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**Electromagnetic coupling by a wire through a cavity-backed
circular aperture in an infinite screen**

Wright, Diana Beth, M.S.

The University of Arizona, 1988

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**ELECTROMAGNETIC COUPLING
BY A WIRE
THROUGH A CAVITY-BACKED CIRCULAR APERTURE
IN AN INFINITE SCREEN**

by
Diana Beth Wright

**A Thesis Submitted to the Faculty of the
DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING
In Partial Fulfillment of the Requirements
For the Degree of
MASTER OF SCIENCE
WITH A MAJOR IN ELECTRICAL ENGINEERING
In the Graduate College
THE UNIVERSITY OF ARIZONA**

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ABSTRACT

The problem of a wire penetrating a circular aperture in an infinite screen and coupling energy into a cavity behind that screen is considered. We formulate an integral equation in terms of the electric field in the aperture. This integral equation is solved using two approximate methods: a zeroth-order approximation valid at low frequencies, and the method of moments. In addition, we introduce an equivalent circuit model to aid in our physical interpretation of the problem. Numerical results for the interior current on the wire and for the equivalent circuit admittance parameters are presented in order to provide a comparison between the two approximations. Inside the cavity, we examine the components of the electric field as a function of position. Finally, the exterior magnetic field far from the aperture is studied as a function of frequency. We examine the relationship between interior resonance features associated with the presence of the cavity and observations of the exterior field.

CHAPTER 1

INTRODUCTION

Electromagnetic analytical models involving penetrations along wires and through apertures into enclosures evolve naturally from practical considerations involving shielding of electronic equipment from electromagnetic interference. Our purpose in this work is to analyze one such model. We shall consider a geometry (specifically described at the beginning of chapter 2) involving a wire penetrating an aperture in a conducting cavity.

We cite four reasons for studying a canonical model of the wire-aperture cavity penetration problem. First, the model allows us to examine *shielding effectiveness* by comparing external fields and currents with internal fields and currents. Second, the model affords us an opportunity to examine what information is available concerning the interior cavity size and shape from observations at a remote exterior location. Third, the model allows us to provide canonical analytical solutions as a means of validating numerical codes using approximate methods to handle more general problems. Fourth, the model provides a positive analytical checkpoint for researchers involved in designing electromagnetic phenomenological experimental range measurements (Kunz, et al., 1987).

Electromagnetic coupling via apertures has been studied extensively by other investigators. However, the wire penetration problem has not as yet received the same degree of attention. As early as 1897, Lord Rayleigh (1897) obtained a solution to the aperture coupling problem by assuming an infinite wavelength in the

neighborhood of the aperture (static conditions) and solving the problem using potential theory. Bethe (1944) was the first to present a quasi-static approach. Other investigators extended this quasi-static approach to handle aperture coupling into cavities (Chen, 1970). Later, an integral equation approach was used by Butler and Umashankar (1976) in the treatment of two-medium problems. Rahmat-Samii and Mittra (1977) also considered this method, one of their objectives being to obtain a procedure for treating non-separable aperture geometries such as rectangles. A discussion of this method, as well as others, appears in a tutorial paper by Butler, Rahmat-Samii, and Mittra (1978).

More recently, aperture coupling into enclosed regions has been studied by means of the powerful generalized dual series approach. A particular two-dimensional problem considered was that of a plane wave scattered from an infinite circular cylinder having an infinite axial slot (Johnson and Ziolkowski, 1984; Ziolkowski and Johnson, 1987). In three dimensions, the coupling behavior of a circular aperture in a spherical shell was investigated by Ziolkowski, Johnson, and Casey (1984). On the subject of coaxial geometries, a very thorough analytical and numerical treatment of planar discontinuities in coaxial waveguides was performed by Harrison and Butler (1981). The geometries considered are closely related to the geometry of our problem. In the area of wire penetration problems, Casey (1987) has considered at low frequencies a wire penetrating an infinite conducting screen while Lee, Dudley, and Casey (1988) have considered this same problem in the general case.

In the next chapter, we provide a complete description of the problem geometry including the source. The differential equation for the unknown magnetic field in each region of our problem is solved using standard Green's function techniques.

Then, by application of the boundary conditions at the aperture, we produce an integral equation in terms of the aperture electric field. In chapter 3, we discuss and compare the solution of this integral equation by two approximate methods: a zeroth-order approximation valid at low frequencies, and the method of moments. In addition, we introduce an equivalent circuit and present numerical results for these circuit parameters. In chapter 4, we examine the components of the electric field as a function of position inside the cavity. The exterior magnetic field far from the aperture is studied as a function of frequency in chapter 5. Comparisons are made between this exterior field and the interior current of chapter 3. Finally, in chapter 6, we summarize our results and make recommendations for future work.

CHAPTER 2

PROBLEM FORMULATION

In this chapter we begin with a complete description of the problem geometry. We produce the applicable TM mode equations. Then for each of the two regions of our problem, we obtain a differential equation for the magnetic field in that region. The differential equations are solved using the method of Green's functions (Stakgold, 1979). In this method, an associated Green's function problem is solved for each region, and our unknown field is then evaluated from the Green's function and knowledge of the source. Finally, matching fields in the aperture, we produce the integral equation whose solution is discussed in the next chapter.

2.1 Field Equations

The problem to be considered is shown in Fig. 2-1 with reference to a cylindrical coordinate system (ρ, ϕ, z) . An infinitesimally thin conducting screen of infinite extent is located at $z = 0$. In this screen there is a circular aperture of radius c centered at $\rho = 0$. The screen divides the problem into two regions: region 1, consisting of an infinite half space, and region 2, consisting of a cylindrical cavity of radius b and length h centered about the z axis. The walls of the cavity and the screen are perfectly conducting. A wire of radius a is aligned with the z axis. Extending from $z = -\infty$ in region 1, it passes through the center of the aperture into region 2 where it follows the axis of the cavity and is shorted to the end at $z = h$. Finally, we assume both regions 1 and 2 are comprised of free space with permeability μ_0 and permittivity ϵ_0 .

Figure 2-1 Wire penetrating a cavity-backed circular aperture in an infinite screen.

We begin with the differential form of Maxwell's equations given by

$$\nabla \times \vec{E} = -i\omega\mu_0\vec{H} - \vec{M} \quad (2-1)$$

$$\nabla \times \vec{H} = i\omega\epsilon_0\vec{E} + \vec{J} \quad (2-2)$$

where $\vec{E}$ is the vector electric field in volts per meter, $\vec{H}$ is the vector magnetic field in amperes per meter, $\vec{J}$ is the electric current density in amperes per square meter, and $\vec{M}$ is the magnetic current density in volts per square meter. We have also assumed time harmonic fields and suppressed the $e^{i\omega t}$ time dependence. The given geometry is ϕ -symmetric and we will assume a ϕ -independent source. Therefore, $\partial/\partial\phi = 0$ and Maxwell's equations decouple into two independent sets: transverse magnetic (abbreviated TM_z where the subscript indicates transverse to the z -axis) with field components (H_ϕ, E_ρ, E_z) and transverse electric (TE_z) with field components (E_ϕ, H_ρ, H_z) . If we select for our source a ϕ -directed magnetic current source M_ϕ , only the TM_z modes are excited and Maxwell's equations reduce to

$$\frac{\partial E_\rho(\rho, z)}{\partial z} - \frac{\partial E_z(\rho, z)}{\partial \rho} = -i\omega\mu_0 H_\phi(\rho, z) - M_\phi(\rho, z) \quad (2-3)$$

$$E_\rho(\rho, z) = -\frac{1}{i\omega\epsilon_0} \frac{\partial H_\phi(\rho, z)}{\partial z} \quad (2-4)$$

$$E_z(\rho, z) = \frac{1}{i\omega\epsilon_0\rho} \frac{\partial(\rho H_\phi(\rho, z))}{\partial \rho} \quad (2-5)$$

We will assume our source M_ϕ is located in region 1. Taking the derivative of (2-4) with respect to z , and the derivative of (2-5) with respect to ρ , and then substituting into (2-3), we obtain for region 1 the following second order differential equation:

$$\left[-\nabla_{\rho z}^2 + \left(\frac{1}{\rho^2} - k^2 \right) \right] H_{\phi 1} = -i\omega\epsilon_0 M_\phi \quad (2-6)$$

where

$$\nabla_{\rho z}^2 \doteq \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} \right) + \frac{\partial^2}{\partial z^2} \quad (2-7)$$

and

$$k = \omega \sqrt{\mu_0 \epsilon_0} \quad (2-8)$$

In order to solve the differential equation in (2-6) we need to specify the boundary conditions of the problem. First, we know that the components of $\vec{E}$ tangential to the perfectly conducting surfaces are zero. Therefore, from (2-4) and (2-5) we have

$$\lim_{z \rightarrow 0} \frac{\partial H_{\phi 1}(\rho, z)}{\partial z} = 0, \quad \text{for } \rho > c \quad (2-9)$$

$$\lim_{\rho \rightarrow a} \frac{\partial(\rho H_{\phi 1}(\rho, z))}{\partial \rho} = 0, \quad \text{for } 0 \leq z < -\infty \quad (2-10)$$

Second, we assume an infinitesimal amount of loss in region 1 such that

$$\lim_{z \rightarrow -\infty} H_{\phi 1}(\rho, z) = 0 \quad (2-11)$$

$$\lim_{\rho \rightarrow \infty} H_{\phi 1}(\rho, z) = 0 \quad (2-12)$$

With the given boundary conditions, we proceed to solve (2-6) using the method of Green's functions.

We consider the simpler associated Green's function problem described by

$$-(\nabla_{\rho z}^2 - \frac{1}{\rho^2} + k^2)g_1 = \frac{\delta(\rho - \rho')\delta(z - z')}{\rho} \quad (2-13)$$

with boundary conditions

$$\lim_{z \rightarrow -\infty} g_1(\rho, z | \rho', z') = 0 \quad (2-14)$$

$$\lim_{z \rightarrow 0} \frac{\partial g_1(\rho, z | \rho', z')}{\partial z} = 0 \quad (2-15)$$

$$\lim_{\rho \rightarrow \infty} g_1(\rho, z | \rho', z') = 0 \quad (2-16)$$

$$\lim_{\rho \rightarrow a} \frac{\partial(\rho g_1(\rho, z | \rho', z'))}{\partial \rho} = 0 \quad (2-17)$$

Physically, the Green's function g_1 is the response in region 1 to a ring source for the case where the aperture is closed. Once we obtain g_1 , that is, the solution to the associated problem, we are able to solve our original differential equation as follows. We apply Green's theorem (Stratton, 1941) in region 1 to obtain

$$\begin{aligned} \int_{V_1} H_{\phi 1} \left[\nabla_{\rho z}^2 - \left(\frac{1}{\rho^2} - k^2 \right) \right] g_1 dV - \int_{V_1} g_1 \left[\nabla_{\rho z}^2 - \left(\frac{1}{\rho^2} - k^2 \right) \right] H_{\phi 1} dV \\ = \oint_{S_1} \left(H_{\phi 1} \frac{\partial g_1}{\partial n} - g_1 \frac{\partial H_{\phi 1}}{\partial n} \right) dS \end{aligned} \quad (2-18)$$

where V_1 denotes the volume of region 1, S_1 is the surface enclosing this volume, and $\hat{n}$ is the outward pointing normal to the surface. Since there is ϕ -symmetry, the volume integral and the surface integral in (2-18) are degenerate in the sense that the integration over ϕ cancels from the equation. Substitution of (2-6), and (2-9) through (2-17), into (2-18) produces the solution to (2-6) in terms of the known Green's function. The field expression in region 1 is given by

$$\begin{aligned} H_{\phi 1}(\rho, z) = -i\omega\epsilon_0 \int_{source} M_{\phi}(\rho', z') g_1(\rho, z | \rho', z') \rho' d\rho' dz' \\ + \int_a^c g_1(\rho, z | \rho', 0) \frac{\partial H_{\phi 1}(\rho', 0)}{\partial z'} \rho' d\rho' \end{aligned} \quad (2-19)$$

Note that in deriving (2-19), as a final step, we switched primed and unprimed coordinates.

For region 2 we need to solve the homogeneous, second-order differential equation

$$\left[-\nabla_{\rho z}^2 + \left(\frac{1}{\rho^2} - k^2 \right) \right] H_{\phi 2} = 0 \quad (2-20)$$

with boundary conditions

$$\lim_{z \rightarrow 0} \frac{\partial H_{\phi 2}(\rho, z)}{\partial z} = 0, \quad \text{for } c \leq \rho \leq b \quad (2-21)$$

$$\lim_{z \rightarrow h} \frac{\partial H_{\phi 2}(\rho, z)}{\partial z} = 0, \quad \text{for } a \leq \rho \leq b \quad (2-22)$$

$$\lim_{\rho \rightarrow a} \frac{\partial(\rho H_{\phi 2}(\rho, z))}{\partial \rho} = 0, \quad \text{for } 0 \leq z \leq h \quad (2-23)$$

$$\lim_{\rho \rightarrow b} \frac{\partial(\rho H_{\phi 2}(\rho, z))}{\partial \rho} = 0, \quad \text{for } 0 \leq z \leq h \quad (2-24)$$

As in region 1, we instead solve a simpler, associated Green's function problem specified by

$$-(\nabla_{\rho z}^2 - \frac{1}{\rho^2} + k^2)g_2 = \frac{\delta(\rho - \rho')\delta(z - z')}{\rho} \quad (2-25)$$

with boundary conditions

$$\lim_{z \rightarrow 0} \frac{\partial g_2(\rho, z | \rho', z')}{\partial z} = 0, \quad \text{for } a \leq \rho \leq b \quad (2-26)$$

$$\lim_{z \rightarrow h} \frac{\partial g_2(\rho, z | \rho', z')}{\partial z} = 0, \quad \text{for } a \leq \rho \leq b \quad (2-27)$$

$$\lim_{\rho \rightarrow a} \frac{\partial(\rho g_2(\rho, z | \rho', z'))}{\partial \rho} = 0, \quad \text{for } 0 \leq z \leq h \quad (2-28)$$

$$\lim_{\rho \rightarrow b} \frac{\partial(\rho g_2(\rho, z | \rho', z'))}{\partial \rho} = 0, \quad \text{for } 0 \leq z \leq h \quad (2-29)$$

