} dA \quad (2.6)$$

where A is the area of the fluid with non-zero vorticity.

Figure 2.1 Illustration of the vector quantities used in the Biot-Savart law.

The first condition mentioned previously for the use of the Biot-Savart law excludes the presence of any rigid internal or external boundaries. The effect of the no-penetration condition on these surfaces is produced through the use of a distribution of bound vorticity, usually in the form of a vortex sheet. The region enclosed by this vortex distribution is assumed to be occupied by a fluid identical to that of the surrounding region. Furthermore, since all solid bodies in this work are assumed to be stationary, the fluid within is also considered to have zero velocity. The bound vortex sheet attached to the cylinder surfaces allows for a discontinuity in the tangential velocity at these points.

In this particular study, the flow will be viewed as it moves relative to the array of cylinders which appear to be stationary. In such a scheme, the fluid far from the cylinders appears to move from left to right with an undisturbed velocity U . The fluid velocity in this body-fixed coordinate system then becomes $\vec{V} = \vec{V}_p + \vec{U}$. The induced velocity at point P is thus expressed as,

$$\vec{V}_p = -\frac{1}{2\pi} \iint \frac{\vec{\omega}_o \times (\vec{r}_o - \vec{r}_p)}{|\vec{r}_o - \vec{r}_p|^2} dA - \frac{1}{2\pi} \oint \frac{\vec{\gamma}_q \times (\vec{r}_q - \vec{r}_p)}{|\vec{r}_q - \vec{r}_p|^2} dl + \vec{U} \quad (2.7)$$

where the second term represents the contribution from the vortex sheet (Figure 2.2).

Evaluation of the two integrals in Equation 2.7 is discussed in Appendices A and B. The determination of γ , required for evaluation of the second integral, will be presented shortly.

Figure 2.2 Graphical representation of the contribution of the bound and free vorticities to the velocity at a point P .

Coordinate System

A Cartesian coordinate system is employed in this work. The circular cylinders are arranged so that the distance from cylinder to cylinder along the same row, and from one row to the other, is the same. Thus, if the center to center distance along one row is 2δ , then the height of one row above and below the other is $\delta \cdot \tan(\pi/3)$. In this way, the centers of each cylinder lie at the apexes of regular triangles. The coordinate system used is shown in Figure 2.3. Note that on the cylinder surface-angles are measured counterclockwise from the horizontal position.

Bound Vorticity Distribution

As mentioned previously, the Biot-Savart law requires an unbounded fluid. This restriction is overcome in the present work by the replacement of all solid boundaries with a distribution of bound vortices, γ . The method of evaluation of this distribution is now presented.

The free vorticity per unit volume, ω , has the usual dimension of 1/time while the bound vorticity, γ , or total vorticity per unit surface area has units of length/time, or velocity (Milne-Thomson, 1966). In two-dimensional flow, both $\vec{\gamma}$ and $\vec{\omega}$ have a direction perpendicular to the flow paths, and are assumed to be positive in a counterclockwise sense. As shown in Figure 2.4, a positive γ induces a negative (from right to left on the upper portion) velocity component on the exterior surface of the body and a positive velocity on the interior surface.

Figure 2.3 Coordinate system for cylinder array.

Figure 2.4 Contribution to the velocity from the bound vorticity at the point where it is applied.

Referring to Figures 2.4 and 2.5, the velocity at point P, within an infinitesimal distance ϵ below the surface, can be written as,

$$\vec{V}_p = U\vec{i} + \frac{1}{2\pi} \iint \frac{\vec{\omega}_o \times \vec{r}_{op}}{|\vec{r}_{op}|^2} dA + \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \oint \frac{\vec{\gamma}_q^n \times \vec{r}_{qp}^n}{|\vec{r}_{qp}^n|^2} dl$$

$$- \frac{\gamma_p}{2} \vec{e}_\theta = 0 \quad (2.8)$$

where $\vec{i}$ is a unit vector in the positive x direction, and $\vec{r}_{ab} = \vec{r}_b - \vec{r}_a$. The unit vector $\vec{e}_\theta$ is tangential to the surface of the cylinder. The first two integrals represent the induced velocities, while $\frac{\gamma_p}{2} \vec{e}_\theta$ is the contribution of the vortex at point P (Karamcheti, 1966, p. 508). Note the summation sign before the second integral sign, which is required due to the presence of an infinite number of bodies (and therefore bound vorticity distributions) in the flow field. The entire expression is set to zero since the body, and its accompanying internal fluid, is considered to be stationary with respect to the external flow.

The following dimensionless quantities are now introduced:

$$\omega^* = \omega/(U/D)$$

$$\gamma^* = \gamma/U$$

$$u^* = u/U$$

$$v^* = v/U$$

$$x^* = x/D$$

$$y^* = y/D$$

Figure 2.5 Graphical representation of the contribution of the bound and free vorticities to the velocity of a point on the surface of the cylinder.

$$p^* = p / \left(\frac{1}{2} \rho U^2 \right)$$

$$\tau^* = \tau / \left(\frac{1}{2} \rho U^2 \right)$$

$$t^* = tU/D$$

Henceforth, since only dimensionless variables will be employed unless otherwise stated, the star suffix will be dropped for convenience.

Taking the scalar component of Equation 2.8 along a direction tangent to the surface of the body, and following the procedure carried out in detail in Appendix A, the following equation for the bound vorticity distribution is obtained,

$$\begin{aligned} \gamma(\phi) &= \frac{1}{\pi} \sum_{r_1=-\infty}^{\infty} \oint \frac{\vec{\gamma}_q^n(\theta) \times \vec{r}_{pq}^n}{|\vec{r}_{pq}^n|^2} d\mathbf{l} \cdot \vec{e}_\theta \\ &= -2 \left[\frac{1}{2\pi} \iint \frac{\vec{\omega}_o \times \vec{r}_{op}}{|\vec{r}_{op}|^2} dA \cdot \vec{e}_\theta + \sin\phi \right] \end{aligned} \quad (2.9)$$

where ϕ and θ are determined as shown in Figure 2.6. It is well known, Hess and Smith (1967) and Wu (1976), that enforcement of the condition given by Equation 2.9 also implies that the velocity component normal to the body also vanishes.

Equation 2.9 is a linear Fredholm integral equation of the second kind and has the general form,

$$\gamma(\phi) - \oint K(\phi, \theta) \gamma(\theta) d\theta = f(\phi) \quad (2.10)$$

Figure 2.6 Angles used to identify points on cylinder surfaces.

The non-homogeneous term, $f(\phi)$, and the kernel of the integral $K(\phi, \theta)$ are known functions in the domain of integration D . These functions are assumed to be quadratically summable on D , i.e.,

$$\int |f(\phi)|^2 d\phi \leq M$$

$$\int |K(\phi, \theta)|^2 d\phi d\theta \leq N$$

where M and N are finite constants. An analysis of these types of equations may be found in several books (Kellogg, 1929; Mikhlin, 1957; Pogorzelski, 1956).

Martensen (1959) showed that, in general, Equation 2.9 possesses a particular and complementary solution. The former is the solution to the complete equation, while the latter is the solution to the homogeneous $[f(\phi) = 0]$ equation. He further proved that the homogeneous equation has a nontrivial solution. This implies that the system is singular. In addition, he demonstrated that this nontrivial solution is unique to a multiplicative constant, and that the integral of this solution does not vanish over the contour of the cylinder. Thus, the general solution becomes

$$\gamma(\phi) = \gamma(\phi)_{\text{part}} + \alpha \gamma(\phi)_{\text{comp}} \quad (2.11)$$

The additional constraint necessary for the determination of the constant α in Equation (2.11) involves enforcement of a global conservation principle, which requires that the total vorticity be conserved for all time. The development of this principle is resumed after consideration of the vorticity dynamics.

This concludes the discussion of the kinematical portion of the problem. This has encompassed the determination of the bound vorticity distribution and the velocity field induced by the free and bound vorticities. To round out this theoretical presentation, it is necessary to understand the method by which free vorticity is generated and diffused and the role that the viscosity of the fluid plays in preventing the slippage of the fluid over the surface of the body and in the diffusion of vorticity into the flow field. These topics are covered in the next few sections.

Vorticity Dynamics

All constant property viscous flows are governed by the Navier-Stokes and continuity equations. The vector form of these equations is, in dimensional form,

$$\frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla) \vec{q} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{q} \quad (2.12)$$

$$\nabla \cdot \vec{q} = 0 \quad (2.13)$$

Here, $\vec{q}$ is the fluid velocity relative to the body.

Letting $\vec{\omega} = \text{curl } \vec{q}$ be the free vorticity in the flow field and taking the curl ($\nabla \times$) of Equation 2.12, the vorticity transport equation is obtained as follows:

$$\frac{\partial \vec{\omega}}{\partial t} + (\vec{q} \cdot \nabla) \vec{\omega} - (\vec{\omega} \cdot \nabla) \vec{q} = \nu \nabla^2 \vec{\omega} \quad (2.14)$$

In the case of two-dimensional flow, the only non-zero component of vorticity is in a direction perpendicular to the plane of

the flow. Therefore, the third term in Equation 2.14 can be dropped, leaving,

$$\frac{\partial \vec{\omega}}{\partial t} + (\vec{q} \cdot \nabla) \vec{\omega} = \nu \nabla^2 \vec{\omega} \quad (2.15)$$

The two terms on the left-hand side of Equation 2.15 represent the storage and convection of vorticity, respectively, while the right-hand side represents the diffusion of vorticity. Using the dimensionless quantities defined in the previous section and the following definition for the Reynolds number,

$$Re = UD/\nu$$

Equation 2.15 can be transformed into the dimensionless form of

$$\frac{\partial \vec{\omega}}{\partial t} + (\vec{q} \cdot \nabla) \vec{\omega} = \frac{1}{Re} \nabla^2 \vec{\omega} \quad (2.16)$$

Initially, the entire flow field is at rest and vorticity is zero everywhere. At $t = 0^+$, a uniform onset flow is impulsively started while the cylinders remain at rest. At this initial instant in time, the fluid slips tangentially along the cylinders as though it lacked viscosity. Therefore, at time $t = 0^+$, the velocity field can be determined through the use of potential flow methods or from the previously developed expressions with $\omega = 0$. This tangential velocity along the surface of the cylinders violates the no-slip condition for viscous flows and therefore cannot be allowed to persist in time. Lighthill (Rosenhead, 1963) argued that the slip velocity can be nullified by generating the right amount of vorticity along the

solid surface. This vorticity, which is generated during a sufficiently small time interval, then diffuses and is convected into the surrounding fluid in accordance with Equation 2.16. It must be noted that the vorticity is neither created nor destroyed in the flow field, and that viscosity provides a mechanism whereby this vorticity is redistributed throughout the flow field.

Lighthill argued verbally that the flux of vorticity at the solid surface, integrated over a small time increment, Δt , is equal to the slip velocity component (bound vorticity) γ . Cielak (1976) deduced this mathematically through the solution of the one-dimensional vorticity transport equation in the vicinity of a boundary for the case where there is a constant flux of vorticity from the wall to the surrounding fluid.

In dimensional form, this process of vorticity production is described by,

$$v \int_t^{t+\Delta t} \left. \frac{\partial \omega}{\partial r} \right|_{r=1/2} dt' = -\gamma(\theta, t) \quad (2.17)$$

where cylindrical coordinates have been employed due to the fact that the bodies in this problem are assumed to be circular cylinders, and $1/2$ is the radius of the cylinders. In dimensionless form, Equation 2.17 reduces to,

$$\frac{1}{Re} \int_t^{t+\Delta t} \left. \frac{\partial \omega}{\partial r} \right|_{r=1/2} dt' = -\gamma(\theta, t) \quad (2.18)$$

At each instant in time, the distribution of the bound vorticity, γ , is determined from the Fredholm integral equation, Equation 2.9. If γ is nonzero, new free vorticity enters the surrounding fluid over the next time increment, Δt , according to Equation 2.18. In this manner, the apparent slip velocity is reduced to zero. A numerical solution of Equation 2.16 is employed to determine the vorticity field. Once this vorticity field is known, the velocity field can be determined with the aid of the Biot-Savart law and the equation of continuity. This cyclic procedure is then repeated for each time step.

In order to complete the analysis, a procedure is needed to render unique the solution for the bound vorticity distribution, as given in Equation 2.11. This procedure can be established from the observation that the pressure on the surface of each cylinder must be a continuous and single-valued function. Thus, the cyclic integral of the pressure gradient along the body surface must vanish.

