AEROELASTIC ANALYSIS OF ROTOR BLADES USING THREE
DIMENSIONAL MULTIBODY DYNAMIC ANALYSIS

by

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TABLE OF CONTENTS

LIST OF FIGURES.....	9
LIST OF TABLES.....	14
ABSTRACT.....	15
1. INTRODUCTION	16
2. EQUATION OF MOTION.....	30
2.1 Floating frame of reference kinematics.....	30
2.2. Principle of Virtual Work.....	36
2.2.1. Virtual work by inertial forces.....	37
2.2.2 Virtual work by internal forces.....	39
2.2.3 Virtual work by external forces.....	43
2.3. Equations of motion.....	45
3. ELEMENT LIBRARY.....	47
3.1. Triangular Plate Element.....	51
3.1.1 Mass Matrix.....	53
3.1.2. Tangent Stiffness matrix.....	55
3.1.2.1 Linear stiffness matrix.....	60
3.1.2.2 Geometric Stiffness Matrix.....	67
3.2 Stiffened Plate Element.....	71
3.2.1 Mass Matrix.....	77
3.2.2 Tangent Stiffness Matrix.....	82
3.2.2.1 Linear Stiffness Matrix.....	87

TABLE OF CONTENTS - *Continued*

3.2.2.2 Geometric Stiffness Matrix.....	91
3.3 Solid Wedge Element.....	93
3.4 One Dimensional Elements.....	96
4. KINEMATIC CONSTRAINTS.....	99
4.1. Rigid Clams.....	101
4.2. Spherical Joint.....	103
4.3. Revolute Joint.....	105
5. NUMERICAL ANALYSIS.....	108
5.1. Implicit integration algorithms.....	109
5.2. System of unconstrained bodies.....	110
5.3. System of constrained bodies.....	114
6. ROTORCRAFT AEROELASTICITY.....	119
6.1. Quasi-steady Rotorcraft Aerodynamics.....	121
6.2. Lumping of aerodynamic load.....	124
6.3. Trim Analysis.....	125
7. NUMERICAL RESULTS.....	128
7.1 Static Analysis.....	128
7.1.1. Box Beam – Linear Analysis.....	129
7.1.2 Composite Beam – Nonlinear Analysis.....	137
7.2 Dynamic Analysis.....	143
7.2.1 Box Beam – Natural Frequency.....	143

TABLE OF CONTENTS - *Continued*

7.2.2 Rapid Spin-up Maneuver.....	148
7.2.3 Propeller.....	150
7.3 Multibody Dynamic Analysis.....	154
7.3.1 Four-Bar Mechanism.....	154
7.3.2 Lateral Buckling of a Thin Plate.....	160
7.3.3 Unfolding Circular Strip.....	163
7.3.4 Buckling of a Cylindrical Shell.....	167
7.3.5 Rotor blade with Trailing Edge Flap in Vacuum.....	176
7.4 Rotorcraft Aeroelasticity.....	180
7.4.1 Articulated Blade.....	182
7.4.2 Hingeless Blade.....	189
8. FINAL REMARKS AND FUTURE WORK.....	193
APPENDIX.....	201
REFERENCES.....	207

LIST OF FIGURES

Figure 1.1.	A multibody system consisting of rigid and flexible bodies.....	17
Figure 1.2.	A typical rotor blade with pitch control mechanism and a trailing edge flap.	24
Figure 1.3.	Beam elements used to model rotor blades.	26
Figure 2.1.	Kinematics of a point in an element.....	33
Figure 2.2.	Displacements and rotations and the corresponding forces and moments on a node.....	44
Figure 3.1.	Geometry and material layout of the plate element with positive directions of nodal displacements and rotations.....	52
Figure 3.2.	The nodal displacements for (a) LST6/12C parent and the resulting (b) LST3/9R ANDES membrane element.....	59
Figure 3.3.	Nodal displacements for (a) parent element with mid-side nodes and the resulting (b) MIN3 element with corner nodes only.....	59
Figure 3.4.	Geometric description of the stiffened plate element: (a) the reference plane; and (b) the cross section.....	74
Figure 4.1.	Schematic of a rigid clamp attached to a flexible body.....	100
Figure 4.2.	Kinematics of a point attached to a flexible body on the rigid clamp...	100
Figure 4.3.	A spherical joint.....	105
Figure 4.4.	A revolute joint.....	107
Figure 5.1.	Graphical representation of the incremental equilibrium equations for a system of two bodies.....	116

LIST OF FIGURES - *Continued*

Figure 5.2.	Graphical representation of the rearranged and reduced form of the incremental equilibrium equations for a system of two bodies.....	117
Figure 6.1.	Flow field in the vicinity of a rotor disk.....	120
Figure 6.2.	An aerodynamic segment.....	120
Figure 6.3.	Sectional velocities and forces.....	123
Figure 6.4.	Pressure distribution over an airfoil and the resulting forces and moment.....	127
Figure 6.5.	Stepwise pressure resulting in a) resultant forces and b) moment.....	127
Figure 7.1.	Mesh comprising of plate elements for a box-beam.....	132
Figure 7.2.	Bending slope under unit tip bending of Type1A beam.....	132
Figure 7.3.	Twist under unit tip torque of beam Type1B beam.....	133
Figure 7.4.	Bending slope under unit tip bending of Type 2A beam.....	133
Figure 7.5.	Twist under unit tip bending of Type 2A beam.....	134
Figure 7.6.	Bending slope under unit tip bending of Type 2B beam.....	134
Figure 7.7.	Twist under unit tip bending of Type 2B beam.....	135
Figure 7.8.	Bending slope under unit tip bending of Type 2C beam.....	135
Figure 7.9.	Twist under unit tip bending of Type 2C beam.....	136
Figure 7.10.	Bending slope under unit tip bending of Type 3A beam.....	136
Figure 7.11.	Bending slope under unit tip bending of Type 3B beam.....	137
Figure 7.12.	Deformed configurations of the BT beam under various load levels.....	141
Figure 7.13.	Variation of displacements with applied weight for Type NC beam.....	141

LIST OF FIGURES - *Continued*

Figure 7.14.	Variation of displacements with applied weight for Type BT beam.....	142
Figure 7.15.	Variation of displacements with applied weight for Type ET beam.....	142
Figure 7.16.	Fundamental frequencies of non-rotating symmetrical box beams.....	146
Figure 7.17.	Fundamental frequencies of symmetrical box beams rotating at 1000 RPM.....	146
Figure 7.18.	Fundamental frequencies of non-rotating antisymmetrical box beams.....	147
Figure 7.19.	Fundamental frequencies of antisymmetrical box beams rotating at 1000 RPM.....	147
Figure 7.20.	Axial (a) and transverse (b) tip displacements of a narrow rotating plate.....	149
Figure 7.21.	Schematics of a propeller.....	152
Figure 7.22.	FE mesh of a propeller with twisted blades.....	152
Figure 7.23.	Deformed configuration of the propeller in the body fixed frame.....	153
Figure 7.24.	Time variation of the displacements of point P in the body fixed frame.....	153
Figure 7.25.	Schematics of a flexible four bar mechanism.....	156
Figure 7.26.	Variation of the moment applied to the crack shaft.....	157
Figure 7.27.	The pinned and tangent frame.....	157
Figure 7.28.	Transverse displacement of the midpoint of the connecting rod.....	158
Figure 7.29.	Deformed configurations of a flexible four bar mechanism.....	159

LIST OF FIGURES - *Continued*

Figure 7.30.	Schematics for lateral buckling of a thin plate.....	161
Figure 7.31.	Transverse displacement of the plate mid-point.....	161
Figure 7.32.	Deformed configurations of thin plate undergoing lateral buckling.....	162
Figure 7.33.	Schematics of a circular strip in (a) folded and (b) unfolded configurations.....	164
Figure 7.34.	The unfolding circular strips at several time intervals.....	165
Figure 7.35.	Time variation of the position of point P in the inertia frame for (a) rigid and (b) flexible systems.....	166
Figure 7.36.	A cylindrical shell connected to a crack and link mechanism.....	168
Figure 7.37.	Variation of transverse displacement in the absence of stiffeners.....	170
Figure 7.38.	Variation of transverse displacement in the presence of stiffeners.....	170
Figure 7.39.	Variation of transverse velocity in the absence of stiffeners.....	171
Figure 7.40.	Variation of transverse velocity in the presence of stiffeners.....	171
Figure 7.41.	Deformed of the stiffened cylindrical shell at various time intervals.....	172
Figure 7.42.	Variation of transverse displacement in the absence of stiffeners.....	173
Figure 7.43.	Variation of transverse displacement in the presence of stiffeners.....	173
Figure 7.44.	Variation of transverse velocity in the absence of stiffeners.....	174
Figure 7.45.	Variation of transverse velocity in the presence of stiffeners.....	174
Figure 7.46.	Deformed of the stiffened cylindrical shell at various time intervals.....	175
Figure 7.47.	Schematics of a blade with a trailing edge flap.....	176

LIST OF FIGURES - *Continued*

Figure 7.48.	Deformed configuration of the blade in the body-fixed frame.....	178
Figure 7.49.	Deformed configuration of the blade in the body-fixed frame.....	178
Figure 7.50.	Variation of normalized deformation, $100 \times (u'_x / R_b)$	179
Figure 7.51.	Variation of normalized deformation, $100 \times (u'_y / R_b)$	179
Figure 7.52.	Description of the torsion angle, θ_{tr}	180
Figure 7.53.	Variation of torsion angle, θ_{tr} , in degrees.....	180
Figure 7.54.	Cross section of a (a) box-beam and (b) D spar.....	181
Figure 7.55.	The mesh for the (a) entire blade (b) box-beam (c) skin and (d) core....	184
Figure 7.56.	Hinges and pitch control mechanism.....	184
Figure 7.57.	Time variation of root flap angle.....	186
Figure 7.58.	Steady state root flap angle.....	186
Figure 7.59.	Steady state root lag angle.....	187
Figure 7.60.	Pitch Link Load.....	187
Figure 7.61.	Flap Bending Moment at $r/R=0.72$	188
Figure 7.62.	Mesh of a hingeless blade with a swept tip.....	190
Figure 7.63.	Schematics of the main blade and the root flexure.....	190
Figure 7.64.	The lag flexure of a hingeless rotor blade.....	191
Figure 7.65.	The flap flexure of a hingeless rotor blade.....	191
Figure 7.66.	Flapping angle variation for a trimmed hingeless rotor.....	192
Figure 7.67.	Flap and lag displacement variation for a trimmed hingeless rotor.....	192

LIST OF TABLES

Table 5.1.	A family of implicit time integrators.....	110
Table 7.1.	Box–beam geometry and material properties.....	130
Table 7.2.	Layup sequence and loading.....	130
Table 7.3.	Laminate Properties.....	140
Table 7.4.	Properties of the Graphite/Epoxy ply.....	140
Table 7.5.	Ply properties of Graphite/Epoxy, Kevlar/Epoxy and Glass/Epoxy.....	145
Table 7.6.	Stacking sequence for symmetric and antisymmetric box beams.....	145
Table 7.7.	Geometrical and material properties used for the four–bar mechanism.....	156
Table 7.8.	Various configurations for buckling of shell.....	169

ABSTRACT

This study presents an approach based on the floating frame of reference method to model complex three-dimensional bodies in a multibody system. Unlike most of the formulations based on the floating frame of reference method, which assume small or moderate deformations, the present formulation allows large elastic deformations within each frame by using the co-rotational form of the updated Lagrangian description of motion. The implicit integration scheme is based on the Generalized- α method, and kinematic joints are invoked in the formulation through the coordinate partitioning method. The resulting numerical scheme permits the usage of relatively large time steps even though the flexible bodies may experience large elastic deformations. A triangular element, based on the first order shear deformable theory, has been developed specifically for folded plate and shell structures. The plate element does not suffer from either shear or aspect-ratio locking under transverse and membrane bending, respectively. A stiffened plate element has been developed that combines a shear deformable plate with a Timoshenko beam. A solid element, that utilized the isoparametric formulation along with incompatible modes, and one-dimensional elements are also included in the element library. The tools developed in the present work are then utilized for detailed rotorcraft applications. As opposed to the conventional approach of using beam elements to represent the rotor blade, the current approach focuses on detailed modeling of the blade using plate and solid elements. A quasi-steady model based on lifting line theory is utilized to compute the aerodynamic loads on the rotor blade in order to demonstrate the capabilities of the proposed tool to model rotorcraft aeroelasticity.

1. INTRODUCTION

Historically, engineering applications that involve both structural dynamic and fluid dynamic phenomenon have been treated as two separate problems. The decoupling approach gave rise to fairly efficient computational models. In recent years, with the advent of high speed computers, coupled fluid–structures interaction analysis has been employed for a wide range of applications such as flutter analyses in aeroelasticity [Farhat and Lesoinne (2000)], inflation of parachutes [Stein et al. (2001)], deformation of structures by explosions [Casadei et al. (2001)], design of sails [Charvet et al. (1996)] and the dynamics of heart valves [Hart et al. (2003)]. These applications require expensive computational fluid dynamic (CFD) based Navier-Stokes solvers and there is need for an equally detailed computational structural dynamic (CSD) model. Such a CSD model should be able to capture large rigid body motion accompanied by large elastic deformations, model material anisotropy and shear deformations, be computationally efficient for three-dimensional modeling and possess multibody dynamic capabilities.

Dynamic simulation of multibody systems, shown in Fig. 1.1, which initially concerned interconnected rigid bodies, has currently reached a state that can analyze flexible bodies undergoing nonlinear elastic deformation. Schiehlen (1997) presented a detailed review on the subject of rigid multibody dynamics, and Wasfy and Noor (2003) and Shabana (1997) devoted their reviews to flexible multibody dynamics. Based on the choice of the reference frame, a flexible multibody dynamic formulation can broadly be categorized into the floating frame, co–rotational frame, or inertia frame approach [Schiehlen (1997)]. Among these three approaches, the floating frame approach is by far the most popular

among the three. This is due to the fact that the floating frame is a direct extension of the concepts used in rigid multibody dynamics. In spite of its popularity, the use of floating frame of reference has mostly been restricted to small deformation. Moreover, the focus has mainly been on developing different types of beam type finite elements to model slender structures. Utilizing the modal reduction technique, several practical problems that exhibit large reference translation and rotation were solved in the past with limited computational resources. However, it was observed by Kane et al. (1987) that these techniques, based on the assumption of small elastic deformations, lead to a dynamic softening effect that eventually result in instability. The progress of the development of multibody dynamic analysis methods with linear elasticity along with methods to accommodate geometric stiffening has been described by Banerjee (2003). Geometric stiffening that occurs due to the coupling of axial and bending deformation has been studied by various researchers [Bakr and Shabana (1986); Ider and Amirouche(1989); Banerjee and Dickens (1990); Wallrapp and Schwertassek (1991); Mayo and Dominguez (1996); Ryu et al. (1997)]. Wallrapp and Schwertassek (1991) incorporated geometric stiffening through the concept of pre-stress, which gives rise to a simplified linear term. The axial-bending coupling can also be obtained by retaining the nonlinear terms from the strain-displacement relationship in the strain energy expression [Bakr and Shabana (1986)]. In the case of moderate deformations, the first nonlinear term, which is the coupling of the linear and nonlinear strains in the strain energy expression, is sufficient whereas the higher order terms are also required for large deformations [Mayo and Dominguez (1996); Mayo et al. (2004)]. Sharf (1995) reviewed several approaches that

incorporate geometric stiffening. They originate from the combination of various techniques that describe the kinematics with two different strain energy formulations.

Fig 1.1 A multibody system consisting of rigid and flexible bodies.

Although these methods can capture geometric stiffening, they can only be employed in the presence of only moderate deformations. Large deformations can be analyzed using the floating frame of reference by dividing a single body into several substructures, each having a reference frame of its own [Liu and Liew (1994), Wu and Haug (1988)]. The drawback of this method stems from the imposition of nonlinear constraint equations to satisfy displacement continuity [Shabana (2005)] and a decrease in computational efficiency due to an increase in the number of variables [Mayo et al. (2004)].

Although numerous formulations for beam-type configurations have been developed in the past, flexible multibody analysis using plate type finite elements has not received sufficient attention. Plate elements that are based on the floating frame of reference

formulations can only handle small or moderate elastic deflection [Banerjee and Kane (1989); Banerjee and Dickens (1990); Chang and Shabana (1990a,b); Boutaghou (1992); Kremer (1993)]. These previous methods incorporated geometric stiffening through various approaches also in the presence of only moderate deflections. However, they cannot capture the effect of motion-induced stiffness due to large elastic deformations. Unlike these studies, Madenci and Barut (1996) and Barut et al. (1996) successfully utilized the floating frame of reference method to develop a shell element capable of including large elastic deformations. The present study extends the formulation by Barut et al. (1996) to analyze complex structures with multiple components connected through kinematic joints. The focus of this study is to develop a detailed three-dimensional modeling capability using plate and solid elements that can capture large reference motion accompanied by large elastic deformations. Within each element, the strain is assumed to be small and the stress-strain relationship is linear. The formulation utilizes the simplicity of the floating frame of reference method to capture the rigid-body motion exactly and the capability of the nonlinear finite element method to model large elastic deformations. The derivation of equation of motion is based on the principle of virtual displacements in conjunction with the co-rotational form of the updated Lagrangian description of motion. The co-rotational approach developed by Rankin and Brogan (1986) is used only to compute the stresses and internal forces. It should not be confused with the convected coordinate kinematic description (1973). An approach similar to the intermediate joint coordinate system has been adopted to facilitate the derivation of these constraint equations arising from the presence of kinematic joints. The coordinate

partitioning method developed by Wehage and Haug (1982) has been used to alleviate the difficulties arising from the inclusion of constraint equations. An implicit time integration algorithm based on the Generalized- α methods [Chung and Hulbert (1993)] has been incorporated to facilitate the use of relatively large time steps.

As part of the present study, a plate element has been developed that is intended specifically for the analysis of folded plates and shell structures. Folded plates and shell structures are extensively used for a wide range of applications in ship building, automobiles and aeronautical structures. For example, the box-beam structure of a rotor blade is composed of folded plates and the D spar is made of a combination of folded plates and shell type structures. In order to analyze shells or folded plate structures that exhibit coupling due to the complex geometric configuration and material anisotropy, the finite element method becomes the natural choice. Niyogi et al. (1999) carried out the vibration analysis of folded laminates using the first order plate theory and later Pal et al. (2008) extended this work to analyze stiffened laminated composite and sandwich folded plate. The nine node plate element utilized the Lagrangian interpolation functions and due to the absence of drilling rotations in the element formulation, an artificial stiffness was added to eliminate the rank deficiency of the global stiffness matrix. Lee et al. (2004) focused on the development of an element based on a third order laminate formulation [Reddy (1997)]. The goal of using a higher order theory was to eliminate the computation of shear correction factor associated with lower order shear deformable plate theory. Similarly, an artificial stiffness was also employed in this study for drilling degrees of freedom. Suresh and Malhotra (1998) studied the free vibration of damped

composite box-beams using a four node plate element. Selective integration was used to avoid over stiffening due to the parasitic shear strain for thin plates. Unfortunately, selective integration can lead to the introduction of spurious or zero energy modes. Note that, as opposed to unfolded plates, the walls of folded plates undergo substantial amount of transverse as well as membrane bending. For instance, under flap bending, the flange and web of a spar in a rotor blade experience transverse and membrane bending, respectively and the opposite occurs under lag bending. Therefore, the FE model should utilize elements that are equally robust for both transverse and membrane bending. The existing methods for folded plate analysis do not include this consideration during the element development stage. This leads to the usage of a very refined mesh in order to achieve satisfactory accuracy. In order to gain computational efficiency, Samanta and Mukhopadhyay, (1999) developed an equivalent orthotropic model that includes both extensional and bending rigidities to model corrugated sheets. Although such a model would be suitable for corrugated sheets, which are plate like structures, its usage for an arbitrarily folded structure is questionable.

The simplest form of triangular membrane element is the constant strain triangle (CST) that was first introduced by Turner et al. (1956). Unfortunately, the CST element does not perform well under bending load due to the presence of spurious shear strain. Although acceptable results can be obtained using the CST formulation, the mesh in such a case has to be very refined. Such an approach would yield a model that would be too expensive for nonlinear dynamic analysis. The addition of drilling degrees of freedom can improve the performance of membrane elements. One such element is the LST–

3/9R-ANDES presented by Felippa (2003) which gives exact results under pure bending conditions. Moreover, the LST-3/9R-ANDES element provides accurate results even for elements with a high aspect ratio. Note that for slender structures like a helicopter blade, maintaining an aspect ratio close to unity would require a very refined mesh. Therefore, the LST-3/9R-ANDES formulation has been adopted for the membrane stiffness of the plate element utilized in the present study. Since these plate elements are used to analyze isotropic plates as well as composite laminates, they should be capable of accommodating shear deformation. Moreover, the plate element has to perform efficiently for thin as well as thick plates. The plate theory for the transverse bending formulation is based on the first order shear deformation theory. Unfortunately, elements based on lower order interpolation functions suffer from shear locking for thin plates. Fortunately, Tessler and Hughes (1985) presented the MIN3 element based on anisotropic interpolation functions that does not suffer from shear locking. By satisfying kinematic compatibility and using a shear correction factor, the MIN3 element performs efficiently for both thick and thin plates. Combining the LST-3/9R-ANDES and MIN3 formulations the resulting element possesses the capabilities to capture both membrane and transverse bending accurately. This triangular element has three corner nodes with six degrees of freedom at each node.

Stiffened plate and shell structures are often used in various engineering applications. Stiffened shell structures can be analyzed using 1) equivalent orthotropic shell models, 2) plate and shell elements for models with detailed discretization, 3) conventional beam and shell finite element models, and 4) stiffened shell element models. An equivalent

orthotropic shell model can approximate the actual stiffened shell behavior when stiffeners are closely placed. However, when the spacing of stiffeners becomes large enough in order to allow the skin to buckle at loadings lower than their ultimate load, the equivalent orthotropic shell model becomes questionable in representing the true structural behavior of stiffened shells [Ji and Yeh (1990)]. Finite element modeling of stiffened panels can be created by discretizing both the skin and the stiffeners with plate and shell elements [Starnes et al. (1985;1995)]. However, in the absence of repeated symmetry conditions the finite element method becomes very demanding because of the extensive number of elements. The beam-shell model requires compatible displacement fields for both the beam and shell elements [Shastry et al. (1976); Koko and Olson (1991); Liao and Reddy (1990); Jiang and Olson (1994)]. However, as discussed by Jiang and Chernuka (1995) this approach imposes extensive modeling requirements. The stiffened plate and shell [Mukhopadhyay and Mukherjee (1990); Goswami and Mukhopadhyay (1995); Chattopadhyay et al. (1996)] element does not suffer from the deficiencies of the other methods. The beam kinematic field is described by the plate kinematic field and therefore only the master plate element is needed to discretize the entire stiffened plate structure. Barut et al. (2000) presented a geometrically nonlinear shear deformable stiffened shell element that accounts for large displacements and rotations for static problems. In the present study, a simplified form the stiffened shell formulation presented by Barut et al. has been used to develop a stiffened plate element experiencing large dynamic motions as well as large elastic deformations.

Since the present study focuses on three dimensional modeling, it is necessary to include a solid element in the element library. Although slender structures can be modeled using beam and plate elements, a bulky structure requires discretization using solid elements. Solid elements can be formulated using isoparametric formulation [Cook et al. (2002)]. However, they can perform overly stiff due to the presence of spurious shear strain. Although, the insertion of mid-side nodes can eliminate this drawback, mesh generation for such an element would introduce additional complexities. The aim of the present study is to develop elements that have nodes only at the corners. Therefore, the solid element developed as a part of the present study utilizes the incompatible mode method that was introduced by Wilson et al. (1973). As the name suggests, the inclusion of incompatible modes violates displacement continuity along the element boundary. In spite of this inconsistency, the incompatible mode method provides the “flexibility” required to avoid locking. Therefore, a solid wedge element, consisting of six nodes with three translational degrees of freedom at each node, is developed using the isoparametric formulation while invoking the incompatible modes methods.

Fig. 1.2 A typical rotor blade with pitch control mechanism and a trailing edge flap.

Apart from analyzing folded plates, stiffened plates and shell structures, the present methodology can be employed to devise an alternative approach to model a helicopter blade shown in Fig. 1.2. Helicopter blades are slender structures that are traditionally modeled using beam elements. The necessity to analyze rotor blades with various levels of complexities has been the source of motivation for the evolution of beam theories. Houbolt and Brooks (1958) extended earlier work on beam theories to account for coupled bending and torsion deformations. Although linear in nature, it included bending–torsion effects in twisted blades and the presence of axial pre–stress. The next step was to include nonlinearities based on an order analysis which retained some of the nonlinear terms (Hodges and Dowell (1974); Kaza and Kvaternik (1977), Rosen and Friedmann (1978) and Crespo da Silva (1981)). Furthermore, geometrically exact theories were developed that were not restricted to moderate rotations (Hodges (1985); Simo and Vu–Quoc (1988); Hodges (1990)). With the advent of rotor blades, made of composite materials, beam theories became more sophisticated in an attempt to capture the shear deformation and coupling due to material anisotropy. Material coupling was incorporated in the beam analysis by Hong and Chopra (1986) and Libove (1987). Several theories were then introduced that incorporated beam warping deformation in various forms. The performance of Euler–Bernoulli beams was improved by introducing transverse shear and a set of orthonormal eigenwarpings by Bauchau (1985) and Bauchau and Hong (1987a, 1987b, 1988). It was assumed that the cross section remained rigid in its own plane and therefore inplane warping was neglected. Kosmatka and Friedmann (1988) allowed both inplane and out of plane warping for the computation of the cross

section properties; however, the one dimensional beam equations did not take into account the presence of inplane warping. Instead of using warping modes, in the method developed by Stemple and Lee (1988,1989), additional nodes were placed on the cross section to capture out of plane warping. Analysis based on the beam theories often originate from a three-dimensional elasticity [Berdichevsky(1982)], which is then split into 1) a linear two-dimensional problem which spans over the beam cross section providing the cross section properties and 2) a nonlinear one-dimensional problem along the beam reference line that computes the nonlinear transient response. As shown in Fig. 1.3, several beam elements with different number of degrees of freedom have been developed with different levels of refinement [Chandra and Chopra (1992); Yuan, P.P. Friedmann (1995); Kim and White (1997); Centolanza and Smith (1998)]. The performance of a beam theory has been attributed to the accuracy of the computed cross section properties [Volovoi (2001)]. Notable contributions in this field have been the work by Giavotto et.al. (1983) and Hodges and his coworkers [Hodges et al. (1992); Cesnik et al. (1996); Cesnik and Hodges (1997); Popescu and Hodges (1999); Popescu et al. (2000)].

Based on different beam theories a number analysis programs have emerged for structural dynamic and comprehensive analysis of rotor blade with various levels of refinement [Johnson (1986); Kunz (2005)]. These programs take advantage of the simplicity of the beam theories and there is a significant reduction in the number of degrees of freedom as compared to a three-dimensional model without any reduction. However, the complete three-dimensional models do not involve any assumptions related to modeling of the

cross section. The major drawback of a three dimensional analysis is the cost required to process computations involving millions of degrees of freedom. The present study, which is based on a method presented by Das et al. (2006 and 2007), aims at achieving a good balance between accuracy and computational efficiency by using a combination of plate and solid elements to model the rotor blade while using a moderate number of degrees of freedom. Rotor blades with advanced geometries can be modeled with relative ease. Note that curvature, twist, taper and any innovative shape can be incorporated into the model directly by including these details in the mesh. A comprehensive analysis based on the present computational structural dynamic (CSD) model would not be restricted to the computation of the displacements. Details such as inter-laminar stresses in the spar and the skin can also be extracted from the present model. Moreover, with the focus of rotorcraft aerodynamic analysis shifting towards Computational Fluid Dynamic tools [Guruswamy (2008),], the present model serves as a perfect candidate to complement a CFD program for computational rotorcraft aeroelastic analysis.

The spar of a rotor blade which is made of either metal or composite laminates is modeled using plate elements and a similar approach is used for the skin. The usage of plate elements results in a substantial decrease in the number of DOFs compared to a full three-dimensional model having several layers of solid elements for each ply in the laminate. The core, which is homogeneous in nature, is modeled using solid elements. The leading edge balance weight is incorporated using a simple lumped mass technique. One dimensional spring and damper are also available to model the pitch link and the

lead–lag damper. A wind tunnel trim procedure is used to search for the control settings and to calculate the steady state periodic solution to the nonlinear equation of motion. The trim procedure involves the search for appropriate values of the control parameters that eliminates 1/rev blade flapping and achieves a specified value of thrust.

Fig.1.3 Beam elements used to model rotor blades.

The description of motion using the floating frame of reference is presented in Chapter 2. Also described is the evaluation of virtual work due to inertial, internal, and external forces. A framework for the construction of the mass, tangent stiffness matrix and the internal force vector is also developed. Chapter 3 provides the details of the development of a plate, stiffened plate, solid and one–dimensional elements. The triangular plate element is based on the first order shear deformable plate theory. The plate element is specifically developed for folded plate and shell structures. The ability to model folded

plate and shell structures makes this element suitable for modeling rotor blades. The triangular stiffened plate element is also based on the first order shear deformable plate theory and the stiffeners are modeled using the Timoshenko beam element. The coupling between different bodies, through a variety of kinematic joints, is implemented by employing a set of constraint equations; the details are presented in Chapter 4. The spherical and revolute joints are discussed in details. In Chapter 5, the numerical integration scheme, along with the implementation of the coordinate partitioning method, is discussed. A quasi-steady model based on the lifting line theory has been employed to calculate the aerodynamic loads. The details of the aerodynamic model and the trim analysis are presented in Chapter 6. In Chapter 7, several numerical examples are considered to validate and demonstrate the capabilities of the current approach. Prior to analyzing the dynamic motion of multibody systems, the static response of the plate element is also validated. The capabilities of the analysis method developed in the present study are demonstrated by investigating the aeroelastic response of articulated and hingeless rotor blades. A summary, conclusion, and future applications of this study are presented in Chapter 8.

2. EQUATION OF MOTION

The first step in the process of formulating a flexible multibody dynamic is to select the kinematic description of motion. As described in Chapter 1, the present study is based on the floating frame of reference. The governing equations are derived using the concept of virtual work. The virtual work due to inertial, internal and external forces is used to formulate the expressions for the various matrices and vectors that constitute the equation of motion. The following sections deal with the development of a general framework to obtain these matrices and vectors.

2.1 Floating frame of reference kinematics

The description of the motion of a point can be simplified by employing appropriate number of coordinate frames. As shown in Fig. 2.1, the inertial coordinate system that remains fixed to the ground is represented by (x, y, z) . The floating frame coordinate system and an element coordinate system that is attached to an element are denoted by (x', y', z') and (x'', y'', z'') , respectively. Therefore, a vector with a single prime or double prime is defined with respect to the base vector of the floating frame or element coordinate system, respectively. The element coordinate system is used only for specific types of elements. For instance a solid element, with only translation degrees of freedom, can be formulated using the isoparametric formulations and the displacement field is described in the floating frame. Therefore, a separate element coordinate system is not

required for the solid element. On the other hand, an element coordinate system can significantly simplify the finite element formulation for a plate bending element.

The kinematic description of a point P between time t and $t + \Delta t$ is illustrated in Fig. 2.1. The configuration at time $t + \Delta t$ represents the unknown current equilibrium configuration. The last known equilibrium configuration is at time t , and it is referred to as the updated configuration. Based on the kinematic motion illustrated in Fig. 2.1, the position of point P within an element corresponding to the n^{th} body at time $t + \Delta t$ is expressed as

$${}^{t+\Delta t}\mathbf{R}^n = {}^{t+\Delta t}\mathbf{R}_0^n + {}^{t+\Delta t}\mathbf{r}^n \quad (2.1)$$

where ${}^{t+\Delta t}\mathbf{R}_0^n$ describes the coordinates of the origin of the floating frame of the n^{th} body at time $t + \Delta t$ and ${}^{t+\Delta t}\mathbf{r}^n$ is the position of the point P relative to ${}^{t+\Delta t}\mathbf{R}_0^n$. In the remaining expressions, the left and right superscripts are dropped for simplicity unless they are different from $t + \Delta t$ and n , respectively. Utilizing a transformation matrix, $\mathbf{T}$, to transform a vector from the floating frame to the inertial frame, Eq. (2.1) can be rewritten as

$$\mathbf{R} = \mathbf{R}_0 + \mathbf{T}\mathbf{r}'. \quad (2.2)$$

In the case of rigid bodies, the vector $\mathbf{r}'$ is constant since the position of point P relative to the origin of the floating frame remains fixed. Applying the variational operator to Eq. (2.2) leads to

$$\delta\mathbf{R} = \delta\mathbf{R}_0 + \delta\mathbf{T}\mathbf{r}' + \mathbf{T}\delta\mathbf{r}'. \quad (2.3)$$

Using the property of the rotation matrix, $\delta\mathbf{T}$ can be expressed as

$$\delta \mathbf{T} = \delta \tilde{\boldsymbol{\omega}} \mathbf{T} \quad (2.4)$$

where the tilde indicates a skew symmetric matrix and $\delta \boldsymbol{\omega}$ is the virtual rigid-body rotation of the floating frame. Therefore, $\delta \tilde{\boldsymbol{\omega}}$ has the following form

$$\delta \tilde{\boldsymbol{\omega}} = \begin{bmatrix} 0 & -\delta \omega_3 & \delta \omega_2 \\ \delta \omega_3 & 0 & -\delta \omega_1 \\ -\delta \omega_2 & \delta \omega_1 & 0 \end{bmatrix}. \quad (2.5)$$

Substituting Eq. (2.4) into Eq. (2.3) and utilizing the transformation

$$\mathbf{r} = \mathbf{T} \mathbf{r}' \quad (2.6)$$

yield the following form

$$\delta \mathbf{R} = \delta \mathbf{R}_0 + \delta \tilde{\boldsymbol{\omega}} \mathbf{r} + \mathbf{T} \delta \mathbf{r}' \quad (2.7)$$

The property of a skew symmetric matrix leads to the following matrix manipulation

$$\delta \tilde{\boldsymbol{\omega}} \mathbf{r} = -\tilde{\mathbf{r}} \delta \boldsymbol{\omega} \quad (2.8)$$

Substituting Eq. (2.8) into Eq. (2.7), $\delta \mathbf{R}$ can be expressed as

$$\delta \mathbf{R} = \delta \mathbf{R}_0 - \tilde{\mathbf{r}} \delta \boldsymbol{\omega} + \mathbf{T} \delta \mathbf{r}'. \quad (2.9)$$

In the case of deformable bodies, $\mathbf{r}'$ can be decomposed as follows

$$\mathbf{r}' = {}^t \mathbf{r}' + \mathbf{u}' \quad (2.10)$$

where ${}^t \mathbf{r}'$ and $\mathbf{u}'$ are the rigid and flexible component, respectively. Based on the assumption of small deformations, the rigid component, ${}^t \mathbf{r}'$ is fixed (i.e., ${}^t \mathbf{r}' = {}^0 \mathbf{r}'$) in reference to the floating frame. In the current study, the deformable bodies undergo large elastic deformation even though the strain is assumed to remain small. The assumption of the current orientation of the element within the floating frame remaining constant can lead to inaccurate computation of the mass and tangent stiffness matrices. Therefore, in

accordance with the Updated Lagrange formulation, ${}^t\mathbf{r}'$ is updated after each iteration.