Then, by application of Green's theorem, we are able to produce a field expression for the region inside the cavity which is given by

$$H_{\phi 2}(\rho, z) = - \int_a^c g_2(\rho, z | \rho', 0) \frac{\partial H_{\phi 2}(\rho', 0)}{\partial z'} \rho' d\rho' \quad (2-30)$$

Finally, with the help of (2-4), we rewrite both (2-19) and (2-30) in terms of the electric field in the aperture as

$$\begin{aligned} H_{\phi 1}(\rho, z) = & -i\omega\epsilon_0 \int_{source} M_{\phi}(\rho', z') g_1(\rho, z | \rho', z') \rho' d\rho' dz' \\ & - i\omega\epsilon_0 \int_a^c g_1(\rho, z | \rho', 0) E_{\rho 1}(\rho', 0) \rho' d\rho' \end{aligned} \quad (2-31)$$

$$H_{\phi 2}(\rho, z) = i\omega\epsilon_0 \int_a^c g_2(\rho, z | \rho', 0) E_{\rho 2}(\rho', 0) \rho' d\rho' \quad (2-32)$$

2.2 Green's Function for Region 1

We now proceed to evaluate the Green's function of region 1 (Stakgold, 1979). We consider the differential equation in (2-13) with associated boundary conditions given in (2-14) through (2-17). The corresponding geometry is shown in Fig. 2-2. The spectral representation of $\delta(z - z')$ for the operator $-d^2/dz^2$ with the boundary conditions of (2-14) and (2-15) is given by the Fourier cosine integral, viz:

$$\delta(z - z') = \frac{2}{\pi} \int_0^{\infty} \cos \gamma z' \cos \gamma z d\gamma \quad (2-33)$$

Therefore, we expand g_1 as

$$g_1 = \frac{2}{\pi} \int_0^{\infty} G_1(\gamma, \rho) \cos \gamma z d\gamma \quad (2-34)$$

Substitution of (2-33) and (2-34) into (2-13) yields

$$\left[\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} - \frac{1}{\rho^2} + k^2 - \gamma^2 \right] G_1(\rho) = -\frac{\delta(\rho - \rho')}{\rho} \cos \gamma z' \quad (2-35)$$

If we let

$$\alpha_1(\rho) = \frac{G_1(\rho)}{\cos \gamma z'} \quad (2-36)$$

and

$$\lambda_1 = \sqrt{k^2 - \gamma^2}, \quad \text{Im}(\lambda_1) < 0 \quad (2-37)$$

we have

$$\left[\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} - \frac{1}{\rho^2} + \lambda_1^2 \right] \alpha_1(\rho) = -\frac{\delta(\rho - \rho')}{\rho} \quad (2-38)$$

From (2-16) and (2-17) we determine boundary conditions on $\alpha_1(\rho)$. The differential equation of (2-38) along with its corresponding boundary conditions is a standard Green's function problem. It is solved by invoking the boundary conditions and applying both the continuity and jump conditions at $\rho = \rho'$. We find that

$$\alpha_1(\rho) = \begin{cases} -\frac{\pi}{2} \frac{H_1^{(2)}(\lambda_1 \rho')}{H_0^{(2)}(\lambda_1 a)} B(\lambda_1 \rho) & \rho < \rho' \\ -\frac{\pi}{2} \frac{H_1^{(2)}(\lambda_1 \rho)}{H_0^{(2)}(\lambda_1 a)} B(\lambda_1 \rho') & \rho > \rho' \end{cases} \quad (2-39)$$

where

$$B(\lambda_1 \rho) = [J_1(\lambda_1 \rho) Y_0(\lambda_1 a) - J_0(\lambda_1 a) Y_1(\lambda_1 \rho)] \quad (2-40)$$

Combining (2-34) and (2-36), we obtain the Green's function for region 1 as

$$g_1(\rho, z | \rho', z') = \frac{2}{\pi} \int_0^\infty \alpha_1(\rho) \cos \gamma z' \cos \gamma z d\gamma \quad (2-41)$$

where α_1 is given in (2-39).

Figure 2-2 Geometry describing the Green's function problem for region 1.

Alternately, we may spectrally expand g_1 as a function of ρ and then solve a standard Green's function problem in z . The spectral representation of $\delta(\rho - \rho')/\rho$ for our case may be evaluated using the result in (2-39) (Lee, 1988; Felsen and Marcuvitz, 1973). We have

$$\frac{\delta(\rho - \rho')}{\rho} = \int_0^\infty \psi(\gamma\rho) \psi^*(\gamma\rho') d\gamma \quad (2-42)$$

where

$$\psi(\gamma\rho) = \frac{\sqrt{\gamma}}{H_0^{(1)}(\gamma a)} B(\gamma\rho) \quad (2-43)$$

and $*$ denotes complex conjugation. We then expand g_1 as

$$g_1 = \int_0^\infty \tilde{G}_1(\gamma, z) \psi(\gamma\rho) d\gamma \quad (2-44)$$

Combining (2-13), (2-42), and (2-44), we obtain

$$\left[\frac{\partial^2}{\partial z^2} + \lambda_1^2 \right] \tilde{\alpha}_1(z) = -\delta(z - z') \quad (2-45)$$

where

$$\tilde{\alpha}_1(z) = \frac{\tilde{G}_1(z)}{\psi^*(\gamma\rho')} \quad (2-46)$$

and λ_1 is given in (2-37). The boundary conditions on $\tilde{\alpha}_1$ are evaluated from (2-14) and (2-15). This closed form Green's function problem has the solution

$$\tilde{\alpha}_1(z) = -\frac{i}{\lambda_1} \begin{cases} e^{i\lambda_1 z} \cos \lambda_1 z' & z < z' \\ e^{i\lambda_1 z'} \cos \lambda_1 z & z > z' \end{cases} \quad (2-47)$$

Using (2-46) in (2-44), we obtain

$$g_1(\rho, z | \rho', z') = \int_0^\infty \tilde{\alpha}_1(z) \frac{B(\gamma\rho)B(\gamma\rho')}{H_0^{(1)}(\gamma a)H_0^{(2)}(\gamma a)} \gamma d\gamma \quad (2-48)$$

where B is defined in (2-40). Equations (2-41) and (2-48) give two different representations for the Green's function of region 1. When performing the integrations in (2-31), either form of g_1 is correct. We select the form given in (2-48) because the singularities of its integrand can be handled much more readily than those of (2-41). This subject will be discussed further in chapter 3.

2.3 Green's Function for Region 2

Next we consider the Green's function problem for the cavity of region 2. The corresponding geometry is shown in Fig. 2-3. Given the differential equation in (2-25) with the boundary conditions of (2-26) to (2-29), we evaluate g_2 by two methods similar to those described in the previous section for g_1 . First, we spectrally expand in terms of z solving closed form in ρ . The spectral representation of $\delta(z-z')$ for the operator $-d^2/dz^2$ with the boundary conditions of (2-26) and (2-27) is given by the Fourier cosine series, viz:

$$\delta(z-z') = \sum_{n=0}^{\infty} \frac{\epsilon_n}{h} \cos \frac{n\pi z}{h} \cos \frac{n\pi z'}{h} \quad (2-49)$$

where

$$\epsilon_n = \begin{cases} 1, & \text{if } n = 0 \\ 2, & \text{if } n > 0 \end{cases} \quad (2-50)$$

Therefore, we expand g_2 as

$$g_2 = \sum_{n=0}^{\infty} G_2(\rho) \sqrt{\frac{\epsilon_n}{h}} \cos \frac{n\pi z}{h} \quad (2-51)$$

Substituting (2-49) and (2-51) into (2-25), we obtain

$$\left[\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} - \frac{1}{\rho^2} + \tilde{\lambda}_2^2 \right] \alpha_2(\rho) = -\frac{\delta(\rho - \rho')}{\rho} \quad (2-52)$$

where

$$\alpha_2(\rho) = \frac{G_2(\rho)}{\sqrt{\frac{\epsilon_n}{h}} \cos \frac{n\pi z'}{h}} \quad (2-53)$$

and

$$\tilde{\lambda}_2 = \sqrt{k^2 - \left(\frac{n\pi}{h} \right)^2} \quad \text{Im}(\tilde{\lambda}_2) < 0 \quad (2-54)$$

From (2-28) and (2-29) we may determine boundary conditions on $\alpha_2(\rho)$. Once again, this differential equation with its corresponding boundary conditions is a standard Green's function problem. We find that

$$\alpha_2(\rho) = \begin{cases} \frac{\pi}{2} \frac{F(\rho', b)F(\rho, a)}{J_0(\tilde{\lambda}_2 b)Y_0(\tilde{\lambda}_2 a) - J_0(\tilde{\lambda}_2 a)Y_0(\tilde{\lambda}_2 b)} & \rho < \rho' \\ \frac{\pi}{2} \frac{F(\rho, b)F(\rho', a)}{J_0(\tilde{\lambda}_2 b)Y_0(\tilde{\lambda}_2 a) - J_0(\tilde{\lambda}_2 a)Y_0(\tilde{\lambda}_2 b)} & \rho > \rho' \end{cases} \quad (2-55)$$

where

$$F(x, y) = J_1(\tilde{\lambda}_2 x)Y_0(\tilde{\lambda}_2 y) - J_0(\tilde{\lambda}_2 y)Y_1(\tilde{\lambda}_2 x) \quad (2-56)$$

Using (2-51) and (2-53), we obtain

$$g_2(\rho, z | \rho', z') = \sum_{n=0}^{\infty} \alpha_2(\rho) \frac{\epsilon_n}{h} \cos \frac{n\pi z'}{h} \cos \frac{n\pi z}{h} \quad (2-57)$$

where α_2 is defined in (2-55).

Figure 2-3 Geometry describing the Green's function problem for region 2.

The second method for evaluating the Green's function in region 2 consists in spectrally expanding g_2 as a function of ρ and then solving a standard Green's function problem in z . Using the result of (2-55), we evaluate the spectral representation of $\delta(\rho - \rho')/\rho$ (Zhang and Dudley, 1987) as

$$\frac{\delta(\rho - \rho')}{\rho} = \sum_{n=0}^{\infty} S_2(\gamma_n \rho) S_2(\gamma_n \rho') \quad (2-58)$$

where

$$S_2(\gamma_n \rho) = \begin{cases} \frac{1}{\rho \sqrt{\ln \frac{b}{a}}} & \text{for } n = 0 \\ \frac{\pi \gamma_n}{\sqrt{2}} \frac{B(\gamma_n \rho)}{\sqrt{\frac{J_0^2(\gamma_n a)}{J_0^2(\gamma_n b)} - 1}} & \text{for } n > 0 \end{cases} \quad (2-59)$$

and B is defined in (2-40). For $n > 0$, γ_n is the solution to the transcendental equation

$$J_0(\gamma_n b) Y_0(\gamma_n a) - J_0(\gamma_n a) Y_0(\gamma_n b) = 0 \quad (2-60)$$

Also, we may show that for $n \gg 1$,

$$\gamma_n \simeq \frac{n\pi}{(b-a)} \quad (2-61)$$

The appropriate expansion is

$$g_2 = \sum_{n=0}^{\infty} \tilde{G}_2(z) S_2(\gamma_n \rho) \quad (2-62)$$

From (2-25), (2-58), and (2-62), we produce

$$\left[\frac{\partial^2}{\partial z^2} + \lambda_2^2 \right] \tilde{\alpha}_2(z) = -\delta(z - z') \quad (2-63)$$

where

$$\tilde{\alpha}_2(z) = \frac{\tilde{G}_2(z)}{S_2(\gamma_n \rho')} \quad (2-64)$$

and

$$\lambda_2 = \sqrt{k^2 - \gamma_n^2} \quad (2-65)$$

The appropriate boundary conditions are derived from (2-28) and (2-29). The solution to (2-63) is

$$\tilde{\alpha}_2(z) = -\frac{1}{\lambda_2 \sin \lambda_2 h} \begin{cases} \cos \lambda_2 z \cos \lambda_2 (h - z') & z < z' \\ \cos \lambda_2 z' \cos \lambda_2 (h - z) & z > z' \end{cases} \quad (2-66)$$

Combining (2-62) and (2-64), we obtain

$$g_2(\rho, z \mid \rho', z') = \sum_{n=0}^{\infty} \tilde{\alpha}_2(z) S_2(\gamma_n \rho) S_2(\gamma_n \rho') \quad (2-67)$$

with $\tilde{\alpha}_2$ defined by (2-66). Once again, we have two representations for the Green's function. We select the form given in (2-67) because we are interested in a representation in terms of the radial modes (i.e., the TM_z modes discussed earlier). In the summation of (2-67), n is the radial mode index.

2.4 Specification of the Source

Having evaluated the Green's function for each region, our aim now is to more completely describe the source for our problem. We have stated that

our source is a ϕ -directed magnetic current source M_ϕ and that this source is ϕ -independent. The source we select is identical to that used by Lee (1988). Lee first considered the sheet source

$$M_\phi(\rho, z) = \begin{cases} \frac{M_0 \delta(z+d)}{\rho} & \rho \in (a, c) \\ 0 & \text{otherwise} \end{cases} \quad (2-68)$$

where M_0 is a constant, d is the distance from the aperture to the source in region 1, and c is finite and greater than a . The source can be visualized as a coaxial line opening into a half-space with its inner conductor extending into that half-space as shown in Fig. 2-4. In such a model, M_0 is proportional to the magnitude of E_ρ at the opening of the coaxial line. Lee then took the limit as $c \rightarrow \infty$ which produces the TEM solution. We rewrite (2-68) as

$$M_\phi(\rho, z) = \frac{M_0 \delta(z+d)}{\rho} \quad \rho \in (a, \infty) \quad (2-69)$$

Although this idealized source is not physically realizable (having infinite extent and infinite energy), Lee points out several advantages in using it.

First, it allows the first term on the right hand side of (2-31) (which we will call I_s) to be evaluated analytically rather than numerically as in the case where c is finite. As shown by Lee,

$$\begin{aligned} I_s &= -i\omega\epsilon_0 \int_{-\infty}^0 \int_a^\infty g_1(\rho, z | \rho', z') \frac{M_0 \delta(z' + d)}{\rho'} \rho' d\rho' dz' \\ &= -\frac{M_0 e^{-ikd}}{\eta\rho} \cos kz \end{aligned} \quad (2-70)$$

Essentially, I_s represents the field contribution by the source in the geometry of Fig. 2-2. The second advantage in using the idealized source is that it simplifies the

Figure 2-4 Coaxial line opening into a half-space with its inner conductor extending into that region.

analysis of the results obtained because we do not have to contend with higher order modes in the incident field. Finally, Lee showed that the idealized source of (2-69) is a good approximation to the realizable source of (2-68) at lower frequencies and in regions where $\rho \ll z$. We control the separation between source and aperture, and since we are primarily interested in the fields near the wire, we assert that the approximation is valid.