The procedure is to integrate both sides of the momentum equation in the θ direction. The dimensional form of the θ -component of the Navier-Stokes equation for cylindrical coordinates is (Schlichting, 1979, p. 67)

$$\begin{aligned} \rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z} \right) \\ = - \frac{1}{r} \frac{\partial p}{\partial \theta} + \mu \left(\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r^2} + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} \right. \\ \left. + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2} \right) \end{aligned} \quad (2.19)$$

Using the no slip conditions on the surface of the cylinders, this equation reduces to,

$$\begin{aligned} \left. \frac{1}{r\rho} \frac{\partial p}{\partial \theta} \right|_{r=1/2} &= \nu \left[\frac{\partial^2}{\partial r^2} \frac{v_\theta}{r^2} + \frac{1}{r} \frac{\partial v_\theta}{\partial r} \right] \bigg|_{r=1/2} \\ &= \left[\frac{\nu}{r} \frac{\partial}{\partial r} \frac{(r \partial v_\theta)}{\partial r} \right] \bigg|_{r=1/2} \end{aligned} \quad (2.20)$$

The dimensionless form of this equation is

$$\left. \frac{1}{r} \frac{\partial p}{\partial \theta} \right|_{r=1/2} = \frac{1}{Re} \left[\frac{1}{r} \frac{\partial}{\partial r} \frac{(r \partial v_\theta)}{\partial r} \right] \bigg|_{r=1/2} \quad (2.21)$$

Vorticity, expressed in plane cylindrical coordinates, is defined as,

$$\omega = \frac{1}{r} \left[\frac{\partial}{\partial r} (r v_\theta) - \frac{\partial v_r}{\partial \theta} \right] = \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} - \frac{1}{r} \frac{\partial v_r}{\partial \theta} \quad (2.22)$$

from which one obtains

$$\frac{\partial \omega}{\partial r} = \frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r^2} - \frac{\partial}{\partial r} \left[\frac{1}{r} \frac{\partial v_r}{\partial \theta} \right] \quad (2.23)$$

Since at the surface of the cylinders v_θ and v_r are zero,

then,

$$\left. \frac{\partial \omega}{\partial r} \right|_{r=1/2} = \left[\frac{1}{r} \frac{\partial}{\partial r} \frac{(r \partial v_\theta)}{\partial r} \right] \bigg|_{r=1/2} \quad (2.24)$$

Comparing Equations 2.21 and 2.24, one concludes that

$$\left. \frac{1}{r} \frac{\partial p}{\partial \theta} \right|_{r=1/2} = \frac{1}{Re} \left. \frac{\partial \omega}{\partial r} \right|_{r=1/2} \quad (2.25)$$

Now eliminate $\left. \frac{\partial \omega}{\partial r} \right|_{r=1/2}$ between Equations 2.25 and 2.19. The result is

$$\int_t^{t+\Delta t} \left[\frac{1}{r} \frac{\partial p}{\partial \theta} \right]_{r=1/2} dt' = -\gamma(\theta, t) \quad (2.26)$$

The integration of this equation along the surface yields,

$$\int_t^{t+\Delta t} [p(\theta, 1/2) - p(0, 1/2)] dt' = -(1/2) \int_0^\theta \gamma(\theta, t) d\theta \quad (2.27)$$

The value of the reference pressure $p(0, 1/2)$, is not essential since a constant pressure does not affect the lift, drag, and moment on a rigid structure.

Since the pressure must be single-valued over each cylinder at all instants of time, $p(2\pi, 1/2)$ must equal $p(0, 1/2)$, and,

$$\int_0^{2\pi} \gamma(\theta, t) d\theta = 0 \quad (2.28)$$

This is then the additional constraint necessary to obtain a unique solution for the bound vorticity distribution. The actual application of Equation 2.28 is discussed in Chapter 3.

Continuity Equation

The determination of the velocity components by application of the Biot-Savart law requires the evaluation of integrals. This evaluation, which is performed numerically in the present work, can be extremely time consuming. In practice, therefore, only one velocity component is calculated by means of the Biot-Savart law while

the transverse component is computed through the use of the continuity equation. The procedure is extremely effective in reducing the computational cost per time step.

In a cylindrical coordinate system, Equation 2.13 may be restated in the form,

$$\frac{\partial}{\partial r} (rv_r) + \frac{\partial}{\partial \theta} v_\theta = 0 \quad (2.29)$$

Integration of Equation 2.29 with respect to r at a fixed value of θ yields,

$$[rv_r]_{r_2} - [rv_r]_{r_1} = - \frac{\partial}{\partial \theta} \int_{r_1}^{r_2} v_\theta \, dr \quad (2.30)$$

Symmetry and Spatial Periodicity in the Flow Field

One of the most critical aspects of this problem involved the development of the flow network. The intent was to take advantage of symmetry and spatial periodicity in order to carry out all computations in the primary computational region shown in Figure 2.7.

Thus, the flow leaving at point a in Figure 2.7 is considered to re-enter the primary region at point b. Likewise, flow leaving at point c re-enters at point d. In this way, the flow boundary conditions are spatially periodic. Furthermore, taking a look at point e and the corresponding "mirror image" point f in Figure 2.7, one finds that the value of the free vorticity at these points is equal and opposite in sign. The velocities in the y-direction at points e and f must also be equal and opposite in sign, while in the x-direction,

Figure 2.7 Flow geometry with primary computational region shown.

they must have the same magnitude and sign. Thus, there is top-bottom antisymmetry between adjacent regions.

It is recognized that by enforcing this symmetry and periodicity in the flow field, that it is impossible for perturbations to occur in only one of the flow regions. Although this is a slight disadvantage, it is felt that the consequent reduction in the size of the flow region was necessary to make the computational costs manageable.

Before adopting this assumption, however, a great deal of effort was expended in order to insure that there was inherent symmetry and spatial periodicity in the flow from region to region. In order to verify this feature, the computational flow field was expanded to three adjacent flow regions, as shown in Figure 2.8. Flow variables in this new computational region were allowed to develop independently of each other around the three cylinders. It was still necessary to employ a periodically recurring flow pattern for the cluster of three cylinders, however, in order to carry out the numerical computations. Within the accuracy of the computer employed, it was determined that there were no significant deviations in the symmetry of the flow from region to region. Details of these computations may be found in Appendix C.

Utilizing the symmetry and spatial periodicity of the flow, the velocity at an arbitrary point may be expressed as

$$\vec{V}_p = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \iint_A \frac{\vec{\omega}_o^n \times \vec{r}_{op}^n}{[\vec{r}_{op}^n]^2} dA + \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \oint \frac{\vec{\gamma}_q^n \times \vec{r}_{qp}^n}{[\vec{r}_{qp}^n]^2} dl \quad (2.31)$$

Figure 2.8 Flow geometry with expanded primary computational region shown.

where n corresponds to a particular computational region. Moving the summations inside the integral signs, one obtains

$$\vec{V}_p = \frac{1}{2\pi} \iint_A \sum_{n=-\infty}^{\infty} \frac{\vec{\omega}_o^n \times \vec{r}_{op}^n}{(r_{op}^n)^2} dA + \frac{1}{2\pi} \oint \sum_{n=-\infty}^{\infty} \frac{\vec{\gamma}_q^n \times \vec{r}_{qp}^n}{(r_{qp}^n)^2} dl \quad (2.32)$$

Due to spatial periodicity, Equation 2.32 may be simplified by noting that

$$\omega_o^1 = \omega_o^2 = \omega_o^3 = \dots = \omega_o$$

$$\gamma_q^1 = \gamma_q^2 = \gamma_q^3 = \dots = \gamma_q$$

with a sign change between adjacent regions which will presently be accounted for. Denoting

$$d\kappa_\omega \equiv \omega_o dA$$

$$d\kappa_\gamma \equiv \gamma_q dl$$

where $d\kappa_\omega$ and $d\kappa_\gamma$ are "vortex strengths", then

$$\begin{aligned} \vec{\omega}_o^n \times \vec{r}_{op}^n dA &= (\omega_o \vec{k}) \times \vec{r}_{op}^n dA = (\omega_o dA) \vec{k} \times \vec{r}_{op}^n \\ &= (d\kappa_\omega) \vec{k} \times \vec{r}_{op}^n \end{aligned} \quad (2.33)$$

and

$$\vec{\gamma}_q^n \times \vec{r}_{qp}^n dl = (\gamma_q dl) \vec{k} \times \vec{r}_{qp}^n = (d\kappa_\gamma) \vec{k} \times \vec{r}_{qp}^n \quad (2.34)$$

Thus,

$$\vec{V}_p = \frac{1}{2\pi} \iint_A d\kappa_\omega \left[\sum_{n=-\infty}^{\infty} \frac{\vec{k} \times \vec{r}_{op}^n}{(r_{op}^n)^2} \right] + \frac{1}{2\pi} \oint d\kappa_\gamma \left[\sum_{n=-\infty}^{\infty} \frac{\vec{k} \times \vec{r}_{qp}^n}{(r_{qp}^n)^2} \right] \quad (2.35)$$

and one can see that the terms in brackets are the same. Therefore, consider only one of the terms for the case of an infinite array of row vortices, as shown in Figure 2.9. Taking the x-component, we note that

$$\frac{\vec{k} \times \vec{r}_{op}^n}{(r_{op}^n)^2} \cdot \vec{i} = \frac{r_{op}}{(r_{op})^2} \sin\theta = \frac{y}{r_{op}^2} \quad (2.36)$$

and thus

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \left[\frac{\vec{k} \times \vec{r}_{op}^n}{(r_{op}^n)^2} \cdot \vec{i} \right] &= -\frac{y}{y^2+x^2} - \frac{y}{y^2+(x-a)^2} - \dots \\ &\quad - \frac{y}{y^2+(x-na)^2} - \dots - \frac{y}{y^2+(x+a)^2} - \dots \\ &\quad - \frac{y}{y^2+(x+na)^2} - \dots \end{aligned} \quad (2.37)$$

or

$$\begin{aligned} S \equiv \sum_{n=-\infty}^{\infty} \left[\frac{\vec{k} \times \vec{r}_{op}^n}{(r_{op}^n)^2} \cdot \vec{i} \right] &= -\frac{y}{y^2+x^2} \\ &\quad - \sum_{n=1}^{\infty} \left[\frac{y}{y^2+(x-na)^2} + \frac{y}{y^2+(x+na)^2} \right] \end{aligned} \quad (2.38)$$

Figure 2.9 Representation of infinite row of vortices.

In order to continue, we make use of complex variable theory. Observe that

$$\frac{1}{z} = \frac{x-iy}{y^2+x^2}$$

and

$$\frac{1}{y-na} = \frac{(x-na)-iy}{y^2+(x-na)^2}$$

and so forth. Therefore

$$S = \text{Imag} \left\{ \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z-na} + \frac{1}{z+na} \right) \right\}$$

where "Imag" denotes the imaginary part of the complex quantity in curly brackets. Simplifying the above, one obtains

$$\begin{aligned} S &= \text{Imag} \left\{ \frac{1}{z} + \sum_{n=1}^{\infty} \left[\frac{2z}{z^2 - n^2 a^2} \right] \right\} = \text{Imag} \left\{ \frac{d}{dz} \left[\ln(z) \right. \right. \\ &\quad \left. \left. + \sum_{n=1}^{\infty} \ln(z^2 - n^2 a^2) \right] \right\} \\ &= \text{Imag} \left\{ \frac{d}{dz} \ln [z(z-a^2)(z-2^2a^2) \dots (z-n^2a^2) \dots] \right\} \end{aligned}$$

As shown by Milne-Thomson (1966),

$$\begin{aligned} &\frac{d}{dz} \{ \ln [z(z-a^2)(z-2^2a^2) \dots (z-n^2a^2) \dots] \} \\ &= \frac{d}{dz} \left[\ln \left(\sin \frac{\pi z}{a} \right) \right] \end{aligned}$$

and therefore,

$$\begin{aligned}
S &= \text{Imag} \left\{ \frac{d}{dz} \left[\ln \left(\sin \frac{\pi Z}{a} \right) \right] \right\} \\
&= \frac{\pi}{a} \text{Imag} \left\{ \cos \frac{\pi Z}{a} / \sin \frac{\pi Z}{a} \right\}
\end{aligned} \tag{2.39}$$

Now,

$$\sin \frac{\pi Z}{a} = \sin \frac{\pi X}{a} \cosh \frac{\pi Y}{a} + i \cos \frac{\pi X}{a} \sinh \frac{\pi Y}{a}$$

and

$$\cos \frac{\pi Z}{a} = \cos \frac{\pi X}{a} \cosh \frac{\pi Y}{a} - i \sin \frac{\pi X}{a} \sinh \frac{\pi Y}{a}$$

Substitution of these two relations into Equation 2.39, and resolution of the expression into real and imaginary components yields the form,

$$S = - \frac{\pi}{a} \left\{ \left(\sinh \frac{2\pi Y}{a} / \left[\cosh \frac{2\pi Y}{a} - \cos \frac{2\pi X}{a} \right] \right) \right\}$$

In summary, then

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \left[\frac{\vec{k} \times \vec{r}_{op}^n}{(r_{op}^n)^2} \cdot \vec{i} \right] &= - \frac{\pi}{a} \left\{ \sinh \frac{2\pi y_p}{a} / \left(\cosh \frac{2\pi y_p}{a} \right. \right. \\
&\quad \left. \left. - \cos \frac{2\pi x_p}{a} \right) \right\}
\end{aligned} \tag{2.40}$$

The above result applies to the case where the row of vortices lies on the y-axis and one of the vortices is located at the origin. Next consider the case shown in Figure 2.10. In this case, we have

Figure 2.10 Representation of infinite row of vortices offset from x -axis.

$$\begin{aligned}
& \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \iint_A \left[\frac{\vec{k} \times \vec{r}_{op}^n}{(r_{op}^n)^2} \cdot \vec{i} \right] \omega_o \, dA = \\
& - \frac{1}{2\pi} \frac{\pi}{a} \iint_A \left\{ \left[\sinh \frac{2\pi(y_p - y_o)}{a} \right] / \left[\cosh \frac{2\pi(y_p - y_o)}{a} \right] \right. \\
& \left. - \cos \frac{2\pi(x_p - x_o)}{a} \right\} \omega_o \, dx_o \, dy_o \quad (2.41)
\end{aligned}$$

In the present problem, symmetry and spatial periodicity assumptions dictate two infinite vortex arrays, as shown in Figure 2.11. For this case, we have

$$\begin{aligned}
& \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \iint_A \left[\frac{\vec{k} \times \vec{r}_{op}^n}{(r_{op}^n)^2} \cdot \vec{i} \right] \omega_o \, dA = \\
& - \frac{1}{2\pi} \frac{\pi}{a} \iint_A \left\{ \left[\sinh \frac{2\pi(y_p - y_o)}{a} \right] / \left[\cosh \frac{2\pi(y_p - y_o)}{a} \right] \right. \\
& \left. - \cos \frac{2\pi(x_p - x_o)}{a} \right\} - \left[\sinh \frac{2\pi(y_p + y_o)}{a} \right] / \left[\cosh \frac{2\pi(y_p + y_o)}{a} \right] \\
& \left. - \cos \frac{2\pi(x_p + x_o)}{a} \right\} \omega_o \, dx_o \, dy_o \quad (2.42)
\end{aligned}$$

Note that the negative sign preceding the second term in the integral accounts for the change in sign of vorticity at corresponding points in adjacent regions. The y-component of the induced velocity is determined in a like manner and may be expressed in the form

Figure 2.11 Representation of double infinite row of vortices.

$$\begin{aligned}
& \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \iint_A \left[\frac{\vec{k} \times \vec{r}_{op}^n}{(r_{op}^n)^2} \cdot \vec{j} \right] \omega_o \, dA \\
&= \frac{1}{2\pi} \frac{\pi}{a} \iint_A \left\{ \left[\sin \frac{2\pi(x_p - x_o)}{a} \right] \left/ \left[\cosh \frac{2\pi(y_p - y_o)}{a} \right] \right. \right. \\
&\quad - \left. \cos \frac{2\pi(x_p - x_o)}{a} \right] + \left[\sin \frac{2\pi(x_p + x_o)}{a} \right] \left/ \left[\cosh \frac{2\pi(y_p + y_o)}{a} \right] \right. \\
&\quad \left. - \cos \frac{2\pi(x_p + x_o)}{a} \right] \Bigg\} \omega_o \, dx_o \, dy_o \quad (2.43)
\end{aligned}$$

A similar set of expressions may be written for the velocity induced by the bound vorticity distribution.

