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TOPOLOGICAL AND GEOMETRICAL CONSIDERATIONS
FOR MAXWELL'S EQUATIONS ON UNSTRUCTURED
MESHES

by
Cynthia Christine Kaus

A Dissertation Submitted to the Faculty of the
DEPARTMENT OF MATHEMATICS
In Partial Fulfillment of the Requirements
For the Degree of
DOCTOR OF PHILOSOPHY
In the Graduate College
THE UNIVERSITY OF ARIZONA

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entitled TOPOLOGICAL AND GEOMETRICAL CONSIDERATIONS FOR MAXWELL'S
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SIGNED: Cynthia Kow

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TABLE OF CONTENTS

LIST OF FIGURES	7
LIST OF TABLES	8
ABSTRACT	9
CHAPTER 1. INTRODUCTION	11
1.1. Background of Problem	11
1.2. The Approach to the Problem	12
CHAPTER 2. DIFFERENTIAL FORMS - THE CONTINUUM	16
2.1. Differential Forms and the Exterior Derivative	17
2.2. The Metric	19
2.2.1. Volumes	21
2.2.2. The Hodge Star Operator	22
2.3. Stokes's Theorem	23
2.4. The Adjoint of the Exterior Derivative	24
2.5. The Laplacian	26
2.6. DeRham Cohomology and Hodge Theory	28
2.6.1. DeRham Cohomology	28
2.6.2. Hodge Theory for Differential Forms	29
CHAPTER 3. CHAINS AND COCHAINS - THE DISCRETE	31
3.1. Cellular Decomposition and Chains	31
3.2. The Boundary Operator	33
3.3. Cochains and the Metric	36
3.3.1. The Coboundary Operator	37
3.4. Hodge Star on Cochains	38
3.5. Cohomology	38
3.6. Relationship Between Cochains and Differential Forms	39
3.6.1. Extension of Stokes's Theorem to a Cell Complex	41
3.7. Combinatorial Laplacian	42
3.8. Hodge Theory for Chains and Cochains	43
CHAPTER 4. THE DUAL OF THE CELL COMPLEX	45
4.1. Obtaining the Dual Complex	45
4.2. Example on an Orthogonal Complex	47
4.3. Examples on Non-Orthogonal Complexes	50

TABLE OF CONTENTS—*Continued*

4.3.1. Parallelepiped Complex	51
4.3.2. Trapezoid Complex	54
4.3.3. Tetrahedral Complex	56
CHAPTER 5. CHAINS, COCHAINS AND MAXWELL'S EQUATIONS	58
5.1. The Inner Product and Integral Relations	58
5.2. Applications to Maxwell's Equations	59
5.2.1. Differential Forms and Maxwell's Equations	59
5.2.2. Chains, Cochairs and Maxwell's Equations	62
5.2.3. Divergence Free Properties	64
CHAPTER 6. ALGORITHMS	67
6.1. Discrete Differential Form Algorithm	67
6.2. Update Equations for Various Cell Complexes	75
6.2.1. Cubical Complex	75
6.2.2. Parallelepiped Complex with One Angle Skewed-TE Wave	79
6.2.3. Parallelepiped Cell Complex with Three Angles Skewed	82
6.2.4. Trapezoid Complex	87
6.3. Yee's Algorithm	91
6.4. Discrete Surface Integral Algorithm	93
CHAPTER 7. CONCLUSION	96
7.1. Further Work	96
REFERENCES	98

LIST OF FIGURES

FIGURE 3.1.	a. Orientation of a 1-cell. b. Orientation of a 2-cell.	33
FIGURE 3.2.	a. The Transpose of the Boundary Operator Applied to a 0-cell. b. The Transpose of the Boundary Operator Applied to a 1-cell.	35
FIGURE 4.1.	An Orthogonal Cell Complex and Its Dual.	48
FIGURE 4.2.	The Oriented Faces and Edges of a Cubical 3-Cell.	49
FIGURE 4.3.	A Parallelepiped Complex.	51
FIGURE 4.4.	A Parallelepiped Complex and Its Dual.	52
FIGURE 4.5.	A Trapezoidal Complex.	54
FIGURE 4.6.	A Dual Trapezoid Complex.	55
FIGURE 4.7.	Triangular Complex and Its Dual.	56
FIGURE 6.1.	A 2-cell Model for the Discrete Differential Form Algorithm. . .	68
FIGURE 6.2.	Parallelepiped Complex With One Angle Skewed.	69
FIGURE 6.3.	Cubical Complex for Solving Faraday's Law.	75
FIGURE 6.4.	Dual Cubical Complex for Solving Ampere's Law.	77
FIGURE 6.5.	Neighboring Cells in a Parallelepiped Cell Complex with One Angle Skewed.	79
FIGURE 6.6.	Three-Dimensional Parallelepiped Complex.	83
FIGURE 6.7.	Trapezoid Complex.	88
FIGURE 6.8.	Electric and Magnetic Field Components on a Unit Cell.	92
FIGURE 6.9.	Neighboring Cells in a Non-Orthogonal Complex.	94

LIST OF TABLES

TABLE 2.1. The Hodge Star Operator on Forms in $\mathbb{R}^3$ in Rectangular, Cartesian Coordinates.	23
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ABSTRACT