2.5 Integral Equation

Let us consider the boundary conditions in the aperture. The tangential electric field must be continuous in the aperture. Hence, we define the aperture field E_a as

$$E_a(\rho) \doteq E_{\rho 1}(\rho, 0) = E_{\rho 2}(\rho, 0) \quad a < \rho < c \quad (2-71)$$

Also, the tangential magnetic field must be continuous in the aperture. Equating (2-31) and (2-32) at $z = 0$, we obtain

$$i\omega\epsilon_0 \int_a^c [g_1(\rho, 0 | \rho', 0) + g_2(\rho, 0 | \rho', 0)] E_a(\rho') \rho' d\rho' = -\frac{M_0 e^{-ikd}}{\eta\rho} \quad (2-72)$$

This is our integral equation which must be solved to determine E_a . We should consider for a moment the behavior of the integrand in (2-72). We may show that the kernel of the integral, which is simply the sum of the two Green's functions, has a logarithmic singularity at $\rho = \rho'$. The procedure involves subtracting out the asymptotic form and evaluating it analytically. The details of this procedure are algebraically tedious and, therefore, are omitted. The main point is that this type of singularity is weak enough to be integrable. However, we also know that at a sharp edge, such as at $\rho' = c$, the aperture field $E_a(\rho')$ becomes singular. The order

of the singularity is dictated by the edge condition (Meixner, 1972). If we consider the static solution for a circular aperture (Jackson, 1975) we might infer that our aperture field behaves as

$$E_a(\rho') \propto \frac{1}{\rho' \sqrt{c^2 - \rho'^2}} \quad (2-73)$$

as $\rho' \rightarrow c$.

The solution of the integral equation in (2-72) is discussed in the next chapter. Once the aperture electric field is known, the magnetic fields in each region may be evaluated from (2-31) and (2-32). Finally, using (2-4) and (2-5), we may determine the electric fields in both regions.

CHAPTER 3

SOLUTION TO THE INTEGRAL EQUATION

Having set up our integral equation, we now turn our attention to its solution. We solve the integral equation using first a zeroth order (ZO) approximation and then the method of moments (MOM) (Harrington, 1968). In either case, we must be concerned with any singularities that exist and the speed of convergence of the solution. Results obtained using each approach are presented following the formulation of both methods. Before proceeding with the evaluation of the aperture electric field E_a by these two methods, we first describe our choice of an equivalent circuit model for the problem.

3.1 Equivalent circuit model

In this section we introduce a simple, zero-dimensional, equivalent circuit model for our problem. For this simple circuit model, we define a current source in parallel with two admittances as shown in Fig. 3-1 (Casey, 1987). From Ampere's Law, we know that the current on the wire is given by

$$I(z) = (2\pi a)[\hat{n} \times \hat{\phi} H_{\phi 2}(a, z)] \cdot \hat{z} \quad (3-1)$$

Substituting (2-32) into (3-1), we obtain the following result for the current inside the cavity:

$$I(z) = -i \frac{2\pi k}{\eta} \int_a^c \left\{ \frac{\cos k(h-z)}{k \sin kh} \frac{1}{\rho' \ln \frac{b}{a}} \right. \\ \left. + \sum_{n=1}^{\infty} \frac{\cos \lambda_2(h-z)}{\lambda_2 \sin \lambda_2 h} \frac{\pi \gamma_n B(\gamma_n \rho')}{\left[\frac{J_0^2(\gamma_n a)}{J_0^2(\gamma_n b)} - 1 \right]} \right\} E_a(\rho') \rho' d\rho' \quad (z \geq 0) \quad (3-2)$$

To obtain a current expression valid outside the cavity we would substitute (2-31) into (3-1). The field expression of (2-31) consists of two terms. The first term is the field due to the source in the presence of a shorting plate at $z = 0$. The second term is the perturbation to that field caused by the aperture. Let us consider only the source term and its contribution to the current expression. The current produced by this source term is composed of both a forward and a backward travelling wave. We identify the forward travelling wave as the incident current on the wire, namely,

$$I^{inc} = I_0 e^{-ikz} \quad \text{where} \quad I_0 \doteq -\frac{M_0 \pi}{\eta} e^{-ikd} \quad (3-3)$$

The current source for our model is simply the short circuit current at $z = 0^-$, that is, the current at $z = 0^-$ if the aperture were closed. From (3-3), we see that it is just $2I_0$. If we define the voltage across the aperture as

$$V_0 = \int_a^c E_a(\rho') d\rho', \quad (3-4)$$

then the equivalent circuit admittances are described by

$$Y_1 = \frac{2I_0 - I(0)}{V_0} \quad (3-5)$$

$$Y_2 = \frac{I(0)}{V_0} \quad (3-6)$$

where Y_1 corresponds to region 1 and Y_2 corresponds to region 2. We can separate each admittance into its real and imaginary parts. The conductance accounts for

radiation losses. The susceptance for region 1 may be thought of as primarily representing the capacitive effects between the wire and the screen, while the susceptance for region 2 is both capacitive and inductive.

The equivalent circuit described above is by no means the only possible model for our problem. It was chosen because, in spite of its simplicity, it produces useful insights as we will see shortly.

3.2 Zeroth Order Approximation

The first method we use in solving the integral equation in (2-72) is referred to as the zeroth order (ZO) approximation. This method produces an approximate solution for E_a valid at low frequencies. We are unable to obtain a closed form quasi-static solution because of the difficulty in analytically evaluating the integral with the edge condition built in. The ZO approximation allows us to evaluate E_a more quickly and easily than does MOM. Also, it provides a useful means of checking the results obtained with MOM.

In this formulation, we approximate the aperture field by

$$E_a(\rho') = \frac{C}{\rho'} \quad (3-7)$$

where C is some constant to be determined. Although (3-7) does not correctly represent the field locally, it still yields good results at lower frequencies. Since we are integrating $E_a(\rho')$, we are not as concerned with the exact shape of a curve of $E_a(\rho')$ vs. ρ' as we are with the area under that curve. The results obtained are better at lower frequencies because there is less variation of $E_a(\rho')$ as a function of ρ' .

Figure 3-1 Equivalent circuit model.

In our integral equation, we wish to eliminate the variation in ρ , therefore we integrate both sides of (2-72) with respect to ρ from a to c . We have

$$\begin{aligned} i\omega\epsilon_0 \int_a^c \int_a^c \left[\int_0^\infty \frac{B(\gamma\rho)B(\gamma\rho')}{i\lambda_1 H_0^{(1)}(\gamma a) H_0^{(2)}(\gamma a)} \gamma d\gamma - \sum_{n=0}^\infty \frac{S_2(\gamma_n \rho) S_2(\gamma_n \rho')}{\lambda_2 \tan \lambda_2 h} \right] E_a(\rho') \rho' d\rho' d\rho \\ = -\frac{M_0 e^{-ikd}}{\eta} \ln \frac{c}{a} \end{aligned} \quad (3-8)$$

Substituting (3-7) into (3-8) and bringing both the ρ and ρ' integrations inside the integral and the summation of the Green's functions, we can evaluate the unknown as

$$C = \frac{I_0 \eta \ln \frac{c}{a}}{\pi k \mathcal{F}} \quad (3-9)$$

where I_0 is given in (3-3),

$$\begin{aligned} \mathcal{F} = \int_0^\infty \frac{A^2(\gamma c)}{\lambda_1 \gamma H_0^{(1)}(\gamma a) H_0^{(2)}(\gamma a)} d\gamma \\ - i \left[\frac{1}{k \tan(kh)} \frac{\ln^2 \frac{c}{a}}{\ln \frac{b}{a}} + \sum_{n=1}^\infty \frac{1}{\lambda_2 \tan(\lambda_2 h)} \frac{\pi^2}{2} \frac{A^2(\gamma_n c)}{\left[\frac{J_0^2(\gamma_n a)}{J_0^2(\gamma_n b)} - 1 \right]} \right] \end{aligned} \quad (3-10)$$

and

$$A(\gamma c) = J_0(\gamma c) Y_0(\gamma a) - J_0(\gamma a) Y_0(\gamma c) \quad (3-11)$$

Let us rewrite (3-10) as

$$\mathcal{F} = I + S \quad (3-12)$$

where I is the integral and S is the sum (including the $n = 0$ term). The integration and summation in (3-10) have to be performed numerically. We must examine the behavior of each more closely beginning with the integral.

First, we use the variable substitution $\gamma = k\xi$ in order to make the variable of integration dimensionless. We obtain

$$I = \int_0^\infty F(\xi) d\xi = \int_0^\infty \frac{A^2(\xi kc)}{k\xi\sqrt{1-\xi^2} H_0^{(1)}(\xi ka) H_0^{(2)}(\xi ka)} d\xi \quad (3-13)$$

The integrand $F(\xi)$ of (3-13) has singularities at $\xi = 0$ and $\xi = 1$. Another difficulty is that we cannot numerically evaluate to the upper limit of integration, which is infinity. Let us consider each of these difficulties in turn.

We begin with the singularity at $\xi = 0$. We define a new integral I_1 as

$$I_1 = \int_0^\delta F(\xi) d\xi \quad (3-14)$$

where $\delta \leq \min(\frac{1}{ka}, 1)$. For small ξ we may approximate the integrand $F(\xi)$ as

$$F(\xi) \simeq \frac{\ln^2(\frac{\xi}{a})}{k\xi \left[\frac{\pi^2}{4} + \ln^2(\frac{\xi ka}{2} + \gamma) \right]} \quad (3-15)$$

Integrating (3-15), we obtain

$$\int_0^\delta F(\xi) d\xi \simeq \frac{\ln^2(\frac{\xi}{a})}{k} \left\{ \frac{2}{\pi} \tan^{-1} \left[\frac{2}{\pi} \left(\ln \left(\frac{\delta ka}{2} \right) + \gamma \right) \right] + 1 \right\} \quad (3-16)$$

Now we subtract (3-15) from the integrand in (3-14), but add (3-16) to the integral.

The combined result is

$$I_1 = \int_0^\delta \left\{ \frac{A^2(\xi kc)}{k\xi\sqrt{1-\xi^2} H_0^{(1)}(\xi ka) H_0^{(2)}(\xi ka)} - \frac{\ln^2(\frac{\epsilon}{a})}{k\xi \left[\frac{\pi^2}{4} + \ln^2(\frac{\xi ka}{2} + \gamma) \right]} \right\} d\xi$$

$$+ \frac{\ln^2(\frac{\epsilon}{a})}{k} \left\{ \frac{2}{\pi} \tan^{-1} \left[\frac{2}{\pi} \left(\ln \left(\frac{\delta ka}{2} \right) + \gamma \right) \right] + 1 \right\} \quad (3-17)$$

We have subtracted off the singularity at $\xi = 0$ and dealt with it analytically.

It should be noted that in computing (3-15) the following small argument approximation is used:

$$Y_0(\xi ka) \simeq \frac{2}{\pi} \left[\ln(\xi ka) - \ln 2 + \tilde{\gamma} \right] \quad (3-18)$$

where $\tilde{\gamma}$ is Euler's constant. Simply using the first term on the right in (3-18) is not a good enough approximation. Subtracting from the integrand the asymptotic form produced by this one-term approximation, we are still unable to compute the integral because we get numerical overflow errors in the limit as $\xi \rightarrow 0$.

To handle the singularity at $\xi = 1$, we define the following two integrals:

$$I_2 = \int_\delta^1 F(\xi) d\xi \quad (3-19)$$

$$I_3 = \int_1^2 F(\xi) d\xi \quad (3-20)$$

Following the method presented by Johnson and Dudley (1983), we make the variable substitutions $\xi = \sin \theta$ in I_2 and $\xi = \sec \theta$ in I_3 to obtain

$$I_2 = \int_{\sin^{-1} \delta}^{\frac{\pi}{2}} \frac{A^2(kc \sin \theta)}{k \sin \theta H_0^{(1)}(ka \sin \theta) H_0^{(2)}(ka \sin \theta)} d\theta \quad (3-21)$$

$$I_3 = \int_0^{\frac{\pi}{3}} \frac{iA^2(kc \sec \theta)}{k H_0^{(1)}(ka \sec \theta) H_0^{(2)}(ka \sec \theta)} d\theta \quad (3-22)$$

Essentially, we have removed the singularity from our path of integration.

Finally, we attend to the matter of the infinite upper limit of integration. On a computer we cannot integrate to infinity so we must truncate the integral at a point at which the remainder of the integral makes an insignificant contribution to the total result (assuming that the integral converges properly). We define a new integral I_4 as follows:

$$I_4 = \int_2^{\infty} F(\xi) d\xi \quad (3-23)$$

In a manner similar to that discussed for I_1 , we find an approximation to $F(\xi)$ for large ξ , namely,

$$F(\xi) \simeq \frac{i2}{\pi ck^2} \frac{\sin^2[\xi k(c-a)]}{\xi^3} \quad (3-24)$$

Next we are able to integrate (3-24) with the following result:

$$\begin{aligned} \int_2^{\infty} F(\xi) d\xi \simeq \frac{i2}{\pi ck^2} \left\{ \frac{\sin^2[2k(c-a)]}{8} + \frac{k(c-a) \sin[4k(c-a)]}{4} \right. \\ \left. - k^2(c-a)^2 \text{Ci}[4k(c-a)] \right\} \end{aligned} \quad (3-25)$$

where Ci is the cosine integral. Subtracting (3-24) from the integrand in (3-23) and then adding back (3-25) to the integral, we have

$$\begin{aligned} I_4 = \int_2^{\infty} \left\{ \frac{A^2(\xi kc)}{k\xi \sqrt{1-\xi^2} H_0^{(1)}(\xi ka) H_0^{(2)}(\xi ka)} - \frac{i2}{\pi ck^2} \frac{\sin^2[\xi k(c-a)]}{\xi^3} \right\} d\xi \\ + \frac{i2}{\pi ck^2} \left\{ \frac{\sin^2[2k(c-a)]}{8} + \frac{k(c-a) \sin[4k(c-a)]}{4} \right. \\ \left. - k^2(c-a)^2 \text{Ci}[4k(c-a)] \right\} \end{aligned} \quad (3-26)$$

In so doing, we improve the convergence of the integral such that when numerically evaluating the integral we may truncate at a lower value of ξ , thus saving computation time.

