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LOW PECLET NUMBER HEAT TRANSFER IN A LAMINAR TUBE FLOW
SUBJECTED TO AXIALLY VARYING WALL HEAT FLUX

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LOW PECLET NUMBER HEAT TRANSFER IN A LAMINAR TUBE FLOW
SUBJECTED TO AXIALLY VARYING WALL HEAT FLUX

by

Brian Paul Dempsey

A Thesis Submitted to the Faculty of the
DEPARTMENT OF AEROSPACE AND MECHANICAL ENGINEERING

In Partial Fulfillment of the Requirements
For the Degree of

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WITH A MAJOR IN MECHANICAL ENGINEERING

In the Graduate College
THE UNIVERSITY OF ARIZONA

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STATEMENT BY AUTHOR

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ABSTRACT

Herein is described an analytical solution technique suitable for handling a variety of low Peclet number forced convection heat transfer problems in conduits subjected to axially varying heat flux or temperature at the conduit wall. A generalized superposition principle is used to deal with the effects of both upstream and downstream axial conduction. Results are presented for several different wall heat flux distributions including sinusoidal and hyperbolic tangent type profiles. Hyperbolic tangent profiles will better approximate conduction through tube walls than discontinuous flux distributions used in previous analyses. Results show that axial conduction effects are important not only near the inlet to a heated section but also in any part of the heated section where the wall heat flux varies rapidly with axial position.

CHAPTER 1

LOW PECLET NUMBER HEAT TRANSFER

Introduction

The problem of heat transfer to a fluid flowing in a duct has been analyzed in a wide variety of ways over the last 100 years. A very common assumption in these analyses has been to neglect the axial conduction term in the energy equation. This assumption dates back to Graetz [1] and has led to some useful and well known results [2].

Within the last 30 years interest in liquid metal heat transfer has grown enormously. Liquid metal cooled nuclear reactors are one obvious example. Any component of the reactor which experiences laminar flow may also experience low Peclet number flow. This situation can occur in a loss of coolant accident where flow is slowed by the loss of a pump or by pipe blockage. Compact heat exchangers are another area in which liquid metals are being used. Low Peclet numbers may occur here depending on the application. Currently there are many proposals calling for the use of liquid metal coolants in various space based hardware including the space station and space power systems. Once again, low Peclet numbers can be expected in some of these applications.

Since the analysis of Schneider [3] demonstrating the importance of axial conduction in the thermal entry region for low Peclet number flows, several investigators have undertaken to include

the axial conduction term. Most of these investigators have simplified the problem by assuming slug flow [3-9] or piecewise constant boundary conditions at the wall(s) [9-26]. The slug flow analyses seem to be correct; however, four of the Poiseuille flow analyses are incorrect. Travelho and Dias [22] and Pirkle and Sigillito [23] apply an initial condition to the elliptic energy equation, resulting in a mathematically ill-posed problem. Singh [24] and Hsu [25] both use an incorrect orthogonality relationship for the eigenfunctions arising in this problem.

There has been at least one analytical attempt to allow for both axial conduction and axially varying wall boundary conditions. Nagasue [26] has obtained a closed form solution to this problem by representing the boundary conditions with a power series expansion. Unfortunately, this representation makes his solution computationally impractical for sinusoidal, exponential, and other practically important boundary conditions. Other investigators have been successful in using a numerical method to solve this problem [9,27-29]. Unfortunately, results obtained by numerical methods usually have a heavy computational burden as well as practical limitations on accuracy.

Thus, there do not appear to be any reliable results or practical methods available for the solution of the important problem of fully developed flow in a conduit with axially varying wall heat flux. It is the purpose of this work to develop such a method. The method's usefulness is demonstrated by the results presented for several wall heat flux distributions of interest.

Problem Formulation

We begin with the constant property energy equation

$$\rho c_p v \frac{\partial T}{\partial z} = k \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{\partial^2 T}{\partial z^2} \right]$$

for a fully developed laminar flow of a constant property Newtonian fluid in a circular tube. The viscous dissipation term has been neglected because it is usually insignificant for low Peclet number flows; however, this term could be easily included if necessary. The analysis can also be modified to accommodate different conduit cross-sections. For a tube of circular cross-section and a laminar velocity profile the energy equation becomes

$$2\rho c_p v_0 \left[1 - (r/r_0)^2 \right] \frac{\partial T}{\partial z} = k \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{\partial^2 T}{\partial z^2} \right] \quad (1a)$$

The laminar velocity profile is the logical choice for low Peclet number flow; however, turbulent profiles present no additional complications.