A discrete differential form approach to solving Maxwell's equations numerically on unstructured meshes is presented. A differential form representation of Maxwell's equations provides a natural and coordinate-free means of studying these equations and their solutions in the presence of curved objects. We begin by reviewing basic properties of differential forms and the operators associated with them for their use in describing electromagnetic fields and sources. Because we are interested in numerically solving Maxwell's equations on unstructured meshes, we introduce discrete representations of these differential forms and the underlying manifolds. This allows a discrete representation of Maxwell's equations in terms of chains and cochains on an arbitrary polyhedral cell complex. The discrete boundary operator, coboundary operator and star operator on cochains are constructed and shown to maintain divergence-free regions. The constructions of the dual of a polyhedral cell complex and the star operator, which give a one-to-one correspondence between the primary and the dual cell complexes, are introduced. This star operation gives the relationship between the magnetic field 1-cochain on the dual cell complex and the magnetic flux 2-cochain on the primary cell complex, and the relationship between the electric field 1-cochain on the primary cell and the electric flux 2-cochain on the dual cell complex. With the construction of these operators, the dual cell complex, and the associated cochains, we have determined the corresponding numerical update equations for the electromagnetic fields on unstructured meshes. The numerical update equations provided by the discrete differential form approach are determined explicitly for cubical, parallelepiped, tetrahedral, and trapezoid cell complexes. For the special cases of an orthogonal complex and a parallelepiped complex, these discrete differential form update equations recover those provided by both Yee's algorithm and the discrete surface integral (DSI) algorithm. It is demonstrated that the discrete differ-

ential form update equations differ from those obtained with the DSI approach on the more irregular trapezoid cell complex and, hence, may overcome the known late-time instabilities associated with the DSI approach applied to such highly unstructured meshes.

Chapter 1

INTRODUCTION

1.1 Background of Problem

There is a great deal of interest in computational electromagnetics (CEM) to develop methods for solving Maxwell's equations numerically on non-orthogonal, unstructured meshes. Unstructured meshes enable curved geometries to be approximated more accurately. K.S. Yee [32] first introduced the finite difference-time domain (FDTD) method for solving Maxwell's equations on orthogonal grids in 1966. The Yee algorithm uses central differencing on a staggered mesh to approximate the first order derivatives in Maxwell's equations and an explicit leap-frog time integrator. The mesh is called staggered because of the staggering of the electric and magnetic field components at half time and space steps. This algorithm is of second order accuracy when the computational domain does not have a curved boundary. In order to conform to curved geometries, the Yee algorithm uses a stair-stepped approach to approximate the boundary. This stair-stepping provides a good approximation if a fine enough discretization is used near the boundary. However, a fine discretization can be very costly in terms of computer memory and run time. Thus, the development of a numerical method for solving Maxwell's equations which allows for non-orthogonal, unstructured meshes became desirable.

Since the introduction of the Yee algorithm, there have been many attempts to develop algorithms based on non-orthogonal unstructured meshes. One approach, [5, 28], was to develop a discrete version of Maxwell's equations based on concepts from algebraic topology. Another approach, [30], also used concepts from algebraic topology, however the applications were for general relativity rather than electromagnetics. Although these methods were beneficial for introducing algebraic topology to

the electromagnetics community, explicit update equations for Maxwell's equations were not determined. Other methods for solving Maxwell's equations in the time-domain on unstructured, non-orthogonal meshes have also been developed. Some of these methods use the finite element approach [4, 19, 2] instead of finite differencing, and a few use concepts from differential geometry, specifically Whitney forms [4, 2].

Previous attempts by N.K. Madsen and R.W. Ziolkowski at developing algorithms for unstructured meshes [19, 18, 17] lead to the development of the discrete surface integral (DSI) algorithm in 1995 [16]. This algorithm allowed for the geometrical region to be discretized using convex polyhedral cells which share a common face and edge. Allowing for non-orthogonality and unstructuredness were useful for conforming to complex geometric shapes. The DSI algorithm, which is sometimes referred to as the generalized Yee algorithm, is based on solving the integral form of Maxwell's equations. This algorithm introduces the concept of a primary mesh and a dual mesh, where the electric field components lie on the edges of the primary mesh and the magnetic field components lie on the edges of the dual mesh. Upon examination, the primary and dual meshes are equivalent to the staggered mesh approach introduced by Yee. When an orthogonal mesh is chosen, the DSI algorithm reduces to the Yee algorithm. Although the DSI algorithm allows for arbitrary cell shapes, it has been shown that after a large number of time steps, this algorithm can become unstable [22, 23].

1.2 The Approach to the Problem

In order to develop an algorithm for solving Maxwell's equations numerically on unstructured grids, it is necessary to develop the algorithm in the most general and global manner possible. This is achieved for the continuum by considering Maxwell's equations on a differentiable manifold with the electric and magnetic fields being represented by differential forms [8]. In order to understand Maxwell's equations in

terms of differential forms, it was necessary to provide background material on the space of differential forms in Chapter 2. Included in this background material are the following operators: the exterior derivative, the Hodge star operator, the adjoint to the exterior derivative, and the Laplacian. The adjoint to the exterior derivative is determined explicitly in terms of the exterior derivative and the Hodge star operator. Also included in Chapter 2 are concepts from DeRham cohomology, and Hodge theory.