It should also be mentioned that due to the oscillatory nature of the integrand in (3-23), we perform the integration in portions. We integrate between zero-crossings of the integrand and compare each new portion to the total integral evaluated thus far. When the contribution of the newest portion is insignificant compared to the present total, we stop integrating. Having successfully dealt with both singularities and the infinite upper limit of our integral, we combine the individual pieces to obtain the total integral as

$$I = I_1 + I_2 + I_3 + I_4 \quad (3-27)$$

Now let us consider the sum of (3-10). The $n = 0$ term (TEM) has been extracted. The remaining sum, which has an infinite number of terms, is treated similarly to I_4 . We evaluate the asymptotic form of the summand and subtract it term by term in order to make the sum converge more rapidly. We analytically evaluate the sum of the asymptotic terms and add this result back outside the original sum. The combined result is

$$\begin{aligned} S = & -\frac{i}{k} \frac{\ln^2(\frac{c}{a})}{\tan kh \ln(\frac{b}{a})} \\ & - \sum_{n=1}^{\infty} \left\{ \frac{i}{\lambda_2 \tan \lambda_2 h} \frac{\pi^2}{2} \frac{A^2(\gamma_n c)}{\left[\frac{J_0^2(\gamma_n a)}{J_0^2(\gamma_n b)} - 1 \right]} + \frac{i 2(b-a)^2}{\pi^3 c} \left[\frac{\sin^2[n\pi(\frac{c-a}{b-a})]}{n^3} \right] \right\} \\ & - \frac{i(b-a)^2}{\pi^3 c} \left\{ \frac{\varphi^2}{2} \ln \varphi - \frac{3}{4} \varphi^2 - \sum_{k=1}^{\infty} \frac{(-1)^{k-1} B_{2k} \varphi^{2(k+1)}}{(2k)(2k+2)!} \right\} \end{aligned} \quad (3-28)$$

where

$$\varphi = 2\pi \left(\frac{c-a}{b-a} \right) \quad (3-29)$$

Using these revised forms for I and S , we have improved convergence and no longer need be concerned with unbounded behavior caused by singularities of the Green's functions. We use these new expressions for the integral and sum in evaluating C from (3-9). The aperture field then readily follows from (3-7).

To determine an expression for the current inside the cavity, we substitute (3-7) into (3-2) and integrate with respect to ρ' . We have

$$I(z) = -i \frac{2\pi k C}{\eta} \left\{ \frac{\cos k(h-z)}{k \sin kh} \frac{\ln \frac{c}{a}}{\ln \frac{b}{a}} - \sum_{n=1}^{\infty} \frac{\cos \lambda_2(h-z)}{\lambda_2 \sin \lambda_2 h} \frac{\pi A(\gamma_n c)}{\left[\frac{J_0^2(\gamma_n a)}{J_0^2(\gamma_n b)} - 1 \right]} \right\} \quad (z \geq 0) \quad (3-30)$$

We need to be concerned with the convergence of the sum, particularly in the region where z is small. As before, we evaluate the asymptotic form of the summand and subtract it term-by-term to improve convergence. The sum of the asymptotic terms, which is of the form $\sum_{n=1}^{\infty} \frac{e^{-n\theta}}{n^2}$ with θ complex, is evaluated analytically as discussed in Appendix A. The result is added back in to preserve the equality. The

new expression for the current is

$$\begin{aligned}
 I(z) = & -i \frac{2\pi k C}{\eta} \left\{ \frac{\cos k(h-z)}{k \sin kh} \frac{\ln \frac{c}{a}}{\ln \frac{b}{a}} \right. \\
 & - \sum_{n=1}^{\infty} \left\{ \frac{\cos \lambda_2(h-z)}{\lambda_2 \sin \lambda_2 h} \frac{\pi A(\gamma_n c)}{\left[\frac{J_0^2(\gamma_n a)}{J_0^2(\gamma_n b)} - 1 \right]} \right. \\
 & \quad \left. + \frac{2}{\pi^2} \sqrt{\frac{a}{c}} (b-a) e^{-n\theta_2} \frac{\sin(n\theta_1)}{n^2} \right\} \\
 & + \frac{2}{\pi^2} \sqrt{\frac{a}{c}} (b-a) \left\{ -\frac{\pi}{2} \theta_2 + \frac{\theta_1 \theta_2}{2} - \frac{\theta_1}{2} \ln(\theta_1^2 + \theta_2^2) \right. \\
 & \quad \left. + \theta_2 \tan^{-1} \frac{\theta_2}{\theta_1} + \theta_1 \right. \\
 & \quad \left. + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} B_{2k}}{(2k)! 2k(2k+1)} (\theta_1^2 + \theta_2^2)^{\frac{2k+1}{2}} \right. \\
 & \quad \left. \cdot \cos \left[(2k+1) \tan^{-1} \frac{\theta_2}{\theta_1} \right] \right\} \quad (z \geq 0) \quad (3-31)
 \end{aligned}$$

where

$$\theta_1 = \pi \left(\frac{c-a}{b-a} \right), \quad \theta_2 = \pi \left(\frac{z}{b-a} \right) \quad (3-32)$$

The form of $I(z)$ given in (3-31) is valid only for $z < z_m$ (specified in Appendix A). This does not present a problem, however, because for $z \geq z_m$ we are able to use (3-30) to evaluate $I(z)$ since z is large enough that the sum converges rapidly.

To evaluate the current at $z = 0$, we use (3-31). Substituting (3-7) into (3-4), we obtain the voltage across the aperture in terms of C as

$$V_0 = C \ln \left(\frac{c}{a} \right) \quad (3-33)$$

With the current and voltage known, we then evaluate the admittances from (3-5) and (3-6). Our equivalent circuit using the ZO approximation is complete. Numerical examples are presented in Sec 3.4, following the MOM formulation in the next section.

3.3 Method of Moments

The second approach we use to evaluate the aperture field is to numerically solve the integral equation in (2-72) using the method of moments (MOM). Herein, we present only an outline of the procedure. For a more thorough description of the method, the reader is referred to Harrington (1968).

In this method, we essentially transform our integral equation into a matrix equation. We begin by dividing the aperture into N subintervals with respect to ρ' . As shown in Fig. 3-2, the subinterval width becomes narrower as we approach $\rho' = c$. The first subinterval nearest the wire at $\rho' = a$ has width Δ , the second has width $\tau\Delta$, the third $\tau^2\Delta$, and so on, to the last which has width $\tau^{(N-1)}\Delta$, where

$$\Delta = \frac{c - a}{\sum_{j=1}^N \tau^{(j-1)}} \quad (3 - 34)$$

and τ is an adjustable parameter with the restriction $0 < \tau \leq 1$. Let us define

$$E_a(\rho') = \frac{\hat{E}_a(\rho')}{\rho'} \quad (3 - 35)$$

Then we expand in a series of non-uniform width pulse functions as follows:

$$\hat{E}_a(\rho') = \sum_{m=1}^N \alpha_m P_m(\rho') \quad (3 - 36)$$

where

$$P_m = \begin{cases} 1 & a + \Delta \sum_{j=1}^{m-1} \tau^{(j-1)} < \rho' < a + \Delta \sum_{j=1}^m \tau^{(j-1)} \\ 0 & \text{otherwise} \end{cases} \quad (3-37)$$

and the α_m 's are unknowns to be determined. The impetus for using successively narrower pulses is to represent the unbounded behavior of the field at the aperture edge without requiring an excessively large number of unknowns (Butler, 1984).

In order to avoid the spatial singularity at the edge, we use point matching. Our weighting functions are delta functions positioned at the center, V_ℓ , of the subinterval, that is

$$w_\ell = \frac{\delta(\rho - V_\ell)}{k\rho} \quad (3-38)$$

where

$$V_\ell = a + \Delta \sum_{j=1}^{\ell-1} \tau^{(j-1)} + \frac{1}{2} \Delta \tau^{(\ell-1)} \quad (3-39)$$

Taking the inner product of w_ℓ with both sides of the integral equation, we obtain

$$\left\langle \frac{M_0 e^{-ikd}}{\eta\rho}, w_\ell \right\rangle = \left\langle -i\omega\epsilon_0 \int_a^c \left[g_1(\rho, 0 | \rho', 0) + g_2(\rho, 0 | \rho', 0) \right] \hat{E}_a(\rho') d\rho', w_\ell \right\rangle \quad (3-40)$$

or in matrix form

$$\begin{pmatrix} T_{11} & T_{12} & \dots & T_{1m} \\ T_{21} & T_{22} & \dots & T_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ T_{\ell 1} & T_{\ell 2} & \dots & T_{\ell m} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_\ell \end{pmatrix} \quad (3-41)$$

Figure 3-2 Division of aperture in MOM formulation.

where

$$F_\ell = \frac{M_0 e^{-ikd}}{\eta k V_\ell} \quad (3-42)$$

and

$$\begin{aligned} T_{\ell m} = & \frac{1}{\eta} \left\{ \int_0^\infty \frac{B(\gamma V_\ell) [A(\gamma R_m) - A(\gamma Q_m)]}{\lambda_1 H_0^{(1)}(\gamma a) H_0^{(2)}(\gamma a)} d\gamma \right. \\ & + i \left[\frac{1}{k \tan(kh)} \frac{\ln \frac{R_m}{Q_m}}{V_\ell \ln \frac{b}{a}} \right. \\ & \left. \left. - \sum_{n=1}^\infty \frac{\pi^2 \gamma_n}{2\lambda_2 \tan(\lambda_2 h)} \frac{B(\gamma V_\ell) [A(\gamma R_m) - A(\gamma Q_m)]}{\left[\frac{J_0^2(\gamma_n a)}{J_0^2(\gamma_n b)} - 1 \right]} \right] \right\} \quad (3-43) \end{aligned}$$

In (3-43), we have used the following shorthand notation:

$$R_m = a + \Delta \sum_{j=1}^m \tau^{(j-1)} \quad (3-44)$$

$$Q_m = a + \Delta \sum_{j=1}^{(m-1)} \tau^{(j-1)} \quad (3-45)$$

Note that

$$Q_m = \begin{cases} R_{m-1} & m > 1 \\ a & m = 1 \end{cases} \quad (3-46)$$

We take advantage of this recursive property when evaluating each matrix element $T_{\ell m}$. If we rewrite (3-43) as

$$\begin{aligned} T_{\ell m} &= \tilde{T}_{\ell m}(R_m) - \tilde{T}_{\ell m}(Q_m) \\ &= \tilde{T}_{\ell m}(R_m) - \tilde{T}_{\ell m}(R_{m-1}) \end{aligned} \quad (3-47)$$

we find that it is only necessary to actually evaluate the first term, $\tilde{T}_{\ell m}(R_m)$, and save this value for the calculation of the next matrix element.

The integration and summation for the Green's functions in (3-43) must be done numerically. Therefore, just as in the previous section, we must concern ourselves with the same singularities in the integral, and the convergence of both the integral and the sum. If we perform an analysis similar to what was done for the ZO case, we obtain

$$\tilde{T}_{\ell m} = \frac{1}{\eta} \left\{ \tilde{I}_{\ell m} + \tilde{S}_{\ell m} \right\} \quad (3-48)$$

where

$$\begin{aligned} \tilde{I}_{\ell m} = \int_0^\delta \left\{ \frac{B(\xi k V_l) A(\xi k R_m)}{\sqrt{1 - \xi^2} H_0^{(1)}(\xi k a) H_0^{(1)}(\xi k a)} + \frac{\ln R_m}{\xi k V_l \left[\frac{\pi^2}{4} + \ln^2 \left(\frac{\xi k a e^\gamma}{2} \right) \right]} \right\} d\xi \\ - \frac{\ln R_m}{k V_l} \left\{ \frac{2}{\pi} \tan^{-1} \left[\frac{2}{\pi} \left(\ln \frac{\delta k a}{2} + \gamma \right) \right] + 1 \right\} \\ + \int_{\sin^{-1} \delta}^{\frac{\pi}{2}} \frac{B(k V_l \sin \theta) A(k R_m \sin \theta)}{H_0^{(1)}(k a \sin \theta) H_0^{(2)}(k a \sin \theta)} d\theta \\ + i \int_0^{\frac{\pi}{2}} \frac{B(k V_l \sec \theta) A(k R_m \sec \theta)}{H_0^{(1)}(k a \sec \theta) H_0^{(2)}(k a \sec \theta)} \sec \theta d\theta \\ + i \int_2^\infty \left\{ \frac{B(\xi k V_l) A(\xi k R_m)}{\sqrt{\xi^2 - 1} H_0^{(1)}(\xi k a) H_0^{(1)}(\xi k a)} \right. \\ \left. + \frac{2}{\pi k \sqrt{V_l}} \frac{\cos[\xi k (V_l - a)] \sin[\xi k (R_m - a)]}{\xi^2 \sqrt{R_m}} \right\} d\xi \\ - \frac{i}{\pi k \sqrt{V_l R_m}} \left\{ \sin[2k(R_m - a)] \cos[2k(V_l - a)] \right. \\ \left. - k[R_m - V_l] \text{Ci}[2k(R_m - V_l)] \right. \\ \left. - k[R_m + V_l - 2a] \text{Ci}[2k(R_m + V_l - 2a)] \right\} \end{aligned} \quad (3-49)$$

$$\begin{aligned}
\tilde{S}_{lm} = & \frac{i}{k \tan kh} \frac{\ln R_m}{V_l \ln \frac{b}{a}} \\
& - \sum_{n=1}^{\infty} \left\{ \frac{i}{\lambda_2 \tan \lambda_2 h} \frac{\pi^2 \gamma_n B(\gamma_n V_l) A(\gamma_n R_m)}{2 \left[\frac{J_0^2(\gamma_n a)}{J_0^2(\gamma_n b)} - 1 \right]} \right. \\
& \quad \left. - i \frac{2(b-a)}{\pi^2 \sqrt{V_l R_m}} \frac{\cos \left[n\pi \left(\frac{V_l - a}{b-a} \right) \right] \sin \left[n\pi \left(\frac{R_m - a}{b-a} \right) \right]}{n^2} \right\} \\
& - i \frac{(b-a)}{\pi^2 \sqrt{V_l R_m}} \left\{ \varphi_1 (1 - \ln \varphi_1) + \varphi_2 (1 - \ln \varphi_2) \right. \\
& \quad \left. + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k)!} B_{2k} \frac{(\varphi_1 + \varphi_2)^{2k+1}}{2k(k+1)} \right\} \quad (3-50)
\end{aligned}$$

$$\varphi_1 = \pi \left(\frac{R_m + V_l - 2a}{b-a} \right), \quad \varphi_2 = \pi \left(\frac{R_m - V_l}{b-a} \right) \quad (3-51)$$

and Ci is the cosine integral. As before, the integral is separated into four pieces and the first term of the sum is extracted. The appropriate asymptotic expressions are included and the same variable transformations are made.

