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**Acoustic wave propagation in a cylindrical borehole with
fractures**

Spring, Christopher Todd, M.S.

The University of Arizona, 1990

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ACOUSTIC WAVE PROPAGATION
IN A CYLINDRICAL BOREHOLE
WITH FRACTURES

by

Christopher Todd Spring

A Thesis Submitted to the Faculty of the
DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING
In Partial Fulfillment of the Requirements
For the Degree of
MASTER OF SCIENCE
WITH A MAJOR IN ELECTRICAL ENGINEERING
In the Graduate College
THE UNIVERSITY OF ARIZONA

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ABSTRACT

We study the problem of acoustic wave propagation in a cylindrical borehole possessing a finite number of transverse discontinuities. We model the field behavior through Green's function techniques. We formulate an integral equation whose solution will enable us to solve for the acoustic field everywhere within our structure. We investigate asymptotic forms to speed the numerical convergence of our solution. To solve the integral equation we employ both the method of moments and the low frequency approximation. We study the reflection coefficient in the time and frequency domains. Finally after presenting solutions for the one and two fracture case, we generalize our analysis for many fractures.

CHAPTER 1

INTRODUCTION

The problem of acoustic wave propagation in a borehole with discontinuities has been studied extensively for geophysical applications. Most investigators have used a transmission line approach, with acoustic impedances representing the discontinuities. This approach is valid for low frequencies, but it breaks down as frequencies increase. In particular we extend here the results of Hornby, et al. (1987) for higher frequency ranges. They considered only the fundamental mode of propagation in their analysis. For our higher frequency analysis we must consider the higher order modes present since these modes characterize the fine structure of the field distribution. A more accurate representation is possible by construction of an integral equation that models the precise wave behavior.

The geometry of interest, shown in Fig. (1-1), is a fluid-filled cylindrical borehole of infinite extent existing in a solid medium, bisected at the origin by a horizontal fluid-filled fracture of infinite extent. In cylindrical coordinates, assuming angular symmetry, we define two regions:

Region 1, the borehole: $\rho \in [0, a]$, $z \in (-\infty, \infty)$;

Region 2, the fracture: $\rho \in [a, \infty)$, $z \in [0, b]$.

A variety of acoustic waves may exist in this structure, such as transverse shear waves or longitudinal dilatational waves in the borehole walls (Skudrzyk, 1971). We will be concerned with only the "tube" waves that propagate in the fluid medium inside the borehole, also known as "hydro" waves (Galperin, 1978)

Figure 1-1 The Single-Fracture Borehole Geometry

or “Stoneley” waves (Hornby, et al., 1987). This form of acoustic radiation acts as a simple acoustic wave satisfying the Helmholtz equation that moves within the borehole at the speed of sound in the fluid.

For our mathematical model of a practical borehole problem, we assume the hard surface boundary condition which assumes the normal acoustic impedance $Z_n = p/v_n$ is infinite at the walls. The fluid viscosity is neglected. The disparity in material properties such as density and bulk modulus between the fluid and the borehole walls makes these reasonable assumptions. This insures that the normal derivative of the velocity potential is zero at the boundaries (the Neumann condition). A source of positive-going acoustic energy is generated in the borehole at some negative position $z = -L$. The acoustic wave travels in the positive z direction in the borehole until it encounters the fracture. Here the wave is split into three components: the positive-going transmitted wave and the negative-going reflected wave, both in the borehole; and the radially-outward wave in the fracture. Because the borehole and the fracture are infinite in length, the acoustic waves travel without reflection once they have passed the interface between the two regions.

The second chapter introduces the velocity potential as the acoustic parameter of interest in the problem. The frequency domain differential equations governing its behavior are presented. For the single fracture case, Green’s functions associated with the velocity potentials in each region are developed. With Green’s theorem the respective velocity potentials are constructed in terms of Green’s functions. With the proper continuity conditions at the interface between the borehole and the fracture, an integral equation is built which will allow the characterization of the fields everywhere within the region of interest.

The solution to the integral equation is investigated in the third chapter. Attention is paid to the infinite sums which are present in the integrands of the equation, particularly to their asymptotic forms. Through the addition and subtraction of the asymptotic forms of the summand, the singularities present in the equation can be observed and the convergence rate of the sums can be increased. Specific asymptotic forms are chosen to optimize convergence rates. The method of moments is applied next to numerically evaluate the integral equation by transforming it into a matrix equation.

In chapter four the solution to the integral equation is used to explore the reflection characteristics of the single fracture case. An expression for the frequency domain reflection coefficient is developed from the velocity potential equations. A low frequency approximation to the acoustic wave equation is made to expedite the numerical computation for certain parameter ranges. Then a time domain transient acoustic pulse is applied to the system to observe its reflection characteristics.

The problem is next extended to a two fracture borehole in the fifth chapter. Much of the work in previous chapters is applicable. After the reformulation into a new integral equation, the method of moments is again employed as the solution technique. The reflection analysis is presented with special attention to the interaction of the two fractures.

The concluding chapter summarizes some of the important results and a generalization to additional numbers of fractures is made. Finally, recommendations are made for future work.

CHAPTER 2

PROBLEM FORMULATION

In this chapter the wave behavior in a borehole with a single fracture is studied. After the characterization of acoustic wave propagation in terms of velocity potentials, Green's functions are created and used to construct an integral equation that models the wave behavior everywhere in the borehole and fracture.

2.1 The Acoustic Wave Equation

The scalar time-domain acoustic wave equation is given by

$$F(\vec{r}, t) = -\nabla^2 U(\vec{r}, t) + \frac{1}{v^2} \frac{\partial^2 U(\vec{r}, t)}{\partial t^2} \quad (2-1)$$

where v is the mean velocity of sound in the medium in meters per second, $\vec{r}$ is the position in meters, t is time in seconds, F is the driving function in Hertz, and U is the velocity potential in meters squared per second. The use of velocity potential as the acoustic variable of interest is a matter of convenience. It is related to the other familiar acoustic quantities of velocity $\vec{V}$ and pressure P by

$$\vec{V} = \nabla U, \quad (2-2)$$

$$P = -\rho_0 \frac{\partial U}{\partial t}, \quad (2-3)$$

where ρ_0 is the mean density in kilograms per cubic meter. With the assumption of time-harmonic behavior ($e^{j\omega t}$ time dependence), the quantities in equation (2-1)

may be replaced by their standard Fourier transform pairs to form the frequency domain equation

$$f(\vec{r}, \omega) = -\nabla^2 u(\vec{r}, \omega) - k^2 u(\vec{r}, \omega) \quad (2-4)$$

where $k = \omega/v$, the wave number.

To find the velocity potential at every point in the two regions, we decompose it into two functions: $u_1(\rho, z)$, existing in Region 1, and $u_2(\rho, z)$, existing in Region 2. Placing a forcing function $f(\rho, z)$ in the borehole, and defining the operator $L = -(\nabla^2 + k^2)$, we must solve two differential equations,

$$Lu_1 = f; \quad (2-5)$$

$$Lu_2 = 0. \quad (2-6)$$

The specific accompanying boundary conditions adapted from the hard surface conditions and from the Sommerfeld radiation condition are in Region 1,

$$\frac{\partial u_1(a, z)}{\partial \rho} = 0, \quad z \notin [0, b]; \quad (2-7)$$

$$\lim_{z \rightarrow \pm\infty} \left[\frac{\partial u_1(\rho, z)}{\partial z} \pm iku_1(\rho, z) \right] = 0, \quad \rho \in [0, a]; \quad (2-8)$$

and in Region 2,

$$\frac{\partial u_2(\rho, 0)}{-\partial z} = 0, \quad \rho \in [a, \infty); \quad (2-9)$$

$$\frac{\partial u_2(\rho, b)}{\partial z} = 0, \quad \rho \in [a, \infty); \quad (2-10)$$

$$\lim_{\rho \rightarrow \infty} \sqrt{\rho} \left[\frac{\partial u_2(\rho, z)}{\partial \rho} + iku_2(\rho, z) \right] = 0, \quad z \in [0, b]. \quad (2-11)$$

These boundary conditions match the geometry shown in Figure (1-1).

2.2 Green's Function Problem

To solve for u_1 and u_2 , the Green's functions g_1 and g_2 , respectively, will be used in a standard solution technique (Stakgold, 1979). Because the wave number k is real, the operator L is formally self-adjoint. This allows us to use the same L operator on the Green's functions. The solutions will also show that the Green's functions are symmetric. Using the same operator and applying delta function sources, we solve two associated differential equations,

$$Lg_1 = \delta(z - z') \frac{\delta(\rho - \rho')}{\rho}; \quad (2-12)$$

$$Lg_2 = \delta(z - z') \frac{\delta(\rho - \rho')}{\rho}. \quad (2-13)$$

The Green's functions correspond to the response of the system to ring source excitation. The boundary conditions for the Green's functions are as follows:

$$\frac{\partial g_1(a, z)}{\partial \rho} = 0, \quad \forall z; \quad (2-14)$$

$$\lim_{z \rightarrow \pm\infty} \left[\frac{\partial g_1(\rho, z)}{\partial z} \pm ikg_1(\rho, z) \right] = 0, \quad \rho \in [0, a]; \quad (2-15)$$

$$\frac{\partial g_2(\rho, 0)}{-\partial z} = 0, \quad \rho \in [a, \infty); \quad (2-16)$$

$$\frac{\partial g_2(\rho, b)}{\partial z} = 0, \quad \rho \in [a, \infty); \quad (2-17)$$

$$\frac{\partial g_2(a, z)}{-\partial \rho} = 0, \quad z \in [0, b]; \quad (2-18)$$

$$\lim_{\rho \rightarrow \infty} \sqrt{\rho} \left[\frac{\partial g_2(\rho, z)}{\partial \rho} + ikg_2(\rho, z) \right] = 0, \quad z \in [0, b]. \quad (2-19)$$

These boundary conditions are crafted to ignore the interface between the two regions, as if they were physically separated into a uniform borehole and an infinite

toroid. The equivalent geometry is shown in Figure (2-1). This choice becomes useful later when the velocity potentials are matched at the interface.