Aerodynamic Forces and Design Parameters

Considering a two-dimensional circular cylinder, as shown in Figure 2.12. With $\vec{e}_r$ and $\vec{e}_\theta$ being the unit vectors for a cylindrical coordinate system (r, θ) , the lift, drag, and pitching moment per unit of span length, generated by the dimensional pressure and friction forces are:

$$L = L_p + L_f = -\oint P(\theta) R d\theta (\vec{e}_r \cdot \vec{j}) + \oint \tau_o(\theta) R d\theta (\vec{e}_\theta \cdot \vec{j}) \quad (2.44)$$

$$D = D_p + D_f = -\oint P(\theta) R d\theta (\vec{e}_r \cdot \vec{i}) + \oint \tau_o(\theta) R d\theta (\vec{e}_\theta \cdot \vec{i}) \quad (2.45)$$

$$M = M_f = +\oint \tau_o(\theta) R^2 d\theta (\vec{e}_r \times \vec{e}_\theta) \quad (2.46)$$

Figure 2.12 Aerodynamic forces on a cylinder.

The dimensionless form of these equations is,

$$\begin{aligned} CL &= CL_p + CL_f = L_p / [(1/2)\rho DU^2] + L_f / [(1/2)\rho DU^2] \\ &= -\oint P(\theta)(.5d\theta)(\vec{e}_r \cdot \vec{j}) + \oint \tau_o(\theta)(.5d\theta)(\vec{e}_\theta \cdot \vec{j}) \end{aligned} \quad (2.47)$$

$$\begin{aligned} CD &= CD_p + CD_f = D_p / [(1/2)\rho DU^2] + D_f / [(1/2)\rho DU^2] \\ &= -\oint P(\theta)(.5d\theta)(\vec{e}_r \cdot \vec{i}) + \oint \tau_o(\theta)(.5d\theta)(\vec{e}_\theta \cdot \vec{i}) \end{aligned} \quad (2.48)$$

$$CM = CM_f = M_f / [(1/2)\rho D^2 U^2] = +\oint \tau_o(\theta)(.25d\theta)(|\vec{e}_r \times \vec{e}_\theta|) \quad (2.49)$$

Now,

$$\vec{e}_r = \cos\theta \vec{i} + \sin\theta \vec{j}$$

and

$$\vec{e}_\theta = -\sin\theta \vec{i} + \cos\theta \vec{j}$$

Thus,

$$\vec{e}_r \cdot \vec{i} = \cos\theta$$

$$\vec{e}_r \cdot \vec{j} = \sin\theta$$

$$\vec{e}_\theta \cdot \vec{i} = -\sin\theta$$

$$\vec{e}_\theta \cdot \vec{j} = \cos\theta$$

$$\vec{e}_r \times \vec{e}_\theta = \vec{k}$$

Therefore, Equations 2.47, 2.48 and 2.49 may be written in the form

$$\begin{aligned} CL = CL_p + CL_f = & -\int_0^{2\pi} P(\theta) \sin\theta (.5d\theta) \\ & + \int_0^{2\pi} \tau_o(\theta) \cos\theta (.5d\theta) \end{aligned} \quad (2.50)$$

$$\begin{aligned} CD = CD_p + CD_f = & -\int_0^{2\pi} P(\theta) \cos\theta (.5d\theta) \\ & - \int_0^{2\pi} \tau_o(\theta) \sin\theta (.5d\theta) \end{aligned} \quad (2.51)$$

$$CM = CM_f = \int_0^{2\pi} \tau_o(\theta) (.25d\theta) \quad (2.52)$$

where it is understood that the moment coefficient has a direction associated with it which is perpendicular to the plane of the flow. Therefore, the unit vector $\vec{k}$ has been omitted.

CHAPTER 3

NUMERICAL ANALYSIS

The fundamental principles underlying the formulation of the velocity field and the equations governing the generation and transport of vorticity for the present problem were discussed in Chapter 2. This chapter is devoted to the numerical techniques used to evaluate those fundamental equations in this work. The vorticity transport equation, in finite difference form, is derived. The boundary condition on the surface of all cylinders whereby bound vorticity is generated and then diffused into the flow field is discussed. The numerical solution of the Fredholm integral equation, which determines the bound vorticity distribution, is subsequently covered. A discussion of the numerical methods employed to evaluate the integrals in the velocity calculations follows. Finally, the numerical evaluation of force and moment coefficients concludes the chapter.

Flow Network

A flow network of eighty by twenty nine nodes was chosen for the final numerical analysis, as shown in Figure 3.1. Several approaches were employed in order to generate a computational grid which was suitable for the analysis.

One of the most important prerequisites for this grid was the ability of the primary computational region to match up with its inverted "mirror image" on the boundary, as shown in Figure 3.1.

Figure 3.1 Grid distribution in computational regions.

Techniques of grid generation, based on conformal mapping were investigated. It was found that this requirement could not be met. Therefore, a method developed by Thames (1975) was adopted which involves the generation of curvilinear coordinates as the solution of two elliptic partial differential equations with Dirichlet boundary conditions. One of these curvilinear coordinates is specified to have a different but constant value along the inner and outer boundaries of the flow region. A transformation is then performed by the selection of a generating elliptic partial-differential system, of which the natural coordinates are a solution in the physical flow plane. The generating system is transformed by an interchange of the dependent and independent variables. This produces a set of quasi-linear elliptic equations for the Cartesian coordinates as a function of the curvilinear coordinates. The advantage of the system is that these equations, although not linear, are transformed into a rectangular plane with boundary conditions specified on straight boundaries.

The curvilinear coordinate lines are labeled as shown in Figure 3.1 with the constant η lines somewhat equivalent to what might be considered "constant potential" lines, while the constant ξ lines could be likened to "constant flow" lines.

It is important to note that since the intersection of the η - and ξ - lines are specified on the boundaries, this is not an orthogonal grid system. The system does allow for the flexibility of controlling the radial spacing of the constant η -lines and of the incidence of angle of the constant ξ -lines at the boundaries. This is

an especially important feature which allows for concentration of grid cells in regions where the flow is expected to change rapidly, and for a smooth transition of the flow from region to region. For additional information, the reader is highly encouraged to read Thames' work.

The index I, which increases clockwise from the positive y-axis, indicates the node number in the ξ -direction. The index J which increases radially from the inner cylindrical boundary, identifies constant η -lines.

A typical control volume is shown in Figure 3.2. As shown in the figure, vorticity is calculated at the centroid of each control volume. The velocity components in the middle of the left and right sides (lines of constant ξ) are calculated through the use of the Biot-Savart law, while the remaining velocity components in the middle of the top and bottom sides are calculated through the use of the equation of continuity. These velocity components are necessary for the solution of the vorticity transport equation. The bound vorticity is calculated on body surfaces at points equidistant between adjacent constant ξ -lines.

Free Vorticity Field

The vorticity transport equation (2.16) is implemented in conservation form and written employing finite difference schemes. Thus, conservation of total vorticity over an arbitrary control volume, $A_{i,j}$, may be written as,

$$\text{Rate of vorticity storage in C.V.} = \text{Vorticity IN} - \text{Vorticity OUT} \quad (3.1)$$

A few observations merit mentioning at this point about Figure 3.2. The vorticity variables shown are evaluated at the node points and are considered to be constant over their corresponding control volumes. Although the u and v velocities are evaluated at a point, they are assumed constant over the sides of the control volumes. DHR, DWU, DHL, and DWL are the lengths of each side of the computational cell, and RR, RU, RL, and RB are the distances to adjacent vorticity nodes.

Variables with the superscript $+$ are evaluated at time $t + \Delta t$, while variables lacking a superscript are evaluated at time t . Weighting factors $F1$, $F2$, $F3$, and $F4$ are associated with the vorticity nodes $(i+1,j)$, $(i,j+1)$, $(i-1,j)$, and $(i,j-1)$ respectively. These factors have values of zero or one. They are employed in order to determine the direction of convection across a boundary. For example, if $F1 = 1$, then $u_{i+(1/2),j}$ is considered positive and $u_{i+(1/2),j} \omega_{i,j}^{DHR_{i,j}}$ is convected out across the right hand side of the computational cell. If $F1 = 0$, then $u_{i+(1/2),j}$ is negative and $u_{i+(1/2),j} \omega_{i+1,j}^{DHR_{i,j}}$ is considered to be convected in across the right hand boundary. The magnitude of the weighting factors is thus coupled to the direction of the velocities across the control volume surfaces. Mathematically, they are expressed as

$$F1 = .5[1 + \text{sign}(u_{i+(1/2),j})]$$

$$F2 = .5[1 + \text{sign}(v_{i,j+(1/2)})]$$

$$F3 = .5[1 + \text{sign}(u_{i-(1/2),j})]$$

$$F4 = .5[1 + \text{sign}(v_{i,j-(1/2)})]$$

where $\text{sign}(u) = 1$ if u is positive and $\text{sign}(u) = -1$ if u is negative. In order to model this convection mathematically, we replace the convection expression on the right hand surface of the control volume in Figure 3.2, for example, with the term,

$$u_{i+(1/2),j} [F1 \omega_{i,j} + (1-F1) \omega_{i+1,j}] \text{DHR}_{i,j}$$

This upwind method is used to guarantee the convection of vorticity from upwind nodes to downwind nodes (Gosman et al., 1969).

The individual contributions to the formulation of the finite difference vorticity transport equation thus become,

$$\text{Rate of vorticity storage in C.V.} = \frac{\omega_{i,j}^+ - \omega_{i,j}^-}{\Delta t} A_{i,j} \quad (3.2)$$

$$\begin{aligned} \text{Vorticity IN} = & u_{i-(1/2),j} [F3 \omega_{i-1,j} \\ & + (1-F3) \omega_{i,j}] \text{DHL}_{i,j} \\ & + v_{i,j-(1/2)} [F4 \omega_{i,j-1} + (1-F4) \omega_{i,j}] \text{DWL}_{i,j} \\ & + \frac{1}{\text{Re}} \left\{ \frac{\omega_{i-1,j} - \omega_{i,j}}{\text{RL}_{i,j}} \text{DHL}_{i,j} \right. \\ & \left. + \frac{\omega_{i,j-1} - \omega_{i,j}}{\text{RB}_{i,j}} \text{DWL}_{i,j} \right\} \end{aligned} \quad (3.3)$$

$$\begin{aligned}
\text{Vorticity OUT} = & u_{i+(1/2),j} [F1 \omega_{i,j} \\
& + (1-F1) \omega_{i+1,j}] \text{DHL}_{i,j} \\
& + v_{i,j+(1/2)} [F2 \omega_{i,j} + (1-F2) \omega_{i,j+1}] \text{DWU}_{i,j} \\
& + \frac{1}{\text{Re}} \left\{ \frac{\omega_{i,j} - \omega_{i+1,j}}{\text{RR}_{i,j}} \text{DHR}_{i,j} \right. \\
& \left. + \frac{\omega_{i,j} - \omega_{i,j+1}}{\text{RU}_{i,j}} \text{DWU}_{i,j} \right\} \quad (3.4)
\end{aligned}$$