Instead of the definition given in (3-35), we might have started by extracting the edge singularity analytically as in (2-73):

$$E_a(\rho') = \frac{\hat{E}_a(\rho')}{\rho' \sqrt{c^2 - \rho'^2}} \quad (3-52)$$

We would then expand $\hat{E}_a(\rho')$ as in (3-36). With the singularity removed, fewer terms would be needed in the expansion to accurately represent the field. We would also no longer be prohibited from matching at $\rho' = c$. The difficulty with this method is that the integration in ρ' could not be handled analytically. Each matrix element would require a numerical integration with respect to ρ' . We stay with our

original method because, first, the integration in ρ' can be performed analytically. Second, with our non-uniform pulse expansion functions, we are able to closely approximate the field near the edge without requiring an excessively large number of unknowns. Finally, even though we avoid the singularity at the edge by weighting with delta functions in the center of the subintervals, we can weight progressively closer to the edge by increasing N .

We now proceed to evaluate our equivalent circuit parameters, starting with the current. Using (3-35) and (3-36) in (3-2) and integrating with respect to ρ' , we obtain

$$I(z) = -i \frac{2\pi k}{\eta} \sum_{m=1}^N \alpha_m \left\{ \frac{\cos k(h-z)}{k \sin kh} \frac{\ln \frac{R_m}{Q_m}}{\ln \frac{b}{a}} + \sum_{n=1}^{\infty} \frac{\cos \lambda_2(h-z)}{\lambda_2 \sin \lambda_2 h} \frac{\pi [A(\gamma_n R_m) - A(\gamma_n Q_m)]}{\left[\frac{J_0^2(\gamma_n a)}{J_0^2(\gamma_n b)} - 1 \right]} \right\} \quad (3-53)$$

We may rewrite this as

$$I(z) = \sum_{m=1}^N \alpha_m [\tilde{I}(z, R_m) - \tilde{I}(z, R_{m-1})] \quad (3-54)$$

For large z , we use (3-53) because the sum over n will converge quickly. When this is not the case, we employ the following expression in which the asymptotics are

included:

$$\begin{aligned}
\tilde{I}(z, R_m) = & -i \frac{2\pi k}{\eta} \left\{ \frac{\cos k(h-z)}{k \sin kh} \frac{\ln R_m}{\ln \frac{b}{a}} \right. \\
& - \sum_{n=1}^{\infty} \left\{ \frac{\cos \lambda_2(h-z)}{\lambda_2 \sin \lambda_2 h} \frac{\pi A(\gamma_n R_m)}{\left[\frac{J_0^2(\gamma_n a)}{J_0^2(\gamma_n b)} - 1 \right]} \right. \\
& \quad \left. + \frac{2}{\pi^2} \sqrt{\frac{a}{R_m}} (b-a) e^{-n\theta_2} \frac{\sin(n\theta_1)}{n^2} \right\} \\
& + \frac{2}{\pi^2} \sqrt{\frac{a}{R_m}} (b-a) \left\{ -\frac{\pi}{2} \theta_2 + \frac{\theta_1 \theta_2}{2} - \frac{\theta_1}{2} \ln(\theta_1^2 + \theta_2^2) \right. \\
& \quad \left. + \theta_2 \tan^{-1} \frac{\theta_2}{\theta_1} + \theta_1 \right. \\
& \quad \left. + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} B_{2k}}{(2k)! 2k(2k+1)} (\theta_1^2 + \theta_2^2)^{\frac{2k+1}{2}} \right. \\
& \quad \left. \cdot \cos \left[(2k+1) \tan^{-1} \frac{\theta_2}{\theta_1} \right] \right\} \Bigg\} \quad (z \geq 0) \quad (3-55)
\end{aligned}$$

where

$$\theta_1 = \pi \left(\frac{R_m - a}{b - a} \right), \quad \theta_2 = \pi \left(\frac{z}{b - a} \right) \quad (3-56)$$

Substituting this form into (3-54), we have an expression for the current inside the cavity which is valid under the restriction specified in Appendix A. We use this form when we are close to the aperture in order to insure rapid convergence of the summations in n .

To calculate the voltage V_0 we use (3-35) and (3-36) in (3-4) and integrate.

The result is

$$V_0 = \sum_{m=1}^N \alpha_m \ln \left(\frac{R_m}{Q_m} \right) \quad (3-57)$$

Once again, Y_1 and Y_2 are determined from (3-5) and (3-6). This concludes the MOM formulation with its corresponding circuit parameters. In the next section we present some numerical examples. We compare ZO and MOM results and discuss the insights provided by the equivalent circuit model.

3.4 Numerical Examples

Let us begin with a brief discussion of the numerical facilities available to us. All numerical computation is done on a VAX 11/750 using 32-bit single-precision arithmetic. The method of integration that we use throughout is based on a combination of Simpson's rule and Newton's 3/8th's rule. We utilize several software packages. For evaluating Bessel functions, we employ efficient and accurate routines contained in the SLATEC math library from Sandia National Laboratories. The matrix equation in (3-41) is solved with LINPACK, a standard package for analyzing and solving systems of linear equations; in particular, we use routines SGECCO and SGECL. Finally, our figures are made using FANCYPLOT, a graphics package developed at Lawrence Livermore National Laboratory.

Before proceeding to the analysis of the equivalent circuit parameters for several numerical examples, we consider the aperture field computed using MOM. Specifically, we wish to demonstrate the advantage of using pulses of non-uniform width in our expansion. We choose a cavity with outer radius $b = 5a$ and length $h = 15a$, where a is the wire radius. Then for an aperture size we select $c = 2a$. In Fig. 3-3, we show plots of $|\rho E_\rho(\rho)|$ as a function of $k\rho$ for this cavity obtained using both uniform and non-uniform width pulses. We also plot the quasi-static solution for comparison. To obtain the quasi-static form we first determine the

value of $|\rho' E_a(\rho')|$ at $\rho' = a$ from the MOM results. Then using this value in (3-52), we solve for $|\hat{E}_a(\rho' = a)|$. In Fig. 3-3, the quasi-static form that we plot is given by $|\hat{E}_a(\rho' = a)|/\sqrt{c^2 - \rho'^2}$. We see that the non-uniform case with τ set to 0.6 provides a better representation of the aperture field using only eight pulses than does the uniform case with 20 pulses. Both MOM curves provide a much better representation than the quasi-static form. The non-uniform case models the edge singularity well, yet the size of the matrix that we are required to solve is significantly reduced thus saving computation time.

Now we study several numerical examples. Examining both the current on the wire and the admittances of the equivalent circuit, we compare ZO versus MOM. In addition, we attempt to uncover the physical significance of the equivalent circuit parameters.

Although the cavity in our problem has an opening, it will prove valuable to draw upon our knowledge of the behavior of closed cavities. Starting with the known resonance frequencies of a closed coaxial cavity, we may track resonances as we open our aperture wider and wider. Also, we specify modes inside our cavity as we would in a closed cavity. We use the notation TM_{npq} where n , p , and q are the indices corresponding to the ρ , ϕ , and z directions, respectively. Since the problem is ϕ -symmetric, p is always zero. We consider three cavity sizes: a "long" cavity, a "short" cavity, and a "large" cavity. The long cavity has radius $b = 5a$ and length $h = 15a$. The short cavity has dimensions $b = 15a$ and $h = 6a$. Finally, the large cavity is specified by $b = 40a$ and $h = 100a$.

In all cases, we examine a frequency range in which only the first few modes may exist. The resonance pattern becomes much too difficult to analyze at higher

Figure 3-3 Normalized aperture field evaluated by MOM: uniform versus non-uniform width pulses.

frequencies; individual mode resonances are nearly impossible to identify. The first few resonant modes of the three corresponding closed coaxial cavities are listed in Tables 3-1 to 3-3 along with their associated resonance frequencies. As suggested above, we may refer to these tables to identify resonances observed in the cavity of our problem.

We begin by discussing the results obtained for the long cavity. The first aperture size that we consider is $c = 2a$. We evaluate the current on the wire inside the cavity from (3-31) (ZO) and (3-55) (MOM). The current at the aperture ($z = 0$) is shown in Fig. 3-4. For this cavity size, as we increase in frequency, we expect to see the first few axial modes before the first radial mode. This is, indeed, the case. Using Table 3-1 we easily identify the TM_{001} , TM_{002} , TM_{003} , TM_{004} , TM_{100} , TM_{101} , TM_{102} , and TM_{103} modes. An additional peak in the current at $ka = 0.045$ is not readily identifiable. In order to explain its occurrence we return to our equivalent circuit model.

Let us consider the total admittance of the circuit expressed as

$$Y_T = \frac{2I_0}{V_0} \quad (3-58)$$

Using the ZO approximation and substituting (3-9) and (3-33) into (3-58), we obtain

$$Y_T = \frac{2\pi k}{\eta \ln^2(\frac{c}{a})} \left\{ \int_0^\infty \frac{A^2(\gamma c)}{\gamma \sqrt{k^2 - \gamma^2} H_0^{(1)}(\gamma a) H_0^{(2)}(\gamma a)} d\gamma \right. \\ \left. - \frac{i}{k \tan kh} \frac{\ln^2(\frac{c}{a})}{\ln(\frac{b}{a})} \right. \\ \left. - \sum_{n=1}^\infty \frac{i}{\lambda_2 \tan \lambda_2 h} \frac{\pi^2}{2} \frac{A^2(\gamma_n c)}{\left[\frac{J_0^2(\gamma_n a)}{J_0^2(\gamma_n b)} - 1 \right]} \right\} \quad (3-59)$$

Table 3-1 Resonance frequencies of lowest TM_z modes in a closed cylindrical cavity with dimensions: $b = 5a$ and $h = 15a$.

mode	ka
TM_{001}	0.209
TM_{002}	0.419
TM_{003}	0.628
TM_{100}	0.763
TM_{101}	0.792
TM_{004}	0.838
TM_{102}	0.871
TM_{103}	0.989
TM_{005}	1.047

Table 3-2 Resonance frequencies of lowest TM_z modes in a closed cylindrical cavity with dimensions: $b = 15a$ and $h = 6a$.

mode	ka
TM_{100}	0.210
TM_{200}	0.438
TM_{001}	0.524
TM_{101}	0.564
TM_{300}	0.664
TM_{201}	0.682
TM_{301}	0.846
TM_{400}	0.890
TM_{401}	1.033

Table 3-3 Resonance frequencies of lowest TM_z modes in a closed cylindrical cavity with dimensions: $b = 40a$ and $h = 100a$.

mode	ka
TM_{001}	0.031
TM_{002}	0.063
TM_{100}	0.073
TM_{101}	0.079
TM_{003}	0.094
TM_{102}	0.096
TM_{103}	0.119

Figure 3-4 Current on the wire at $z = 0$; $b/a = 5$, $c/a = 2$, $h/a = 15$.

In the low frequency limit, this becomes

$$Y_T \simeq \frac{2\pi}{\eta} \left\{ \int_0^k \frac{k d\gamma}{\gamma \sqrt{k^2 - \gamma^2} \left[\frac{\pi^2}{4} + \ln^2(\gamma a) \right]} + i \int_k^\infty \frac{k d\gamma}{\gamma^2 \left[\frac{\pi^2}{4} + \ln^2(\gamma a) \right]} \right. \\ \left. - \frac{i}{kh \ln(\frac{b}{a})} + \sum_{n=1}^{\infty} \frac{ik}{\gamma_n \tanh \gamma_n h} \frac{\pi^2}{2} \frac{A^2(\gamma_n c)}{\ln^2(\frac{c}{a}) \left[\frac{J_0^2(\gamma_n a)}{J_0^2(\gamma_n b)} - 1 \right]} \right\} \quad (3-60)$$

We note that since $\left[\frac{J_0^2(\gamma_n a)}{J_0^2(\gamma_n b)} - 1 \right]$ may be either positive or negative depending on the values of a and b , the final term in (3-60) may look either capacitive or inductive. For the parameters we are using in this case, it is capacitive. Therefore, in terms of individual circuit elements, we have

$$Y_T = G_1 + i\omega C_1 + \frac{1}{i\omega L_2} + i\omega C_2 \quad (3-61)$$

where

$$G_1 = \frac{2\pi}{\eta} \int_0^k \frac{k d\gamma}{\gamma \sqrt{k^2 - \gamma^2} \left[\frac{\pi^2}{4} + \ln^2(\gamma a) \right]} \quad (3-62)$$

$$C_1 = 2\pi\epsilon_0 \int_k^\infty \frac{d\gamma}{\gamma^2 \left[\frac{\pi^2}{4} + \ln^2(\gamma a) \right]} \quad (3-63)$$

$$C_2 = \epsilon_0 \sum_{n=1}^{\infty} \frac{\pi^3}{\gamma_n \tanh \gamma_n h} \frac{A^2(\gamma_n c)}{\ln^2(\frac{c}{a}) \left[\frac{J_0^2(\gamma_n a)}{J_0^2(\gamma_n b)} - 1 \right]} \quad (3-64)$$

$$L_2 = \frac{\mu_0}{2\pi} h \ln \frac{b}{a} \quad (3-65)$$

The equivalent circuit is illustrated in Fig. 3-5. The physical significance of each circuit element is as follows: as stated earlier, G_1 represents the radiation losses in

the exterior region; C_1 is approximately one half of the "gap" capacitance of the aperture; C_2 is the other half of the gap capacitance combined with the capacitance of the cavity; and finally, L_2 is simply the inductance per unit length of a coaxial cable having inner radius a and outer radius b , multiplied by the cable length h . Now expressing the current in terms of Y_T , we obtain

$$I(0) = 2I_0 \left(\frac{Y_2}{Y_T} \right) \quad (3-66)$$

We can see that when

$$\omega = \frac{1}{\sqrt{L_2(C_1 + C_2)}} \quad (3-67)$$

Y_T is at a minimum resulting in a maximum of $I(0)$. This is the maximum that we observe in Fig. 3-4.