2.2.1 Green's Function: Region 1

To simplify the problem, cross-sectional eigenfunctions are used to expand the Green's function. Because of the angular symmetry, the natural expansion is

$$g_1(\rho, z|\rho', z') = \sum_{n=0}^{\infty} A_n(z) J_0(\chi_n \rho), \quad \chi_n = \frac{x'_{0n}}{a} = \frac{x_{1n}}{a}, \quad (2-20)$$

where x'_{0n} is the n^{th} zero of the derivative of the zero-order Bessel function. This satisfies

$$\frac{\partial g_1}{\partial \rho} \Big|_{\rho=a} = 0. \quad (2-21)$$

Expanding the operator L on this form of g_1 , we find that (2-12) becomes

$$Lg_1 = - \sum_{n=0}^{\infty} \left[\frac{\partial^2}{\partial z^2} + (k^2 - \chi_n^2) \right] A_n(z) J_0(\chi_n \rho) \quad (2-22)$$

Defining

$$k_{nz} = \sqrt{k^2 - \chi_n^2}, \quad \text{Im}(k_{nz}) < 0, \quad (2-23)$$

we arrive at a system of equations

$$- \sum_{n=0}^{\infty} \left[A_n''(z) + k_{nz}^2 A_n(z) \right] J_0(\chi_n \rho) = \delta(z - z') \frac{\delta(\rho - \rho')}{\rho}. \quad (2-24)$$

Figure 2-1 The Equivalent Geometry for the Green's Functions

Using the orthogonal properties of the Bessel functions, we multiply each equation by $J_0(\chi_m \rho)$ and integrate over the radius to eliminate one dimension, as follows:

$$\int_0^a L g_1(\rho, z) J_0(\chi_m \rho) \rho d\rho = \int_0^a \delta(z - z') \delta(\rho - \rho') J_0(\chi_m \rho) d\rho. \quad (2-25)$$

This equation collapses to

$$-\left(\frac{\partial^2}{\partial z^2} + k_{mz}^2\right) A_m(z) = Q \delta(z - z'), \quad (2-26)$$

where

$$Q = \frac{2J_0(\chi_m \rho')}{a^2 J_0^2(\chi_m a)}. \quad (2-27)$$

This one dimensional Green's function is easily solved. For boundedness at infinity and in accordance with the choice of $Im(k_{nz}) < 0$, we choose

$$A_n(z) = \begin{cases} B e^{ik_{nz} z}, & z < z'; \\ C e^{-ik_{nz} z}, & z > z', \end{cases} \quad (2-28)$$

Establishing the continuity

$$B e^{ik_{nz} z'} = C e^{-ik_{nz} z'} \quad (2-29)$$

and jump conditions

$$-ik_{nz}(C e^{-ik_{nz} z'} + B e^{ik_{nz} z'}) = -Q \quad (2-30)$$

at $z = z'$ to solve for B and C , we find

$$B = \frac{Q}{2ik_{nz}} e^{-ik_{nz} z'} \quad (2-31)$$

$$C = \frac{Q}{2ik_{nz}} e^{ik_{nz} z'}. \quad (2-32)$$

Combining with (2-28), we arrive at the expression

$$A_n(z|\rho', z') = \frac{J_0(\chi_n \rho')}{ik_{nz} a^2 J_0^2(\chi_n a)} e^{-ik_{nz}|z-z'|}, \quad (2-33)$$

where g_1 is found by substituting (2-33) into (2-20). Note the symmetry of the Green's function with respect to the primed and unprimed coordinate systems.

Equation (2-33) describes the mode behavior of the waves in the borehole. The exponential function has a wave number k_{nz} that determines the cutoff for each mode of propagation. The n^{th} mode is cut off below frequencies where $k_{nz} = 0$, or in terms of frequency, $f_{cn} = x'_{0n} v / 2\pi a$.

2.2.2 Green's Function: Region 2

Similarly for the Green's function in Region 2, we substitute the cross-sectional eigenfunctions to represent the Green's function as mentioned before. The natural expansion is

$$g_2(\rho, z|\rho', z') = \sum_{n=0}^{\infty} B_n(\rho) \cos\left(\frac{n\pi z}{b}\right); \quad (2-34)$$

which satisfies

$$\frac{\partial g_2}{\partial z} \Big|_{z=0,b} = 0. \quad (2-35)$$

Expanding the L operator, we find

$$Lg_2 = - \sum_{n=0}^{\infty} \left[\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} \right) + k_{n\rho}^2 \right] B_n \cos\left(\frac{n\pi z}{b}\right), \quad (2-36)$$

where

$$k_{n\rho} = \sqrt{k^2 - (n\pi/b)^2}, \quad \text{Im}(k_{n\rho}) < 0. \quad (2-37)$$

Utilizing the orthogonal properties of cosine, we can again collapse the infinite sum to a single equation

$$-\left[\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial B_n}{\partial \rho} \right) + k_{n\rho}^2 B_n \right] = R \frac{\delta(\rho - \rho')}{\rho}, \quad (2-38)$$

where

$$R = \frac{\epsilon_n}{b} \cos\left(\frac{n\pi z'}{b}\right). \quad (2-39)$$

This differential equation is a form of Bessel's equation. Our choice of the branch of $k_{n\rho}$ demands that the positive-going wave function be the Hankel function of the second kind. The unknown is then expanded as

$$B_n(\rho) = \begin{cases} CH_0^{(1)}(k_{n\rho}\rho) + DH_0^{(2)}(k_{n\rho}\rho), & a \leq \rho \leq \rho'; \\ EH_0^{(2)}(k_{n\rho}\rho), & \rho' \leq \rho. \end{cases} \quad (2-40)$$

Applying the continuity and jump conditions as in Region 1, we solve for $B_n(\rho)$,

$$B_n(\rho) = \frac{\pi\epsilon_n}{4ib} \frac{H_0^{(2)}(k_{n\rho}\rho_{>})}{H_1^{(2)}(k_{n\rho}a)} W(\rho_{<}) \cos\left(\frac{n\pi z'}{b}\right), \quad (2-41)$$

where

$$W(\rho) = H_1^{(2)}(k_{n\rho}a)H_0^{(1)}(k_{n\rho}\rho) - H_1^{(1)}(k_{n\rho}a)H_0^{(2)}(k_{n\rho}\rho), \quad (2-42)$$

$$\rho_{>} = \max(\rho, \rho'), \quad (2-43)$$

$$\rho_{<} = \min(\rho, \rho'), \quad (2-44)$$

$$\epsilon_n = \begin{cases} 1, & n = 0; \\ 2, & n \neq 0. \end{cases} \quad (2-45)$$

The Green's function g_2 in Region 2 is found by substituting (2-41) into (2-34). As with the first Green's function, there is symmetry with respect to the primed and unprimed coordinate systems.

2.3 Integral Equation

Using Green's Theorem, we can construct the velocity potentials from their respective Green's functions. Defining an inner product in Region 1

$$\langle X, Y \rangle_1 = \int_{-\infty}^{\infty} \int_0^a XY \rho' d\rho' dz'. \quad (2-46)$$

and performing the inner product of g_1 with Lu_1 , we find

$$u_1(\rho, z) = \langle g_1, f \rangle_1 + \oint_{S_1} \left(g_1 \frac{\partial u_1}{\partial n'} - u_1 \frac{\partial g_1}{\partial n'} \right) dS'. \quad (2-47)$$

The surface S_1 bounds Region 1, existing at $\rho = a$, $z \in (-\infty, \infty)$, and at $z = \pm\infty$, $\rho \in [0, a]$. The surface integral at infinity is zero because of the radiation condition. The other surface integral is zero nearly everywhere; only between $0 < z < b$ is the normal derivative of the velocity potential u_1 not zero. This is the justification for choosing the boundary conditions on the Green's function slightly differently from those on its associated velocity potential; g_1 was specifically chosen to be zero here as it is elsewhere. By choosing the forcing function

$$f(\rho', z') = 2ik\delta(z') \quad (2-48)$$

the source term reduces to a positive-going tube wave, and u_1 reduces to

$$u_1(\rho, z) = e^{-ikz} + a \int_0^b g_1(\rho, z|a, z') \frac{\partial u_1(a, z')}{\partial \rho'} dz'. \quad (2-49)$$

To construct u_2 we proceed as before, this time applying Green's Theorem in Region 2. Because there is no source in this region, the velocity potential is equal only to the surface integral

$$u_2(\rho, z) = \oint_{S_2} (g_2 \frac{\partial u_2}{\partial n'} - u_2 \frac{\partial g_2}{\partial n'}) dS'. \quad (2-50)$$

Once again, all the terms vanish except for the surface connecting the fracture with the borehole. Note the sign change due the reversal in the direction of the outward normals, as follows:

$$u_2(\rho, z) = -a \int_0^b g_2(\rho, z|a, z') \frac{\partial u_2(a, z')}{\partial \rho'} dz'. \quad (2-51)$$

The integral equations of (2-49) and (2-51) must be self-consistent. A ρ' -derivative of the velocity potential appears under the integral in both equations, leading us to perform a self-consistency check by differentiating the equations by ρ . The result should be that the integral collapses into the partial derivative of u with respect to ρ . This is readily demonstrable in Region 2 by differentiating the u_2 integral equation with respect to ρ . Remembering the Wronskian identity

$$H_0^{(1)}(z)H_1^{(2)}(z) - H_1^{(1)}(z)H_0^{(2)}(z) = 4i/\pi z \quad (2-52)$$

and the spectral representation of the delta function

$$\delta(z - z') = \sum_{n=0}^{\infty} \frac{\epsilon_n}{b} \cos\left(\frac{n\pi z}{b}\right) \cos\left(\frac{n\pi z'}{b}\right) \quad (2-53)$$

we arrive at

$$\frac{\partial u_2(a, z)}{\partial \rho} = -a \int_0^b \frac{\partial g_2(a, z|a, z')}{\partial \rho} \frac{\partial u_2(a, z')}{\partial \rho'} dz' \quad (2-54)$$

$$= \int_0^b \left[\sum_{n=0}^{\infty} \frac{\epsilon_n}{b} \cos\left(\frac{n\pi z'}{b}\right) \cos\left(\frac{n\pi z}{b}\right) \right] \frac{\partial u_2(a, z')}{\partial \rho'} dz' \quad (2-55)$$

$$= \int_0^b \delta(z - z') \frac{\partial u_2(a, z')}{\partial \rho'} dz', \quad (2-56)$$

which is identically equal.