Combining Eq. (2.9) with Eq. (2.10) and using the transformation law

$$\tilde{\mathbf{r}} = \mathbf{T}\tilde{\mathbf{r}}'\mathbf{T}^T \quad (2.11)$$

result in

$$\delta\mathbf{R} = \delta\mathbf{R}_0 - \mathbf{T}\tilde{\mathbf{r}}'\mathbf{T}^T \delta\boldsymbol{\omega} + \mathbf{T}\delta\mathbf{u}' . \quad (2.12)$$

Since the vector ${}^t\mathbf{r}'$ is constant between the known and the unknown configuration, its variation vanishes.

Fig. 2.1 Kinematics of a point in an element

The expression for the velocity at point P is obtained from Eq. (2.2) as

$$\dot{\mathbf{R}} = \dot{\mathbf{R}}_0 + \dot{\mathbf{T}}\mathbf{r}' + \mathbf{T}\dot{\mathbf{r}}'. \quad (2.13)$$

By using the relation

$$\dot{\mathbf{T}}\mathbf{r}' = \tilde{\boldsymbol{\Omega}}\mathbf{T}\mathbf{r}' \quad (2.14)$$

the velocity at a point can be rewritten as

$$\dot{\mathbf{R}} = \dot{\mathbf{R}}_0 + \tilde{\boldsymbol{\Omega}}\mathbf{T}\mathbf{r}' + \mathbf{T}\dot{\mathbf{u}}' \quad (2.15)$$

where $\boldsymbol{\Omega}$ is the angular velocity vector of the n^{th} flexible body arising from either a prescribed slew rate, external forces, and/or the kinematic coupling with other flexible bodies. The first two terms in Eq. (2.15) arise due to the motion of the origin of the floating frame. Whereas the first term is due to the translational motion, the second term originates due to the rotation of the floating frame. The last term in Eq. (2.15), which is characteristic to flexible bodies, arises due to the variation in the relative position of the point P and the origin of the floating frame.

The velocity given in Eq. (2.15) can be expressed in an alternative form as

$$\dot{\mathbf{R}} = \dot{\mathbf{R}}_0 - \mathbf{T}\tilde{\mathbf{r}}'\mathbf{T}^T\boldsymbol{\Omega} + \mathbf{T}\dot{\mathbf{u}}' \quad (2.16)$$

The acceleration of the point P can be obtained by from Eqs. 2.14 and 2.15 as

$$\ddot{\mathbf{R}} = \ddot{\mathbf{R}}_0 + \left(-\mathbf{T}\tilde{\mathbf{r}}'\mathbf{T}^T\dot{\boldsymbol{\Omega}} + \tilde{\boldsymbol{\Omega}}\tilde{\boldsymbol{\Omega}}\mathbf{T}\mathbf{r}' + \tilde{\boldsymbol{\Omega}}\mathbf{T}\dot{\mathbf{u}}'\right) + \left(\tilde{\boldsymbol{\Omega}}\mathbf{T}\dot{\mathbf{u}}' + \mathbf{T}\ddot{\mathbf{u}}'\right). \quad (2.17)$$

Rearranging the terms in Eq.(2.16) leads to

$$\ddot{\mathbf{R}} = \ddot{\mathbf{R}}_0 - \mathbf{T}\tilde{\mathbf{r}}'\mathbf{T}^T\dot{\boldsymbol{\Omega}} + 2\tilde{\boldsymbol{\Omega}}\mathbf{T}\dot{\mathbf{u}}' + \tilde{\boldsymbol{\Omega}}\tilde{\boldsymbol{\Omega}}\mathbf{T}\mathbf{r}' + \mathbf{T}\ddot{\mathbf{u}}'. \quad (2.18)$$

The first two terms in Eq. (2.18) is due to the translation and rotation of the floating frame. The third and forth terms are termed as the Coriolis and centrifugal accelerations,

respectively. The last term is attributed to the acceleration of the point P within the floating frame.

In previous studies, the flexible component, $\mathbf{u}'$, was often expressed in terms of vibration modes. Such an approach is applicable only if the elastic deformations are in the linear range. The usage of vibration modes can significantly reduce the size of the finite element model. In the current analysis, due to the presence of large deformations, modal reduction techniques cannot be used. Therefore, the incremental displacement vector, $\mathbf{u}'$, is expressed based on the finite element displacement approximation in terms of the incremental total nodal vector, $\mathbf{v}'$, and the interpolation matrix, $\mathbf{N}'$, as

$$\mathbf{u}' = \mathbf{N}'\mathbf{v}' \quad (2.19)$$

The form of the interpolation function and the definition of the terms in the incremental total nodal vector depend on the type of element chosen. The interpolation matrix, $\mathbf{N}'$, is defined in the floating frame and it remains constant for small deformation analysis. In the current analysis, due to large rotations within the floating frame, the interpolation matrix varies as the orientation of the elements changes significantly. Although the interpolation matrix is continuously updated, its rate of change is assumed to be negligible and therefore

$$\dot{\mathbf{u}}' = \mathbf{N}'\dot{\mathbf{v}}' \quad (2.20a)$$

and

$$\ddot{\mathbf{u}}' = \mathbf{N}'\ddot{\mathbf{v}}' . \quad (2.20b)$$

Based on the form of the displacement field provided in Eq. (2.12), the variation of the position of point P can be rewritten as

$$\delta \mathbf{R} = \begin{bmatrix} \mathbf{I} & -\mathbf{T}\tilde{\mathbf{r}}'\mathbf{T}^T & \mathbf{T}\mathbf{N}' \end{bmatrix} \begin{Bmatrix} \delta \mathbf{R}_0 \\ \delta \boldsymbol{\omega} \\ \delta \mathbf{v}' \end{Bmatrix}. \quad (2.21)$$

Similarly, the acceleration of the point P can be expressed as

$$\ddot{\mathbf{R}} = \begin{bmatrix} \mathbf{I} & -\mathbf{T}\tilde{\mathbf{r}}'\mathbf{T}^T & \mathbf{T}\mathbf{N}' \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{R}}_0 \\ \dot{\boldsymbol{\Omega}} \\ \ddot{\mathbf{v}}' \end{Bmatrix} + \mathbf{f}_{R1} + \mathbf{f}_{R2} \quad (2.22)$$

where

$$\mathbf{f}_{R1} = 2\tilde{\boldsymbol{\Omega}}\mathbf{T}\mathbf{N}\dot{\mathbf{v}}' \quad (2.23a)$$

and

$$\mathbf{f}_{R2} = \tilde{\boldsymbol{\Omega}}\tilde{\boldsymbol{\Omega}}\mathbf{T}\mathbf{r}' \quad (2.23b)$$

are the force vectors arising from Coriolis and centrifugal effects, respectively. Note that the unknowns related to the displacements and rotations of the floating frame are expressed in the inertial coordinate system and the nodal deformations are expressed in the floating frame.

2.2. Principle of Virtual Work

The governing equations are obtained by applying the principle of virtual work, in which the work done by the external, internal, and inertial forces over the virtual displacements (deformations) applied to the body at $t + \Delta t$ must vanish. By using the updated Lagrangian description of motion, where all the kinematic variables associated with the configuration at time $t + \Delta t$ are mapped to the known configuration at time t , the virtual work due to inertial, internal, and external forces is expressed as

$$\int_V \rho \delta \mathbf{R}^T \ddot{\mathbf{R}} dV + \int_V \delta {}_t \mathbf{E}^T {}_t \mathbf{S} dV = \delta W_p \quad (2.24)$$

where the vectors ${}_t \mathbf{E} = ({}^{t+\Delta t}{}_t \mathbf{E})$ and ${}_t \mathbf{S} = ({}^{t+\Delta t}{}_t \mathbf{S})$ contain the unknown components of the Green–Lagrangian strain and the (2nd–order) Piola–Kirchhoff stress tensors that are evaluated at time $t + \Delta t$ while they are referred to the configuration at time t . The left subscript t indicates the configuration by which the strain and the stress vectors, ${}_t \mathbf{E}$ and ${}_t \mathbf{S}$, are measured. The left subscript t for the position vector $\mathbf{R}$ is simply dropped because the direction and magnitude of a material point is invariant regardless of the choice of the material configuration. The mass density of the body is represented by ρ , and its magnitude changes with the configuration such that the total mass of the body is conserved.

2.2.1. Virtual work by inertial forces

The first integral in Eq. (2.24), representing the virtual work done by the inertial forces, is utilized to obtain the mass matrix and the force vector arising from Coriolis and centrifugal effects in the following manner:

$$\int_V \rho \delta \mathbf{R}^T \ddot{\mathbf{R}} dV = \delta \mathbf{b}^T \begin{bmatrix} \mathbf{M}_{RR} & \mathbf{M}_{R\omega} & \mathbf{M}_{Rv'} \\ & \mathbf{M}_{\omega\omega} & \mathbf{M}_{\omega v'} \\ sym. & & \mathbf{M}_{v'v'} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{R}}_0 \\ \dot{\boldsymbol{\Omega}} \\ \ddot{\mathbf{v}}' \end{Bmatrix} + \delta \mathbf{b}^T \begin{Bmatrix} \mathbf{f}_1 + \mathbf{f}_2 \\ \mathbf{g}_1 + \mathbf{g}_2 \\ \mathbf{h}_1 + \mathbf{h}_2 \end{Bmatrix} \quad (2.25)$$

where

$$\mathbf{b}^T = \left\{ \mathbf{R}_0^T \quad \boldsymbol{\omega}^T \quad \mathbf{v}'^T \right\}. \quad (2.26)$$

Substituting expressions for $\delta \mathbf{R}$ and $\ddot{\mathbf{R}}$ from Eq. (2.21) and Eq. (2.22), respectively, into the first integral in Eq. (2.24), the form of each of the sub-matrices of the mass matrix in Eq. (2.25) can be expressed as

$$\mathbf{M}_{\mathbf{RR}} = \rho^t V \mathbf{I} \quad (2.27a)$$

$$\mathbf{M}_{\mathbf{R}\omega} = -\rho \mathbf{T} \int_{t_V} \tilde{\mathbf{r}}' d^t V \mathbf{T}^T \quad (2.27b)$$

$$\mathbf{M}_{\mathbf{R}\mathbf{v}'} = \rho \mathbf{T} \int_{t_V} \mathbf{N}' d^t V \quad (2.27c)$$

$$\mathbf{M}_{\omega\omega} = -\rho \mathbf{T} \int_{t_V} \tilde{\mathbf{r}}' \tilde{\mathbf{r}}' d^t V \mathbf{T}^T \quad (2.27d)$$

$$\mathbf{M}_{\omega\mathbf{v}'} = \rho \mathbf{T} \int_{t_V} \tilde{\mathbf{r}}' \mathbf{N}' d^t V \quad (2.27e)$$

$$\mathbf{M}_{\mathbf{v}\mathbf{v}'} = \rho \int_{t_V} \mathbf{N}'^T \mathbf{N}' d^t V \quad (2.27f)$$

The sub-matrices $\mathbf{M}_{\mathbf{RR}}$ and $\mathbf{M}_{\omega\omega}$ represent the mass matrix associated with the translation and rotational coordinates of the body reference, respectively. The matrix $\mathbf{M}_{\mathbf{R}\omega}$ represents the inertia coupling between the rigid body translation and the rigid body rotation. In rigid body dynamics, this matrix disappears when the floating frame is located at the mass center of the body. The matrix $\mathbf{M}_{\mathbf{v}\mathbf{v}'}$ is associated with the finite element displacement degrees of freedom and this matrix typically appears in the finite element structural dynamic formulation.

The force vectors due to the Coriolis and centrifugal effect have the following form

$$\mathbf{f}_1 = 2\rho \tilde{\mathbf{\Omega}} \mathbf{T} \int_{t_V} \mathbf{N}' d^t V \dot{\mathbf{v}} \quad (2.28a)$$

$$\mathbf{g}_1 = 2\rho \mathbf{T} \int_{t_V} \tilde{\mathbf{r}}' \mathbf{T}^T \tilde{\mathbf{\Omega}} \mathbf{T} \mathbf{N}' d^t V \dot{\mathbf{v}}' \quad (2.28b)$$

$$\mathbf{h}_1 = 2\rho \int_{\mathcal{V}} \mathbf{N}'^T \mathbf{T}^T \tilde{\mathbf{\Omega}} \mathbf{T} \mathbf{N}' d^tV \dot{\mathbf{v}}' \quad (2.28c)$$

$$\mathbf{f}_2 = \rho \tilde{\mathbf{\Omega}} \tilde{\mathbf{\Omega}} \mathbf{T} \int_{\mathcal{V}} \mathbf{r}' d^tV \quad (2.28d)$$

$$\mathbf{g}_2 = \rho \mathbf{T} \int_{\mathcal{V}} \tilde{\mathbf{r}}'^T \mathbf{T}^T \tilde{\mathbf{\Omega}} \tilde{\mathbf{\Omega}} \mathbf{T} \mathbf{r}' d^tV \quad (2.28e)$$

$$\mathbf{h}_2 = \rho \int_{\mathcal{V}} \mathbf{N}'^T \mathbf{T}^T \tilde{\mathbf{\Omega}} \tilde{\mathbf{\Omega}} \mathbf{T} \mathbf{r}' d^tV \quad (2.28f)$$

2.2.2 Virtual work by internal forces

The second integral term of the virtual work expression in Eq. (2.24) represents the virtual work of the internal forces, δW_σ , for arbitrary virtual strains ($\delta {}_t\mathbf{E}$). It is defined as

$$\delta W_\sigma = \int_{\mathcal{V}} \delta {}_t\mathbf{E}^T {}_t\mathbf{S} dV. \quad (2.29)$$

In order to achieve the solution in an incremental form, the Piola–Kirchhoff stress vector, ${}_t\mathbf{S}$, is represented in incremental form as

$${}_t\mathbf{S} = {}^t\boldsymbol{\sigma} + {}_t\Delta\boldsymbol{\sigma} \quad (2.30)$$

in which ${}^t\boldsymbol{\sigma}$ denotes the Cauchy (true) stress vector evaluated at time t and ${}_t\Delta\boldsymbol{\sigma}$ represents the incremental unknown Piola–Kirchhoff stress vector occurring during the time increment Δt . The Green–Lagrangian strain vector, ${}_t\mathbf{E}$, can be decomposed into its linear and nonlinear components as

$${}_t\mathbf{E} = {}_t\mathbf{E}_L + {}_t\mathbf{E}_{NL} \quad (2.31)$$

with

$$\delta_t \mathbf{E} = \delta_t \mathbf{E}_L + \delta_t \mathbf{E}_{NL}. \quad (2.32)$$

Substituting Eq. (2.30) and Eq. (2.32) into the internal virtual work expression given in Eq. (2.29), while expanding the resulting terms, leads to

$$\delta W_\sigma = \int_{V'} \delta_t \mathbf{E}_L^T \mathbf{Q}_t \mathbf{E}_L d^t V + \int_{V'} \delta_t \mathbf{E}_{NL}^T {}^t \boldsymbol{\sigma} d^t V + \int_{V'} \delta_t \mathbf{E}_L^T {}^t \boldsymbol{\sigma} d^t V. \quad (2.33)$$

Note that prior to the derivation Eq. (2.33), the following linearization is performed on the integral terms involving the higher-order terms of the unknown incremental vectors, i.e.,

$$\delta_t \mathbf{E}^T {}_t \Delta \boldsymbol{\sigma} \cong \delta_t \mathbf{E}_L^T {}_t \Delta \boldsymbol{\sigma} \quad (2.34a)$$

and

$${}_t \Delta \boldsymbol{\sigma} = \mathbf{Q}_t \mathbf{E} \cong \mathbf{Q}_t \mathbf{E}_L \quad (2.34b)$$

where $\mathbf{Q}$ denotes the linear constitutive matrix for an orthotropic material relating the incremental strains to incremental stresses. The linear component of the Green strain tensor, which is expressed as

$${}_t \mathbf{E}_L = \mathbf{B} \mathbf{v}' \quad (2.35)$$

is substituted into the first term of Eq. (2.33), resulting in

$$\int_{V'} \delta_t \mathbf{E}_L^T \mathbf{Q}_t \mathbf{E}_L d^t V = \delta \mathbf{v}'^T \mathbf{k}_L \mathbf{v}' \quad (2.36)$$

where $\mathbf{k}'_L$ is the linear stiffness matrix

$$\mathbf{k}'_L = \int_{V'} \mathbf{B}^T \mathbf{Q} \mathbf{B} d^t V. \quad (2.37)$$

The next step is to obtain the geometric stiffness matrix. This requires the nonlinear part of the Green strain tensor which can be expressed as

$${}^t\mathbf{E}_{NL} = \frac{1}{2} \mathbf{A} \boldsymbol{\Theta} . \quad (2.38)$$

The nonlinear part of the Green strain consists of square of the derivatives of the displacement terms which is split into matrix $\mathbf{A}$, and vector, $\boldsymbol{\Theta}$. The form and components of $\mathbf{A}$ and $\boldsymbol{\Theta}$ depend on the nature of the finite element. The vector $\boldsymbol{\Theta}$ can be expressed in terms of the incremental nodal displacements, $\mathbf{v}'$, and the derivatives of the interpolation functions, $\mathbf{G}$, as

$$\boldsymbol{\Theta} = \mathbf{G} \mathbf{v}' . \quad (2.39)$$

Based on certain algebraic manipulations, irrespective of the type of element, the following properties are observed:

$$\delta \mathbf{A} \boldsymbol{\Theta} = \mathbf{A} \delta \boldsymbol{\Theta} \quad (2.40)$$

and

$$\mathbf{A}^T {}^t\boldsymbol{\sigma} = \mathbf{M}_\sigma \boldsymbol{\Theta} = \mathbf{M}_\sigma \mathbf{G} \mathbf{v}' \quad (2.41)$$

where the matrix $\mathbf{M}_\sigma$ consists of components of either stress or stress resultants, depending on the type of element. Substituting the nonlinear strain vector into the second integral in Eq. (2.33) and using the properties shown in Eq. (2.41) result in

$$\int_{V'} \delta {}^t\mathbf{E}_{NL}^T {}^t\boldsymbol{\sigma} dV = \delta \mathbf{v}'^T \mathbf{k}_g \mathbf{v}' \quad (2.42)$$

where

$$\mathbf{k}'_g = \int_{V'} \mathbf{G}^T \mathbf{M}_\sigma \mathbf{G} dV \quad (2.43)$$

is the geometric stiffness matrix, which takes into account the effect of stress stiffening.

Finally, the last term in Eq. (2.33) can be rewritten in the following manner:

$$\int_{t_V} \delta {}_t\mathbf{E}_L^T {}_t\boldsymbol{\sigma} dV = \delta \mathbf{v}'^T \mathbf{k}'_L \mathbf{v}'^* \quad (2.44)$$

where $\mathbf{v}'^*$ consists of the deformation part of the total displacement between the initial and updated configuration at time t . The computation of the vector $\mathbf{v}'^*$ is based on the element free co-rotational procedure presented by Rankin and Brogan (1986). The term co-rotational implies that, while the element deforms, the initial configuration of the element rigidly translates and rotates to coincide with the reference plane of the element. To prevent confusion, this configuration is referred to as “rigid” from here on. Thus, the difference between the deformed and rigid configurations of the element yields the elastic deformations of the element. Although the difference in translational displacements between the two configurations (deformed and rigid) can be directly obtained by simple vector subtraction, the difference in orientations of nodal points requires special treatment of the nodal rotations of the element. In order to find the exact orientations of the element after deformation, a triad is attached to each structural node. After deformation, the orientation of each nodal triad is determined by a (3×3) rotation matrix that can be expressed in terms of three independent quantities (pseudo rotations), as explained by Rankin and Brogan (1986). After each successive rotation increment, these quantities are updated to give the current orientation of a node. Also, the orientation of the rigid configuration of the element can be easily determined from the plane formed by the corner nodes of the element. Thus, the angle difference between the orientations of deformed and rigid configurations of the element (which is assumed to be fairly small so that it allows vector operations) yields the deformational rotations of the element at nodal

points. Whereas the deformational part of the displacements and rotations of the element is measured between the current and initial time steps in the work by Rankin and Brogan (1986), in the present analysis the nodal quantities are measured between two successive iterations. The deformational part of the incremental displacements and rotations is then stored in the appropriate locations in vector, $\mathbf{v}^*$.

Substituting Eq. (2.37), Eq. (2.43), and Eq. (2.44) into the internal virtual work expression, (25), yields the final form

$$\int_{V'} \delta \mathbf{t}^T \mathbf{E}^{T \ t+\Delta t} \mathbf{S} dV = \delta \mathbf{b}^T \mathbf{f}_b^\sigma \quad (2.45)$$

where the (unknown) internal force vector, $\mathbf{f}_b^\sigma$, is defined as

$$\mathbf{f}_b^\sigma = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{k}'_T \end{bmatrix} \mathbf{b} + \left\{ \begin{array}{c} \mathbf{0} \\ \mathbf{0} \\ \mathbf{k}'_L \mathbf{v}^{*} \end{array} \right\} \quad (2.46)$$

in which $\mathbf{k}'$ is the tangent stiffness matrix given by

$$\mathbf{k}'_T = \mathbf{k}'_L + \mathbf{k}'_g. \quad (2.47)$$

The details of computing the relevant vectors and matrices for a solid wedge, a triangular plate and stiffened plate element are given in Chapter 3.

2.2.3 Virtual work by external forces

The virtual work due to nodal forces and moments at the i^{th} node is expressed as

$$\delta W_i = \delta \mathbf{R}_i^T \mathbf{F}_i + (\delta \boldsymbol{\omega}^T + \delta \boldsymbol{\theta}_i^T) \mathbf{M}_i \quad (2.48)$$

where $\mathbf{F}_i$ and $\mathbf{M}_i$ are the nodal force and moment vectors, respectively, at the i^{th} node, as shown in Fig. 2.2. Substituting for $\delta\mathbf{R}_i$ from Eq. (2.21) and using the property of the transformation matrix, $\mathbf{T}$, result in

Fig. 2.2. Displacements and rotations and the corresponding forces and moments on a node.

$$\delta W_i = \delta \mathbf{R}_0^T \mathbf{F}_i + \delta \boldsymbol{\omega}^T \mathbf{T} (\tilde{\mathbf{r}}_i' \mathbf{F}_i' + \mathbf{M}_i') + \delta \mathbf{u}_i'^T \mathbf{F}_i' + \delta \boldsymbol{\theta}_i'^T \mathbf{M}_i'. \quad (2.49)$$

Adding the contribution from each of the n_e nodes of an element results in the following expression for virtual work due to external loads:

$$\delta W_p = \delta \mathbf{b}^T \mathbf{f}_b^p \quad (2.50)$$

where

$$\mathbf{f}_b^{pT} = \{ \tilde{\mathbf{F}} \quad \tilde{\mathbf{M}} \quad \mathbf{P}' \} \quad (2.51)$$

in which

$$\tilde{\mathbf{F}} = \sum_{i=1}^{n_e} \mathbf{F}_i \quad \tilde{\mathbf{M}} = \mathbf{T} \sum_{i=1}^{n_e} (\tilde{\mathbf{r}}_i' \mathbf{F}_i' + \mathbf{M}_i') \quad (2.52)$$

and

$$\mathbf{P}'^T = \left\{ \mathbf{P}_1'^T \quad \mathbf{P}_2'^T \quad \mathbf{P}_3'^T \right\} \quad (2.53)$$

with

$$\mathbf{P}_i'^T = \left\{ \mathbf{F}_i'^T \quad \mathbf{M}_i'^T \right\}. \quad (2.54)$$

The aerodynamic loads discussed in Chapter 6, can easily be modeled through these external forces and moments.

2.3. Equations of motion

Substituting from Eqs. 2.24, 2.44 and 2.49 into the virtual work expression in Eq. (2.24) and requiring that the virtual nodal unknown vector can be arbitrary result in the dynamic equilibrium equations of the k^{th} element as

$$\mathbf{M}\ddot{\mathbf{b}} = \mathbf{f}_b^p - \mathbf{f}_b^\sigma - \mathbf{f}_b^\rho. \quad (2.55)$$

Furthermore, computing Eq. (2.55) for each element in the n^{th} flexible body and assembling their contribution give rise to the following equation of motion of the n^{th} flexible body

$$\mathbf{M}^n \ddot{\mathbf{b}}_n = \mathbf{f}_b^{pn} - \mathbf{f}_b^{\sigma n} - \mathbf{f}_b^{\rho n}. \quad (2.56)$$

These equations arising from several flexible bodies can further be assembled to construct the equation of motion of the unconstrained flexible multibody system

$$\bar{\mathbf{M}}(\bar{\mathbf{b}}) \ddot{\bar{\mathbf{b}}} = \bar{\mathbf{F}}^p(\bar{\mathbf{b}}) - \bar{\mathbf{F}}^\sigma(\bar{\mathbf{b}}) - \bar{\mathbf{F}}^\rho(\bar{\mathbf{b}}) \quad (2.57)$$

where

$$\bar{\mathbf{M}}(\bar{\mathbf{b}}) = \begin{bmatrix} \mathbf{M}^1 & & & \\ & \mathbf{M}^2 & & \\ & & \ddots & \\ & & & \mathbf{M}^N \end{bmatrix}, \quad (2.58a)$$

$$\bar{\mathbf{b}}^T = \{\bar{\mathbf{b}}_1^T \quad \bar{\mathbf{b}}_2^T \quad \cdots \quad \bar{\mathbf{b}}_N^T\}, \quad (2.58b)$$

$$\bar{\mathbf{F}}^{p^T}(\bar{\mathbf{b}}) = \{\mathbf{f}_b^{p1^T} \quad \mathbf{f}_b^{p2^T} \quad \cdots \quad \mathbf{f}_b^{pN^T}\}, \quad (2.58c)$$

$$\bar{\mathbf{F}}^{\sigma^T}(\bar{\mathbf{b}}) = \{\mathbf{f}_b^{\sigma1^T} \quad \mathbf{f}_b^{\sigma2^T} \quad \cdots \quad \mathbf{f}_b^{\sigma N^T}\}, \quad (2.58d)$$

$$\bar{\mathbf{F}}^{\rho^T}(\bar{\mathbf{b}}) = \{\mathbf{f}_b^{\rho1^T} \quad \mathbf{f}_b^{\rho2^T} \quad \cdots \quad \mathbf{f}_b^{\rho N^T}\}. \quad (2.58e)$$

The vector $\bar{\mathbf{b}}$ contains all the nodal unknowns (displacements and slopes) in the FE model. The matrix $\bar{\mathbf{M}}(\bar{\mathbf{b}})$ in Eq. (2.57) represents the nodal system mass matrix, and it is a function of the unknown solution vector, $\bar{\mathbf{b}}$. On the right-hand side, the vectors $\bar{\mathbf{F}}^\sigma(\bar{\mathbf{b}})$ and $\bar{\mathbf{F}}^\rho(\bar{\mathbf{b}})$ are the nodal force vectors arising from internal, as well as inertial forces, and they are also dependent on the unknown solution vector, $\bar{\mathbf{b}}$. Any external loading arising due to the presence of gravity or aerodynamic forces is represented by the vector $\mathbf{F}^p(\bar{\mathbf{b}})$. In the construction of Eq. (2.57), the nodal unknowns associated with each body are uncoupled among themselves. Therefore, Eq. (2.57) represents the equations of motion of multiple bodies in the absence of the kinematic joints. The coupling occurs due to the presence of kinematic joints, such as spherical, revolute, and universal joints. Introducing the coupling of flexible bodies through constraint equations gives rise to a system of equations representing the motion of a constrained flexible multibody system.

3. ELEMENT LIBRARY

The present study attempts to focus on detailed three dimensional modeling of components that constitute a multibody system. Such an approach should give sufficient importance to ensure accurate representation of the geometry and material distribution while keeping the computational size of the model at a reasonable level. Therefore, special attention has been given to the type of elements that can be used to discretize the flexible bodies. In the following sections the development of a triangular plate, stiffened plate, solid wedge and one-dimensional elements will be discussed. Although several types of formulations are available which are dedicated for plate and solid elements, the procedures utilized in the present study are chosen while keeping in mind the modeling of a rotor blade with various complexities.

As discussed in the previous chapter the addition of any new element to the element library is fairly straightforward. Therefore, the subsequent sections will focus only on development of finite element matrices that are required to obtain the equation of motion. The details regarding time marching and utilizing these elements for multibody analysis will be discussed in later chapters because these procedures are independent of the element type. The development of the mass and stiffness (linear and geometric) matrices for each element is discussed in details. As explained in Chapter 2, the computation of the internal force vector is based on the co-rotational formulation. Whereas solid elements require only the translation part of deformation, the computation of the internal force vector for plate elements requires deformational rotations as well. The procedure discussed in Chapter 2 can readily be applied to both solid and plate elements and

therefore the computation of internal force vector is not discussed in the present Chapter. Similarly, the computation of external force vector due to concentrated force and moment is independent of the type of element used. Hence these details are purposely not included as part of the element development effort. In Chapter 6, the development of external force due to aerodynamic force is explained in detail.

In order to model rotor blades with plate elements special care should be taken to ensure that the element is equally robust for both transverse and membrane bending. Whereas under flap bending, the flange and web of a spar in a rotor blade experience transverse and membrane bending, respectively, the reverse occurs under lag bending. The simplest form of a membrane element is the constant strain element (CST). As the name suggests, the strain field inside the CST element remains constant and therefore several layers of elements are required to capture a varying strain field that occurs in bending dominant problems. Moreover, the CST element suffers from shear locking and convergence is very slow as well. Due to the absence of out of plane drilling rotations, an artificial stiffness need to be added in order to ensure numerical stability. Although the concept of using drilling degrees of freedom was exploited by several researchers, the first successful work was produced by Allman (1984). This element suffered from rank deficiency and the vertex rotations were not the true out of plane rotations. These shortcomings were overcome in a later development work by the same author [Allman (1988)]. Furthermore, several approaches were presented based on the concept of drilling rotations [Frey (1989); Hughes and Brezzi (1989); Hughes et al. (1995)]. For instance Piltner and Taylor (2000) utilized enhanced strain in order to develop an improved

triangular element with drilling degrees of freedom. Although these elements are more efficient in comparison to the CST element, their performance degrades as the aspect ratio increases. While summarizing earlier work on the extended free formulation and Assumed Natural DEviatoric Strain (ANDES) formulation [Alvin et al. (1992); Felippa (1992a); Felippa (1992b)], an LST-3/9R (linear strain triangle with three nodes and nine degrees of freedom that includes rotational degrees of freedom) optimal membrane element was presented within the framework of template development by Felippa (2003). The LST-3/9R-ANDES element captures the exact response for a rectangular mesh under pure bending irrespective of the aspect ratio of the elements. This feature makes the LST-3/9R-ANDES element a perfect candidate for slender structural members experiencing membrane bending. Therefore, the plate element developed for the present analysis incorporates the ANDES formulation presented in Felippa (2003).

As for transverse bending of thick plates or composite laminates, it is crucial to incorporate transverse shear in the FE model. One of the major difficulties associated with the development of a shear deformable plate bending element is the isoparametric representation of the element displacement interpolation functions that lead to locking for low-order elements when the element thickness becomes small. The locking difficulty can be described as the excessive stiffening of an element due to over stiff representation of shear stiffness terms relative to bending stiffness terms. This is mainly caused by the kinematic incompatibility of the terms representing the linear part of the strains, since these terms are used in the generation of the linear element stiffness matrix. In order to eliminate this problem without resorting to reduced integration techniques, Tessler and

Hughes (1985) suggested the use of different orders of interpolation functions for out-of-plane displacements and normal rotations. Furthermore, a shear correction factor is calculated based on residual bending flexibility technique. While convergence is rapid, the element yields good results for both thick and thin plates as well.

An alternative approach to the beam–plate finite element models, where both beam and plate elements are separately used in the discretization of stiffened structures is to utilize the stiffened plate elements directly. A stiffened plate element includes the kinematics of both the beam (stiffener) and plate components within the element formulation. In other words, the beam kinematic field is defined in terms of the kinematic field of the host plate element through certain constraint conditions. The stiffened plate element is then constructed by assembling both components into the stiffness matrix associated with the stiffened plate element. Because the kinematics of the beam is described by the kinematics of the host plate element, the resulting element matrix contains the degrees of freedom associated with the plate element only. In this case, only the host plate elements are needed to discretize the entire stiffened plate domain. The use of host plate elements for discretization of stiffened structures automatically eliminates the modeling difficulties attributed to the conventional beam–plate finite element models. The finite element formulation of a stiffened plate element based on the combination of plate and beam elements is utilized in the present study.

A solid element with six corner nodes has been developed for modeling bodies for which beam or plate elements are not a suitable choice. Due to their inability to capture bending deformations, solid elements based on isoparametric formulations have a tendency to

lock due to the presence of parasitic shear strain. In order to capture bending deformation with high accuracy a very fine mesh is required. Unfortunately, any such detailed model will result in extensive computational time especially for nonlinear transient analysis. Therefore, the method of incompatible modes is utilized in the present study which provides additional modes to capture bending deformations accurately even with a fairly coarse mesh.