We will consider (1a) with the Neumann boundary conditions

$$k \frac{\partial T(z, r_0)}{\partial r} = \begin{cases} 0 & -\infty < z < 0 \\ q(z) & 0 \leq z \leq L \\ q_\infty & L < z < \infty \end{cases} \quad (1b)$$

and

$$\lim_{z \rightarrow -\infty} T(z, r) = T_0 \quad (1c)$$

The method presented can also be used to solve the corresponding Dirichlet problem, namely,

$$T(z, r_0) = \begin{cases} T_0 & -\infty < z < 0 \\ T_1(z) & 0 \leq z \leq \sigma \\ T_\infty & \sigma < z < \infty \end{cases} \quad (1d)$$

Equations (1a-c) can be nondimensionalized by

$$\sigma = L/(r_0 Pe), \quad \eta = z/(r_0 Pe), \quad \xi = r/r_0, \quad \gamma = q_\infty/q_{\max}$$

$$b(\eta) = q(z)/q_{\max} \text{ and } \theta(\eta, \xi) = k(T - T_0)/(q_{\max} r_0),$$

where $q_{\max}$ is the maximum of $|q(z)|$ in the interval $[0, L]$. After substitution, (1a-c) become

$$(1 - \xi^2) \frac{\partial \theta}{\partial \eta} = \frac{1}{\xi} \frac{\partial}{\partial \xi} \left(\xi \frac{\partial \theta}{\partial \xi} \right) + \frac{1}{Pe^2} \frac{\partial^2 \theta}{\partial \eta^2} \quad (2a)$$

$$\frac{\partial \theta(\eta, 1)}{\partial \xi} = \begin{cases} 0 & -\infty < \eta < 0 \\ b(\eta) & 0 \leq \eta \leq \sigma \\ \gamma & \sigma < \eta < \infty \end{cases} \quad (2b)$$

and

$$\lim_{\eta \rightarrow -\infty} \theta(\eta, \xi) = 0, \quad (2c)$$

where

$$Pe = \frac{2\rho c_p V_0 r_0}{k}$$

The other boundary condition in the η direction is that, at large η , the solution becomes fully developed. We assume that in the interval $[0, \sigma]$, $b(\eta)$ has a finite number of discontinuities, say m . We let $b^+(\eta_j)$ and $b^-(\eta_j)$ be the right and left limits of $b(\eta)$ at the j^{th} discontinuity and write

$$\begin{aligned} b(\eta) &= b_1(\eta) + \sum_{j=0}^m [b^+(\eta_j) - b^-(\eta_j)] U(\eta - \eta_j) \\ &= b_1(\eta) + b_2(\eta) \end{aligned}$$

where $U(y)$ is the unit step function

$$U(y) = \begin{cases} 0 & y < 0 \\ 1 & y \geq 0 \end{cases}$$

Thus $b_1(\eta)$ is continuous, piecewise differentiable, and $b_1(0) = 0$. [Note that $b_1(\eta)$ may be defined on a piecewise basis, e.g., $b_1(\eta) = \sin \eta$ for $0 < \eta \leq \pi$ and $b_1(\eta) = \eta - \pi$ for $\pi < \eta$.]

We now decompose $\theta(\eta, \xi)$ into three parts

$$\theta(\eta, \xi) = \sum_{i=1}^3 \theta_i(\eta, \xi) \quad (3)$$

where each θ_i satisfies (2a and c) and

$$\frac{\partial \theta_k(\eta, 1)}{\partial \xi} = \begin{cases} 0 & -\infty < \eta < 0 \\ b_k(\eta) & 0 \leq \eta \leq \sigma \\ 0 & \sigma < \eta < \infty \end{cases} \quad k = 1, 2 \quad (4a)$$

and

$$\frac{\partial \theta_3(\eta, 1)}{\partial \xi} = \begin{cases} 0 & -\infty < \eta < 0 \\ 0 & 0 \leq \eta \leq \sigma \\ \gamma & \sigma < \eta < \infty \end{cases} \quad (4b)$$

The linearity and homogeneity of the energy equation ensures that the sum of θ_1 , θ_2 , and θ_3 satisfies (2a-c).

Construction of the Solution

Consider, for $0 < \eta^* < \sigma$, the problem

$$(1-\xi^2) \frac{\partial \theta^*(\eta, \xi; \eta^*)}{\partial \eta} = \frac{1}{\xi} \frac{\partial}{\partial \xi} \left[\xi \frac{\partial \theta^*(\eta, \xi; \eta^*)}{\partial \xi} \right] + \frac{1}{Pe^2} \frac{\partial^2 \theta^*(\eta, \xi; \eta^*)}{\partial \eta^2} \quad (5a)$$

which represents the response to a step in heat flux beginning at $\eta = \eta^*$ and ending at $\eta = \sigma$. This equation is subject to the piecewise constant boundary conditions

$$\frac{\partial \theta^*(\eta, 1; \eta^*)}{\partial \xi} = \begin{cases} 0 & -\infty < \eta < \eta^* & (5b,i) \\ 1 & \eta^* \leq \eta \leq \sigma & (5b,ii) \\ 0 & \sigma < \eta < \infty & (5b,iii) \end{cases}$$