Now we consider the other resonances (those associated with the closed cavity) in terms of the equivalent circuit model. Let us examine the admittance of region 2. From (3-6), (3-30) and (3-33), we produce

$$Y_2 = -\frac{i2\pi}{\eta} \left\{ \frac{1}{\ln \frac{b}{a} \tan kh} - \sum_{n=1}^{\infty} \frac{\pi k}{\lambda_2 \tan \lambda_2 h} \frac{A(\gamma_n c)}{\ln(\frac{c}{a}) \left[\frac{J_0^2(\gamma_n a)}{J_0^2(\gamma_n b)} - 1 \right]} \right\} \quad (3-68)$$

In the low frequency limit, this becomes

$$Y_2 \simeq -\frac{2\pi}{\eta} \left\{ \frac{i}{kh \ln \frac{b}{a}} - \sum_{n=1}^{\infty} \frac{ik}{\gamma_n \tanh \gamma_n h} \frac{\pi A(\gamma_n c)}{\ln(\frac{c}{a}) \left[\frac{J_0^2(\gamma_n a)}{J_0^2(\gamma_n b)} - 1 \right]} \right\} \quad (3-69)$$

Figure 3-5 Equivalent circuit model in the low frequency limit.

We note that although the first term on the right is the same as $\frac{1}{i\omega L_2}$ in (3-61), the second term is of a slightly different form than $i\omega C_2$. This means that the summation of Y_T is not exactly Y_2 . However, at low frequencies, we can prove numerically that they are very close as shown in Fig. 3-6.

Examining just the first term on the right of (3-68), we see that when $k = n\pi/h$ ($n = 0, 1, 2, \dots$), $Y_2 \rightarrow \infty$. At these same frequencies, $Y_T \rightarrow \infty$ as well, but not as rapidly. Therefore, the ratio in (3-66) remains finite, but $I(0)$ does have a maximum. These frequencies correspond to the TEM resonances. Since the first term of (3-68) contains this resonant information, it is not adequate to model it as simply an inductance as in the low frequency limit. It must be represented by a combination of inductance and capacitance. Our equivalent circuit model becomes more complicated in this case. Examining the second term on the right of (3-68), we see that the higher order modes are resonant at $k = \sqrt{\gamma_n^2 + (\frac{n\pi}{h})^2}$. In summary, the first resonance in the current must be explained in terms of the total admittance Y_T while the other resonances can be identified in Y_2 . The first peak in the current is essentially an effect of the aperture and the cavity combined while the others are more closely related to just the cavity.

Returning to the current plot of Fig. 3-4, we make two final observations. At resonance, the current magnitude actually exceeds the short circuit current $2I_0$. Also, ZO and MOM results agree favorably.

The current at three other positions on the wire ($z = \frac{h}{2}$, $z = \frac{2h}{3}$, and $z = h$) is shown in Figs. 3-7 to 3-9. To explain the maxima, minima, and nulls of each we

Figure 3-6 Susceptance of region 2: true B_2 versus the summation in Y_T .

consider a simple ray optic formulation for the current given by

$$|I(z)| = |A_0 T [1 + e^{i2k(h-z)}] \sum_{n=0}^{\infty} R^n e^{in2kh}| \quad (3-70)$$

where A_0 is some constant, R is the reflection coefficient and T is the transmission coefficient characterizing the $z = 0$ plane as seen from inside the cavity. We caution that this description only pertains to a TEM wave travelling along the wire. It is not valid at frequencies in which higher order modes propagate. Using this ray optic formulation of the current, we see that maxima occur at $k = m\pi/h$, minima at $k = (2m - 1)\pi/2h$, and nulls at $k = (2m - 1)\pi/2(h - z)$ where $m = 1, 2, 3, \dots$. For example, at $z = 2h/3$, we would predict maximum current at $ka = 0.209, 0.419$, and 0.628 , minima at $ka = 0.105, 0.314$, and 0.524 , and nulls at $ka = 0.314, 0.471$, and 0.785 . In Fig. 3-8, we verify all but the second null. One possible explanation is that this null is too close to the second maximum and, therefore, is in some way cancelled out.

Let us next consider the same cavity dimensions, but with a smaller aperture size. Fig. 3-10 shows the current at the end of the cavity for the case where $c = 1.01a$. As we would expect, the resonance peaks are sharper since our geometry is looking more like a closed cavity. Also, there is excellent agreement between ZO and MOM results over this same frequency range because the aperture is smaller in terms of wavelengths. This means there is less variation in the field across the aperture, therefore, the ZO approximation is better.

The current at $z = h$ is plotted for three different aperture sizes in Fig. 3-11. We observe that as aperture size increases, the resonance peaks broaden and shift away from the closed cavity values in the direction of higher frequency. One

Figure 3-7 Current on the wire at $z = h/2$; $b/a = 5$, $c/a = 2$, $h/a = 15$.

Figure 3-8 Current on the wire at $z = 2h/3$; $b/a = 5$, $c/a = 2$, $h/a = 15$.

Figure 3-9 Current on the wire at $z = h$; $b/a = 5$, $c/a = 2$, $h/a = 15$.

Figure 3-10 Current on the wire at $z = h$; $b/a = 5$, $c/a = 1.01$, $h/a = 15$.

possible explanation is as follows. The gap capacitance decreases as we enlarge the aperture. Assuming the inductance is not strongly dependent on the aperture size (in the low frequency limit, it is completely independent), the resonance frequency increases with decreasing capacitance.

We proceed now to the evaluation of the equivalent circuit parameters Y_1 and Y_2 . The conductivities G_1 and G_2 account for radiation losses in the two regions. With the exception of the $c = 2a$ case which we will discuss momentarily, we observe that G_1 increases smoothly with frequency and with decreasing aperture size as shown in Fig. 3-12. This may be interpreted as follows. In the open region, the ideal source launches a TEM wave down the wire (Lee, 1988). No radiation loss is associated with this mode. At the aperture there is mode conversion. It is the higher order modes produced at the aperture that are responsible for the radiation losses. As the separation between the wire and screen decreases, or as frequency increases, there is greater mode conversion, more scattering, and, therefore, higher conductivity.

Inside the cavity, we would expect zero conductivity since for the closed region, there can be no radiation loss. Note that although the cavity is not completely closed, we consider radiation loss through the aperture as loss in region 1. Fig. 3-13 shows that G_2 is indeed negligible except when $c = 2a$. For this aperture size, both G_1 and G_2 seem to be affected by the radial mode resonances. This prompts us to reexamine our equivalent circuit model. The implication is that our zeroth-order, equivalent circuit model works well for TEM, but starts to break down when higher order modes are introduced. Our definitions of current and voltage are essentially mode specific. With the exception of this resonant behavior, there is excellent agreement between ZO and MOM results.

Figure 3-11 Current on the wire at $z = h$ for three aperture sizes: $c/a = 1.01$, 1.1, and 2; $b/a = 5$, $h/a = 15$.

Figure 3-12 Conductance of region 1 for three aperture sizes: $c/a = 1.01, 1.1$, and 2 ; $b/a = 5$, $h/a = 15$.

Figure 3-13 Conductance of region 2 for three aperture sizes: $c/a = 1.01, 1.1$, and 2 ; $b/a = 5$, $h/a = 15$.

The susceptance of region 1 for various aperture sizes is shown in Fig. 3-14. In this region, the susceptance primarily represents the capacitance between the wire and screen. Therefore, as frequency increases or aperture size decreases, B_1 increases. We observe that for the larger aperture size, B_1 senses the radial mode resonances just like G_1 and G_2 . In fact, a slight resonance effect can even be seen in the $c = 1.1a$ case. For region 2, the susceptance depends not only on the capacitance between the wire and screen, but on the capacitance between the wire and cavity walls as well. Although we again observe that the susceptance increases with frequency and with decreasing aperture size, it is the resonant behavior of the cavity that dominates (Fig. 3-15).

Two other cavity sizes remain to be considered. We present representative plots of the current and the admittances for each. Figs. 3-16 to 3-20 correspond to the short cavity having dimensions $b = 15a$ and $h = 6a$ with an aperture size $c = 2a$. For this cavity size, the first few radial modes turn on before the axial modes. Using Table 3-2, we can identify the various resonant modes in the current at $z = h$ (Fig. 3-16). The ZO curve agrees with the MOM curve at lower frequencies. As we move higher in frequency, they begin to diverge, particularly near radial resonances. The admittance parameters for this case shown in Figs. 3-17 through 3-20 have the same features as those of the long cavity. Since we are considering a frequency range in which a number of radial modes are present, the equivalent circuit model is not particularly useful in this case.

Plots of current and admittance for the large cavity ($b = 40a$, $h = 100a$) having an aperture radius $c = 2a$ are shown in Figs. 3-21 to 3-25. Even though we have significantly increased the overall cavity dimensions with reference to the

Figure 3-14 Susceptance of region 1 for three aperture sizes: $c/a = 1.01, 1.1$, and 2 ; $b/a = 5, h/a = 15$.

Figure 3-15 Susceptance of region 2 for three aperture sizes: $c/a = 1.01$, 1.1 , and 2 ; $b/a = 5$, $h/a = 15$.

Figure 3-16 Current on the wire at $z = h$; $b/a = 15$, $c/a = 2$, $h/a = 6$.

Figure 3-17 Conductance of region 1; $b/a = 15$, $c/a = 2$, $h/a = 6$.

Figure 3-18 Conductance of region 2; $b/a = 15$, $c/a = 2$, $h/a = 6$.

Figure 3-19 Susceptance of region 1; $b/a = 15$, $c/a = 2$, $h/a = 6$.

Figure 3-20 Susceptance of region 2; $b/a = 15$, $c/a = 2$, $h/a = 6$.

wire radius, the behavior of the current and admittance parameters is essentially the same as in the previous two cases studied. We do make one new observation, however. Whereas we are looking at a frequency range in which the first radial modes appear, and we are considering an aperture size used in both prior cases, the admittance parameters seem less sensitive to the radial mode resonances. This is not surprising, since for this case the aperture size is smaller compared to the cavity size. The implication is that the size of the aperture relative to the size of the cavity is just as important as the aperture size relative to frequency.

The above examples clearly demonstrate the utility of the ZO approximation. At low frequencies, it produces nearly identical results to MOM. We have also been successful in identifying resonances by tracking those of the closed cavity. As we increased the aperture size, the resonances shifted higher in frequency. Finally, we have demonstrated the usefulness of an equivalent circuit model, in as much as it enhances our physical understanding of the problem.

Figure 3-21 Current on the wire at $z = h$; $b/a = 40$, $c/a = 2$, $h/a = 100$.

Figure 3-22 Conductance of region 1; $b/a = 40$, $c/a = 2$, $h/a = 100$.

Figure 3-23 Conductance of region 2; $b/a = 40$, $c/a = 2$, $h/a = 100$.

Figure 3-24 Susceptance of region 1; $b/a = 40$, $c/a = 2$, $h/a = 100$.

Figure 3-25 Susceptance of region 2; $b/a = 40$, $c/a = 2$, $h/a = 100$.

CHAPTER 4

FIELDS INSIDE THE CAVITY

In the last chapter, we investigated the frequency response of the cavity by evaluating both the current and equivalent admittance parameters inside. In this chapter, we develop field expressions valid inside the cavity and examine these fields as a function of position. We present contour plots for both components of the electric field at various resonance and non-resonance frequencies.

4.1 Field Expressions

The magnetic field inside the cavity is obtained by using the MOM expression for the aperture field and the Green's function of (2-67) in (2-32), viz:

$$H_{\phi 2}(\rho, z) = -i\omega\epsilon_0 \sum_{m=1}^N \alpha_m [\tilde{H}_{\phi 2}(\rho, z, R_m) - \tilde{H}_{\phi 2}(\rho, z, R_{m-1})] \quad (4-1)$$

where

$$\begin{aligned} \tilde{H}_{\phi 2}(\rho, z, R_m) = & \frac{\cos k(h-z)}{k \sin kh} \frac{\ln R_m}{\rho \ln \frac{b}{a}} \\ & - \frac{\pi^2}{2} \sum_{n=1}^{\infty} \frac{\cos \lambda_2(h-z)}{\lambda_2 \sin \lambda_2 h} \frac{\gamma_n B(\gamma_n \rho) A(\gamma_n R_m)}{\left[\frac{J_0^2(\gamma_n a)}{J_0^2(\gamma_n b)} - 1 \right]} \end{aligned} \quad (4-2)$$