Once the continuous representations are understood, the mathematically correct discretizations of the manifold and differential forms can be determined. Several different interpretations of this discretization are given in [30, 28, 17, 5]. The discrete analog of the manifold can be represented by a polyhedral cellular decomposition, i.e. the faces are polyhedrals, and the discrete analog of the differential forms can be represented by cochains. In Chapter 3, these discrete representations are introduced. The relationship between cochains and differential forms through integration was developed to provide insight into the process of going from the continuum to the discrete. Also included in Chapter 3 is the development of various operators on the space of chains and cochains. Specifically, the boundary operator, the transpose of the boundary operator, the coboundary operator, a discrete version of the Hodge star operator on cochains, and the combinatorial Laplacian are developed. Instrumental in the application of Maxwell's equations to a cell complex in Chapter 5 are the relationships between the adjoint of the boundary operator applied to a cochain, the exterior derivative applied to the corresponding differential form, and the extension of Stokes's theorem to a cell complex. Also included in this chapter are concepts from cellular homology and cohomology, and Hodge theory for chains and cochains.

In Chapter 4, the dual complex associated with a polyhedral cell complex is introduced. This dual complex is determined by connecting the barycenters of adjacent n -cells, where n is the dimension of our manifold, in the primary (initial) complex and setting up a one-to-one correspondence between k -cells in the primary complex and $(n - k)$ -cells in the dual complex. The construction of the dual complex relies

on the definition of the one-to-one correspondence described above, a star operator, $\star : C_k(M, \mathbb{R}) \rightarrow \underline{C_{n-k}(M, \mathbb{R})}$, where the underline denotes the dual complex. Also included in this chapter is the relationship between the boundary operator on the primary complex and the transpose of the boundary operator on the dual complex.

In Chapter 5, the applications of the mathematics in the first three chapters is applied to Maxwell's equations. First, Maxwell's equations are written in terms of differential forms and differentiable manifolds. The electric and magnetic fields are represented by differential 1-forms on $\mathbb{R}^3$. Applying the Hodge star operator to the electric and magnetic field 1-forms, E and H , we obtain the electric and magnetic flux 2-forms on $\mathbb{R}^3$, D and B , respectively. Once the continuum version of Maxwell's equations is given, then we are able to write Maxwell's equations in terms of chains and cochains. In this case, the electric and magnetic fields are represented by 1-cochains generated by the differential forms E and H and the electric and magnetic fluxes are represented by 2-cochains generated by the differential forms D and B . In the last section of this chapter, it is shown that the representation of Maxwell's equations in terms of cochains and chains maintains their divergence-preserving properties.

From this discrete representation of Maxwell's equations on a cell complex, a discrete differential form algorithm for solving Maxwell's equations on unstructured meshes is developed in Chapter 6. The first steps in the algorithm involve choosing an appropriate cell decomposition, defining the dual complex, and determining the metric on both the primary and dual complexes. The second step of the algorithm involves writing Maxwell's equations in terms of the inner product. In the third and fourth steps, we invoke the divergence-preserving properties to define the boundary operators on the cell complex and to make numerical choices. In the final step, we write the update equations for the electric and magnetic fields in terms of our choices from step four. Thus, we have realized the corresponding numerical update equations on arbitrary grids. In Chapter 6, we also show that the discrete differential form algorithm reduces to the Yee algorithm on orthogonal grids and to the DSI algorithm

on parallelepiped grids. The update equations obtained from the discrete differential form algorithm reveal potential defects in previous algorithms, such as the discrete surface integral algorithm.

Chapter 2

DIFFERENTIAL FORMS - THE CONTINUUM

The laws of physics are described well using integrals. An integral consists of an integrand, which is called an exterior differential form of rank k or a k -form, and a domain of integration, which is a k -dimensional manifold with boundary which may be empty. For example in $\mathbb{R}^3$, the integrand in the line integral

$$\int_{\gamma} A dx + B dy + C dz,$$

where A , B , and C are functions on x, y, z , is the 1-form

$$\omega = A dx + B dy + C dz$$

where γ is a path in $\mathbb{R}^3$. Likewise, the integrand in the surface integral

$$\int_S A dy dz + B dz dx + C dx dy$$

is the 2-form

$$\omega = A dy dz + B dz dx + C dx dy,$$

where S is a surface in $\mathbb{R}^3$ and the integrand in the volume integral

$$\int_V A dx dy dz$$

is the 3-form

$$\omega = A dx dy dz,$$

where V is a volume in $\mathbb{R}^3$. The 0-forms are just the smooth functions on the manifold.

In this chapter, differential forms, their properties and their operators will be discussed.