$$\lim_{\eta \rightarrow -\infty} \theta^*(\eta, \xi; \eta^*) = 0 \quad (5c)$$

We then represent θ^* as

$$\theta^*(\eta, \xi; \eta^*) = \begin{cases} \theta_1^*(\eta, \xi; \eta^*) & -\infty < \eta < \eta^* \\ \theta_2^*(\eta, \xi; \eta^*) + 4(\eta - \eta^*) + \xi^2 - \xi^4/4 + 8/Pe^2 - 7/24 & \eta^* \leq \eta \leq \sigma \\ \theta_3^*(\eta, \xi; \eta^*) + 4(\sigma - \eta^*) & \sigma < \eta < \infty \end{cases} \quad (6a)$$

and require that each θ_i^* satisfy (5a). The terms added to θ_2^* and θ_3^* satisfy the nonhomogeneous boundary condition (5b,ii), and represent the fully developed solution far downstream, respectively. These terms can be developed by starting with (5a) and assuming a solution, far downstream, of the form $\theta_{fd}(\eta, \xi) = C_1 \eta + \psi(\xi)$. After substitution into (2a) we are left with an ordinary differential equation which can be solved by integrating twice. The resulting constants can be evaluated by applying the boundary conditions

$$\theta_{fd}(\eta, 0) \quad \text{bounded}$$

and

$$\frac{\partial \theta_{fd}(\eta, 1)}{\partial \xi} = 1$$

and an integral auxiliary condition

$$-\eta = \int_0^1 (1-\xi^2) \xi \theta_{fd} d\xi - \frac{1}{2Pe^2} \frac{\partial \theta_{fd}}{\partial \eta}$$

which can be found by extending an energy balance [33] to the case where axial conduction is included. Also appended to (5a) are

$$\frac{\partial \theta_i^*(\eta, 1; \eta^*)}{\partial \xi} = 0 \quad i = 1, 2, 3 \quad (6b)$$

and

$$\lim_{\eta \rightarrow -\infty} \theta^*(\eta, \xi; \eta^*) = 0 \quad (6c)$$

along with continuity of temperature

$$\theta_1^*(\eta^*, \xi; \eta^*) = \theta_2^*(\eta^*, \xi; \eta^*) + \xi^2 - \xi^4/4 + 8/Pe^2 - 7/24 \quad (6d)$$

$$\theta_2^*(\sigma, \xi; \eta^*) + \xi^2 - \xi^4/4 + 8/Pe^2 - 7/24 = \theta_3^*(\sigma, \xi; \eta^*) \quad (6e)$$

and heat flux

$$\frac{\partial \theta_1^*(\eta^*, \xi; \eta^*)}{\partial \eta} = \frac{\partial \theta_2^*(\eta^*, \xi; \eta^*)}{\partial \eta} + 4 \quad (6f)$$

$$4 + \frac{\partial \theta_2^*(\sigma, \xi; \eta^*)}{\partial \eta} = \frac{\partial \theta_3^*(\sigma, \xi; \eta^*)}{\partial \eta} \quad (6g)$$

at $\eta = \eta^*$ and $\eta = \sigma$.

From [12], we know that the functions θ_i^* can be expanded as

$$\theta_1^*(\eta, \xi; \eta^*) = \sum_{n=1}^{\infty} A_{n1}(\eta^*) e^{\lambda_n \eta} R_n(\xi) \quad (7a)$$

$$\theta_2^*(\eta, \xi; \eta^*) = \sum_{n=-\infty}^{\infty} A_{n2}(\eta^*) e^{\lambda_n \eta} R_n(\xi) \quad (7b)$$

$$\theta_3^*(\eta, \xi; \eta^*) = \sum_{n=-\infty}^{-1} A_{n3}(\eta^*) e^{\lambda_n \eta} R_n(\xi) \quad (7c)$$

Upon substitution we obtain the eigenvalue problem

$$R_n''(\xi) + \frac{1}{\xi} R_n'(\xi) + \lambda_n \left[\frac{\lambda_n}{Pe^2} - (1-\xi^2) \right] R_n(\xi) = 0 \quad (8a)$$

$$R_n'(0) = 0 \quad (8b)$$

$$R_n'(1) = 0 \quad (8c)$$

The boundary conditions (8b) and (8c) arise from a boundedness requirement at $r=0$, and (6b), respectively. Pearlstein [30] has shown that the system (8a-c) has only real eigenvalues λ_n .

The validity of the expansions (7a-c) depends on the completeness of the eigenfunctions, and in (7a,c) on the completeness of two subsets corresponding to positive and negative eigenvalues, respectively. Although no explicit results are available regarding the completeness of the eigenfunctions of (8a-c), Deavours [17] has shown that

the eigenfunctions of a related problem for heat transfer to a fluid in Poiseuille flow between plane parallel plates are complete.