3.1. Triangular Plate Element

The construction of the triangular plate element with three nodes having six degrees of freedom at each node is shown in Fig. 3.1. The displacement at a point inside the triangular plate element, based on the first order shear deformable plate theory, is expressed as

$$\mathbf{u}'' = \begin{Bmatrix} u(x'', y'', z'') \\ v(x'', y'', z'') \\ w(x'', y'', z'') \end{Bmatrix} = \begin{Bmatrix} u_0(x'', y'') \\ v_0(x'', y'') \\ w_0(x'', y'') \end{Bmatrix} + z'' \begin{Bmatrix} \theta_{y''}(x'', y'') \\ \theta_{x''}(x'', y'') \\ 0 \end{Bmatrix} \quad (3.1)$$

or

$$\mathbf{u}'' = \mathbf{u}_0'' + z'' \boldsymbol{\theta}_0'' \quad (3.2)$$

in which u_0 , v_0 , and w_0 represent mid-surface displacements. The in-plane displacement components in the x'' and y'' directions are denoted by u and v , respectively. The out-of-plane deflection is represented by w . The bending rotations about the x'' and y'' axes are denoted by $\theta_{x''}$ and $\theta_{y''}$, respectively.

Fig. 3.1 Geometry and material layout of the plate element with positive directions of nodal displacements and rotations.

3.1.1 Mass Matrix

The incremental displacement and rotation vectors associated with the mid-surface, $\mathbf{u}_0''$ and $\boldsymbol{\theta}_0''$, are approximated in terms of the nodal unknowns and linear interpolation function as follows

$$\begin{Bmatrix} u_0(x'', y'') \\ v_0(x'', y'') \\ w_0(x'', y'') \end{Bmatrix} = \sum_{i=1}^3 \xi_i \begin{Bmatrix} u_{0i}(x'', y'') \\ v_{0i}(x'', y'') \\ w_{0i}(x'', y'') \end{Bmatrix} \quad (3.3a)$$

and

$$\begin{Bmatrix} \theta_{x''} \\ \theta_{y''} \\ \theta_{z''} \end{Bmatrix} = \sum_{i=1}^3 \xi_i \begin{Bmatrix} \theta_{x''i} \\ \theta_{y''i} \\ \theta_{z''i} \end{Bmatrix} \quad (3.3b)$$

where

$$\xi_i = \frac{(c_i + b_i x'')}{2A} \quad (3.4)$$

with

$$b_i = y_i'' - y_k'' \quad (3.5a)$$

$$c_i = x_j'' y_k'' - x_k'' y_j'' \quad (i = 1, 2, 3; j = 2, 3, 1; k = 3, 1, 2) \quad (3.5b)$$

and A is the in-plane area of the plate element.

Combining the finite element approximation given in Eq. (3.3) along with the displacement field defined in Eq. (3.2), the displacement field inside the plate can be approximated as

$$\mathbf{u}'' = \mathbf{N}_m'' \mathbf{v}'' \quad (3.6)$$

where

$$\mathbf{N}_m'' = \mathbf{N}_{mu}'' + z'' \mathbf{N}_{m\theta}'' \quad (3.7)$$

is the matrix of interpolation functions and $\mathbf{v}_p''$ is the vector of nodal unknowns associated with the plate.

Introducing a transformation matrix, $\mathbf{Z}$, between the element coordinate system and the body-fixed frame, the element displacement field, $\mathbf{u}''$, is transformed to the (x', y', z') frame as

$$\mathbf{u}' = \mathbf{Z}\mathbf{u}'' = \mathbf{Z}\mathbf{N}_m'' \mathbf{v}'' . \quad (3.8)$$

Furthermore, using the same transformation matrix $\mathbf{Z}$, the displacements and rotations of the element at each of the three nodes can be transformed to the (x', y', z') frame with a compact transformation matrix, $\mathbf{Z}^*$, in the form

$$\mathbf{v}' = \mathbf{Z}^* \mathbf{v}'' \quad (3.9)$$

in which $\mathbf{Z}^*$ is a square transformation matrix of order 18 containing the components of the element transformation matrix associated with nodal displacement rotation vectors defined at each node of the element. Note that the transformation matrix $\mathbf{Z}^*$ is orthonormal. Hence, inverting (3.9) and substituting into (3.8) for $\mathbf{v}''$ lead to

$$\mathbf{N}_m' = \mathbf{Z}\mathbf{N}_m'' \mathbf{Z}^{*T} . \quad (3.10)$$

The element mass matrices and the load vectors arising from the presence of Coriolis and centrifugal forces appearing in the expression for the virtual work due to inertial forces are expressed as

$$\begin{aligned}
\mathbf{M}_{RR} &= \rho^t V \mathbf{I} , & \mathbf{M}_{R\omega} &= -\rho \mathbf{T} \int_{t_V} \tilde{\mathbf{r}}' d^t V \mathbf{T}^T , \\
\mathbf{M}_{Rv'} &= \rho \mathbf{T} \mathbf{Z} \int_{t_V} \mathbf{N}_m'' d^t V \mathbf{Z}^{*T} , & \mathbf{M}_{\omega\omega} &= -\rho \mathbf{T} \mathbf{Z} \int_{t_V} \tilde{\mathbf{r}}'' \tilde{\mathbf{r}}'' d^t V \mathbf{Z}^T \mathbf{T}^T , \\
\mathbf{M}_{\omega v'} &= \rho \mathbf{T} \mathbf{Z} \int_{t_V} \tilde{\mathbf{r}}'' \mathbf{N}_m'' d^t V \mathbf{Z}^{*T} , & \mathbf{M}_{v'v'} &= \rho \mathbf{Z}^* \int_{t_V} \mathbf{N}_m''^T \mathbf{N}_m'' d^t V \mathbf{Z}^{*T} , \\
& & & (3.11a-l) \\
\mathbf{f}_1 &= 2\rho \mathbf{T} \mathbf{Z} \tilde{\mathbf{\Omega}}'' \int_{t_V} \mathbf{N}_m'' d^t V \dot{\mathbf{v}}'' , & \mathbf{f}_2 &= \rho \mathbf{T} \tilde{\mathbf{\Omega}}'' \tilde{\mathbf{\Omega}}'' \int_{t_V} \tilde{\mathbf{r}}' d^t V , \\
\mathbf{g}_1 &= 2\rho \mathbf{T} \mathbf{Z} \int_{t_V} \tilde{\mathbf{r}}'' \tilde{\mathbf{\Omega}}'' \mathbf{N}_m'' d^t V \dot{\mathbf{v}}'' , & \mathbf{g}_2 &= \rho \mathbf{T} \mathbf{Z} \int_{t_V} \tilde{\mathbf{r}}'' \tilde{\mathbf{\Omega}}'' \tilde{\mathbf{\Omega}}'' \tilde{\mathbf{r}}'' d^t V , \\
\mathbf{h}_1 &= 2\rho \mathbf{Z}^* \int_{t_V} \mathbf{N}_m''^T \tilde{\mathbf{\Omega}}'' \mathbf{N}_m'' d^t V \dot{\mathbf{v}}'' , & \mathbf{h}_2 &= \rho \mathbf{Z}^* \int_{t_V} \mathbf{N}_m''^T \tilde{\mathbf{\Omega}}'' \tilde{\mathbf{\Omega}}'' \mathbf{r}'' d^t V .
\end{aligned}$$

3.1.2. Tangent Stiffness matrix

The plate element is constructed from a laminate having layers of orthotropic materials with arbitrary ply orientations; therefore, Eq. 2.32 is rewritten with layerwise representation as

$$\begin{aligned}
\delta W_\sigma &= \sum_{k=1}^K \int_{t_A} \int_{z_{k-1}}^{z_k} \delta_t \mathbf{E}_L^T \bar{\mathbf{Q}}^{(k)} \mathbf{E}_L dz d^t A + \sum_{k=1}^K \int_{t_A} \int_{z_{k-1}}^{z_k} \delta_t \mathbf{E}_{NL}^T \mathbf{\sigma}^{(k)} dz d^t A \\
&\quad + \sum_{k=1}^K \int_{t_A} \int_{z_{k-1}}^{z_k} \delta_t \mathbf{E}_L^T \mathbf{\sigma}^{(k)} dz d^t A
\end{aligned} \tag{3.12}$$

where K denotes the total number of layers. The position of the layers in reference to the mid-plane is specified by z_k . The thickness of the k^{th} layer is given by $t_k = z_k - z_{k-1}$.

At time t , the element reference area is denoted by ${}^t A$. The stress and strain components are related through the relation $\mathbf{\sigma}^{(k)} = \bar{\mathbf{Q}}^{(k)} \mathbf{E}_L$, in which $\bar{\mathbf{Q}}^{(k)}$ is the orthotropic material property of the k^{th} ply after being transformed to the local element coordinate system, and it has the following form

$$\bar{\mathbf{Q}}^{(k)} = \begin{bmatrix} \bar{\mathbf{Q}}_m^{(k)} & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{Q}}_s^{(k)} \end{bmatrix} \quad (3.13)$$

where $\bar{\mathbf{Q}}_m^{(k)}$ and $\bar{\mathbf{Q}}_s^{(k)}$ are associated with membrane and transverse strain, respectively.

Each layer is assumed to be homogeneous, elastic, and orthotropic with elastic moduli, E_L and E_T ; shear modulus, G_{LT} ; and Poisson's ratio, ν_{LT} . The subscripts L and T specify the longitudinal and transverse directions relative to the fiber in the layer.

In accordance with the modified form of the von-Karman large-deflection plate theory, the linear and nonlinear parts of the Green-Lagrangian strain vector, ${}_t\mathbf{E}_L$ and ${}_t\mathbf{E}_{NL}$, during the time interval between t and $t + \Delta t$ are defined in the local element coordinate system (x'', y'', z'') as

$${}_t\mathbf{E}_L = \begin{Bmatrix} u_{0,x''} \\ v_{0,y''} \\ u_{0,y''} + v_{0,x''} \\ w_{0,x''} + \theta_{y''} \\ w_{0,y''} + \theta_{x''} \end{Bmatrix} + z'' \begin{Bmatrix} \theta_{y'',x''} \\ \theta_{x'',y''} \\ \theta_{x'',x''} + \theta_{y'',y''} \\ 0 \\ 0 \end{Bmatrix} \quad (3.14a)$$

$${}_t\mathbf{E}_{NL} = \frac{1}{2} \begin{Bmatrix} u_{0,x''}^2 + v_{0,x''}^2 + w_{0,x''}^2 \\ u_{0,y''}^2 + v_{0,y''}^2 + w_{0,y''}^2 \\ 2(u_{0,x''}u_{0,y''} + v_{0,x''}v_{0,y''} + w_{0,x''}w_{0,y''}) \\ 0 \\ 0 \end{Bmatrix}. \quad (3.14b)$$

Substituting for the linear and nonlinear parts of the Green-Lagrangian strain vector and performing through-the-thickness integration of Eq. (3.12) lead to

$$\delta W_\sigma = \int_{t_A} \delta {}_t\mathbf{e}_L^T \mathbf{C}_t \mathbf{e}_L d^t A + \int_{t_A} \delta {}_t\mathbf{e}_{NL}^T {}^t\mathbf{s} d^t A + \int_{t_A} \delta {}_t\mathbf{e}_L^T {}^t\mathbf{s} d^t A \quad (3.15)$$

where ${}^t\mathbf{s}$ represents the stress resultant vector known from the previous (known) time step. It is defined in terms of the in-plane stress resultant, ${}^t\mathbf{s}_\varepsilon$, bending moment, $\mathbf{s}_\kappa^t$, and shear stress resultant, ${}^t\mathbf{s}_\gamma$, vectors as

$${}^t\mathbf{s}^T = \left\{ {}^t\mathbf{s}_\varepsilon \quad {}^t\mathbf{s}_\kappa \quad {}^t\mathbf{s}_\gamma \right\}. \quad (3.16)$$

with

$${}^t\mathbf{s}_\varepsilon = \begin{Bmatrix} N_{x''x''} \\ N_{y''y''} \\ N_{x''y''} \end{Bmatrix} = \sum_{n=1}^N \begin{Bmatrix} \sigma_{x''x''} \\ \sigma_{y''y''} \\ \sigma_{x''y''} \end{Bmatrix}^n (z_n'' - z_{n-1}'') \quad (3.17a)$$

$${}^t\mathbf{s}_\kappa = \begin{Bmatrix} M_{x''x''} \\ M_{y''y''} \\ M_{x''y''} \end{Bmatrix} = \frac{1}{2} \sum_{n=1}^N \begin{Bmatrix} \sigma_{x''x''} \\ \sigma_{y''y''} \\ \sigma_{x''y''} \end{Bmatrix}^n (z_n''^2 - z_{n-1}'')^2 \quad (3.17b)$$

$${}^t\mathbf{s}_\gamma = \begin{Bmatrix} Q_{x''z''} \\ Q_{y''z''} \end{Bmatrix} = \sum_{n=1}^N \begin{Bmatrix} \sigma_{x''z''} \\ \sigma_{y''z''} \end{Bmatrix}^n (z_n'' - z_{n-1}'') \quad (3.17c)$$

The vector ${}^t\mathbf{e}_L$ contains the mid-plane strain, curvatures, and transverse shear components. It is defined as

$${}^t\mathbf{e}_L^T = \left\{ {}^t\mathbf{e}_{\varepsilon L}^T \quad {}^t\mathbf{e}_{\kappa L}^T \quad {}^t\mathbf{e}_{\gamma L}^T \right\} \quad (3.18)$$

with

$${}^t\mathbf{e}_{\varepsilon L}^T = \left\{ u_{0,x''}, v_{0,y''}, (u_{0,y''} + v_{0,x''}) \right\} \quad (3.19a)$$

$${}^t\mathbf{e}_{\kappa L}^T = \left\{ \theta_{y'',x''}, \theta_{x'',y''}, (\theta_{x'',x''} + \theta_{y'',y''}) \right\} \quad (3.19b)$$

$${}^t\mathbf{e}_{\gamma L}^T = \left\{ (w_{0,x''} + \theta_{y''}), (w_{0,y''} + \theta_{x''}) \right\}. \quad (3.19c)$$

The vector ${}_t\mathbf{e}_{NL}^T$ comprises of the nonlinear part of Green–Lagrangian strain

$${}_t\mathbf{e}_{NL}^T = \frac{1}{2} \left\{ \left(u_{0,x''}^2 + v_{0,x''}^2 + w_{0,x''}^2 \right), \left(u_{0,y''}^2 + v_{0,y''}^2 + w_{0,y''}^2 \right), \right. \\ \left. 2 \left(u_{0,x''} u_{0,y''} + v_{0,x''} v_{0,y''} + w_{0,x''} w_{0,y''} \right) \right\} \quad (3.20)$$

Also in (3.15), the matrix $\mathbf{C}$ represents the material property matrix relating the resultant stress vector, ${}_t\mathbf{s}$, to the vector of strain resultants, ${}_t\mathbf{e}$. The matrix $\mathbf{C}$ is composed of the extensional (**A**) membrane–bending coupling (**B**) bending (**D**) and transverse shear (**G**) stiffness matrices,

$$\mathbf{C} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{0} \\ \mathbf{B} & \mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{G} \end{bmatrix} \quad (3.21)$$

with

$$A_{ij} = \sum_{n=1}^N \bar{Q}_{mij}^{(n)} (z_n'' - z_{n-1}'') \quad (3.22a)$$

$$B_{ij} = \frac{1}{2} \sum_{n=1}^N \bar{Q}_{mij}^{(n)} (z_n''^2 - z_{n-1}''^2) \quad (3.22b)$$

$$D_{ij} = \frac{1}{3} \sum_{n=1}^N \bar{Q}_{mij}^{(n)} (z_n''^3 - z_{n-1}''^3) \quad (3.22c)$$

$$G_{ij} = k_{ij} \sum_{n=1}^N \bar{Q}_{sij}^{(n)} (z_n'' - z_{n-1}'') \quad (3.22d)$$

where k_{ij} is the shear correction factor.

Fig. 3.2 The nodal displacements for (a) LST6/12C parent and the resulting (b) LST3/9R ANDES membrane element.

Fig. 3.3 Nodal displacements for (a) parent element with mid-side nodes and the resulting (b) MIN3 element with corner nodes only.

3.1.2.1 Linear stiffness matrix

The first term in Eq. (3.15) eventually leads to the linear stiffness matrix which consists of the membrane, $\mathbf{K}_m$, membrane–bending coupling, $\mathbf{K}_{mb}$, bending, $\mathbf{K}_b$, and shear, $\mathbf{K}_s$, stiffness matrix. Since these matrices are associated with different sets of degrees of freedom, therefore the vector of nodal unknowns is split into the following form

$$\mathbf{v}''^T = \left\{ \mathbf{v}_m''^T \quad \mathbf{v}_b''^T \right\} \quad (3.23)$$

where

$$\mathbf{v}_m'' = \left\{ u_{01}, v_{01}, \theta_{z^*1}, u_{02}, v_{02}, \theta_{z^*2}, u_{03}, v_{03}, \theta_{z^*3} \right\}^T \quad (3.24a)$$

$$\mathbf{v}_b'' = \left\{ w_{01}, \theta_{x^*1}, \theta_{y^*1}, w_{02}, \theta_{x^*2}, \theta_{y^*2}, w_{03}, \theta_{x^*3}, \theta_{y^*3} \right\}^T. \quad (3.24b)$$

Whereas $\mathbf{K}_m$ is associated with $\mathbf{v}_m''$, which consists of the in-plane displacements and drilling rotation, $\mathbf{K}_b$ and $\mathbf{K}_s$ operate on $\mathbf{v}_b''$ which comprises of transverse displacement and bending rotations. The stiffness matrix, $\mathbf{K}_{mb}$ which appears for unsymmetrical laminates, couples the membrane and bending degrees of freedom.

The membrane stiffness matrix, $\mathbf{K}_m$, is based on the ANDES template formulation proposed by Felippa (2003). The membrane element has drilling degrees of freedom in addition to the translation degrees of freedom. The parent LST–6/12C (linear strain triangle with six nodes and twelve degrees of freedom that include only conventional degrees of freedom) and the resulting LST–3/9R ANDES (linear strain triangle with three nodes with nine degrees of freedom that includes rotational degrees of freedom and based on Assumed Natural DEviatoric Strain) element is shown in Fig. 3.2. Based on the

decomposition introduced by Bergan and Nygard (1984), the membrane stiffness matrix can be decomposed as

$$\mathbf{K}_m = \mathbf{K}_{m-b} + \mathbf{K}_{m-h} \quad (3.25)$$

where $\mathbf{K}_{m-b}$ and $\mathbf{K}_{m-h}$ are the basic stiffness matrix and higher order stiffness matrix, respectively. Whereas the matrix $\mathbf{K}_{m-b}$ is constructed for convergence, the matrix $\mathbf{K}_{m-h}$ ensures stability and accuracy. The basic stiffness matrix can be expressed as

$$\mathbf{K}_{m-b} = \frac{1}{A} \mathbf{L} \mathbf{A} \mathbf{L}^T \quad (3.26)$$

where $\mathbf{L}$ is the force-lumping matrix, that maps the boundary traction to the nodal forces and its explicit form is given in Appendix A.

The higher order stiffness matrix given by Felippa (2003) has a form similar to the one presented by Felippa and Militello (1992) although the set of free parameters are chosen to obtain the optimal stiffness matrix. The first task is to obtain the higher order stiffness matrix, $\mathbf{K}_{\tilde{\theta}}$, in terms of the hierarchical rotations which are defined as

$$\tilde{\theta}_{zi}'' = \theta_{zi}'' - \theta_{z0}'' \quad (3.27)$$

where θ_{z0}'' is the mean out of plane rotation. Combining both bending and torsional strain fields, the higher order natural strains at the i -th node is expressed as $\boldsymbol{\varepsilon}_i = \mathbf{Q}_i \tilde{\boldsymbol{\theta}}$. Therefore, the higher order natural strain field inside the element can be expressed in terms of linear interpolation functions as

$$\boldsymbol{\varepsilon} = (\mathbf{Q}_1 \xi_1 + \mathbf{Q}_2 \xi_2 + \mathbf{Q}_3 \xi_3) \tilde{\boldsymbol{\theta}} \quad (3.28)$$

where the explicit form of the matrices $\mathbf{Q}_1$, $\mathbf{Q}_2$ and $\mathbf{Q}_3$ is given in Appendix A.

The exact integration of $\mathbf{K}_{\tilde{\theta}}$ requires the computation of $\mathbf{Q}_i$ at the mid point of the sides which gives

$$\mathbf{Q}_4 = \frac{1}{2}(\mathbf{Q}_1 + \mathbf{Q}_2) \quad \mathbf{Q}_5 = \frac{1}{2}(\mathbf{Q}_3 + \mathbf{Q}_3) \quad \mathbf{Q}_6 = \frac{1}{2}(\mathbf{Q}_3 + \mathbf{Q}_1) \quad (3.29)$$

The higher order stiffness can now be expressed as

$$\mathbf{K}_{\tilde{\theta}} = \sum_{i=4}^6 \mathbf{Q}_i^T \mathbf{A}_{nat} \mathbf{Q}_i \quad (3.30)$$

where $\mathbf{A}_{nat}$ is the natural resultant stress strain matrix which is defined as

$$\mathbf{A}_{nat} = \mathbf{T}_e^T \mathbf{A} \mathbf{T}_e \quad (3.31)$$

with $\mathbf{T}_e$ being the transformation matrix which relates the Cartesian strains to the natural strain, and matrix $\mathbf{A}$ is the resultant stress–strain relationship matrix defined in Eq. (3.22a). The explicit form of matrix $\mathbf{T}_e$ is given in Appendix A.

The final form of the higher order stiffness matrix, $\mathbf{K}_{m-h}$, is obtained through the following transforming

$$\mathbf{K}_{m-h} = \frac{3}{4} \beta_0 \mathbf{T}_{\tilde{\theta}u}^T \mathbf{K}_{\tilde{\theta}} \mathbf{T}_{\tilde{\theta}u} \quad (3.32)$$

where β_0 is a scaling factor and matrix $\mathbf{T}_{\tilde{\theta}u}$ transforms nodal unknowns to hierarchical rotations and its explicit form is given in Appendix A.

At this stage the scaling factor, α_b , the overall scaling coefficient, β_0 , and the free parameters $\beta_i (i=1-9)$ are yet to be determined. Based on the choice of these parameters, a library of elements can be constructed. However, in order to enforce the

higher order strains to be energy orthogonal, $\mathbf{Q}_1\zeta_1 + \mathbf{Q}_2\zeta_2 + \mathbf{Q}_3\zeta_3$ should vanish at the centroid, which can be satisfied provided that

$$\beta_1 + \beta_5 + \beta_9 = \beta_2 + \beta_6 + \beta_7 = \beta_3 + \beta_4 + \beta_8 = 0 \quad (3.33)$$

Fellipa (2003) obtained the following values that lead to the optimal element under pure bending

$$\alpha_b = 1.5, \quad \beta_0 = 0.5(1 - 4\nu^2),$$

$$\beta_1 = \beta_3 = \beta_5 = 1, \quad \beta_2 = 2, \quad \beta_4 = 0, \quad \beta_6 = \beta_7 = \beta_8 = -1, \quad \beta_9 = -2. \quad (3.34a-g)$$

Note that these values satisfy Eq. (3.33) and therefore the higher order strain field is energy orthogonal. These values ensure that for isotropic materials the strain energy absorbed by the finite elements is same as the exact solution and therefore optimal performance is guaranteed. For non-isotropic materials the ratio of the strain energy for finite elements and exact solution is independent of the aspect ratio, but is not equal to one. Therefore, only a suboptimal performance can be obtained for non-isotropic materials. Moreover, Felippa (2003) suggested the following value for β_0 to be used for non-isotropic materials:

$$\beta_0 = \max\left(\frac{256 \times \det(\mathbf{A})}{W} - \frac{3}{2}, 0.01\right) \quad (3.35)$$

where the expression for W is given in Appendix A. Although optimal performance is not guaranteed, the element is not prone to aspect ratio locking.

Combining Eqs. (3.26) and (3.32) the complete form of the membrane stiffness matrix becomes

$$\mathbf{K}_m = \frac{1}{A} \mathbf{L} \mathbf{A} \mathbf{L}^T + \frac{3}{4} \beta_0 \mathbf{T}_{\tilde{\theta}u}^T \mathbf{K}_{\tilde{\theta}} \mathbf{T}_{\tilde{\theta}u} \quad (3.36)$$

The transverse bending stiffness matrix is based on the C^0 –anisoparametric formulation proposed by Tessler and Hughes (1985). The incremental in–plane, transverse, and out–of–plane rotational components on the mid–surface of the element are approximated as

$$w_0 = \sum_{i=1}^3 N_i w_{0i} + \sum_{i=4}^6 N_i w_{0i}^+ \quad (3.37)$$

where

$$N_i = \xi_i (2\xi_i - 1) \quad N_{i+3} = 4\xi_i \xi_k \\ (i = 1, 2, 3; \quad k = 2, 3, 1) \quad (3.38)$$

and

$$\begin{Bmatrix} \theta_{x''} \\ \theta_{y''} \end{Bmatrix} = \sum_{i=1}^3 \xi_i \begin{Bmatrix} \theta_{x''i} \\ \theta_{y''i} \end{Bmatrix}. \quad (3.39)$$

The transverse displacement field, w_0 , is approximated by a quadratic interpolation function, N_i , with w_{0i} and w_{0i}^+ representing the nodal values at the vertices and the mid–side of the edges, respectively. The normal rotational components, $\theta_{x''}$ and $\theta_{y''}$, are approximated by linear interpolation functions, ξ_i , with $\theta_{x''i}$ and $\theta_{y''i}$ representing nodal values at the vertices. The choice of the order of interpolation functions satisfies the kinematic compatibility requirements. Since the expression for transverse shear strain involves the normal rotations and the derivative of transverse displacement, the order of polynomials for w_0 is one degree higher than those for $\theta_{x''}$ and $\theta_{y''}$.

At this stage, based on Eqs. (3.37) and (3.39), the element matrices involve degrees of freedom associated with the mid-side nodes. In order to gain nodal simplicity, Tessler and Hughes (1985) performed condensation of the mid-side degrees of freedom. The condensation is achieved by enforcing continuous shear strain constraints at every element edge [Tessler (1981); Tessler and Hughes (1983)] as follows

$$\frac{\partial}{\partial s} \gamma_{sz}^{(k)}(s) = \frac{\partial}{\partial s} \left[\frac{\partial}{\partial s} w_0(s) + \theta_n(s) \right]^{(k)} = 0$$

$$(i = 1, 2, 3) \quad (3.40)$$

where s denotes the edge coordinate and θ_n is the tangential edge rotation. Enforcement of the constraint condition in Eq. (3.40) results in an expression relating the mid-side degrees of freedom with the ones at the corner nodes such that

$$w_{0i+3}^+ = \frac{1}{2}(w_i + w_j) + \frac{1}{8} \left[b_k (\theta_{x^*i} - \theta_{x^*j}) + a_k (\theta_{y^*j} - \theta_{x^*i}) \right] \quad (3.41)$$

$$(i = 1, 2, 3; j = 2, 3, 1; k = 3, 1, 2)$$

Substituting Eq. (3.41) into Eq. (3.37), the transverse displacement, w_0 , can now be expressed in terms of only the corner degrees of freedom

$$w_0 = \sum_{i=1}^3 (\xi_i w_{0i} + L_i \theta_{x^*i} + M_i \theta_{y^*i}) \quad (3.42)$$

where

$$L_i = \frac{1}{8} (b_k N_{i+3} - b_j N_{k+3}) \quad M_i = \frac{1}{8} (a_j N_{k+3} - a_k N_{i+3})$$

$$(i = 1, 2, 3; j = 2, 3, 1; k = 3, 1, 2). \quad (3.43)$$

The nodal degrees of freedom associated with the parent element with the mid-side nodes and the final MIN3 element with only corner nodes is shown in Fig3.3.

The displacement variation in Eq. (3.42) is C^0 continuous and the following conditions ensure capturing of the constant strain field

$$\sum_{i=1}^3 \xi_i = 1 \quad \sum_{i=1}^3 L_i = 0 \quad \sum_{i=1}^3 M_i = 0. \quad (3.44)$$

Based on the form of displacement variation given in Eqs (3.39) and (3.42), the curvature and transverse shear strain can be expressed as

$${}_t \mathbf{e}_{L\kappa} = \mathbf{B}_{\kappa} \mathbf{v}_b'' \quad \text{and} \quad {}_t \mathbf{e}_{L\gamma} = \mathbf{B}_{\gamma} \mathbf{v}_b'' \quad (3.45a,b)$$

where $\mathbf{B}_{\kappa}$ and $\mathbf{B}_{\gamma}$ are the strain-displacement relationship matrices given in Appendix

A. Utilizing the strain-displacement matrix, the stiffness matrix associated with $\mathbf{v}_b''$ can be readily obtained as

$$\mathbf{K}_{bs} = \mathbf{K}_b + \mathbf{K}_s \quad (3.46)$$

where

$$\mathbf{K}_b = \int_{{}_t A} \mathbf{B}_{\kappa}^T \mathbf{D} \mathbf{B}_{\kappa} dA \quad (3.47a)$$

$$\mathbf{K}_s = \int_{{}_t A} \mathbf{B}_{\gamma}^T \mathbf{G} \mathbf{B}_{\gamma} dA \quad (3.47b)$$

Although the form of displacement given in Eqs (3.39) and (3.42) satisfy the kinematic compatibility, further improvements can be achieved with the proper choice of the shear correction factor, k_{ij} , given in Eq. (3.22d).

The concept that was utilized by Tessler and Hughes (1983 and 1985) is similar to the energy balance [Fried (1974)] and residual bending flexibility [MacNeal (1978)] techniques. The shear correction factor comprises of both classical (through-the-thickness), k_{ij}^* , and finite element (in-plane), ϕ^2 , corrections to the transverse shear stress resultant and energy. It is expressed as

$$k_{ij} = k_{ij}^* \phi^2 \quad (3.48)$$

where

$$\phi^2 = \frac{1}{1 + \alpha / 2} \quad (3.49)$$

and α , which is the ratio of the diagonal shear and bending stiffness coefficients associated with the rotational degrees of freedom, is defined as

$$\alpha = \frac{\sum_{i=j}^{j+1} k_{sii}}{\sum_{i=j}^{j+1} k_{bii}} \quad \text{with } (j = 4, 10, 16) \quad (3.50)$$

The matrix $\mathbf{B}$ in Eq. (3.22b), which couples the membrane strain and curvatures deformations, does not exist in the case of isotropic plates and symmetrical laminates. For unsymmetrical laminates, the matrix $\mathbf{B}$ becomes non-zero and so does the corresponding stiffness matrix, $\mathbf{K}_{mb}$, which couples the membrane and bending degrees of freedom. The stiffness matrix $\mathbf{K}_{mb}$ has the following form

$$\mathbf{K}_{mb} = \mathbf{K}_{bm}^T = \int_A \mathbf{B}_\varepsilon^T \mathbf{B} \mathbf{B}_\kappa dA \quad (3.51)$$

where $\mathbf{B}_\varepsilon$ and $\mathbf{B}_\kappa$ are the membrane and bending strain–displacement relationship matrices, respectively. Note that the closed form expression of the membrane stiffness matrix does not utilize the conventional strain–displacement relationship matrix. Similar to the stiffness matrix, the membrane strain–displacement relationship matrix comprises of the basic and higher order terms. Therefore, the matrix $\mathbf{B}_\varepsilon$ can be written as

$$\mathbf{B}_\varepsilon = \mathbf{B}_{\varepsilon-b} + \mathbf{B}_{\varepsilon-h} \quad (3.52)$$

where $\mathbf{B}_{\varepsilon-b}$ and $\mathbf{B}_{\varepsilon-h}$ are the basic and higher order parts of matrix $\mathbf{B}_\varepsilon$, respectively, and they have the following form

$$\mathbf{B}_{\varepsilon-b} = \frac{\mathbf{L}^T}{tV} \quad (3.53)$$

$$\mathbf{B}_{\varepsilon-h} = \frac{3}{2} \sqrt{\beta_0} \mathbf{T}_e (Q_1 \xi_1 + Q_2 \xi_2 + Q_3 \xi_3). \quad (3.54)$$

Utilizing the expressions given in Eqs. (3.45) and (3.52), the stiffness matrix in Eq. (3.51) can readily be obtained by using numerical integration.