In Region 1, the equality is not as apparent because the derivative of the Green's function is not clearly a delta function

$$\frac{\partial u_1(a, z)}{\partial \rho} = \lim_{\rho \rightarrow a} \int_0^b \left[\sum_{n=0}^{\infty} \frac{-\chi_n J_1(\chi_n \rho)}{ik_{nz} a J_0(\chi_n a)} e^{-ik_{nz}|z-z'|} \right] \frac{\partial u_1(a, z')}{\partial \rho'} dz'. \quad (2-57)$$

If an alternate spectral representation of the Green's function is chosen as an integral rather than a summation

$$\tilde{g}_1(\rho, z) = \int_{-\infty}^{\infty} C(\rho, \lambda) e^{-i\lambda z} d\lambda, \quad (2-58)$$

the delta function surfaces. This form of the Green's function appears as

$$\begin{aligned} \tilde{g}_1(\rho, z) = & \int_{-\infty}^{\infty} \frac{1}{4} e^{-i\lambda(z-z')} \frac{J_0(\gamma \rho_{<})}{J_1(\gamma a)} \\ & \times \left[Y_1(\gamma a) J_0(\gamma \rho_{>}) - J_1(\gamma a) Y_0(\gamma \rho_{>}) \right] d\lambda, \end{aligned} \quad (2-59)$$

where

$$\gamma = \sqrt{k^2 - \lambda^2}, \quad \text{Im}(\gamma) < 0. \quad (2-60).$$

Once the derivative of the velocity potential with respect to ρ is taken, we recall the Wronskian identity

$$J_1(z)Y_0(z) - J_0(z)Y_1(z) = 2/\pi z \quad (2-61)$$

and the spectral expansion for our operator

$$\delta(z - z') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda(z-z')} d\lambda \quad (2-62)$$

to arrive the identity

$$\frac{\partial u_1(a, z)}{\partial \rho} = \int_0^b \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda(z-z')} d\lambda \right] \frac{\partial u_1(a, z')}{\partial \rho'} dz' \quad (2-63)$$

$$= \int_0^b \delta(z - z') \frac{\partial u_1(a, z')}{\partial \rho'} dz'. \quad (2-64)$$

To unite the integral equations of (2-49) and (2-51), we equate the velocity potentials themselves at the interface

$$u_1(a, z) = u_2(a, z), \quad z \in [0, b]. \quad (2-65)$$

Also, since the normal partial derivative of the velocity potential physically represents the normal velocity vector of the acoustic wave, we may define

$$\frac{\partial u_1(a, z)}{\partial \rho} = \frac{\partial u_2(a, z)}{\partial \rho} \equiv v(z) \quad z \in [0, b] \quad (2-66)$$

to establish continuity. The result is the following integral equation:

$$\begin{aligned} -e^{-ikz} = \int_0^b v(z') \sum_{n=0}^{\infty} \left[\frac{e^{-ik_{nz}|z-z'|}}{ik_{nz}a} + \right. \\ \left. \frac{\epsilon_n}{bk_{n\rho}} \frac{H_0^{(2)}(k_{n\rho}a)}{H_1^{(2)}(k_{n\rho}a)} \cos\left(\frac{n\pi z}{b}\right) \cos\left(\frac{n\pi z'}{b}\right) \right] dz'. \end{aligned} \quad (2-67)$$

Our task is to determine $v(z)$. Once this is done, the velocity potential $u(z)$ can be calculated everywhere.

CHAPTER 3

SOLUTION TO THE INTEGRAL EQUATION

In this chapter techniques used to solve the integral equation are explored. Various asymptotic forms are employed to isolate singularities and speed the convergence of the infinite sums. A brief introduction to the method of moments is provided. Then the method is applied to solve the integral equation approximately by transforming it to a matrix equation.

3.1 Asymptotic Forms

The solution to the integral equation (2-67) is complicated by the presence of the infinite sums. Obviously in the computational phase of this project, the sums will be truncated at some value of the index. The question of how many terms are necessary for reasonable accuracy and speed is important. A tool often used to resolve this problem is the use of asymptotic forms of the summand. For large values of the index, the summands of the integral equation approaches a simpler form which can often be summed in closed form. These closed form results often give useful insight into any singular behavior of the sum. Defining

$$S_{n1} = \frac{e^{-ik_{nz}|z-z'|}}{ik_{nz}a}, \quad (3-1)$$

$$S_{n2} = \frac{2}{bk_{n\rho}} \frac{H_0^{(2)}(k_{n\rho}a)}{H_1^{(2)}(k_{n\rho}a)} \cos\left(\frac{n\pi z}{b}\right) \cos\left(\frac{n\pi z'}{b}\right), \quad n > 0, \quad (3-2)$$

from the integral equation and noting the behavior of the zeros x'_{0n} and the Hankel function ratio for n large (Abramowitz and Stegun, 1972)

$$S_{n1} \sim S'_{n1} = \frac{e^{-n\pi|z-z'|/a}}{n\pi} \quad (3-3)$$

$$S_{n2} \sim S'_{n2} = \frac{2}{n\pi} \cos\left(\frac{n\pi z}{b}\right) \cos\left(\frac{n\pi z'}{b}\right) \quad (3-4)$$

we find that these simpler, asymptotic expressions have the following closed forms when summed:

$$\sum_{n=1}^{\infty} S'_{n1} = -\frac{1}{\pi} \ln [1 - e^{-\pi|z-z'|/a}] \quad (3-5)$$

$$\sum_{n=1}^{\infty} S'_{n2} = -\frac{1}{\pi} \ln [4 \sin\left(\frac{\pi|z-z'|}{2b}\right) \sin\left(\frac{\pi(z+z')}{2b}\right)]. \quad (3-6)$$

The techniques used to sum these series are presented in the appendix. With small argument approximations to the exponential and sine function, both (3-5) and (3-6) can be shown to have the singular form $\ln |z-z'|$. The closed forms of the sums show the logarithmic singularity in both terms as z approaches z' . The important result from this observation is that since the singularity is logarithmic, it is integrable.

3.2 The Method of Moments

To solve the integral equation, the method of moments (MoM) is employed as an approximate solution technique (Harrington, 1967, 1968). The aperture is divided into N subintervals, E_p , spaced symmetrically with respect to the center point. The center of each subinterval is defined as z_p . The center subinterval at $z = b/2$ has length Δ . As we proceed toward the ends of the aperture, the subinterval sizes vary in a geometric progression from $\Delta\tau$ to $\Delta\tau^{N/2}$, where τ is a scaling

coefficient between zero and one in a technique suggested in (Butler, 1984). Since τ is less than or equal to one, the subintervals become smaller as they near the ends of the aperture. This allows for greater resolution of the aperture distribution $v(z)$ at the ends, where field singularities may occur. The center subinterval length is carefully chosen such that the sum of the subintervals spans the entire aperture. Defining the left and right sides of the p^{th} subinterval

$$E_p \in [L_p, R_p] \quad (3-7)$$

and the center point

$$z_p = (R_p + L_p)/2 \quad (3-8)$$

we find that for N even,

$$\Delta = \frac{b}{2} \frac{1 - \tau}{1 - \tau^{N/2}}, \quad (3-9)$$

$$L_p = \frac{b}{2} \pm \Delta \frac{1 - \tau^{|p-1-N/2|}}{1 - \tau}, \quad (3-10)$$

$$R_p = \frac{b}{2} \pm \Delta \frac{1 - \tau^{|p-N/2|}}{1 - \tau}, \quad (3-11)$$

and for N odd,

$$\Delta = b \frac{1 - \tau}{1 + \tau - 2\tau^{(N+1)/2}}, \quad (3-12)$$

$$L_p = \frac{b}{2} \pm \Delta \left(\frac{1 - \tau^{|p-(N+1)/2| + \frac{1}{2} \mp \frac{1}{2}}}{1 - \tau} - \frac{1}{2} \right), \quad (3-13)$$

$$R_p = \frac{b}{2} \pm \Delta \left(\frac{1 - \tau^{|p-(N+1)/2| + \frac{1}{2} \pm \frac{1}{2}}}{1 - \tau} - \frac{1}{2} \right). \quad (3-14)$$

The top sign is chosen for subintervals left of the center subinterval and the bottom sign for those to the right. Figure (3-1) portrays the discretization scheme.

Figure 3-1 Method of Moments Discretization Scheme

With these parameters established, we expand the aperture distribution as a sum of weighted pulse functions,

$$v(z) = \sum_{p=1}^N \alpha_p f_p(z) \quad (3-15)$$

where

$$f_p(z) = \begin{cases} 1, & z \in E_p \\ 0, & \text{otherwise.} \end{cases} \quad (3-16)$$

Next we apply point matching by choosing a weight function as a delta function at the center point z_q of the q^{th} subinterval, $q \in [1, N]$,

$$w_q(z) = \delta(z - z_q), \quad (3-17)$$

and define an inner product

$$\langle X, Y \rangle_z = \int_0^b XY dz. \quad (3-18)$$

By taking the inner product of the weight function with both sides of the integral equation from (2-67), we find

$$-e^{-ikz_q} = \sum_{p=1}^N \alpha_p \int_{E_p} [g_1(0, z_q | 0, z') + g_2(0, z_q | 0, z')] dz'. \quad q \in [1, N] \quad (3-19)$$

In this manner, the integral equation has been changed to a matrix equation

$$\underline{f} = Y \underline{a}, \quad (3-20)$$

where

$$f_q = -e^{-ikz_q}; \quad (3-21)$$

$$a_p = \alpha_p; \quad (3-22)$$

$$y_{qp} = g_{qp}^{(1)} + g_{qp}^{(2)}; \quad (3-23)$$

$$g_{qp}^{(m)} = \int_{E_p} g_m(0, z_q | 0, z') dz', \quad m = 1, 2. \quad (3-24)$$

Y can be evaluated numerically, as can $\underline{f}$; therefore if Y is not singular, the aperture distribution coefficients α_p are given by

$$\underline{a} = Y^{-1} \underline{f}. \quad (3-25)$$

Once the aperture distribution coefficients are computed, they can be substituted into equations (2-49) and (2-51) for u_1 and u_2 , respectively, to calculate the velocity potential at any point within the borehole or fracture.

