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EFFECT OF GRAVITY MODULATION ON THE STABILITY OF
A HORIZONTAL DOUBLE-DIFFUSIVE LAYER

by
Wen-Yau Chen

A Dissertation Submitted to the Faculty of the
DEPARTMENT OF AEROSPACE AND MECHANICAL ENGINEERING
In Partial Fulfillment of the Requirements
For the Degree of
DOCTOR OF PHILOSOPHY
WITH A MAJOR IN MECHANICAL ENGINEERING
In the Graduate College
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requirement for the Degree of Doctor of Philosophy

Chuan F. Chen
Chuan F. Chen

Nov 6, 2001
Date

Thomas F. Balsa
Thomas F. Balsa

11/6/01
Date

Cholik Chan
Cholik Chan

11/6/01
Date

Date

Date

Final approval and acceptance of this dissertation is contingent upon
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Chuan F. Chen
Dissertation Director Chuan F. Chen

Nov 26, 2001
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DEDICATION

This dissertation and the effort I have put is dedicated to God for His faith and unfailing love.

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ABSTRACT

The effect of gravity modulation on the instability onset in an infinite horizontal layer of double-diffusive fluid is investigated in this dissertation. The spectral-Galerkin method is used to transform the linearized perturbation equations to the system of time-periodic ordinary differential equations. The Chebyshev expansion method (Sinha and Wu, 1991) is applied to calculate the fundamental matrix which is used to determine the stability of the system according to the Floquet theory. Fluids of Prandtl number $Pr = 0.01, 1$, and 7 are investigated. The instability onsets in one of three modes, synchronous, subharmonic, and quasi-periodic mode. In the synchronous mode, the instability oscillates at the same frequency as the gravity modulation Ω , in the subharmonic mode, the instability oscillates at $\Omega/2$, while in the quasi-periodic mode, the oscillation frequency of the instability is different from the above two. The quasi-periodic mode onsets at the same thermal Rayleigh number, R_T as that of instability onset under steady gravity. The subharmonic mode onsets with wave numbers in the neighboring region of k where the oscillation frequency of the instability onset in the steady-g case, ω , equals to half of the modulation frequency, $\Omega/2$. Similarly, the synchronous mode onsets at the neighboring of k where the ω equals to Ω . The onset R_T for quasi-periodic mode is not changed by the modulation frequency Ω and the relative amplitude of the modulation, h . For the synchronous and subharmonic modes, destabilization increases with increasing h . If h is large enough, the subharmonic mode will be more unstable than the synchronous and quasi-periodic mode, so the instability mode will be switched by increasing h . For a given h with varying Ω , the resonance effect occurs in the neighborhood of $\Omega \approx 2\omega_{cr}$, i.e., twice the critical oscillation frequency of the instability in the steady-g

case associated with the critical R_T . The resonant phenomena is found for fluids with $Pr = 0.01, 1$, and 7 , and the effect diminishes as the Prandtl number increases. The effect of gravity modulation is asymptotic to zero when the modulated frequency Ω approaches zero and infinitely large. For the case of Prandtl number, $Pr, = 0.01$, it is found that the critical thermal Rayleigh number R_T is reduced from the steady-g value of 2183 by 4% , 41% , and 86% as h is increased from 0.01 , 0.1 and 0.2 . In fact when $h = 0.22737$, the layer of fluid is destabilized at $\Omega = 9.1$ with $R_T = 0$, i.e., without heating from below. This is analogous to the case in the research of Gresho and Sani (1970) that a horizontal layer being heated from above can be destabilized by the oscillation of the layer. In this dissertation the stabilization effect caused by the modulation is found at some cases of Ω , which is analogous to the stability of motion of the pendulum with pivot in oscillation as discussed in Gresho and Sani (1970).

Chapter 1

INTRODUCTION

1.1 Motivation

In a space station, low-amplitude gravity perturbations caused by crew motions, equipments, solar drag, and other sources are experienced. The maximum G-jitter amplitudes generated by these perturbations are of order $10^{-3} g_o$ for equipment and $10^{-2} g_o$ for crew motions (W.Knabe and D.Eilers, 1982) where the g_o is the terrestrial gravity. Compared to the space-station gravity of $10^{-6} g_o$, these amplitudes are significant, so it is of fundamental interest to understand the effect of these perturbations on the system's stability, specially the effect on the thermosolutal convection in the material processing in space base. In fact, the effect could be very large because the ratio of the maximum jitter amplitude generated by perturbations to the constant gravity at space base is of many orders in magnitude. This high ratio of gravity will strongly enhance the buoyancy force to overcome the viscous resistance (see the Navier-Stokes equation in below), and may cause the onset of instability. However the gravity oscillation will reverse the direction of the buoyancy force in each cycle, the 180° change in the direction of the buoyancy force may possibly stabilize the unstable convection. This possibility is dependent on the ratio of time scale of the gravity oscillation to the period of the propagating instability wave and it can be conjectured by the simple experiment of inverted and hanging pendulum whose pivot is oscillated in vertical direction. Whether the G-jitter is to stabilize or destabilize the system has caused many workers to investigate the effect of the jitter. These studies include linear stability analysis, fully nonlinear simulation, in two- or

three-dimension domain, in an infinite vertical slot, horizontal layer, or an enclosure.

In the Navier-Stokes equation, the gravity term can be regarded as the sum of constant background gravity and the oscillatory G-jitter term. The G-jitter term is usually simulated by sinusoidal time varying function (it's called gravity modulation) or by a random time function.

In the following paragraphs, previous research results of the effect of gravity modulation on the two-dimensional instability onset of convection and thermosolutal convection, and the influence of random G-jitter on the flow are reviewed. The motivation of the dissertation is to understand the effect of the gravity modulation on the two-dimensional thermosolutal convection in an infinite layer, because it is very important to know the conditions of gravity modulation under which the system will become unstable. The understanding will be helpful to the material processing in space base. There are three modes of instability onset of thermosolutal convection under the effect of gravity modulation according to Joseph (1976). These are synchronous, subharmonic, and quasi-periodic mode. In the synchronous mode the instability at onset oscillates with the same frequency as that of gravity modulation, in the subharmonic mode the instability oscillates with half the frequency as that of gravity modulation, and all other cases are known as quasi-periodic mode. In the linear stability analysis of the gravity-modulation effect, the one-term or multiple-term spectral function expansion of Galerkin method is usually used to transform the linearized governing equations to the time-periodic system of ordinary differential equations. Various techniques (see below) are used to determine the critical conditions for the instability onset.

1.2 Effect of Gravity Modulation On a Single-Diffusive Layer

Gresho and Sani (1970) examined the stability of Boussinesq fluid contained in two infinite horizontal (x-variable) plates at h apart being vibrated sinusoidally in the vertical direction (z-variable). The amplitude of the vibration is δh . They considered the layer being heated either from below or from above. The boundary conditions are zero velocity and constant temperature on both the top and bottom plates. Based on the scheme of spectral Galerkin method, they use an one-term spectral expansion in the z-direction to qualitatively study the linear stability analysis. With the approximation, the linearized governing equations are transformed to the Mathieu equation. The stability behavior of Mathieu equation is then applied to determine the instability onset of the convection under the influence of gravity modulation. The obtained equation is analogous to that of small-angle motion of a pendulum with viscous damping whose pivot point is sinusoidally oscillated in the vertical direction. To such a pendulum, the vertical oscillations of pivot could destabilize a stable hanging pendulum and stabilize the unstable inverted pendulum (Stoker,1950).

Gresho and Sani (1970), presented the stability characterization of a fluid with $Pr=7$, and $\delta Fr = 10^{-5}$. $P_r = \frac{\nu}{\kappa}$ with ν the kinetic viscosity and κ the thermal diffusivity. δ is the nodimensional displacement of vibration in units of h , and the Fr is the Froude number defined by $\frac{\kappa^2}{gh^3}$ with g the constant gravity. Applying the analogy to the problem, they found that gravity modulation can stabilize the horizontal layer being heated from below (analogous to unstable inverted pendulum) and destabilize the layer being heated from above (analogous to stable hanging pendulum).

For the case of heated from below, with the chosen P_r and δFr , they found

that the instability is in the synchronous mode for the nondimensional modulation frequency ω less than 1250, and the instability is in subharmonic mode for ω larger than 1250. The transition point, or the mode-switching frequency at $\omega = 1250$, denoted by ω^* . The effect of ω on the critical condition for the onset of instability is small for ω less than 1100. In this range, the critical Rayleigh number R_c is slightly larger than 1825, the asymptotic value when $\omega \rightarrow 0$. The value of 1825 is the R_c in steady g (denoted by R_o) which is calculated by the approximation of one-term spectral expansion. It is noted that the exact value of R_o is 1707 as given by Chandrasekhar (1961). As ω is increased from 1100 to 1250, R_c increases very rapidly until a maximum is reached. After 1250, R_c will begin to drop slowly and asymptotically to a limit ($R_{c\infty}$) when ω is large enough. Gresho and Sani (1970) reported the limit $R_{c\infty}$ will be smaller than R_o if Pr and δF_r are large enough according to an approximate formula, $R_{c\infty} \approx \frac{0.4739}{Pr}(\delta F_r)^{-1}$. They only examine the group where $Pr = 7$ and $\delta F_r = 10^{-5}$. In summary, for heating from below, the gravity modulation will enhance the stability of the layer if ω is increased from zero up to 1250 with the instability in the synchronous mode. In the zone of subharmonic mode, the layer is still stabilized by gravity modulation unless Pr and δF_r are large enough. The zone of destabilization by gravity modulation must occur in subharmonic mode. The nondimensionalized wave number k_c shows a discontinuity at the $\omega = 1250$, where the dominant mode is switched. The local k_c of synchronous mode is always smaller than subharmonic mode.

For being heating from above, when ω is less than about 400, the critical frequency, the layer is always stable. In $400 < \omega < 422$, the synchronous mode dominates. When ω is larger than 422, the subharmonic mode dominates. At about 422, there is a mode-switching transition ω^* , so the k_c is discontinuous there. The

value of $-R_c$ (minus sign means heated from above) drops very rapidly from ∞ down to a very low value in a very short frequency range of $400 < \omega < 450$. When larger than 450, $-R_c$ decreases very slowly, and then asymptots to the limit value $R_{c\infty}$. Gresho and Sani (1970) reported that both of the limit $R_{c\infty}$ and the critical frequency will decrease if δF_r is increased. This means that at larger modulation amplitude the layer will become unstable at a lower ω and a lower $-R_c$. The synchronous mode dominates only in a very small range of ω , and subharmonic mode will prevail in most of ω .

Clever et al. (1993) used spectral Galerkin method to study the effect of gravity modulation on the instability onset of convection and obtained the nonlinear solutions of the two-dimesional oscillatory motion. The geometry and boundary conditions of the layer are the same as in Gresho and Sani (1970) with heating either from above or below. Using the same nondimensionalization parameters as in Gresho and Sani (1970), Clever et al. (1993) investigated the parameter range of $Pr = 0.7-50$, $\omega = 30-6000$, $\delta F_r \omega^2 = 1-10^4$. They used the multi-terms spectral expansion along the time and the z axis. The spectral functions used in z direction and the numerical scheme used are different from those used by Gresho and Sani (1970).