Combining Eqs. (3.36), (3.46) and (3.51), the linear stiffness matrix can be expressed as

$$\mathbf{K}_L'' = \begin{bmatrix} \mathbf{K}_m & \mathbf{K}_{mb} \\ \mathbf{K}_{mb}^T & \mathbf{K}_b + \mathbf{K}_s \end{bmatrix}. \quad (3.55)$$

3.1.2.2 Geometric Stiffness Matrix

The vector ${}_t\mathbf{e}_{NL}$ contains the nonlinear part of the in–plane strain resultants and is defined in the form

$${}^t\mathbf{e}_{NL} = \begin{Bmatrix} {}^t\mathbf{E}_{NL} \\ \mathbf{0} \end{Bmatrix} = \frac{1}{2} \mathbf{A}_{NL} \boldsymbol{\theta}_{NL} \quad (3.56)$$

where

$$\boldsymbol{\theta}_{NL}^T = \{u_{0,x''} \quad v_{0,x''} \quad w_{0,x''} \quad u_{0,y''} \quad v_{0,y''} \quad w_{0,y''}\} \quad (3.57)$$

and

$$\mathbf{A}_{NL} = \begin{bmatrix} u_{0,x''} & v_{0,x''} & w_{0,x''} & 0 & 0 & 0 \\ 0 & 0 & 0 & u_{0,y''} & v_{0,y''} & w_{0,y''} \\ u_{0,y''} & v_{0,y''} & w_{0,y''} & u_{0,x''} & v_{0,x''} & w_{0,x''} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (3.58)$$

Moreover, they possess the following properties

$$\delta \mathbf{A}_{NL} \boldsymbol{\theta}_{NL} = \mathbf{A}_{NL} \delta \boldsymbol{\theta}_{NL} \quad \text{and} \quad \mathbf{A}_{NL}^T {}^t\mathbf{s}_\varepsilon = {}^t\mathbf{S} \boldsymbol{\theta}_{NL} \quad (3.59)$$

with

$${}^t\mathbf{S} = \begin{bmatrix} \mathbf{N}_{x''x''} \mathbf{I} & \mathbf{N}_{x''y''} \mathbf{I} \\ \mathbf{N}_{y''x''} \mathbf{I} & \mathbf{N}_{y''y''} \mathbf{I} \end{bmatrix}. \quad (3.60)$$

where $\mathbf{I}$ is a 3×3 identity matrix. Substituting for the nonlinear part of the Green strain vector and utilizing the properties stated in (3.59), the second integral in (3.15) can be written as

$$\int_{{}^tA} \delta {}^t\mathbf{e}_{NL}^T {}^t\mathbf{s}_m d^t a = \int_{{}^tA} \delta \boldsymbol{\theta}_{NL}^T {}^t\mathbf{S}_m \boldsymbol{\theta}_{NL} d^t a. \quad (3.61)$$

The vector $\boldsymbol{\theta}_{NL}$ can be expressed in the form

$$\boldsymbol{\theta}_{NL} = \mathcal{B} \mathbf{u}_0'' \quad (3.61)$$

with the matrix differential operator

$$\mathcal{B} = \begin{bmatrix} \mathbf{I} \frac{\partial}{\partial x''} \\ \mathbf{I} \frac{\partial}{\partial y''} \end{bmatrix}. \quad (3.62)$$

Similar to the approximation made for the computation of the mass matrix, the displacement field is assumed to vary linearly inside the element. Therefore, the vector $\boldsymbol{\theta}_{NL}$ in Eq (3.61) can be expressed in terms of nodal unknowns as

$$\boldsymbol{\theta}_{NL} = \mathcal{B} \mathbf{N}_{mu}'' \mathbf{v}'' \quad (3.63)$$

Finally, substituting Eq. (3.63) into Eq. (3.61) leads to

$$\int_{t_A} \delta_t \mathbf{e}_{NL}^T {}^t \mathbf{s} d^t a = \delta \mathbf{v}''^T \mathbf{k}_g'' \mathbf{v}'' \quad (3.64)$$

where the geometric stiffness matrix, $\mathbf{k}_g''$, is defined by

$$\mathbf{k}_g'' = \int_{t_A} \mathbf{N}_{mu}^T \mathcal{B}^T {}^t \mathbf{S} \mathcal{B} \mathbf{N}_{mu} d^t a. \quad (3.65)$$

Combining the linear stiffness and geometric stiffness matrix, the tangential stiffness matrix, $\mathbf{k}_T''$, can be defined as

$$\mathbf{k}_T'' = \mathbf{k}_L'' + \mathbf{k}_g''. \quad (3.66)$$

Note that the geometric stiffness matrix in Eq. (3.66) is expressed in the element coordinate system and therefore in order to use the form given in Eq. (2.47), they have to be transformed to the floating frame of reference coordinate as follows

$$\mathbf{k}_T' = \mathbf{Z}^* \mathbf{k}_T'' \mathbf{Z}^{*T}. \quad (3.67)$$

The tangent stiffness matrix $\mathbf{k}'_T$ can be computed for each plate element and assembled into the global stiffness matrix which is defined in the floating frame of reference coordinate.

3.2 Stiffened Plate Element

The geometric description of the stiffened plate element is depicted in Fig. 3.4. As shown in Fig. 3.4, two Cartesian coordinate systems are used to define the kinematics of the stiffened plate element and to derive the element matrices and vectors associated with this element. The element matrices and vectors belonging to the plate element are derived with respect to the (x'', y'', z'') element coordinate system. In order to carry out integrations of matrices belonging to the beam element, a local coordinate system situated on the reference surface of the plate element is utilized. This coordinate system is established by projecting the longitudinal axis of the beam element to the reference plane. Also, this local coordinate system is represented by the axes $\tilde{x}''$, $\tilde{y}''$ and $\tilde{z}''$. Note that the $(\tilde{x}'' - \tilde{y}'')$ plane of the local coordinate system lies on the same plane as the $(x'' - y'')$ plane of the plate element. Hence, a simple plane transformation is sufficient to transform variables between these two coordinate systems. The z'' – and the $\tilde{z}''$ – axes of these two coordinate systems are parallel to each other. Thus, the same vertical axis can be used to define the mid-heights and the transverse displacements in both elements. The position and the orientation of the local coordinate system $(\tilde{x}'', \tilde{y}'', \tilde{z}'')$ are identified by the vector $\mathbf{r}_b$ and the angle ϕ_b between the $\tilde{x}''$ – and x'' – axes, respectively. Therefore, the

following transformation is used to relate the (x'', y'', z'') and $(\tilde{x}'', \tilde{y}'', \tilde{z}'')$ coordinate systems

$$\begin{Bmatrix} x''(\tilde{x}'', \tilde{y}'') \\ y''(\tilde{x}'', \tilde{y}'') \\ z''(\tilde{x}'', \tilde{y}'') \end{Bmatrix} = \begin{bmatrix} \cos \phi_b & -\sin \phi_b & 0 \\ \sin \phi_b & \cos \phi_b & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \tilde{x}'' \\ \tilde{y}'' \\ \tilde{z}'' \end{Bmatrix} + \begin{Bmatrix} r_{bx} \\ r_{by} \\ 0 \end{Bmatrix}. \quad (3.68)$$

At $\tilde{y}'' = 0$, the coordinates x'' and y'' define a line passing through the $\tilde{x}''$ – axis:

$$x''(\tilde{x}'', 0) = \cos \phi_b \tilde{x}'' + r_{bx} \quad (3.69a)$$

$$y''(\tilde{x}'', \tilde{y}'') = \sin \phi_b \tilde{y}'' + r_{by}. \quad (3.69b)$$

The distance between the neutral axis of the beam and the plate mid–surface is indicated by the eccentricity parameter, $\tilde{z}_{ec}''$. The positive directions of displacement and rotation field in both elements are shown in Fig. 3.4.

In accordance with Mindlin's theory, the incremental displacement components of the plate element, u_p'' , v_p'' and w_p'' in the x'' –, y'' –, and z'' – directions, are expressed as

$$\mathbf{u}_p'' = \begin{Bmatrix} u_p(x'', y'', z'') \\ v_p(x'', y'', z'') \\ w_p(x'', y'', z'') \end{Bmatrix} = \begin{Bmatrix} u_{p0}(x'', y'') \\ v_{p0}(x'', y'') \\ w_{p0}(x'', y'') \end{Bmatrix} + z'' \begin{Bmatrix} \theta_{py''}(x'', y'') \\ \theta_{px''}(x'', y'') \\ 0 \end{Bmatrix} \quad (3.70)$$

or

$$\mathbf{u}_p'' = [\mathbf{I} \quad z''\mathbf{I}] \begin{Bmatrix} \mathbf{u}_{p0}'' \\ \boldsymbol{\theta}_{p0}'' \end{Bmatrix} \quad (3.71)$$

where $\mathbf{I}$ is a 3×3 identity matrix.

The functions u_{p0} and v_{p0} represent the in–plane displacements and w_{p0} the out–of–plane displacements on the mid–surface of the plate element. The bending (normal)

rotations about the x'' – and y'' – axes are denoted by $\theta_{px''}$ and $\theta_{py''}$, respectively. The element coordinate system is chosen such that the positive x'' – axis points in the direction from node 1 to node 2 of the element. As shown in Fig. 3.4, the element reference plane coincides with the $(x''-y'')$ plane. The components of the incremental displacements and bending rotations at each node of the element, with positive sign conventions, are shown in Fig. 3.4.

For the beam element, the displacements at a point are expressed based on the Timoshenko beam theory, which assumes constant transverse shear deformations across the thickness of the beam. Considering the interaction between the plate and the stiffener, it is also assumed that the in-plane shear and in-plane bending stiffness of the plate element are much higher than those of the beam element. Although this assumption seems somewhat restrictive, it is acceptable for laminated stiffened construction in aerospace applications, in which stiffeners are designed to carry axial forces and bending moments and the skin section is designed to resist in-plane normal and shear forces. Thus, the slope of the beam element associated with the in-plane bending deformations of the plate element (the drilling rotation) can be disregarded. The resulting expressions for the displacements at a point in the stiffener can be written as

$$\tilde{\mathbf{u}}_b'' = \begin{Bmatrix} \tilde{u}_b(\tilde{x}'', \tilde{y}'', \tilde{z}'') \\ \tilde{v}_b(\tilde{x}'', \tilde{y}'', \tilde{z}'') \\ \tilde{w}_b(\tilde{x}'', \tilde{y}'', \tilde{z}'') \end{Bmatrix} = \begin{Bmatrix} \tilde{u}_{b0}(\tilde{x}'') \\ \tilde{v}_{b0}(\tilde{x}'') \\ \tilde{w}_{b0}(\tilde{x}'') \end{Bmatrix} + \begin{bmatrix} 0 & (z'' - z_{ec}'') & 0 \\ (z'' - z_{ec}'') & 0 & 0 \\ \tilde{y}'' & 0 & 0 \end{bmatrix} \begin{Bmatrix} \tilde{\theta}_{bx''}(\tilde{x}'') \\ \tilde{\theta}_{by''}(\tilde{x}'') \\ 0 \end{Bmatrix} \quad (3.72)$$

or

(a)

(b)

Fig. 3.4 Geometric description of the stiffened plate element: (a) the reference plane; and (b) the cross section.

$$\tilde{\mathbf{u}}_b'' = \begin{bmatrix} \mathbf{I} & \tilde{\mathbf{Z}} \end{bmatrix} \begin{Bmatrix} \tilde{\mathbf{u}}_{b0}'' \\ \tilde{\boldsymbol{\theta}}_{b0}'' \end{Bmatrix}. \quad (3.73)$$

The matrix $\tilde{\mathbf{Z}}$ in Eq. (3.73) is defined as

$$\tilde{\mathbf{Z}} = \begin{bmatrix} 0 & (z'' - z_{ec}'') & 0 \\ (z'' - z_{ec}'') & 0 & 0 \\ \tilde{y}'' & 0 & 0 \end{bmatrix} \quad (3.74)$$

where $\tilde{u}_{b0}$, $\tilde{v}_{b0}$ and $\tilde{w}_{b0}$ are the displacements defined along the longitudinal axis of the beam ($\tilde{y}'' = 0$) and $\tilde{\theta}_{bx''}$ and $\tilde{\theta}_{by''}$ represent the rotations about the $\tilde{x}''$ - and $\tilde{y}''$ - axes, respectively. The sign conventions for the displacements and rotations are illustrated in Fig. 3.4.

The interface conditions between the stiffener (beam) and the plate elements require the enforcement of the continuity of displacements and stress components. However, this is difficult to achieve with the beam and the plate element formulations. Instead of applying these continuity conditions along the interface, the approach proposed by Barut et al. (2000) has been adopted. This approach requires that the cross section of the beam and the section of the plate before deformation remain co-planar after deformation (i.e., the cross section of the beam lies on the same plane as that of the plate after deformation), thus leading to constraint equations between the displacement and rotation fields of these elements. Based on this approach, the following constraint conditions are, therefore, introduced:

$$\tilde{\theta}_{bx''}(\tilde{x}'') = -\tilde{\theta}_{px''}(\tilde{x}'') \quad (3.75a)$$

$$\tilde{\theta}_{by''}(\tilde{x}'') = \tilde{\theta}_{py''}(\tilde{x}'') \quad (3.75b)$$

$$\tilde{w}_{b0}(\tilde{x}'') = \tilde{w}_{p0}(\tilde{x}'') \quad (3.75c)$$

in which $\tilde{w}_{p0}$ is the transverse displacement defined along the longitudinal axis of the beam and $\tilde{\theta}_{px''}$ and $\tilde{\theta}_{py''}$ represent the rotations about the $\tilde{x}''$ – and $\tilde{y}''$ – axes. Utilizing the constraint conditions in Eqs. (3.75a) and (3.75b), two additional constraint equations can be obtained by relating the displacements of the beam element in terms of the displacements and rotations of the plate element as

$$\tilde{u}_{b0}(\tilde{x}'') = \tilde{u}_{p0}(\tilde{x}'') + \tilde{z}_{ec}'' \tilde{\theta}_{py''}(\tilde{x}'') \quad (3.76a)$$

$$\tilde{v}_{b0}(\tilde{x}'') = \tilde{v}_{p0}(\tilde{x}'') + \tilde{z}_{ec}'' \tilde{\theta}_{px''}(\tilde{x}'') \quad (3.76b)$$

where $\tilde{u}_{b0}$ and $\tilde{v}_{b0}$ are the displacement components of the beam element in $\tilde{x}''$ – and $\tilde{y}''$ – directions, respectively. The constraint conditions in Eqs. (3.75) and (3.76) are accurate, as long as the overall thickness of the stiffened-plate is maintained at a moderate range.

In matrix notation, Eqs. (3.75) and (3.76) can be rearranged and written as

$$\begin{Bmatrix} \tilde{\mathbf{u}}_{b0}'' \\ \tilde{\boldsymbol{\theta}}_{b0}'' \end{Bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{z}_{ec}'' \\ \mathbf{0} & \bar{\mathbf{I}} \end{bmatrix} \begin{Bmatrix} \tilde{\mathbf{u}}_{p0}'' \\ \tilde{\boldsymbol{\theta}}_{p0}'' \end{Bmatrix} \quad (3.77)$$

where $\mathbf{I}$ is a 3×3 identity matrix and $\mathbf{z}_{ec}''$ and $\bar{\mathbf{I}}$ are defined as

$$\mathbf{z}_{ec}'' = z_{ec}'' \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (3.78a)$$

and

$$\bar{\mathbf{I}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (3.78b)$$

Utilizing the constraint given in Eq. (3.77), the displacement at a point in the stiffeners can be defined in terms of unknowns associated with the plate as follows

$$\tilde{\mathbf{u}}_b'' = \begin{bmatrix} \mathbf{I} & \mathbf{z}_{ec}'' + \tilde{\mathbf{Z}}\bar{\mathbf{I}} \end{bmatrix} \begin{Bmatrix} \tilde{\mathbf{u}}_{p0}'' \\ \tilde{\boldsymbol{\theta}}_{p0}'' \end{Bmatrix}. \quad (3.79)$$

3.2.1 Mass Matrix

The construction of the mass matrix is based on the following approximation of the displacement field

$$\begin{Bmatrix} u_{p0}(x'', y'') \\ v_{p0}(x'', y'') \\ w_{p0}(x'', y'') \end{Bmatrix} = \sum_{i=1}^3 \xi_i \begin{Bmatrix} u_{p0i}(x'', y'') \\ v_{p0i}(x'', y'') \\ w_{p0i}(x'', y'') \end{Bmatrix} \quad (3.80a)$$

and

$$\begin{Bmatrix} \theta_{px''} \\ \theta_{py''} \\ \theta_{pz''} \end{Bmatrix} = \sum_{i=1}^3 \xi_i \begin{Bmatrix} \theta_{px''i} \\ \theta_{py''i} \\ \theta_{pz''i} \end{Bmatrix}. \quad (3.80b)$$

where

$$\xi_i = \frac{(c_i + b_i x'')}{2A_p} \quad (3.81)$$

with

$$b_i = y_i'' - y_k'' \quad (3.82)$$

$$c_i = x_j'' y_k'' - x_k'' y_j''$$

$$(i = 1, 2, 3; j = 2, 3, 1; k = 3, 1, 2) \quad (3.83)$$

and A_p is the in-plane area of the plate element.

Combining the finite element approximation given in Eq. (3.80) along with the displacement field defined in Eq. (3.71), the displacement field inside the plate can be approximated as

$$\mathbf{u}''_p = \mathbf{N}''_{mp} \mathbf{v}''_p \quad (3.84)$$

where $\mathbf{N}''_{mp}$ is the matrix of interpolation functions and $\mathbf{v}''_p$ is the vector of nodal unknowns associated with the plate. In the finite element field approximations of the stiffener component, the interpolation functions approximating the displacements and rotations of the triangular plate element, Eq. (3.84), will be utilized via the constraint conditions given in Eq. (3.79) and appropriate nodal vector transformations between the $(\tilde{x}'' - \tilde{y}'')$ and $(x'' - y'')$ planes. The shape matrices in Eq. (3.84) are constructed by choosing the element coordinate system $(x'' - y'')$ as the basis for the interpolation function. If the $(\tilde{x}'' - \tilde{y}'')$ coordinate system is chosen as the basis for these interpolation functions, Eq. (3.84) can be rewritten as

$$\tilde{\mathbf{u}}''_p = \tilde{\mathbf{N}}''_{mp} \tilde{\mathbf{v}}''_p \quad (3.85)$$

In Eq. (3.85), the shape matrices, $\tilde{\mathbf{N}}''_{mp}$, are defined exactly in the same manner as $\mathbf{N}''_{mp}$ and the only difference between these two appears in the choice of their base coordinate systems.

The nodal unknown vector, $\tilde{\mathbf{v}}_p''$ is also transformed from the $(\tilde{x}'' - \tilde{y}'')$ plane to the $(x'' - y'')$ plane in order to assemble them with the plate element. This is achieved through the transformation

$$\tilde{\mathbf{v}}_p'' = \mathbf{A}_v \mathbf{v}_p'' \quad (3.86)$$

where the components of $\mathbf{A}_v$ are given as

$$\mathbf{A}_v = \begin{bmatrix} \mathbf{A}_{v6} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{v6} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{v6} \end{bmatrix} \quad (3.87)$$

with

$$\mathbf{A}_{v6} = \begin{bmatrix} \mathbf{A}_{v3} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{v3}^T \end{bmatrix} \quad (3.88)$$

and

$$\mathbf{A}_{v3} = \begin{bmatrix} \cos \phi_b & \sin \phi_b & 0 \\ -\sin \phi_b & \cos \phi_b & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (3.89)$$

Combining Eqs. (3.79), (3.85) and (3.86), and utilizing the following transformation

$$\tilde{\mathbf{u}}_b'' = \mathbf{A}_{v3} \mathbf{u}_b'' \quad (3.90)$$

the displacement field inside the plate can be approximated as

$$\mathbf{u}_b'' = \mathbf{N}_{mb}'' \mathbf{v}_p'' \quad (3.91)$$

where

$$\mathbf{N}_{mb}'' = \mathbf{A}_{v3}^T \left[\mathbf{I} \quad \mathbf{z}_{ec}'' + \tilde{\mathbf{Z}} \bar{\mathbf{I}} \right] \tilde{\mathbf{N}}_{mp}'' \mathbf{A}_v \quad (3.92)$$

Having defined the displacements inside stiffened plate, which comprises of the plate and the stiffeners, in terms of the unknowns associated with the plate only, the work done by the inertial, internal and external forces can be expressed solely as a function of $\mathbf{v}_p''$. Therefore, any additional unknowns for the beam element are not required in the present formulation.

In Eqs. (3.84) and (3.91) the displacements which are expressed in the element coordinate system can also be expressed in the body-fixed frame as

$$\mathbf{u}'_p = \mathbf{N}'_{mp} \mathbf{v}'_p \quad (3.93a)$$

and

$$\mathbf{u}'_b = \mathbf{N}'_{mb} \mathbf{v}'_p \quad (3.93b)$$

where $\mathbf{N}'_{mp}$ and $\mathbf{N}'_{mb}$ are the matrix of interpolation functions for the plate and beam, respectively, defined in the body-fixed system. Introducing a transformation matrix, $\mathbf{Z}$, between the element coordinate system and the body-fixed frame, the element displacement field, $\mathbf{u}''_p$ and $\mathbf{u}''_b$, is transformed to the (x', y', z') frame as

$$\mathbf{u}'_p = \mathbf{Z} \mathbf{u}''_p = \mathbf{Z} \mathbf{N}''_{mp} \mathbf{v}''_p \quad (3.94a)$$

and

$$\mathbf{u}'_b = \mathbf{Z} \mathbf{u}''_b = \mathbf{Z} \mathbf{N}''_{mb} \mathbf{v}''_p. \quad (3.94b)$$

Furthermore, using the same transformation matrix $\mathbf{Z}$, the displacements and rotations of the element at each of the three nodes can be transformed to the (x', y', z') frame with a compact transformation matrix, $\mathbf{Z}^*$, in the form

$$\mathbf{v}'_p = \mathbf{Z}^* \mathbf{v}''_p \quad (3.95)$$

in which $\mathbf{Z}^*$ is a square transformation matrix of order 18 containing the components of the element transformation matrix associated with nodal displacement rotation vectors defined at each node of the element. Note that the transformation matrix $\mathbf{Z}^*$ is orthonormal. Hence, inverting (3.95) and substituting into (3.94) leads to

$$\mathbf{N}'_{mp} = \mathbf{Z} \mathbf{N}''_{mp} \mathbf{Z}^{*T} \quad (3.96a)$$

and

$$\mathbf{N}'_{mb} = \mathbf{Z} \mathbf{N}''_{mb} \mathbf{Z}^{*T}. \quad (3.96b)$$

The element mass matrices and the load vectors arising from the presence of Coriolis and centrifugal forces appearing in the expression for the virtual work due to inertial forces are expressed as

$$\mathbf{M} = \mathbf{M}^p + \mathbf{M}^b \quad (3.97)$$

where

$$\mathbf{M}^p = \begin{bmatrix} \mathbf{M}^p_{RR} & \mathbf{M}^p_{R\omega} & \mathbf{M}^p_{Rv'} \\ & \mathbf{M}^p_{\omega\omega} & \mathbf{M}^p_{\omega v'} \\ sym. & & \mathbf{M}^p_{v'v'} \end{bmatrix} \quad (3.98a)$$

And

$$\mathbf{M}^b = \begin{bmatrix} \mathbf{M}^b_{RR} & \mathbf{M}^b_{R\omega} & \mathbf{M}^b_{Rv'} \\ & \mathbf{M}^b_{\omega\omega} & \mathbf{M}^b_{\omega v'} \\ sym. & & \mathbf{M}^b_{v'v'} \end{bmatrix} \quad (3.98b)$$

with

$$\begin{aligned}
\mathbf{M}_{RR}^p &= \rho {}^tV_p \mathbf{I} , & \mathbf{M}_{R\omega}^p &= -\rho \mathbf{T} \int_{{}^tV_p} \tilde{\mathbf{r}}' d{}^tV \mathbf{T}^T , \\
\mathbf{M}_{Rv'}^p &= \rho \mathbf{T} \mathbf{Z} \int_{{}^tV_p} \mathbf{N}_{mp}'' d{}^tV \mathbf{Z}^{*T} , & \mathbf{M}_{\omega\omega}^p &= -\rho \mathbf{T} \mathbf{Z} \int_{{}^tV_p} \tilde{\mathbf{r}}'' \tilde{\mathbf{r}}'' d{}^tV \mathbf{Z}^T \mathbf{T}^T , \\
\mathbf{M}_{\omega v'}^p &= \rho \mathbf{T} \mathbf{Z} \int_{{}^tV_p} \tilde{\mathbf{r}}'' \mathbf{N}_{mp}'' d{}^tV \mathbf{Z}^{*T} , & \mathbf{M}_{v'v'}^p &= \rho \mathbf{Z}^* \int_{{}^tV_p} \mathbf{N}_{mp}''^T \mathbf{N}_{mp}'' d{}^tV \mathbf{Z}^{*T} ,
\end{aligned} \tag{3.99a-l}$$

$$\begin{aligned}
\mathbf{f}_1^p &= 2\rho \mathbf{T} \mathbf{Z} \tilde{\mathbf{\Omega}}'' \int_{{}^tV_p} \mathbf{N}_{mp}'' d{}^tV \dot{\mathbf{v}}'' , & \mathbf{f}_2^p &= \rho \mathbf{T} \tilde{\mathbf{\Omega}}'' \tilde{\mathbf{\Omega}}'' \int_{{}^tV_p} \tilde{\mathbf{r}}' d{}^tV , \\
\mathbf{g}_1^p &= 2\rho \mathbf{T} \mathbf{Z} \int_{{}^tV_p} \tilde{\mathbf{r}}'' \tilde{\mathbf{\Omega}}'' \mathbf{N}_{mp}'' d{}^tV \dot{\mathbf{v}}'' , & \mathbf{g}_2^p &= \rho \mathbf{T} \mathbf{Z} \int_{{}^tV_p} \tilde{\mathbf{r}}'' \tilde{\mathbf{\Omega}}'' \tilde{\mathbf{\Omega}}'' \tilde{\mathbf{r}}'' d{}^tV , \\
\mathbf{h}_1^p &= 2\rho \mathbf{Z}^* \int_{{}^tV_p} \mathbf{N}_{mp}''^T \tilde{\mathbf{\Omega}}'' \mathbf{N}_{mp}'' d{}^tV \dot{\mathbf{v}}'' , & \mathbf{h}_2^p &= \rho \mathbf{Z}^* \int_{{}^tV_p} \mathbf{N}_{mp}''^T \tilde{\mathbf{\Omega}}'' \tilde{\mathbf{\Omega}}'' \mathbf{r}'' d{}^tV .
\end{aligned}$$

$$\begin{aligned}
\mathbf{M}_{RR}^b &= \rho {}^tV_b \mathbf{I} , & \mathbf{M}_{R\omega}^b &= -\rho \mathbf{T} \int_{{}^tV_b} \tilde{\mathbf{r}}' d{}^tV \mathbf{T}^T , \\
\mathbf{M}_{Rv'}^b &= \rho \mathbf{T} \mathbf{Z} \int_{{}^tV_b} \mathbf{N}_{mb}'' d{}^tV \mathbf{Z}^{*T} , & \mathbf{M}_{\omega\omega}^b &= -\rho \mathbf{T} \mathbf{Z} \int_{{}^tV_b} \tilde{\mathbf{r}}'' \tilde{\mathbf{r}}'' d{}^tV \mathbf{Z}^T \mathbf{T}^T , \\
\mathbf{M}_{\omega v'}^b &= \rho \mathbf{T} \mathbf{Z} \int_{{}^tV_b} \tilde{\mathbf{r}}'' \mathbf{N}_{mb}'' d{}^tV \mathbf{Z}^{*T} , & \mathbf{M}_{v'v'}^b &= \rho \mathbf{Z}^* \int_{{}^tV_b} \mathbf{N}_{mb}''^T \mathbf{N}_{mb}'' d{}^tV \mathbf{Z}^{*T} ,
\end{aligned} \tag{3.100a-l}$$

$$\begin{aligned}
\mathbf{f}_1^b &= 2\rho \mathbf{T} \mathbf{Z} \tilde{\mathbf{\Omega}}'' \int_{{}^tV_b} \mathbf{N}_{mb}'' d{}^tV \dot{\mathbf{v}}'' , & \mathbf{f}_2^b &= \rho \mathbf{T} \tilde{\mathbf{\Omega}}'' \tilde{\mathbf{\Omega}}'' \int_{{}^tV_b} \tilde{\mathbf{r}}' d{}^tV , \\
\mathbf{g}_1^b &= 2\rho \mathbf{T} \mathbf{Z} \int_{{}^tV_b} \tilde{\mathbf{r}}'' \tilde{\mathbf{\Omega}}'' \mathbf{N}_{mb}'' d{}^tV \dot{\mathbf{v}}'' , & \mathbf{g}_2^b &= \rho \mathbf{T} \mathbf{Z} \int_{{}^tV_b} \tilde{\mathbf{r}}'' \tilde{\mathbf{\Omega}}'' \tilde{\mathbf{\Omega}}'' \tilde{\mathbf{r}}'' d{}^tV , \\
\mathbf{h}_1^b &= 2\rho \mathbf{Z}^* \int_{{}^tV_b} \mathbf{N}_{mb}''^T \tilde{\mathbf{\Omega}}'' \mathbf{N}_{mb}'' d{}^tV \dot{\mathbf{v}}'' , & \mathbf{h}_2^b &= \rho \mathbf{Z}^* \int_{{}^tV_b} \mathbf{N}_{mb}''^T \tilde{\mathbf{\Omega}}'' \tilde{\mathbf{\Omega}}'' \mathbf{r}'' d{}^tV .
\end{aligned}$$

3.2.2 Tangent Stiffness Matrix

Based on the co-rotational form of the updated Lagrangian description of motion, the principle of virtual work for the stiffened plate element is written as

$$\delta W_\sigma = \int_{{}^tV_p} \delta {}_t\mathbf{E}_p^T {}_t\mathbf{S}_p dV_p + \int_{{}^tV_b} \delta {}_t\mathbf{E}_b^T {}_t\mathbf{S}_b dV_b \tag{3.101}$$

The Green-Lagrangian strain vectors, $\mathbf{E}_p$ and $\mathbf{E}_b$, are associated with the plate and beam elements, respectively. The components of the Piola-Kirchhoff stress tensor for both

elements are contained in vectors denoted by ${}_t\mathbf{S}_p$ for the plate element and ${}_t\mathbf{S}_b$ for the beam element. The volumes of the elements at current configuration are denoted by tV_p for the plate element and tV_b for the beam element. The virtual work done by the internal forces in the plate and the beam elements and are represented by

$$\delta W_\beta = \int_{{}^tV_\beta} \delta {}_t\mathbf{E}_\beta^T {}_t\mathbf{S}_\beta dV_\beta \quad (3.102)$$

where the subscript β identifies the element type, and no summation is intended on β . In order to achieve the solution in an incremental form, the Piola–Kirchhoff stress vector, ${}_t\mathbf{S}_\beta$, is represented in incremental form as

$${}_t\mathbf{S}_\beta = {}^t\boldsymbol{\sigma}_\beta + {}_t\Delta\boldsymbol{\sigma}_\beta \quad (3.103)$$

in which ${}^t\boldsymbol{\sigma}_\beta$ denotes the Cauchy (true) stress vector evaluated at time t and ${}_t\Delta\boldsymbol{\sigma}_\beta$ represents the incremental unknown Piola–Kirchhoff stress vector occurring during the time increment Δt . The Green–Lagrangian strain vector, ${}_t\mathbf{E}_\beta$, can be decomposed into its linear and nonlinear components as

$${}_t\mathbf{E}_\beta = {}_t\mathbf{E}_{\beta L} + {}_t\mathbf{E}_{\beta NL} \quad (3.104)$$

with

$$\delta {}_t\mathbf{E}_\beta = \delta {}_t\mathbf{E}_{\beta L} + \delta {}_t\mathbf{E}_{\beta NL} . \quad (3.105)$$

Substituting (3.103) and (3.105) into the internal virtual work expression given in (3.102), while expanding the resulting terms, leads to

$$\delta W_\beta = \int_{{}^tV_\beta} \delta {}_t\mathbf{E}_{\beta L}^T \mathbf{Q}_\beta {}_t\mathbf{E}_{\beta L} d{}^tV_\beta + \int_{{}^tV_\beta} \delta {}_t\mathbf{E}_{\beta NL}^T {}^t\boldsymbol{\sigma}_\beta d{}^tV_\beta + \int_{{}^tV_\beta} \delta {}_t\mathbf{E}_{\beta L}^T {}_t\boldsymbol{\sigma}_\beta d{}^tV_\beta \quad (3.106)$$

Note that prior to the derivation (3.106), the following linearization is performed on the integral terms involving the higher-order terms of the unknown incremental vectors, i.e.,

$$\delta_t \mathbf{E}_t^T \Delta \boldsymbol{\sigma} \cong \delta_t \mathbf{E}_{L_t}^T \Delta \boldsymbol{\sigma} \quad (3.107)$$

and

$$_t \Delta \boldsymbol{\sigma} = \mathbf{Q}_t \mathbf{E} \cong \mathbf{Q}_t \mathbf{E}_L \quad (3.108)$$

where $\mathbf{Q}$ denotes the linear constitutive matrix for an orthotropic material relating the incremental strains to incremental stresses.