3.3 Convergence

Calculation of the Y matrix still requires the evaluation of two infinite sums for each entry; therefore, the speed at which the sums converge to their final value is an important matter. We turn to asymptotic analysis again to hasten the computation of the sums.

Since the Green's functions g_1 and g_2 are infinite series, we may interchange the order of integration and summation in (3-24) to represent the subinterval contributions $g_{qp}^{(1)}$ and $g_{qp}^{(2)}$ as a sum of integrals rather than an integral of a series:

$$g_{qp}^{(m)} = \sum_{n=0}^N G_{nqp}^{(m)}, \quad m = 1, 2 \quad (3-26)$$

From the borehole Green's function, we find that upon evaluation of the integrals

$$G_{nqp}^{(1)} = \begin{cases} \frac{e^{-iknz\theta_3} - e^{-iknz\theta_2}}{k_{nz}^2 a}, & p > q; \\ 2 \frac{e^{-iknz\theta_3} - 1}{k_{nz}^2 a}, & p = q; \\ \frac{e^{iknz\theta_2} - e^{iknz\theta_3}}{k_{nz}^2 a}, & p < q; \end{cases} \quad (3-27)$$

and from the fracture Green's function, we find that for all p and q

$$G_{nqp}^{(2)} = \begin{cases} \frac{\theta_3 - \theta_2}{kb} \frac{H_0^{(2)}(ka)}{H_1^{(2)}(ka)}, & n = 0; \\ \frac{1}{n\pi k_{np}} \frac{H_0^{(2)}(k_{np}a)}{H_1^{(2)}(k_{np}a)} \sum_{i=1}^4 (-1)^{(i+1)} \sin\left(\frac{n\pi\theta_i}{b}\right), & n \neq 0; \end{cases} \quad (3-28)$$

where

$$\theta_1 = R_p + z_q \quad (3-29)$$

$$\theta_2 = L_p - z_q \quad (3-30)$$

$$\theta_3 = R_p - z_q \quad (3-31)$$

$$\theta_4 = L_p + z_q. \quad (3-32)$$

The terms in (3-27) and (3-28) have asymptotic forms for $n \geq 1$:

$$\hat{G}_{nqp}^{(1)} = \begin{cases} \frac{e^{-n\pi\theta_3/a} - e^{-n\pi\theta_2/a}}{(n\pi/a)^2 a}, & p > q; \\ 2 \frac{e^{-n\pi\theta_3/a} - 1}{(n\pi/a)^2 a}, & p = q; \\ \frac{e^{-n\pi\theta_2/a} - e^{n\pi\theta_3/a}}{(n\pi/a)^2 a}, & p < q; \end{cases} \quad (3-33)$$

and for all p and q

$$\hat{G}_{nqp}^{(2)} = \frac{b}{(n\pi)^2} \sum_{i=1}^4 (-1)^{(i+1)} \sin\left(\frac{n\pi\theta_i}{b}\right), \quad (3-34)$$

respectively. Through slightly different manipulation, the term $G_{nqp}^{(2)}$ can be shown to have another asymptotic form, namely,

$$\check{G}_{nqp}^{(2)} = \frac{b}{\pi^2(n^2 + (kb/\pi)^2)} \sum_{i=1}^4 (-1)^{(i+1)} \sin\left(\frac{n\pi\theta_i}{b}\right). \quad (3-35)$$

The matrix equation in (3-25) will be unchanged if the asymptotic forms are added and then subtracted term by term from the original terms, as follows:

$$y_{qp} = y_{qp}^{(1)} + y_{qp}^{(2)} + y_{qp}^{(3)} + y_{qp}^{(4)} \quad (3-36)$$

where

$$y_{qp}^{(1)} = G_{0qp}^{(1)} + \sum_{n=1}^{\infty} \hat{G}_{nqp}^{(1)}, \quad (3-37)$$

$$y_{qp}^{(2)} = \sum_{n=1}^{\infty} (G_{nqp}^{(1)} - \hat{G}_{nqp}^{(1)}), \quad (3-38)$$

$$y_{qp}^{(3)} = G_{0qp}^{(2)} + \sum_{n=1}^{\infty} \tilde{G}_{nqp}^{(2)}, \quad (3-39)$$

$$y_{qp}^{(4)} = \sum_{n=1}^{\infty} (G_{nqp}^{(2)} - \tilde{G}_{nqp}^{(2)}), \quad (3-40)$$

and the tilde represent the choice of the alternate forms of the asymptotic of $G_{nqp}^{(2)}$. This step, however, will be beneficial computationally. First, the asymptotic sums $y_{qp}^{(1)}$ and $y_{qp}^{(3)}$ are much simpler in format. This fact may allow a closed form calculation of the sum. Second, the difference terms $y_{qp}^{(2)}$ and $y_{qp}^{(4)}$ become smaller quickly as the original terms approach the asymptotic terms; therefore fewer terms

are required to compute this sum to any degree of accuracy. We have hastened the summing process greatly, provided we can compute the asymptotic sums quickly.

The element $y_{qp}^{(1)}$ is of the form $\sum_{n=1}^{\infty} e^{-nx}/n^2$, which has the benefit of exponential decay to aid in its convergence for large values of the argument. Using complex analysis and geometric series as demonstrated in the Appendix, we can increase the convergence rate for small arguments. With knowledge that

$$\sum_{n=1}^{\infty} \frac{e^{-nx}}{n^2} = \frac{\pi^2}{6} - \frac{x^2}{4} + x(\ln x - 1) - \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k+1)!} x^{2k+1}, \quad (3-41)$$

where B_{2k} are the even Bernoulli numbers (Zhang and Dudley, 1987), one infinite sum can be replaced by constants plus another infinite sum. The number of terms required to compute this new sum for small values is decreased dramatically from the original sum, particularly due to the fast decrease in the Bernoulli numbers and in the factorial. For an accuracy of one part in 10^{-7} of the sum, over five hundred terms are required to sum the original series for $x = 0.001$. The new series only requires a single term. For larger arguments the original series is more efficient; a pivot point of $x = 2.1$ is chosen to alternate between the series.

The element $y_{qp}^{(3)}$ is of the form $\sum_{n=1}^{\infty} \sin nx/n^2$ or $\sum_{n=1}^{\infty} \sin nx/(n^2 - d^2)$, depending on which of the asymptotic forms is chosen. The first form of the sum is Clausen's integral in summation form (Abramowitz and Stegun, 1972),

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^2} = \phi(1 - \ln \phi) - \sum_{k=1}^{\infty} \frac{|B_{2k}|}{2k(2k+1)!} \phi^{2k+1}, \quad (3-42)$$

$$\phi = \begin{cases} x, & x \leq \pi \\ 2\pi - x, & x > \pi. \end{cases} \quad (3-43)$$

The second form of the sum is

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^2 - d^2} = \sum_{k=1}^{\infty} \frac{\sin kx}{k^2 - d^2} \left(1 - \tanh(kp)\right) + \frac{\pi}{2d} \tanh(pd) \frac{\sin(\pi - x)d}{\sin \pi d} - \frac{\pi}{p} \sum_{k=1}^{\infty} \frac{1}{\left(\frac{(2k-1)\pi}{2p}\right)^2 + d^2} \frac{\sinh\left[\frac{(2k-1)\pi}{2p}(\pi - x)\right]}{\sinh\left[\frac{(2k-1)\pi^2}{2p}\right]}, \quad (3-44)$$

where p is a parameter chosen for optimal convergence (Collin, 1960; Nabulsi, 1984).

These two forms of $y_{qp}^{(3)}$ are interesting with regard to the regions for which they converge faster, shown in Fig. (3-2). The second sum converges quickly for x in the neighborhood of π , while the first sum converges more rapidly for x near 0 and 2π . We can use this fact to our advantage. By alternating between the two asymptotic forms of $y_{qp}^{(3)}$ depending on the argument, we can evaluate this fourth term of the matrix with greater speed than if we only used a single form. For example, on the interval $[0, 2\pi]$ for an accuracy of 10^{-7} the first form of the sum requires up to fourteen terms to converge. The second form can require over twenty terms. By carefully choosing between these forms, no more than five terms are required to compute the sum. This is a huge improvement over the three thousand terms necessary to compute the original sum without the benefit of these asymptotic aids.

3.4 Numerical Results

The computational phase of this thesis was performed with eight byte, double precision accuracy on a VAX 11/750. Much supplementary software was accessed: SLATEC math library for Bessel functions, LINPACK for matrix operation routines, SIG for signal processing routines, and FANCYPLOT and CricketGraph

Figure 3-2 Convergence Rates of Sums (3-42) and (3-44)

for all plotting and graphing routines. This manuscript was prepared with the computer typesetting system, $\text{\TeX}$. All other software was created for this project.

Clearly computation in the method of moments matrix is very parameter dependent. The chief parameters effecting its computation are the frequency, the fracture size, and the number of subintervals used to discretize the aperture.

To adequately represent the field behavior, the subintervals must be chosen carefully. We must optimize accuracy while maintaining a reasonable matrix size. The initial studies of this problem were performed with uniformly spaced subintervals in the MoM; consequently, large numbers of subintervals were required to characterize the edge singularities at either end of the aperture. This was poor computationally, because in order to observe the edge singularities, the center portion of the aperture had to be excessively discretized. The change to nonuniform spacing allows for many fewer subintervals for similar characterization of the aperture field. This significantly reduces computational time. Figure (3-3) shows that even at ten subintervals the difference in the edge singularity discretization is apparent between the uniform and nonuniform schemes. Figure (3-4) shows this difference is even more apparent at forty subintervals. The edge singularities are much larger and therefore more accurately represented in the nonuniform plot.