For heating from below, at $Pr=7$ and $\delta F_r = 10^{-5}$, essentially they found the similar R_c vs ω curve as in Gresho and Sani (1970). They examined cases where $Pr = 0.71-50$, and $\delta F_r = 3.162 \times 10^{-5}, 10^{-4}, 3.162 \times 10^{-4}, 10^{-3}$. The results show that the features of R_c vs ω curve for these cases are simialr. Gresho and Sani (1970) only examined one case with $Pr = 7$ and $\delta F_r = 10^{-5}$ and predicted the asymptotic result $R_{c\infty} \approx \frac{0.4739}{Pr}(\delta F_r)^{-1}$. Clever et al. (1993) proved this formula by investigating broad cases as in above. In Gresho and Sani (1970), the $R_{c\infty}$ is larger than R_o but in Clever et al. (1993) the $R_{c\infty}$ is shown to be smaller than R_o when the δF_r or Pr

is large enough. It means the layer heated from below generally will be stabilized by gravity modulation, but can be destabilized if the Pr and δF_r are very large when $\omega \gg \omega^*$. In all the cases considered the local k_c of synchronous mode is always smaller than the subharmonic mode. For heating from above, Clever et al. (1993) examined $\frac{\epsilon}{\omega^2}$ ($= \delta F_r$ in Gresho and Sani, 1970) $= 10^{-5}, 10^{-4}, 10^{-3}$, and the results show the excellent agreement with the theory of Gresho and Sani (1970).

1.3 Effect of Gravity Modulation On a Double-Diffusive Layer

In Saunders et al. (1992), a linear stability analysis with Boussinesque approximation was applied to investigate the effect of vertical gravity modulation on the onset of thermosolutal convection in an infinite horizontal layer with stress-free boundaries. Sinusoidal functions are used for the spectral expansion in the z direction to transform the linearized governing equations which are coupled partial differential equations to a time-periodic system of ordinary differential equations. They use high-order Adams method in the numerical integration to solve for the fundamental matrix.

For the diffusive regime, that is heating from below of a stable solute gradient, one example is calculated in which $Pr = 0.1$, $Sc = 10$, $R_c = -3 \times 10^5$, $R_T = -1000$, where the Schmidt number Sc is $\frac{\nu}{\kappa_s}$, ν is the kinetic viscosity, κ_s is the mass diffusivity of the salt, R_c is the Rayleigh number based on salt concentration, R_T is the Rayleigh number based on temperature. In the nomenclature of Saunders et al. (1992), negative R_c stabilizes the layer; negative R_T destabilizes the layer. The stability results are presented in a graph with the dimensionless amplitude of gravity modulation, ϵ , as the vertical axis, and $1/\omega$ as the horizontal axis. The regions of

instability appear in nonintersecting lobe-shaped regions and there is only one instability mode in each lobe. The quasi-periodic mode is in the first lobe located in the high ω , and the subharmonic and the synchronous modes alternate among the rest of lobes as ω is reduced. There are 3 lobes intersected by the line $\epsilon = 0.5$, and 5 lobes by $\epsilon = 0.7$...etc, finite numbers of lobes intersected by horizontal line of constant ϵ , because the value of ϵ at the minimum of lobes of subharmonic and synchronous modes increases as $1/\omega$ increases. These lobes are referred to as resonant bands by Saunders et al. (1992). The highest-frequency resonant band has a minimum of $\epsilon \approx 10^{-2}$ at $\omega \approx 2\sigma \approx 190$ where the σ is the oscillatory frequency of the instability onset at steady g. The minimum of the m_{th} band occurs at $\omega \approx 2\sigma/m$.

Terrones and Chen (1993) investigated the effect of gravity modulation on the onset of instability in an infinite horizontal layer of double-diffusive fluid with and without cross-diffusive effects. They investigated the effect of free-free and rigid-rigid boundary conditions by a finite-term expansion in the series of the spectral expansion. Their emphasis was on the characterization of the marginal stability curve. They presented a number of examples of selected sets of parameters to illustrate the nature of the instability at the onset condition caused by the gravity modulation. Two examples are cited here because they are of interest to this dissertation. The first example is for a free-free layer with $P_r = 7$, $r = \frac{\kappa_1}{\kappa} = 0.8$, $R_c = -2700$, and $\Omega = 10$. In the absence of gravity modulation, the nature of the marginal stability curve which shows the variation of the critical thermal Rayleigh number R_T as a function of the wave number k of the perturbations, consists of an oscillatory branch connected by two bifurcating points to a steady convection neutral branch. When the gravity modulation is introduced, the oscillating branch is replaced by a banana-shaped subharmonic branch, and the steady branch is replaced by a synchronous branch. These

two branches are connected by a quasi-periodic branch. The second example is for a liquid metal layer with rigid-rigid boundaries where $Pr = 0.01$, $r = 3 \times 10^{-4}$, $R_c = -3200$, and $\Omega = 10$. The neutral curve consists of the quasi-periodic and the subharmonic branches. In this case, the quasi-periodic branch being the most critical one. However a slight increase in the solute stabilization to $R_c = -3300$, the subharmonic branch becomes the most critical one. For code validation purposes, Terrones and Chen (1993) performed the stability analysis for a layer of single-diffusive fluid with $Pr = 7.1$ and rigid-rigid boundaries. For a given value of gravity modulation amplitude they obtained the critical value of R_T as a function of Ω . Comparison of the results with Gresho and Sani (1970) was not straight forward because of the different nondimensional procedures. But for one case in which all parameters are equal, they predicted that the values of critical thermal Rayleigh number R_T and k are the same as those read off a graph in Gresho and Sani (1970).

1.4 Influence of Random G Jitter On the Flow

In this section, it will be introduced the influence of the G jitter which is simulated by random function or other forms (see below) rather than sinusoidal modulation, because these function approximate the real disturbance in low-gravity environment more precisely than the sinusoidal function.

Biringen et al. (1990) calculated the influence of G jitter simulated by random function on the instability of three dimensional Benard convection in fully numerical calculation by use of semi-implicit pseudospectral method. Results show that the random gravity variation is more unstable than the sinusoidal modulation.

Drolet et al. (1998) used a Gaussian, and white stochastic model for G-

jitter to study the fluid motion that is induced. Considering two particles suspended in an incompressible fluid of different density from that of the particles and assuming the fluid is enclosed in a rigid box being vibrated, the G-jitter can induce an effective attractive force between the two spherical particles if the particles are separated at a distance larger than their radius. The time rate of change of the inter-particle separation is inversely proportional to the cubic power of the separation. The attractive force can become a repulsive one at very short inter-particle separations. The definition of Gaussian and white is $\langle g(t) \rangle = 0$, $\langle g(t)g(t') \rangle = 2D\delta(t - t')$, where the D is the maximum amplitude of G-jitter fluctuation and $\langle \rangle$ means the time average.

Researches on G-jitter induced convection in single-component flow system suggest that the convection in the low-gravity environment is related to the magnitude and frequency of the G-jitter and to the alignment of the gravity field with respect to the direction of the temperature gradient (Alexander, 1989, 1991). The results show that the orientation of the G-jitter oscillation relative to the density gradient determines whether a mean flow is induced in the fluid. de Groh et al. (2001) used finite element method to investigate the double-diffusive convection driven by G-jitter in microgravity environment. The Boussinesq approximation is assumed, and the fluid is contained in a square rigid-wall cavity. The temperature is high on the left wall and the concentration is high on the right wall. The flux of the heat and concentration are assumed to be zero on the top and bottom walls. The G-jitter and the background gravity are along vertical direction. The true G-jitter is random but can be decomposed to a synthesized time Fourier series. Results show that the temperature field is almost unaffected but the contours of isoconcentration are significantly distorted. There is an induced secondary flow oscillating with twice the frequency of the G-jitter.

1.5 Outline of the Dissertation

In this dissertation, linear stability analysis is systematically applied to understand the effect of gravity modulation on the onset of diffusive instabilities in an infinite horizontal double-diffusive layer with rigid-rigid boundaries. Boussinesque approximation is assumed in the equations where the physical properties of fluid are constant except for the density in the buoyancy term in the Navier-Stokes equation. The density is assumed to be linear with the temperature and solute concentration. A multiple-term spectral expansion is used to transform the governing equations to a periodic system of ordinary differential equations based on the Galerkin method. The fundamental matrix is calculated by the Chebyshev expansion method (Sinha and Wu, 1991). In the beginning, the basic-state solution and the nondimensionalized perturbation equations are introduced. Then the mathematical analysis is presented which includes the use of spectral expansion and the Galerkin method to transform the governing equations to a system of ordinary differential equations and the use of the Chebyshev expansion method to calculate the fundamental matrix. The Floquet theory is introduced next to determine the stability of the time-periodic system. The results are presented for both steady and unsteady gravity and the quantitative analysis for asymptotic cases is described in the appendix.

Chapter 2

MATHEMATICAL EQUATIONS

2.1 Governing Equations

Assume a Boussinesque fluid contained in a two-dimensional infinite horizontal (x axis) slot with height H . The gravity is along the vertical direction (z axis). The temperature and concentration are maintained at different constant values on the top and bottom rigid boundaries. The double-diffusive fluid is assumed in the diffusive regime without considering cross diffusion. The gravity is assumed to be :

$$g(t) = g_o + g'_1 \cos(\Omega t) \quad (2.1)$$

where g_o is the constant gravity, and the g'_1 is the amplitude of gravity modulation. The density is assumed to be linear with the temperature T and concentration C as follows :

$$\rho = \rho_o [1 + \beta_T (T - T_{bottom}) + \beta_C (C - C_{bottom})] \quad (2.2)$$

$$\beta_T = \frac{1}{\rho_o} \left(\frac{\partial \rho}{\partial T} \right), \quad \beta_C = \frac{1}{\rho_o} \left(\frac{\partial \rho}{\partial C} \right) \quad (2.3)$$

where the β_T is negative and the β_C is positive, and ρ_o is the density at the bottom plate. The continuity, momentum, energy, and the concentration equations are :

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \quad (2.4)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = - \frac{\partial p}{\rho_o \partial x} + \nu \nabla^2 u \quad (2.5)$$

$$\begin{aligned} \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} = & - \frac{\partial p}{\rho_o \partial z} + \nu \nabla^2 w \\ & - g(t) [1 + \beta_T (T - T_{bottom}) + \beta_C (C - C_{bottom})] \end{aligned} \quad (2.6)$$

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + w \frac{\partial T}{\partial z} = \kappa_T \nabla^2 T \quad (2.7)$$

$$\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} + w \frac{\partial C}{\partial z} = \kappa_C \nabla^2 C \quad (2.8)$$

u and w are the x - and z - components of velocity which are zero on the top and bottom rigid boundaries. T and C are temperature and solute concentration. On the top and bottom boundaries, T are assumed to be T_{top} and T_{bottom} , and C are C_{top} and C_{bottom} . The κ_T and κ_C are the thermal and solutal diffusivity, and ν is the kinematic viscosity.