Substitution of Equation 3.2, 3.3 and 3.4 into Equation 3.1 and solving the resulting equation for $\omega_{i,j}^+$ yields the final form,

$$\begin{aligned}
\omega_{i,j}^+ = & \omega_{i,j} + \frac{\Delta t}{A_{i,j}} \left\{ \omega_{i,j} (1-F3) u_{i-(1/2),j} \text{DHL}_{i,j} \right. \\
& + (1-F4) v_{i,j-(1/2)} \text{DWL}_{i,j} - F1 u_{i+(1/2),j} \text{DHR}_{i,j} \\
& - F2 v_{i,j+(1/2)} \text{DWU}_{i,j} - \frac{1}{\text{Re}} \left[\frac{\text{DHL}_{i,j}}{\text{RL}_{i,j}} + \frac{\text{DWL}_{i,j}}{\text{RB}_{i,j}} \right. \\
& \left. + \frac{\text{DHR}_{i,j}}{\text{RR}_{i,j}} + \frac{\text{DWU}_{i,j}}{\text{RU}_{i,j}} \right] + \omega_{i+1,j} \left[-(1-F1) u_{i+(1/2),j} \text{DHR}_{i,j} \right. \\
& \left. + \frac{\text{DHR}_{i,j}}{\text{Re RR}_{i,j}} \right] + \omega_{i,j+1} \left[-(1-F2) v_{i,j+(1/2)} \text{DWU}_{i,j} \right. \\
& \left. + \frac{\text{DWU}_{i,j}}{\text{Re RU}_{i,j}} \right] + \omega_{i-1,j} \left[F3 u_{i-(1/2),j} \text{DHL}_{i,j} + \frac{\text{DHL}_{i,j}}{\text{Re RL}_{i,j}} \right] \\
& \left. + \omega_{i,j-1} \left[F4 v_{i,j-(1/2)} \text{DWL}_{i,j} + \frac{\text{DWL}_{i,j}}{\text{Re RB}_{i,j}} \right] \right\} \quad (3.5)
\end{aligned}$$

For control volumes adjacent to a body surface, a production term of total vorticity must be accounted for. During each time step, vorticity enters the fluid by diffusion from the solid boundary. This vorticity production can be related to the bound vorticity distribution as shown in Chapter 2. The relationship is given in Equation 2.19, which is rewritten here in the form,

$$\int_t^{t+\Delta t} \left. \frac{\partial \omega}{\partial r} \right|_{r=1/2} = -\text{Re } \gamma(t) \quad (3.6)$$

For a sufficiently small Δt , Equation 3.6 becomes

$$\left. \frac{\partial \omega}{\partial r} \right|_{r=1/2} \cong -\text{Re } \frac{\gamma}{\Delta t} \quad (j=1) \quad (3.7)$$

After substituting Equation 3.7 into Equation 3.5, keeping in mind that the velocity normal to the cylinder is zero, Equation 3.5 becomes for $j=1$,

$$\begin{aligned} \omega_{i,1}^+ &= \omega_{i,1} + \frac{\Delta t}{A_{i,1}} \left\{ \omega_{i,1} \left[(1-F3) u_{i-(1/2),1} \text{DHL}_{i,1} \right. \right. \\ &\quad - F1 u_{i+(1/2),1} \text{DHR}_{i,1} - F2 v_{i,1+(1/2)} \text{DWU}_{i,1} \\ &\quad \left. \left. - \frac{1}{\text{Re}} \left(\frac{\text{DHL}_{i,1}}{\text{RL}_{i,1}} + \frac{\text{DHR}_{i,1}}{\text{RR}_{i,1}} + \frac{\text{DWU}_{i,1}}{\text{RU}_{i,1}} \right) \right] \right. \\ &\quad + \omega_{i+1,1} \left[-(1-F) u_{i+(1/2),1} \text{DHR}_{i,1} + \frac{\text{DHR}_{i,1}}{\text{Re RR}_{i,1}} \right] \\ &\quad \left. + \omega_{i,1+1} \left[-(1-F2) v_{i,1+(1/2)} \text{DWU}_{i,1} + \frac{\text{DWU}_{i,1}}{\text{Re RU}_{i,1}} \right] \right\} + \end{aligned}$$

$$\begin{aligned}
& + \omega_{i-1,1} \left[F3u_{i-(1/2),1} \text{DHL}_{i,1} + \frac{\text{DHL}_{i,1}}{\text{Re } \text{RL}_{i,1}} \right] \\
& + \gamma_i \text{DWL}_{i,1}/A_{i,1}
\end{aligned} \tag{3.8}$$

At time $t = 0$, the vorticity is everywhere zero in the fluid. Thus, all terms vanish on the right-hand side of Equation 3.8, except the last one. After the first time step, therefore, the vorticity is everywhere zero except in the first row of control nodes ($j = 1$) surrounding all cylinders. For this row, the vorticity becomes,

$$[\omega_{i,1}]^{t=\Delta t} = \frac{\text{DWL}_{i,1}}{A_{i,1}} [\gamma_i]^{t=0^+} \tag{3.9}$$

where $[\gamma_i]^{t=0^+}$ is the bound vorticity immediately following the impulsive motion of the fluid.

This shows that the bound vorticity, initially in the form of a sheet, diffuses into the first row of cells during the first time step. In subsequent time steps, it is redistributed through the mechanisms of convection and diffusion, and new free vorticity is produced as subsequent distributions of bound vorticity are obtained at the new time levels.

Bound Vorticity

Employing standard numerical techniques for the solution of Fredholm integral equations (Mikhlin, 1957, p. 286), Equation 2.9 can be approximated by the equation,

$$\gamma(\phi_i) - \frac{1}{\pi} \sum_{j=1}^N K(\phi_i, \theta_j) \gamma(\theta_j) d\theta = -2[u_\omega(\phi_i) + \sin\phi_i] \quad (3.10)$$

where $u_\omega(\phi)$ is defined by Equation A.13. It is the tangential velocity induced at the cylinder surfaces by the free vorticity in the fluid. The function $K(\phi, \theta)$ is evaluated according to Equations 2.42 and 2.43.

This constitutes a system of N equations for N unknowns, where N is the total number of points on the surface of the cylinder at which γ is calculated. In the present work, N was chosen to be eighty. In matrix form, this equation is written as,

$$[A][\gamma] = [B] \quad (3.11)$$

As discussed in Chapter 2, however, this system of equations is expected to be singular if a solution exists to the homogeneous equations. It is found that the rank is indeed $N-1$, and therefore, this system of equations must be reduced by one.

If $\gamma(\phi_{80})$ is assigned an arbitrary value of one, then the coefficient matrix $[A]$ of Equation 3.11 has the form as shown on the following page. Alternatively, one can suppose that all the equations are divided by $\gamma(\phi_{80})$, with the same result. Next, it is important to note that since the system of equations has been reduced in size by

$$\begin{bmatrix}
 1 - \frac{K}{\pi} (\phi_1, \theta_1)_{\Delta\theta} & -\frac{K}{\pi} (\phi_1, \theta_2)_{\Delta\theta} & \dots\dots\dots & -\frac{K}{\pi} (\phi_1, \theta_{N-1})_{\Delta\theta} \\
 \\
 -\frac{K}{\pi} (\phi_2, \theta_1)_{\Delta\theta} & 1 - \frac{K}{\pi} (\phi_2, \theta_2)_{\Delta\theta} & \dots\dots\dots & -\frac{K}{\pi} (\phi_2, \theta_{N-1})_{\Delta\theta} \\
 \\
 \cdot & \cdot & \cdot & \cdot \\
 \cdot & \cdot & \cdot & \cdot \\
 \cdot & \cdot & \cdot & \cdot \\
 \\
 -\frac{K}{\pi} (\phi_{N-1}, \theta_1)_{\Delta\theta} & -\frac{K}{\pi} (\phi_{N-1}, \theta_2)_{\Delta\theta} & \dots\dots\dots & 1 - \frac{K}{\pi} (\phi_{N-1}, \theta_{N-1})_{\Delta\theta}
 \end{bmatrix}$$

one, the term $\frac{K}{\pi} (\phi_i, \theta_N) \Delta\theta$ must now be added to the right hand side of Equation 3.11 for the determination of the homogeneous solution.

The two systems of equations are then solved. The distribution of γ is determined from the addition of the homogeneous to the particular solution. Since the homogeneous solution remains valid if multiplied by any arbitrary constant, Equation 2.12 may be rewritten as,

$$\gamma(\phi_i) = \underset{\text{comp}}{\alpha \gamma(\phi_i)} + \underset{\text{part}}{\gamma(\phi_i)} \quad (3.12)$$

Utilizing the additional constraint provided by Equation 2.28, we obtain the relationship,

$$\int_0^{2\pi} \gamma(\phi_i) d\phi = \alpha \int_0^{2\pi} \underset{\text{comp}}{\gamma(\phi_i) d\phi} + \int_0^{2\pi} \underset{\text{part}}{\gamma(\phi_i) d\phi} = 0 \quad (3.13)$$

which can be solved for α to give

$$\alpha = - \int_0^{2\pi} \underset{\text{part}}{\gamma(\phi_i) d\phi} / \int_0^{2\pi} \underset{\text{comp}}{\gamma(\phi_i) d\phi} \quad (3.14)$$

Once α is determined, then Equation 3.12 is employed to provide the bound vorticity distribution.

Velocity Field

As discussed in Chapter 2, the velocity field is determined through a combination of the Biot-Savart law of induced velocities and the equation of continuity. This approach results in sharply reduced computational costs by minimizing the time consuming Biot-Savart velocity integrations while guaranteeing satisfaction of the continuity equation for each cell.

Figure 3.2 reveals the points at which velocity components are calculated using these two methods. Equation 2.7, which in dimensionless form is expressed as,

$$\vec{V}_p = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \oint \frac{\vec{\gamma}_q^n \times (\vec{r}_p - \vec{r}_q^n)}{[\vec{r}_p - \vec{r}_q^n]^2} dl + \frac{1}{2\pi} \iint \frac{\vec{\omega}_0 \times (\vec{r}_p - \vec{r}_0)}{[\vec{r}_p - \vec{r}_0]^2} + \vec{i} \quad (3.15)$$

is evaluated numerically. The treatment of the first and second integrals is explained in detail in Appendices A and B, respectively. The integrals cover an infinite number of cylinders and a flow field of infinite extent. However, because the domain is spatially periodic, the integrals may be recast into an infinite series of single integrals over a single cylinder and the primary computational region, as explained in Chapter 2.

The evaluation of the integral involving the free vorticity in Equation 3.15 is the most time consuming operation performed during each time step. In the case where there are N computational cells, the contribution of N cell vorticities must be summed for each point at which the velocity is desired.

As stated previously, the vorticity is assumed to be a constant throughout each cell. This approximation allows the moving of the vorticity out of the integral sign, such that,

$$\frac{1}{2\pi} \iint_A \frac{\vec{\omega}_o \times (\vec{r}_p - \vec{r}_o)}{[\vec{r}_p - \vec{r}_o]^2} dA \approx \sum_{n=1}^N \vec{\omega}_n \times \frac{1}{2\pi} \iint_{\Delta A_n} \frac{[\vec{r}_p - \vec{r}_o^n]}{[\vec{r}_p - \vec{r}_o]^2} dA \quad (3.16)$$

The integrals on the right hand side of Equation 3.16 involve only the geometry. For each A_n these are referred to as a geometric coefficient in the present work. They are evaluated and stored before the main program is run. Thus, if the velocity at a point due to the free vorticity is desired, then N free vorticity values are multiplied by the geometric coefficients appropriate to that point, and the products are summed together.

Essentially the same approach is employed for the calculation of the induced velocity due to the bound vorticity at any point in the flow field. The above mentioned geometric coefficients are evaluated using Gauss quadrature, and symmetry is employed wherever possible in order to reduce the number of coefficients to be stored.

Referring to Figure 3.2, the remaining velocity components are calculated by rewriting Equation 2.21 in the form,

$$\begin{aligned} v_{i,j+(1/2)} = & (u_{i-(1/2),j}^{DHL_{i,j}} + v_{i,j-(1/2)}^{DWL_{i,j}} \\ & - u_{i+(1/2),j}^{DWL_{i,j}}) / DWU_{i,j} \end{aligned} \quad (3.17)$$

One begins with the zero value of transverse velocity at points on the

cylinder ($j=1$). Thus, $v_{i,(1/2)} = 0$ in Equation 3.17. One then progresses away from the cylinder.

Force and Moment Coefficients

The evaluation of the design parameters is performed in accordance with Equations 2.50, 2.51, and 2.52. The pressure distribution on the surface of each cylinder is evaluated employing Equation 2.27, which may be rewritten in the form

$$p(\theta, 1/2) - p(0, 1/2) = - \frac{1/2}{\Delta t} \int_0^\theta \gamma(\theta', t) d\theta' \quad (3.18)$$

To determine the shear stress, we begin with the dimensional definition,

$$\tau_0 = \mu \left. \frac{\partial u}{\partial r} \right|_{r=1/2} \quad (3.19)$$

which in dimensionless form becomes,

$$\tau_0 = 2 \text{ Re } \left. \frac{\partial u}{\partial r} \right|_{r=1/2} \quad (3.20)$$

Since the fluid velocity is known only at discrete points, a second degree polynomial is fitted using three radial points at a given angular location on the surface of the cylinder. It should be noted that the computational grid is set up so that these three points are aligned along lines perpendicular to the body surface. These include a point on the surface and two nodal points adjacent to the surface.

Define the velocity function as,

$$u(r, \theta) = a_0 + a_1(r-.5) + a_2(r-.5)^2 \quad (3.21)$$

Since $u = 0$ at $r = 1/2$, then $a_0 = 0$. The other conditions are

$u(r_1, \theta) = u_1$ and $u(r_2, \theta) = u_2$. These yield,

$$a_1 = \frac{u_1 - a_2(r_1-.5)^2}{(r_1-.5)} \quad (3.22)$$

and

$$a_2 = \frac{u_2(r_1-.5) - u_1(r_2-.5)}{(r_1-.5)(r_2-.5)(r_2-r_1)} \quad (3.23)$$

Finally, a combination of Equations 3.21 and 3.22 yields the desired result,

$$\tau_0(\theta) = \frac{2a_1(\theta)}{Re} \quad (3.24)$$

CHAPTER 4

RESULTS AND DISCUSSION

In this chapter, the results of computations for flow over the infinite array of cylinders are presented and discussed. A short discussion on spatial and time increments begins the chapter, followed by a discussion of the particular geometry which is modeled. Finally, the numerical results are presented and compared with those for flow over a single cylinder at the same Reynolds number. Preliminary work with single cylinders is discussed in Appendix C.