From (6d,f) and (7a,b) we have

$$\sum_{n=1}^{\infty} A_{n1} (\eta^*) e^{\lambda_n \eta^*} R_n(\xi) = \sum_{n=-\infty}^{\infty} A_{n2} (\eta^*) e^{\lambda_n \eta^*} R_n(\xi) + \xi^2 - \xi^4/4 + 8/Pe^2 - 7/24 \quad (9a)$$

and

$$\sum_{n=1}^{\infty} A_{n1} (\eta^*) \lambda_n e^{\lambda_n \eta^*} R_n(\xi) = \sum_{n=-\infty}^{\infty} A_{n2} (\eta^*) \lambda_n e^{\lambda_n \eta^*} R_n(\xi) + 4 \quad (9b)$$

Similarly, (6e,g) require

$$\begin{aligned} \sum_{n=-\infty}^{\infty} A_{n2} (\eta^*) e^{\lambda_n \sigma} R_n(\xi) + \xi^2 - \xi^4/4 + 8/Pe^2 - 7/24 \\ = \sum_{n=-\infty}^{-1} A_{n3} (\eta^*) e^{\lambda_n \sigma} R_n(\xi) \end{aligned} \quad (9c)$$

and

$$\sum_{n=-\infty}^{\infty} A_{n2} (\eta^*) \lambda_n e^{\lambda_n \sigma} R_n(\xi) + 4 = \sum_{n=-\infty}^{-1} A_{n3} (\eta^*) \lambda_n e^{\lambda_n \sigma} R_n(\xi) \quad (9d)$$

Equations (9a-d) can be written more compactly as

$$\sum_{n=1}^{\infty} [A_{n1}(\eta^*) - A_{n2}(\eta^*)] e^{\lambda_n \eta^*} \begin{bmatrix} R_n(\xi) \\ \lambda_n R_n(\xi) \end{bmatrix} - \sum_{n=-\infty}^0 A_{n2}(\eta^*) e^{\lambda_n \eta^*} \begin{bmatrix} R_n(\xi) \\ \lambda_n R_n(\xi) \end{bmatrix} =$$

$$\begin{bmatrix} \xi^2 - \xi^4/4 + 8/Pe^2 - 7/24 \\ 4 \end{bmatrix} \quad (10a)$$

$$\sum_{n=-\infty}^{-1} [A_{n3}(\eta^*) - A_{n2}(\eta^*)] e^{\lambda_n \sigma} \begin{bmatrix} R_n(\xi) \\ \lambda_n R_n(\xi) \end{bmatrix} - \sum_{n=0}^{\infty} A_{n2}(\eta^*) e^{\lambda_n \sigma} \begin{bmatrix} R_n(\xi) \\ \lambda_n R_n(\xi) \end{bmatrix} =$$

$$\begin{bmatrix} \xi^2 - \xi^4/4 + 8/Pe^2 - 7/24 \\ 4 \end{bmatrix} \quad (10b)$$

It has been pointed out on numerous occasions that the eigenfunctions defined by (8a-c) are not orthogonal with respect to the usual inner product. They are, however, orthogonal with respect to several other inner products, as shown by Pearlstein [30], and Smith et al. [10]. We follow the method of Smith et al. and take the scalar product of

$$\xi R_m(\xi) \begin{bmatrix} \lambda_m/Pe^2 - (1-\xi^2) \\ 1/Pe^2 \end{bmatrix} \quad (11)$$

with (10a) and (10b) to obtain

$$[A_{m1}(\eta^*) - A_{m2}(\eta^*)] e^{\lambda_m \eta^*} B_m = C_m \quad 1 \leq m \quad (12a)$$

$$- A_{m2}(\eta^*) e^{\lambda_m \eta^*} B_m = C_m \quad 0 \geq m \quad (12b)$$

$$[A_{m3}(\eta^*) - A_{m2}(\eta^*)] e^{\lambda_m \sigma} B_m = C_m \quad 0 \geq m \quad (12c)$$

$$A_{m2}(\eta^*) e^{\lambda_m \sigma} B_m = C_m \quad 1 \leq m \quad (12d)$$

where

$$B_m = \int_0^1 \left[2\lambda_m / Pe^2 - (1 - \xi^2) \right] R_m^2(\xi) \xi d\xi \quad (13a)$$

$$C_m = \int_0^1 \left[\left(\lambda_m / Pe^2 - 1 + \xi^2 \right) \left(\xi^2 - \xi^4 / 4 + 8 / Pe^2 - 7 / 24 \right) + 4 / Pe^2 \right] R_m(\xi)$$

$\xi d\xi$

(13b)

Equations (12a-d) can be solved to obtain

$$A_{m1}(\eta^*) = F_m \left[e^{-\lambda_m \eta^*} - e^{-\lambda_m \sigma} \right] \quad (14a)$$

$$A_{m2}(\eta^*) = \begin{cases} -F_m e^{-\lambda_m \eta^*} & m \leq 0 \\ -F_m e^{-\lambda_m \sigma} & m \geq 1 \end{cases} \quad (14b)$$

$$(14c)$$

and

$$A_{m3}(\eta^*) = F_m \left[e^{-\lambda_m \sigma} - e^{-\lambda_m \eta^*} \right] \quad (14d)$$

where

$$F_m = C_m / B_m. \quad (15)$$

Thus the solution $\theta^*(\eta, \xi; \eta^*)$ of (5a-c) is given by (6a), (7a-c) and (13-15).