2.2 Basic State

The horizontal layer is assumed to be infinite in x , so the effect from both ends is neglected. The temperature and concentration are constant on the top and bottom plates, so $\frac{\partial}{\partial t} = \frac{\partial}{\partial x} = 0$ is applied to the basic state of temperature and concentration. Let u_b and w_b be the basic-state velocity components in the x and z

direction. The circulation flow in the central part of the slot is ignored because the horizontal dimension is assumed to be in infinite length, so assume the $u_b = 0$. By the use of zero u_b , the continuity equation and the no-slip conditions, w_b will be derived to be zero. The governing equations of basic state are:

$$0 = -\frac{\partial P_b}{\rho_o \partial x} \quad (2.9)$$

$$0 = -\frac{\partial P_b}{\rho_o \partial z} - g(t)[1 + \beta_T(T_b - T_{bottom}) + \beta_C(C_b - C_{bottom})] \quad (2.10)$$

$$0 = \kappa_T \frac{\partial^2 T_b}{\partial z^2} \quad (2.11)$$

$$0 = \kappa_C \frac{\partial^2 C_b}{\partial z^2} \quad (2.12)$$

By use of the constant temperature and concentration on boundaries, the steady-state solutions for basic-state temperature and concentration, T_b, C_b , are the linear distribution as follows:

$$\frac{dT_b}{dz} = \frac{T_{top} - T_{bottom}}{H} \quad (2.13)$$

$$\frac{dC_b}{dz} = \frac{C_{top} - C_{bottom}}{H} \quad (2.14)$$

2.3 First-Order Perturbation Equations

To investigate the stability of the field equations with respect to small disturbances, consider the following linear perturbations :

$$u = u'$$

$$w = w'$$

$$p = P_b + p'$$

$$T = T_b + T'$$

$$C = C_b + C'$$

Combining with the basic-state equations, and substituting of these perturbations into the governing equations, it results with the first-order perturbation equations :

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0 \quad (2.15)$$

$$\frac{\partial u'}{\partial t} = -\frac{\partial p'}{\rho_o \partial x} + \nu \nabla^2 u' \quad (2.16)$$

$$\frac{\partial w'}{\partial t} = -\frac{\partial p'}{\rho_o \partial z} + \nu \nabla^2 w' - g(t)[\beta_T T' + \beta_C C'] \quad (2.17)$$

$$\frac{\partial T'}{\partial t} + w' \frac{\partial T_b}{\partial z} = \kappa_T \nabla^2 T' \quad (2.18)$$

$$\frac{\partial C'}{\partial t} + w' \frac{\partial C_b}{\partial z} = \kappa_C \nabla^2 C' \quad (2.19)$$

2.4 Nondimensionalization of the Perturbation Equations

Introducing the following nondimensional quantities, which are denoted by $\sim$ except for T' and C' , they are denoted by S_T and S_C

$$(x, z) = (\tilde{x}, \tilde{z}) H$$

$$t = \tilde{t} \frac{H^2}{\kappa_T}$$

$$(u', w') = (\tilde{u}, \tilde{w}) \frac{\kappa_T}{H}$$

$$p' = \tilde{p} \rho_o \frac{\nu}{H} \frac{\kappa_T}{H}$$

$$T' = \left(\frac{g_o \beta_T H^3}{\kappa_T \nu} \right)^{-1} S_T$$

$$C' = \left(\frac{g_o \beta_C H^3}{\kappa_T \nu} \right)^{-1} S_C$$

$$\Omega = \tilde{\Omega} \frac{\kappa_T}{H^2}$$

For convenience, drop the symbol $\sim$, the nondimensionalized perturbation equations are obtained :

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \quad (2.20)$$

$$\frac{1}{Pr} \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \nabla^2 u \quad (2.21)$$

$$\begin{aligned} \frac{1}{Pr} \frac{\partial w}{\partial t} = & -\frac{\partial p}{\partial z} + \nabla^2 w - \\ & (1 + h \cos \Omega t) [S_T + S_C] \end{aligned} \quad (2.22)$$

$$\frac{\partial S_T}{\partial t} + R_T w = \nabla^2 S_T \quad (2.23)$$

$$\frac{\partial S_C}{\partial t} + R_C w = r \nabla^2 S_C \quad (2.24)$$

where $Pr = \frac{\nu}{\kappa_T}$, $r = \frac{\kappa_C}{\kappa_T}$, $h = \frac{g'}{g_o}$. The Rayleigh number based on temperature and concentration R_T and R_C are defined as:

$$R_T = \frac{g_o \beta_T \delta_T H^3}{\kappa_T \nu} \quad R_C = \frac{g_o \beta_C \delta_C H^3}{\kappa_T \nu}$$

where $\delta_T = T_{top} - T_{bottom}$ and $\delta_C = C_{top} - C_{bottom}$. It's noted that $R_T \propto \beta_T \delta_T \propto \rho_{top} - \rho_{bottom}$, so the layer is destabilized if $R_T > 0$, and stabilized if $R_T < 0$. , similarly for the R_C . In the double-diffusive case, the heating from below destabilizes the layer because both of the β_T and δ_T are negative, so R_T is positive. The higher salt concentration in bottom stabilizes the layer because the β_C is positive and δ_C is negative, so R_C is negative.

2.5 Use of Stream Function

Because of the 2-D continuity equation, the stream function can be applied to the perturbation equations to combine x-component and z-component momentum equations to one equation without the pressure term. By the use of $u = \frac{\partial \psi}{\partial z}$ and $w = -\frac{\partial \psi}{\partial x}$ where ψ is the stream function, and elimination of the pressure term between the equations (2.21 and 2.22), the perturbation equations become :

$$\frac{1}{Pr} \frac{\partial}{\partial t} \nabla^2 \psi - \nabla^4 \psi = (1 + h \cos \Omega t) \left(\frac{\partial S_r}{\partial x} + \frac{\partial S_c}{\partial x} \right) \quad (2.25)$$

$$\frac{\partial S_r}{\partial t} = R_T \frac{\partial \psi}{\partial x} + \nabla^2 S_r \quad (2.26)$$

$$\frac{\partial S_c}{\partial t} = R_C \frac{\partial \psi}{\partial x} + r \nabla^2 S_c \quad (2.27)$$

with boundary conditions of $S_r = S_c = \psi = D\psi = 0$ at $z = \pm \frac{1}{2}$ where $D \equiv \frac{\partial}{\partial z}$.

Chapter 3

MATHEMATICAL TRANSFORMATION

3.1 Expansions to Multiple-Term Spectral Functions

In equations (2.25 ~ 2.27), Fourier spectral expansion is made in the x variable because x is infinite, the equations are linear, and physically the perturbation terms are wave-like in x . Multiple-term expansion in the Chandrasekhar (1961) spectral function for the stream function and sinusoidal spectral function for temperature and concentration are applied to the z variable. These spectral function are chosen based on the boundary conditions. For detail reasons, see Chandrasekhar (1961). The spectral expansions for stream function and temperature and concentration are shown as below:

$$\psi(x, z, t) = [\sum_{n=1}^N a_n(t) \phi_n(z)] e^{ikx} \quad (3.1)$$

$$S_T(x, z, t) = [\sum_{n=1}^N b_n(t) T_n(z)] e^{ikx} \quad (3.2)$$

$$S_C(x, z, t) = [\sum_{n=1}^N c_n(t) T_n(z)] e^{ikx} \quad (3.3)$$

where the trial functions $\phi_n(z)$ (Chandrasekhar, 1961) and $T_n(z)$ are :

$$\phi_n(z) = \frac{\cosh(\rho_n z)}{\cosh(\rho_n/2)} - \frac{\cos(\rho_n z)}{\cos(\rho_n/2)} \quad \text{if } n = \text{odd}$$

$$\text{where } \tanh(\rho_n/2) + \tan(\rho_n/2) = 0$$

$$\phi_n(z) = \frac{\sinh(\mu_n z)}{\sinh(\mu_n/2)} - \frac{\sin(\mu_n z)}{\sin(\mu_n/2)} \quad \text{if } n = \text{even}$$

$$\text{where } \coth(\mu_n/2) + \cot(\mu_n/2) = 0$$

$$T_n(z) = \cos n\pi z \quad \text{if } n = \text{odd}$$

$$T_n(z) = \sin n\pi z \quad \text{if } n = \text{even}$$

3.2 Transformation by the Use of the Galerkin Method

Substituting the above spectral expansions into the perturbation equations (2.25 ~ 2.27), then applying the Galerkin method, the governing equations will be transformed to a system of ordinary differential equations as follows:

$$\begin{aligned}
\frac{d}{dt} \vec{y} = & \begin{bmatrix} \frac{\bar{C}}{\bar{P}_r} & 0 & 0 \\ 0 & \bar{F} & 0 \\ 0 & 0 & \bar{F} \end{bmatrix}^{-1} \times \\
& \begin{bmatrix} \bar{D} & (1 + h \cos \Omega t) k^2 \bar{H}^T & (1 + h \cos \Omega t) k^2 \bar{H}^T \\ R_T \bar{H} & \bar{G} & 0 \\ R_C \bar{H} & 0 & r \bar{G} \end{bmatrix} \vec{y} \\
\equiv & \mathbf{A}(t) \vec{y}
\end{aligned} \tag{3.4}$$

where $\mathbf{A}(t)$ is a $3N \times 3N$ time periodic matrix with period of $\frac{2\pi}{\Omega}$ and the $\bar{C}$, $\bar{F}$, $\bar{D}$, $\bar{H}$, $\bar{H}^T$, $\bar{G}$ are the $N \times N$ submatrix as defined as follows :

$$\begin{aligned}
\bar{C}_{ji} &\equiv - \int_{-1/2}^{1/2} \phi_j (D^2 - k^2) \phi_i dz \equiv - \langle \phi_j, (D^2 - k^2) \phi_i \rangle \\
\bar{D}_{ji} &\equiv - \langle \phi_j, (D^2 - k^2)^2 \phi_i \rangle \\
\bar{H}_{ji} &\equiv \langle T_j, \phi_i \rangle \\
\bar{H}^T &\equiv \text{transpose of } \bar{H} \\
\bar{G}_{ji} &\equiv \langle T_j, (D^2 - k^2) T_i \rangle \\
\bar{F}_{ji} &\equiv \langle T_j, T_i \rangle
\end{aligned}$$

where $D \equiv \frac{d}{dz}$ and the vector $\vec{y}$ is defined as a $3N \times 1$ column vector:

$$\vec{y} = \begin{bmatrix} ika_1 \\ ika_2 \\ \vdots \\ ika_N \\ b_1 \\ b_2 \\ \vdots \\ b_N \\ c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix}$$

3.3 Limiting Cases of Ω

1. When Ω approaches zero:

The term $1 + h \cos \Omega t$ in equation (2.25) becomes a constant. The system is as like as in steady g.

2. When Ω approaches infinitely large.

From the equation (3.4):

$$\mathbf{A}(t) = \mathbf{A}_0 + \mathbf{A}_1 h \cos \Omega t \quad (3.5)$$

where $\mathbf{A}_0$ and $\mathbf{A}_1$ are constant matrix. If h is zero, it becomes the steady-g case and the equation (3.4) reduces to:

$$\frac{d}{dt} \bar{\mathbf{y}} = \mathbf{A}_0 \bar{\mathbf{y}} \quad (3.6)$$

By using the equation (3.4), let $\bar{\mathbf{y}}(o)$ be the initial condition of $\bar{\mathbf{y}}$

$$\begin{aligned} \bar{\mathbf{y}} &= \bar{\mathbf{y}}(o) + \int (\mathbf{A}_0 + \mathbf{A}_1 h \cos \Omega t) \bar{\mathbf{y}} dt \\ &= \bar{\mathbf{y}}(o) + \int \mathbf{A}_0 \bar{\mathbf{y}} dt + h \mathbf{A}_1 \int \bar{\mathbf{y}} \cos \Omega t dt \end{aligned} \quad (3.7)$$

By nature assume the instability mode oscillates at finitely-large frequency because the celerity of instability mode is assumed to be finite. The infinite celerity means if suddenly the disturbance is imposed on one point in the fluid layer, it will simultaneously appear on the other point far away. Let the $\bar{\mathbf{y}}$ be a good function of time. If Ω goes to infinite large, the last term in the equation (3.7) will vanish because of the cancellation caused by the integral of $\cos \Omega t$. Also using the Riemann-Lebesgue lemma that guarantees :

$$\int_a^b f(t) e^{i\Omega t} dt \Rightarrow 0 \quad \text{if } \Omega \Rightarrow \infty \quad (3.8)$$

provided that the $\int_a^b \| f(t) \| dt$ exists. From the Floquet solution in equation (4.1), the $\int \| \bar{\mathbf{y}} \| dt$ is bounded at the neutral onset, so the last term in the equation (3.7)

must vanish for the case of infinitely large Ω . The equation (3.7) will become :

$$\vec{y} = \vec{y}(o) + \int \mathbf{A}_o \vec{y} dt$$

so, it is equivalent to

$$\frac{d}{dt} \vec{y} = \mathbf{A}_o \vec{y}$$

This is exactly equivalent to the equation (3.6) , the steady-g case.