2.1 Differential Forms and the Exterior Derivative

The differential forms discussed above were 0, 1, 2 and 3-forms on $\mathbb{R}^3$ in rectangular Cartesian coordinates. For an arbitrary n -dimensional manifold, M , a general k -form can be expressed in local coordinates $(x^1, x^2, \dots, x^n)$, as

$$\omega = \sum_{i_1 < \dots < i_k} \omega_{i_1 \dots i_k}(x) dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

where $1 \leq i_1 < \dots < i_k \leq n$, the coefficients $\omega_{i_1 \dots i_k}(x)$ are smooth functions of the variables $x^1, \dots, x^n$ and $dx^{i_1} \wedge \dots \wedge dx^{i_k}$ form a basis in the space of k -forms on M . In other words, the space of k -forms on M , $\Omega^k(M)$, is a module over the space of smooth functions. The space of differential forms is given by the following direct sum.

$$\Omega(M) = \bigoplus_k \Omega^k(M).$$

The wedge product is an operation on the space of forms such that if $\omega = \sum_I \omega_I(x) dx^I$ is a k -form and $\mu = \sum_J \mu_J(x) dx^J$ is a p -form, then

$$\omega \wedge \mu = \sum_{I, J} \omega_I(x) \mu_J(x) dx^I \wedge dx^J$$

is a $(k + p)$ -form. If $k + p > n$ then $\omega \wedge \mu = 0$. Note that the wedge product may be omitted as it was in the above integral equations. The wedge product has the following properties,

1. Associativity: $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$.
2. Distributivity: $(\alpha_1 + \alpha_2) \wedge \beta = \alpha_1 \wedge \beta + \alpha_2 \wedge \beta$ and $\alpha \wedge (\beta_1 + \beta_2) = \alpha \wedge \beta_1 + \alpha \wedge \beta_2$.
3. Skew-Commutativity: $\alpha \wedge \beta = (-1)^{km} \beta \wedge \alpha$ where $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^m(M)$,
(Note that $\alpha \wedge \alpha = 0$ for $\alpha \in \Omega^1(M)$).

Another operator on the space of forms is the exterior derivative, d_k , which is defined to be the unique map

$$d_k : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$$

such that the following properties hold:

1. $d_k(\omega + \mu) = d_k\omega + d_k\mu$ and $d_k(c\omega) = cd_k\omega$ for all $\omega, \mu \in \Omega^k(M)$, $c \in \mathbb{R}$.
2. $d_{k+p}(\omega \wedge \mu) = d_k\omega \wedge \mu + (-1)^k\omega \wedge d_p\mu$ for all $\omega \in \Omega^k(M)$ and $\mu \in \Omega^p(M)$.
3. $(d_{k+1}d_k)(\omega) = 0$ for all $\omega \in \Omega^k(M)$.
4. For each $f \in \Omega^0(M)$, $d_0f = \sum \frac{\partial f}{\partial x^i} dx^i$ where $(x^1, x^2, \dots, x^n)$ are local coordinates.
5. For $k \geq n$, $d_k\omega = 0$.

The subscript on d is sometimes omitted when the rank of the form is implied. In local coordinates $(x^1, \dots, x^n)$, the exterior derivative acting on a k -form (for $k < n$) is defined as,

$$d_k\omega = \sum_{j=1}^n dx^j \wedge \frac{\partial \omega}{\partial x^j}. \quad (2.1.1)$$

For example in rectangular, Cartesian coordinates, if $\omega \in \Omega^0(\mathbb{R}^3)$ then (2.1.1) gives

$$d_0\omega = \frac{\partial \omega(x, y, z)}{\partial x} dx + \frac{\partial \omega(x, y, z)}{\partial y} dy + \frac{\partial \omega(x, y, z)}{\partial z} dz$$

which is similar to the gradient operator. If $\omega \in \Omega^1(\mathbb{R}^3)$ given by $\omega = \omega_x(x, y, z)dx + \omega_y(x, y, z)dy + \omega_z(x, y, z)dz$, then

$$d_1\omega = \left(\frac{\partial \omega_y}{\partial x} - \frac{\partial \omega_x}{\partial y} \right) dx dy + \left(\frac{\partial \omega_x}{\partial z} - \frac{\partial \omega_z}{\partial x} \right) dz dx + \left(\frac{\partial \omega_z}{\partial y} - \frac{\partial \omega_y}{\partial z} \right) dy dz \quad (2.1.2)$$

which is similar to the curl operator. If $\omega \in \Omega^2(\mathbb{R}^3)$ given by $\omega = \omega_x(x, y, z)dydz + \omega_y(x, y, z)dzdx + \omega_z(x, y, z)dxdy$ then

$$d_2\omega = \left(\frac{\partial \omega_x}{\partial x} + \frac{\partial \omega_y}{\partial y} + \frac{\partial \omega_z}{\partial z} \right) dxdydz \quad (2.1.3)$$

which is similar to the divergence operator.