We now proceed to show how the solutions $\theta_k(\eta, \xi)$ ($k=1,2,3$) in (3) can be constructed using $\theta^*(\eta, \xi; \eta^*)$. We begin with $\theta_2(\eta, \xi)$, for which it is clear from (1a,b), (4a) and (5b) that

$$\theta_2(\eta, \xi) = \sum_{j=0}^m [b^+(\eta_j) - b^-(\eta_j)] \theta^*(\eta, \xi; \eta_j) \quad (16)$$

Here θ_2 results from the superposition of the heat flux jumps in (4a); substitution of (6a) into (16) gives

$$\theta_2(\eta, \xi) = \left\{ \begin{array}{l} \sum_{j=0}^m [b^+(\eta_j) - b^-(\eta_j)] \left[\sum_{n=1}^{\infty} A_{n1}(\eta_j) e^{\lambda_n \eta} R_n(\xi) \right] \\ -\infty < \eta \leq \eta_j \end{array} \right. \quad (17a)$$

$$\theta_2(\eta, \xi) = \left\{ \begin{array}{l} \sum_{j=0}^m [b^+(\eta_j) - b^-(\eta_j)] \left[\sum_{n=-\infty}^{\infty} A_{n2}(\eta_j) e^{\lambda_n \eta} R_n(\xi) \right. \\ \left. + 4(\eta - \eta_j) + \xi^2 - \xi^4/4 + 8/Pe^2 - 7/24 \right] \\ \eta_j \leq \eta \leq \sigma \end{array} \right. \quad (17b)$$

$$\theta_2(\eta, \xi) = \left\{ \begin{array}{l} \sum_{j=0}^m [b^+(\eta_j) - b^-(\eta_j)] \left[\sum_{n=-\infty}^{-1} A_{n3}(\eta_j) e^{\lambda_n \eta} R_n(\xi) + 4(\sigma - \eta_j) \right] \\ \sigma < \eta \leq \infty \end{array} \right. \quad (17c)$$

where A_{n1} , A_{n2} and A_{n3} are given by (14a-d).

The response, $\theta_1(\eta, \xi)$, to the continuous part of the heat flux in $[0, \sigma]$ is found similarly. For $\theta_1(\eta, \xi)$, however, we begin with the general superposition principle

$$\theta_1(\eta, \xi) = - \int_0^\sigma b_1(\eta) \left[\frac{\partial \theta^*(\eta, \xi; \eta^*)}{\partial \eta^*} \right] d\eta^* \quad (18)$$

which can be found in, e.g., Ozisik [31]. (Duhamel's theorem cannot be used here because it is applicable to parabolic rather than elliptic problems.) Integrating by parts, we obtain

$$\begin{aligned} \theta_1(\eta, \xi) = & -\theta^*(\eta, \xi; \sigma)b_1(\sigma) + \theta^*(\eta, \xi; 0)b_1(0) \\ & + \int_0^\sigma \frac{db_1(\eta^*)}{d\eta^*} \theta^*(\eta, \xi; \eta^*) d\eta^* \end{aligned} \quad (19)$$

Using (6a) we easily find the solution in the intervals $(-\infty, 0]$ and $[\sigma, \infty)$ to be

$$\theta_1(\eta, \xi) = \begin{cases} \int_0^\sigma \frac{db_1(\eta^*)}{d\eta^*} \theta_1^*(\eta, \xi; \eta^*) d\eta^* & -\infty < \eta \leq 0 \quad (20a) \\ \int_0^\sigma \frac{db_1(\eta^*)}{d\eta^*} [\theta_3^*(\eta, \xi; \eta^*) + 4(\sigma - \eta^*)] d\eta^* & \sigma \leq \eta < \infty \quad (20b) \end{cases}$$

To develop a representation for θ_1 in the interval $[0, \sigma]$ we realize that the solution at any point η consists of a contribution from the wall heat flux variation upstream of η along with a contribution from

the downstream wall heat flux variation. Thus, in the heated region,

(19) becomes

$$\begin{aligned}
 \theta_1(\eta, \xi) = & \int_0^\eta \frac{db_1(\eta^*)}{d\eta^*} \left[\theta_2^*(\eta, \xi; \eta^*) + 4(\eta - \eta^*) + \xi^2 - \xi^4/4 + 8/Pe^2 - 7/24 \right] d\eta^* \\
 & + \int_\eta^\sigma \frac{db_1(\eta^*)}{d\eta^*} \theta_1^*(\eta, \xi; \eta^*) d\eta^* - b_1(\eta^*) \theta_1^*(\eta, \xi; \eta^*) \Big|_0^\eta \\
 & - b_1(\eta^*) \left[\theta_2^*(\eta, \xi; \eta^*) + \xi^2 - \xi^4/4 - 7/24 + 8/Pe^2 + 4(\eta - \eta^*) \right]_\eta^\sigma \\
 & \qquad \qquad \qquad 0 \leq \eta \leq \sigma \qquad (20c)
 \end{aligned}$$