Chapter 4

MATHEMATICAL ANALYSIS

4.1 Floquet Solution

In the dynamical system equation (3.4), $\mathbf{A}(t)$ is a $3N \times 3N$ time-periodic matrix because of the modulation term $\cos(\Omega t)$ in the equation, and the system is a time-periodic nonautonomous dynamical system. The period of the $\mathbf{A}(t)$ is $\frac{2\pi}{\Omega}$. By Floquet theory, Yakubovich(1975), the solution to the dynamic system (equation 3.4) will be in the form of :

$$\vec{y}(t) = \mathbf{P}(t) e^{\mathbf{B}t} \mathbf{P}^{-1}(o) \vec{y}(o) \quad (4.1)$$

which is also called Floquet solution where $\mathbf{P}(t)$ is a periodic nonsingular matrix with same period of $\mathbf{A}(t)$, $\frac{2\pi}{\Omega}$. $\mathbf{B}$ is an unknown constant matrix. $\vec{y}(o)$ is an arbitrary initial condition. $\vec{y}(t)$ and $\vec{y}(o)$ are $3N \times 1$ column vector, and the matrices $\mathbf{B}$, $\mathbf{P}(t)$ and $\mathbf{P}^{-1}(o)$ are $3N \times 3N$ where N is the number of spectral functions in the spectral expansion(see equations 3.1, 3.2, 3.3). Because the matrix $e^{\mathbf{B}t}$ is nonsingular, therefore the matrix $\mathbf{P}(t) e^{\mathbf{B}t} \mathbf{P}^{-1}(o)$ is also nonsingular, then the $\vec{y}(t)$ is unique for any a given initial $\vec{y}(o)$. The stability of the dynamic system (equation 3.4) is defined by the stability of the solution $\vec{y}(t)$. If $\vec{y}(t)$ becomes larger and larger in time evolution, then $\vec{y}(t)$ and the dynamic system are unstable. If $\vec{y}(t)$ is asymptotic to zero or periodic in the time evolution, then $\vec{y}(t)$ and the dynamic system are stable. By means of the form of the Floquet solution in equation (4.1), the stability of $\vec{y}(t)$ and

the dynamic system will be determined by the stability of the term $e^{\mathbf{B}t}$ because the matrix $\mathbf{P}(t)$ is periodic so its magnitude is bound, a finite-order term. The stability of the $e^{\mathbf{B}t}$ will be determined by the real part of the eigenvalues of the constant matrix $\mathbf{B}$ by means of the principle of the Jordan normal form. The detail interpretation is shown in the following section.

4.2 Jordan Normal Form

For any given $3N \times 3N$ constant matrix $\mathbf{B}$, there always can be found an unitary matrix $\mathbf{U}$, such that $\mathbf{B}$ is equal to :

$$\mathbf{B} = \mathbf{U}^\dagger \mathbf{\Lambda} \mathbf{U} \quad (4.2)$$

Where the $\mathbf{\Lambda}$ is defined as following:

$$\mathbf{\Lambda} \equiv \begin{bmatrix} [\lambda]_1 & \mathbf{O} & \cdots & \mathbf{O} \\ \mathbf{O} & [\lambda]_2 & \cdots & \mathbf{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{O} & \mathbf{O} & \cdots & [\lambda]_m \end{bmatrix}$$

where the $\mathbf{O}$ is a zero submatrix, the $[\lambda]_1, [\lambda]_2, \cdots, [\lambda]_m$ are the Jordan blocks of eigenvalues of matrix $\mathbf{B}$ and the total sum of row dimension of each of these Jordan blocks are equal to $3N$, the row dimension of $\mathbf{B}$. The Jordan block $[\lambda]_j$, $j = 1, 2, \cdots, m$, with $m \leq 3N$ has the following form:

$$[\lambda]_j \equiv \begin{bmatrix} \lambda_j & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_j & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_j & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \cdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \lambda_j & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_j & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \lambda_j \end{bmatrix}$$

where λ_j is the eigenvalue of matrix $\mathbf{B}$.

By use of the Jordan block and equation(4.2), $e^{\mathbf{B}t}$ can be equated to :

$$\begin{aligned} e^{\mathbf{B}t} &= \mathbf{U}^\dagger e^{\mathbf{\Lambda}t} \mathbf{U} \\ &= \mathbf{U}^\dagger \begin{bmatrix} e^{[\lambda]_1 t} & \mathbf{O} & \cdots & \mathbf{O} \\ \mathbf{O} & e^{[\lambda]_2 t} & \cdots & \mathbf{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{O} & \mathbf{O} & \cdots & e^{[\lambda]_m t} \end{bmatrix} \mathbf{U} \end{aligned} \quad (4.3)$$

where $e^{[\lambda]_j t}$ has the following form :

$$e^{[\lambda]_j t} = \begin{bmatrix} e^{\lambda_j t} & te^{\lambda_j t} & \frac{t^2}{2!}e^{\lambda_j t} & \frac{t^3}{3!}e^{\lambda_j t} & \cdots \\ 0 & e^{\lambda_j t} & te^{\lambda_j t} & \frac{t^2}{2!}e^{\lambda_j t} & \cdots \\ \vdots & \cdots & \cdots & \cdots & \vdots \\ 0 & \cdots & 0 & e^{\lambda_j t} & te^{\lambda_j t} \\ 0 & 0 & 0 & \cdots & e^{\lambda_j t} \end{bmatrix}$$

The row dimension of each block is finite, so the power of the polynomial of t is finite.

If the real part of the eigenvalue λ_j is negative, then $e^{\lambda_j t}$ will decay to zero faster than the polynomial of t as t becomes larger and larger, such that the Jordan block $e^{[\lambda]_j t}$ will be asymptotic to zero, which is a stable matrix for all time. Similarly the

block will be unstable if the real part of the eigenvalue λ_j is positive. Therefore, the $e^{\mathbf{B}t}$ will be stable if the real part of all eigenvalues of the matrix $\mathbf{B}$ are negative, and it will be unstable if any one of them is positive. From the previous section, it is known that the stability of $\vec{\mathbf{y}}(t)$ and the dynamical system(equation 3.4) will be determined by the stability of the $e^{\mathbf{B}t}$, so they will be determined by the real part of the eigenvalues of matrix $\mathbf{B}$.

4.3 Fundamental Matrix and Similarity Transform

From the previous sections and equation (3.4), it is known that the stability of $\vec{\mathbf{y}}(t)$ and the dynamic system (equation 3.4) will be determined by $e^{\mathbf{B}t}$ but not by $\mathbf{P}(t)$ according to equation (4.1). The stability of $e^{\mathbf{B}t}$ is determined by the real part of the eigenvalues of $\mathbf{B}$. Therefore the real part of the eigenvalues of $\mathbf{B}$ determine the stability of $\vec{\mathbf{y}}(t)$ and the dynamic system (equation 3.4). But because there is no way to obtain $\mathbf{B}$ and $\mathbf{P}(t)$, the stability of the dynamic system can not be determined by way of solving for $\mathbf{B}$ in order to obtain its eigenvalues. The similarity transform is another way to calculate the eigenvalues without knowing $\mathbf{B}$ which is described below.

For the system of equation (3.4), it is explained in previous sections there will be a one-to-one solution $\vec{\mathbf{y}}(t)$ for any given initial condition $\vec{\mathbf{y}}(o)$, because the $\mathbf{P}(t)$, $\mathbf{P}^{-1}(o)$ and $e^{\mathbf{B}t}$ in equation (4.1) are all nonsingular. It is convenient to choose the $3N$ columns of the $3N \times 3N$ identity matrix to be $3N$ initial conditions of the equation (3.4), denoted as $\vec{\mathbf{y}}_1(0)$, $\vec{\mathbf{y}}_2(0)$, $\dots$, $\vec{\mathbf{y}}_{3N}(0)$, where the $\vec{\mathbf{y}}_j(0)$ is a $3N \times 1$ vector whose only nonzero element is 1 on the j_{th} row. There is always a nonsingular $3N \times 3N$ matrix whose $3N$ columns are the $3N$ solutions of equation (3.4) corresponding

to the above $3N$ initial conditions, which are denoted by $\vec{y}_1(t)$, $\vec{y}_2(t)$, $\dots$, $\vec{y}_{3N}(t)$. Denote the nonsingular matrix as :

$$\mathbf{Y}(t) \equiv [\vec{y}_1(t), \vec{y}_2(t), \dots, \vec{y}_{3N}(t)] \quad (4.4)$$

if the time is chosen at $t = \frac{2\pi}{\Omega} \equiv T$, by use of the equation (4.1), it becomes:

$$\begin{aligned} \mathbf{Y}(T) &= \mathbf{P}(T) e^{\mathbf{B}T} \mathbf{P}^{-1}(o) I \\ &= \mathbf{P}(T) e^{\mathbf{B}T} \mathbf{P}^{-1}(o) \end{aligned} \quad (4.5)$$

where the I is the $3N \times 3N$ identity matrix and the $3N \times 3N$ $\mathbf{Y}(T)$ is called fundamental matrix. It is noted in equation (4.1) $\mathbf{P}(t)$ is a periodic matrix with same period of $\mathbf{A}(t)$, $\frac{2\pi}{\Omega}$, so $\mathbf{P}(T) = \mathbf{P}(o)$, and it is obtained that :

$$\mathbf{Y}(T) = \mathbf{P}(o) e^{\mathbf{B}T} \mathbf{P}^{-1}(o) \quad (4.6)$$

By use of the similarity transform and the result of Jordan normal form, from the above equation, the eigenvalues of fundamental matrix $\mathbf{Y}(T)$ will equal to the eigenvalues of the matrix $e^{\mathbf{B}T}$; that is $\sigma_j = e^{\lambda_j T}$ $j = 1, 2, 3, \dots, 3N$ where the σ_j are eigenvalues of $\mathbf{Y}(T)$ and λ_j are eigenvalues of $\mathbf{B}$. Therefore by use of this relationship, the eigenvalues of $\mathbf{B}$ is able to be obtained once the fundamental matrix is obtained.

From the above interpretations, by use of the Floquet theory (equation 4.1), the stability of the dynamic system (equation 3.4) is determined by the eigenvalues of $\mathbf{B}$. As it's mentioned in the above paragraphs, eventhough there is no way

to calculate $\mathbf{B}$, it is of no need to know $\mathbf{B}$ in order to obtain its eigenvalues. Based on the similarity transform, these eigenvalues now can be found by knowing of the the fundamental matix $\mathbf{Y}(T)$. The methods of calculating the foundamnetal matrix $\mathbf{Y}(T)$ are generally the 4th-order Runge-Kutta, high-order Adams method, , or by the Chebyshev expansion method (Sinha and Wu, 1991). The numerical methods and the difference of efficiency between these methods will be explained in the next section.

Chapter 5

NUMERICAL METHODS

5.1 Time-Integration Methods

As mentioned in the previous section, calculating the fundamental matrix $\mathbf{Y}(T)$ is the way to obtain the eigenvalues of $\mathbf{B}$ to determine the stability of the dynamic system (equation 3.4) without need of calculating $\mathbf{B}$ directly. The methods of calculating $\mathbf{Y}(T)$ are divided into two kinds : (1) time-integration method, (2) Chebyshev-expansion method. The first one has been used for last decades, and the Chebyshev method developed by Sinha and Wu (1991) was first applied to fluid dynamics by Chen and Chen (1999) in the investigations of the effect of the gravity modulation on the onset of the convection instability in an infinite vertical slot. In the research of Chen and Chen (1999) and this dissertation, it has been experienced that the Chebyshev method is much more efficient than the conventional time-integration method. The time-integration method will be introduced in this section and the Chebyshev method in the next.