2.2 The Metric

The exterior derivative does not depend on how distances are defined on a manifold. However, there are operators on differential forms which do depend on how the distance is defined. In Euclidean n -dimensional space $\mathbb{R}^n$, we can define the distance between two points A and B as

$$\text{dist}(A, B) = \inf\{l(\gamma(t)) : \gamma(0) = A \text{ and } \gamma(1) = B\}$$

where the curve $\gamma(t) = \{x^1(t), \dots, x^n(t)\}$ is defined in terms of the coordinates $x^1, \dots, x^n$ and the length of the segment

$$l(\gamma(t)) = \int_0^1 \sqrt{\left(\frac{dx^1}{dt}\right)^2 + \dots + \left(\frac{dx^n}{dt}\right)^2} dt. \quad (2.2.1)$$

Since a general manifold does not have a global coordinate system, (2.2.1) is not valid on every M . In order to compute distances on a manifold, we need to define an inner product $(,)$. In a vector space $\hat{V}$, the inner product is defined on $\hat{V} \times \hat{V}$. Since manifolds in general are not vector spaces, the inner product needs to be defined on a vector space associated with the manifold. The tangent space to M at a point p , $T_p(M)$, serves as this vector space. The tangent space to M at the point p consists of all of the tangent vectors at $p \in M$. With the tangent space defined, we can now define the inner product on a manifold as follows.

Definition 2.2.2. *A semi-Riemannian metric on a smooth manifold is a correspondence which associates to each point $p \in M$ an inner product $(,)_p$ that is symmetric, bilinear, and nondegenerate and which varies smoothly with p .*

A choice of local coordinates $x = (x^1, \dots, x^n)$ about a point p on M induces a basis $v_j = \left\{ \frac{\partial}{\partial x^j} \right\}$ in the tangent space to p . The inner product at a point q close to p is then defined by its action on these basis vectors:

$$g_{ij}(q) = \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right)_q. \quad (2.2.3)$$

The g_{ij} 's form a matrix. $(g_{ij}(x))$, called the metric tensor. So, if $\gamma(t)$ is a curve lying entirely in a local coordinate system $(x^1, \dots, x^n)$ then

$$l(\gamma) = \int_0^1 \sqrt{\sum_{ij} g_{ij}(x(t)) \frac{dx^i}{dt} \frac{dx^j}{dt}} dt. \quad (2.2.4)$$

The metric also defines an isomorphism between $T_p M$ and the cotangent space to M at p , $T_p^* M$ (whose elements are differential forms). Since (g_{ij}) is a non-singular matrix, then (g_{ij}) is invertible and we denote the inverse matrix as (g^{ij}) . The metric gives a one-to-one correspondence between vectors and covectors (1-forms) in the following manner. If we let $\{e_i\}$ be a basis in $T_p M$ such that $g_{ij} = (e_i, e_j)$, then any vector $v \in T_p M$ can be written as $v = \sum_i v^i e_i$. The basis $\{e_i\}$ gives rise to the basis $\{e^i\}$ in $T_p^* M$ such that $(e^i, e_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta. Therefore, the corresponding covector has the form $\hat{v} = \sum_k v_k e^k$ where

$$v_k = \sum_i g_{ik} v^i$$

and

$$v^i = \sum_k g^{ik} v_k,$$

which is often referred to as the lowering and raising of indices for obvious reasons. In this manner, the metric and its inverse allow us to turn vector fields into 1-forms and vice-versa.

The inner product of two differential 1-forms ω and μ is given by

$$(\omega, \mu) = \sum_{ij} g^{ij} \omega_i \mu_j. \quad (2.2.5)$$

For example, if we let $\omega = \omega_x dx + \omega_y dy + \omega_z dz$ be a 1-form on $\mathbb{R}^3$ in rectangular, Cartesian coordinates, then

$$(\omega, \omega) = \omega_x^2 + \omega_y^2 + \omega_z^2.$$

Due to the bilinearity of the inner product, it suffices to define the inner product of k -forms as the inner product of the wedge product of 1-forms. Suppose $\alpha^1, \alpha^2, \dots, \alpha^k$ and $\beta^1, \beta^2, \dots, \beta^k$ are 1-forms on M . Then

$$(\alpha^1 \wedge \alpha^2 \wedge \dots \wedge \alpha^k, \beta^1 \wedge \beta^2 \wedge \dots \wedge \beta^k) = \det[(\alpha^i, \beta^j)]. \quad (2.2.6)$$

2.2.1 Volumes

The metric not only allows us to measure distances on a manifold, but also to measure volumes and to determine orientations. A volume form, V , on M is defined as a nowhere vanishing n -form. The standard volume form in rectangular, Cartesian coordinates on $\mathbb{R}^3$ is given by

$$V = dx \wedge dy \wedge dz.$$

A manifold M is orientable if there exists a volume form on M . When a manifold is orientable there are two choices of orientation. Making a choice for the orientation then gives us an oriented manifold. For the standard volume form in rectangular, Cartesian coordinates on $\mathbb{R}^3$, the elements of one of the equivalence classes are the volume forms which are an even permutation of $dx \wedge dy \wedge dz$ (the right-hand rule) and the elements of the other equivalence class consists of those volume forms which are an odd permutation of $dx \wedge dy \wedge dz$ (the left-hand rule).

A volume form on an oriented manifold M with metric g_{ij} can be written in local coordinates $(x^1, x^2, \dots, x^n)$, compatible with the orientation, as

$$V = \sqrt{|\det(g_{ij})|} dx^1 \wedge \dots \wedge dx^n. \quad (2.2.1.1)$$

If we let $R \subset M$ be a region (an open connected set) whose closure is compact, then the volume of R is given by

$$\text{vol}(R) = \int \dots \int \sqrt{|\det(g_{ij})|} dx^1 \wedge \dots \wedge dx^n.$$

This expression does not depend on the choice of coordinate system. This implies the existence of a volume form on all of M . The volume form can also be thought of as a measure which allows us to define integration on manifolds.