It is clear from the definition of $b_1(\eta)$ and $\theta^*(\eta, \xi; \eta^*)$ that for η in the interval $(-\infty, 0]$ or $[\sigma, \infty)$ the integrated part of (19) vanishes. For η in the interval $[0, \sigma]$, however, the integrated part vanishes only after application of the matching conditions (6d,e). The three parts of $\theta_1(\eta, \xi)$ are now assembled and (7a-c) are substituted into (20a-c) to obtain the final form

$$\theta_1(\eta, \xi) = \left\{ \begin{array}{ll} \int_0^\sigma \frac{db_1(\eta^*)}{d\eta^*} \sum_{n=1}^{\infty} A_{n1}(\eta^*) e^{\lambda_n \eta} R_n(\xi) d\eta^* & -\infty < \eta \leq 0 \quad (21a) \\ \int_\eta^\sigma \frac{db_1(\eta^*)}{d\eta^*} \sum_{n=1}^{\infty} A_{n1}(\eta^*) e^{\lambda_n \eta} R_n(\xi) d\eta^* \\ + \int_0^\eta \frac{db_1(\eta^*)}{d\eta^*} \sum_{n=-\infty}^{-1} A_{n2}(\eta^*) e^{\lambda_n \eta} R_n(\xi) d\eta^* \\ + \int_0^\eta \frac{db_1(\eta^*)}{d\eta^*} \sum_{n=1}^{\infty} A_{n2}(\eta^*) e^{\lambda_n \eta} R_n(\xi) d\eta^* \\ + \int_0^\eta \frac{db_1(\eta^*)}{d\eta^*} [4(\eta - \eta^*) + \xi^2 - \xi^4/4 - 7/24 + 8/Pe^2] d\eta^* & 0 \leq \eta \leq \sigma \quad (21b) \\ \int_0^\sigma \frac{db_1(\eta^*)}{d\eta^*} \left[\sum_{n=-\infty}^{-1} A_{n3}(\eta^*) e^{\lambda_n \eta} R_n(\xi) + 4(\sigma - \eta^*) \right] d\eta^* & \sigma \leq \eta < \infty \quad (21c) \end{array} \right.$$

where A_{n1} , A_{n2} , and A_{n3} are given by (14a-d).

The development of $\theta_3(\eta, \xi)$ is slightly different from the development of θ_1 and θ_2 . We start by defining $\theta_{3u}(\eta, \xi)$ and $\theta_{3d}(\eta, \xi)$ such that

$$\theta_{3u}(\eta, \xi) \quad \eta \leq \sigma \quad (22a)$$

$$\theta_{3d}(\eta, \xi) \quad \eta \geq \sigma \quad (22b)$$

We proceed as before with an eigenfunction expansion

$$\theta_{3u} = \sum_{n=1}^{\infty} A_{n4} e^{\lambda_n \eta} R_n(\xi) \quad (23a)$$

$$\theta_{3d} = \sum_{n=-\infty}^{-1} A_{n5} e^{\lambda_n \eta} R_n(\xi) + \gamma[4(\eta-\sigma) + \xi^2 - \xi^4/4 - 7/24 + 8/Pe^2] \quad (23b)$$

Matching the temperatures and heat fluxes at σ we obtain

$$\theta_{3u}(\sigma, \xi) = \theta_{3d}(\sigma, \xi) \quad (24a)$$

and

$$\frac{\partial \theta_{3u}(\sigma, \xi)}{\partial \eta} = \frac{\partial \theta_{3d}(\sigma, \xi)}{\partial \eta} \quad (24b)$$

Applying the matching conditions and writing the result in vector form

we have

$$\sum_{n=1}^{\infty} A_{n4} e^{\lambda_n \sigma} \begin{bmatrix} R_n(\xi) \\ \lambda_n R_n(\xi) \end{bmatrix} - \sum_{n=-\infty}^{-1} A_{n5} e^{\lambda_n \sigma} \begin{bmatrix} R_n(\xi) \\ \lambda_n R_n(\xi) \end{bmatrix} = \begin{bmatrix} \gamma(\xi^2 - \xi^4/4 - 7/24 + 8/Pe^2) \\ 4\gamma \end{bmatrix} \quad (25)$$

Taking the scalar product of (11) with (25) we find

$$A_{m4} e^{\lambda_m \sigma} B_m = \gamma C_m \quad m \geq 1 \quad (26a)$$

and

$$-A_{m5} e^{\lambda_m \sigma} B_m = \gamma C_m \quad m \leq -1 \quad (26b)$$

where B_m and C_m are defined by (13a,b), respectively. Equations (26a,b) can be solved to obtain

$$A_{m4} = \gamma F_m e^{-\lambda_m \sigma} \quad m \geq 1 \quad (27a)$$

$$A_{m5} = -\gamma F_m e^{-\lambda_m \sigma} \quad m \leq -1 \quad (27b)$$

where $F_m = C_m/B_m$. Assembling θ_{3u} and θ_{3d} we have

$$\theta_3(\eta, \xi) = \begin{cases} \sum_{n=1}^{\sigma} A_{n4} e^{\lambda_n \eta} R_n(\xi) & \eta \leq \sigma \quad (28a) \\ \sum_{n=-\infty}^{-1} A_{n5} e^{\lambda_n \eta} R_n(\xi) + \\ \gamma[4(\eta - \sigma) + \xi^2 - \xi^4/4 - 7/24 + 8/Pe^2] & \eta \geq \sigma \quad (28b) \end{cases}$$