Substituting the expressions for δW_p and δW_b from Eq. (3.106) into Eq. (3.101) results in

$$\begin{aligned} \delta W_\sigma = & \int_{V_p} \delta_t \mathbf{E}_{pL}^T \mathbf{Q}_p {}_t \mathbf{E}_{pL} dV_p + \int_{V_b} \delta_t \mathbf{E}_{bL}^T \mathbf{Q}_b {}_t \mathbf{E}_{bL} dV_b \\ & \int_{V_p} \delta_t \mathbf{E}_{pNL}^T {}_t \boldsymbol{\sigma}_p dV_p + \int_{V_b} \delta_t \mathbf{E}_{bNL}^T {}_t \boldsymbol{\sigma}_b dV_b + \\ & \int_{V_p} \delta_t \mathbf{E}_{pL}^T {}_t \boldsymbol{\sigma}_p dV_p + \int_{V_b} \delta_t \mathbf{E}_{bL}^T {}_t \boldsymbol{\sigma}_b dV_b \end{aligned} \quad (3.109)$$

Considering the geometrical and material properties for both components of the stiffened plate element, the terms belonging to the plate element in Eq. (3.109) can be represented in double-integral forms, with the integration taken across the surface and through the thickness of the plate element. Similarly, the terms belonging to the beam (stiffener) element can be shown in double-integral forms, with the integration taken along the beam axis and across the beam cross-sectional area. These integral manipulations on Eq. (3.109) leads to

$$\begin{aligned}
\delta W_\sigma = & \sum_{k=1}^{K_p} \int_{A_p} \int_{z_{k-1}}^{z_k} \delta {}_t\mathbf{E}_{pL}^T \bar{\mathbf{Q}}_p^{(k)} {}_t\mathbf{E}_{pL} dz d {}^tA_p + \int_{l_p} \sum_{i=1}^{K_b} \int_{A_b^{(i)}} \delta {}_t\mathbf{E}_{bL}^T \bar{\mathbf{Q}}_b {}_t\mathbf{E}_{bL} d {}^tA_b^{(i)} d\tilde{x}'' + \\
& \sum_{k=1}^{K_p} \int_{A_p} \int_{z_{k-1}}^{z_k} \delta {}_t\mathbf{E}_{pNL}^T {}^t\boldsymbol{\sigma}_p^{(k)} dz d {}^tA_p + \int_{l_p} \sum_{i=1}^{K_b} \int_{A_b^{(i)}} \delta {}_t\mathbf{E}_{bNL}^T {}^t\boldsymbol{\sigma}_b d {}^tA_b^{(i)} d\tilde{x}'' + \quad (3.110) \\
& \sum_{k=1}^{K_p} \int_{A_p} \int_{z_{k-1}}^{z_k} \delta {}_t\mathbf{E}_{pL}^T {}^t\boldsymbol{\sigma}_p^{(k)} dz d {}^tA_p + \int_{l_p} \sum_{i=1}^{K_b} \int_{A_b^{(i)}} \delta {}_t\mathbf{E}_{bL}^T {}^t\boldsymbol{\sigma}_b d {}^tA_b^{(i)} d\tilde{x}''
\end{aligned}$$

The linear and nonlinear parts of the Green–Lagrangian strain vector for the plate element are defined as

$${}_t\mathbf{E}_{pL} = \begin{Bmatrix} u_{p0,x''} \\ v_{p0,y''} \\ u_{p0,y''} + v_{p0,x''} \\ w_{p0,x''} + \theta_{py''} \\ w_{p0,y''} + \theta_{px''} \end{Bmatrix} + z'' \begin{Bmatrix} \theta_{py'',x''} \\ \theta_{px'',y''} \\ \theta_{px'',x''} + \theta_{py'',y''} \\ 0 \\ 0 \end{Bmatrix} \quad (3.111a)$$

and

$${}_t\mathbf{E}_{pNL} = \frac{1}{2} \begin{Bmatrix} u_{p0,x''}^2 + v_{p0,x''}^2 + w_{p0,x''}^2 \\ u_{p0,y''}^2 + v_{p0,y''}^2 + w_{p0,y''}^2 \\ 2(u_{p0,x''}u_{p0,y''} + v_{p0,x''}v_{p0,y''} + w_{p0,x''}w_{p0,y''}) \\ 0 \\ 0 \end{Bmatrix} \quad (3.111b)$$

The plate element is assumed to be much stiffer in in-plane bending and in-plane shear deformations than the beam element. Under this assumption, the resultant transverse shear and bending deformations of the beam associated with the in-plane shear and bending deformations of the plate, respectively can be omitted. Using this assumption and the displacement definition for a stiffener given in Eq. (3.72), the incremental Green strain components for a stiffener can be written as

$${}_t\tilde{\mathbf{E}}''_{bL} = \begin{Bmatrix} \tilde{u}_{b0,\tilde{x}''} \\ 0 \\ \tilde{w}_{b0,\tilde{x}''} + \theta_{by''} \end{Bmatrix} + z'' \begin{Bmatrix} \tilde{\theta}_{by'',\tilde{x}''} \\ -\tilde{\theta}_{bx'',\tilde{x}''} \\ 0 \end{Bmatrix} + \tilde{y}'' \begin{Bmatrix} 0 \\ 0 \\ \tilde{\theta}_{bx'',\tilde{x}''} \end{Bmatrix} \quad (3.112a)$$

$${}_t\tilde{\mathbf{E}}''_{pNL} = \frac{1}{2} \begin{Bmatrix} \tilde{u}_{b0,\tilde{x}''}^2 + \tilde{v}_{b0,\tilde{x}''}^2 + \tilde{w}_{b0,\tilde{x}''}^2 \\ 0 \\ 0 \end{Bmatrix} + z'' \begin{Bmatrix} \tilde{u}_{b0,\tilde{x}''}\tilde{\theta}_{by'',\tilde{x}''} - \tilde{v}_{b0,\tilde{x}''}\tilde{\theta}_{bx'',\tilde{x}''} \\ 0 \\ 0 \end{Bmatrix}. \quad (3.112b)$$

Substituting from Eqs. (3.111) and (3.112) for the incremental strain components associated with the plate and the beam (stiffener) elements, respectively into Eq. (3.110) and performing integrations through the thickness of the plate and across the cross-sectional area of the beam lead to

$$\begin{aligned} \delta W_\sigma = & \int_{t_{A_p}} \delta {}_t\mathbf{e}_{pL}^T \mathbf{C}_p {}_t\mathbf{e}_{pL} d^tA_p + \int_{t_{l_b}} \delta {}_t\tilde{\mathbf{e}}_{bL}^T \tilde{\mathbf{C}}_b {}_t\tilde{\mathbf{e}}_{bL} d\tilde{x}'' + \\ & \int_{t_{A_p}} \delta {}_t\mathbf{e}_{pNL}^T {}^t\mathbf{s}_p d^tA_p + \int_{t_{l_b}} \delta {}_t\tilde{\mathbf{e}}_{bNL}^T {}^t\tilde{\mathbf{s}}_b d\tilde{x}'' + \\ & \int_{t_A} \delta {}_t\mathbf{e}_{pL}^T {}^t\mathbf{s}_p d^tA_p + \int_{t_{l_b}} \delta {}_t\tilde{\mathbf{e}}_{bL}^T {}^t\tilde{\mathbf{s}}_b d\tilde{x}'' + \end{aligned} \quad (3.113)$$

where the vector is ${}_t\mathbf{e}_{pL}$, ${}_t\mathbf{e}_{pNL}$, ${}^t\mathbf{s}_p$ and the material property matrix, $\mathbf{C}_p$, are the same as those given in Eqs. (3.18), (3.20), (3.16) and (3.21), respectively. The incremental resultant strains from the stiffener, ${}_t\tilde{\mathbf{e}}_{bL}$ and ${}_t\tilde{\mathbf{e}}_{bNL}$ are defined as

$${}_t\tilde{\mathbf{e}}_{bL}^T = \left\{ \tilde{u}_{b0,\tilde{x}''}, \quad \tilde{w}_{b0,\tilde{x}''}, \quad \tilde{\theta}_{bx'',\tilde{x}''}, \quad \tilde{\theta}_{by'',\tilde{x}''} \right\} \quad (3.114)$$

$${}_t\tilde{\mathbf{e}}_{bNL}^T = \left\{ \frac{1}{2} \left(\tilde{u}_{b0,\tilde{x}''}^2 + \tilde{v}_{b0,\tilde{x}''}^2 + \tilde{w}_{b0,\tilde{x}''}^2 \right), \quad 0, \quad 0, \quad \tilde{u}_{b0,\tilde{x}''}\tilde{\theta}_{by'',\tilde{x}''} - \tilde{v}_{b0,\tilde{x}''}\tilde{\theta}_{bx'',\tilde{x}''} \right\} \quad (3.115)$$

The resultant stress vector ${}^t\tilde{\mathbf{s}}_b$ for the stiffener is defined as

$${}^t\tilde{\mathbf{s}}_b^T = \left\{ {}^t\tilde{N}_{bx''}, \quad {}^t\tilde{N}_{bz''}, \quad {}^t\tilde{M}_{bx''}, \quad {}^t\tilde{M}_{by''} \right\} \quad (3.116)$$

with

$$\left\{ {}^t\tilde{N}_{bx''}, {}^t\tilde{N}_{bz''} \right\} = \sum_{i=1}^{K_b} \int_{A_b^{(i)}} \left({}^t\tilde{\sigma}_{bx''x''}, {}^t\tilde{\sigma}_{bx''z''} \right) dA_b^{(i)} \quad (3.117)$$

$$\left\{ {}^t\tilde{M}_{bx''}, {}^t\tilde{M}_{by''} \right\} = \sum_{i=1}^{K_b} \int_{A_b^{(i)}} \left(\tilde{y}'' {}^t\tilde{\sigma}_{bx''z''} - \tilde{z}'' {}^t\tilde{\sigma}_{bx''y''}, \tilde{z}'' {}^t\tilde{\sigma}_{bx''x''} \right) dA_b^{(i)} \quad (3.118)$$

The material property matrix for a stiffener can be written as

$$\tilde{\mathbf{C}}_b = \begin{bmatrix} \tilde{C}_{11} & \tilde{C}_{12} & \tilde{C}_{13} & \tilde{C}_{14} \\ & \tilde{C}_{22} & \tilde{C}_{23} & \tilde{C}_{24} \\ & & \tilde{C}_{33} & \tilde{C}_{34} \\ Sym & & & \tilde{C}_{44} \end{bmatrix} \quad (3.119)$$

where the explicit expressions for $\tilde{C}_{mn}$ are defined in Appendix A.

3.2.2.1 Linear Stiffness Matrix

The finite element approximation for the plate element presented in Section 3.1 utilizes the LST-3/9R-ANDES [Felippa (2003)] and MIN3 [Tessler and Hughes (1985)] formulations for the membrane and bending stiffness matrix. Note that the LST-3/9R-ANDES element is based on assumed strains and therefore the finite element approximation for the displacement field inside the plate element is not available. The stiffened plate formulation requires the finite element approximation for the displacement field inside the plate element since the displacements for the beam element are extrapolated from the plate displacements. Therefore, the LST-3/9R-ANDES formulation can not be utilized for stiffened plate. A linearly varying in-plane displacement field is assumed for the stiffened plate such that

$$\begin{Bmatrix} u_{p0}(x'', y'') \\ v_{p0}(x'', y'') \end{Bmatrix} = \sum_{i=1}^3 \xi_i \begin{Bmatrix} u_{p0i}(x'', y'') \\ v_{p0i}(x'', y'') \end{Bmatrix} \quad (3.120)$$

where u_{p0i} and v_{p0i} are the in-plane displacement unknowns at the corners of the plate element.

The transverse displacement and the out-of-plane rotational are approximated based on the C^0 -anisoparametric interpolation functions proposed by Tessler and Hughes (1985). Thus, the incremental transverse and out-of-plane rotational components on the mid-surface of the element are approximated as

$$w_{p0}(x'', y'') = \sum_{i=1}^3 N_i w_{p0i}(x'', y'') + \sum_{i=4}^6 N_i w_{p0i}^+(x'', y'') \quad (3.121)$$

where

$$\begin{aligned} N_i &= \xi_i (2\xi_i - 1) & N_{i+3} &= 4\xi_i \xi_k \\ (i &= 1, 2, 3; \quad k = 2, 3, 1) \end{aligned} \quad (3.122)$$

and

$$\begin{Bmatrix} \theta_{px''}(x'', y'') \\ \theta_{py''}(x'', y'') \end{Bmatrix} = \sum_{i=1}^3 \xi_i \begin{Bmatrix} \theta_{px''i}(x'', y'') \\ \theta_{py''i}(x'', y'') \end{Bmatrix} \quad (3.123)$$

Whereas the nodal rotations, $\theta_{px''i}$ and $\theta_{py''i}$, are defined only at the corner, the transverse displacement are defined at the corners, w_{p0i} , and at the mid-side nodes, w_{p0i}^+ , as well.

As discussed in Section 3.1 the mid-side nodes can be condensed out based on the assumption that the transverse shear strain is constant along the edges resulting in

$$w_{p0} = \sum_{i=1}^3 \left(\xi_i w_{p0i} + L_i \theta_{px''i} + M_i \theta_{py''i} \right) \quad (3.124)$$

where

$$L_i = \frac{1}{8}(b_k N_{i+3} - b_j N_{k+3}) \quad M_i = \frac{1}{8}(a_j N_{k+3} - a_k N_{i+3})$$

$$(i = 1, 2, 3; j = 2, 3, 1; k = 3, 1, 2). \quad (3.125)$$

Combining the finite element approximation given in Eqs. (3.123) and (3.124) along with the displacement field defined in Eq. (3.71), the displacement field inside the plate can be approximated as

$$\mathbf{u}_p'' = \mathbf{N}_p'' \mathbf{v}_p'' \quad (3.126)$$

where $\mathbf{N}_p''$ is the matrix of interpolation functions and $\mathbf{v}_p''$ is the vector of nodal unknowns associated with the plate. The shape matrices in Eq. (3.126) are constructed by choosing the element coordinate system $(x'' - y'')$ as the basis for the interpolation and the corresponding form in the $(\tilde{x}'' - \tilde{y}'')$ coordinate system written as

$$\tilde{\mathbf{u}}_p'' = \tilde{\mathbf{N}}_p'' \tilde{\mathbf{v}}_p'' \quad (3.127)$$

Utilizing Eqs. (3.79), (3.127) the displacement field inside the plate can be approximated as

$$\tilde{\mathbf{u}}_b'' = \tilde{\mathbf{N}}_b'' \mathbf{v}_p'' \quad (3.128)$$

where

$$\tilde{\mathbf{N}}_b'' = [\mathbf{I} \quad \mathbf{z}_{ec}'' + \tilde{\mathbf{Z}}\mathbf{I}] \tilde{\mathbf{N}}_p'' \mathbf{A}_v \quad (3.129)$$

In order to generate the stiffness matrix, the linear part of the strain–displacement relations for both components of the stiffened plate element need to be expressed using

finite element approximations. The linear incremental strain resultant vector, ${}_t\mathbf{e}_{pL}$, for the plate element can be written as

$${}_t\mathbf{e}_{pL} = \mathbf{B}_p'' \mathbf{u}_p'' \quad (3.130)$$

where $\mathbf{B}_p''$ constitutes the strain–displacement transformation matrix. The explicit form of $\mathbf{B}_p''$ is given in Appendix A. Combining Eqs. (3.127) and (3.130) the strain field inside the plate element can be expressed in terms of the nodal unknowns as

$${}_t\mathbf{e}_{pL} = \mathbf{B}_p'' \mathbf{N}_p'' \mathbf{v}_p'' \quad (3.131)$$

For the beam component, the linear incremental resultant strain vector, ${}_t\tilde{\mathbf{e}}_{bL}$, is written in matrix form as

$${}_t\tilde{\mathbf{e}}_{bL} = \tilde{\mathbf{B}}_b'' \tilde{\mathbf{u}}_b'' \quad (3.132)$$

in which the matrix $\tilde{\mathbf{B}}_b''$ is given in Appendix A.

Combining Eqs. (3.128) and (3.132), the strain field inside the beam element can be expressed in terms of the unknowns associated with the plate element as

$${}_t\tilde{\mathbf{e}}_{bL} = \tilde{\mathbf{B}}_b'' \tilde{\mathbf{N}}_b'' \mathbf{v}_p'' \quad (3.133)$$

Utilizing the form of the finite element approximation of the strain field inside the plate and beam elements given in Eqs. (3.131) and (3.133), respectively, the linear stiffness matrix for the stiffened plate element can be expressed as

$$\mathbf{K}_L = \mathbf{K}_{pL} + \mathbf{K}_{bL} \quad (3.134)$$

where

$$\mathbf{K}_{pL} = \int_{A_p} \mathbf{N}_p''^T \mathbf{B}_p''^T \mathbf{C}_p \mathbf{B}_p'' \mathbf{N}_p'' dA_p \quad (3.135a)$$

$$\mathbf{K}_{bL} = \int_{\iota_{l_b}} \tilde{\mathbf{B}}_b''^T \tilde{\mathbf{N}}_b''^T \tilde{\mathbf{C}}_b \tilde{\mathbf{B}}_b'' \tilde{\mathbf{N}}_b'' d\tilde{x}'' . \quad (3.135b)$$

3.2.2.2 Geometric Stiffness Matrix

The geometric stiffness matrix for the plate element, which can be constructed following the procedure presented in Section 3.1.2.2, has the following form

$$\mathbf{k}_{pg}'' = \int_{\iota_{A_p}} \mathbf{N}_{pm}^T \mathcal{B}^T \iota \mathbf{S}_m \mathcal{B} \mathbf{N}_{pm} d\iota A_p \quad (3.136)$$

The derivation of the geometric stiffness matrices belonging to the stiffener (beam) component requires the finite element treatment of the forth integral terms in Eq. (3.113). Similar to the finite element formulation of the plate component, the nonlinear part of the incremental strain resultant vector for the beam component, $\iota \tilde{\mathbf{e}}_{bNL}$, defined by Eq. (3.115), is written in matrix form as

$$\iota \tilde{\mathbf{e}}_{bNL} = \frac{1}{2} \tilde{\mathbf{A}}_{bNL} \tilde{\boldsymbol{\theta}}_{bNL} \quad (3.137)$$

where

$$\tilde{\mathbf{A}}_{bNL} = \begin{bmatrix} \tilde{u}_{b0,\tilde{x}''}'' & \tilde{v}_{b0,\tilde{x}''}'' & \tilde{w}_{b0,\tilde{x}''}'' & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \tilde{\theta}_{by'',\tilde{x}''} & -\tilde{\theta}_{bx'',\tilde{x}''} & 0 & -\tilde{v}_{b0,\tilde{x}''}'' & \tilde{u}_{b0,\tilde{x}''}'' \end{bmatrix} \quad (3.138)$$

and

$$\tilde{\boldsymbol{\theta}}_{bNL}^T = \left\{ \tilde{u}_{b0,\tilde{x}''}'' \quad \tilde{v}_{b0,\tilde{x}''}'' \quad \tilde{w}_{b0,\tilde{x}''}'' \quad \tilde{\theta}_{bx'',\tilde{x}''} \quad \tilde{\theta}_{by'',\tilde{x}''} \right\} \quad (3.139)$$

With $\tilde{\mathbf{A}}_{bNL}$ and $\tilde{\boldsymbol{\theta}}_{bNL}$ processing the following properties

$$\delta \tilde{\mathbf{A}}_{bNL} \tilde{\boldsymbol{\theta}}_{bNL} = \tilde{\mathbf{A}}_{bNL} \delta \tilde{\boldsymbol{\theta}}_{bNL} \text{ and } \tilde{\mathbf{A}}_{bNL}^T {}^t \tilde{\mathbf{s}}_b = {}^t \tilde{\mathbf{S}}_b \tilde{\boldsymbol{\theta}}_{bNL} \quad (3.140)$$

with

$${}^t \tilde{\mathbf{S}}_b = \begin{bmatrix} {}^t \tilde{N}_{bx''} & 0 & 0 & 0 & {}^t \tilde{M}_{by''} \\ 0 & {}^t \tilde{N}_{bx''} & 0 & -{}^t \tilde{M}_{by''} & 0 \\ 0 & 0 & {}^t \tilde{N}_{bx''} & 0 & 0 \\ 0 & -{}^t \tilde{M}_{by''} & 0 & 0 & 0 \\ {}^t \tilde{M}_{by''} & 0 & 0 & 0 & 0 \end{bmatrix} \quad (3.141)$$

Substituting for the nonlinear part of the Green strain vector and utilizing the properties stated in Eq. (3.140), the forth integral in Eq. (3.113) can be written as

$$\int_{l_b} \delta {}^t \tilde{\mathbf{e}}_{bNL}^T {}^t \tilde{\mathbf{s}}_b d\tilde{x}'' = \int_{l_b} \delta \tilde{\boldsymbol{\theta}}_{bNL}^T {}^t \tilde{\mathbf{S}}_b \tilde{\boldsymbol{\theta}}_{bNL} d\tilde{x}'' \quad (3.142)$$

The vector $\tilde{\boldsymbol{\theta}}_{bNL}$ can be expressed in the form

$$\tilde{\boldsymbol{\theta}}_{bNL} = \mathcal{B}_b \tilde{\mathbf{u}}_b'' \quad (3.143)$$

with the matrix differential operator $\mathcal{B}_b$ given in Appendix A.

Similar to the approximation made for the computation of the mass matrix, the displacement field is assumed to vary linearly inside the element. Therefore, the vector $\tilde{\boldsymbol{\theta}}_{bNL}$ in Eq (3.143) can be expressed in terms of nodal unknowns as

$$\tilde{\boldsymbol{\theta}}_{bNL} = \mathcal{B}_b \tilde{\mathbf{N}}_{mb}'' \mathbf{v}_p'' \quad (3.144)$$

where

$$\tilde{\mathbf{N}}_{mb}'' = [\mathbf{I} \quad \mathbf{z}_{ec}'' + \tilde{\mathbf{Z}}\bar{\mathbf{I}}] \tilde{\mathbf{N}}_{mp}'' \mathbf{A}_v. \quad (3.145)$$

Finally, substituting Eq. (3.144) into Eq. (3.142) leads to

$$\int_{l_b} \delta {}^t \tilde{\mathbf{e}}_{bNL}^T {}^t \tilde{\mathbf{s}}_b d\tilde{x}'' = \delta \mathbf{v}_p''^T \mathbf{k}_{bg}'' \mathbf{v}_p'' \quad (3.146)$$

where the geometric stiffness matrix, $\mathbf{k}_{bg}''$, is defined by

$$\mathbf{k}_{bg}'' = \int_{I_b} \tilde{\mathbf{N}}_{mb}''^T \mathcal{B}_b^T \tilde{\mathbf{S}}_b \mathcal{B}_b \tilde{\mathbf{N}}_{mb}'' d\tilde{x}'' . \quad (3.147)$$

3.3 Solid Wedge Element

The isoparametric element formulation is used to represent the displacement field inside the solid wedge element

$$u'_x = \sum_{i=1}^6 h_i v_x'^i \quad u'_y = \sum_{i=1}^6 h_i v_y'^i \quad u'_z = \sum_{i=1}^6 h_i v_z'^i \quad (3.148)$$

where $v_x'^i$, $v_y'^i$, and $v_z'^i$ are the nodal displacements. The interpolation functions used in (3.148) are as follows:

$$\begin{aligned} h_1 &= \xi_1 (1 + \eta), \quad h_2 = \xi_2 (1 + \eta), \quad h_3 = \xi_3 (1 + \eta), \\ h_4 &= \xi_1 (1 - \eta), \quad h_5 = \xi_2 (1 - \eta), \quad h_6 = \xi_3 (1 - \eta) \end{aligned} \quad (3.149a-f)$$

where ξ_i are the area coordinates for a triangle and η is the one-dimensional linear shape function. In matrix form, (3.148) can be expressed as

$$\mathbf{u}' = \mathbf{N}' \mathbf{v}' \quad (3.150)$$

where $\mathbf{N}'$ and $\mathbf{v}'$ are the interpolation matrix and vector of nodal displacements, respectively, expressed in the floating frame. Similarly, the position of a point within the element can also be expressed as

$$\mathbf{r}' = \mathbf{N}' \mathbf{c}' \quad (3.151)$$

where $\mathbf{c}'$ is the vector of nodal locations within the floating frame. The mass matrix, Coriolis force vector, and centrifugal force vector are constructed utilizing the

interpolation matrix, $\mathbf{N}'$, along with position vector, $\mathbf{r}'$. A full integration scheme with six integration points is employed to carry out the volume integrals.

The element tangent stiffness matrix and the internal force vector arise from the virtual work due to internal forces. The displacement field based on linear interpolation functions, given in (3.150), gives rise to shear locking. This is due to the inability of these interpolation functions to represent the bending mode and therefore the occurrence of spurious shear strain. Since the current analysis deals with large deformations and rotations within the floating frame, it is critical for the displacement field to capture the bending mode. The shortcoming of the displacement representation given in (3.150) is overcome by adding the desired higher-order modes. Therefore, in accordance with the method of incompatible modes [Wilson et al. (1973)], quadratic modes can be added to the linear displacement field. Unfortunately, when added along the triangular face of the wedge element, quadratic modes fail to capture the constant stress field. Therefore, the quadratic modes can be added only along the thickness of the wedge element, resulting in the following form of the displacement field:

$$u'_x = \sum_{i=1}^6 h_i v_x^i + (1 - \eta^2) a_x \quad (3.152a)$$

$$u'_y = \sum_{i=1}^6 h_i v_y^i + (1 - \eta^2) a_y \quad (3.152b)$$

$$u'_z = \sum_{i=1}^6 h_i v_z^i + (1 - \eta^2) a_z \quad (3.152c)$$

which in matrix form becomes

$$\mathbf{u}' = [\mathbf{N}' \quad \mathbf{N}_a] \begin{Bmatrix} \mathbf{v}' \\ \mathbf{v}_a \end{Bmatrix} \quad (3.153)$$

where $\mathbf{N}_a$ is the matrix of incompatible shape functions and $\mathbf{v}_a$ is the vector of generalized degrees of freedoms. The strain–displacement relationship can be obtained by taking the derivative of the displacement field, and its final form can be written as

$$\boldsymbol{\varepsilon} = [\mathbf{B}_v \quad \mathbf{B}_a] \begin{Bmatrix} \mathbf{v}' \\ \mathbf{v}_a \end{Bmatrix} \quad (3.154)$$

where $\mathbf{B}_v$ and $\mathbf{B}_a$ are the strain–displacement relationship matrices that operate on the nodal and generalized degrees of freedom, respectively. The first term in Eq. (2.33) is expressed as

$$\int_{V'} \delta_t \mathbf{E}_L^T \mathbf{Q}_t \mathbf{E}_L dV = \begin{Bmatrix} \mathbf{v}' \\ \mathbf{v}_a \end{Bmatrix}^T \begin{bmatrix} \mathbf{k}_{L11}^* & \mathbf{k}_{L12}^* \\ \mathbf{k}_{L12}^{*T} & \mathbf{k}_{L22}^* \end{bmatrix} \begin{Bmatrix} \mathbf{v}' \\ \mathbf{v}_a \end{Bmatrix} \quad (3.155)$$

where

$$\mathbf{k}_{L11}^* = \int_{V'} \mathbf{B}_v^T \mathbf{Q}_t \mathbf{B}_v dV, \quad \mathbf{k}_{L12}^* = \int_{V'} \mathbf{B}_v^T \mathbf{Q}_t \mathbf{B}_a dV, \quad \mathbf{k}_{L11}^* = \int_{V'} \mathbf{B}_a^T \mathbf{Q}_t \mathbf{B}_a dV. \quad (3.156a-c)$$

Although the addition of incompatible modes facilitates in capturing the bending mode, the element fails to represent the constant stress state when the element shape becomes distorted. The constant stress state for distorted elements can be captured by using the value of the Jacobian and its determinant at the center of the element for the computation of $\mathbf{B}_a$, at each of the six integration points. Since no load is associated with the generalized degrees of freedom, the stiffness matrix, $\mathbf{k}_L^*$, can be condensed statistically to a smaller matrix, such that the matrix $\mathbf{k}_L'$ in Eq. (2.37) has the following form:

$$\mathbf{k}'_L = \mathbf{k}^*_{L11} - \mathbf{k}^*_{L12} \mathbf{k}^{*-1}_{L22} \mathbf{k}^{*T}_{L12}. \quad (3.157)$$

The matrix $\mathbf{M}_\sigma$ in Eq. (2.31) has the following form:

$$\mathbf{M}_\sigma = \begin{bmatrix} \sigma_{xx} \mathbf{I} & \tau_{xy} \mathbf{I} & \tau_{xz} \mathbf{I} \\ \tau_{xy} \mathbf{I} & \sigma_{yy} \mathbf{I} & \tau_{yz} \mathbf{I} \\ \tau_{xz} \mathbf{I} & \tau_{yz} \mathbf{I} & \sigma_{zz} \mathbf{I} \end{bmatrix} \quad (3.158)$$

where $\mathbf{I}$ is the identity matrix. The matrix $\mathbf{G}$ is derived from the displacement field given in (3.150) and therefore incompatible modes are not used in the computation of the geometric stiffness matrix.

3.4 One Dimensional Elements

Although the elements developed in the previous sections facilitate in modeling critical components in a detailed manner, it is more practical to use simple and computationally inexpensive one-dimensional elements for certain components. For instance rotor blades have balance weights, pitch links and lead-lag dampers which can be modeled using one-dimensional mass, spring and damper elements, respectively. Therefore, such elements have also been included in the element library.

The one-dimensional elements have two nodes with three translational degrees of freedom at each node. The displacement field for these elements can be approximates as

$$u'_x = \sum_{i=1}^2 h_i v_x^i \quad u'_y = \sum_{i=1}^2 h_i v_y^i \quad u'_z = \sum_{i=1}^2 h_i v_z^i \quad (3.159)$$

where v_x^{ii} , v_y^{ii} , and v_z^{ii} are the nodal displacements and h_i represent the linear interpolation function for line coordinates. Therefore, the vector of unknowns given in Eq. (2.26) has the following form

$$\mathbf{b}^T = \{R_{0x}, R_{0y}, R_{0z}, \omega_x, \omega_y, \omega_z, v_x^{i1}, v_y^{i1}, v_z^{i1}, v_x^{i2}, v_y^{i2}, v_z^{i2}\} \quad (3.160)$$

The mass matrix, Coriolis force vector, and centrifugal force vector are then constructed utilizing the displacement field given in Eq. 3.159. The stiffness matrix of a spring element is also based on the displacement approximation given in Eq. 3.159. The corotational procedure that is used for the computation of the internal force takes a simplified form and it is defined as the product of the spring stiffness and the difference between the current and the original length of the spring. This simple form is valid even when the spring undergoes large rigid body displacements and rotations. The concept of proportional or Rayleigh damping is often used to compute the damping matrix in order to represent viscous damping effects. In the present analysis the damping matrix is assumed to be proportional to the stiffness matrix. The dynamic equilibrium equation for the one-dimensional elements can be expressed as

$$\mathbf{M}\ddot{\mathbf{b}} = \mathbf{f}_b^p - \mathbf{f}_b^\sigma - \mathbf{f}_b^p - \mathbf{f}_b^c \quad (3.161)$$

where $\mathbf{M}$ is the mass matrix and $\mathbf{f}_b^p$ consists of the centrifugal and Coriolis forces. The vectors $\mathbf{f}_b^\sigma$ and $\mathbf{f}_b^p$ represent the force vector due to internal and external forces. The damping force vector, $\mathbf{f}_b^c$, has the following form

$$\mathbf{f}_b^c = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{C} \end{bmatrix} \dot{\mathbf{b}} \quad (3.162)$$

where $\mathbf{C}$ is the structural damping matrix.

4. KINEMATIC CONSTRAINTS

The equation of motion of a single flexible body is given by Eq. (2.56). Thus Eq. (2.56) can be employed to model a body undergoing large rigid body motion along with large elastic deformations. A majority of the engineering applications can be analyzed using the single body approach. However, various mechanical and structural systems such as automobiles, space structures, robotics, mechanisms and rotor blades require each component of the mechanical system to be modeled as a separate body or subsystem. These subsystems are connected to each other through a variety of joints. Therefore, although each subsystem is allowed to undergo large displacement and elastic deformation, their motion is kinematically constrained based on their connectivity to other bodies. The complicated task of analyzing the motion of the multibody system is split into two parts. The first task is to devise an approach that can be used to represent the physics of each subsystem or component. The equations of motion given in Eq. (2.56), which are in differential form, can be used for this purpose. Next, a method to model the joint mechanism has to be developed. The joints dictate the motion of the bodies, which are connected to it, in a specific way. This phenomenon can be incorporated in the model using a set of algebraic equations which are termed as constraint equations. Therefore, the equations of motion, Eq. (2.57), for the individual flexible bodies are coupled by employing constraint equations that can represent a variety of kinematic joints. The following sections deal with the implementation of these constraint equations for joints between flexible bodies.

Fig. 4.1 Schematic of a rigid clamp attached to a flexible body.

Fig 4.2 Kinematics of a point attached to a flexible body on the rigid clamp.

4.1. Rigid Clamps

Multibody dynamic analysis has been mostly utilized to analyze a system of interconnected rigid bodies. The formulation of constraint equations for rigid multibody dynamics is fairly well established. In the present analysis since the joints are to be introduced among flexible bodies, the derivation of the constraint equations can become fairly complicated, especially for certain joints such as revolute and cylindrical joints. Note that the nodal displacements of the flexible bodies are expressed in their respective floating frames. The motion dictated by the joints, however, is prescribed in the inertial frame. Therefore, an additional coordinate system referred to as the intermediate joint coordinate system, which is rigidly attached to an infinitesimal volume on the flexible body, is often used in flexible multibody formulations. A similar approach has been adopted in the current study.

As shown in Fig. 2.1, a rigid, massless clamp is attached to specific nodes of each flexible body prior to introducing kinematic joints between the flexible bodies. As shown in Fig. 2.2, each clamp has its own frame, $(\hat{x}, \hat{y}, \hat{z})$. The clamp position and orientation, with respect to the body, are denoted by $\mathbf{r}'_{cl}$ and $\boldsymbol{\omega}'_{cl}$, respectively. The position of a point B on the clamp can be expressed with respect to the body-fixed frame, (x', y', z') , as

$$\mathbf{r}' = \mathbf{r}'_{cl} + \mathbf{H}\hat{\mathbf{r}} \quad (4.1)$$

where $\hat{\mathbf{r}}$ denotes the position of the point B in the clamp-fixed frame, $(\hat{x}, \hat{y}, \hat{z})$, and $\mathbf{H}$ is the transformation matrix between the clamp-fixed and body-fixed frames.

In order to eliminate any relative rigid-body motion between the rigid clamp and the flexible body, at least three nodes of the solid elements must be attached to the clamp. At each node, three constraint equations are required to ensure that the i^{th} node translates with the corresponding point on the clamp. They are expressed in terms of vectors as

$$\Phi^{(cl)} \equiv \mathbf{r}' - \mathbf{u}'_i = \mathbf{0} . \quad (4.2)$$

Unlike solid elements, a single node is sufficient to rigidly attach the clamp to a plate element since plate elements possess both translational and rotational degrees of freedom. This requires, in addition to Eq. (4.2), three constraint equations to suppress any relative rotation between the clamp and the flexible body. These constraint equations are expressed in terms of vectors as

$$\Phi^{(cr)} \equiv \boldsymbol{\omega}'_{cl} - \boldsymbol{\theta}'_i = \mathbf{0} . \quad (4.3)$$

Introduction of the intermediate clamp-fixed frame facilitates implementation of the kinematic joint constraints. After attaching the clamps to each flexible body, they are attached to each other through the appropriate kinematic joint constraints. The coupling of the equation of motion, Eq. (2.56), with the constraint equations is implemented through the coordinate partitioning methods [Wehage and Haug (1982)]. The details of the coordinate partitioning method along with the time marching algorithm are presented in Chapter 5. As will be shown in Chapter 5, the implementation of the constraint equation requires the Taylor series expansion of these equations. Therefore, the Jacobian matrix that appears in the Taylor series expansion needs to be evaluated, which is expressed as part of the incremental form of the constraint equation. The incremental

form, $\Phi_{,b} \Delta \mathbf{b}^*$, of the constraint equation, Φ , consists of the degrees of freedom involved in a particular constraint, $\Delta \mathbf{b}^*$, and the corresponding Jacobian matrix, $\Phi_{,b}^*$.