2.2.2 The Hodge Star Operator

By defining a volume form on an orientable manifold, we can now define the Hodge star operator.

Definition 2.2.2.1. *The Hodge star operator is the map*

$$\star : \Omega^k(M) \rightarrow \Omega^{n-k}(M)$$

such that

$$(\omega, \mu)V = \omega \wedge \star \mu$$

for all $\omega, \mu \in \Omega^k(M)$ and where V is the volume form on M .

The Hodge star operator has the property that

$$\star \star \omega = (-1)^{k(n-k)+s} \omega \text{ for all } \omega \in \Omega^k(M) \quad (2.2.2.2)$$

where n is the dimension of the manifold and $(s, n-s)$ is the signature of the space. The number of positive eigenvalues of the metric is s and the number of negative eigenvalues of the metric is $(n-s)$. For example in $\mathbb{R}^{3+1}$, Minkowski space, the signature is $(3, 1)$. Explicitly for a 1-form $\omega \in \Omega^1(M)$, where M is a 3-dimensional manifold and $\omega = \omega_\xi d\xi + \omega_\eta d\eta + \omega_\alpha d\alpha$, the Hodge star operator applied to ω is given by

$$\begin{aligned} \star \omega &= (g^{11}\omega_\xi + g^{12}\omega_\eta + g^{13}\omega_\alpha) \sqrt{\det g} d\eta \wedge d\alpha \\ &+ (g^{21}\omega_\xi + g^{22}\omega_\eta + g^{23}\omega_\alpha) \sqrt{\det g} d\alpha \wedge d\xi \\ &+ (g^{31}\omega_\xi + g^{32}\omega_\eta + g^{33}\omega_\alpha) \sqrt{\det g} d\xi \wedge d\eta, \end{aligned} \quad (2.2.2.3)$$

ω	$\star\omega$
$\omega(x, y, z)$	$\omega(x, y, z)dxdydz$
$\omega = \omega_x dx + \omega_y dy + \omega_z dz$	$\omega_x dydz + \omega_y dzdx + \omega_z dxdy$
$\omega_x dydz + \omega_y dzdx + \omega_z dxdy$	$\omega_x dx + \omega_y dy + \omega_z dz$
$\omega dxdydz$	$\omega(x, y, z)$

TABLE 2.1. The Hodge Star Operator on Forms in $\mathbb{R}^3$ in Rectangular. Cartesian Coordinates.

where the inverse of the metric is (g^{ij}) . Suppose M is 3-dimensional Euclidean space. Then the standard volume form, V , in rectangular. Cartesian coordinates is given by $V = dxdydz$. With this volume form, the action of the star operator applied to 0, 1, 2 and 3-forms on $\mathbb{R}^3$ is given in Table 2.1.

2.3 Stokes's Theorem

Because n -forms are just objects that one integrates, tools for integration on manifolds, in particular Stokes's theorem, are needed.

Theorem 2.3.1. *Let M be a compact, oriented n -dimensional manifold with boundary (the boundary may be empty) and let $\omega \in \Omega^{n-1}(M)$. Then*

$$\int_M d\omega = \int_{\partial M} \omega \quad (2.3.2)$$

Note that we can replace the hypothesis that M is compact with the hypothesis that ω has compact support (ω vanishes outside of some compact set).

Stokes's theorem is one theorem that generalizes three well-known theorems from calculus. Recall that 0-forms are just smooth functions on M , the exterior derivative of a 1-form is similar to the curl, as in (2.1.2), and the exterior derivative of a 2-form is similar to the divergence, as in (2.1.3). Then Theorem 2.3.1 is a generalization of the Fundamental theorem of calculus

$$\int_a^b f'(x)dx = f(b) - f(a),$$

the Divergence theorem

$$\int_V \nabla \cdot \vec{A} dv = \oint_S \vec{A} \cdot \vec{ds},$$

and Stokes's theorem in vector notation

$$\int_S (\nabla \times \vec{A}) \cdot \vec{ds} = \oint_C \vec{A} \cdot \vec{dl}.$$

The following are corollaries to Stokes's theorem.

Corollary 2.3.3. *If $\omega \in \Omega^k(M)$ and $\eta \in \Omega^{n-k-1}(M)$ then*

$$\int_M d\omega \wedge \eta = (-1)^{k+1} \int_M \omega \wedge d\eta + \int_{\partial M} \omega \wedge \eta.$$

The previous corollary is an extension of the familiar integration by parts.