Computational Aspects

The eigenvalues of (8a-c) are determined using an Adams method to solve a system of two first order ordinary differential equations. A shooting method was used to find the eigenvalues λ_n . The Adams method used is implemented in a program developed by Shampine [32]. Forty positive and forty negative eigenvalues having ten correct significant figures each were calculated for Peclet numbers 1 and 5. For reasons which will be discussed later, only twenty positive and twenty negative eigenvalues were calculated for Peclet numbers 10, 20 and 50. Using this method on a Cyber 175 it takes less than 400 cpu seconds to calculate eighty eigenvalues. These eigenvalues agree exactly with previous work [12].

The expansion coefficients F_m are calculated using a Simpson composite integration routine. The coefficients for the first twenty positive and first twenty negative eigenvalues were calculated to an accuracy of at least six correct figures. At the 40th positive or negative eigenvalue we maintain at least four correct significant figures.

The accuracy of the total solution is dependent on the type of heat flux distribution and the axial position for which the solution is being obtained. For constant heat flux the solution is generally accurate to much better than 1%, unless the axial position is very close to the beginning of the heated section.

For most axially varying wall heat flux distributions the n th term in the series approaches zero like $(-1)^n \lambda_n^{-1}$, instead of exponentially. Since the magnitude of the eigenvalues increases more slowly for small Peclet numbers than for large Peclet numbers, more eigenvalues are needed at small Peclet numbers to maintain accuracy. The error for axially varying heat fluxes is normally less than 1% but can reach 10% for rapidly changing flux distributions and Peclet numbers less than 5. The calculation of the total solution is very efficient. To calculate $\theta(n, \xi)$ at 800 points in the tube requires between three and ten cpu seconds. The most costly part of the entire calculation is the one-time calculation of the eigenvalues and the expansion coefficients. The fact that the eigenfunctions are orthogonal, and only very small matrices need be handled means that the entire calculation of

$\theta(\eta, \xi)$ can be done to a high degree of accuracy, in a reasonable amount of time, on a personal computer.

Results

Results have been obtained for a variety of wall heat flux distributions (constant and axially varying). The constant case has been included to allow comparison to previous results of other investigators. The axially varying cases were selected to show that this method can handle many physical situations and to show the importance of axial conduction in certain problems.

Figures 1-3 show the effects of a unit step in heat flux at $\eta=0$ on radial temperature profiles and Nusselt numbers. The results shown in Figure 1 for $Pe = 5$ are virtually identical to results obtained by Papoutsakis et al. [11]. Figure 2 shows the effect of increasing the Peclet number to 10. Nusselt number versus axial position for the unit step problem is shown in Figure 3. These results agree with those of Hsu [15] to within one percent.

To test the accuracy with which the matching equations (6d-g) are satisfied, results were obtained using both of the applicable solutions for a unit step at $\eta = 0$. The difference in the two solutions at $\eta=0$ is shown in Figure 4 for $Pe = 1$ and $Pe = 20$. The relative error $(100 - 100 \theta_2/\theta_1)$ at $\eta=0$ is shown in Figure 5 for $Pe = 1$. The relative error for $Pe = 20$ is omitted because the magnitude of the solution near the center line is on the order of 10^{-6} . Thus a small absolute error leads to a large relative error. It should be

Fig. 1. Temperature Profiles for $Pe = 5$ Near the Beginning of Heating for a Unit Step in Heat Flux at the Wall, $\sigma = 2.0$.

Fig. 2. Temperature Profiles for $Pe=10$ Near the Beginning of Heating for a Unit Step in Heat Flux at the Wall, $\sigma = 1.0$.

Fig. 3. Nusselt Number versus Axial Position for a Unit Step in Heat Flux at $\eta = 0$.

Fig. 4. Accuracy of Matching at $\eta = 0.0$ for a Unit Step in Wall Heat Flux.

Fig. 5. Accuracy of Matching at $\eta = 0.0$ for a Unit Step in Wall Heat Flux, $Pe = 1$.

noted that if the wall heat flux is continuous at $\eta = 0$ or $\eta = \sigma$ it can be shown that the matching is exact.

The first axially varying wall heat flux analyzed is $\sin \frac{\pi}{\sigma} \eta$. This function was chosen because of its application to cooling of nuclear reactors. Figures 6-8 show radial temperature profiles near the end of the heated section for Peclet numbers 1, 5 and 10. The aspect ratio (L/r_0) of the heated section remains constant for this and all other axially varying heat flux distributions. The rapid approach towards a flat radial temperature profile at the exit from the heated section for $Pe = 1$ (Figure 6) is the key feature of these figures.