The incremental form of the constraints provided in Eqs. 4.2 and 4.3 can be grouped together such that

$$\Phi_{,b}^{(c)} \Delta \mathbf{b}^* = \begin{bmatrix} \mathbf{I} & -\widetilde{\mathbf{H}\hat{\mathbf{r}}} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \{ \Delta \mathbf{r}'_{cl} \quad \Delta \boldsymbol{\omega}'_{cl} \}^T + \begin{bmatrix} -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix} \{ \Delta \mathbf{u}'_i \quad \Delta \boldsymbol{\theta}'_i \}. \quad (4.4)$$

Whereas Eq. (4.2) and Eq. (4.3) are expressed in the floating frame, the constraint equations for joints are expressed with respect to the inertial frame. Since the clamps are rigid, the conventional kinematic constraint equations for rigid multibody dynamic analysis are utilized by invoking the constraints arising from the kinematic joints. The derivation of the spherical and revolute joints is shown for illustrative purposes. The constraint equations for other types of joints can be formulated in a similar manner.

4.2. Spherical Joint

A spherical joint between two adjacent bodies, n and m , is shown in Fig. 4.3. Combining Eqs. 2.2 and 4.1, the position of point C on a clamp attached to the n^{th} body is expressed in the inertial frame as

$$\mathbf{R}^n = \mathbf{R}_0^n + \mathbf{T}^n \mathbf{r}'_{cl} + \mathbf{T}^n \mathbf{H}^n \hat{\mathbf{r}}^n \quad (4.5)$$

where the right superscript refers to the body number. As shown in Fig. 4.3, let C be the center of the spherical joint such that this point, on bodies n and m , share the same

location in the inertial frame. Therefore, the three constraint equations corresponding to a spherical joint can be expressed as

$$\Phi^{(s)} \equiv \mathbf{R}^n - \mathbf{R}^m = \mathbf{0}. \quad (4.6)$$

The incremental form of the constraints provided in Eq. (4.6) can be expressed in terms of the unknown vector, $\Delta \mathbf{b}^*$, and the Jacobian matrix as

$$\begin{aligned} \Phi_{,b^*}^{(s)} \Delta \mathbf{b}^* = & \begin{bmatrix} \mathbf{I} & \overline{-\mathbf{T}^n \mathbf{r}_{cl}^{n'} + \mathbf{T}^n \mathbf{H}^n \hat{\mathbf{r}}^n} & \mathbf{T}^n & -\mathbf{T}^n \widehat{\mathbf{H}^n \hat{\mathbf{r}}^n} \end{bmatrix} \begin{Bmatrix} \Delta \mathbf{R}_0^n \\ \Delta \boldsymbol{\omega}^n \\ \Delta \mathbf{r}_{cl}^{n'} \\ \Delta \boldsymbol{\omega}_{cl}^{n'} \end{Bmatrix} \\ & + \begin{bmatrix} -\mathbf{I} & \overline{\mathbf{T}^m \mathbf{r}_{cl}^{m'} + \mathbf{T}^m \mathbf{H}^m \hat{\mathbf{r}}^m} & -\mathbf{T}^m & \mathbf{T}^m \widehat{\mathbf{H}^m \hat{\mathbf{r}}^m} \end{bmatrix} \begin{Bmatrix} \Delta \mathbf{R}_0^m \\ \Delta \boldsymbol{\omega}^m \\ \Delta \mathbf{r}_{cl}^{m'} \\ \Delta \boldsymbol{\omega}_{cl}^{m'} \end{Bmatrix}. \end{aligned} \quad (4.7)$$

Fig. 4.3 A spherical joint.

4.3. Revolute Joint

A revolute joint between two adjacent bodies, n and m is shown in Fig. 4.4. Let C be an arbitrary point on the joint axis and impose Eq. (4.6) so that point C on both the bodies translates together in the inertial frame. As shown in Fig. 4.4, the initial orientations of the clamp-fixed frames on bodies n and m are chosen such that one of the unit vectors on each of the bodies, e.g., $\hat{s}_1^{nT} = (1 \ 0 \ 0)$ and $\hat{s}_1^{mT} = (1 \ 0 \ 0)$, is set parallel to the joint axis. The revolute joint requires that the vectors $\hat{s}_1^n$ and $\hat{s}_1^m$ remain parallel, which is achieved through two constraint equations. These two constraints are

used to enforce vectors $\hat{\mathbf{s}}_2^{mT} = (0 \ 1 \ 0)$ and $\hat{\mathbf{s}}_3^{mT} = (0 \ 0 \ 1)$ to remain perpendicular to $\hat{\mathbf{s}}_1^n$ such that

$$\Phi^{(p1)} \equiv \mathbf{s}_1^{nT} \mathbf{s}_2^m = 0 \text{ and } \Phi^{(p2)} \equiv \mathbf{s}_1^{nT} \mathbf{s}_3^m = 0 \quad (4.8)$$

where $\mathbf{s}_1^n$ is the projection of $\hat{\mathbf{s}}_1^n$ in the inertial frame such that

$$\mathbf{s}_1^n = \mathbf{T}^n \mathbf{H}^n \hat{\mathbf{s}}_1^n. \quad (4.9)$$

Similar expressions can be obtained for $\mathbf{s}_2^m$ and $\mathbf{s}_3^m$. The incremental form of Eq. (4.9)

can be expressed as

$$\begin{aligned} \Phi_{,b^*}^{(p)} \Delta \mathbf{b}^* = & \begin{bmatrix} \mathbf{0} & -\mathbf{s}_2^{mT} \tilde{\mathbf{s}}_1^n & \mathbf{0} & -\mathbf{s}_2^{mT} \mathbf{T}^n \widetilde{\mathbf{H}^n \hat{\mathbf{s}}_1^n} \\ \mathbf{0} & -\mathbf{s}_3^{mT} \tilde{\mathbf{s}}_1^n & \mathbf{0} & -\mathbf{s}_3^{mT} \mathbf{T}^n \widetilde{\mathbf{H}^n \hat{\mathbf{s}}_1^n} \end{bmatrix} \begin{Bmatrix} \Delta \mathbf{R}_0^n \\ \Delta \boldsymbol{\omega}^n \\ \Delta \mathbf{r}_{cl}^{n'} \\ \Delta \boldsymbol{\omega}_{cl}^{n'} \end{Bmatrix} \\ & + \begin{bmatrix} \mathbf{0} & -\mathbf{s}_1^{nT} \tilde{\mathbf{s}}_2^m & \mathbf{0} & -\mathbf{s}_1^{nT} \mathbf{T}^m \widetilde{\mathbf{H}^m \hat{\mathbf{s}}_2^m} \\ \mathbf{0} & -\mathbf{s}_1^{nT} \tilde{\mathbf{s}}_3^m & \mathbf{0} & -\mathbf{s}_1^{nT} \mathbf{T}^m \widetilde{\mathbf{H}^m \hat{\mathbf{s}}_3^m} \end{bmatrix} \begin{Bmatrix} \Delta \mathbf{R}_0^m \\ \Delta \boldsymbol{\omega}^m \\ \Delta \mathbf{r}_{cl}^{m'} \\ \Delta \boldsymbol{\omega}_{cl}^{m'} \end{Bmatrix}. \end{aligned} \quad (4.10)$$

Although in a conventional revolute joint the bodies are free to rotate about the joint axis, an actuator assembly can be used to control the angle of rotation through the following constraint

$$\Phi^{(f)} \equiv \mathbf{s}_2^{nT} \mathbf{s}_3^m - \cos(\theta_f) = 0 \quad (4.11)$$

where θ_f is the desired angle between $\mathbf{s}_2^n$ and $\mathbf{s}_3^m$.

Fig. 4.4 A revolute joint.

5. NUMERICAL ANALYSIS

The present study deals with transient analysis of flexible bodies undergoing large rigid body displacements and rotations along with large elastic deformations. Therefore, as stated in previous chapters, a nodal approach has been adopted in the present study as opposed to a modal approach since modal analysis is restricted to linear problems. The task in this chapter is to choose the most appropriate numerical algorithms for time integration and removal of numerical difficulties arising from the presence of constraint conditions leading to differential algebraic equations (DAE). Direct integration schemes are classified into either explicit or implicit algorithms. Explicit direct integration is best suitable for wave propagation problems where the simulation time is small. Explicit integration algorithms are conditionally stable and the maximum size of the time step is dictated by the stability criteria. The small time step and the low cost per time step make explicit time integration routines very attractive for capturing high frequency response that are of interest for a small duration. However, the problems considered in the present study fall under the category of structural dynamics where the lower frequencies are dominant and the simulation last for a fairly long period. For instance, the trim analysis for rotorcraft aeroelasticity requires several runs and in each run the blade response is recorded for as high as ten cycles. Under such circumstances implicit integration algorithms are a better choice for the present analysis. Implicit integration routines are unconditionally stable for linear problems and therefore large time steps can be used. For nonlinear problems implicit algorithms become conditionally stable even although the

critical time step is still considerably larger than that of explicit algorithms. The cost per time step is fairly high for implicit algorithms because a system of uncoupled equation has to be solved at each time step or iteration. Due to the usage of large time steps and the requirement to run the simulation for long duration, an implicit integration algorithm has been adopted for the current analysis.

5.1. Implicit integration algorithms

The simplest of the implicit algorithms is the trapezoidal methods. The trapezoidal rule is unconditionally stable for linear problems and it is an energy conserving method. Note that the equation of motion for a flexible multibody analysis can be stiff and nonlinear. The stiff nature of these equations is due to the presence of high frequencies associated with the flexible bodies. In order to ensure convergence, the numerical integrator should be capable of dissipating high-frequency components while minimizing low-frequency dissipation. Therefore, an energy conserving algorithm such as the trapezoidal rule is inappropriate for stiff problems. On the other hand, the algorithms presented by Houbolt (1950), Newmark (1959) and Bathe and Wilson (1973) process numerical dissipation which damps out any spurious participation of higher modes. Unfortunately, these methods damp out the physically significant lower frequencies as well. Therefore, the presence of artificial damping can lead to the misleading results. In order to mitigate the severe damping of lower frequency modes the HHT- α and WBZ- α algorithms were devised by Hilber et al. (1977) and Wood et al. (1981), respectively. These methods are termed as the α extension to the Newmark family [Newmark (1959)]. The

extra parameter, α , acts on the stiffness and mass matrix in the HHT- α and WBZ- α method, respectively. In spite of the difference in the formulation, the performances of the two methods have been found to be fairly similar [Adams and Wood (1983)]. Chung and Hulbert (1993) combined the HHT- α and WBZ- α methods to develop the generalized- α . As shown in Table 5.1 a family of integrators can be generated from the generalized- α methods. The time integrator for the current analysis is based on the generalized- α method, in conjunction with the Newton-Raphson incremental iterative scheme to solve the nonlinear system of equations.

Table 5.1 A family of implicit time integrators.

	β	γ	α_m	α_f
Trapezoidal	1/4	1/2	0	0
Newmark	✓	✓	0	0
HHT- α	✓	✓	0	✓
WBZ- α	✓	✓	✓	0
Generalized- α	✓	✓	✓	✓

5.2. System of unconstrained bodies

The iterative solution of the unknown vector, $\bar{\mathbf{b}}$, is established by its recursive representation between two consecutive iterations as

$$\bar{\mathbf{b}}_i = \bar{\mathbf{b}}_{i-1} + \Delta \bar{\mathbf{b}} \quad (5.1)$$

where $\bar{\mathbf{b}}_{i-1}$ is the solution vector known from the previous iteration, and it is assumed to be close to the converged solution vector, $\bar{\mathbf{b}}_i$, by an unknown increment of $\Delta\bar{\mathbf{b}}$. In order to determine $\Delta\bar{\mathbf{b}}$, the Taylor's series expansion of Eq. (2.57) is performed about the known solution vector $\bar{\mathbf{b}}_{i-1}$ in conjunction with the decomposition given in Eq. (5.1), resulting in

$$\begin{aligned}
 & (1-\alpha_m) \left[\bar{\mathbf{M}}(\bar{\mathbf{b}}_{i-1}) + \left(\partial \bar{\mathbf{M}} / \partial \bar{\mathbf{b}}_i \right) \Big|_{\bar{\mathbf{b}}_{i-1}} \Delta\bar{\mathbf{b}} + o(\Delta\bar{\mathbf{b}}^2) \right] \left(\ddot{\bar{\mathbf{b}}}_{i-1} + \Delta\ddot{\bar{\mathbf{b}}} \right) + \alpha_m \bar{\mathbf{M}}({}^t\bar{\mathbf{b}}) {}^t\ddot{\bar{\mathbf{b}}} \\
 & + (1-\alpha_f) \left[\bar{\mathbf{F}}^\sigma(\bar{\mathbf{b}}_{i-1}) + \left(\partial \bar{\mathbf{F}}^\sigma / \partial \bar{\mathbf{b}}_i \right) \Big|_{\bar{\mathbf{b}}_{i-1}} \Delta\bar{\mathbf{b}} + o(\Delta\bar{\mathbf{b}}^2) \right] + \alpha_f {}^t\bar{\mathbf{F}}^\sigma \quad (5.2) \\
 & + \left[\bar{\mathbf{F}}^\rho(\bar{\mathbf{b}}_{i-1}) + \left(\partial \bar{\mathbf{F}}^\rho / \partial \bar{\mathbf{b}}_i \right) \Big|_{\bar{\mathbf{b}}_{i-1}} \Delta\bar{\mathbf{b}} + o(\Delta\bar{\mathbf{b}}^2) \right] = \left[\bar{\mathbf{F}}^p(\bar{\mathbf{b}}_{i-1}) + \left(\partial \bar{\mathbf{F}}^p / \partial \bar{\mathbf{b}}_i \right) \Big|_{\bar{\mathbf{b}}_{i-1}} \Delta\bar{\mathbf{b}} + o(\Delta\bar{\mathbf{b}}^2) \right]
 \end{aligned}$$

where α_m and α_f are the additional damping control terms corresponding to the generalized- α algorithm. Substituting from Eq. (2.46) for the gradient of $\mathbf{f}_b^\sigma$ for all elements in the multibody system and assembling for $\bar{\mathbf{F}}^\sigma$ in Eq. (5.2) while retaining the most significant terms in Eq. (5.2) yields

$$\begin{aligned}
 & (1-\alpha_m) \bar{\mathbf{M}}(\bar{\mathbf{b}}_{i-1}) \Delta\ddot{\bar{\mathbf{b}}} + (1-\alpha_f) \bar{\mathbf{K}}_T(\bar{\mathbf{b}}_{i-1}) \Delta\bar{\mathbf{b}} = \bar{\mathbf{F}}^p(\bar{\mathbf{b}}_{i-1}) - \bar{\mathbf{F}}^\rho(\bar{\mathbf{b}}_{i-1}) \\
 & - (1-\alpha_m) \bar{\mathbf{M}}(\bar{\mathbf{b}}_{i-1}) \ddot{\bar{\mathbf{b}}}_{i-1} - (1-\alpha_f) \bar{\mathbf{F}}^\sigma(\bar{\mathbf{b}}_{i-1}) - \alpha_m \bar{\mathbf{M}}({}^t\bar{\mathbf{b}}) {}^t\ddot{\bar{\mathbf{b}}} - \alpha_f {}^t\bar{\mathbf{F}}^\sigma \quad (5.3)
 \end{aligned}$$

in which $\bar{\mathbf{K}}_T$ represents the tangent stiffness matrix of the entire system and can be obtained by assembling the element tangent stiffness matrices in Eq. (2.47). The gradients of the mass matrix and the force vectors arising from external, Coriolis, and

centrifugal forces are assumed to be much smaller than unity between two consecutive iterations.

The solution vector, $\bar{\mathbf{b}}_i$, and its time derivative, $\dot{\bar{\mathbf{b}}}_i$, at the i^{th} iteration during the increment between t and $t + \Delta t$ are approximated as

$$\bar{\mathbf{b}}_i = {}^t\bar{\mathbf{b}} + \Delta t {}^t\dot{\bar{\mathbf{b}}} + \Delta t^2 (1/2 - \beta) {}^t\ddot{\bar{\mathbf{b}}} + \Delta t^2 \beta \ddot{\bar{\mathbf{b}}}_i \quad (5.4a)$$

$$\dot{\bar{\mathbf{b}}}_i = {}^t\dot{\bar{\mathbf{b}}} + \Delta t (1 - \gamma) {}^t\ddot{\bar{\mathbf{b}}} + \Delta t \gamma \ddot{\bar{\mathbf{b}}}_i \quad (5.4b)$$

where β and γ are the Newmark algorithm parameters. Note that, at the beginning of each time increment, the rigid-body configuration of the multibody system is updated as the last deformed configuration for the solution of the unknown configuration for the next time step. In this case, the solution vector, $\bar{\mathbf{b}}$, is given an initial condition that applies only to the beginning of each time increment. Since the last deformed configuration obtained from the previous time increment is frozen in time to act as rigid, $\mathbf{b}_0$ and ${}^t\mathbf{b}$ in Eq. (5.4) automatically vanish before the iterations begin (i.e., $i = 0$), thus yielding

$$\dot{\bar{\mathbf{b}}}_0 = \left(1 - \frac{\gamma}{\beta}\right) {}^t\dot{\bar{\mathbf{b}}} + \Delta t \left((1 - \gamma) - \left(\frac{\gamma(1 - 2\beta)}{2\beta} \right) \right) {}^t\ddot{\bar{\mathbf{b}}} \quad (5.5a)$$

$$\ddot{\bar{\mathbf{b}}}_0 = -\left(\frac{1}{\beta \Delta t} \right) {}^t\dot{\bar{\mathbf{b}}} - \left(\frac{(1 - 2\beta)}{2\beta} \right) {}^t\ddot{\bar{\mathbf{b}}}. \quad (5.5b)$$

The iterative procedure then continues by updating Eq. (5.4) during the time increment with the following substitutions into Eq. (5.3):

$$\Delta \dot{\bar{\mathbf{b}}} = \left(\frac{\gamma}{\beta \Delta t} \right) \Delta \bar{\mathbf{b}} \quad (5.6a)$$

$$\Delta \ddot{\mathbf{b}} = \left(\frac{1}{\beta \Delta t^2} \right) \Delta \bar{\mathbf{b}} \quad (5.6b)$$

leading to a system of equations of the form

$$\bar{\mathbf{J}} \Delta \bar{\mathbf{b}} = \bar{\mathbf{r}} \quad (5.7)$$

where

$$\bar{\mathbf{J}} = \left(\left(\frac{(1 - \alpha_m)}{\beta \Delta t^2} \right) \bar{\mathbf{M}}(\bar{\mathbf{b}}_{i-1}) + (1 - \alpha_f) \bar{\mathbf{K}}_T(\bar{\mathbf{b}}_{i-1}) \right) \quad (5.8a)$$

and

$$\begin{aligned} \bar{\mathbf{r}} = & \bar{\mathbf{F}}^p(\bar{\mathbf{b}}_{i-1}) - \bar{\mathbf{F}}^p(\bar{\mathbf{b}}_{i-1}) - (1 - \alpha_m) \bar{\mathbf{M}}(\bar{\mathbf{b}}_{i-1}) \ddot{\bar{\mathbf{b}}}_{i-1} \\ & - (1 - \alpha_f) \bar{\mathbf{F}}^\sigma(\bar{\mathbf{b}}_{i-1}) - \alpha_m \bar{\mathbf{M}}({}^t\bar{\mathbf{b}}) {}^t\ddot{\bar{\mathbf{b}}} - \alpha_f {}^t\bar{\mathbf{F}}^\sigma. \end{aligned} \quad (5.8b)$$

The system of equation in Eq 5.7 represents a multibody system where the bodies are neither connected to each other nor are they “attached” to any frame. Prior to coupling different bodies through kinematic joints, the motion of each flexible body relative to its frame has to be determined. A crucial step at this stage is the choice of the floating frame. Note that in the floating frame of reference formulation the frame is not unique. The motion of the reference frame is an approximation of rigid body motion and therefore depending on the choice of the frame the total displacement is split between the frame and the nodal DOF at various proportions. The types of formulations defining the relative motion of the flexible body with respect to the frame can be grouped into the moving axis and fixed axis formulations. In the moving axis formulation, the floating frame does not coincide with any specific material point. Two different types of frames are possible; 1) the Tisserand frame [Cavin and Dusto (1977); Agrawal and Shabana

(1985,1986), Nikravesh (2005), Lin and Nikravesh (2005); Nikravesh and Lin (2005)] where the kinetic energy of the flexible motion with respect to an observer stationed at the frame is minimum and 2) the Buckens frame [Canavin and Likins (1977)] where the sum of the squares of the displacements, with respect to an observer stationed at the frame, is minimum. The fixed axis formulation, on the other hand, follows a somewhat finite element type approach. In this case the frame is attached to one or more nodes leading to clamped, roller supported or simply-supported type boundary conditions. The required boundary conditions can be enforced by suppressing a set of degrees of freedom. Due to its simplicity the fixed axis frame has been utilized for the present work. Further details on various types of boundary conditions and the resulting fixed axis frame are given in Section 7.3.1.

A graphical representation of Eq. (5.7), for a system consisting of two bodies, is shown in Fig. 5.1. Also shown in Fig. 5.1 is an example of a set of nodal degrees of freedom that have to be suppressed in order to prescribe a boundary conditions. The elimination of these degrees of freedom gives rise to a reduced system of equation.

5.3. System of constrained bodies.

Although the rigid body modes of the flexible bodies with respect to their reference frames have been eliminated, the resulting equations still do not represent a system of constrained bodies. The constrained multibody system requires the constraint equations derived in Chapter 4, which can be grouped together and written as

$$\Phi(\bar{\mathbf{b}}) = \mathbf{0} . \quad (5.9)$$

Coupling Eq. (5.7) and Eq. (5.8) can be performed through the Lagrange multiplier method. The major drawback of this method is that the final system of equations is in the form of differential algebraic equations (DAEs), which are very stiff in nature. There is no mass associated with the Lagrange multipliers and therefore very large frequencies creep into the solution. Due to the nonlinear nature of the equations, energy is transferred from the high frequency modes to low frequency modes which eventually results in divergence. Another complexity involved with this method is the initial estimation of the Lagrange multipliers. In order to circumvent these issues, the coordinate partitioning method developed by Wehage and Haug (1982) has been utilized in the current analysis. This method is also very efficient in avoiding the constraint violation phenomenon. However, the drawback of this method is that the resulting matrix is fully populated. In order to incorporate the constraint equations in an incremental procedure, a Taylor series expansion of Eq. (5.9) is performed about the known solution vector $\bar{\mathbf{b}}_{i-1}$, resulting in

$$\Phi(\bar{\mathbf{b}}_i) = \Phi(\bar{\mathbf{b}}_{i-1}) + \Phi_{,\bar{\mathbf{b}}}({}^t\bar{\mathbf{b}}_{i-1})\Delta\bar{\mathbf{b}} + o(\Delta\bar{\mathbf{b}}^2) = \mathbf{0} \quad (5.10)$$

where $\Phi_{,\bar{\mathbf{b}}}({}^t\bar{\mathbf{b}}_{i-1})$ represents the Jacobian matrix; the specific form of the sub-matrices corresponding to the spherical and revolute joints is given in Chapter 4.

In accordance with the coordinate partitioning method, the unknown vector, $\bar{\mathbf{b}}$, is partitioned into dependent and independent degrees of freedom, $\hat{\mathbf{b}}^T = (\hat{\mathbf{b}}_{in}^T \ \hat{\mathbf{b}}_{de}^T)$, such that Eq. (5.10) can be written as

$$\Phi(\hat{\mathbf{b}}_{i-1}) + \left(\Phi_{,\hat{\mathbf{b}}_{in}} \ \Phi_{,\hat{\mathbf{b}}_{de}} \right) \left(\Delta\hat{\mathbf{b}}_{in}^T \ \Delta\hat{\mathbf{b}}_{de}^T \right)^T = \mathbf{0}. \quad (5.11)$$

Note that the higher order terms have been neglected in Eq. (5.11). Since the constraint equations are linearly independent, thereby having a full row rank, the dependent set of coordinates, $\hat{\mathbf{b}}_{de}$, can be selected such that $\det(\Phi_{\hat{\mathbf{b}}_{de}}^{-1}) \neq 0$. This enables expressing the dependent coordinates in terms of the independent coordinates as

Fig. 5.1 Graphical representation of the incremental equilibrium equations for a system of two bodies.

Fig. 5.2 Graphical representation of the rearranged and reduced form of the incremental equilibrium equations for a system of two bodies.

$$\Delta \hat{\mathbf{b}}_{de} = -\Phi_{,\hat{\mathbf{b}}_{de}}^{-1} \Phi_{,\hat{\mathbf{b}}_{in}} \Delta \hat{\mathbf{b}}_{in} - \Phi_{,\hat{\mathbf{b}}_{de}}^{-1} \Phi(\hat{\mathbf{b}}_{i-1}). \quad (5.12)$$

The unknown vector is now expressed in terms of the independent coordinates

$$\Delta \hat{\mathbf{b}} = \begin{pmatrix} \Delta \hat{\mathbf{b}}_{in}^T & \Delta \hat{\mathbf{b}}_{de}^T \end{pmatrix}^T = \mathbf{h} \Delta \hat{\mathbf{b}}_{in} + \mathbf{g} \quad (5.13)$$

where

$$\mathbf{h} = \begin{Bmatrix} \mathbf{I} \\ -\Phi_{,\hat{\mathbf{b}}_{de}}^{-1} \Phi_{,\hat{\mathbf{b}}_{in}} \end{Bmatrix} \quad (5.14a)$$

$$\mathbf{g} = \begin{Bmatrix} \mathbf{0} \\ -\Phi_{,\hat{\mathbf{b}}_{de}}^{-1} \Phi(\hat{\mathbf{b}}_{i-1}) \end{Bmatrix}. \quad (5.14b)$$

Similar to the rearrangement of the constraint equations, the reduced form of incremental equations given in Eq. (5.7) is partitioned into independent and dependent degrees of freedom. The resulting equations are expressed as

$$\hat{\mathbf{J}} \Delta \hat{\mathbf{b}} = \hat{\mathbf{r}} \quad (5.15)$$

and the corresponding graphical representation is shown in Fig. 5.2. Substituting the expression of $\hat{\Delta \mathbf{b}}$ from Eq. (5.13) into Eq. (5.15) and pre-multiplying the resulting equation with $\mathbf{h}^T$ yield

$$\mathbf{h}^T \hat{\mathbf{J}} \mathbf{h} \Delta \hat{\mathbf{b}}_{in} = \mathbf{h}^T \hat{\mathbf{r}} - \mathbf{h}^T \hat{\mathbf{J}} \mathbf{g} \quad (5.16)$$

The modified Jacobian, $\mathbf{h}^T \hat{\mathbf{J}} \mathbf{h}$, results in a smaller system of equations compared to the unconstrained system. Moreover, these equations are of a purely differential form and therefore the difficulties associated with DAEs are avoided. Unfortunately, Eq. 5.16 results in a fully populated matrix and therefore the solvers associated with sparse matrices can no longer be used. The computation of the dependent unknowns is achieved by substituting $\bar{\mathbf{b}}_{in}$ into Eq. (5.12). This ensures that the constraints are satisfied exactly and the instability issues that arise due to constraint violations are avoided. During a specific time increment, Eq. (5.16) is recursively solved until a convergence is reached. The convergence criterion is given by

$$\left(\|\Delta \hat{\mathbf{b}}_i\| / \left\| \sum_{k=1}^i \Delta \hat{\mathbf{b}}_k \right\| \right) \leq \varepsilon \quad (5.17)$$

where typically $\varepsilon = 10^{-4}$.

6. ROTORCRAFT AEROELASTICITY

A complex aerodynamic flow field exists in the vicinity of the rotor disk, as shown in Fig. 6.1. In forward flight, the rotor blades experience time-varying free stream and radial flow with varying inflow, and angles of attack, constantly changing with time. Therefore, the aerodynamic loads vary considerably with the radial and the azimuth location of the blade. The asymmetry of flow field in forward flight results in transonic conditions on the advancing side and reverse flow on the retreating side. Due to a rapid increase in the angle of attack, the rotor can encounter stall in the retreating side, where excess vortices build up on the leading edge, then detach and are shed into the wake. This time-varying phenomenon of flow separation and reattachment is termed as dynamic stall. In addition, in many flight conditions, a rotor blade may encounter the wake or a tip vortex of another rotor blade or of itself on a previous revolution. This phenomenon, which is called blade vortex interaction (BVI), is a major cause of vibration in the flexible rotor blade.

An aerodynamic model based on a lifting surface theory or computational fluid dynamic (CFD) is an appropriate choice while considering a three-dimensional computational structural dynamic (CSD) model for an aeroelastic analysis. However, as a first step the present computational structural dynamic (CSD) program is tested by coupling it with an aerodynamic model that is based on a lifting line theory. Although the rotor blade is subjected to a complex aerodynamic environment, most of the characteristic features can be captured with simple analytical models. These models provide the aerodynamic lift,

Fig.6.1 Flow field in the vicinity of a rotor disk.

Fig. 6.2. An aerodynamic segment.

drag and radial forces along with the moment at the aerodynamic center of the airfoil cross-section.

6.1. Quasi-steady Rotorcraft Aerodynamics

A quasi-steady model based on the lifting line theory has been employed to calculate the aerodynamic loads. The quasi-steady nature of the model is due to the dependence of the aerodynamic loads on the current state of the input variable, such as angle of attack, Mach number and pitch rate. Since the time history of these variables is not taken into account, therefore, these models can not be termed as unsteady.

Rotor blades are fairly slender structures and therefore the high aspect ratio allows the lifting line problem to be split into two parts: 1) an inner problem that considers the aerodynamics of the two-dimensional wing section and 2) an outer problem where the induced velocity is obtained. Note the similarity of the aerodynamic model with the structural models based on beam elements. The aerodynamic model requires the blade to be divided into a finite number of aerodynamic segments or elements, as shown in Fig. 6.2. The forces and moments that are computed at the center of each of these segments are then distributed over the surface of the segment based on an assumed variation for pressure distribution.

The aerodynamics only take into account the “wind” velocity, including the helicopter forward and climb speed and the rotor rotation, and the motion of the blades relative to the hub. The velocities are measured with respect to the floating frame that is attached to the hub, and rotates with an angular velocity of Ω_b . The relative velocity of air with

respect to the blade has components, u_T , u_P , and u_R , as shown in Fig. 6.3. The dominant contributor to in-plane components, u_T is the rotational motion of the blade which gives rise to a radially varying velocity, $\Omega_b r$. The forward velocity, V_h , of the helicopter gives rise to a azimuthally varying component, $V_h \sin \psi$ and $V_h \cos \psi$, to u_T and u_R , respectively. The elastic deformation of the blade with respect to the floating frame results in additional relative velocity of the blade with respect to the wind. The transverse velocity, u_P , is due to the rotor disk tilt, component of u_R normal to the blade, flapping motion of the blade and the induced inflow velocity. The transverse velocity due to flapping, which includes both rigid body motion and elastic deformation, is obtained from the finite element solution. A linear induced inflow model is used to obtain the induced inflow distribution over the rotor disk. The linear inflow is an extension of the uniform inflow predicted by momentum theory. The induced inflow is adjusted to account for variations in lift over the rotor disk in forward flight. The inflow varies linearly in the longitudinal and lateral directions,

$$\lambda_i = \frac{C_T}{2\sqrt{\mu^2 + \lambda^2}} (1 + k_x r \cos \psi + k_y r \sin \psi) \quad (6.1)$$

where C_T is the coefficient of thrust, λ is the total inflow due to induced effect and forward flow and μ is the non-dimensional forward speed. The terms k_x and k_y are zero for hover and depend on flight conditions for forward flight [Johnson (1980a); Johnson (1980b)]. Due to the nonlinear nature of Eq. (6.1), the induced inflow velocity has to be computed iteratively.

The blade pitch θ_b is the angle that the chord makes with the hub plane. Note that θ_b includes the contribution from built-in twist, elastic torsion, and pitch control angles. The angle between the resultant velocity at a blade section, $U_b = (u_T^2 + u_P^2)^{1/2}$, and the hub plane is defined as the induced angle, ϕ_b . The angle of attack is defined as $\alpha_b = \theta_b - \phi_b$.

Fig. 6.3. Sectional velocities and forces.

Given the velocities at a blade section and the angle of attack, the next step is to calculate the airloads, which are defined in reference to the blade section coordinates. The sectional lift, drag, and radial forces are

$$F_L = \rho_a U_b^2 c_b c_L / 2 + F_L^{US} \quad (6.2a)$$

$$F_D = \rho_a U_b^2 c_b c_D / 2 \quad (6.2b)$$

$$F_R = \rho_a U_b u_R c_b c_D / 2 \quad (6.2c)$$

and the moment is specified as

$$M = \rho_a U_b^2 c_b^2 c_M / 2 + M^{US} \quad (6.2d)$$

where ρ_a is air density and F_L^{US} and M^{US} are the unsteady terms arising from the pitch rate and plunging motion of the blade. The sectional lift, drag, and moment coefficients, c_L , c_D , and c_M , are functions of the angle of attack and the Mach number. These coefficients are obtained from static two-dimensional airfoil experiments, which are conducted for a range of angles of attack and Mach numbers. Dynamic stall, which occurs due to high angle of attack in the retreating side, has not been considered in the aerodynamic model.

6.2. Lumping of aerodynamic load

A typical pressure distribution on an airfoil is shown in Fig. 6.4a and the corresponding forces, normal and parallel to the resultant velocity, and moment are shown in Fig. 6.4b. When the blade is modeled with beam elements, these forces and moments are lumped directly on a single node that lies on the shear center of the airfoil cross-section. The

CSD tool considered for the present analysis utilizes plate elements to model the blade surface. Therefore, a pressure distribution, similar to the one shown in Fig. 6.4a, which can typically be obtained from a CFD analysis can be applied directly on the plate elements. However, such pressure distribution is not currently available from the present aerodynamic model. Therefore, the section forces and moments are distributed along the top and bottom blade surfaces based on an assumed variation of the pressure distribution. As shown in Fig. 6.5a, a stepwise pressure distribution is applied to reproduce the effect of the sectional forces, normal to the chord, while having zero moment. Similarly, the moment about the aerodynamic center is also applied through a stepwise pressure distribution, as shown in Fig. 6.4b, which creates zero net force and a moment equivalent to the sectional moment. The element nodal forces are then computed based on these pressure distributions.