As mentioned in the previous section, it's convenient to choose the $3N$ columns of the $3N \times 3N$ identity matrix to be the $3N$ initial conditions, denoted as $[\tilde{\mathbf{y}}_1(0), \tilde{\mathbf{y}}_2(0), \dots, \tilde{\mathbf{y}}_{3N}(0)]$, where the $\tilde{\mathbf{y}}_j(0)$ is $3N \times 1$ vector with only nonzero element 1 on the j_{th} row. Let the $\tilde{\mathbf{y}}_j(T)$ be the solution of equation (3.4) corresponding to the initial condition $\tilde{\mathbf{y}}_j(0)$ at time $t = \frac{2\pi}{\Omega}$, so

$$\tilde{\mathbf{y}}_j(T) = \tilde{\mathbf{y}}_j(0) + \int_0^T \mathbf{A}(t) \tilde{\mathbf{y}}_j(t) dt \quad (5.1)$$

To calculate the integral, based on the numerical scheme, divide the period T into M subintervals, $M = 500, 1000, \text{ or } 10000 \dots$, let ΔT be equal to $\frac{T}{M}$, and $\vec{y}_j^{(n)}$ be the numerically computed value at $t = n \Delta T$, so $\vec{y}_j^{(o)}$ is exactly the true value of $\vec{y}_j(0)$, therefore,

$$\begin{aligned}
 \vec{y}_j^{(1)} &= \vec{y}_j^{(0)} + \Gamma(\vec{y}_j^{(0)}, \mathbf{A}, \Delta T) \\
 \vec{y}_j^{(2)} &= \vec{y}_j^{(1)} + \Gamma(\vec{y}_j^{(0)}, \vec{y}_j^{(1)}, \mathbf{A}, \Delta T) \\
 &\dots \\
 &\vdots \\
 \vec{y}_j^{(M)} &= \vec{y}_j^{(M-1)} + \Gamma(\vec{y}_j^{(0)}, \vec{y}_j^{(1)}, \dots, \vec{y}_j^{(M-1)}, \mathbf{A}, \Delta T)
 \end{aligned} \tag{5.2}$$

where the function Γ is dependent on the numerical method, such as the Runge-Kutta method, or the advanced Adams method. According to the scheme shown in equation (5.2), the following factors will affect the efficiency of computing the fundamental matrix $\mathbf{Y}(T)$.

1. The dimension N

N is the number of the spectral functions in the expansions (equations 3.1, 3.2, 3.3), and it is dependent on the problem. As mentioned before, it needs

to calculate $3N$ solutions of the equation (3.4) to consist of the $3N$ columns of the fundamental matrix $\mathbf{Y}(T) = [\vec{y}_1(T), \vec{y}_2(T), \dots, \vec{y}_{3N}(T)]$. These solutions are all $3N \times 1$ vectors, based on the $3N$ initial conditions which are exactly the $3N$ columns of identity matrix. Therefore, more the N terms of the spectral functions in the spectral expansion, higher the dimension of columns of the fundamental matrix, so the time consumed in the computation will be higher.

2. Period T

In the scheme shown in (5.2), larger the time period T of the gravity modulation, higher M is required, so the time consumed in the calculation is more.

3. Numerical scheme

For higher accuracy of calculation in each subinterval ΔT , the higher-order numerical scheme is used in the Γ and it consumes more time than the lower-order method.

5.2 Chebyshev Expansion Method

The Chebyshev method developed by Sinha and Wu (1991) was first applied to fluid dynamics by Chen and Chen (1999). In this dissertation, it is used to investigate the effect of the gravity modulation on the instability onset of the thermosolutal convection with higher efficiency than the time-integration method. The Chebyshev method is developed to solve the solutions of the time-periodic dynamic system of equation (3.4). When it is applied to calculate the $3N \times 3N$ fundamental matrix, there is no need to calculate the $3N$ $\vec{y}_j$ one by one and step by step on each time subinterval. Let the time scale t is normalized by $T = \frac{2\pi}{\Omega}$, $t^* \equiv \frac{t}{T}$, so by a change of variable, equation (3.4) will become

$$\frac{d}{dt^*} \bar{\mathbf{y}} = T \mathbf{A}(t^*) \bar{\mathbf{y}}(t^*) \quad (5.3)$$

where the $\mathbf{A}(t^*)$ is a time-periodic matrix with period of 1. This can be seen in the equation (3.4) where the term $\cos(\Omega t)$ becomes $\cos(2\pi t^*)$ after t is normalized by T so it make $\mathbf{A}(t^*)$ periodic with period of 1. Therefore the fundamental matrix is changed as $\mathbf{Y}(1) = [\bar{\mathbf{y}}_1(1), \bar{\mathbf{y}}_2(1), \dots, \bar{\mathbf{y}}_{3N}(1)]$. Each $\bar{\mathbf{y}}_j(1)$ is the solution of equation (5.3) at $t^* = 1$ corresponding to the initial condition which is the j_{th} column of the $3N \times 3N$ identity matrix. It is solved by

$$\bar{\mathbf{y}}_j(1) = \bar{\mathbf{y}}_j(0) + T \int_0^1 \mathbf{A}(t^*) \bar{\mathbf{y}}(t^*) dt^* \quad (5.4)$$

5.2.1 Properties of the Shifted Chebyshev Polynomials of First Kind

Define the shifted Chebyshev polynomials of first kind (Luke, 1969) as $T^*(t^*)$, $0 \leq t^* \leq 1$, so the orthogonality is :

$$\int_0^1 T_r^*(t^*) T_k^*(t^*) w(t^*) dt^* = \begin{cases} 0 & \text{when } r \neq k \\ \pi/2 & \text{when } r = k \neq 0 \\ \pi & \text{when } r = k = 0 \end{cases} \quad (5.5)$$

where the weighting function $w(t^*)$ is equal to $(t^* - t^{*2})^{-1/2}$.

The integration properties of the shifted Chebyshev polynomials of first kind are :

$$\int_0^{t^*} T_r^*(s)ds = \frac{1}{4} \left[\frac{T_{r+1}^*(t^*)}{r+1} - \frac{T_{r-1}^*(t^*)}{r-1} \right] - \frac{(-1)^r}{2(r^2-1)}, \quad r = 0, 2, 3, 4, \dots \quad (5.6)$$

$$\int_0^{t^*} T_1^*(s)ds = (T_2^*(t^*) - T_0^*(t^*)) / 8 \quad (5.7)$$

where $0 \leq t^* \leq 1$

The above integrations are the key advantage of applying this Chebyshev method to calculate the fundamental matrix in place of step-by-step integration one because the closed-form formula in equation (5.6) and (5.7) will give the integration value faster without need of step by step calculation over the time period. The equation (5.6 and 5.7) can also be written as :

$$\int_0^{t^*} \{ \mathbf{T}^*(s) \} ds = \{ \mathbf{T}^*(t^*) \} \mathbf{G}^T \quad (5.8)$$

where the $\mathbf{G}^T$ is the transpose of constant matrix $\mathbf{G}$ constructed by using equations (5.6), and (5.7). It is noted that in Sinha and Wu (1991), the term $T_m^*(t^*)/4m$ is inadvertently dropped in the expression of $\mathbf{G}$. The $\{ \mathbf{T}^*(t^*) \}$ is defined as row vector $\{ T_0^*(t^*), T_1^*(t^*), \dots, T_{m-1}^*(t^*) \}$, so the transpose of the $\{ \mathbf{T}^*(t^*) \}$ is $\{ \mathbf{T}^*(t^*) \}^T$, a column vector :

$$\{ \mathbf{T}^*(t^*) \}^T = \begin{Bmatrix} T_0^*(t^*) \\ T_1^*(t^*) \\ \vdots \\ T_{m-1}^*(t^*) \end{Bmatrix} \quad (5.9)$$

$\mathbf{G}$ is defined by the following constant matrix (more clearly described than in the Sinha and Wu (1991))

$$\mathbf{G} = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 & \dots & \dots & 0 & 0 \\ -1/8 & 0 & 1/8 & 0 & 0 & \dots & \dots & 0 & 0 \\ -1/6 & -1/4 & 0 & 1/12 & 0 & \dots & \dots & 0 & 0 \\ 1/16 & 0 & -1/8 & 0 & \dots & \dots & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \dots & & & & & \\ \frac{-(-1)^{m-1}}{2(m-1)(m-3)} & 0 & 0 & \dots & \dots & 0 & \frac{-1}{4(m-3)} & 0 & \frac{1}{4(m-1)} \\ \frac{-(-1)^{m-1}}{2m(m-2)} & 0 & 0 & \dots & \dots & 0 & 0 & \frac{-1}{4(m-2)} & 0 \end{bmatrix} \quad (5.10)$$

By use of the multiplication formula :

$$T_r^*(t^*)T_k^*(t^*) = \frac{1}{2}(T_{r+k}^*(t^*) + T_{|r-k|}^*(t^*)), \quad r, k = 0, 1, 2, 3, \dots \quad (5.11)$$

the multiplication of $\{\mathbf{T}^*(t^*)\}^T$ and $\{\mathbf{T}^*(t^*)\}$ will be equal to:

$$\begin{aligned}
\{ \mathbf{T}^*(t^*) \}^T \{ \mathbf{T}^*(t^*) \} &= \begin{Bmatrix} T_0^*(t^*) \\ T_1^*(t^*) \\ \vdots \\ T_{m-1}^*(t^*) \end{Bmatrix} \{ T_0^*(t^*), T_1^*(t^*), \dots, T_{m-1}^*(t^*) \} \\
&= \begin{bmatrix} T_0^* & T_1^* & T_2^* & \cdots & T_{m-1}^* \\ T_1^* & \frac{T_0^*+T_2^*}{2} & \frac{T_1^*+T_3^*}{2} & \cdots & \cdots \\ T_2^* & \frac{T_1^*+T_3^*}{2} & \frac{T_0^*+T_4^*}{2} & \cdots & \cdots \\ \vdots & \cdots & \cdots & \cdots & \cdots \\ T_{m-1}^* & \cdots & \cdots & \cdots & \frac{T_0^*+T_{2m-2}^*}{2} \end{bmatrix} \quad (5.12)
\end{aligned}$$

In Sinha and Wu (1991), there is a mistake in this formula. Let any two scalar functions $R(t^*)$ and $z(t^*)$ be represented by series expansions in the shifted Chebyshev polynomials.:

$$\begin{aligned}
R(t^*) &= \sum_0^{n-1} a_r T_r^*(t^*) \\
&= \{ a_0, a_1, \dots, a_{m-1} \} \begin{Bmatrix} T_0^*(t^*) \\ T_1^*(t^*) \\ \vdots \\ T_{m-1}^*(t^*) \end{Bmatrix} \\
&\equiv \{ \mathbf{a} \} \{ \mathbf{T}^*(t^*) \}^T \quad (5.13)
\end{aligned}$$

$$\begin{aligned}
z(t^*) &= \sum_0^{m-1} b_r T_r^*(t^*) \\
&= \{ b_0, b_1, \dots, b_{m-1} \} \begin{Bmatrix} T_0^*(t^*) \\ T_1^*(t^*) \\ \vdots \\ T_{m-1}^*(t^*) \end{Bmatrix} \\
&\equiv \{ \mathbf{b} \} \{ \mathbf{T}^*(t^*) \}^T \quad (5.14)
\end{aligned}$$

the coefficients a_r, b_r are constant and the $\{\mathbf{a}\}$ and $\{\mathbf{b}\}$ define the row vectors $\{a_0, a_1, \dots, a_{m-1}\}$ and $\{b_0, b_1, \dots, b_{m-1}\}$, so the transpose of $\{\mathbf{a}\}$ $\{\mathbf{b}\}$ are :

$$\{\mathbf{a}\}^T = \begin{Bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{m-1} \end{Bmatrix}$$

$$\{\mathbf{b}\}^T = \begin{Bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{m-1} \end{Bmatrix}$$

Because the $T_0^*(t^*) = 1$ for all t^* , any constant, for example the initial condition of $z(t^*)$, $z(0)$, can be also represented as above,

$$\begin{aligned} z(0) &= \sum_0^{m-1} c_r T_r^*(t^*) \\ &= \{1, 0, 0, \dots, 0\} \begin{Bmatrix} T_0^*(t^*) \\ T_1^*(t^*) \\ \vdots \\ T_{m-1}^*(t^*) \end{Bmatrix} \end{aligned} \quad (5.15)$$

The multiplication of $\mathbf{R}$ and $\mathbf{z}$ can be written as:

$$\begin{aligned} R(t^*)z(t^*) &= \{a_0, a_1, \dots, a_{m-1}\} \{\mathbf{T}^*(t^*)\}^T \{\mathbf{T}^*(t^*)\} \{b_0, b_1, \dots, b_{m-1}\}^T \\ &= \{\mathbf{a}\} \{\mathbf{T}^*(t^*)\}^T \{\mathbf{T}^*(t^*)\} \{\mathbf{b}\}^T \end{aligned} \quad (5.16)$$

By some rearrangements and the use of recursion relation, equations (5.12), the above formula is reduced to:

$$\begin{aligned} R(t^*)z(t^*) &= \{ \mathbf{T}^*(t^*) \} \mathbf{Q} \{ b_0, b_1, \dots, b_{m-1} \}^T \\ &= \{ \mathbf{T}^*(t^*) \} \mathbf{Q} \{ \mathbf{b} \}^T \end{aligned} \quad (5.17)$$

Where the $\mathbf{Q}$ is a constant matrix which consists of coefficients a_r .