Substituting this result into (2-4) and (2-5) and differentiating, we obtain expressions for the electric field components inside the cavity. In order to improve numerical convergence in the summation of each expression, we subtract the asymptotic

form term-by-term in the sum, evaluate the sum of the asymptotic terms analytically, and add this asymptotic sum back in to preserve the equality just as we did for the current. The final forms for $E_{\rho 2}$ and $E_{z 2}$ are as follows:

$$E_{\rho 2}(\rho, z) = \sum_{m=1}^N \alpha_m [\tilde{E}_{\rho 2}(\rho, z, R_m) - \tilde{E}_{\rho 2}(\rho, z, R_{m-1})] \quad (4-3)$$

where

$$\begin{aligned} \tilde{E}_{\rho 2}(\rho, z, R_m) = & \frac{\sin k(h-z)}{\sin kh} \frac{\ln R_m}{\rho \ln \frac{b}{a}} \\ & - \sum_{n=1}^{\infty} \left\{ \frac{\pi^2}{2} \frac{\sin \lambda_2(h-z)}{\sin \lambda_2 h} \frac{\gamma_n B(\gamma_n \rho) A(\gamma_n R_m)}{\left[\frac{J_0^2(\gamma_n a)}{J_0^2(\gamma_n b)} - 1 \right]} \right. \\ & \quad \left. - \frac{e^{-n\theta_2}}{\pi \sqrt{\rho R_m}} \left[\frac{\sin(n\theta_4)}{n} + \frac{\sin(n\theta_3)}{n} \right] \right\} \\ & - \frac{1}{\pi \sqrt{\rho R_m}} \left[\frac{(\theta_3 + \theta_4)}{2} \right. \\ & \quad \left. + \tan^{-1} \left(\frac{\tanh \frac{\theta_2}{2}}{\tanh \frac{\theta_3}{2}} \right) + \tan^{-1} \left(\frac{\tanh \frac{\theta_2}{2}}{\tanh \frac{\theta_4}{2}} \right) \right] \quad (4-4) \end{aligned}$$

and

$$E_{z 2}(\rho, z) = \sum_{m=1}^N \alpha_m [\tilde{E}_{z 2}(\rho, z, R_m) - \tilde{E}_{z 2}(\rho, z, R_{m-1})] \quad (4-5)$$

where

$$\begin{aligned}
\tilde{E}_{z2}(\rho, z, R_m) = & \sum_{n=1}^{\infty} \left\{ \frac{\pi^2}{2} \frac{\cos \lambda_2(h-z)}{\lambda_2 \sin \lambda_2 h} \frac{\gamma_n^2 A(\gamma_n \rho) A(\gamma_n R_m)}{\left[\frac{J_0^2(\gamma_n a)}{J_0^2(\gamma_n b)} - 1 \right]} \right. \\
& + \frac{e^{-n\theta_2}}{\pi \sqrt{\rho R_m}} \left[\frac{\cos(n\theta_4)}{n} - \frac{\cos(n\theta_3)}{n} \right] \left. \right\} \\
& + \frac{1}{2\pi \sqrt{\rho R_m}} \left[\ln \left(\sin^2 \frac{\theta_4}{2} + \sinh^2 \frac{\theta_2}{2} \right) \right. \\
& \left. - \ln \left(\sin^2 \frac{\theta_3}{2} + \sinh^2 \frac{\theta_2}{2} \right) \right] \quad (4-6)
\end{aligned}$$

In the above expressions we have used the following definitions:

$$\begin{aligned}
\theta_2 &= \pi \left(\frac{z}{b-a} \right) \\
\theta_3 &= \pi \left(\frac{R_m + \rho - 2a}{b-a} \right) \quad \theta_4 = \pi \left(\frac{R_m - \rho}{b-a} \right) \quad (4-7)
\end{aligned}$$

Having produced our field expressions, we next present a specific numerical example in which we consider the fields as a function of position inside the cavity.

4.2 Numerical Example .

The cavity we select has an outer wall radius $b = 5a$ and length $h = 15a$. The aperture radius is chosen as $c = 2a$. We examine the fields inside the cavity at three different frequencies: (1) at a non-resonance location; (2) at the TM_{001} resonance (first axial mode); and (3) at the TM_{100} resonance (first radial mode). If we envision slicing the cavity along the axis at some fixed angle of ϕ , we obtain a ρz plane cross-sectional view of the fields inside. We present contour plots of the

electric field components in this plane. In Fig. 4-1, $|E_{\rho 2}|$ is shown for $ka = 0.349$. This is a non-resonant location below the cutoff frequency of the first radial mode. We verify that the field goes to zero at the front and back walls of the cavity. The maximum value of $|E_{\rho 2}|$ is in the plane of the aperture. There is also a "hot spot" on the wire at $z = 10.5a$. In Fig. 4-2, $|E_{z 2}|$ is shown for this same non-resonance frequency. Since we are below the cutoff frequency of all non-TEM modes, the field is essentially zero inside the cavity except near the aperture.

In order to see the true behavior of the field near the edge at this relatively high frequency, we need to examine a very small area around the edge to insure enough resolution. In Fig. 4-3, $|E_{\rho 2}|$ is plotted over the range: $1.9a \leq \rho \leq 2.1a, 0 \leq z \leq 0.2a$. Let us consider the static case (Jackson, 1975). We would expect $|E_{\rho 2}| \rightarrow \infty$ as we approach the edge in the plane $z = 0$. Off this plane, as we draw nearer the edge, we would predict that $|E_{\rho 2}| \rightarrow 0$. This is the behavior that we see in the contour plot for our time-dependent case. Referring to statics again, as the distance between observation point and edge decreases, we would expect $|E_{z 2}| \rightarrow 0$ in the $z = 0$ plane and $|E_{z 2}| \rightarrow \infty$ off this plane. This agrees with Fig. 4-4 which shows $|E_{z 2}|$ plotted over the same range.

We now consider the fields at a resonance frequency. It should be noted that the exact location of a resonance is difficult to determine. We can estimate the frequency from the known closed cavity resonance locations, but as shown previously, the presence of the aperture does shift these resonances. Alternately, we can refer to current versus frequency plots from the previous chapter or field versus frequency plots obtained from the above field expressions. The difficulty with this approach is that the resonances shift slightly as we change location inside the

Figure 4-1 Contour plot of $|E_{\rho 2}|$ at $ka = 0.349$ (non-resonant location); $b/a = 5$, $c/a = 2$, $h/a = 15$.

Figure 4-3 Contour plot of $|E_{\rho 2}|$ close to the aperture for $ka = 0.349$; $b/a = 5$, $c/a = 2$, $h/a = 15$.

Figure 4-2 Contour plot of $|E_{z2}|$ at $ka = 0.349$ (non-resonant location); $b/a = 5$, $c/a = 2$, $h/a = 15$.

Figure 4-4 Contour plot of $|E_{z2}|$ close to the aperture for $ka = 0.349$; $b/a = 5$, $c/a = 2$, $h/a = 15$.

cavity as demonstrated in our previous current plots. We elect to use the current at the back of the cavity (Fig. 3-10) as our reference. According to this plot, the TM_{001} resonance is at approximately $ka = 0.243$. The field magnitudes for this frequency are shown in Fig. 4-5 and 4-6. We observe a significant increase in $|E_{\rho 2}|$ at this resonance. However, $|E_{z2}|$ is still negligible since the radial modes are cut off. Again, we observe a hot spot of $|E_{\rho 2}|$ on the wire.

In Figs. 4-7 and 4-8, $|E_{\rho 2}|$ and $|E_{z2}|$ are shown for $ka = 0.770$. This is our best guess of the resonance frequency for the first radial mode TM_{100} . The radial mode resonances are sharper and therefore even more difficult to pinpoint than those of the axial modes. There is greater variation in the field throughout the cavity as we would expect at a higher mode, but the magnitude is less than that observed for the TM_{001} mode. Possibly, we are slightly off resonance. Multiple hot spots are observed on the wire as well as one in the upper left quadrant of the mapping. Now that the first non-TEM mode has been excited, a z -component of the field exists inside the cavity. We verify that E_{z2} is zero at the surface of the wire and on the surface at $\rho = b$. A hot spot is observed on the back wall of the cavity at $\rho = 2.75$.

In summary, we have mapped the field throughout the cavity at both resonance and non-resonance frequencies. As in chapter 3, we tracked and identified the resonances from the closed cavity case. Also, from our field contour plots we identified hot spots inside and examined the field behavior near the edge.

Figure 4-5 Contour plot of $|E_{\rho 2}|$ at $ka = 0.243$ (TM_{001} resonance); $b/a = 5$, $c/a = 2$, $h/a = 15$.

Figure 4-6 Contour plot of $|E_{z2}|$ at $ka = 0.243$ (TM_{001} resonance); $b/a = 5$, $c/a = 2$, $h/a = 15$.

Figure 4-7 Contour plot of $|E_{\rho 2}|$ at $ka = 0.770$ (TM_{100} resonance); $b/a = 5$, $c/a = 2$, $h/a = 15$.

Figure 4-8 Contour plot of $|E_{z2}|$ at $ka = 0.770$ (TM_{100} resonance); $b/a = 5$, $c/a = 2$, $h/a = 15$.

CHAPTER 5

FIELDS OUTSIDE THE CAVITY

In this chapter, we consider the fields outside the cavity, in particular, the fields at a large distance from the aperture. This allows us to employ a far-field approximation and evaluate the integrals in our field expressions analytically. We examine the fields as a function of frequency for various observation points. We are interested in the information content concerning the interior of the cavity which is present in these exterior fields.

5.1 Far-Field Approximation

The magnetic field in region 1 is obtained using (2-31) with the variable substitution $\gamma = k\xi$. We obtain

$$H_{\phi 1}(\rho, z) = \frac{k}{\eta} \sum_{m=1}^N \alpha_m \int_0^\infty \frac{e^{ikz\sqrt{1-\xi^2}}}{\sqrt{1-\xi^2}} \frac{B(\xi k \rho) [A(\xi k R_m) - A(\xi k Q_m)]}{H_0^{(1)}(\xi k a) H_0^{(2)}(\xi k a)} d\xi$$

$$- \frac{M_0 e^{-ikd}}{\eta \rho} \cos kz \quad (5-1)$$

where A is given in (3-11). We rewrite this as

$$H_{\phi 1}(\rho, z) = H_{\phi 1}^a(\rho, z) + H_{\phi 1}^s(\rho, z) \quad (5-2)$$

where the superscript s denotes the field resulting from the source in the presence of a shorting plate at $z = 0$ and the superscript a identifies the field perturbation

due to the presence of the aperture. If we consider only $H_{\phi 1}^a$, we see that there is a singularity at $\xi = 0$. Therefore, we separate $H_{\phi 1}^a$ into two integrals as follows:

$$H_{\phi 1}^a(\rho, z) = \frac{k}{\eta} \sum_{m=1}^N \alpha_m \left\{ \int_0^{\delta} F_H(\xi) d\xi + \int_{\delta}^{\infty} F_H(\xi) d\xi \right\} \quad (5-3)$$

where δ is small. In the first integral, we may use small argument approximations for the Bessel functions as well as the variable substitution $x = \xi ka$ to obtain

$$\int_0^{\delta} F_H(\xi) d\xi \simeq -\frac{e^{ikz}}{k\rho} \ln\left(\frac{R_m}{Q_m}\right) \int_0^{\hat{\delta}} \frac{dx}{x \ln^2 x} \quad (5-4)$$

where $\hat{\delta} = \delta ka$. It may easily be shown that for small δ , the contribution to $H_{\phi 1}^a$ from this integral is negligible.

Now we proceed to evaluate the second integral in (5-3). One possible method is to rewrite the integral as

$$\int_{\delta}^{\infty} F_H(\xi) d\xi = \int_{-\infty}^{\infty} F_H(\xi) d\xi - \int_{-\infty}^{\delta} F_H(\xi) d\xi \quad (5-5)$$

The first term on the right of (5-5) can be evaluated using the method of steepest descent (Wait, 1985). It may then be shown that the contribution of the second integral is negligible. Another approach, the one we will use here, is to employ the method of stationary phase (Bender and Orszag, 1978) to evaluate the second integral in (5-3). Expressing the integral in a slightly different form, we have

$$\begin{aligned} \int_{\delta}^{\infty} F_H(\xi) d\xi &= -\frac{i}{2} \int_{\delta}^{\infty} \frac{e^{ikz\sqrt{1-\xi^2}}}{\sqrt{1-\xi^2}} \left[\frac{H_1^{(2)}(\xi k\rho)}{H_0^{(2)}(\xi ka)} - \frac{H_1^{(1)}(\xi k\rho)}{H_0^{(1)}(\xi ka)} \right] \\ &\quad \cdot [A(\xi k R_m) - A(\xi k Q_m)] d\xi \\ &\doteq \int_{\delta}^{\infty} [F_H^1(\xi) + F_H^2(\xi)] d\xi \end{aligned} \quad (5-6)$$

Let us investigate the first term on the right side of (5-6). For $k\rho$ large, we may use the large argument approximation for the Hankel function, namely,

$$H_1^{(2)}(\xi k\rho) \simeq \sqrt{\frac{2}{\pi \xi k\rho}} e^{-i(\xi k\rho - \frac{3\pi}{4})} \quad (5-7)$$

If, in addition, we make the variable substitution $\xi = \cos \alpha$, we obtain

$$\int_{\delta}^{\infty} F_H^1(\xi) d\xi \simeq \frac{e^{i\frac{3\pi}{4}}}{i\sqrt{2\pi k\rho}} \int_{-i\infty}^{\frac{\pi}{2}-\epsilon} e^{ik(z \sin \alpha - \rho \cos \alpha)} \cdot \frac{[A(kR_m \cos \alpha) - A(kQ_m \cos \alpha)]}{\sqrt{\cos \alpha} H_0^{(2)}(ka \cos \alpha)} d\alpha \quad (5-8)$$

where ϵ is some small number and the path of integration is shown in Fig 5-1.

Next we invoke a change of coordinates, from cylindrical to spherical, using the definitions:

$$\begin{aligned} \rho &= +R \cos \Omega \\ z &= -R \sin \Omega \end{aligned} \quad (5-9)$$

This new coordinate system is illustrated in Fig 5-2. We obtain

$$\int_{\delta}^{\infty} F_H^1(\xi) d\xi = \frac{e^{i\frac{3\pi}{4}}}{i\sqrt{2\pi kR \cos \Omega}} \int_{-i\infty}^{\frac{\pi}{2}-\epsilon} e^{iR\psi(\alpha)} G(\cos \alpha) d\alpha \quad (5-10)$$

where

$$\psi(\alpha) = -k \cos(\Omega - \alpha) \quad (5-11)$$

and $G(\cos \alpha)$ is a shorthand notation representing all the terms inside the integral excluding the exponential. We restrict ourselves to values of Ω away from 0 or $\frac{\pi}{2}$ (i.e., grazing angles).

Figure 5-1 Complex integration path.

Figure 5-2 Cylindrical versus spherical coordinate systems.