6.3. Trim Analysis

Accurate prediction of the rotor control settings, rotor disk orientation and helicopter orientation are required for a prescribed flight condition. The procedure of calculating these parameters is termed as trim. Propulsive trim is employed to simulate the free flight conditions of a helicopter. As the name suggests, the wind tunnel trim is used to test rotors in a wind tunnel. The present study incorporates a wind tunnel trim procedure to search for the control settings and to calculate the steady state periodic solution to the nonlinear equation of motion. The blade pitch variation as a function of the azimuth angle, ψ is given as

$$\hat{\theta} = \hat{\theta}_0 + \hat{\theta}_{1s} \sin(\psi) + \hat{\theta}_{1c} \cos(\psi) \quad (6.3)$$

where $\hat{\theta}_0$ is the collective pitch and $\hat{\theta}_{1s}$ and $\hat{\theta}_{1c}$ are the longitudinal and lateral cyclic pitch respectively. The trim procedure involves the search for appropriate values of these control parameters that would eliminate 1/rev blade flapping and achieve specified values for C_T/σ , where C_T is the coefficient of thrust and σ is the rotor solidity. Hence, the task at hand is to satisfy the following:

$$\mathbf{F}(\hat{\boldsymbol{\theta}}) = \boldsymbol{\Gamma}_{\text{current}} - \boldsymbol{\Gamma}_{\text{target}} \quad (6.4)$$

where $\hat{\boldsymbol{\theta}}^T = \{\hat{\theta}_0 \quad \hat{\theta}_{1s} \quad \hat{\theta}_{1c}\}$ and $\boldsymbol{\Gamma}_{\text{current}}^T = \{C_T/\sigma \quad \beta_{1s} \quad \beta_{1c}\}$. The trim controls are obtained in an iterative manner, and the process starts with an initial guess. A Newton-type method is used to update the trim controls after each cycle of rotation

$$\hat{\boldsymbol{\theta}}^{i+1} = \hat{\boldsymbol{\theta}}^i - \left[\partial \mathbf{F} / \partial \hat{\boldsymbol{\theta}} \right]_i^{-1} \mathbf{F}^i \quad (6.5)$$

The Jacobian matrix in Eq. (6.5) is obtained by giving small perturbations to each component of $\hat{\boldsymbol{\theta}}$ about the current estimate and monitoring the corresponding change in $\mathbf{F}(\hat{\boldsymbol{\theta}})$. The Jacobian matrix can either be updated or kept constant, depending on the rate of convergence.

Fig. 6.4 Pressure distribution over an airfoil and the resulting forces and moment.

Fig. 6.5. Stepwise pressure resulting in a) resultant forces and b) moment.

7. NUMERICAL RESULTS

The methodology formulated and developed during the course of this study is specifically for simulating three dimensional flexible multibody dynamic systems experiencing large elastic deformation. Moreover, including the effect of material anisotropy on coupling of different deformation modes is also an integral part of this work. Along with demonstrating the capabilities of the present methodology, several validation problems are also considered to establish its fidelity. The type of problems considered in this chapter range from linear static to nonlinear multibody dynamic. Some of the problems are chosen in order to check the feasibility of using the current method for rotorcraft aeroelasticity. This method is finally used to model rotor blades in a detailed manner.

7.1 Static Analysis

Because the present study concerns modeling the rotor blade with plate elements instead of beam elements, it is necessary to first check the accuracy of these elements for static analysis. The specific problems considered in this section deal with laminates with various stacking sequences and their effect on the structural response. Aeroelastic tailoring through directional stiffening can yield significant improvement in aeroelastic performance [Shirk and Hertz (1986)]. The layup sequence has a marked effect on the type of coupling. Couplings such as extension–twist and bending–twist can be utilized during the design of rotor blades in such a fashion so as to affect the structural and

aerodynamic performance in a beneficial way. Therefore, a simulation program should be capable of accurately predicting the effect due to material coupling.

7.1.1. Box Beam – Linear Analysis

A set of experiments were conducted by Chandra et al. (1990) to investigate the response of composite box beams with coupling due to different laminate stacking sequences. In their study dealing with an analytical model for box-beams, Smith and Chopra (1991) compared their results with the 1) experimental results of Chandra et al. (1990) and 2) a model based on beam elements developed by Stemple and Lee (1988;1989). The results from the beam FE analysis had a better correlation than the analytical results and therefore, the results from the current analysis are compared with the experimental and beam FE analysis. As shown in Fig 7.1, each wall of the box beam is modeled using plate elements. Note that the walls undergo both transverse and membrane deformation. When box-beams are modeled using beam elements, the stiffness due to all of the walls is lumped to a single node and this approach does not face the challenge of the detailed modeling of transverse and membrane deformations. However, the approach adopted in the present analysis demands an element that is efficient for both transverse and membrane bending. The plate element developed in Section 3.1 is very suitable for such a task. Whereas the MIN3 formulation helps to captures the transverse shear and bending deformation, the ANDES formulation ensures that the membrane deformations are computed accurately.

The details of the geometry of the box beam and the material properties of Graphite/Epoxy that were used for the analysis are given in Table 7.1. As shown in Table 7.2., a variety of stacking sequences were used in order to fabricate beams that display 1) no coupling (cross ply), 2) bending–torsion and extension–shear coupling (symmetric) and 3) extension–twist and bending–shear coupling (anti–symmetric).

Table 7.1. Box–beam geometry and material properties.

	L/d = 29	L/d = 56
L (in)	30	30
d (in)	1.025	0.537
c (in)	2.060	0.953
Ply Thickness (in)	0.005	0.005
Wall Thickness (in)	0.030	0.030
E_L (msi)	20.59	20.59
E_T (msi)	1.42	1.42
G_{LT} (msi)	0.87	0.87
ν_{LT} (msi)	0.42	0.42

Table 7.2 Layup sequence and loading.

Beam Type		L/d	Top and Bottom	Sides	Loading
Type 1A	Cross–Ply	29	(0/90) ₃	(0/90) ₃	Bending
Type 1B	Cross–Ply	29	(0/90) ₃	(0/90) ₃	Torsion
Type 2A	Symmetric	56	(15) ₆	(15/–15) ₃	Bending
Type 2B	Symmetric	56	(30) ₆	(30/–30) ₃	Bending
Type 2C	Symmetric	56	(45) ₆	(45/–45) ₃	Bending
Type 3A	Anti–Symmetric	56	(15) ₆	(15) ₆	Bending
Type 3B	Anti–Symmetric	56	(45) ₆	(45) ₆	Bending

A box beam with the cross–ply stacking sequence that does not exhibit any coupling is considered first. The variation of bending slope subjected to a unit tip bending load and a

twist angle under tip torque is shown in Fig 7.2 and 7.3, respectively. The present method underpredicts the bending slope in comparison to the experimental results but the correlation for twist angle is rather acceptable. A set of symmetric beams under tip bending load are analyzed next. The variation of bending slope and twist is shown in Figs 7.4–7.9. The results from the present analysis are in good agreement with the experimental results in general. In all the cases, the predictions from the present approach are either equally good or better than the results obtained from the beam FEM analysis. Finally, antisymmetric beams under tip bending load are considered and the results are shown in Figs. 7.10 and 7.11. In this case, the results from the analysis by Stemple and Lee (1988;1989) are unavailable and therefore the results presented by Smith and Chopra (1991) have been considered for the sake of comparison. The results from the present analysis are in good agreement with the experimental results. Although the results obtained by Smith and Chopra (1991) for Type 3A beam appear to be much stiffer than the experimental results, their analysis shows the importance of including transverse shear effects in the formulation.

Fig. 7.1. Mesh comprising of plate elements for a box-beam.

Fig. 7.2. Bending slope under unit tip bending of Type1A beam.

Fig. 7.3. Twist under unit tip torque of beam Type1B beam.

Fig. 7.4. Bending slope under unit tip bending of Type 2A beam.

Fig. 7.5. Twist under unit tip bending of Type 2A beam.

Fig. 7.6. Bending slope under unit tip bending of Type 2B beam.

Fig. 7.7. Twist under unit tip bending of Type 2B beam.

Fig. 7.8. Bending slope under unit tip bending of Type 2C beam.

Fig. 7.9. Twist under unit tip bending of Type 2C beam.

Fig. 7.10. Bending slope under unit tip bending of Type 3A beam.

Fig. 7.11. Bending slope under unit tip bending of Type 3B beam.

7.1.2 Composite Beam – Nonlinear Analysis

The problems considered in the previous section validate the capabilities of the present method only for a linear analysis i.e. in the small deflection regime. Unfortunately, a linear analysis might often lead to misleading results even if the displacements might appear to be “small”. In several practical applications, flexible bodies while experience large deflection might undergo a noticeable change in their shape thereby inheriting a much stiffer or softer property compared to their original configuration. Therefore, it is crucial to validate the capabilities of the current method to capture the correct response in the large deflection regime. In other words, the performance of the co-rotational

formulation in conjunction with the updated Lagrangian description of motion needs to be verified.

Experimental studies such as the Princeton beam test [Dowell and Traybar (1975)] have been conducted to investigate the response of beams made of isotropic material experiencing large deformation. The experiments conducted by Minguet and Dugundji (1990a and 1990b), on the other hand, dealt with composite beams exhibiting various types of coupling due to different ply layup. The nonlinear capabilities of the present method are therefore validated against the experimental results obtained by Minguet and Dugundji (1990a).

In the experimental studies, three types of beams were considered. The stacking sequence, type of coupling and laminate thickness of each of these beams are given in Table 7.3. The laminates are composed of AS4/3501-6 Graphite/Epoxy, the material properties for which are provided in Table 7.4. The beams are 560 mm long and 30 mm wide. The load is applied in terms of weights (grams) at a distance 550 mm from the root and the displacements are measured at a distance 500 mm from the root. The FE mesh along with deformed configuration of the BT beam is shown in Fig. 7.12. The same mesh is used for all the beams.

The transverse, axial and inplane deflections of the Type NC beam as a function of the applied weight are shown in Fig. 7.13. Clearly the displacements are in the nonlinear range. Due to the absence of any coupling, the beam does not experience any deformation in the inplane direction. Axial foreshortening, which is the occurrence of axial displacement due to transverse displacement, is captured by the present analysis and

the results are in good agreement with the experimental results. In the case of transverse displacement, the present analysis predicts a slightly stiffer response in comparison to the experimental observations.

The results for the Type BT beam experiencing bending–twist coupling is shown in Fig 7.14. In this case the results obtained using VABS (Variationally Asymptotic Beam Sectional Analysis) by Cesnik and Hodges (1997) is also included for the purpose of comparison. Under the influence of transverse load, the beam twists, thereby swaying the tip in the inplane direction. The inplane displacements obtained from both the numerical approaches i.e. VABS and the present analysis, are in good agreement with the experimental results. A similar trend is observed in the case of transverse displacement as well. However, the present analysis under predicts the axial displacements. Therefore, in order to gain confidence in the present analysis comparison was made against the VABS results. As shown in Fig. 7.14 the axis displacement obtained from VABS are also smaller than the experimental results and in a better agreement with the results obtained from the present analysis.

Finally the Type ET beam is considered and the results are shown in Fig. 7.15. In this case, the inplane and axial displacements obtained from both the analyses and the experiments are in good agreement. The present analysis predicts a stiffer response for the transverse displacement. This deviation from the experimental results is observed in the case of VABS as well.

Table 7.3. Laminate Properties.

Beam Type	Layup	Coupling	Thickness
Type NC	$[0^\circ / 90^\circ]_{3s}$	None	1.49 mm
Type BT	$[45^\circ / 0^\circ]_{3s}$	Bending–Twist	1.47 mm
Type ET	$[20^\circ / -70^\circ / -70^\circ / 20^\circ]_{2a}$	Extension–Twist	1.92 mm

Table 7.4. Properties of the Graphite/Epoxy ply.

AS4/3501–6 Ply Properties	
E_L (GPa)	142.0
E_T (GPa)	9.8
G_{LT} (GPa)	6.0
ν_{LT} (GPa)	0.3
ρ (kg/m ³)	1580.0
t_{PLY} (mm)	0.134

Fig. 7.12. Deformed configurations of the BT beam under various load levels.

Fig. 7.13. Variation of displacements with applied weight for Type NC beam.

Fig. 7.14. Variation of displacements with applied weight for Type BT beam.

Fig. 7.15. Variation of displacements with applied weight for Type ET beam.

7.2 Dynamic Analysis

The problems in this section concern the validation and demonstration of the dynamic analysis. This section specifically consider the transient responses of a single body experiencing either linear or nonlinear deformation.

7.2.1 Box Beam – Natural Frequency

Prior to investigating the transient response, the natural frequency of box beams made of composite laminates is considered first. Accurate computation of fundamental frequency is a crucial part of validating a structural dynamic model. Moreover, very often due to high rotational speed the flexible bodies exhibit a stiffer response resulting in a higher value for the fundamental frequency, and this phenomenon should be captured accurately by the structural dynamic model.

Chandra and Chorpa (1992) carried out an experimental investigation of free vibration characteristics of thin-walled composite box beams with bending–twist and extension–twist coupling under both non–rotating and rotating conditions. Glass/Epoxy, Kevlar/Epoxy and Graphite/Epoxy symmetric and antisymmetric box beams were tested in an in– vacuo rotor test facility for their vibration characteristics. The in–vacuo test results provide a means to validate the structural dynamic model in the absence of aerodynamic load. The box beams have a length of 33.25 in and the internal dimension of the width and height is 0.893 in and 0.477 in, respectively. The properties of Glass/Epoxy, Kevlar/Epoxy and Graphite/Epoxy lamina are given in Table 7.5. Six different configurations were considered and the details are given in Table 7.6. In order

to compute the natural frequencies of non-rotating beams, using the present analysis, a step load was applied at the tip of the beam and the transient response was recorded. Care was taken to ensure that the displacements were in the linear range. The fundamental frequency was then computed using the Fast Fourier Transformation. Armanios and Badir (1995) computed the natural frequencies of non-rotating beams based on the variational asymptotic approach and their results have been considered for the sake of comparison.

The non-rotating natural frequency of the symmetric Graphite/Epoxy $[30^\circ]_6$, Kevlar/Epoxy $[45^\circ]_2$ and Glass/Epoxy $[45^\circ]_2$ box beams is shown in Fig 7.16. The results from the current analysis are in good agreement with the experimentally obtained natural frequencies. In Fig. 7.17 the natural frequencies of the symmetric box beams rotating at 1000 RPM is shown. As expected, due to the presence of centrifugal forces, the stiffness of the box beams increases thereby resulting in a higher fundamental frequency. This phenomenon is observed in both the experimental results and the present analysis.

The non-rotating fundamental frequency of the antisymmetric box beams made of Graphite/Epoxy with stacking sequence of $[15^\circ]_6$, $[0^\circ/30^\circ]_3$ and $[0^\circ/45^\circ]_3$ is shown in Fig. 7.18. Although a good correlation is observed in the case of $[15^\circ]_6$ laminate, the fundamental frequency predicted by the present analysis is higher than the experimental results for the $[0^\circ/30^\circ]_3$ and $[0^\circ/45^\circ]_3$ laminates. However, the results reported by Armanios and Badir (1995) show a similar behavior and their results are in a good

agreement with the fundamental frequencies computed by the present analysis. The natural frequencies of the antisymmetric box beams rotating at 1000 RPM is shown in Fig 7.19. Similar to the observation made in the case of symmetric box beam, the fundamental frequency increases due to centrifugal stiffening. Since the fundamental frequencies predicted by the present analysis were higher than the experimental observation for the non-rotating beams, a similar trend is observed in the case of rotating beams as well.

Table 7.5. Ply properties of Graphite/Epoxy, Kevlar/Epoxy and Glass/Epoxy.

Material	Graphite/Epoxy	Kevlar/Epoxy	Glass/Epoxy
E_L (10^6 psi)	20.59	11.0	7.0
E_T (10^6 psi)	1.42	0.80	2.10
G_{LT} (10^6 psi)	0.89	0.34	0.80
ν_{LT}	0.42	0.34	0.26
ρ (10^{-3} lbs ² /in ⁴)	0.1352	0.1042	0.1676
t_{PLY} (in)	0.005	0.010	0.016

Table 7.6. Stacking sequence for symmetric and antisymmetric box beams.

Configuration	Material	Flanges		Web	
		Top	Bottom	Left	Right
Symmetric	Graphite/Epoxy	$[30^\circ]_6$	$[30^\circ]_6$	$[\pm 30^\circ]_3$	$[\pm 30^\circ]_3$
	Kevlar/Epoxy	$[45^\circ]_2$	$[45^\circ]_2$	$\pm 45^\circ$	$\pm 45^\circ$
	Glass/Epoxy	$[45^\circ]_2$	$[45^\circ]_2$	$\pm 45^\circ$	$\pm 45^\circ$
Anti-symmetric	Graphite/Epoxy	$[15^\circ]_6$	$[-15^\circ]_6$	$[15^\circ]_6$	$[-15^\circ]_6$
	Graphite/Epoxy	$[0^\circ / 30^\circ]_3$	$[0^\circ / -30^\circ]_3$	$[0^\circ / 30^\circ]_3$	$[0^\circ / -30^\circ]_3$
	Graphite/Epoxy	$[0^\circ / 45^\circ]_3$	$[0^\circ / -45^\circ]_3$	$[0^\circ / -45^\circ]_3$	$[0^\circ / 45^\circ]_3$

Fig. 7.16. Fundamental frequencies of non-rotating symmetrical box beams.

Fig. 7.17. Fundamental frequencies of symmetrical box beams rotating at 1000 RPM.

Fig. 7.18. Fundamental frequencies of non-rotating antisymmetrical box beams.

Fig. 7.19. Fundamental frequencies of antisymmetrical box beams rotating at 1000 RPM.

7.2.2 Rapid Spin-up Maneuver

The next validation problem, considered previously by Simo and Vu-Quoc (1986), involves a flexible beam subjected to a prescribed spin-up maneuver at one end. The beam is subjected to clamped boundary conditions with a spin-up maneuver given by

$$\Omega_z = \begin{cases} \frac{\Omega_0}{T_0} \left[t - \frac{T_0}{2\pi} \sin\left(\frac{2\pi t}{T_0}\right) \right] & , \quad t < T_0 \\ \Omega_0 & , \quad t > T_0 \end{cases} \quad (7.1)$$

where the steady-state angular velocity, Ω_0 , and the spin-up time, T_0 , are taken to be 6.0 rad/s and 15.0 s, respectively. Plate and solid elements are used separately to model the beam as a narrow plate with density $\rho = 15.492 \text{ lbs}^2/\text{in}^4$, Young's modulus $E = 3.615 \times 10^8 \text{ psi}$, and Poisson's ratio $\nu = 0.0$. The length, $L = 10.0 \text{ in}$, width, $W = 1.0 \text{ in}$, and thickness, $h = 0.07746 \text{ in}$, define the geometry of the plate. A constant time step of $\Delta t = 0.1 \text{ s}$ was chosen for the simulation. Temporal axial and transverse displacements at the tip of the plate are shown in Fig. 7.20 and a comparison of these results with those presented by Simo and Vu-Quoc (1986) yields a good agreement. The steady-state axial extension of the free end has a value of $5.14 \times 10^{-4} \text{ in}$, which is the same as reported by Simo and Vu-Quoc (1986). Although the plate is relatively thin, the model based on solid elements does not lock due to the presence of higher-order incompatible modes.

(a)

(b)

Fig. 7.20. Axial (a) and transverse (b) tip displacements of a narrow rotating plate.

7.2.3 Propeller

The three-dimensional nature of the current analysis is demonstrated by considering a rotating propeller with a complex geometry. Kosmatka and Friedmann (1989) developed a beam element to determine the natural frequencies of composite turbopropellers. Models based on beam elements can give acceptable results for slender blades which do not exhibit chordwise bending. In the case of a propeller blade which experiences significant chordwise bending, plate elements should be used in order to capture the chordwise deformation modes accurately.

The top view of the propeller with two blades is shown in Fig. 7.21. As shown in the figure, a body-fixed coordinate system is attached at the center of the hub. The profile of the blade, shown in Fig. 7.22, is defined as

$$z_b = \frac{z_0}{2} \left(1 + \frac{\theta}{\theta_0} \right) \left(\frac{r - r_i}{r_0 - r_i} \right)^2 \quad \begin{array}{l} -\theta_0 \leq \theta \leq \theta_0 \\ r_i \leq r \leq r_0 \end{array} \quad (7.2)$$

where r is the radial distance from the center, $z_0 = 0.4$ m, $\theta_0 = \pi/6$, $r_i = 0.2$ m, and $r_0 = 1.0$ m. The aerodynamic forces are approximated by applying a uniform pressure field, p , acting on the bottom surface of the blades. The time variation of the spin-up maneuver and the amplitude of the pressure field are specified as

$$\left\{ \frac{\Omega_z}{\Omega_0}, \frac{p}{p_0} \right\} = \left\{ \begin{array}{l} \left[\frac{t}{T_0} - \frac{1}{2\pi} \sin \left(\frac{2\pi t}{T_0} \right) \right] \\ 1.0 \end{array} \right. , \quad \begin{array}{l} t < T_0 \\ t > T_0 \end{array} \quad (7.3)$$

where the steady-state angular velocity, Ω_0 , the spin-up time, T_0 , and the steady-state pressure amplitude, p_0 , are specified to be π rad/s, 15.0 secs, and 6.0 Nm^{-2} , respectively. The blades that are modeled using plate elements have a density $\rho = 1500 \text{ kg/m}^3$, Young's modulus $E = 2.0 \text{ GPa}$, Poisson's ratio $\nu = 0.3$, and thickness $h = 1.2 \text{ mm}$. Solid elements are used for modeling the hub that has a density $\rho = 2000 \text{ kg/m}^3$, Young's modulus $E = 10.0 \text{ GPa}$, and Poisson's ratio $\nu = 0.3$. A constant time step of $\Delta t = 0.1 \text{ sec}$ was chosen for the simulation.

Large deformations experienced by the blades are evident from Fig. 7.23, which shows the deformed configuration of the blade, in the body-fixed coordinate system, at various time intervals. The deformation of a point, P , located at the blade tip, in the body fixed frame is shown in Fig. 7.24. Under the action of normal pressure, the blades have a tendency to flap upwards whereas the centrifugal forces apply a restoring moment that opposes this motion. Therefore, a natural spring like phenomenon is created. The pressure, p , acting on the blade surface is not dependent on the velocity of the blade and therefore does not possess the aerodynamic damping property. Hence the blades continue to oscillate even after the angular velocity and the amplitude of pressure become constant.

Fig. 7.21. Schematics of a propeller.

Fig. 7.22. FE mesh of a propeller with twisted blades.

Fig. 7.23. Deformed configuration of the propeller in the body fixed frame.

Fig. 7.24. Time variation of the displacements of point P in the body fixed frame.

7.3 Multibody Dynamic Analysis

In this section the transient response of interconnected flexible bodies undergoing large deformation is analyzed. Both two and three dimensional problems have been considered. One of the salient features of this work is to utilize the floating frame of reference formulation to model flexible bodies experiencing large elastic deformations. Therefore, results from the present analysis are compared with the ones previously solved using other large deformation formulations such as the geometrically exact shell model [Simo and Fox (1989) and Simo et al. (1989)] and Absolute Nodal Coordinate [Shabana (2005)] formulation. Moreover, the robustness of the triangular element developed in the present study is demonstrated by analyzing shell structures and special geometries which cannot be discretized using rectangular elements.

7.3.1 Four-Bar Mechanism

The four-bar linkage is one of the most common mechanisms used in the multibody dynamics. In most of the cases the analysis is based on the assumption that the bars are either rigid or experience small elastic deformation. Based on the Absolute Nodal Coordinate formulation, a four-bar linkage experiencing large deformations was previously analyzed by Yakoub and Shabana (1999), Berzeri et al. (2000,2001) and Iwai and Kobayashi (2003). The schematics of the mechanism is shown in Fig. 7.25 and the details of the bars are given in Table 7.7. Note that the coupler is made of a very flexible material and has a very low bending stiffness. The bars are connected to each other

through revolute joints. The effect of gravity is taken into consideration. The crankshaft is driven by an applied moment defined as

$$M(t) = \begin{cases} M_1 \sin(\omega t) & , \quad t < T_0 \\ M_2 e^{-t/\tau} & , \quad t > T_0 \end{cases} \quad (7.4)$$

where $M_1 = 10 \text{ Nm}$, $M_2 = 466.907 \text{ Nm}$, $\omega = 3\pi$, $\tau = 0.06122914 \text{ sec}$ and $T_0 = 0.2778 \text{ sec}$. The variation of the applied with time is shown in Fig. 7.26. The crankshaft, follower and coupler are modeled using two, eight and thirty-six plate elements. A constant time step of $\Delta t = 5.0 \times 10^{-4} \text{ s}$ was chosen for the simulation

A crucial aspect of this problem is the choice of the floating frame. The fixed-axis frame has been utilized in the present analysis. As shown in Fig. 7.27 two possible choices for the fixed-axis frame are the pinned and tangent frames. The pinned frame has one node pinned to the beam and another node is on roller support. In case of the tangent, the beam is clamped at one of the nodes and all the other nodes are set free. Therefore, different frames can be obtained based on various combinations of nodes that are picked to “attach” the frame to the beam. In Fig. 7.27 only two such frames are shown. From the point of view of numerical stability, in the case of small deformation analysis, the choice of frame is not as significant as in the case of large deformation analysis. This is due to the fact that, irrespective of the frame used, all the nodal deformations remain small in linear deformation analysis. In large deformation analysis the magnitude of the FE nodal displacements depend on the choice of frame. For instance, the maximum transverse displacement in the tangent frame shown in Fig. 7.27 is much larger than the maximum displacement in the pinned frame. Therefore, the

pinned frame would require fewer elements to produce a converged solution. Hence in the present analysis a pinned frame is used for each member comprising the four-bar mechanism. The transverse displacement of the mid-point of the coupler is shown in Fig. 7.28 and the results from the current analysis are in good agreement with the results obtained using Absolute Nodal Coordinate formulation. The deformed configuration at various time intervals is shown in Fig. 7.29. Note that the coupler goes through very large deformations and its deformed configurations are fairly different in comparison to its initial configuration.

Table 7.7. Geometrical and material properties used for the four-bar mechanism.

Body	m (kg)	A (m ²)	I (m ⁴)	l (m)	E (Pa)
Crankshaft	0.6881	1.257×10^{-3}	1.257×10^{-7}	0.2	1.0×10^9
Coupler	2.4740	1.960×10^{-3}	3.068×10^{-3}	0.9	5.0×10^6
Follower	1.4700	7.068×10^{-4}	3.976×10^{-8}	0.5196174	5.0×10^8

Fig. 7.25. Schematics of a flexible four bar mechanism.

Fig. 7.26. Variation of the moment applied to the crack shaft.

Fig. 7.27. The pinned and tangent frame.

Fig. 7.28. Transverse displacement of the midpoint of the connecting rod.

Fig. 7.29. Deformed configurations of a flexible four bar mechanism.

7.3.2 Lateral Buckling of a Thin Plate

The validation of the multibody analysis involving plate elements is established by studying the dynamic buckling of a plate previously considered by Bauchau et al. (2002). As shown in Fig. 7.30, the cantilevered plate is acted upon by a crank and link mechanism. The plate has a thickness of 2.0 mm; the remaining dimensions are given in Fig. 7.30. In the study by Bauchau et al. (2002), a beam, which acts as a stiffener, is attached on the right side of the plate whereas in the current analysis a rigid clamp has been used. A constant angular velocity of 1.0 rad/s is applied to the crank, which is connected to the ground and the link, at its two ends, by means of revolute joints. The link is connected to the plate through a spherical joint. The crank and link are assumed to be rigid whereas the plate is flexible. The Young's modulus and Poisson ratio of the plate are $E = 210 \text{ GPa}$ and $\nu = 0.25$, respectively. All components have a density of $\rho = 7870 \text{ kg/m}^3$. A constant time step of $\Delta t = 0.001 \text{ s}$ was chosen for the simulation. The lateral displacement of the mid-point of the plate obtained from the current analysis is compared with the energy decaying scheme used by Bauchau et al. (2002), and the results are in good agreement, as shown in Fig. 7.31. Since no imperfections have been added, the plate is not biased to buckle in any particular direction. The deformed configurations at various time intervals are shown in Fig. 7.32.

Fig. 7.30. Schematics for lateral buckling of a thin plate.

Fig. 7.31. Transverse displacement of the plate mid-point.

Fig. 7.32. Deformed configurations of thin plate undergoing lateral buckling.

7.3.3 Unfolding Circular Strip

The next problem involves the unfolding of three circular strips that are connected to each other through revolute joints. The schematics of circular strips in folded and unfolded (assuming rigid) configurations are shown in Fig. 7.33. The top plate has one of its ends clamped whereas the other two bodies are free to exhibit rigid-body motion. The relative angle between the plates about the revolute joint is specified as

$$\theta_{motor} = \begin{cases} \frac{\theta_0}{T_0} \left[3 \times \left(\frac{t}{T_0} \right)^2 - 2 \times \left(\frac{t}{T_0} \right)^3 \right] & , \quad t < T_0 \\ \theta_0 & , \quad t > T_0 \end{cases} \quad (7.5)$$

where the steady-state relative angle, θ_0 , and the spin-up time, T_0 , are taken to be π rad and 5.0 s, respectively. The plates have a density $\rho = 7800 \text{ kg/m}^3$, Young's modulus $E = 200 \text{ GPa}$, Poisson's ratio $\nu = 0.3$, and thickness $h = 2.0 \text{ mm}$. A rigid plate is also considered for the purpose of comparison. A constant time step of $\Delta t = 0.05 \text{ sec}$ was chosen for the simulation. The deformed configurations of the flexible unfolding system at different time intervals are shown in Fig. 7.34. During the unfolding stage, the plates undergo both rigid-body motion and elastic deformations whereas after $t = 5.0 \text{ sec}$ only the elastic deformations prevail. The positions of a point P , shown in Fig. 7.33b, in the inertial coordinate systems for rigid and flexible systems are shown in Fig. 7.35. Whereas the rigid system unfolds into a flat circular strip, the flexible system continues to experience large deformations.

Fig. 7.33. Schematics of a circular strip in (a) folded and (b) unfolded configurations.

Fig. 7.34. The unfolding circular strips at several time intervals.

(a)

(b)

Fig. 7.35. Time variation of the position of point P in the inertia frame for (a) rigid and (b) flexible systems.

7.3.4 Buckling of a Cylindrical Shell

The snap through phenomenon is observed in the case of cylindrical shells under unstable conditions. A cylindrical shell connected to a crank and link mechanism is shown in Fig. 7.36. The shell consists of a 60° sector of a cylinder and has a radius of curvature of 5.0 m. The two straight sides of the shell are simply supported and the other two curved edges are set free. One of the curved edges is reinforced with a beam, as shown in Fig. 7.36. The shell has a thickness of 10.0 mm and the stiffener has a height and breadth of 15 mm and 30 mm, respectively. As shown in Fig. 7.36 the shell is connected to a crank and link mechanism at point C by means of revolute joints. The crank and the link are located directly below point C and have a length of 0.25 m and 1.0 m, respectively. Whereas the cylindrical shell and the link are modeled as flexible bodies, the crank is assumed to be rigid. All the components are made of aluminum with Young's modulus, $E = 73.0 \text{ GPa}$, Poisson ratio, $\nu = 0.3$, and density, $\rho = 2700.0 \text{ kg/m}^3$.

The following rotating is prescribed to the crank

$$\theta_c = \begin{cases} \frac{\pi}{2}(1 - \cos 2\pi t / T_0) & , \quad t < T_0 / 2 \\ \pi & , \quad t > T_0 / 2 \end{cases} \quad (7.6)$$

where $T_0 = 6.0 \text{ sec}$. As shown in Table 7.8. four different configurations are investigated that consist of two different height of the cylindrical shell. Moreover, the effect due to the presence and absence of the stiffeners is also investigated.

The transverse displacement of Points A, B and C is shown in Fig. 7.37 for Case 1. Since the height of the cylindrical shell is fairly short therefore it behaves more like a curved

beam rather than a plate as is evident from the similarity in the displacements experienced by the points B and C. Moreover, the cylindrical shell does not experience any kind of instability. The addition of a stiffener has a marked influence on the displacement variation, as shown in Fig. 7.38. The shell buckles at 1.5 sec and keeps vibrating thereafter. The variation of velocities in Case 1 and 2, as shown in Figs. 7.39 and 7.40, further clarifies the presence of the vibrations due to the snap-through phenomenon. The deformed configuration of the stiffened shell is shown in Fig. 7.41.

Fig. 7.36. A cylindrical shell connected to a crack and link mechanism.

Table 7.8. Various configurations for buckling of shell.

	Length, H	Stiffeners Present
Case1	0.5 m	No
Case2	0.5 m	Yes
Case3	2.5 m	No
Case4	2.5 m	Yes

The transverse displacement of Points A, B and C for Case 3 and 4 is shown in Fig. 7.42 and 7.43, respectively. The displacements for points B and C are fairly different from each other and therefore a plate like behavior is observed in these two cases. Unlike Case 1, in Case 3 the cylindrical shell without the stiffeners also undergoes snap through buckling. The presence of beams makes the shell stiffer as is evident from the reduction in amplitude of vibration. This observation is further seen in the variation of velocities, as shown in Fig. 7.44 and 7.45. The deformed configuration of the stiffened shell is shown in Fig. 7.46.

Fig. 7.37. Variation of transverse displacement in the absence of stiffeners.

Fig. 7.38. Variation of transverse displacement in the presence of stiffeners.

Fig. 7.39. Variation of transverse velocity in the absence of stiffeners.

Fig. 7.40. Variation of transverse velocity in the presence of stiffeners.

Fig. 7.41. Deformed of the stiffened cylindrical shell at various time intervals.