A choice of forty subintervals with a spacing coefficient of $\tau = 0.80$ provides adequate resolution of the edges singularities for large fractures without excessive oversampling of aperture distribution. Figure (3-5) shows the aperture distribution at various number of discretizations. In the central portion of the aperture the discretization is fairly accurate regardless of how many subintervals are chosen. At the edges the singularity is represented accurately only at a minimum of $N = 40$ subintervals. A larger number of subintervals would provide better discretization

Figure 3-3 Aperture Distribution $v(z)$: Uniform vs. Nonuniform Discretization, $a = 0.1\text{m}$, $b = 0.1\lambda$, $N = 10$

Figure 3-4 Aperture Distribution $v(z)$: Uniform vs. Nonuniform Discretization, $a = 0.1\text{m}$, $b = 0.1\lambda$, $N = 40$

of the aperture, but we wish to minimize the matrix dimension. An increase in the number of subintervals does not improve the model of the field appreciably. Values of τ less than this choice spaces the subintervals at the ends too close together, creating numerical singularity of the MoM matrix. A larger τ does not provide enough discretization of the edges.

Our frequency range includes all values below the first cutoff mode of the acoustic radiation. This is a broad band of frequencies. At the higher frequencies the matrix takes longer to compute. At the lower frequencies the matrix becomes close to numerical singularity and is more difficult to compute accurately, as shown in Figure (3-6). The magnitude of the aperture distributions have been normalized to the same level. At frequencies of 0.01 Hz and above, the $v(z)$ is smooth and well defined. When we reach $f = 10^{-4}$ Hz, the matrix cannot be computed accurately. Here the aperture distribution is jagged and unreliable, which is logical since the computer program was set for an accuracy of one part in 10^{-5} . Both of these problems must be carefully addressed to insure the efficient and accurate computation of the matrix. The frequency used in the most of the analysis was half the first higher order cutoff, which for the choice of $a = 0.1$ meters and $c = 1460$ meters/sec is 4450 Hz. The next section will highlight an alternate technique for the field computation at low frequencies.

The size of the fracture is important as well. Because our frequency range is below the first cutoff, the zero-order mode alone propagates, the electromagnetic equivalent of the TEM mode. This allows more power to be coupled from the main borehole to the fracture. With small fractures this effect is noticeable as a slight asymmetry in the field distribution with clearly defined edge singularities, shown in Figure (3-7). With fractures of the same order as the wavelength, more power

Figure 3-5 Aperture Distribution $v(z)$ for Various Numbers of Subintervals:
 $a = 0.1\text{m}$, $f = 0.5f_c$, $b = 0.1\lambda$, $\tau = 0.8$

Figure 3-6 Aperture Distribution $v(z)$ at Various Frequencies: $a = 0.1m$, $N = 40$, $b = 0.1m$, $\tau = 0.8$

leaks into the fracture. The central portion of the aperture field develops ripples, as shown in Figures (3-8), (3-9), and (3-10). Between $b = 10^{-7}\lambda$ and $b = 2.0\lambda$ the singularity magnitude drops from 200 m/s to 40 m/s, indicating that the edge singularity is not as strongly defined at wider fractures. The effect of numerical sampling on the definition of the singularity is present, with the larger fractures are sampled at larger intervals when the number of subintervals is kept constant. While this does effect the strength of the calculated singularity, the primary contribution to the reduction of the edge singularity is from the size of the fracture draining the signal rather than numerical artifacts.

Figure 3-7 Aperture Distribution $v(z)$: $a = 0.1\text{m}$, $b = 1 \times 10^{-5}\lambda$, $f = 0.5f_c$,
 $N = 40$, $\tau = 0.8$

Figure 3-8 Aperture Distribution $v(z)$: $a = 0.1\text{m}$, $b = 0.1\lambda$, $f = 0.5f_c$, $N = 40$, $\tau = 0.8$

Figure 3-9 Aperture Distribution $v(z)$: $a = 0.1\text{m}$, $b = 1.0\lambda$, $f = 0.5f_c$, $N = 40$, $\tau = 0.8$

Figure 3-10 Aperture Distribution $v(z)$: $a = 0.1\text{m}$, $b = 2.0\lambda$, $f = 0.5f_c$,
 $N = 40$, $\tau = 0.8$

CHAPTER 4

REFLECTION ANALYSIS

In this chapter reflection behavior in the borehole is computed first by the method of moments and then by a low frequency approximation. Using these reflection coefficients, we investigate the reflection of a time domain transient pulse.

4.1 Reflection Coefficient

An important physical quantity pertinent to this problem is the reflection coefficient. As stated earlier, our system is driven by a source located in the negative z -region of the borehole. When the tube wave arrives at the fracture, the wave is broken into reflected and transmitted components. If we can calculate the reflection coefficient for a particular fracture, we can more readily model the wave behavior.

The velocity potential in Region 1 was given by (2-49) as a sum of a positive-going tube wave and an integral over the product of the Green's function and $v(z)$, g_1 being a function composed of a sum of Bessel and exponential functions. The argument of the exponential function shows the modal behavior of the waves; a different wave number exists for each of the infinite terms of the sum. The $n = 0$ term corresponds to the fundamental mode of the tube wave since the wave number is simply $k_{0z} = \omega/c = k$. The first higher order mode occurs when $k_{1z} = 0$, or $ka = x'_{01} \simeq 3.83$.

By driving the source at frequencies below the cutoff frequency of the first higher order mode, only the fundamental mode will propagate. All the higher order

modes will be exponentially attenuated rather than propagated. Therefore, with a choice of the frequency

$$ka < x'_{01} \quad (4-1)$$

and an observation point located as far from the fracture as $z < -a$, only the $n = 0$ term of the sum is significant. The velocity potential in Region 1 reduces to

$$u_1 \doteq e^{-ikz} + \Gamma e^{ikz}, \quad z < -a, \quad (4-2)$$

where the reflection coefficient is given by

$$\Gamma = \frac{1}{ika} \int_0^b e^{-ikz'} v(z') dz'. \quad (4-3)$$

Once the aperture distribution, $v(z)$, is known, the reflection coefficient is quickly calculated. Although the higher order terms have been discarded, their contribution to the result is still present in the aperture field $v(z)$. Indeed, incorporating the MoM into this result, we find

$$\Gamma = \frac{1}{k^2 a} \sum_{p=1}^N \alpha_p [e^{-ikR_p} - e^{-ikL_p}]. \quad (4-4)$$

Figure (4-1) shows the MoM reflection coefficient up to 4451 Hz, which is half the first higher order mode cutoff. In the limit as the frequency becomes zero, the reflection coefficient goes to -1 , indicating total reflection of the incident wave. The fractures displayed are small with respect to the given frequencies, since the electrical length $k_{max}b = 2\pi 10^{-7}$ and $k_{max}b = 2\pi 10^{-1}$; therefore the curve is smooth and flat at the higher frequencies. Notice that the magnitude of the reflection coefficient is directly proportional to the size of the fracture. The fracture

$b = 10^{-7}\lambda_{min}$ has a reflection coefficient whose high frequency limit approaches 10^{-7} , while the $b = 0.1\lambda_{min}$ fracture has a reflection coefficient that approaches 0.1. These results compare well with theoretical results in (Hornby, et al., 1989). In Figure (4-2) the fractures of the order of one wavelength are displayed. The effects of multiple reflection within the fracture are apparent now as the flat surface develops depressions. The number of depressions is directly proportional to the number of minimum half wavelengths present in the fracture width; i.e. the $b = 1.5\lambda_{min}$ reflection coefficient has three abrupt drops in magnitude at 1500 Hz, 3000 Hz, and 4500 Hz.

It should be noted the reflection coefficient must be calculated anew at each frequency of interest. The MoM matrix must be loaded and inverted for every frequency point. This can be numerically intensive and time consuming.

4.2 Low Frequency Approximation

At low enough frequencies, the reflection coefficient may be approximated without computation and inversion of the entire method of moments matrix. Since the far field quantities are relatively insensitive to the form chosen for the aperture distribution (Zhang and Dudley, 1987; Marcuvitz, 1951), the aperture distribution may be approximated as a constant,

$$v(z) = \alpha. \quad (4-5)$$

This is equivalent to a one-term Method of Moments expansion. The integral equation of (3-19) produces

$$\alpha = -e^{-ikb/2} / \int_0^b [g_1(0, \frac{b}{2}|0, z') + g_2(0, \frac{b}{2}|0, z')] dz'. \quad (4-6)$$

Figure 4-1 ReflectionCoefficient $\Gamma(f)$ at Various Fracture Widths in terms of Fractions of λ_{min} : $a = 0.1\text{m}$, $N = 40$, $\tau = 0.8$

Figure 4-2 Reflection Coefficient $\Gamma(f)$ at Various Fracture Widths in terms of Fractions of λ_{min} : $a = 0.1\text{m}$, $N = 40$, $\tau = 0.8$

Substituting the Green's functions, we find

$$\alpha = -e^{-ikb/2} / \frac{H_0^{(2)}(ka)}{kH_1^{(2)}(ka)} + 2 \sum_{n=0}^{\infty} \frac{(e^{-ik_{nz}b/2} - 1)}{k_{nz}^2}. \quad (4-7)$$

So long as

$$ka < 2x'_{01} \quad (4-8)$$

we need only consider the first term in the sum since the other terms will be exponentially attenuated as k_{nz} becomes imaginary, and α becomes

$$\alpha = -1 / \left(\frac{2}{k^2 a} (1 - e^{+ikb/2}) + \frac{H_0^{(2)}(ka)}{kH_1^{(2)}(ka)} e^{+ikb/2} \right). \quad (4-9)$$

If we further assume that the fracture is small,

$$kb \ll 1, \quad (4-10)$$

to justify our choice of a constant aperture distribution, the reflection coefficient from (4-4) becomes

$$\Gamma_{LFA} = -1 / \left(\frac{ka e^{ikb/2}}{(e^{-ikb} - 1)} \frac{H_0^{(2)}(ka)}{H_1^{(2)}(ka)} + 2 \frac{(1 - e^{-ikb/2})}{(e^{-ikb} - 1)} \right). \quad (4-11)$$

When both the fracture and frequency satisfy the above criteria, the low frequency approximation is an excellent substitute for the MoM. Figures (4-3) and (4-4) shows the MoM reflection coefficient versus the low frequency approximation for small fractures, $b = 10^{-7} \lambda_{min}$ and $b = 10^{-1} \lambda_{min}$. The resultant reflection coefficient is as accurate, and the computation time is very small in comparison to the MoM solution. As the fracture size increases the low frequency approximation diverges

from the MoM solution, as is expected, shown in Figure (4-5). The Hankel function ratio in Equation (4-11) is often modified into a logarithmic form, shown in Figures (4-3) and (4-4). This approximation is clearly invalid for any frequencies close to the first higher order cutoff frequency of the system, as in our cases, but for frequencies below one percent of the first higher order mode cutoff it is acceptable.