Corollary 2.3.4. *If the boundary of M is empty, that is $\partial M = \{\emptyset\}$, then*

$$\int_M d\omega = 0.$$

2.4 The Adjoint of the Exterior Derivative

Using the exterior derivative and Stokes's theorem, we can define the adjoint of the exterior derivative in terms of the Hodge star operator. The adjoint of the exterior derivative on $(k+1)$ -forms is defined as the map $d_k^* : \Omega^{k+1}(M) \rightarrow \Omega^k(M)$ such that

$$\langle d_k \omega, \mu \rangle = \langle \omega, d_k^* \mu \rangle \text{ for all } \omega \in \Omega^k(M) \text{ and } \mu \in \Omega^{k+1}(M) \quad (2.4.1)$$

where the global inner product, $\langle, \rangle$, for $\alpha, \beta \in \Omega^p(M)$ is given by

$$\langle \alpha, \beta \rangle = \int_M (\alpha, \beta) V = \int_M \alpha \wedge \star \beta. \quad (2.4.2)$$

We can determine the adjoint of the exterior derivative in terms of the Hodge star operator and the exterior derivative as follows. Let M be a compact, orientable

manifold of dimension n , $\omega \in \Omega^{k-1}(M)$ and $\mu \in \Omega^k(M)$. Then by (2.4.2), we have that

$$\langle d_{k-1}\omega, \mu \rangle = \int_M (d_{k-1}\omega, \mu) V.$$

Using Definition 2.2.2.1 for the Hodge star operator,

$$\langle d_{k-1}\omega, \mu \rangle = \int_M d_{k-1}\omega \wedge \star \mu.$$

Leibniz's rule for the exterior derivative is given by $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$, where p is the rank of the form ω . From this rule, we obtain

$$\langle d_{k-1}\omega, \mu \rangle = \int_M d_{n-1}(\omega \wedge \star \mu) - (-1)^{k-1} \int_M (\omega \wedge d_{n-k} \star \mu) \quad (2.4.3)$$

where the first term on the right-hand side is equal to zero due to Stokes's theorem. Now, the right-hand side of (2.4.1) gives

$$\langle \omega, d_{k-1}^* \mu \rangle = \int_M (\omega, d_{k-1}^* \mu) V.$$

Again, using Definition 2.2.2.1 for the Hodge star operator, we obtain

$$\langle \omega, d_{k-1}^* \mu \rangle = \int_M \omega \wedge \star d_{k-1}^* \mu. \quad (2.4.4)$$

Equating (2.4.3) and (2.4.4) and looking at the integrands in those two equations, the adjoint of the exterior derivative and the exterior derivative are related by

$$(-1)^k d_{n-k} \star \mu = \star d_{k-1}^* \mu. \quad (2.4.5)$$

Applying the star operator to both sides of (2.4.5) and recalling (2.2.2.2), we obtain

$$d_{k-1}^* \mu = (-1)^{(k-1)(k-n)+1-s} \star d_{n-k} \star \mu \quad (2.4.6)$$

where $k \geq 1$, $\mu \in \Omega^k(M)$ and $(s, n-s)$ is the signature of the metric on the space. We define $d_k^* = 0$ for $k < 0$. In particular, if $k = 1$, then

$$d_0^* \mu = (-1)^{1-s} \star d_{n-1} \star \mu$$

and if $k = n$, then

$$d_{n-1}^* \mu = (-1)^{1-s} \star d_0 \star \mu.$$

The following properties hold for the adjoint of the exterior derivative:

1. $(d_{k-1}^* d_k^*)(\omega) = 0$ for all $\omega \in \Omega^{k+1}(M)$.
2. $\star d_k \omega = (-1)^{k+1} d_{n-k-1}^* \star \omega$ for all $\omega \in \Omega^k(M)$.
3. $\star d_{k-1}^* \omega = (-1)^k d_{n-k} \star \omega$ for all $\omega \in \Omega^k(M)$.

2.5 The Laplacian

Many of the problems in mathematical physics involve the solution of partial differential equations. In particular, the Laplacian, which is a second order differential operator, is part of several important equations in physics: Laplace's equation, the wave equation and the diffusion equation. For a real valued function f , the Laplacian is given by

$$\Delta(f) = -\text{div}(\text{grad}(f)).$$

The map, $\Delta_k : \Omega^k(M) \rightarrow \Omega^k(M)$, defined by $\Delta_k = d_k^* d_k + d_{k-1} d_{k-1}^*$, for $k > 0$, is called the Laplacian on differential forms. The Laplacian is a linear and self-adjoint ($\Delta_k^* = \Delta_k$) operator which is non-negative when the metric on the space is positive definite. Linearity follows from the linearity of the exterior derivative and its adjoint. Non-negativity follows from the following:

$$(\Delta_k \omega, \omega) = (d_k^* d_k \omega, \omega) + (d_{k-1} d_{k-1}^* \omega, \omega) = (d_k \omega, d_k \omega) + (d_{k-1}^* \omega, d_{k-1}^* \omega)$$

where $\omega \in \Omega^k(M)$ and both of the terms are greater than or equal to zero because the metric is positive definite, so their sum is greater than or equal to zero. Thus, the Laplacian is non-negative for a positive definite metric. Also,

$$(\Delta_k \omega, \omega) = 0 \Leftrightarrow (d_k \omega, d_k \omega) = 0 \text{ and } (d_{k-1}^* \omega, d_{k-1}^* \omega) = 0 \quad (2.5.1)$$

which implies that

$$(\Delta_k \omega, \omega) = 0 \Leftrightarrow d_k \omega = 0 \text{ and } d_{k-1}^* \omega = 0. \quad (2.5.2)$$

Thus, the Laplacian is non-negative for a positive definite metric. Defining $\mathcal{H}^k(M) = \{\omega \in \Omega^k(M) : \Delta_k \omega = 0\}$ to be the space of Harmonic forms. then from (2.5.1) and (2.5.2) we see that $\mathcal{H}^k(M) = \ker \Delta_k = \ker d_k \cap \ker d_{k-1}^*$.