5.2.2 Scheme of Applying Chebyshev Expansion Method to Solve the Differential Equation

In the following, an example of using the Chebyshev expansion method to solve the differential equation for a scalar function is shown. The system of differential equations with dimension of $3N \times 3N$ such as in the present dissertation, is solved in the similar way. For given two scalar functions where $R(t^*)$ is in time-explicit form and $z(t^*)$ is unknown.

$$\frac{d}{dt^*} z(t^*) = R(t^*) z(t^*) \quad (5.18)$$

$$z(t^*) = z(o) + \int_0^{t^*} R(s)z(s)ds \quad (5.19)$$

By using the equations (5.14, 5.15, 5.17), the above equation will be equal to :

$$\begin{aligned}
 \{\mathbf{T}^*(t^*)\}\{\mathbf{b}\}^T &= \{\mathbf{T}^*(t^*)\}\begin{Bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{Bmatrix} + \int_0^{t^*} \{\mathbf{T}^*(s)\}\mathbf{Q}\{\mathbf{b}\}^T ds \\
 &= \{\mathbf{T}^*(t^*)\}\begin{Bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{Bmatrix} + \int_0^{t^*} \{\mathbf{T}^*(s)\}ds\mathbf{Q}\{\mathbf{b}\}^T \\
 &= \{\mathbf{T}^*(t^*)\}\begin{Bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{Bmatrix} + \{\mathbf{T}^*(t^*)\}\mathbf{G}^T\mathbf{Q}\{\mathbf{b}\}^T \quad (5.20)
 \end{aligned}$$

Using identity matrix, it becomes

$$\{\mathbf{T}^*(t^*)\}\mathbf{I}\{\mathbf{b}\}^T = \{\mathbf{T}^*(t^*)\}\begin{Bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{Bmatrix} + \{\mathbf{T}^*(t^*)\}\mathbf{G}^T\mathbf{Q}\{\mathbf{b}\}^T \quad (5.21)$$

where the $\mathbf{I}$ is the identity matrix whose dimension is $m \times m$.

Arrange LHS and RHS of the above equation, so

$$\{\mathbf{T}^*(t^*)\}(\mathbf{I} - \mathbf{G}^T \mathbf{Q})\{\mathbf{b}\}^T = \{\mathbf{T}^*(t^*)\} \begin{Bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{Bmatrix} \quad (5.22)$$

Because the $\{\mathbf{T}^*(t^*)\}$ is function of time t^* , by use of the orthogonality, the equation to solve the unknown vector $\{\mathbf{b}\}^T$ is obtained as follows :

$$(\mathbf{I} - \mathbf{G}^T \mathbf{Q})\{\mathbf{b}\}^T = \begin{Bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{Bmatrix} \quad (5.23)$$

so,

$$\{\mathbf{b}\}^T = (\mathbf{I} - \mathbf{G}^T \mathbf{Q})^{-1} \begin{Bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{Bmatrix} \quad (5.24)$$

If the inverse matrix exists, then the $\{\mathbf{b}\}^T$ is obtained. The Chebyshev expansion method can be applied to any good function $R(t^*)$ by using the equation (5.5) to evaluate the expansion coefficients a_r for R (Eq.5.13). Meanwhile, once the expansion coefficients b_r in equation (5.14), are solved, then z is obtained for time $0 \leq t^* \leq 1$, but not only $z(1)$. Similarly, the above scheme can be applied to equation (5.4) to solve the $\tilde{y}_j(1)$. (Actually, one can solve $\tilde{y}_j(t^*)$ at any t^* when the coefficients are obtained)

Chapter 6

RESULTS

As mentioned in the previous sections, the stability of the dynamic system, equation (3.4), is determined by the real part of the eigenvalues of matrix $\mathbf{B}$. The eigenvalues of $\mathbf{B}$ are λ_j , with $j = 1, 2, 3, \dots, 3N$. Let $\hat{\lambda}$ be the eigenvalue with the maximum real part compared with other λ_j . The $\hat{\lambda}$ could be complex with conjugate denoted by $\hat{\lambda}^*$. Let $\hat{\lambda} = \alpha + i\beta$, and $\hat{\lambda}^* = \alpha - i\beta$. The system will be stable if α is negative and unstable if positive. There are 3 modes of instability onset ($\alpha = 0$), which are the synchronous mode when β is zero, the subharmonic mode when $\beta = \Omega/2$ where Ω is the modulation frequency, and the quasi-periodic mode for other cases (Joseph, 1976).

6.1 $Pr = 0.01$

In this section to be discussed are the results of steady-g and gravity modulation cases for a fluid of $Pr = 0.01$, $r = 3 \times 10^{-4}$, $R_c = -3200$. Negative R_c means the concentration gradient stabilizes the layer because of heavier concentration fluid at the bottom. These parameters are typical of liquid metal.

6.1.1 Steady Gravity

If h is zero, then equation (2.25) becomes the stability equation for the steady-g case. The stability analysis is carried out by using 6 terms in the Galerkin expansion when converged results are obtained. The threshold thermal Rayleigh number R_T for each wave number k is determined by the eigenvalues which are calculated by solving the algebraic eigen-equation without need of numerical time-integration method or Chebyshev expansion method. The results are shown in figure (6.1) in which the $k - R_T$ curve of the marginal onset is presented. The curve displays the thermal Rayleigh number R_T at which the system becomes unstable for a given mode of perturbation of the wave number k . Heating from below is characterized by positive R_T which destabilizes the layer because the heating lessens the fluid density near the bottom. In this figure, in comparison with other modes, it's found that the mode at $k = 3.36$ onsets at the minimum R_T of 2183. This special k is the critical mode, denoted by k_{cr} , and the R_T corresponding to the k_{cr} is denoted by $R_{T_{cr}}$. The $R_{T_{cr}}$ in this steady-g case is 2183. It is seen that R_T increases very sharply beyond 6×10^4 when k is less than 0.3 and larger than 15.

FIGURE 6.1. Marginal stability curve for steady gravity. $P_r = 0.01$, $r = 3 \times 10^{-4}$, and $R_c = -3200$

For the steady-g case, the eigenvalues at onset are purely imaginary with conjugate for each mode, so the instability is in the oscillatory mode. Shown in figure (6.2) is the $k - \omega$ curve, which is known as the dispersion curve. ω is the oscillating frequency for each mode in the figure (6.1). At $k = k_{cr} = 3.36$, the $\omega(k_{cr})$ is 3.96. This particular ω is defined as the natural frequency, denoted by ω_{cr} . The ω_{cr} in this steady-g case is equal to 3.96. ω is a function of k and its range is $0.47 \leq \omega \leq 5.5$ for $0.3 \leq k \leq 15$.

FIGURE 6.2. Dispersion curve, $k - \omega$, for steady gravity, $Pr = 0.01$, $r = 3 \times 10^{-4}$, and $R_c = -3200$

6.1.2 Gravity Modulation

As seen from the stability results at steady g , there is a frequency of oscillation associated with each mode of the perturbation. It is then reasonable to expect that resonant phenomena would occur when gravity is modulated. In this section, the effect of gravity modulation is studied systematically to examine the possible effects of resonant interactions. Three amplitudes of gravity modulation are investigated; $h = \frac{g_1}{g_0} = 0.01, 0.1, \text{ and } 0.2$. For each h , the following eight frequencies of gravity modulation Ω usually set at the harmonics of ω_{cr} are studied: $3.96(=\omega_{cr})$, $7.92(=2\omega_{cr})$, 9.0 , 10.0 , $11.88(=3\omega_{cr})$, $15.84(=4\omega_{cr})$, $19.80(=5\omega_{cr})$ and $23.76(=6\omega_{cr})$. For each set of given h and Ω , marginal stability curve and the dispersion curve are obtained. The range of the wave number k of the perturbation is $1 < k < 9$.

6.1.2.1 $h = 0.01$

Shown in figures (6.3) and (6.4) are the $k - R_T$ marginal curves for the eight different values of Ω at $h = 0.01$. It is noted that the solid line is the $k - R_T$ marginal curve in steady- g case, figure (6.1). The general response of the system to the gravity modulation is that the instability onsets in almost all cases with the quasi-periodic mode which is denoted by $+$ sign in these two figures. The onset R_T of quasi-periodic mode is almost the same as in the steady- g case. But the oscillation frequency for each quasi-periodic mode is different from the ω in steady g case. In the cases of $\Omega = 3.96, 7.92, 9.00$ and 10.00 in figure (6.3a), the subharmonic modes are denoted by the symbol $\bullet$. The subharmonic mode is defined when the value of β in the Floquet solution is equal to $\Omega / 2$. The R_T of subharmonic mode is slightly lower than that in steady- g case because the amplitude of gravity modulation is not

large enough to destabilize the system. As the Ω increases, the values of k for the subharmonic modes are also increased, and the decrease of R_T from that of steady-g case is much more discernible. But there is no subharmonic mode observed when Ω is larger than 11 as in cases of $\Omega = 11.88, 15.84, 19.80, 23.76$ in figure (6.4). In the case of $\Omega = 3.96$, synchronous modes are observed and denoted by the symbol $\blacktriangle$ where the β is equal to Ω . There is no synchronous mode when the Ω is larger than 5.5 within the k range studied. In the next paragraph based on the results in figure (6.5) and (6.6), it will be explained why the synchronous modes are observed in $\Omega = 3.96$ rather than in Ω larger than 5.5, and the subharmonic mode in $\Omega = 3.96, 7.92, 9.00$ and 10.00 when $0.3 \leq k \leq 15$.

FIGURE 6.3. Marginal stability curves for gravity modulation at $h = 0.01$ for $\Omega = 3.96, 7.92, 9.00, 10.00$. + quasi-periodic mode, • subharmonic mode, ▲ synchronous mode

FIGURE 6.4. Marginal stability curves for gravity modulation at $h = 0.01$ for $\Omega = 11.88, 15.84, 19.80, 23.76$. + quasi-periodic mode, • subharmonic mode, ▲ synchronous mode

The dispersion curves of $k - \beta$ for the same h and values of Ω in the figures (6.3) and (6.4) are shown in figures (6.5) and (6.6) for $1 < k < 9$. The solid line is the dispersion curve of $k - \omega$ in the steady-g case, figure (6.2). On the solid line, the ω is in range of $0.47 \leq \omega \leq 5.5$ for $0.3 \leq k \leq 15$. In these eight cases of Ω it is seen that the β for quasi-periodic modes denoted by $+$ sign are same with the ω on the solid line. The ω is the oscillation frequency of the instability onset in the steady-g case. The β is the imaginary part of the eigenvalue $\hat{\lambda}$ as described in above. By definition it is equal to $\Omega / 2$ for subharmonic mode, Ω for synchronous mode, and other cases for quasi-periodic mode. In $0.3 \leq k \leq 15$, the subharmonic mode ($\beta = \Omega / 2$) will occur only when the $\Omega / 2$ is equal to some ω in the range of $0.47 \leq \omega \leq 5.5$. Therefore the subharmonic modes denoted by symbol $\bullet$ will occur in cases of Ω where the $\Omega / 2$ is in $(0.47, 5.5)$. For the cases of $\Omega = 3.96, 7.92, 9.00$, and 10.00 , $\Omega / 2$ is respectively $1.98, 3.96, 4.50, 5.00$. The subharmonic mode region can be observed in $1 < k < 9$ for these cases because their $\Omega / 2$ are within the ω range $1.5 < \omega < 5.5$ for $1 < k < 9$. There is no subharmonic-mode region observed in $1 < k < 9$ for the cases of $\Omega = 11.88, 15.84, 19.80, 23.76$ because their $\Omega / 2$ are out of the range $(1.5, 5.5)$. Similarly, by the definition, in the $0.3 \leq k \leq 15$, the synchronous mode will be found only when Ω is in the range of $0.47 \leq \omega \leq 5.5$. In these eight Ω of figures (6.5) and (6.6), for $0.3 \leq k \leq 15$, the synchronous mode denoted by $\blacktriangle$ is only found in the case of $\Omega = 3.96$, but it cannot be found in other values of Ω because their Ω are all larger than 5.5 .