4.3 Transient Analysis

To obtain information concerning the structure, we send a time-domain signal down the borehole and observe the reflection. A transient pulse

$$p(t) = \sin(2\pi f_0 t) \sin^2(\alpha t), \quad t \in [0, \pi/\alpha], \quad (4-12)$$

that characterizes a typical transducer pulse is applied to the borehole at $z = -L$. The central frequency f_0 is chosen as one tenth of the cutoff frequency of the first higher order mode, which occurs at $f_c = x'_{01} v / 2\pi a$. The variable α is chosen to provide an envelope of three cycles of the f_0 sine wave. The pulse and its spectrum are shown in Figures (4-6) and (4-7).

The Fourier transform of $p(t)$, $P(f)$, is multiplied by the reflection coefficient $\Gamma(f)$, and the inverse Fourier transform of the product is taken to observe the reflected time-domain pulse, shown in Figures (4-8) and (4-9) respectively. Observing the differences in the pulse spectrum magnitudes in Figure (4-7), we may truncate the spectrum at five times the central frequency. Because there is a 40 db difference in the magnitudes at these two points, all contributions above our upper limit are negligible. The higher frequency ranges can therefore be zero padded for sufficient time domain resolution without loss of accuracy from the truncation. This truncation of higher frequency content allows the calculation of a smaller range

Figure 4-3 Reflection Coefficient Solutions: Method of Moments, Low Frequency Approximation, and Logarithmic Approximation; $a = 0.1\text{m}$, $b = 10^{-7}\lambda_{\min}$, $N = 40$

Figure 4-4 Reflection Coefficient Solutions: Method of Moments, Low Frequency Approximation, and Logarithmic Approximation; $a = 0.1m$, $b = 0.1\lambda_{min}$, $N = 40$

Figure 4-5 Reflection Coefficient Solutions: Method of Moments and Low Frequency Approximation; $a = 0.1\text{m}$, $N = 40$, $b = 0.51\lambda_{\min}$

Figure 4-6 Incident Pulse, $p(t)$: $f_0 = 0.1f_c$, $\alpha = f_0\pi/3$

Figure 4-7 Incident Pulse Spectrum, $P(f)$: $f_0 = 0.1f_c$, $\alpha = f_0\pi/3$

of frequency values of the reflection coefficient. As was remarked earlier, every frequency point of the reflection coefficient must be calculated separately, so by minimizing the number of frequencies spanned, the computation time is reduced.

The amplitude of reflected pulse is proportional to the size of the fracture. Intuitively it makes sense that a small fracture will reflect only a small amount of the signal. For these small fractures, the reflected pulse is nearly identical in shape to the negative of the incident pulse. The phase inversion is because the DC reflection coefficient is -1 . Figures (4-10) and (4-11) show the pulses reflected from two cases with small fractures, comparing the MoM solution and Low Frequency Approximation. Clearly the smaller fracture in Figure (4-10) produces a smaller amplitude. Because these fractures were small, relatively, the MoM and LFA solutions were very similar, with only slightly phase and magnitude differences. In Figures (4-12) and (4-13) two pulses are shown from fractures that are large enough to reflect most of the incident pulse. The LFA solution begins to diverge significantly in magnitude and phase from the MoM solution; the fractures are too large for the LFA to be accurate. Since the fracture cannot reflect more than was incident, the amplitudes of any pulses reflected from larger fractures will peak at this point.

Figure 4-8 Reflected Pulse Spectrum: $P \times \Gamma$, $b = 0.1\lambda_0 = 0.5\lambda_{min}$

Figure 4-9 Reflected Pulse: $\text{IFFT}[P \times \Gamma]$, $b = 0.1\lambda_0 = 0.5\lambda_{\min}$

Figure 4-10 Reflected Pulses, MoM versus Low Frequency Approximation: $a = 0.1\text{m}$, $b = 0.02\lambda_0 = 0.1\lambda_{\min}$, $N = 40$

Figure 4-11 Reflected Pulses, MoM versus Low Frequency Approximation: $a = 0.1m$, $b = 0.1\lambda_0 = 0.5\lambda_{min}$, $N = 40$

Figure 4-12 Reflected Pulses, MoM versus Low Frequency Approximation: $a = 0.1\text{m}$, $b = 0.2\lambda_0 = 1.0\lambda_{\min}$, $N = 40$

Figure 4-13 Reflected Pulses, MoM versus Low Frequency Approximation: $a = 0.1\text{m}$, $b = 0.3\lambda_0 = 1.5\lambda_{\min}$, $N = 40$

CHAPTER 5

TWO-FRACTURE CASE

The problem is generalized to a two-fracture case in this chapter. Relying on previous developments, we highlight the differences introduced to the mathematics by the additional fracture. The proper integral equation is created, optimized, and solved as shown before. With the solution of the integral equation to model the field behavior, the reflection parameters of the two-fracture case are analyzed.

5.1 Problem Formulation

When adding a second fracture, we define

Region 3, $\rho \in [a, \infty]$, $z \in [c, c + d]$.

This region is just a scaled and shifted version of Region 2; therefore the Green's function and velocity potential are easily found from (2-34) and (2-51) by replacing z with $z - c$ and b with d :

$$g_3(\rho, z|\rho', z') = \sum_{n=0}^{\infty} \frac{\pi \epsilon_n}{4id} \frac{H_0^{(2)}(k_{n\rho}\rho)}{H_1^{(2)}(k_{n\rho}a)} W(\rho_{<}) \\ \times \cos\left(\frac{n\pi(z-c)}{b}\right) \cos\left(\frac{n\pi(z'-c)}{d}\right), \quad (5-1)$$

$$u_3(\rho, z) = -a \int_c^{c+d} g_3(\rho, z|a, z') \frac{\partial u_3(a, z')}{\partial \rho'} dz'. \quad (5-2)$$

Because the functionality is the same as in Region 2, the discussion of asymptotic extraction and singularities remains the same with the noted variable changes.

Figure 5-1 The Two-Fracture Borehole Geometry

The formulation of g_1 , g_2 , and u_2 is unchanged. The only modification needed is an alteration of the boundary conditions of u_1 to account for the additional aperture. Because of this change in the main borehole, the result of Green's theorem is slightly different. The two discontinuities in Region 1 create two surfaces over which the derivative of the velocity potential with respect to ρ is not zero. Integration over the surface produces two integrals plus the source term:

$$u_1(\rho, z) = e^{-ikz} + a \int_0^b g_1(\rho, z|a, z') \frac{\partial u_1(a, z')}{\partial \rho'} dz' + a \int_c^{c+d} g_1(\rho, z|a, z') \frac{\partial u_1(a, z')}{\partial \rho'} dz'. \quad (5-3)$$

Continuity must be established at both apertures to unite the velocity potentials in all three regions. We define

$$v(z) = \begin{cases} \frac{\partial u_1(a, z')}{\partial \rho'} = \frac{\partial u_2(a, z')}{\partial \rho'}, & z \in [0, b] \\ \frac{\partial u_1(a, z')}{\partial \rho'} = \frac{\partial u_3(a, z')}{\partial \rho'}, & z \in [c, c+d] \end{cases} \quad (5-4)$$

as the aperture distribution over both fractures. By equating

$$u_1(a, z) = u_2(a, z), \quad z \in [0, b], \quad (5-5)$$

$$u_1(a, z) = u_3(a, z), \quad z \in [c, c+d], \quad (5-6)$$

we form two independent equations, one for each fracture:

$$-e^{ikz} = a \int_0^b [g_1 + g_2] v(z') dz' + a \int_c^d g_1 v(z') dz', \quad z \in [0, b]; \quad (5-7)$$

$$-e^{ikz} = a \int_0^b g_1 v(z') dz' + a \int_c^{c+d} [g_1 + g_3] v(z') dz', \quad z \in [c, c+d]. \quad (5-8)$$

5.2 Solution to the Integral Equation

The method of moments can still be used to expand $v(z)$ as a sum of weighted pulse functions. We note that when N subintervals are chosen in the expansion, a break point BR must be selected such that the integers $p \in [1, BR]$ partition the aperture $z \in [0, b]$, and the integers $p \in [BR+1, N]$ partition the aperture $z \in [c, c+d]$. To provide equal discretization of both fractures, the simplest choice is BR as half the value of N . The two integral equations become a matrix equation

$$\begin{aligned}
 -e^{ikz_q} &= a \sum_{p=1}^{BR} \alpha_p \int_{E_p} [g_1 + g_2] dz' + a \sum_{p=BR+1}^N \alpha_p \int_{E_p} g_1 dz', \quad q \in [1, BR]; \quad (5-9) \\
 -e^{ikz_q} &= a \sum_{p=1}^{BR} \alpha_p \int_{E_p} g_1 dz' + a \sum_{p=BR+1}^N \alpha_p \int_{E_p} [g_1 + g_3] dz', \quad q \in [BR+1, N].
 \end{aligned}$$

(5-10)

The position variables z_p and E_p are chosen nonuniformly as in the single fracture case to be more densely spaced at the edges of the apertures than at the centers. After adding and subtracting the proper singularities and choosing the proper asymptotic forms for each term, we load and invert the matrix to solve for the aperture distributions. The matrix appears like:

$$y_{qp} = \begin{pmatrix} G_1 + G_2 & \cdots & G_1 + G_2 & G_1 & \cdots & G_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ G_1 + G_2 & \cdots & G_1 + G_2 & G_1 & \cdots & G_1 \\ G_1 & \cdots & G_1 & G_1 + G_3 & \cdots & G_1 + G_3 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ G_1 & \cdots & G_1 & G_1 + G_3 & \cdots & G_1 + G_3 \end{pmatrix} \quad (5-11)$$

From Equation (5-11) we notice the contribution of the borehole Green's function G_1 throughout the entire matrix, while the contributions of the fracture Green's functions G_2 and G_3 appear only on the diagonal submatrices. The upper right and lower left corners of the matrix, where the borehole Green's function alone is present, act to couple the fields between the two fractures. The upper right corner elements represent the induced field at the second fracture due to excitation at the first fracture from (5-9), while the lower left matrix elements represent coupling the induced field at the first fracture due to excitation at the second fracture from (5-10).