Fig. 7.42. Variation of transverse displacement in the absence of stiffeners.

Fig. 7.43. Variation of transverse displacement in the presence of stiffeners.

Fig. 7.44. Variation of transverse velocity in the absence of stiffeners.

Fig. 7.45. Variation of transverse velocity in the presence of stiffeners.

Fig. 7.46. Deformed of the stiffened cylindrical shell at various time intervals.

Fig. 7.47. Schematics of a blade with a trailing edge flap.

7.3.5 Rotor blade with Trailing Edge Flap in Vacuum

A dynamic response of a rotor blade with a trailing edge flap in vacuum, as shown in Fig. 7.47, is considered next. The blade radius, chord, and linear twist are specified as $R_b = 5.16 \text{ m}$, $c_b = 25.4 \text{ cm}$ and $\theta_{tw} = 10^\circ$, respectively. The flap has a length and chord of $R_f = 80.5 \text{ cm}$ and $c_f = 8.7 \text{ cm}$, respectively. A symmetric laminate with a stacking sequence $[(0_4^o/90_4^o/0_4^o)_2]_s$ is used for the blade, flap, and stiffeners, where 0° corresponds to the radial direction and each ply has a thickness of 0.127 mm . The graphite/epoxy ply has elastic moduli $E_{11} = 138.0 \text{ GPa}$ and $E_{22} = 8.96 \text{ GPa}$, shear moduli $G_{12} = G_{13} = 7.10 \text{ GPa}$ and $G_{23} = 3.45 \text{ GPa}$, Poisson's ratio $\nu_{12} = 0.3$, and density $\rho = 1600.0 \text{ kg/m}^3$. The spin-up maneuver imparted to the blade hub is specified as

$$\Omega_b(t) = \begin{cases} \Omega_0 \left[6(t/T_0)^5 - 15(t/T_0)^4 + 10(t/T_0)^3 \right], & t \leq T_0 \\ \Omega_0, & t \leq T_0 \end{cases} \quad (7.7)$$

where $\Omega_0 = 10\pi$ rads/sec and $T_0 = 0.1$ sec. The variation of angular velocity is shown in Fig. 7.48. Therefore, the blade experiences angular acceleration for a short time period and thereafter has a constant angular velocity of 300 RPM. The flap motion controlled by the actuator assembly is specified as

$$\theta_f = \hat{\theta}_f \times \sin(\Omega_f \times t) \quad (7.8)$$

where $\hat{\theta}_f = 5^\circ$ and $\Omega_f = 4\Omega_0$.

The blade is modeled with 42 nodes and 92 elements, whereas the flap has 4 nodes and 2 elements. A constant time step of $\Delta t = 5 \times 10^{-4}$ is used throughout the analysis. The blade is cantilevered at the root and the aerodynamic load has been neglected. The deformed configuration of the blade in the body-fixed coordinate at three different time intervals is shown in Fig. 7.49. It is evident that the blade experiences substantial deformation in the lag, flap, and torsion modes. This is due to the fairly small value chosen for T_0 , which results in a very rapid motion initially, accompanied by a high angular acceleration. The leading and trailing edge deformations of the blade tip are shown in Figs. 7.50 and 7.51, respectively. The time scale has been normalized by factor of $\Omega_0/2\pi$ and deformation has been represented in terms of percentage of blade radius. The deformation along the x' -axis, which is representative of the lag motion, and the flapping motion represented by u'_y have peak-to-peak values of around 9% and 16%, respectively. In Fig. 7.52, torsion angle θ_{tr} is depicted; its variation with time is shown in Fig. 7.53. The torsion angle θ_{tr} varies very rapidly and reaches a peak value of as

much as 40° . These deformational characteristics can be used to compute the natural frequencies of the rotor blade which can then be compared with experimental values.

Fig. 7.48. Deformed configuration of the blade in the body-fixed frame.

Fig. 7.49. Deformed configuration of the blade in the body-fixed frame.

Fig. 7.50. Variation of normalized deformation, $100 \times (u'_x/R_b)$.

Fig. 7.51. Variation of normalized deformation, $100 \times (u'_y/R_b)$.

Fig. 7.52. Description of the torsion angle, θ_{tr} .

Fig. 7.53. Variation of torsion angle, θ_{tr} , in degrees.

7.4 Rotorcraft Aeroelasticity

In this section the aeroelastic response of rotor blades is investigated using the detailed modeling techniques developed in the present study. As mentioned earlier, the difference between the various approaches for rotor blade structural modeling is in the manner in which the cross-section is represented. Models based on beam elements rely on various techniques to extract the cross-section properties and lump these values on a single node. In the present study, each component of the cross-section is modeled using either plate or

solid elements. The cross section of a rotor blade with a box-beam and D spar is shown in Fig. 7.54. The spar which is made of either isotropic or composite laminates can be modeled using plate elements. Although solid elements can also be used, such a model would be very expensive for a nonlinear dynamic analysis. Note that the composite laminates are made of several layers and each layer would require two to four layers of solid elements. Moreover, since the ply are very thin therefore the size of the solid elements in the radial and chord wise directions has to kept small in order to maintain the aspect ratio in an acceptable range. On the other hand a single layer of plate elements can be used to model the composite laminates irrespective of their thickness. The cross-section also has a core which is typically made of isotropic materials or orthotropic materials such as honeycomb. Solid elements can be used for the core and since the core is homogeneous therefore only a single layer of element will be sufficient for this purpose.

Fig. 7.54. Cross section of a (a) box-beam and (b) D spar.

7.4.1 Articulated Blade

The capabilities of present analysis are demonstrated by performing aeroelastic analysis of an articulated blade. The articulated blade has the geometry, structural and mass properties similar to the UH60 blade [Davis (1981); Hilbert (1984)]. The details of the blade geometry are given in Table 7.9. The spar of the rotor blade has a box beam structures, as shown in Fig. 7.54a. The spar is made of titanium with density $\rho = 4500.0 \text{ kg/m}^3$, elastic modulus $E = 114.0 \text{ GPa}$, and Poisson's ratio $\nu = 0.31$. The core is made of foam with density $\rho = 80.0 \text{ kg/m}^3$, elastic modulus $E = 0.861 \text{ GPa}$, and Poisson's ratio $\nu = 0.3$. A layer of fiberglass skin with density $\rho = 2440.0 \text{ kg/m}^3$, elastic modulus $E_{11} = 54.0 \text{ GPa}$, $E_{22} = E_{33} = 18.0 \text{ GPa}$, shear modulus $G_{12} = G_{13} = G_{23} = 9.0 \text{ GPa}$ and Poisson's ratio $\nu_{12} = 0.25$ covers the entire assembly. The skin has a stacking sequence $[0^\circ / 90^\circ]_5$, where 0° corresponds to the circumferential direction. Note that the titanium spar is the main structural member of the blade and the skin and core together form a sandwich type structure and therefore provide additional stiffness. A balance weight made of tungsten is lumped on the leading edge, as shown in Fig. 7.54a. The balance weight has a mass per unit length of $\rho A = 1.93 \text{ kg/m}$. The fundamental difference between the present approach and the beam models is evident from the details of the mesh shown in Fig. 7.55. Each wall of the box-beam spar is modeled using plate elements. Plate elements are used for the composite skin as well and the core is modeled using solid elements. Note that the transverse direction of the solid

elements is aligned along the radius of the blade in order to accurately capture the flap and lag bending. As shown in Fig. 7.56 the flap and lag hinges coincide at the same location which has an offset of 0.38 m. The flap segment is attached to the lag segment which in turn is attached through the pitch bearing to the pitch control mechanism. The pitch link has stiffness of $k_{PITCH} = 1.25 \times 10^6 \text{ Nm}^{-1}$. As shown in Fig. 7.56 the displacement at the bottom of the pitch link is controlled externally which results in the variation of the blade pitch. For a desired pitch variation the prescribed displacement is calculated in an approximate manner assuming that the blade does experience flap or lag motion. A triad is attached at the location of blade attachment. Note that the sequence of rotations in this particular case is lead–lag followed by flap and pitch. The orientation of the triad is monitored and then the flap, lag and pitch rotations are calculated based on the sequence of rotation. As shown in Fig. 7.56 a lag damper with $c_{LAG} = 4.0 \times 10^5 \text{ Nm}^{-1}\text{s}^{-1}$ is attached to the lag segment.

Table 7.9. Geometry of the blade.

Blade Geometry	
Blade radius	8.18 m
Blade chord	0.531 m
Linear twist	16°
Hinge offset	38.0 cm
Spar outer width	38.9 cm
Spar outer height	6.5 cm
Spar thickness	4.5 mm
Skin thickness	1.25 mm

Fig. 7.55. The mesh for the (a) entire blade (b) box-beam (c) skin and (d) core.

Fig. 7.56. Hinges and pitch control mechanism.

Three different cases are considered:

$$\text{Case 1 – Hover} \quad \mu = 0.0, \quad \theta_o = 8.9^\circ, \theta_s = 0.0^\circ, \theta_c = 0.0^\circ$$

$$\text{Case 2 – Untrimmed} \quad \mu = 0.25, \quad \theta_o = 6.45^\circ, \theta_s = 0.0^\circ, \theta_c = 0.0^\circ$$

$$\text{Case 3 – Trimmed} \quad \mu = 0.25, \quad \theta_o = 7.95^\circ, \theta_s = -4.86^\circ, \theta_c = 0.98^\circ.$$

For each of these cases the rotor RPM is set at 260.0 and a value of 0.07 is used for C_T / σ . The variation of the root flap angle for the first five cycles is shown in Fig. 7.57.

After some initial transient motion the blade attains the steady state within three cycles.

The root flap and lag angles are shown in Fig. 7.58 and 7.59, respectively. The flap angle for Case 1 settles at 4° whereas the untrimmed rotor has a huge 1/rev component.

The trimmed rotor is devoid of any 1/rev flapping component and exhibits higher harmonics of flapping. The lag is fairly unaffected due to the trimming of the rotor, as shown in Fig. 7.59. In Fig. 7.60 the pitch link load is shown and the flap bending moment at a radial section $r/R=0.72$ is shown in Fig. 7.61.

Fig. 7.57. Time variation of root flap angle.

Fig. 7.58. Steady state root flap angle.

Fig. 7.59. Steady state root lag angle.

Fig. 7.60. Pitch Link Load.

Fig. 7.61. Flap Bending Moment at $r/R=0.72$.

7.4.2 Hingeless Blade

The aeroelastic response of a hingeless rotor has been considered next. The pitch motion is applied by rotating the origin of the body-fixed frame. The blade radius, chord, and linear twist are specified as $R_b = 4.94\text{ m}$, $c_b = 39.52\text{ cm}$ and $\theta_{tw} = -9^\circ$, respectively. A swept-tip at an angle of 15° is located at a radius of 0.85 as shown in Fig. 7.62. Advanced geometry at the blade tip is used mainly to reduce the vibration due to aerodynamic loading. These advanced tips can also have significant influence on the structural dynamic response [Epps and Chandra (1996)]. In the present analysis these geometrical modifications can be implement directly from mesh generation and a separate formulation dedicated to model advanced geometries is not required. All the components of the blade are made of Graphite/Epoxy. A constant angular velocity, $\Omega_b = 360\text{ rev/min}$, is applied, and the shaft angle is set to $\alpha_s = 4^\circ$. The blade is modeled with 76 nodes and 146 elements. The salient feature of a hingeless rotor is the flexure assembly at the root as shown in Fig. 7.63. As shown in Fig. 7.64 the lag flexure is modeled with a collection of vertical plate elements. Similarly, a set of horizontal plate elements are used to model the flap flexure and the zone in-between these two flexures exhibits coupled flap-lag flexure as shown in Fig. 7.65. Note that any type of flap and lag flexure design can be modeled using the current approach with relative ease. The results from a trim analysis for an advanced ratio of $\mu = 0.25$ and $C_T/\sigma = 0.07$ is shown in Figs. 7.66–7.67. The trim analysis is similar to the one performed in the previous problem. As shown in Fig. 7.67 the flapping is devoid of $1/\text{rev}$ component.

Fig. 7.62. Mesh of a hingeless blade with a swept tip.

Fig 7.63. Schematics of the main blade and the root flexure.

Fig. 7.64. The lag flexure of a hingeless rotor blade.

Fig. 7.65. The flap flexure of a hingeless rotor blade.

Fig. 7.66. Flapping angle variation for a trimmed hingeless rotor.

Fig. 7.67. Flap and lag displacement variation for a trimmed hingeless rotor.

8. FINAL REMARKS AND FUTURE WORK

A three-dimensional approach for analyzing multibody systems undergoing large displacement and elastic deformations has been introduced in the present study. The flexible multibody formulation is based on the floating frame of reference frame. Instead of confining to linear elastic deformations with respect to the floating, the present study allows large elastic deformations. The equation of motion is obtained through the principle of virtual work. The virtual work due to inertial, internal and external work has been utilized to compute the mass matrix, tangent stiffness matrix, internal force vector and the external force vector. Due to the presence of large elastic deformations, the mass matrix is not constant and it takes into account the changing shape of the flexible bodies. The tangent stiffness matrix and the external force vector are obtained using the corotational form of the updated Lagrange formulation. The tangent stiffness matrix comprises of the linear and geometric stiffness matrices. The process of computing the stresses and internal forces has been significantly simplified using the element-free corotational formulation. A fixed axis frame has been utilized to dictate the motion of the flexible bodies with respect to the floating frame. Due to the presence of plate and solid elements a rigid link is attached to the flexible bodies prior to connecting the flexible bodies to each other. A spherical and revolute joint has been implemented in the multibody analysis tool. The resulting constraint equations have been incorporated using the coordinate partitioning method. The usage of the coordinate partitioning methods provides a pure differential form of the final system of equations. Moreover, constraint violation is also avoided. A library of implicit time integrators has been incorporated for

marching in time. These integrators range from energy conserving to controlled energy decaying methods. The generalized- α method enables to dissipate the spurious high frequency components while minimizing the decay of the low frequency response.

Since three-dimensional modeling is an integral aspect of the present work, therefore, a plate, stiffened plate and solid element has been developed. The triangular plate element with three corner nodes is based on the first order shear deformable theory. The transverse bending stiffness matrix is based on the MIN3 element which is free from shear locking. The membrane stiffness matrix is based on the ANDES formulation, which utilized drilling degrees of rotations and is well suited for elements with a high aspect ratio. The plate element, therefore, had three translational and three rotational degrees of freedom at each node. The stiffened plate element comprises of a plate element based on the first order shear deformable theory and a beam element based on the Timoshenko beam theory. Constraint equations are used to relate the beam displacement field with the plate displacement field. Since the displacements along the beam are expressed in terms of the nodal unknowns associated with the plate element, the combined stiffness matrix has the same size as the parent plate element. The membrane stiffness matrix is based on constraint strain and drilling degrees of rotation are not included. Therefore, the corresponding beam element cannot experience inplane bending. A solid wedge element has been developed, based on the isoparametric formulation. Incompatible modes have been superimposed on the linear interpolation functions in order to capture bending deformation and avoid shear locking.

A quasi-steady aerodynamic model based on the lifting line theory has been utilized to compute the aerodynamic load. Complex rotorcraft aerodynamic phenomenon such as blade vortex interaction and dynamic stall has been disregarded. A wind tunnel trim analysis has been utilized to compute the control setting that are required to produce a specific thrust and eliminate the 1/rev flapping motion.

Several problems have been analyzed in order to validate the present tool for flexible multibody and rotorcraft analysis. The plate element performs efficiently for both linear and nonlinear static problems. The results from the present analysis compare well with previously published experimental results conducted on laminated box-beams and beams with a rectangular cross-section. These problems also prove the capability of the plate element to capture various types of coupling due to material anisotropy. Although the box-beam problem can be solved using regular plate elements, the size of such a model would be considerably large due to usage of a refined mesh. The plate element developed in the present work is well suited for folded plates since it is equally efficient for transverse and membrane bending. Hence the present plate element is an ideal candidate for modeling helicopter blades. Published experimental data on natural frequencies of non-rotating and rotating composite box-beams were considered to validate the structural dynamic model based on plate element. Here as well the robustness of the plate element to model folded plate structures is established. The accuracy of the nonlinear dynamic response of the plate element, stiffener component of the stiffened plate element and the solid element were confirmed by analyzing the rapid spin-up maneuver of a rotating beam. The nonlinear effect due to centrifugal stiffening

is captured by each of these elements. Two problems involving multibody systems have been considered to check the validity of the present tool. The first problem deals with a four bar mechanism where the coupler has a very low bending stiffness, because of which large elastic deformations are observed. The second problem is concerned with the lateral buckling of a thin plate. Here as well large elastic deformations are observed and the results are in good agreement with published data based on numerical models.

A propeller with a complex shape is considered to demonstrate the capability of the present analysis to model three-dimensional bodies. Note that models based on beam elements are unable to capture chordwise bending and this drawback often leads to incorrect computation of the higher natural frequencies. The robustness of the triangular element is shown through a problem dealing with the unfolding of interconnected circular strips. The responses of the flexible and rigid strips are significantly different. The buckling of cylindrical shells is considered next. The effect of the cylinder height and the presence of a stiffener on the buckling response is investigated. The multibody dynamic capabilities of the present tool are employed to analyze a rotor blade with a trailing edge flap. The effect of aerodynamic load was disregarded in this particular problem. A detailed analysis of an articulated blade is considered in the next problem. Each member of the cross-section is modeled individually. Whereas the titanium box-beam and the Graphite-Epoxy skin are modeled using plate elements, the homogeneous core is modeled using solid elements. Lumped mass is used to represent the balance weight, which is made of tungsten. A trim analysis is conducted to eliminate the 1/rev flapping motion. A similar analysis is carried out with hingeless rotor blades as well. In this case

the flap and lead–lag flexures at the root are modeled using plate elements as opposed to using rotational spring elements.

Several modifications and improvements can be implemented to enhance the efficiency of the present work. The generalized- α method has been used for time integration in order to ensure controlled energy dissipation and therefore provide a stable scheme and accurate results. In recent years several integration schemes have been developed specifically for nonlinear and stiff problems that appear in multibody systems [Bottasso and Borri (1997); Bottasso et al. (2001); Bauchau (1988); Bauchau et al. (2003)]. The integration scheme developed by Bauchau et al. (2003) is unconditionally stable for nonlinear problems. Moreover, apart from dissipating high frequencies this scheme ensures the exact satisfaction of the constraint equations.

As mentioned in Chapter 5, the fixed axis frame has been used to govern the relative motion of the flexible bodies with respect to the floating frame. It is worth investigating the added advantage of the moving axis frames such as the Tisserand frame [Cavin and Dusto (1977); Agrawal and Shabana (1985,1986), Nikraves (2005), Lin and Nikraves (2005); Nikraves and Lin (2005)] and the Buckens frame [Canavin and Likins (1977)]. Note that with the fixed frame the largest displacement with respect to an observer stationed at the frame can be minimized. The moving reference frames on the other hand attempt to minimize either the kinetic energy or the squares of the displacements with respect to the floating frame. The usage of the moving frame for the improvement of stability characteristics for multibody systems experiencing large elastic deformations should be investigated.

The plate element used in the present work is based on the first order shear deformable theory. Although the element is free from shear locking for thin plates, it is advisable to use an element based on Kirchhoff plate theory to analyze very thin membranes. The discrete Kirchhoff triangular (DKT) developed by Batoz et al. (1980) and Batoz (1982) and the free-formulation based element developed by Madenci and Barut (1994) can be used for this purpose. Due to the combination of the ANDES formulation for membrane bending and an efficient transverse bending element based on the Kirchhoff plate, problems related to thin materials such as wrinkling in fabrics can be investigated.

The stiffened plate element developed in the present work assumes constant inplane strain inside the element and drilling degrees of freedom are not included in the element formulation. Therefore, the performance of the element for problems with significant membrane bending is not satisfactory. The present formulation for the membrane stiffness matrix should be replaced with an element that utilizes the drilling degrees of freedom [Allman (1984); Allman (1988); Frey (1989); Hughes and Brezzi (1989); Hughes et al. (1995)]. Note that the stiffened plate element can be converted into a beam element by disregarding the plate stiffness. Currently, the resulting beam element can only experience transverse bending. With the inclusion of the drilling degrees of freedom, the beam element would be free to bend in both the transverse and inplane directions.

The stiffened plate element can be utilized to model insect wings. A typical insect wing consists of the membrane and a network of veins. The thin membrane provides nominal stiffness to the wing and its main objective is to create aerodynamic lift. The veins on the other hand are the primary supporting structures in the wing. The vein network varies

widely among insects and the functional significance of the diversity in the venation pattern is still unknown [Combes and Daniel (2003)]. The stiffened plate element is an excellent choice to understand the deformational characteristics of various venation patterns.

The solid element developed as a part of the present work is based on isoparametric formulation. Incompatible modes have been used to eliminate shear locking. Unfortunately, the higher order incompatible mode can be added only along the thickness direction and therefore the element does not possess good inplane bending characteristics. The solid-shell element [Ahmad et al. (1970); Bathe (1982); Hauptmann and Schweizerhof (1998)] exhibits good bending characteristics and for thick shells its performance is either comparable or better than shell elements based on single layer theory. Solid-shell elements form a class of finite element models intermediate between thin shell and conventional solid elements. Therefore, an element based on the solid-shell formulation can be added to the element library in order to circumvent the drawbacks of the present solid element.

Several problems were considered to check the feasibility of using plate element for rotorcraft computational structural dynamics. The results from the present analysis were compared with previously conducted experimental results. The experiments dealt with simple geometries such as a beam with rectangular cross section and a box-beam. It is crucial to validate the present analysis tool against experiments where full-scale rotor blades are tested either in wind tunnels or under actual flight conditions. One source of such a set of experiments is the U.S.Army/NASA-Ames UH-60A Black Hawk Airloads

Program [Kufeld (1994)]. The data from this test program includes 1) pressure measurements at various radial and chordwise locations 2) flap, chord and torsion bending moment 3) pitch link load and 4) blade root angles. The aerodynamic loads can be applied to a detailed rotor model based on the present method and the computed moments and forces can be compared to the experimental data. Upon successful validation, the present approach can be employed to investigate the effect of blade damage due to ply failure, dents and punctures on the aeroelastic response.

In order to complement the present three-dimensional structural model an efficient aerodynamic model is required for realistic comprehensive analysis. Most of the available comprehensive tools use aerodynamic models based on lifting line theory. These models try to incorporate various phenomena relevant to rotorcraft aerodynamics based on ad-hoc methods. Although these models are able to capture most of the characteristic features, the load predictions are not accurate enough. Recently some studies have made an attempt to couple computational structural dynamic tools with sophisticated computational fluid dynamic solvers [Leishman and Bagai (2002); Potsdam et.al (2004); Datta et.al (2006)]. These studies have proven that CFD offers key improvements over the lifting line theories.

APPENDIX

Assuming constant stress inside the element, the expression for the force–lumping matrix $\mathbf{L}$ given in Eq. (3.26), that maps the boundary traction to the nodal forces has the following form

$$\mathbf{L} = \frac{1}{2} \begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ \frac{1}{6}\alpha_b y_{23}(y_{13} - y_{21}) & \frac{1}{6}\alpha_b x_{32}(x_{31} - x_{12}) & \frac{1}{3}\alpha_b(x_{31}y_{13} - x_{12}y_{21}) \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ \frac{1}{6}\alpha_b y_{31}(y_{21} - y_{32}) & \frac{1}{6}\alpha_b x_{13}(x_{12} - x_{23}) & \frac{1}{3}\alpha_b(x_{12}y_{21} - x_{23}y_{32}) \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \\ \frac{1}{6}\alpha_b y_{12}(y_{32} - y_{12}) & \frac{1}{6}\alpha_b x_{21}(x_{23} - x_{31}) & \frac{1}{3}\alpha_b(x_{23}y_{32} - x_{31}y_{13}) \end{bmatrix} \quad (\text{A.1})$$

where α_b is a scaling factor associated with the shape functions that relate the boundary normal displacements to the corner rotations [Bergan and Felippa (1985)]. In case of $\alpha_b = 0$ the effect due to corner rotations is eliminated and the nodal forces are associated only with translation degrees of freedom. This eventually leads to Eq. (A.1) and Eq. (3.26) taking the form of the force–lumping matrix and stiffness matrix for a constant strain triangle, respectively.

The matrices $\mathbf{Q}_i$ in Eq. (3.28) that relate the natural strains at the i –th node to the hierarchical rotations have the following form

$$\mathbf{Q}_1 = \frac{2A}{3} \begin{bmatrix} \frac{\beta_1}{l_{21}^2} & \frac{\beta_2}{l_{21}^2} & \frac{\beta_3}{l_{21}^2} \\ \frac{\beta_4}{l_{32}^2} & \frac{\beta_5}{l_{32}^2} & \frac{\beta_6}{l_{32}^2} \\ \frac{\beta_7}{l_{13}^2} & \frac{\beta_8}{l_{13}^2} & \frac{\beta_9}{l_{13}^2} \end{bmatrix} \quad (\text{A.2a})$$

$$\mathbf{Q}_2 = \frac{2A}{3} \begin{bmatrix} \frac{\beta_9}{l_{21}^2} & \frac{\beta_7}{l_{21}^2} & \frac{\beta_8}{l_{21}^2} \\ \frac{\beta_3}{l_{32}^2} & \frac{\beta_1}{l_{32}^2} & \frac{\beta_2}{l_{32}^2} \\ \frac{\beta_6}{l_{13}^2} & \frac{\beta_4}{l_{13}^2} & \frac{\beta_5}{l_{13}^2} \end{bmatrix} \quad (\text{A.2b})$$

$$\mathbf{Q}_3 = \frac{2A}{3} \begin{bmatrix} \frac{\beta_5}{l_{21}^2} & \frac{\beta_6}{l_{21}^2} & \frac{\beta_4}{l_{21}^2} \\ \frac{\beta_8}{l_{32}^2} & \frac{\beta_9}{l_{32}^2} & \frac{\beta_7}{l_{32}^2} \\ \frac{\beta_2}{l_{13}^2} & \frac{\beta_3}{l_{13}^2} & \frac{\beta_1}{l_{13}^2} \end{bmatrix} \quad (\text{A.2c})$$

The transformation matrix $\mathbf{T}_e$ in Eq. (3.31) which relates the Cartesian strains to the natural strains has the following form

$$\mathbf{T}_e = \frac{1}{4A^2} \begin{bmatrix} y_{23}y_{13}l_{21}^2 & y_{31}y_{21}l_{32}^2 & y_{12}y_{32}l_{13}^2 \\ x_{23}x_{13}l_{21}^2 & x_{31}x_{21}l_{32}^2 & x_{12}x_{32}l_{13}^2 \\ (y_{23}x_{31} + x_{32}y_{13})l_{21}^2 & (y_{31}x_{12} + x_{13}y_{21})l_{32}^2 & (y_{12}x_{23} + x_{21}y_{32})l_{13}^2 \end{bmatrix}. \quad (\text{A.3})$$

The matrix $\mathbf{T}_{\tilde{\theta}_u}$ in Eq. (3.32) which transforms nodal unknowns to hierarchical rotations has the following form

$$\mathbf{T}_{\tilde{\theta}u} = \frac{1}{4A} \begin{bmatrix} x_{32} & y_{32} & 4A & x_{13} & y_{13} & 0 & x_{21} & y_{21} & 0 \\ x_{32} & y_{32} & 0 & x_{13} & y_{13} & 4A & x_{21} & y_{21} & 0 \\ x_{32} & y_{32} & 0 & x_{13} & y_{13} & 0 & x_{21} & y_{21} & 4A \end{bmatrix}. \quad (\text{A.4})$$

The factor W in Eq. (3.35) can be expressed in terms of the invariants of the elasticity tensor $\mathbf{A}$. The complete expression for this factor has the following form

$$\begin{aligned} W = & -6A_{12}^3 + 5A_{11}^2A_{22} - 5A_{12}^2A_{22} - A_{22}(75A_{13}^2 + 14A_{13}A_{23} + 3A_{23}^2) \\ & + 2A_{12}(7A_{13}^2 + 46A_{13}A_{23} + 7A_{23}^2) \\ & - A_{11}(5A_{12}^2 + 3A_{13}^2 - 6A_{12}A_{22} - 5A_{22}^2 + 14A_{13}A_{23} + 75A_{23}^2) \\ & + A_{33}(3A_{11}^2 + 82A_{11}A_{22} + 3A_{22}^2 - 4(6A_{12}^2 + 5A_{13}^2 - 6A_{13}A_{23} + 5A_{23}^2)) \\ & + 4A_{33}^2(5A_{11} - 6A_{12} + 5A_{22}) \end{aligned} \quad (\text{A.5})$$

The matrices $\mathbf{B}_b$ and $\mathbf{B}_s$ in Eq. (3.45) which relate the curvatures and shear strain to the nodal unknowns can be expressed as

$$\mathbf{B}_b = \begin{bmatrix} 0 & 0 & \xi_{1,x''} & 0 & 0 & \xi_{2,x''} & 0 & 0 & \xi_{3,x''} \\ 0 & \xi_{1,y''} & 0 & 0 & \xi_{2,y''} & 0 & 0 & \xi_{3,y''} & 0 \\ 0 & \xi_{1,x''} & \xi_{1,y''} & 0 & \xi_{2,x''} & \xi_{2,y''} & 0 & \xi_{3,x''} & \xi_{3,y''} \end{bmatrix} \quad (\text{A.6a})$$

$$\mathbf{B}_s = \begin{bmatrix} \xi_{1,x''} & L_{1,x''} & M_{1,x''} + \xi_1 & \xi_{2,x''} & L_{2,x''} & M_{2,x''} + \xi_2 \\ \xi_{1,y''} & L_{1,y''} + \xi & M_{1,y''} & \xi_{2,y''} & L_{2,y''} + \xi_2 & M_{2,y''} \\ & & & \xi_{3,x''} & L_{3,x''} & M_{3,x''} + \xi_3 \\ & & & \xi_{3,y''} & L_{3,y''} + \xi_3 & M_{3,y''} \end{bmatrix}. \quad (\text{A.6b})$$

In Eq. (3.119) the components $(\tilde{C}_{mn})$ of the material property matrix, $\tilde{\mathbf{C}}_b$, are defined as follows:

$$(\tilde{C}_{11}, \tilde{C}_{12}, \tilde{C}_{22}) = \sum_{i=1}^{K_b} \int_{A_b^{(i)}} (\tilde{Q}_{11}, \tilde{Q}_{13}, \tilde{Q}_{33}) d^t A_b^{(i)} \quad (\text{A.7a})$$

$$\tilde{C}_{13} = \sum_{i=1}^{K_b} \int_{A_b^{(i)}} \left(\tilde{y}'' \tilde{Q}_{14} - z'' \tilde{Q}_{13} \right) d^t A_b^{(i)} \quad (\text{A.7b})$$

$$\tilde{C}_{23} = \sum_{i=1}^{K_b} \int_{A_b^{(i)}} \left(\tilde{y}'' \tilde{Q}_{44} - z'' \tilde{Q}_{34} \right) d^t A_b^{(i)} \quad (\text{A.7c})$$

$$\tilde{C}_{14} = \sum_{i=1}^{K_b} \int_{A_b^{(i)}} z'' \tilde{Q}_{11} d^t A_b^{(i)} \quad (\text{A.7d})$$

$$\tilde{C}_{24} = \sum_{i=1}^{K_b} \int_{A_b^{(i)}} z'' \tilde{Q}_{14} d^t A_b^{(i)} \quad (\text{A.7e})$$

$$\tilde{C}_{33} = \sum_{i=1}^{K_b} \int_{A_b^{(i)}} \left(-2\tilde{y}'' z'' \tilde{Q}_{34} + \tilde{y}''^2 \tilde{Q}_{44} - z''^2 \tilde{Q}_{33} \right) d^t A_b^{(i)} \quad (\text{A.7f})$$

$$\tilde{C}_{34} = \sum_{i=1}^{K_b} \int_{A_b^{(i)}} \left(\tilde{y}'' z'' \tilde{Q}_{14} - z''^2 \tilde{Q}_{13} \right) d^t A_b^{(i)} \quad (\text{A.7g})$$

$$\tilde{C}_{44} = \sum_{i=1}^{K_b} \int_{A_b^{(i)}} z''^2 \tilde{Q}_{11} d^t A_b^{(i)} \quad (\text{A.7h})$$

in which $\tilde{Q}_{mn}$ ($m, n = 1, 3, 4, 6$) are the components of the material property matrix.

The strain–displacement transformation matrix, $\mathbf{B}_p''$, in Eq. (3.130) is given as

$$\mathbf{B}_p'' = \begin{bmatrix} \frac{\partial}{\partial x''} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial y''} & 0 & 0 & 0 & 0 \\ \frac{\partial}{\partial y''} & \frac{\partial}{\partial x''} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial}{\partial x''} & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial y''} & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial x''} & \frac{\partial}{\partial y''} & 0 \\ 0 & 0 & \frac{\partial}{\partial x''} & 0 & 1 & 0 \\ 0 & 0 & \frac{\partial}{\partial y''} & 1 & 0 & 0 \end{bmatrix}. \quad (\text{A.8})$$

The matrix $\tilde{\mathbf{B}}_b''$ in Eq. (3.132) is given as

$$\tilde{\mathbf{B}}_b'' = \begin{bmatrix} \frac{\partial}{\partial \tilde{x}''} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial}{\partial \tilde{x}''} & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial \tilde{x}''} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial}{\partial \tilde{x}''} & 0 \end{bmatrix}. \quad (\text{A.9})$$

The vector $\tilde{\boldsymbol{\theta}}_{bNL}$ can be expressed in the form

$$\tilde{\boldsymbol{\theta}}_{bNL} = \mathcal{B}_b \tilde{\mathbf{u}}_b'' \quad (3.143)$$

The matrix differential operator $\mathcal{B}_b$ given in Eq. (3.143) has the following form

$$\mathcal{B}_b = \begin{bmatrix} \frac{\partial}{\partial \tilde{x}''} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial \tilde{x}''} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial}{\partial \tilde{x}''} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial \tilde{x}''} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial}{\partial \tilde{x}''} & 0 \end{bmatrix}. \quad (\text{A.10})$$

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