FIGURE 6.5. $k - \beta$ curves for gravity modulation at $h = 0.01$ for $\Omega = 3.96, 7.92, 9.00, 10.00$, $+$ quasi-periodic mode, $\bullet$ subharmonic mode, $\blacktriangle$ synchronous mode

(a) $\Omega = 11.88$ (c) $\Omega = 19.80$ (b) $\Omega = 15.84$ (d) $\Omega = 23.76$

FIGURE 6.6. $k - \beta$ curves for gravity modulation at $h = 0.01$ for $\Omega = 11.88, 15.84, 19.80, 23.76$, + quasi-periodic mode, • subharmonic mode, ▲ synchronous mode

6.1.2.2 $h = 0.1$

Shown in figures (6.7) and (6.8) are the $k - R_T$ marginal curves for $h = 0.1$ at the same values of Ω as in figures (6.3) and (6.4) with the same symbols. The solid line is the $k - R_T$ marginal stability curve in steady-g case, the figure (6.1). In these two figures, it is noted that in almost all cases the instability onsets with the quasi-periodic mode denoted by + sign. The onset R_T of the quasi-periodic perturbation is not changed much by gravity modulation when $h = 0.1$, and is almost the same with that of the oscillatory mode in steady-g case as in the case of $h = 0.01$. Similarly to the case of $h = 0.01$, the subharmonic modes, denoted by symbol $\bullet$, occur in $\Omega = 3.96, 7.92, 9.00, 10.00$ but not in $\Omega = 11.88, 15.84, 19.80, 23.76$ for $0.3 \leq k \leq 15$. The subharmonic-mode region in the case of $h = 0.1$ is more enlarged than in $h = 0.01$. The onset R_T for subharmonic mode is reduced much more than those shown in figures (6.3) and (6.4). Define a symbol $R_{T_{mini-sub}}$ to be the minimum R_T in the subharmonic-mode region for a given Ω . $R_{T_{mini-sub}}$ for $\Omega = 3.96$ is 3869 which is higher than the $R_{T_{cr}} = 2183$ in the steady-g case. The value of $R_{T_{cr}} = 2183$ is also the minimum onset R_T of quasi-periodic mode. The $R_{T_{mini-sub}}$ for $\Omega = 7.92$ and 9.00 are 1310 and 1373, which become the $R_{T_{cr}}$ because they are more unstable than the R_T of the most unstable quasi-periodic mode, 2183. In $\Omega = 7.92$, and 9.00 at $h = 0.1$, their $R_{T_{cr}}$ are respectively 40% and 37% reduction from 2183, and the instability onsets with subharmonic mode. But the $R_{T_{mini-sub}}$ at $\Omega = 10.00$ is higher than 2183, so the instability onsets with quasi-periodic mode. As the Ω is increased the wave number range of the subharmonic-mode region is increased. When Ω is over than 11, these instability modes cannot be observed in the range of $0.3 \leq k \leq 15$. Similarly, define $R_{T_{mini-syn}}$ to be the minimum R_T in the synchronous-mode region which is decreased from 2182 at $h = 0.01$ to 2140 at $h = 0.1$, but the k of the onset

synchronous mode at $R_{T_{mini-syn}}$ is not changed. The $R_{T_{cr}}$ for $\Omega = 3.96$ at $h = 0.1$ is 2140 in place of 2183 with reduction of only 2% from 2183. However, the instability onsets with synchronous mode in $\Omega = 3.96$.

FIGURE 6.7. Marginal stability curves for gravity modulation at $h = 0.1$ for $\Omega = 3.96, 7.92, 9.00, 10.00$, + quasi-periodic mode, • subharmonic mode, ▲ synchronous mode

FIGURE 6.8. Marginal stability curves for gravity modulation at $h = 0.1$ for $\Omega = 11.88, 15.84, 19.80, 23.76$, + quasi-periodic mode, • subharmonic mode, ▲ synchronous mode

The dispersion curve $k - \beta$ for the same h and same values of Ω in figures (6.7) and (6.8) are shown in figures (6.9) and (6.10) using the same symbols as before. The solid line is the dispersion curve of $k - \omega$ in steady-g case, the figure (6.2). The β of quasi-periodic modes denoted by $+$ sign are almost the same as the ω along the solid line. In figure (6.9), it is clearly shown the increase of the subharmonic region as Ω is increased. At the same k , outside of subharmonic-mode region (denoted by $\bullet$) and synchronous-mode region (denoted by $\blacktriangle$), both β and onset R_T of quasi-periodic modes are equal to ω and onset R_T of the instability mode in steady-g case. Referred to the explanation in figures (6.5) and (6.6), in these eight Ω , for $0.3 \leq k \leq 15$, the subharmonic-mode region occurs in $\Omega = 3.96, 7.92, 9.00, 10.00$ but not in $11.88, 15.84, 19.80, 23.76$. The synchronous-mode only occurs in case of $\Omega = 3.96$ because the Ω is in the range of $(0.47, 5.5)$. The subharmonic-mode region is enlarged than in $h = 0.01$.

(a) $\Omega = 3.96$ (c) $\Omega = 9.00$ (b) $\Omega = 7.92$ (d) $\Omega = 10.00$

FIGURE 6.9. $k - \beta$ curves for gravity modulation at $h = 0.1$ for $\Omega = 3.96, 7.92, 9.00, 10.00$, + quasi-periodic mode, • subharmonic mode, ▲ synchronous mode

FIGURE 6.10. $k - \beta$ curves for gravity modulation at $h = 0.1$ for $\Omega = 11.88, 15.84, 19.80, 23.76$, + quasi-periodic mode, • subharmonic mode, ▲ synchronous mode

6.1.2.3 $h = 0.2$

The marginal stability curves $k - R_T$ in $h = 0.2$ for the same values of Ω in figures (6.7) and (6.8) with same symbols are shown in figures (6.11) and (6.12). As before, the solid line is the marginal stability curve $k - R_T$ for steady-g case in figure (6.1). The quasi-periodic modes denoted by + sign onset at same R_T with the oscillatory mode in steady-g case as well as the cases $h = 0.01$, and 0.1 . Similar to the above figures, for $0.3 \leq k \leq 15$, the subharmonic modes denoted by symbol $\bullet$ occur in $\Omega = 3.96, 7.92, 9.00, 10.00$ but not in $11.88, 15.84, 19.80, 23.76$. The subharmonic-mode region at $h = 0.2$ is much more discernible than at $h = 0.1$, and is enlarged when Ω is increased from 3.96 to 10.00 . The $R_{T_{mini-sub}}$ at $\Omega = 7.92, 9.00$ at $h = 0.2$ are reduced to $439, 297, 80\%$ and 86% from 2183 , larger than the reduction of 40% and 37% at $h = 0.1$. In $\Omega = 10$, the dominating mode of the instability onset is switched from the quasi-periodic mode to subharmonic mode because the $R_{T_{mini-sub}}$ is reduced from 2488 at $h = 0.1$ to 842 at $h = 0.2$, lower than 2183 . As before, the subharmonic mode is not found in $0.3 \leq k \leq 15$ for Ω larger than 11 . The synchronous-mode region is found only in the case of $\Omega = 3.96$ for $0.3 \leq k \leq 15$, but not in other seven cases of Ω because their Ω are beyond the range of $(0.47, 5.5)$. The $R_{T_{mini-syn}}$ is decreased from 2140 at $h = 0.1$ to 2014 at $h = 0.2$, 7% reduction from 2183 , so the instability onset is dominated by synchronous mode. But the k associated with the $R_{T_{mini-syn}}$ is not changed. Let $k_{mini-sub}$ be defined as the k associated with the $R_{T_{mini-sub}}$. In figure (6.13), it shows the $k_{mini-sub}$ vs Ω at (6.13a) $h = 0.01$, (6.13b) $h = 0.1$, and (6.13c) $h = 0.2$, in use of symbol $*$ for $\Omega = 3.96$, $\bullet$ for $\Omega = 7.92$, $\blacksquare$ for $\Omega = 9.00$, and $\blacktriangle$ for $\Omega = 10.00$. The $k_{mini-sub}$ is also shown in the following table. It can be seen that the $k_{mini-sub}$ is not changed much as h is increased from 0.01 to 0.2 .

TABLE 6.1. $k_{mini-sub}$ vs Ω , for $h = 0.01, 0.1, 0.2$

h	$\Omega = 3.96$	7.92	9.00	10.00
0.01	1.31	3.33	4.43	6.42
0.1	1.31	3.33	4.43	6.15
0.2	1.31	3.33	4.43	6.06

FIGURE 6.11. Marginal stability curves for gravity modulation at $h = 0.2$ for $\Omega = 3.96, 7.92, 9.00, 10.00$, + quasi-periodic mode, • subharmonic mode, ▲ synchronous mode

FIGURE 6.12. Marginal stability curves for gravity modulation at $h = 0.2$, for $\Omega = 11.88, 15.84, 19.80, 23.76$, + quasi-periodic mode, $\bullet$ subharmonic mode, $\blacktriangle$ synchronous mode

(a) $h = 0.01$ (b) $h = 0.1$ (c) $h = 0.2$

FIGURE 6.13. $k_{mini-sub}$ vs Ω , using symbol * for $\Omega = 3.96$, • for 7.92 , ■ for 9.00 , ▲ for 10.00 , for (a) $h = 0.01$, (b) $h = 0.1$, and (c) $h = 0.2$

The $k - \beta$ curves for gravity modulation at $h = 0.2$ are shown in figures (6.14) and (6.15) with the same symbols as before. The β of quasi-periodic modes denoted by + sign are the same with the ω of the oscillatory mode of steady-g case, the solid line. The solid line is the $k - \omega$ dispersion curve of steady-g case, the figure(6.2). As explained before, it shows that the subharmonic-mode region is observed in cases of $\Omega = 3.96, 7.92, 9.00$ and 10.00 , and synchronous-mode region in case of 3.96 for $0.3 \leq k \leq 15$. The increase of the subharmonic region over the wavenumber region is clearly exhibited. The following table shows the k range of the subharmonic-mode region of $\Omega = 7.92, 9.00$, and 10.00 , for $h = 0.1$ and 0.2 . For example, the k range of the subharmonic-mode region for $\Omega = 10$ at $h = 0.2$ is from 5.34 to 8.60 .