For the far-field approximation in which R becomes very large, we can show that the contribution of the integral from $-\infty$ to 0 is negligible because the exponential term in (5-10) behaves as $e^{-kR \sin \Omega \sinh |\alpha|} \rightarrow 0$ as $R \rightarrow \infty$. We are left with only the integration along the real axis. As R becomes very large, the exponential in (5-10) oscillates rapidly and adjacent subintervals effectively cancel except near the stationary point $\alpha = \Omega$. Negligible error is introduced, therefore, if we change the limits of integration to be $\Omega - \tilde{\epsilon}$ to $\Omega + \tilde{\epsilon}$, where $\tilde{\epsilon}$ is simply some small perturbation from Ω . Likewise, since the major contribution is in the vicinity of Ω , we can change our integration limits to be $-\infty$ to $+\infty$, again without adding any significant error. Then, employing the following approximations

$$G(\cos \alpha) \simeq G(\cos \Omega) \quad (5-12)$$

$$\psi(\alpha) \simeq \psi(\Omega) + \psi'(\Omega)(\alpha - \Omega) + \psi''(\Omega) \frac{(\alpha - \Omega)^2}{2} \quad (5-13)$$

we obtain

$$\int_{\delta}^{\infty} F_H^1(\xi) d\xi \simeq - \frac{G(\cos \Omega) e^{-ikR}}{i\sqrt{\cos \Omega} kR} \quad (5-14)$$

The second term on the right side of (5-6) does not have a stationary point, so it essentially makes no contribution. Therefore, (5-14) and (5-6) are equivalent. Using this result in (5-3), we obtain

$$H_{\phi 1}^a(R, \Omega) = i\omega\epsilon \frac{e^{-ikR}}{kR} \sum_{m=1}^N \alpha_m \frac{[A(kR_m \cos \Omega) - A(kQ_m \cos \Omega)]}{\cos \Omega H_0^{(2)}(ka \cos \Omega)} \quad (5-15)$$

We apply the same coordinate transformation specified in (5-9) to the source term in (5-1). Combining this result with (5-15), we have

$$\begin{aligned}
H_{\phi 1}(R, \Omega) = i\omega\epsilon \frac{e^{-ikR}}{kR} \sum_{m=1}^N \alpha_m \frac{[A(kR_m \cos \Omega) - A(kQ_m \cos \Omega)]}{\cos \Omega H_0^{(2)}(ka \cos \Omega)} \\
- \frac{M_0 e^{-ikd}}{\eta R \cos \Omega} \cos(kR \sin \Omega)
\end{aligned} \tag{5-16}$$

This is our far-field approximation for the total magnetic field in region 1. The θ -directed electric field differs only by a factor of η . Having completed the analysis, we now present a few numerical examples.

5.2 Numerical Examples

We select the same cavity dimensions as in the previous chapter: $b = 5a$ and $h = 15a$. However, in this section, we consider two different aperture sizes: $c = 1.01a$ and $c = 2a$. Let us examine just the field component associated with the aperture, that is, $H_{\phi 1}^a(R, \Omega)$. In Fig. 5-3, for the case where $c = 1.01a$, we compare this exterior field to an interior current, specifically, the current at the end of the cavity ($z = h$). In order to plot both curves on the same scale, it is necessary to normalize $I(z = h)$ by a constant. We are not concerned with actual magnitudes, but rather we are interested in comparing the frequency responses. We observe the same resonant features in both the field and the current with what appear to be slight frequency shifts at the higher resonances. If we plot the frequency range where these higher resonances occur with greater resolution as in Fig. 5-4, we see that, in fact, the resonances align almost exactly. Although, we would ideally like to use very high resolution at all times, we must point out that this greatly increases computation time. In Fig. 5-5, we compare the field to the current at $z = 0$.

Again, the resonances align very closely. For $c = 2a$, the field is also compared to the current at $z = h$ and $z = 0$ as shown in Figs. 5-6 and 5-7. Once again, there is a one-to-one correspondence between the field and current resonances.

In Fig. 5-8, we show $H_{\phi 1}^a(R, \Omega)$ at various distances from the aperture ($c = 2a$). Not surprisingly, as R increases, the magnitude of the field decreases, but the frequency response does not change. This is the behavior that we would predict from (5-15). In Fig. 5-9, $H_{\phi 1}^a(R, \Omega)$ is plotted for various observation angles. As we move towards the wire, the field strength increases. The frequency response is again unchanged.

The existence of the resonances in the field plots is a very important result. It means that we can obtain information about the interior of the cavity at an exterior location far from the aperture. The resonance pattern has also been shown to be independent of observation angle. This concept has great potential for diagnostic applications.

Let us assume a measurement is made of the total field. If we short out the aperture and make a second measurement of just the source term, the difference of the two produces $H_{\phi 1}^a$, which contains the interior information. It may then be shown that we can estimate the general size of the cavity based solely on an exterior observation, namely, the number of resonances in a given bandwidth.

Figure 5-3 External magnetic field versus normalized current as a function of frequency; $z = h$, $R/a = 200$, $\Omega = \pi/4$, $b/a = 5$, $c/a = 1.01$, $h/a = 15$.

Figure 5-4 External magnetic field versus normalized current as a function of frequency; $z = h$, $R/a = 200$, $\Omega = \pi/4$, $b/a = 5$, $c/a = 1.01$, $h/a = 15$.

Figure 5-5 External magnetic field versus normalized current as a function of frequency; $z = 0$, $R/a = 200$, $\Omega = \pi/4$, $b/a = 5$, $c/a = 1.01$, $h/a = 15$.

Figure 5-6 External magnetic field versus normalized current as a function of frequency; $z = h$, $R/a = 200$, $\Omega = \pi/4$, $b/a = 5$, $c/a = 2$, $h/a = 15$.

Figure 5-7 External magnetic field versus normalized current as a function of frequency; $z = 0$, $R/a = 200$, $\Omega = \pi/4$, $b/a = 5$, $c/a = 2$, $h/a = 15$.

Figure 5-8 External magnetic field versus distance from the aperture as a function of frequency; $\Omega = \pi/4$, $b/a = 5$, $c/a = 2$, $h/a = 15$.

Figure 5-9 External magnetic field versus observation angle as a function of frequency; $R/a = 200$, $b/a = 5$, $c/a = 2$, $h/a = 15$.

CHAPTER 6

CONCLUSIONS AND RECOMMENDATIONS

In this thesis we have presented the solution to the problem of a wire penetrating a circular aperture in an infinite screen and coupling energy into a cavity behind that screen. Beginning with the TM mode equations, we produced a differential equation for the azimuthal magnetic field in both the exterior and the interior regions. We solved both differential equations using the method of Green's functions. Applying the boundary conditions in the aperture, we obtained an integral equation in terms of the aperture electric field. We solved this integral equation using two approximate methods. The zeroth order approximation provided very good results at low frequencies while dramatically reducing computation time. Using the method of moments (MOM), we obtained greater accuracy over a larger frequency range, but at the expense of increased computation time. We showed, however, that by employing non-uniform width pulse functions in the MOM expansions, we could reduce the size of the matrix to be solved and thus improve computation speed. In both methods, we effectively treated any singularities that existed and improved the speed of convergence by considering the asymptotic limits.

An equivalent circuit model consisting of a current source in parallel with two admittances was introduced in order to aid in our physical interpretation of the problem. The current on the wire inside the cavity and the two admittance parameters were examined as a function of frequency for three cavity sizes. In each case, we considered only a frequency range in which the first few modes were above cutoff. From the known resonance frequencies of a closed coaxial cavity,

we were able to identify all but one of the resonances appearing in the current plots. The discovery of this unexpected resonance, lower in frequency than the others, prompted us to reexamine our equivalent circuit. It was shown that this first resonance was an effect produced by the aperture and cavity combination. Whereas the other resonances could be explained in terms of the admittance of the cavity alone, it was necessary to consider the total admittance of the circuit in order to explain this lowest resonance.

We also examined the current inside the cavity at various positions on the wire. Employing a ray-optic description, we were successful in predicting the locations of the various maxima, minima, and nulls. It was observed that as aperture size increased the resonance peaks in the current broadened and shifted away from the closed cavity values in the direction of higher frequency. This effect was attributed to the decreasing gap capacitance of the aperture.

The admittance parameters were studied next. In the exterior region, the real part of the admittance, representing the radiation losses for the region, was shown to depend on the amount of mode conversion at the aperture. That is, the conductivity increased with increasing frequency, or with decreasing aperture size. Inside the cavity, as expected, there was no radiation loss, and, therefore, zero conductivity. The imaginary part of the admittance, or susceptance, for the exterior region was described as representing the capacitance between the wire and screen. As frequency increased, or the aperture size decreased, the susceptance increased. For the interior region, the susceptance was shown to depend not only on the gap capacitance of the aperture, but even more so on the capacitance associated with the cavity. The resonances of the cavity were clearly visible in the susceptance curves for that region.

Although the equivalent circuit parameters provide valuable physical insights, we discovered that there are certain limitations associated with use of the model. For a larger aperture size, that is, large relative to frequency or relative to the cavity size, the admittance parameters outside the cavity are affected by the radial mode resonances. In addition, the conductivity inside the cavity, which should be zero, exhibits resonant behavior at these same frequencies. Essentially, our zeroth-order, equivalent circuit model works well for TEM, but begins to break down when higher order modes are introduced.

In addition to examining the current and admittance for the cavity region, we presented contour plots of the electric field components inside. We evaluated the field components at a non-resonance location, at a TEM resonance, and at a radial mode resonance. Besides mapping the field throughout the cavity, we also observed the field behavior very near the aperture edge and made comparisons to the static case.

Outside the cavity, we evaluated the magnetic field using a far-field approximation. This allowed the integrals in the field expression to be evaluated analytically. We examined the field as a function of frequency for different observation points. Although the magnitude varied, the frequency response was unaffected by changes in distance from the aperture or changes in observation angle. The exterior field was compared to an interior current and the following important observation was made: the same resonance features appeared in both curves. The implication is that information concerning the interior of the cavity can be obtained at an exterior location far from the aperture. In addition, the acquisition of such information is relatively insensitive to observer position as long as we remain in the far field. This idea has significant potential as a diagnostic tool.

There are a number of possible extensions to the work presented here. We might consider this same problem in the time domain. Given a pulse excitation, we could observe the multiple reflections inside the cavity. A second extension would be to place dielectric or conductive fill inside the cavity and observe how the coupling between the exterior and interior regions changes. This would not require any modifications to the existing numerical code. Another option would be to consider our problem geometry with a different source excitation. In particular, we might examine the case of a plane wave obliquely incident on the plane of the aperture. Such an extension would involve the coupling of TE_z and TM_z mode solutions. Finally, as suggested by Wait (1988), we might also consider a dielectric coated wire. In this manner, we could insure that only a single mode propagates down the wire toward the aperture, thus eliminating the need for our ideal source definition.

APPENDIX A

ANALYTIC EVALUATION OF THE SUM OF THE ASYMPTOTICS

In the analysis pertaining to the asymptotic forms for the current inside the cavity, we are concerned with the summation of the following series

$$S = \sum_{n=1}^{\infty} \frac{e^{-n\theta}}{n^2} \quad (A-1)$$

where

$$\theta = \theta_r + i\theta_i \quad \theta_r, \theta_i \in \mathbb{R} \quad (A-2)$$

In this appendix, we present an extension to the work done by Zhang and Dudley (1987) in which they considered the summation for θ real. The derivation is essentially the same except that the restriction placed on the region of validity is different.

Zhang and Dudley begin by considering an integral similar to Clausen's integral, namely,

$$I = \int \ln \left(2 \sinh \frac{\theta}{2} \right) d\theta \quad (A-3)$$

They expand $2 \sinh(\frac{\theta}{2})$ in a Maclaurin series as follows:

$$\ln \left(2 \sinh \frac{\theta}{2} \right) = \ln \theta + \ln(1 + \varphi) \quad (A-4)$$

where

$$\varphi = \sum_{n=1}^{\infty} \frac{(\frac{\theta}{2})^{2n}}{(2n+1)!} \quad (A-5)$$

The next step is to use

$$\ln(1 + \varphi) = \varphi - \frac{\varphi^2}{2} + \frac{\varphi^3}{3} - \dots \quad (A-6)$$

Here is where the difference lies between the real and complex cases. The restriction on (A-6) for φ complex is given by

$$|\varphi| \leq 1 \quad \text{and} \quad \varphi \neq -1 \quad (A-7)$$

(Abramowitz and Stegun, 1972). This is different than the restriction for real φ . Given (A-7), we will determine the restriction on θ presently. Let us first complete the evaluation of (A-1) as described by Zhang and Dudley. Using (A-5) and (A-6) in (A-4), we find that

$$I = \theta(\ln \theta - 1) + \sum_{k=1}^{\infty} \frac{B_{2k} \theta^{2k+1}}{(2k)! (2k)(2k+1)} \quad (A-8)$$

where B_{2k} are Bernoulli numbers. Finally, the sum in (A-1) is written as follows:

$$\begin{aligned} S &= \sum_{n=1}^{\infty} \frac{1}{n^2} - \int \sum_{n=1}^{\infty} \frac{e^{-n\theta}}{n} d\theta \\ &= \frac{\pi^2}{6} - \int \left[\frac{\theta}{2} - \ln \left(2 \sinh \frac{\theta}{2} \right) \right] d\theta \\ &= \frac{\pi^2}{6} - \frac{\theta^2}{4} + I \end{aligned} \quad (A-9)$$

where I is given in (A-8).

Now returning to the determination of the restriction on θ , we have from (A-4) the relationship:

$$\varphi = \frac{2 \sinh \frac{\theta}{2}}{\theta} - 1 \quad (A-10)$$

Applying the condition in (A-7) given (A-10), we find after some algebraic manipulation, that in terms of θ (i.e., θ_r and θ_i) this translates to the following inequality:

$$\begin{aligned} \sinh\left(\frac{\theta_i}{2}\right) \left[\sinh\left(\frac{\theta_i}{2}\right) - \theta_i \cos\left(\frac{\theta_r}{2}\right) \right] \\ + \sin\left(\frac{\theta_r}{2}\right) \left[\sin\left(\frac{\theta_r}{2}\right) - \theta_r \cosh\left(\frac{\theta_i}{2}\right) \right] < 0 \quad (A-11) \end{aligned}$$

This transcendental equation can be solved numerically. For a given θ_i , we determine the restriction on θ_r , such that (A-9) is valid. In evaluating the asymptotic forms, we observe that θ_i is a function of the aperture size while θ_r is a function of the position along the wire. Hence, for a given aperture size, we solve (A-11) to determine how far from the aperture we may be such that the asymptotic form is still valid. We refer to this distance as z_m .

Due to the relative complexity of (A-11), we are unable to make any general statements regarding the behavior of θ_r as a function of θ_i . In our numerical computations, for a given aperture size and wire location, we simply test the inequality. If it is true, we employ the current expression which includes the asymptotic terms.

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