The features noted in the one fracture case for frequency ranges, fracture sizes, and discretization remain true for the two-fracture case. For low frequencies, the matrix becomes numerically singular, while at high frequencies computation time increases. Nonuniform subintervals need to be used to characterize the edge singularities of both fractures. Large fractures produce greatly asymmetric aperture field distributions. Also, if the first fracture is large with respect to wavelength, less of the acoustic wave will reach the second fracture due to power loss in the first fracture. The resultant field distribution in the second fracture will therefore be reduced as well.

Figure (5-2) shows the aperture distribution of two distantly spaced, small, equal fractures. Since the fractures are small, the distributions are smooth and nearly identical. In Figure (5-3), however, the equal apertures are of the order of a wavelength. Significant power losses have occurred at the first aperture, leaving little for the second aperture. The result is a distorted distribution at the second aperture. The anomaly at the second fracture is noted and unexplained. With increasing fracture separations the anomaly moves across the aperture distribution

to the right until it vanishes for large separations. With increasing fracture sizes the anomaly moves to the left, shown in Figure (5-4). In any case its behavior is too minor to have any effect on the subsequent reflection coefficients.

5.3 Reflection Analysis

With two fractures, the reflection coefficient becomes slightly more complicated. From (5-3) one can see how the velocity potential behaves in the borehole. If we again drive the source below the first cutoff frequency, which is independent of the number of fractures, only the zero order terms of the sums in the Green's functions survive. The result is a reflected tube wave as in (4-2) with a reflection coefficient given by

$$\Gamma = \frac{1}{ika} \left[\int_0^b e^{-ikz'} v(z') dz' + \int_c^{c+d} e^{-ikz'} v(z') dz' \right] \quad (5-12)$$

With the application of the MoM to discretize the aperture distribution, the resultant reflection coefficient is the same equation (4-4) found in the single fracture case, since the spacing discontinuity is taken into account by the left and right end subinterval variables.

Rather than a smooth monotonically decreasing line present in the single fracture case, the two-fracture reflection coefficient goes through a series of nulls and maxima, shown in Figure (5-5). This is due to the field interference of the two fractures. The period of the nulls Δf is directly related to the fracture separation. The time necessary for the wave to travel from the first fracture to the second and then back is the reciprocal of Δf ; therefore $\Delta f = v/2c$. For instance from Figure (5-5), $\Delta f = 1490/0.4\lambda_0 = f_c/4 = 2200Hz$ and the round-trip transit time

Figure 5-2 Aperture Distribution $v(z)$: $a = 0.1\text{m}$, $b = d = 0.1\lambda$, $c = 10.0\lambda$, $f = 0.5f_c$, $N = 80$, $\tau = 0.8$

Figure 5-3 Aperture Distribution $v(z)$: $a = 0.1\text{m}$, $b = d = 1.0\lambda$, $c = \lambda$, $f = 0.5f_c$, $N = 80$, $\tau = 0.8$

Figure 5-4 Aperture Distribution $v(z)$: $a = 0.1\text{m}$, $b = d = 2.5\lambda$, $c = \lambda$, $f = 0.5f_c$, $N = 80$, $\tau = 0.8$

is 0.45ms. Large separations cause the pattern to be more densely packed as the transit time increases; a separation three times as great has three times the number of nulls.

For the low frequency approximation, the apertures are assumed to have a constant field distribution. We then assume both fractures are small with respect to wavelength, $kb \ll 1$, and $kd \ll 1$, as well as the criteria $ka < 2x'_{01}$. This produces a reflection coefficient

$$\Gamma_{LFA} = -\left(e^{-ikb} - 1 + e^{-ik2c}(e^{-ikd} - 1)\right) / \left(\frac{kaH_0^{(2)}(ka)}{H_1^{(2)}(ka)} + e^{-ikb} + e^{-ikb} - 2\right). \quad (5-13)$$

Figure (5-6) illustrates that the low frequency approximation retains the interference patterns found in the MoM solution. Again, for the proper frequency ranges and fracture sizes, the approximation is excellent in terms of accuracy and time-saving.

An important problem in the two-fracture case is one of system identification: when two fractures are very close together, is it possible to resolve the reflected wave into a reflection from each fracture? By causality the reflection from the second fracture must arrive after the first reflection. When the two fractures are separated by a large enough distance, a reflection is clearly visible from each fracture. Figures (5-7), (5-8), and (5-9) demonstrate the case of widely separated fractures. As the two fractures grow closer, the reflected pulses from each fracture overlap until it is no longer clear whether there are two closely spaced fractures or one larger fracture. Figure (5-10) compares the pulses reflected from a two-fracture borehole with closely spaced fractures and from a one fracture borehole whose fracture is as large as the sum of the two fractures in the previous case. The two are virtually identical. A look at the frequency domain in Figure (5-11) may help to

Figure 5-5 Reflection Coefficient at Various Fracture Separations: $a = 0.1\text{m}$, $N = 80$, $b = d = 0.002\lambda_0$

Figure 5-6 Reflection Coefficient, MoM vs. Low Frequency Approx: $a = 0.1m$,
 $N = 80$, $b = d = 0.002\lambda_0$

resolve this problem, if a null can be detected to characterize the two-fracture case. A drop in the two-fracture case is evident from the figure at higher frequencies, but it is not clear whether this will become a null. When the separations are that small, the nulls will be very widely spaced from our previously stated result $\Delta f = v/2c$ and will therefore not occur in the limited frequency range we have plotted.

Figure 5-7 Reflected Pulse: $a = 0.1\text{m}$, $N = 40$, $b = d = 0.002\lambda_0 = 0.01\lambda_{\min}$,
 $c = 1.8\lambda_0 = 9\lambda_{\min}$

Figure 5-8 Reflected Pulse: $a = 0.1\text{m}$, $N = 40$, $b = d = 0.002\lambda_0 = 0.01\lambda_{\min}$, $c = 1.2\lambda_0 = 6\lambda_{\min}$

Figure 5-9 Reflected Pulse: $a = 0.1\text{m}$, $N = 40$, $b = d = 0.002\lambda_0 = 0.01\lambda_{\min}$, $c = 0.6\lambda_0 = 3\lambda_{\min}$

Figure 5-10 Reflected Pulses, Two Closely Spaced Fractures, $b = d = 0.002\lambda_0 = 0.01\lambda_{min}$, $c = 0.04\lambda_0 = 0.2\lambda_{min}$ versus One Fracture, $b = 0.004\lambda_0 = 0.02\lambda_{min}$: $a = 0.1m$, $N = 40$.

Figure 5-11 Reflection Coefficient, Two Closely Spaced Fractures, $b = d = 0.002\lambda_0 = 0.01\lambda_{min}$, $c = 0.04\lambda_0 = 0.2\lambda_{min}$ versus One Fracture, $b = 0.004\lambda_0 = 0.02\lambda_{min}$; $a = 0.1m$, $N = 40$.

CHAPTER 6

CONCLUSIONS

This thesis has presented the solution to the acoustic wave equation in a cylindrical borehole with fractures using integral equation techniques. After a brief study of the acoustic variables of interest such as velocity potential, the scalar wave equation was presented with the appropriate boundary conditions. The total velocity potential was divided into velocity potentials valid in the subregions of the geometry, namely the borehole and the fracture. Associated Green's functions were presented and derived for both regions. Through Green's theorem these functions were unified in integral equations to form the respective velocity potentials. Continuity of the fields at the aperture between the fracture and the borehole was established to produce a single integral equation in terms of the normal velocity field. The solution allows the calculation of the velocity potential everywhere within the geometry.

The form of the integral equation was studied closely before any solution techniques were attempted. From the Green's functions two infinite series were introduced into the integral equation. The asymptotic forms of the summands were extracted and analyzed to show that a logarithmic singularity occurs in both series. The asymptotic terms were added and subtracted to the original summands to form two types of sums: the asymptotic sums that contain the singularities, and the difference sums that converge rapidly.

The integration of the asymptotic sums was a critical topic. The sum from the borehole Green's function contained exponential terms over the index squared.

An alternate form was found that converged quickly for small arguments. The sum from the fracture Green's function contained sinusoidal terms over the index squared. This sum proved to be very interesting, in that different asymptotic forms could be used to compute the sum for different argument values to minimize the necessary calculation.

The method of moments was used to solve the integral equation approximately. To allow for proper characterization of the field singularities at the edge of the aperture, the velocity field was nonuniformly discretized for fine edge spacing and coarse central spacing. The integral equation was transformed in this manner into a matrix equation. For frequencies small with respect to the first higher order mode cutoff mode of the system, the matrix to be inverted becomes numerically singular. For large frequencies the matrix takes longer to compute. For small fractures the velocity field is a symmetric trough with sharply defined edge singularities. At fracture sizes approaching a wavelength the velocity field becomes highly asymmetric as more acoustic energy is drawn into the fracture. Eventually the edge singularities become undetectable.

At operating frequencies less than the first cutoff, the reflection coefficient was derived from the borehole velocity potential integral equation. Since the reflection coefficient is a function of frequency, the MoM matrix had to be packed and inverted at every frequency step of the reflection coefficient. For fractures small with respect to wavelength, a low frequency approximation was made to eliminate the need for the time and labor necessary for the MoM computation. The approximation compared well with the MoM solution for the given parameter ranges and significantly reduced calculation speeds.

The reflection coefficients were multiplied by the Fourier transform of a time domain pulse, then inverse-transformed to observe the time domain characteristics of the reflected pulse. Because the DC value of the reflection coefficient is -1 , the reflected pulses resembled the negative of the incident pulse scaled in magnitude by the fracture width.