Three separate flows are modeled. These are as follows:

1. Double row of circular cylinders, $Re = 100$, center-to-center distance = 3 diameters
2. Double row of circular cylinders, $Re = 1000$, center-to-center distance = 3 diameters
3. Double row of circular cylinders, $Re = 10000$, center-to-center distance = 3 diameters.

Choice of Finite Increments

The solution of the Fredholm integral equation is performed at every time step. As discussed in Chapter 3, the evaluation of this equation involves a system of N linear equations in N unknowns, where N is the number of bound vortex points on the cylinder surface. The more vortex points employed, the greater the accuracy of the solution. This must be balanced with the computational cost, which increases

approximately as N^3 . Previous experience obtained solving the same Fredholm integral equation (Taslim, 1981) indicated that very satisfactory results are obtained with the value of $N = 80$, which was ultimately chosen for this study.

The time-explicit nature of the boundary condition, which is used in the solution of the vorticity transport equation, implies a limit on Δt . The assumption is made that the vorticity, which is diffused from the surface of the cylinders, does not pass through the first row of control volumes in a single time step Δt . In addition, there is the consideration of the stability of the solution of Equation 3.5, which also places a restriction on Δt .

In order to select a suitable time step, Equation 3.5 may be written in the form

$$\begin{aligned} \omega_{i,j}^+ = & F \omega_{i,j} + \frac{\Delta t}{A_{i,j}} \left\{ \omega_{i+1,j} \left[-(1-F1) u_{i+(1/2),j} \right. \right. \\ & \cdot \left. \left. \frac{DHR_{i,j}}{Re RR_{i,j}} \right] + \omega_{i,j+1} \left[-(1-F2) \right. \right. \\ & \cdot \left. \left. v_{i,j+(1/2)} \frac{DWU_{i,j}}{Re RU_{i,j}} \right] \right. \\ & + \omega_{i-1,j} \left[F3 u_{i-(1/2),j} \frac{DHL_{i,j}}{Re RL_{i,j}} \right] \\ & \left. + \omega_{i,j-1} \left[F4 v_{i,j-(1/2)} \frac{DWL_{i,j}}{Re RB_{i,j}} \right] \right\} \quad (4.1) \end{aligned}$$

where

$$\begin{aligned}
F = 1 + \frac{\Delta t}{A_{i,j}} \left\{ \left[(1-F3) u_{i-(1/2),j} \text{DHL}_{i,j} \right. \right. \\
+ (1-F4) v_{i,j-(1/2)} \text{DWL}_{i,j} \\
- F1 u_{i+(1/2),j} \text{DHR}_{i,j} - F2 v_{i,j+(1/2)} \text{DWU}_{i,j} \\
\left. \left. - \frac{1}{\text{Re}} \left(\frac{\text{DHL}_{i,j}}{\text{RL}_{i,j}} + \frac{\text{DWL}_{i,j}}{\text{RB}_{i,j}} + \frac{\text{DHR}_{i,j}}{\text{RR}_{i,j}} + \frac{\text{DWU}_{i,j}}{\text{RU}_{i,j}} \right) \right] \right\} \quad (4.2)
\end{aligned}$$

A necessary condition for a stable solution (Torrance and Rockett, 1969) is that the coefficient F must be greater than or equal to zero at each fluid cell for every Δt . This condition then relates the time step to the grid parameters. The procedure then becomes one of selecting a grid spacing and Reynolds number, and then insuring that the coefficient F in Equation 4.1 is greater than zero. Note that near the surface of the cylinder, the velocities are very small and diffusive transport predominates. A stable Δt from Equation 4.2 implies that vorticity diffused from the surface of the cylinders does not penetrate beyond the first layer of fluid cells.

As can be seen from Equation 4.2, an increase in Δt generally must be coupled with an increase in cell area. As the cell size is increased, however, more information about the flow is "lost" and the resulting accuracy of the flow modeling suffers.

In an attempt to balance these competing interests between Δt and grid size, the present center-to-center distance of three diameters

between cylinders was chosen. The network of eighty by thirty nodes was selected under these criteria. The first two rows of cells were spaced concentrically around each cylinder with a radial spacing of .016 diameters. This figure was chosen from preliminary studies conducted for flows over single cylinders.

Array of Cylinders

Three cases with Reynolds numbers of 10^2 , 10^3 , and 10^4 were run for this geometry. The center-to-center distance between adjacent cylinders was fixed at three cylinder diameters.

Figure 4.1 compares the development of the drag coefficient with time for a single cylinder and for an infinite number of cylinders at $Re = 100$. There are similarities in the initial behavior of the drag coefficients. However, the drag calculated for a cylinder in the array drops rapidly to a much lower value at the end of six non-dimensional time units. The drag due to pressure for a cylinder in the array is shown as a dashed line. Plots of the lift and moment coefficients are shown in Figures 4.2 and 4.3 respectively. The velocity profiles at the boundary of the computational region are shown at non-dimensional times of 0.25, 0.50, 3.0, and 6.0 in Figures 4.4 through 4.7. Only components perpendicular to the boundary are displayed. A vector of unit length is shown beneath the region for comparison of velocity values to the flow far from the array. The velocity components on the top and bottom portions of the boundary are magnified by a factor of ten. The decrease in magnitude of these particular components

Figure 4.1 Variation of drag coefficient with time at $Re = 100$ for flow over a single cylinder and flow over an array of cylinders. Variation of pressure drag with time for a cylinder in the array is shown with a dashed line.

Figure 4.2 Variation of lift coefficient with time at $Re = 100$ for flow over an array of cylinders.

Figure 4.3 Variation of moment coefficient with time at $Re = 100$ for flow over an array of cylinders.

Figure 4.4 Velocity profile at boundary, $Re = 100$, $t = 0.25$.

Figure 4.5 Velocity profile at boundary, $Re = 100$, $t = 0.5$.

Figure 4.6 Velocity profile at boundary, $Re = 100$, $t = 3.0$.

Figure 4.7 Velocity profile at boundary, $Re = 100$, $t = 6.0$.

as the flow develops in time is noteworthy. Reversal of the direction of the flow is shown behind the cylinder in Figure 4.6.

A history of the development of vorticity in the flow field for $Re = 100$ is shown in Figures 4.8 through 4.13. Figure 4.8 corresponds to a non-dimensional time of 0.5 units, with the remaining figures spaced one time unit apart. As a measure of the extent of viscous effects, it is noted that at the end of six time units, the maximum value of free vorticity in the flow field was 7.0. The evolution of the zero vorticity contour is noteworthy in these figures. Initially it extends from cylinder to cylinder between the upper and lower rows. As time progresses, this contour separates into one continuous curve between the cylinder rows, and curves which connect cylinder to cylinder in the same row. The movement of points of zero bound vorticity and vanishing shear stress ($\omega = 0$) on the cylinder with time is shown in Figure 4.14. The points of vanishing bound vorticity coincide with points of local maxima or minima in the pressure distribution on the surface. In the vicinity of the leading edge ($\theta = 0$), points of zero bound vorticity and shear almost coincide at all instants in time. This is clearly a stagnation point. At approximately $\pi/2$ and $3\pi/2$, points of zero bound vorticity indicate the beginning of an adverse pressure gradient (local pressure minimum). The interesting behavior of the points of zero shear stress on the rear of the cylinder is due to the alternate connection and separation of the trailing zero vorticity contour line with vorticity "bubbles" forming on the rear

Figure 4.8 Constant vorticity contours for flow over an array of cylinders at $Re = 100$, $t = 0.5$.

Figure 4.9 Constant vorticity contours for flow over an array of cylinders at $Re = 100$, $t = 1.5$.

Figure 4.10 Constant vorticity contours for flow over an array of cylinders at $Re = 100$, $t = 2.5$.

Figure 4.11 Constant vorticity contours for flow over an array of cylinders at $Re = 100$, $t = 3.5$.

Figure 4.12 Constant vorticity contours for flow over an array of cylinders at $Re = 100$, $t = 4.5$.

Figure 4.13 Constant vorticity contours for flow over an array of cylinders at $Re = 100$, $t = 5.5$.

Figure 4.14 Movement of points of zero bound vorticity (...) and zero wall shear (—) on cylinder surface with time. $Re = 100$. Angles measured as shown on inset.

side of the cylinder. This behavior is graphically portrayed in Figures 4.11 through 4.13.

Figures 4.15 through 4.17 respectively show the drag, lift and moment coefficients as a function of time for $Re = 10^3$. Once again the drag coefficient is compared with that for a single cylinder. Velocity profiles at the boundary of the computational region are shown in Figures 4.18 through 4.21 at various time units. Once again, components on the top and bottom portions of the boundary are magnified by a factor of ten.

The time history of the vorticity development is shown in Figures 4.22 through 4.27. Figure 4.22 corresponds to a non-dimensional time of 0.75 units, while the remainder of the pictures are once again spaced one time unit apart. At six time units, the maximum value of the free vorticity was found to be 15.5. The development of the zero vorticity contour is noteworthy once more. Also note that the vorticity contours corresponding to $\omega = -2.0$ and -1.0 break up as they are shed into the flow field from the trailing side of each cylinder. The movement of points of zero bound vorticity and vanishing shear stress on the cylinder with time is shown in Figure 4.28. The leading edge again shows evidence of a stagnation point which moves slowly in a clockwise direction with time. Adverse pressure gradients begin at about $\pi/2$ and $3\pi/2$, with points of vanishing shear stress occurring soon thereafter towards the rear of the cylinder. The mechanism of interaction between the generation of vorticity "bubbles" and the

Figure 4.15 Variation of drag coefficient with time at $Re = 1000$ for flow over a single cylinder and flow over an array of cylinders. Variation of pressure drag with time for a cylinder in the array is shown with a dashed line.

Figure 4.16 Variation of lift coefficient with time at $Re = 1000$ for flow over an array of cylinders.

Figure 4.17 Variation of moment coefficient with time at $Re = 1000$ for flow over an array of cylinders.

Figure 4.19 Velocity profile at boundary, $Re = 1000$, $t = 0.5$.

Figure 4.20 Velocity profile at boundary, $Re = 1000$, $t = 3.0$.

Figure 4.21 Velocity profile at boundary, $Re = 1000$, $t = 6.0$.

Figure 4.22 Constant vorticity contours for flow over an array of cylinders at $Re = 1000$, $t = .75$.

Figure 4.23 Constant vorticity contours for flow over an array of cylinders at $Re = 1000$, $t = 1.75$.

Figure 4.24 Constant vorticity contours for flow over an array of cylinders at $Re = 1000$, $t = 2.75$.

Figure 4.25 Constant vorticity contours for flow over an array of cylinders at $Re = 1000$, $t = 3.75$.

Figure 4.26 Constant vorticity contours for flow over an array of cylinders at $Re = 1000$, $t = 4.75$.

Figure 4.27 Constant vorticity contours for flow over an array of cylinders at $Re = 1000$, $t = 5.75$.

Figure 4.28 Movement of points of zero bound vorticity (...) and zero wall shear (—) on cylinder surface with time. $Re = 1000$. Angles measured as shown on inset.

trailing zero vorticity contour accounts for the activity near the rear of the cylinder.

The design parameters for $Re = 10^2$ and 10^3 were found to be changing slowly at the end of six non-dimensional time units. It was, therefore, decided to run one of the cases out to steady state. Thus, for $Re = 10^4$, the case of flow over an infinite number of cylinders was run out to $t = 14.5$. At this point, changes in lift and drag seemed to be minimal, while the moment coefficient seemed to be close to a steady value.

Figure 4.29 is a plot of drag coefficient versus time for the single cylinder and an array of cylinders out to a time of six non-dimensional units. Figures 4.30 through 4.32 then show the drag, lift and moment coefficients, respectively, for $Re = 10^4$ out to $t = 14.5$. Velocity profiles at the boundary of the computational region are shown in Figures 4.33 through 4.36 at various time units. Once again, components on the time and bottom portions of the boundary are magnified by a factor of ten.

Figures 4.37 through 4.42 show the time history of the vorticity development beginning at $t = 0.75$ and spaced at one time unit apart. The same observations apply as for previous vorticity plots. At six time units, the maximum value of the free vorticity was found to be 37.0, while at 14.5 units, it was 20.8. At this relatively high Reynolds number, the sharp changes in direction of vorticity lines a short distance behind each cylinder are interesting. This behavior seems to be due to the coarseness of the grid in this part of the

Figure 4.29 Variation of drag coefficient with time at $Re = 10000$ for flow over a single cylinder and flow over an array of cylinders. Variation of pressure drag with time for a cylinder in the array is shown with a dashed line.

Figure 4.30 Variation of total drag coefficient (solid line) and pressure drag coefficient (dashed line) with time at $Re = 10000$ for flow over array of cylinders.

Figure 4.31 Variation of lift coefficient with time at $Re = 10000$ for flow over an array of cylinders.

Figure 4.32 Variation of moment coefficient with time at $Re = 10000$ for flow over an array of cylinders.

Figure 4.33 Velocity profile at boundary, $Re = 10000$, $t = 0.25$.

Figure 4.34 Velocity profile at boundary, $Re = 10000$, $t = 0.5$.

Figure 4.35 Velocity profile at boundary, $Re = 10000$, $t = 3.0$.

Figure 4.36 Velocity profile at boundary, $Re = 10000$, $t = 6.0$.

Figure 4.37 Constant vorticity contours for flow over an array of cylinders at $Re = 10000$, $t = 0.75$.

Figure 4.38 Constant vorticity contours for flow over an array of cylinders at $Re = 10000$, $t = 1.75$.