TABLE 6.2. k range of subharmonic-mode region of $\Omega = 7.92, 9.00$, and 10.00 , for $h = 0.1$ and 0.2

Ω	$h = 0.1$	$h = 0.2$
7.92	3.3 - 3.5	3.08 - 3.72
9.00	4.3 - 4.7	3.98 - 5.24
10.00	5.9 - 7.3	5.34 - 8.60

(a) $\Omega = 3.96$ (c) $\Omega = 9.00$ (b) $\Omega = 7.92$ (d) $\Omega = 10.00$

FIGURE 6.14. $k - \beta$ curves for gravity modulation at $h = 0.2$ for $\Omega = 3.96, 7.92, 9.00, 10.00$, + quasi-periodic mode, • subharmonic mode, ▲ synchronous mode

(a) $\Omega = 11.88$ (c) $\Omega = 19.80$ (b) $\Omega = 15.84$ (d) $\Omega = 23.76$

FIGURE 6.15. $k - \beta$ curves for gravity modulation at $h = 0.2$ for $\Omega = 11.88, 15.84, 19.80, 23.76$, + quasi-periodic mode, • subharmonic mode, ▲ synchronous mode

6.1.2.4 Summary of $h = 0.01, 0.1, 0.2$

In the range of the wave number $0.3 \leq k \leq 15$, when h is ≤ 0.2 , there is no subharmonic mode for Ω larger than 11 because the half of Ω is larger than 5.5, and there is no synchronous mode for Ω larger than 5.5. The quasi-periodic mode always onsets at the same R_T with the same marginal curve as that of oscillatory modes in the steady-g case. The dominating mode of the instability onset for these h can be quasi-periodic, subharmonic, and synchronous mode. For $h \leq 0.2$, the dominating mode for Ω larger than 11 is only the quasi-periodic mode. For $\Omega/2$ in the range of (0.47, 5.5), as the modulation amplitude h is increased large enough, the dominant mode of the instability onset will switch from the quasi-periodic mode to the subharmonic mode. In $\Omega = 3.96$, the dominant mode is always synchronous mode unless the h is very large such that $R_{T_{mini-sub}}$ is largely reduced to be lower than the $R_{T_{mini-syn}}$.

Figure (6.16) shows the effect of the Ω on the $R_{T_{cr}}$ for the fluid of $Pr = 0.01$, $r = 3 \times 10^{-4}$, $R_c = -3200$ at three different modulation amplitudes $h = 0.01, 0.1, 0.2$. At a given h , it shows the $R_{T_{cr}}$ of the marginal stability curve for each Ω . At $h = 0.01, 0.1$, and 0.2 , when Ω is in the neighborhood of $2\omega_{cr}$, these Ω have the most effect on the $R_{T_{cr}}$. This special region is the resonant region. The ω_{cr} for this fluid is 3.96. Define the $R_{T_{cr-mini}}$ to be the minimum $R_{T_{cr}}$ of all Ω at a given h . This $R_{T_{cr-mini}}$ is a function of h , which is : 2094 occurring at $\Omega = 2\omega_{cr} = 7.92$ for $h = 0.01$, 1282 at $\Omega = 8.3$ for $h = 0.1$, 297 at $\Omega = 9.0$ for $h = 0.2$. The reduction from 2183, the critical thermal Rayleigh number in the steady-g case, for these three cases are respectively 4%, 41%, and 86%. It is noted that $R_{T_{cr}}$ would be asymptotic to 2183 for the limiting cases of Ω at each h . The value of 2183 is the $R_{T_{cr}}$ in steady-g case denoted by the dashed horizontal line. Since when the Ω and $\Omega/2$ are away from

(0.47 , 5.5), as explained before, only quasi-periodic mode occurs on the marginal stability curve and onsets at same R_T with the oscillatory perturbation in steady-g case at each k . Therefore, the asymptotic $R_{T_{cr}}$ for both of very large and closed-to-zero Ω is exactly the $R_{T_{cr}}$ for steady-g case. It has been shown quantitatively in the previous section that the time-periodic dynamic system equation (3.4) equates to a steady-g system at infinitely large and closed-to-zero Ω .

It is noted here that resonant interaction occurring in a horizontal diffusive-regime layer with rigid boundaries is quite different from that in a layer with free boundaries. For rigid boundaries it is found that the resonance only occurs in the neighborhood of $\Omega = 2\omega_{cr}$, while Saunders et al. (1992) found multiple resonant regions in a free-free layer as the gravity modulation is decreased.

FIGURE 6.16. Effect of Ω on the $R_{T_{cr}}$ for $h = 0.01, 0.1, 0.2$ for $Pr = 0.01$, $r = 3 \times 10^{-4}$, $R_c = -3200$

In a fixed Ω , increasing h will destabilize the subharmonic mode and thus decrease the $R_{T_{mini-sub}}$, so the subharmonic mode will dominate the instability if h is large enough. That is why the resonant region is formed. Since $R_{T_{mini-sub}}$ decreases with increasing h , it is expected that by increasing h one can reach a critical state of zero $R_{T_{cr}}$. This is indeed the case as shown in figure (6.17). For $h = 0.22737$, the instability onsets with subharmonic mode at zero $R_{T_{cr}}$ when $\Omega = 9.1$. This means that the instability can occur in a stable stratified horizontal layer of fluid by means of oscillation without heating from below. This is analogous to the case of a horizontal fluid layer being heated from above can be destabilized by oscillation as shown by Gresho and Sani (1970).

FIGURE 6.17. Effect of Ω on the $R_{T_{cr}}$ for $h = 0.22737$ for $Pr = 0.01$, $r = 3 \times 10^{-4}$, $R_c = -3200$

6.2 $Pr = 1$ and 7

It's of interest to extend the investigation on the effect of gravity modulation on the instability onset to fluids with $Pr = 1$ and 7. For $Pr = 1$, the other parameters are kept the same as those used previously. For $Pr = 7$, the parameter r is changed to a value appropriate for water-salt solution. The results are shown in figures (6.18) and (6.19). The resonant region occurs for these two different fluids with the same feature as in figure (6.16), the fluid of $Pr = 0.01$.

In the figure (6.18), for $Pr = 1$, $R_{T_{cr}}$ for steady-g case is 3818 shown as the horizontal dashed line and the ω_{cr} is 24.386. The resonant region is in the neighborhood of $2\omega_{cr} = 48.772$. The $R_{T_{cr-mini}}$ is 3657 occurring at $\Omega = 48.772$ for $h = 0.1$, 3179 at $\Omega = 52$ for $h = 0.4$, 2859 at $\Omega = 54$ for $h = 0.6$. The reduction from the 3818 for these three cases are 4%, 16%, and 25%. Slight stabilization can be obtained for some values of Ω . For example when $\Omega = \omega_{cr}/2 = 12.193$, the $R_{T_{cr}}$ is 3857 and 3863 for $h = 0.4$, and 0.6, and when $\Omega = 3\omega_{cr} = 73.158$, the $R_{T_{cr}}$ is 3829 and 3844 for $h = 0.4$ and 0.6. As explained before for $Pr = 0.01$, at the limiting cases of Ω , very large or closed-to-zero, the $R_{T_{cr}}$ is asymptotic to 3818 for each h . The stabilization by the gravity modulation is just like the investigation by Gresho and Sani (1970) analogous to the motion of pendulum with the pivot sinusoidally oscillating in vertical direction.

FIGURE 6.18. Effect of Ω on the $R_{T_{cr}}$ for $h = 0.1, 0.4, 0.6$ for $Pr = 1$, $r = 3 \times 10^{-4}$, $R_C = -3200$

For the case of $Pr = 7$, $r = 0.01$, and $R_c = -3200$, the effects of gravity modulation for $h = 0.1, 0.2, 0.4$ are shown in the figure (6.19). The applied Ω is from $\omega_{cr}/2$ to $5\omega_{cr}$ with $\omega_{cr} = 27.13$, the natural frequency of the perturbation in steady-g case. $R_{T_{cr}}$ for the steady-g case is 4758. The resonant region near $2\omega_{cr}$ is also found as in the previous figures. The $R_{T_{cr-mini}}$ is 4616 occurring at $\Omega = 2\omega_{cr} = 54.26$ for $h = 0.1$, 4478 at $\Omega = 2\omega_{cr}$ for $h = 0.2$, 4214 at $\Omega = 2\omega_{cr}$ for $h = 0.4$. The reduction are 3%, 6% and 11% from 4758 for these three cases. The resonant region is observed even though the fluid properties are changed, but the relative reduction of $R_{T_{cr-mini}}$ is smaller than the cases of smaller Pr , 0.01, and 0.1. The resonant region is therefore expected to be a general phenomena based on the results on figures (6.16), (6.18), (6.19). At $h = 0.4$ and $\Omega = 3\omega_{cr}$ and $4\omega_{cr}$ and $5\omega_{cr}$, the layer is stabilized very slightly by the gravity modulation, but no such stabilization effect is evident when $h = 0.1$ and 0.2 .

FIGURE 6.19. Effect of Ω on the $R_{T_{cr}}$ for $h = 0.1, 0.2, 0.4$, for $Pr = 7$, $r = 0.01$, $R_C = -3200$

Prandtl number Pr is the ratio of kinematic viscosity to thermal diffusivity, so higher Pr means relatively high viscosity. The effect of Pr on the resonant region at $h = 0.1$ is presented in figure(6.20) in which the normalized $R_{T_{cr}}$ is a function of the normalized modulation frequency. The normalized modulation frequency is defined by $\frac{\Omega}{\omega_{cr}}$ where the ω_{cr} is the natural frequency of the perturbation in steady-g case, equal to 3.96 for $Pr = 0.01$, and 24.386 for $Pr = 1.00$. The normalized $R_{T_{cr}}$ is defined as $R_{T_{cr}}(\text{ with gravity modulation})/R_{T_{cr}}(\text{steady g})$. The $R_{T_{cr}}(\text{steady-g}) = 2183$ for $Pr = 0.01$, and 3818 for $Pr = 1.00$. The figure shows the decrease of the normalized $R_{T_{cr}}$ at $h = 0.1$ is about 40% for $Pr = 0.01$, but only 5% for $Pr = 1.00$. It means that the viscosity is the important resistance to the destabilization caused by gravity modulation.

FIGURE 6.20. Normalized Ω - normalized $R_{T_{cr}}$ at $h = 0.1$, $r = 3 \times 10^{-4}$, $R_c = -3200$

Chapter 7

CONCLUSIONS

The following conclusions are reached.

1. When a horizontal layer of fluid stabilized by a concentration gradient is heated from below under steady gravity, the instability onsets in an oscillatory mode. The frequency of oscillation ω is a function of the wave number k of the perturbation. Within the range $0.3 \leq k \leq 15$, ω varies from $0.47 \leq \omega \leq 5.5$. When gravity contains a modulating component of frequency Ω , the oscillation frequency of the instability at onset will be changed into one of the following three modes. The instability may oscillate in the synchronous mode with frequency $= \Omega$, in the subharmonic mode with frequency $= \Omega / 2$, or in the quasi-periodic mode with frequency other than the previous two.
2. The imposed gravity modulation does not change the thermal Rayleigh number R_T at onset of instability in the quasi-periodic mode from the value for the steady gravity case. However, it reduces the value of onset R_T from that of steady-g case for the instability in the synchronous and subharmonic modes when h , the amplitude of gravity modulation, is increased.
3. For the subharmonic mode, the destabilization due to the resonant effect increases with increasing h . For a $Pr = 0.01$ fluid such effect can be quite dramatic. The reduction of the $R_{T_{cr}}$ from the steady-g value was found to be 4%, 41%, and 86% for $h = 0.01, 0.1$, and 0.2 . However, the critical wave number of subharmonic mode remains essentially the same.
4. When the layer is bound by rigid boundaries, resonance occurs only in a region

near $2\omega_{cr}$. This result is quite different from that for a layer with free boundaries in which multiple resonant interaction zones were found by Saunders et al. (1992) as the modulation frequency is decreased.

5. A horizontal fluid layer stabilized by a concentration gradient can be destabilized by gravity modulation alone without any heating from below. For $Pr = 0.01$ fluid example, such destabilization occurs at $h = 0.22737$. This is analogous to the results of Gresho and Sani (1970) that a fluid layer heated from above can be destabilized by gravity modulation.

6. The Chebyshev expansion method is more efficient than the Runge-Kutta integration method in calculating the fundamental matrix of the Floquet theory.

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