The second axially varying case is a hyperbolic tangent heat flux distribution. This distribution is chosen because it better approximates conduction through tube walls of finite thickness than do discontinuous flux distributions used in previous analyses. Figure 9 shows the bulk temperatures for Peclet numbers 1, 5 and 20 for a hyperbolic tangent distribution which slowly changes from 0 to $(1 + \tanh 1)/2$. This would correspond to a tube wall with a low thermal conductivity. If the fully developed temperature solution discussed below (6a) is integrated to determine the fully developed bulk temperature we find that $\theta_{\text{bulk}} = 2(\eta - \sigma) + 7/32 + 4/Pe^2$; thus we expect to see the large difference between bulk temperatures for $Pe = 1$ and $Pe = 5$ shown in Figure 9. [We note here that in the case with no axial conduction, $\theta_{\text{bulk}}(\eta)$ cannot be obtained by direct integration (in ξ then η) of (2a).] Figures 10-12 show radial temperature profiles for a hyperbolic tangent distribution which changes rapidly from 0

Fig. 6. Radial Temperature Profiles Near the end of Heating for $Pe = 1$, $\sigma = 10.0$ and Heat Flux, q , Indicated.

Fig. 7. Radial Temperature Profiles Near the end of Heating for $Pe = 5$, $\sigma = 2.0$ and Heat Flux, q , Indicated.

Fig. 8. Radial Temperature Profiles Near the end of Heating for $Pe = 10$, $\sigma = 1.0$ and Heat Flux, q , Indicated.

Fig. 9. Bulk Temperature for Peclet Numbers 1, 5, 20 and Heat Flux, q , indicated.

Fig. 10. Radial Temperature Profiles for $Pe = 1$, $\sigma = 10.0$ and Heat Flux, q , Indicated.

Fig. 11. Radial Temperature Profiles for $Pe = 5$, $\sigma = 2.0$ and Heat Flux, q , Indicated.

Fig. 12. Radial Temperature Profiles for $Pe = 20$, $\sigma = 0.5$ and Heat Flux, q , Indicated.

to $(1 + \tanh 5)/2$. This distribution corresponds to a tube wall with high thermal conductivity.

The last variable wall heat flux considered is $\sin^2 \frac{4\pi}{\sigma} \eta$.

Figures 13-15 show radial temperature profiles for Peclet numbers 1, 5 and 50. These figures show the importance of the axial conduction term near the point $\eta = \sigma/4$. Note that Peclet number 50 roughly corresponds to no axial conduction. Figure 16 shows a plot of the bulk temperature for the $\sin^2 \frac{4\pi}{\sigma} \eta$ distribution for Peclet numbers 1, 5 and 10.

Discussion

The analysis herein presents a method of solving the elliptic energy equation for fully developed flow in a tube with axially varying wall heat flux. The computational aspects have proven to be no more difficult or time-consuming than previous analyses which consider only piecewise constant boundary conditions. The accuracy of this method is as good as or better than previous analyses which have excluded axial conduction. As is the case with almost all analytical solutions, this method is faster, more accurate and more versatile than numerical solutions. The results from the test cases compare very well with previous studies. The solutions to variable wall heat flux problems all follow expected trends. In many cases, for Peclet numbers above 30 axial conduction can be neglected. However, since this method is fast and accurate, there is no reason that it should not be used for Peclet numbers up to 100.

Fig. 13. Radial Temperature Profiles for $Pe = 1$, $\sigma = 10.0$ and Heat Flux, q , Indicated.

Fig. 14. Radial Temperature Profiles for $Pe = 5$, $\sigma = 2.0$ and Heat Flux, q , Indicated.

Fig. 15. Radial Temperature Profiles for $Pe = 50$, $\sigma = 0.2$ and Heat Flux, q , Indicated.

Fig. 16. Bulk Temperatures for Peclet Numbers $Pe = 1, 5, 10$ and Heat Flux, q , Indicated

APPENDIX A

NOTATION

A_{n1}, A_{n2}, A_{n3}	expansion functions (7a-c)
B_m	constant resulting from inner product relation (13a)
$b(\eta)$	nondimensional heat flux in $[0, \sigma]$
C_m	constant resulting from inner product relation (13b)
c_p	specific heat of fluid at constant pressure
k	thermal conductivity of fluid
L	length of variably heated section
Pe	Peclet number identified below (2c)
q	dimensional heat flux
r	radial coordinate
T	temperature of fluid
V_o	average velocity of fluid
$v(r)$	velocity of fluid at any point
z	axial coordinate

Greek

γ	heat flux in $[\sigma, \infty)$
η	axial variable
η^*	dummy axial variable (5a)
$\theta_1, \theta_2, \theta_3$	temperatures corresponding to each of the three regions $(-\infty, 0], [0, \sigma], [\sigma, \infty)$ (3)

$\theta_1^*, \theta_2^*, \theta_3^*$ temperature response to heating started at η^* (6a)

θ_{3u}, θ_{3d} upstream and downstream temperature response to γ (22a-b)

λ_n nth eigenvalue

ξ radial variable

ρ dimensional fluid density

σ end of heated section

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