Figure 4.39 Constant vorticity contours for flow over an array of cylinders at $Re = 10000$, $t = 2.75$.

Figure 4.40 Constant vorticity contours for flow over an array of cylinders at $Re = 10000$, $t = 3.75$.

Figure 4.41 Constant vorticity contours for flow over an array of cylinders at $Re = 10000$, $t = 4.75$.

Figure 4.42 Constant vorticity contours for flow over an array of cylinders at $Re = 10000$, $t = 5.75$.

computational region. Figure 4.43 shows the vorticity plot at $t = 14.5$. Figure 4.44 shows the movement of points of zero bound vorticity and vanishing shear stress with time for this Reynolds number. Similar observations as for the previous Reynolds numbers apply. This diagram illustrates the cyclic nature of the flow due to periodic vortex shedding.

An attempt was made after computations were completed to calculate the drag on a cylinder in the array through use of the momentum theorem. The results of this computation would serve to verify the calculation of drag through integration of shear and pressure forces around the cylinder surface. Due to the spatial periodicity of the flow field, the influx and outflux of momentum across the left and right-hand sides of the boundary must cancel out. The same conclusion may be drawn about the pressure forces. This means that the drag on a cylinder is then determined solely by the flux of the x-component of momentum across the upper and lower portions of the boundary. The x-component of velocity was not calculated at the upper and lower boundaries during the course of the calculations. Therefore, this momentum term could not be determined. However, an approximation was used in which this x-component of velocity was set to unity. The drag thus obtained was much larger than that calculated by the surface forces. The validity of this approximation cannot be assessed. Therefore, no independent verification of the drag coefficient is possible without repeating the entire set of calculations.

Figure 4.43 Constant vorticity contours for flow over an array of cylinders at $Re = 10000$, $t = 14.5$.

Figure 4.44 Movement of points of zero bound vorticity (...) and zero wall shear (—) on cylinder surface with time. $Re = 100000$. Angles measured as shown on inset.

APPENDIX A

CALCULATION OF VELOCITY INDUCED BY BOUND VORTICITY

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CALCULATION OF VELOCITY INDUCED BY BOUND VORTICITY

Calculation of the Bound Vorticity

Consider for a moment a single circular cylinder with a uniform onset flow $U\vec{i}$ as shown in Figure A.1. The velocity vector at point P, located an infinitesimal distance below the surface of the cylinder, can be written as,

$$\begin{aligned}\vec{V}_p &= \vec{i} + \frac{1}{2\pi} \int_A \int \frac{\vec{\omega}_o \times \vec{r}_{op}}{[\vec{r}_{op}]^2} dA + \frac{1}{2\pi} \oint \frac{\vec{\gamma}_q \times \vec{r}_{qp}}{[\vec{r}_{qp}]^2} dl \\ &\quad - \frac{\gamma_p}{2} \vec{e}_\theta = 0\end{aligned}\tag{A.1}$$

where $\vec{r}_{ab} = \vec{r}_b - \vec{r}_a$, $\vec{i}$ and $\vec{j}$ are the unit vectors in the x and y directions, respectively, and $\frac{\gamma_p}{2} \vec{e}_\theta$ is the contribution to the velocity of point P by the vortex at that point. This is shown in Figure 2.4 and by Equation 2.9. The term $\vec{i}$ is due to the onset flow velocity U, which has been used to non-dimensionalize the flow variables.

To simplify Equation A.1, we let,

$$\vec{\gamma}_q = \gamma_q \vec{k}\tag{A.2}$$

$$\vec{r}_{qp} = (x_p - x_q) \vec{i} + (y_p - y_q) \vec{j}\tag{A.3}$$

On the surface of the cylinder one has

$$x = .5\cos\theta\tag{A.4}$$

Figure A.1 Contribution of free and bound vorticities to the velocity at a point on the surface of the body.

$$y = .5\sin\theta \quad (\text{A.5})$$

$$\vec{e}_\theta = -\sin\theta \vec{i} + \cos\theta \vec{j} \quad (\text{A.6})$$

where $\vec{e}_\theta$ is a unit vector tangential to the surface of the circular cylinder. Let ϕ denote the angle which identifies point P and θ the angle which identifies point q. Upon substitution of Equations A.4 and A.5 into A.3, there is obtained

$$\vec{r}_{qp} = .5(\cos\phi - \cos\theta) \vec{i} + .5(\sin\phi - \sin\theta) \vec{j} \quad (\text{A.7})$$

and therefore,

$$|\vec{r}_{qp}|^2 = .5^2(\cos\phi - \cos\theta)^2 + .5^2(\sin\phi - \sin\theta)^2 \quad (\text{A.8})$$

In addition,

$$\begin{aligned} \vec{\gamma}_q \times \vec{r}_{qp} &= \gamma(\theta) \vec{k} \times [.5(\cos\phi - \cos\theta) \vec{i} + .5(\sin\phi - \sin\theta) \vec{j}] \\ &= .5\gamma(\theta) [(\cos\phi - \cos\theta) \vec{j} - (\sin\phi - \sin\theta) \vec{i}] \quad (\text{A.9}) \end{aligned}$$

and,

$$dl = .5d\theta \quad (\text{A.10})$$

The second integral in Equation A.1 can thus be written as

$$\frac{1}{2\pi} \oint \frac{\vec{\gamma}_q \times \vec{r}_{qp}}{|\vec{r}_{qp}|^2} dl = \frac{1}{2\pi} \int_0^{2\pi} - \frac{(\sin\phi - \sin\theta)\vec{i} + (\cos\phi - \cos\theta)\vec{j}}{(\cos\phi - \cos\theta)^2 + (\sin\phi - \sin\theta)^2} \gamma(\theta) d\theta \quad (\text{A.11})$$

The component tangential to the surface of the cylinder is,

$$\begin{aligned}
 \frac{1}{2\pi} \oint \frac{\vec{\gamma}_q \times \vec{r}_{qp}}{[\vec{r}_{qp}]^2} d\vec{l} \cdot \vec{e}_\theta &= \frac{1}{2\pi} \int_0^{2\pi} - \frac{(\sin\phi - \sin\theta)\vec{i} + (\cos\phi - \cos\theta)\vec{j}}{(\cos\phi - \cos\theta)^2 + (\sin\phi - \sin\theta)^2} \\
 &\quad \cdot \gamma(\theta) d\theta \cdot [-\sin\phi \vec{i} + \cos\phi \vec{j}] \\
 &= \frac{1}{2\pi} \int_0^{2\pi} \frac{(\sin\phi - \sin\theta)\sin\phi + (\cos\phi - \cos\theta)\cos\phi}{(\cos\phi - \cos\theta)^2 + (\sin\phi - \sin\theta)^2} \\
 &\quad \cdot \gamma(\theta) d\theta \quad (A.12)
 \end{aligned}$$

Let us now define,

$$\frac{1}{2\pi} \iint \frac{\vec{\omega}_o \times \vec{r}_{op}}{[\vec{r}_{op}]^2} dA \cdot \vec{e}_\theta = u_\omega \quad (A.13)$$

Since point P is just inside the surface of the cylinder, the fluid must be at rest at that point. Therefore, $V_{p\theta}$ must be zero, and Equation A.1 can be rearranged into the form,

$$\begin{aligned}
 \gamma_p - \frac{1}{\pi} \int_0^{2\pi} \frac{(\sin\phi - \sin\theta)\sin\phi + (\cos\phi - \cos\theta)\cos\phi}{(\cos\phi - \cos\theta)^2 + (\sin\phi - \sin\theta)^2} \gamma(\theta) d\theta \\
 = -2(u_\omega + \sin\phi) \quad (A.14)
 \end{aligned}$$

Now, u_ω is a function of position on the surface of the cylinder and thus a function of ϕ . Therefore, Equation A.14 can be written in the general form

$$\gamma(\theta) - \frac{1}{\pi} \oint_D K(\phi, \theta) \gamma(\theta) d\theta = f(\phi) \quad (A.15)$$

which is a Fredholm integral equation of the second kind. The right hand side is considered known.

In general, Equation A.15 must be solved numerically. However, in the case of single circular cylinder in the flow field, an analytical solution to $\gamma(\theta)$ is possible (Mikhlin, 1957, pp. 26-30). The kernel of the integral in Equation A.12 can be reduced as follows,

$$\begin{aligned} & \frac{(\sin\phi - \sin\theta)\sin\phi + (\cos\phi - \cos\theta)\cos\phi}{(\cos\phi - \cos\theta)^2 + (\sin\phi - \sin\theta)^2} = \{1 - \cos(\phi - \theta)\} \\ & \quad / \{[-2\sin(\frac{\phi+\theta}{2})\sin(\frac{\phi-\theta}{2})]^2 + [2\cos(\frac{\phi+\theta}{2})\cos(\frac{\phi-\theta}{2})]^2\} \\ & = \frac{1 - \cos(\phi - \theta)}{4\sin^2(\frac{\phi-\theta}{2})} = \frac{1 - \cos(\phi - \theta)}{2[1 - \cos(\phi - \theta)]} = 1/2 \end{aligned}$$

And thus Equation A.14 for the single cylinder reduces to,

$$\gamma(\phi) - \frac{1}{2\pi} \int_0^{2\pi} \gamma(\theta) d\theta = f(\phi) \quad (\text{A.16})$$

If we set

$$\frac{1}{2\pi} \int_0^{2\pi} \gamma(\theta) d\theta = A$$

where A is an arbitrary constant, then

$$\gamma(\phi) = f(\phi) + A$$

Multiplication of both sides of the previous equation by $1/2\pi$, and integration yields

$$\int_0^{2\pi} f(\theta) d\theta = 0 \quad (\text{A.17})$$

Equation A.16 may be rearranged into the form

$$\gamma(\phi) = f(\phi) + \frac{1}{2\pi} \int_0^{2\pi} \gamma(\theta) d\theta \quad (\text{A.18})$$

The method of successive approximations (Mikhlin, 1957) may then be employed to solve for $\gamma(\phi)$. The technique is to assume initially that $\gamma(\phi) = f(\phi)$. This value for $\gamma(\phi)$ is then substituted into the right-hand side of Equation A.18. A new value of $\gamma(\phi)$ is obtained which is once again substituted into the right-hand side of A.18 until this iteration process yields the desired convergence. In the present case, convergence is achieved in two iterations. The result is

$$\gamma(\phi) = f(\phi) + \frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\theta \quad (\text{A.19})$$

Combination of this equation with Equation A.17 yields the final result,

$$\gamma(\phi) = f(\phi) \quad (\text{A.20})$$

For the case of the array of cylinders, spatial periodicity in the flow field dictates that the velocity induced at a point be expressed in a manner similar to that of Equation 2.31. The velocity induced by this infinite series may be evaluated in a closed form through the employment of Equations 2.42 and 2.43.

Velocity Induced by Bound Vorticity for Single Cylinder Case

Consider a point P somewhere in the flow field, as shown in Figure A.2. The induced velocity at that point due to the bound vorticity is,

$$\vec{V}_p = \frac{1}{2\pi} \oint \frac{\vec{\gamma}_q \times \vec{r}_{qp}}{[\vec{r}_{qp}]^2} dl \quad (A.21)$$

Now,

$$\vec{r}_{qp} = (x_p - .5\cos\theta) \vec{i} + (y_p - .5\sin\theta) \vec{j}$$

$$\vec{\gamma}_q = \gamma_q \vec{k}$$

$$dl = .5d\theta$$

Therefore,

$$\frac{1}{2\pi} \oint \frac{\vec{\gamma}_q \times \vec{r}_{qp}}{[\vec{r}_{qp}]^2} dl = \frac{1}{2\pi} \int_0^{2\pi} - \frac{(y_p - .5\sin\theta)\vec{i} + (x_p - .5\cos\theta)\vec{j}}{(x_p - .5\cos\theta)^2 + (y_p - .5\sin\theta)^2} \gamma(\theta) (.5d\theta) \quad (A.22)$$

Once the induced velocity is known at a particular point in the $\vec{i}$ and $\vec{j}$ directions, it is then a simple matter to obtain the velocity at that point in any direction by application of the appropriate scalar products.

The exact solution to uniform inviscid flow over a two-dimensional cylinder is well known (Lamb, 1932; Milne-Thomson, 1966). The velocity components for this solution are easily expressed in polar coordinates as,

Figure A.2 Contribution of the bound vorticity to the velocity at a point in the flow field.

$$V_r = (1 - a^2/r^2) \cos\theta$$

$$V_\theta = -(1 + a^2/r^2) \sin\theta$$

where a is the radius of the cylinder. The induced velocities are obtained through the numerical integration of Equation A.20. These are compared with the exact solution for the case of uniform onset flow in Tables A.1 and A.2. Here the V_θ components of the flow are compared at 80 points around the cylinder at $r = .55$ and $r = .75$. The agreement is generally to seven significant figures.

The spatial periodicity of the flow field, for the case of the array of cylinders, once again dictates the modification of velocities induced at any given point in accordance with Equation 2.31. The infinite series is evaluated in closed form through the use of Equations 2.42 and 2.43.

Table A.1 Comparison of Induced Velocities Due to Bound Vorticity and the Exact Solution for Uniform Onset Flow, Radial Distance $R = 0.55$.