The two-fracture case was derived following the example of the one fracture case. An additional integral equation was necessary to provide continuity at the second fracture, but otherwise the procedure was the same. The MoM again transforms the integral equations into a matrix equation. The fracture Green's functions contribute to the submatrices lying on the diagonal while the coupling terms lie off the diagonal.

Two-fracture reflection proved to be a more challenging topic. For sufficiently spaced fractures, two distinct reflected pulses were observable. When the separation became small, the two reflections overlapped to form a composite pulse until the two were indistinguishable from a single fracture as large as the sum of the two.

Recommendations for future work include development of a multifracture code, and exploration of power flow quantities in the borehole and fractures. Clearly the development could be expanded for more than two fractures. Additional fractures would introduce additional integrals to be coupled into the solution, but the derivation of the two-fracture case outlines the required steps. Another important extension would be the inverse problem: the determination of the size and number of fractures present in the borehole from observation of the input and reflected signals. This would be an interesting problem in system identification with important application to geophysical exploration.

APPENDIX A

EVALUATION OF INFINITE SERIES

The development of the asymptotic forms for the velocity potentials led to three different infinite series:

$$\sum_{n=1}^{\infty} \frac{\sin(n\beta)}{n^2}, \quad (A-1)$$

$$\sum_{n=1}^{\infty} \frac{e^{-n\alpha}}{n^2}, \quad (A-2)$$

$$\sum_{n=1}^{\infty} \frac{\sin(n\beta)}{(n^2 - d^2)}. \quad (A-3)$$

Each of these series is considered in turn below. The reader is referred to the literature (Abramowitz and Stegun, 1972; Collin, 1960; Nabulsi, 1984; Zhang and Dudley, 1987) for a more thorough analysis.

We begin at the well-known result for complex geometric series,

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \quad \begin{cases} |z| < 1 \\ |z| = 1, \text{Arg}(z) \neq m\pi \end{cases} \quad (A-4)$$

Integrating this from zero to z and substituting $z = e^{-w}$ we find

$$\sum_{n=1}^{\infty} \frac{e^{-nw}}{n} = -\ln(1 - e^{-w}), \quad \begin{cases} \text{Re}(w) > 0; \\ \text{Re}(w) = 0, \text{Im}(w) \neq 2m\pi. \end{cases} \quad (A-5)$$

If we allow the real part of w to vanish by defining $w = i\beta$, then by manipulating the logarithm and equating real and imaginary parts we find

$$\sum_{n=1}^{\infty} \frac{\cos(n\beta)}{n} = -\ln |2 \sin(\beta/2)|, \quad (A-6)$$

$$\sum_{n=1}^{\infty} \frac{\sin(n\beta)}{n} = \frac{\pi - \beta}{2}, \quad (A-7)$$

By performing

$$-i \int \sum_{n=1}^{\infty} \frac{e^{-in\beta}}{n} \quad (A-8)$$

we find

$$\sum_{n=1}^{\infty} \frac{e^{-in\beta}}{n^2} = \frac{\beta^2}{4} - \frac{\pi\beta}{2} + C + i \int \ln |2 \sin(\beta/2)| d\beta. \quad (A-9)$$

The logarithmic integral cannot be found in closed form. We therefore must manipulate the integrand into a more suitable form. Expanding the sine function in a Taylor series and factoring a power of β we can rewrite

$$\ln |2 \sin(\beta/2)| = \ln |\beta| + \ln |1 + \gamma| \quad (A-10)$$

where

$$\gamma = \sum_{j=1}^{\infty} \frac{(-i\beta/2)^{2n}}{(2n+1)!}, \quad \gamma \in \Re. \quad (A-11)$$

If $\gamma < 1$, then the logarithm can also be expanded in a Taylor series,

$$\ln |1 + \gamma| = - \sum_{k=1}^{\infty} \frac{(-\gamma)^k}{k}. \quad (A-12)$$

The bound on γ creates a bound on β from (A-10), namely,

$$\frac{\sin \beta/2}{\beta} < 1 \quad (A-13)$$

which is true for all β . When combined with the previous result, we find this produces

$$\ln |2 \sin(\beta/2)| = \ln |\beta| + \sum_{k=1}^{\infty} \frac{|B_{2k}|}{2k} \frac{\beta^{2k}}{(2k)!} \quad (A-14)$$

where B_{2k} are the even Bernoulli numbers. Integrating this result, we arrive at

$$\sum_{n=1}^{\infty} \frac{e^{-in\beta}}{n^2} = \frac{\beta^2}{4} - \frac{\pi\beta}{2} + C + i\beta(\ln |\beta| - 1) + i \sum_{k=1}^{\infty} \frac{|B_{2k}|}{2k} \frac{\beta^{2k+1}}{(2k+1)!}. \quad (A-15)$$

The integration constant can be found by allowing $\beta = 0$ and by using the well-known fact that $\sum_{k=1}^{\infty} 1/k^2 = \pi^2/6 = C$. Equating real and imaginary parts we produce the desired series,

$$\sum_{n=1}^{\infty} \frac{\sin(n\beta)}{n^2} = \beta(1 - \ln |\beta|) - \sum_{k=1}^{\infty} \frac{|B_{2k}|}{2k} \frac{\beta^{2k+1}}{(2k+1)!}, \quad 0 < \beta < \pi \quad (A-16)$$

Although one sum has been replaced by another, this new sum converges very quickly due to the rapid diminuation of the factorials and the Bernoulli numbers. The range on β can be expanded by noting that the sine function on the range from $\pi < \beta < 2\pi$ is the negative of the sine function on the range from $\pi > \beta > 0$. This function is graphed in Figure (A-1).

Figure A-1 The function $\sum_{n=0}^{\infty} \sin n\beta/n^2$ from Equations (A-16) and (A-27)

Substituting $\beta = -i\alpha$ into (A-16) and paying close attention to the branches of the logarithm, we find another of our series

$$\sum_{n=1}^{\infty} \frac{e^{-n\alpha}}{n^2} = \frac{\pi^2}{6} - \frac{\alpha^2}{4} + \alpha(\ln|\alpha| - 1) - \sum_{k=1}^{\infty} \frac{B_{2k}}{2k} \frac{\alpha^{2k+1}}{(2k+1)!}. \quad (A-17)$$

This result may also be found through the techniques used for finding S_1 . A transcendental equation similar to (A-13) arises, bounding $\alpha < 4.355$ (Zhang and Dudley, 1987). Since the original series is as rapidly convergent as the new series for $\alpha = 2.1$, this restriction is easily met. The original series can be used for large arguments, while the new series is superior for small arguments.

The third series demands a different approach. Geometric series are not applicable to this series, but contour integration methods are. The technique presented in (Collin, 1960) requires the summand be an even function of the index. We do this by adding and subtracting a hyperbolic tangent contribution

$$\sum_{n=1}^{\infty} \frac{\sin(n\beta)}{(n^2 - d^2)} = S_1 + S_2, \quad (A-18)$$

where

$$S_1 = \sum_{n=1}^{\infty} \frac{\sin(n\beta)}{(n^2 - d^2)} (1 - \tanh np), \quad (A-19)$$

$$S_2 = \sum_{n=1}^{\infty} \frac{\sin(n\beta)}{(n^2 - d^2)} \tanh np, \quad (A-20)$$

and p is a parameter chosen for convergence. When p is small, S_2 converges faster, while for larger p , S_1 is quicker. The first series will converge quickly since the term $(1 - \tanh np)$ becomes small quickly, so we need not alter its form to speed it. The

second series can be computed with the contour integration method given in Collin since it is now an even function of n .

The series S_2 can be rewritten over positive and negative n :

$$S_2 = \frac{1}{2i} \sum_{-\infty}^{\infty} \frac{e^{in\beta}}{P(n)} \quad (A-21)$$

where

$$P(n) = \frac{n^2 - d^2}{\tanh np}. \quad (A-22)$$

Consider the integral

$$\oint_{C_n} \frac{e^{i\beta z} dz}{P(z)(e^{2\pi iz} - 1)}. \quad (A-23)$$

The poles of the integrand occur at

$$z = \pm d; \quad (A-24)$$

$$z = \pm n; \quad (A-25)$$

$$z = i(2n-1)\pi/2p. \quad (A-26)$$

The contour is large enough to encompass all the poles, so the integral equals zero by Cauchy's Theorem. The evaluation of the residues at $z = \pm n$ generates S_2 ; therefore S_2 equals the negative of the sum of the residues at the other poles. The result is:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\sin(n\beta)}{(n^2 - d^2)} &= \sum_{n=1}^{\infty} \frac{\sin(n\beta)}{(n^2 - d^2)} (1 - \tanh np) + \frac{\pi \tanh pd \sin(\pi - \beta)d}{2d \sin \pi d} \\ &\quad - \frac{\pi}{p} \sum_{n=1}^{\infty} \frac{1}{d^2 + \left(\frac{(2n-1)\pi}{2p}\right)^2} \frac{\sinh\left[\frac{(2n-1)\pi(\pi-\beta)}{2p}\right]}{\sinh\left[\frac{(2n-1)\pi^2}{2p}\right]}. \end{aligned} \quad (A-27)$$

Figure A-2 The contour for the evaluation of S_2

The argument β is constrained to be between 0 and 2π by (Nabulsi, 1984), and d is restricted to be not an integer.

To optimize p the conditions for S_1 and S_2 must be balanced. For sufficient exponential decay in the first series the choice of

$$np = 10/3 \quad (A - 28)$$

is made. For the second series the choice of

$$(2n - 1)\frac{\pi\phi}{2p} = 5 \quad (A - 29)$$

is necessary , where

$$\phi = \begin{cases} \beta, & 0 < \beta < \pi; \\ 2\pi - \beta, & \pi < \beta < 2\pi. \end{cases} \quad (A - 30)$$

Combining both of these into a quadratic equation and solving for the roots, we find the optimal p :

$$p = -\frac{\pi\phi}{20} + \sqrt{\left(\frac{\pi\phi}{20}\right)^2 + \frac{2\pi}{3}} \quad (A - 31)$$

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