RADIAL DISTANCE $R = 0.55$

I	UTHEA	EXACT	I	UTHEA	EXACT
1	0.735581430-13	0.00000000E+00	2	0.143301300+00	0.143301320+00
3	0.235719130+00	0.235719150+00	4	0.423375350+00	0.423375420+00
5	0.554402850+00	0.554402940+00	6	0.693950620+00	0.693950730+00
7	0.829189130+00	0.829189250+00	8	0.954315410+00	0.954315560+00
9	0.107355800+01	0.107355820+01	10	0.118618180+01	0.118618200+01
11	0.129149240+01	0.129149240+01	12	0.138834040+01	0.138834060+01
13	0.147762590+01	0.147762610+01	14	0.155730120+01	0.155730150+01
15	0.162737530+01	0.162737550+01	16	0.163741610+01	0.163741630+01
17	0.173705340+01	0.173705360+01	18	0.177598120+01	0.177598140+01
19	0.180395940+01	0.180395970+01	20	0.182031570+01	0.182031600+01
21	0.182344630+01	0.182344630+01	22	0.182031570+01	0.182031600+01
23	0.180395940+01	0.180395970+01	24	0.177598120+01	0.177598140+01
25	0.173705340+01	0.173705360+01	26	0.163741610+01	0.163741630+01
27	0.162737530+01	0.162737550+01	28	0.155730120+01	0.155730150+01
29	0.147762590+01	0.147762610+01	30	0.138834040+01	0.138834060+01
31	0.129149240+01	0.129149240+01	32	0.118618180+01	0.118618200+01
33	0.107355800+01	0.107355820+01	34	0.954315410+00	0.954315560+00
35	0.829189130+00	0.829189250+00	36	0.693950620+00	0.693950730+00
37	0.554402850+00	0.554402940+00	38	0.423375350+00	0.423375420+00
39	0.235719130+00	0.235719150+00	40	0.143301300+00	0.143301320+00
41	0.149487600-12	0.715926970-13	42	-0.143301300+00	-0.143301320+00
43	-0.235719130+00	-0.235719150+00	44	-0.423375350+00	-0.423375420+00
45	-0.554402850+00	-0.554402940+00	46	-0.693950620+00	-0.693950730+00
47	-0.829189130+00	-0.829189250+00	48	-0.954315410+00	-0.954315560+00
49	-0.107355800+01	-0.107355820+01	50	-0.118618180+01	-0.118618200+01
51	-0.129149240+01	-0.129149240+01	52	-0.138834040+01	-0.138834060+01
53	-0.147762590+01	-0.147762610+01	54	-0.155730120+01	-0.155730150+01
55	-0.162737530+01	-0.162737550+01	56	-0.163741610+01	-0.163741630+01
57	-0.173705340+01	-0.173705360+01	58	-0.177598120+01	-0.177598140+01
59	-0.180395940+01	-0.180395970+01	60	-0.182031570+01	-0.182031600+01
61	-0.182344630+01	-0.182344630+01	62	-0.182031570+01	-0.182031600+01
63	-0.180395940+01	-0.180395970+01	64	-0.177598120+01	-0.177598140+01
65	-0.173705340+01	-0.173705360+01	66	-0.163741610+01	-0.163741630+01
67	-0.162737530+01	-0.162737550+01	68	-0.155730120+01	-0.155730150+01
69	-0.147762590+01	-0.147762610+01	70	-0.138834040+01	-0.138834060+01
71	-0.129149240+01	-0.129149240+01	72	-0.118618180+01	-0.118618200+01
73	-0.107355800+01	-0.107355820+01	74	-0.954315410+00	-0.954315560+00
75	-0.829189130+00	-0.829189250+00	76	-0.693950620+00	-0.693950730+00
77	-0.554402850+00	-0.554402940+00	78	-0.423375350+00	-0.423375420+00
79	-0.235719130+00	-0.235719150+00	80	-0.143301300+00	-0.143301320+00

Table A.2 Comparison of Induced Velocities Due to Bound Vorticity and the Exact Solution for Uniform Onset Flow, Radial Distance $R = 0.75$.

RADIAL DISTANCE $R = 0.75$

I	UTHEA	EXACT	I	UTHEA	EXACT
1	0.110358720-13	0.000000000+00	2	0.113329790+00	0.113329800+00
3	0.225960870+00	0.225960870+00	4	0.337198320+00	0.337198360+00
5	0.446357830+00	0.446357830+00	6	0.552764900+00	0.552764960+00
7	0.655763990+00	0.655764050+00	8	0.754720070+00	0.754720150+00
9	0.849023050+00	0.849023140+00	10	0.938091530+00	0.938091630+00
11	0.102137640+01	0.102137650+01	12	0.109836410+01	0.109836420+01
13	0.116358000+01	0.116358010+01	14	0.123159120+01	0.123159130+01
15	0.128700930+01	0.128700940+01	16	0.133449250+01	0.133449270+01
17	0.137374820+01	0.137374830+01	18	0.140453420+01	0.140453430+01
19	0.142566050+01	0.142566070+01	20	0.143999160+01	0.143999170+01
21	0.144444430+01	0.144444440+01	22	0.143999160+01	0.143999170+01
23	0.142566080+01	0.142566090+01	24	0.140453420+01	0.140453430+01
25	0.137374820+01	0.137374830+01	26	0.133449250+01	0.133449270+01
27	0.128700930+01	0.128700940+01	28	0.123159120+01	0.123159130+01
29	0.116358000+01	0.116358010+01	30	0.109836410+01	0.109836420+01
31	0.102137640+01	0.102137650+01	32	0.938091530+00	0.938091630+00
33	0.849023050+00	0.849023140+00	34	0.754720070+00	0.754720150+00
35	0.655763990+00	0.655764050+00	36	0.552764900+00	0.552764960+00
37	0.446357830+00	0.446357830+00	38	0.337198320+00	0.337198360+00
39	0.225960870+00	0.225960870+00	40	0.113329790+00	0.113329800+00
41	0.123452930-12	0.566190610-13	42	-0.113329790+00	-0.113329800+00
43	-0.225960870+00	-0.225960870+00	44	-0.337198320+00	-0.337198360+00
45	-0.446357830+00	-0.446357830+00	46	-0.552764900+00	-0.552764960+00
47	-0.655763990+00	-0.655764050+00	48	-0.754720070+00	-0.754720150+00
49	-0.849023050+00	-0.849023140+00	50	-0.938091530+00	-0.938091630+00
51	-0.102137640+01	-0.102137650+01	52	-0.109836410+01	-0.109836420+01
53	-0.116358000+01	-0.116358010+01	54	-0.123159120+01	-0.123159130+01
55	-0.128700930+01	-0.128700940+01	56	-0.133449250+01	-0.133449270+01
57	-0.137374820+01	-0.137374830+01	58	-0.140453420+01	-0.140453430+01
59	-0.142566050+01	-0.142566070+01	60	-0.143999160+01	-0.143999170+01
61	-0.144444430+01	-0.144444440+01	62	-0.143999160+01	-0.143999170+01
63	-0.142566080+01	-0.142566090+01	64	-0.140453420+01	-0.140453430+01
65	-0.137374820+01	-0.137374830+01	66	-0.133449250+01	-0.133449270+01
67	-0.128700930+01	-0.128700940+01	68	-0.123159120+01	-0.123159130+01
69	-0.116358000+01	-0.116358010+01	70	-0.109836410+01	-0.109836420+01
71	-0.102137640+01	-0.102137650+01	72	-0.938091530+00	-0.938091630+00
73	-0.849023050+00	-0.849023140+00	74	-0.754720070+00	-0.754720150+00
75	-0.655763990+00	-0.655764050+00	76	-0.552764900+00	-0.552764960+00
77	-0.446357830+00	-0.446357830+00	78	-0.337198320+00	-0.337198360+00
79	-0.225960870+00	-0.225960870+00	80	-0.113329790+00	-0.113329800+00

APPENDIX B

INDUCED VELOCITY DUE TO FREE VORTICITY

APPENDIX B

INDUCED VELOCITY DUE TO FREE VORTICITY

The contribution to the velocity at an arbitrary point P, due to the free vorticity, ω , in a computational cell of size ΔA_0 , can be determined once again through the use of the Biot-Savart law. One has

$$\vec{V}_p = \frac{1}{2\pi} \int_{\Delta} \int_{A_0} \frac{\vec{\omega}_0 \times \vec{r}_{op}}{[\vec{r}_{op}]^2} d A_0 \quad (B.1)$$

where the orientation is as shown in Figure B.1.

It follows that the velocity induced at P by an infinitesimal area $dxdy$ within ΔA_0 is,

$$d^2 \vec{V}_p = \frac{1}{2\pi} \frac{\vec{\omega}_0 \times \vec{r}_{op}}{[\vec{r}_{op}]^2} dxdy \quad (B.2)$$

Please refer to Figure B.2, from which it is seen that

$$\vec{r}_{op} = (x_p - x_0) \vec{i} + (y_p - y_0) \vec{j} \quad (B.3)$$

Also, $\vec{\omega}_0 = \omega_0 \vec{k}$. Substitution of Equations B.3 and $\vec{\omega}_0$ into B.2 yields,

$$d^2 \vec{V}_p = \frac{\omega_0}{2\pi} \left[\frac{-(y_p - y_0) \vec{i}}{r_{op}^2} + \frac{(x_p - x_0) \vec{j}}{r_{op}^2} \right] dxdy \quad (B.4)$$

The scalar components of B.4 are,

$$d^2 u_x = - \frac{\omega_0 (y_p - y_0)}{2\pi r_{op}^2} dxdy \quad (B.5)$$

Figure B.1 Graphical representation of the induced velocity due to the free vorticity.

Figure B.2 Graphical representation of the contribution of an infinitesimal region of free vorticity to the velocity at a point P .

and,

$$d^2 u_y = \frac{\omega_0 (x_p - x_0)}{2\pi r_{op}^2} dx dy \quad (B.6)$$

Equations B.5 and B.6 can be simplified by using a change of variables. Consider a vertical slice of the cross-sectional area of a region of free vorticity, as shown in Figure B.3. Now,

$$r_{op} d\alpha = -dy \sin\alpha \quad (B.7)$$

and,

$$\sin\alpha = (x_0 - x_p)/r_{op} \quad (B.8)$$

$$\cos\alpha = (y_0 - y_p)/r_{op} \quad (B.9)$$

Substitution of Equations B.7, B.8 and B.9 into Equations B.5 and B.6, yields,

$$d^2 u_x = -\frac{\omega_0}{2\pi} \left(\frac{-\cos\alpha}{r_{op}} \right) \left(\frac{-r_{op} d\alpha}{\sin\alpha} \right) dx = -\frac{\omega_0 d\sin\alpha}{2\pi \sin\alpha} dx$$

Next, integrate with respect to α to get

$$\begin{aligned} d u_x &= -\frac{\omega_0}{2\pi} \int_{\alpha_1}^{\alpha_2} \frac{d\sin\alpha}{\sin\alpha} = -\frac{\omega_0}{2\pi} \ln \left(\frac{\sin\alpha_2}{\sin\alpha_1} \right) dx \\ &= -\frac{\omega_0}{2\pi} \ln(R_1/R_2) dx \end{aligned} \quad (B.10)$$

Similarly,

$$d u_y = \frac{\omega_0}{2\pi} (\alpha_2 - \alpha_1) dx \quad (B.11)$$

Figure B.3 Coordinate system used for Biot-Savart integration.

A significant reduction is gained by using this change-of-variable scheme. It is seen that the evaluation of the velocity components due to free vorticity involves single rather than double integrals. The final integration with respect to x is performed numerically. In effect, the finite cell area, ΔA_0 , has been decomposed into vertical strips of width Δx . The velocity components induced by each strip are given by Equations B.10 and B.11. The summation over all strips within ΔA_0 yields the total velocity contribution for that fluid cell.

For the case of the array of cylinders, Equation B.1 must be modified in accordance with Equation 2.31 in order to account for the spatial periodicity in the flow field. The closed form solution of this infinite series is then evaluated as shown in Equations 2.42 and 2.43.

APPENDIX C

PRELIMINARY STUDIES

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PRELIMINARY STUDIES

Flow Over a Single Circular Cylinder

The solution for flow over a single circular cylinder was calculated at Reynolds numbers of 10^2 , 10^3 , 10^4 , and 10^5 in order to gain confidence with the present approach. Also, it provides a useful comparison with the flow over a double row of cylinders.

With a view toward computational economy, the flow region was established as a square, centered on the cylinder, with sides equal to three cylinder diameters. This allows for the resolution of the near field and a satisfactory comparison with the majority of the experimental data. However, as will be pointed out, neglect of the far wake does create some discrepancies.

A plot of drag coefficient as a function of time is shown in Figure C.1. The development of these curves with time follows that shown in many previous studies (Payne, 1957; Loc, 1980). The steady state drag coefficient for a Reynolds number of 100 was determined to be approximately 1.67, which is somewhat higher than the generally accepted experimental value of about 1.35 (Goldstein, 1938). This is felt to be due to the relative thickness of the wake at this low Reynolds number. This vorticity is neglected once it travels 1.5 cylinder diameters downstream in the computational scheme employed. At a Reynolds number of 10^3 , this effect seems to be much less

Figure C.1 Variation of drag coefficient with time for flow over a single cylinder at various Reynolds numbers.

pronounced as evidenced by a steady state drag coefficient of 1.11, which compares very favorably with an experimental value of about 1.0. At a Reynolds number of 10^4 , the value computed is 0.77. This is below the accepted experimental value of 1.1. The reason for this behavior is unclear. However, it is interesting to note that Chorin (1973) obtained similar computational results at this same Reynolds number. He conjectured that his rough representation of the boundary layer, which is not quite the case with the computational grid in the present work, might be triggering premature onset of the drag crisis. This can be accomplished experimentally with a rough boundary or noisy flow. In order to test this hypothesis, the Reynolds number was raised to a value of 10^5 . The drag coefficient computed was approximately 0.61, which seems to be in agreement with experimental values beyond the drag crisis (Schlichting, 1979).

A comparison of the normal derivative of the velocity on the surface of the cylinder obtained from the present study with that computed by Cowley (1983) is shown in Figures C.2, C.3, and C.4. Cowley studied the initial stages of flow development by extending the Blasius time-series expansion to many terms. The dashed line shown on the figures represents his results at $t = 0.75$ time units, non-dimensionalized according to the manner defined in the present work. There is some doubt as to the non-dimensionalization he employed for velocity, since this was not given explicitly. It appears that he used $2U$ as the characteristic velocity. The plots shown are based on this assumption, which leads to the relationship

Figure C.2 Normal derivative of velocity at the wall for $Re = 100$ at various time intervals: — present study; --- Cowley (1983) at $t = 0.75$.

Figure C.3 Normal derivative of velocity at the wall for $Re = 1000$ at various time intervals: — present study; ---Cowley (1983) at $t = 0.75$.

Figure C.4 Normal derivative of velocity at the wall for $Re = 10000$ at various time intervals: — present study, --- Cowley (1983) at time = 0.75.

$$4 \text{ Re}^{1/2} u_{y \text{ Cowley}} = u_{y \text{ Present work}} \quad (\text{C.1})$$

Here, $u_y = u/y$.

Cowley was unable to complete the curve between $\theta = 0.62\pi$ and 0.82π at this particular instant in time due to an insufficient number of terms in his approximation. The location of points of zero shear stress and the magnitude of the wall shear upstream of these points compare favorably with those obtained in this work for Reynolds numbers of 100 and 1000. The reason for the disparity in shear stress on the downstream side of the cylinder is unclear. The difference in values at a Reynolds number of 10000 is most probably due to the coarse grid near the surface employed in the present work.

Steady state contours of constant vorticity for the four Reynolds numbers considered are shown in Figures C.5 through C.8. These show the relatively high degree of resolution obtained for the flow near the cylinder. As a measure of the extent of viscous effects, it should be noted that at six time units, the maximum value of free vorticity on the flow field was 20.5 for $\text{Re} = 10^2$, 54.1 for $\text{Re} = 10^3$, 95.7 for $\text{Re} = 10^4$, and 107.3 for $\text{Re} = 10^5$.

As a further point of comparison with experimental results, Strouhal numbers for the various flows were determined. At a Reynolds number of 100, the calculated value is approximately 0.196. This again compares poorly with an experimental value of about 0.15 (McCroskey, 1977). However, agreement improves markedly at higher values of the Reynolds number. For Reynolds numbers between 400 and 200,000, the accepted value for the Strouhal number is 0.21. In this region, the

Figure C.5 Constant vorticity contours for flow over a single cylinder at $Re = 100$, $t = 6.0$.

Figure C.6 Constant vorticity contours for flow over a single cylinder at $Re = 1000$, $t = 6.0$.

Figure C.7 Constant vorticity contours for flow over a single cylinder at $Re = 10000$, $t = 6.0$

Figure C.8 Constant vorticity contours for flow over a single cylinder at $Re = 100000$, $t = 6.0$.

correspondence is quite good, with computed values of 0.198 for $Re = 10^3$, 0.215 for $Re = 10^4$, and 0.190 for $Re = 10^5$.

Verification of Symmetry Assumptions

Although the spatial assumptions of symmetry discussed in Chapter 2 were thought to be reasonable, a great deal of effort was spent on their verification. The method employed was to expand the computational flow field to three adjacent flow regions, as shown in Figure C.9, and to allow vorticities, free and bound, as well as velocities, to develop independently in each region. The assumption of a recurring flow pattern (about a group of three cylinders) was then again employed in the remainder of the flow region. Although not an absolute proof, the relaxation of symmetry in this newly expanded computational flow region was deemed to be a reasonable test of the assumptions made.

The methods employed in the computations were very similar to those already discussed in Chapter 3. However, the solution for the bound vorticity distribution for several bodies in the flow field has some subtle features which need to be brought out.

The governing equation for the bound vorticity distribution on each cylinder remains similar to Equation 3.10. The only difference is that the kernel $k(\phi, \theta)$ now involves, in addition, the influence of the bound vortex points from one cylinder to another in the expanded computational region. This equation, once again, constitutes n equations in n unknowns in the form

$$a_{i,j} \gamma_j = b_i \quad (C.2)$$

Figure C.9 Computational flow field expanded to three adjacent flow regions.

where n is the total number of points on the surfaces of the cylinders at which γ is calculated. This equation is indeterminate due to reasons presented in Chapter 2. As shown below, the system of equations must now be reduced by an amount equal to the number of bodies present in the flow field, in this case three.

Since the method of solution of this indeterminate system is perfectly general, it can be easily expressed for any number of bodies in the flow field. Let p be the number of bodies present in the flow field, with m bound vortex points located on each cylinder. The number of total bound vortex points to be calculated in the flow field then becomes p multiplied by m , or n . The rank of this system of equations is now $n-p$, since the presence of each body reduces the rank of the matrix by one. This is because the solution is non-unique to within an arbitrary circulation, which must be specified for each cylinder. Kreyszig (1979, pp. 315-316) states that p solutions may be obtained by assigning values to $n-p$ unknowns in an arbitrary fashion and then combining these individual solutions to form a basis of solutions (i.e., linearly independent solutions) for the homogeneous case. The non-uniqueness of the general solution arises from the fact that the homogeneous algebraic equations possess non-trivial solutions. These are complementary solutions, and the number of such solutions equals the number of bodies, p . Each solution corresponds to a pure circulation about each cylinder. Let these individual solutions be designated as $\gamma_j^{(1)}, \gamma_j^{(2)}, \dots, \gamma_j^{(p)}$. The solution of the bound vorticity distribution then becomes

$$\gamma_j = C_1 \gamma_j^{(1)} + C_2 \gamma_j^{(2)} + \dots + C_p \gamma_j^{(p)} + \gamma_j^{(p)} \quad (C.3)$$

where C_j represents a set of arbitrary multiplicative constants, and $\gamma_j^{(p)}$ is the particular solution. The procedure used to find the various solutions is explained below.

In order to solve for the p number of complementary solutions, we begin by choosing a point on each of p cylinders which is to be assigned an arbitrary value. In the present case, $m, 2m, 3m, \dots, n$ are designated to be those particular points. In order to determine $\gamma_j^{(1)}$, without loss of generality, by assigning a value of unity to $\gamma_m^{(1)}$ and zero to $\gamma_{2m}^{(1)}, \gamma_{3m}^{(1)}, \dots, \gamma_n^{(1)}$. The equation to be solved now becomes

$$a_{i,j} \gamma_j^{(1)} = -a_{i,m} \quad (C.4)$$

where i and j vary from 1 to n and there are $n-p$ equations. Note that the i and j indices now may not take the value of $m, 2m, \dots, n$, since those values have been arbitrarily assigned. This system of equations is nonsingularly and a unique solution may be easily determined. A similar procedure is employed to find $\gamma_j^{(2)}$. In that case, $\gamma_{2m}^{(2)}$ is set to unity and $\gamma_m^{(2)}, \gamma_{3m}^{(2)}, \dots, \gamma_n^{(2)}$ are set to zero. This procedure is continued until the p complementary solutions are determined.

Now we turn our attention to finding the particular solution. Again, without loss of generality, $\gamma_m^{(p)}, \gamma_{2m}^{(p)}, \dots, \gamma_n^{(p)}$ may all be set to zero. The resulting equation is identical in form to C.1, where now the indices i and j may not take the value of $m, 2m, \dots, n$.

The particular and complementary solutions may now be combined into a general form in accordance with Equation C.2. The index j may now take any value from 1 through n . Care must be taken to insure that the arbitrarily assigned values of bound vorticity for each complementary solution and the particular solution be included in Equation C.2.

The arbitrary multiplicative constants, C_j , in Equation C.2 still remain to be found. Utilizing the constraint provided by Equation 2.28 for each individual cylinder, the relationship

$$\sum_{j=1}^m \gamma_j = \sum_{j=m+1}^{2m} \gamma_j = \dots = \sum_{j=(p-1)m+1}^n \gamma_j = 0 \quad (C.5)$$

is obtained. This relationship constitutes a system of p simultaneous equations in p unknowns which may be expressed in the form as shown in Equation C.6 on the following page. Once the constants C_j for $j = 1, p$ are determined, the bound vorticity distribution is determined through the use of Equation C.2.

Solutions for the bound vorticity distribution may be greatly facilitated if the inverse of the reduced $a_{i,j}$ matrix is already known. In that case, the various $\gamma_j^{(i)}$ solutions for $i = 1, p$ may be precomputed leaving only the calculation $\gamma_j^{(P)}$ at each time step.

Trial calculations were made for a Reynolds number of 100 for this expanded computational region. Once hundred time steps were executed at a Δt of 0.0002 nondimensional time units. A summary of the corresponding flow variables in the regions surrounding each cylinder is shown in Table C.1. The summary includes bound vorticity, free

$$c_1 \sum_{j=1}^m \gamma_j^{(1)} + c_2 \sum_{j=1}^m \gamma_j^{(2)} + \dots + c_p \sum_{j=1}^m \gamma_j^{(p)} + \sum_{j=1}^m \gamma_j^{(p)} = 0$$

$$c_1 \sum_{j=m+1}^{2m} \gamma_j^{(1)} + c_2 \sum_{j=m+1}^{2m} \gamma_j^{(2)} + \dots + c_p \sum_{j=m+1}^{2m} \gamma_j^{(p)} + \sum_{j=m+1}^{2m} \gamma_j^{(p)} = 0$$

$$\begin{array}{ccccccccc} \cdot & & \cdot & & \cdot & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot & & \cdot & & \cdot \end{array}$$

$$c_1 \sum_{j=(p-1)m+1}^n \gamma_j^{(1)} + c_2 \sum_{j=(p-1)m+1}^n \gamma_j^{(2)} + \dots + c_p \sum_{j=(p-1)m+1}^n \gamma_j^{(p)} + \sum_{j=(p-1)m+1}^n \gamma_j^{(p)} = 0$$

(C.6)

Table C.1 Flow Variables at Corresponding Points in Computational Regions

<u>Gamma Distribution</u>			
<u>Index</u>	<u>Left Region</u>	<u>Center Region</u>	<u>Right Region</u>
1	-0.3173295200E-02	0.3173295208E-02	-0.3173295205E-02
11	-0.1232498867E-02	0.1232498867E-02	-0.1232498867E-02
21	-0.2194030855E-03	0.2194030855E-03	-0.2194030855E-03
31	0.1898834142E-02	-0.1898834142E-02	0.1898834142E-02
41	0.2459731788E-02	-0.2459731788E-02	0.2459731788E-02
51	0.1211730543E-02	-0.1211730543E-02	0.1211730543E-02
61	-0.2616908391E-03	0.2616908391E-03	-0.2616908392E-03
71	-0.9828770354E-05	0.9828770330E-05	-0.9828770351E-05
<u>Free Vorticity: First Row</u>			
<u>Index</u>	<u>Left Region</u>	<u>Center Region</u>	<u>Right Region</u>
1	0.1134129469E+03	-0.1134129468E+03	0.1134129468E+03
11	0.5269848358E+02	-0.5269848358E+02	0.5269848358E+02
21	-0.3620496283E+01	0.3620496284E+01	-0.3620496284E+01
31	-0.8238736688E+02	0.8238736688E+02	-0.8238736688E+02
41	-0.9757257356E+02	0.9757257356E+02	-0.9757257356E+02
51	-0.8343373387E+02	0.8343373387E+02	-0.8343373387E+02
61	0.9190606041E+01	-0.9190606042E+01	0.9190606040E+01
71	0.6456060067E+02	-0.6456060066E+02	0.6456060066E+02
<u>U Velocity Component: First Row</u>			
<u>Index</u>	<u>Left Region</u>	<u>Center Region</u>	<u>Right Region</u>
1	-0.2020198982E+00	0.2020198980E+00	-0.2020198980E+00
11	-0.1036517083E+00	0.1036517083E+00	-0.1036517083E+00
21	-0.4034269289E-02	0.4034269287E-02	-0.4034269286E-02
31	0.1490262841E+00	-0.1490262841E+00	0.1490262841E+00
41	0.1795823632E+00	-0.1795823632E+00	0.1795823632E+00
51	0.1572479294E+00	-0.1572479294E+00	0.1572479294E+00
61	-0.4510593271E-02	0.4510593272E-02	-0.4510593270E-02
71	-0.1122799316E+00	0.1122799316E+00	-0.1122799316E+00
<u>Normal Velocity at Boundary</u>			
<u>Index</u>	<u>Left Region</u>	<u>Center Region</u>	<u>Right Region</u>
1	-0.1000714220E-01	-0.1000714220E-01	-0.1000714220E-01
11	0.5909653939E+00	0.5909653939E+00	0.5909653939E+00
21	0.8192978739E+00	0.8192978739E+00	0.8192978739E+00
31	0.7658834281E+00	0.7658834281E+00	0.7658834281E+00
41	-0.8602677290E-02	-0.8602677292E-02	-0.8602677294E-02
51	-0.5899637911E+00	-0.5899637911E+00	-0.5899637911E+00
61	-0.8191117994E+00	-0.8191117994E+00	-0.8191117994E+00
71	-0.7666071131E+00	-0.7666071131E+00	-0.7666071131E+00

vorticity and u velocity components in the row of cells directly adjacent to the cylinder surfaces, and normal velocity components at the rectangular boundary of the subregion surrounding each cylinder. The index number shown on the table corresponds to the I index in Figure 3.1. Note that the changes in sign from region to region are expected for all variables except normal velocity components. The resulting symmetry is excellent and, in almost all cases, extends beyond ten significant figures.

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