To understand why Δ_k is called the Laplacian it is essential to examine Δ_0 . Let $\alpha \in \Omega^1(M)$ and $h \in \Omega^0(M)$. Then, the Laplacian on $\Omega^0(M)$ is defined as $\Delta_0 = d_0^* d_0$. By the definition of the adjoint of a linear operator, we have

$$\langle d_0^*(\alpha_j(x)dx^j), h(x) \rangle = \langle \alpha_j(x)dx^j, d_0 h(x) \rangle. \quad (2.5.3)$$

Using the definition of the global inner product, (2.5.3) becomes

$$\langle \alpha_j(x)dx^j, d_0 h(x) \rangle = \int_M \alpha_j(x) \frac{\partial h(x)}{\partial x^k} (dx^j, dx^k) \sqrt{g(x)} dx \quad (2.5.4)$$

where $(dx^j, dx^k) = g^{jk}$, $g(x) = \det(g^{jk}(x))$, and $\sqrt{g(x)}dx$ is the volume form. Using integration by parts, (2.5.4) becomes

$$\langle \alpha_j(x)dx^j, d_0 h(x) \rangle = - \int_M h(x) \frac{\partial}{\partial x^k} \left[\alpha_j(x) g^{jk}(x) \sqrt{g(x)} \right] dx. \quad (2.5.5)$$

Equating (2.5.3) and (2.5.5), we can determine d_0^* as

$$d_0^*(\alpha_j(x)dx^j) = - \frac{1}{\sqrt{g(x)}} \frac{\partial(\alpha_j(x) g^{jk}(x) \sqrt{g(x)})}{\partial x^k}$$

which is the divergence of the 1-form α . Therefore, if we let $f(x) \in \Omega^0(M)$ then $d_0 f = \frac{\partial f(x)}{\partial x^j} dx^j$ and

$$d_0^* d_0 f = - \frac{1}{\sqrt{g(x)}} \frac{\partial}{\partial x^k} (g^{jk}(x) \frac{\partial f}{\partial x^j} \sqrt{g(x)})$$

which is the Laplace-Beltrami operator on a compact manifold M with metric g . For example, if $(g^{jk}(x))$ is the Euclidean metric then $g(x) = 1$ and

$$d_0^* d_0 f = - \frac{\partial^2 f}{\partial x^k \partial x^j} \delta^{kj}$$

which is the usual Laplacian.

2.6 DeRham Cohomology and Hodge Theory

2.6.1 DeRham Cohomology

There are two types of differential forms that are of particular interest, the closed differential forms and the exact differential forms. The closed differential forms are those forms for which the exterior derivative is equal to zero. Equations of the form $d\omega = 0$ are independent of a choice of coordinates. Therefore, $d\omega = 0$ is invariant under all coordinate transformations (translation, rotation, Lorentz transformation, etc.). The exact differential forms are those forms which can be written as the exterior derivative of a form of dimension one less. From the property that $dd = 0$, the exact differential forms are also closed differential forms. Suppose $d_k\omega = 0$ and we need to determine a solution to this equation. Since it has been shown that $d_k d_{k-1} = 0$, then any form $\omega = d_{k-1}\eta$ (an exact form) will be a solution. Another possible solution to $d_k\omega = 0$ is the one in which $d_k\omega = 0$ but $\omega \neq d_{k-1}\eta$. The study of deRham Cohomology leads to these solutions.

Let M be a smooth manifold of dimension n and with metric g . The exterior derivatives form a sequence of maps

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d_0} \dots \xrightarrow{d_{k-1}} \Omega^k(M) \xrightarrow{d_k} \Omega^{k+1}(M) \xrightarrow{d_{k+1}} \dots \xrightarrow{d_{n-1}} \Omega^n(M) \xrightarrow{d_n} 0$$

which satisfy the property $d_{k+1}d_k = 0$. The space $Z^k(M) = \{\omega \in \Omega^k(M) : d_k\omega = 0\}$ is called the group of closed forms of rank k . The space $B^k(M) = \{\omega \in \Omega^k(M) : \omega = d_{k-1}\alpha \text{ for } \alpha \in \Omega^{k-1}(M)\}$ is called the group of exact forms of rank k . Since $B^k(M)$ is a subgroup of $Z^k(M)$, then the factor group $H^k(M) = Z^k(M)/B^k(M)$ exists and is called the space of DeRham cohomologies of M of rank k . The study of the space of DeRham cohomologies of M of rank k gives solutions to the equation $d_k\omega = 0$ for $\omega \neq d_{k-1}\eta$.

2.6.2 Hodge Theory for Differential Forms

The Hodge theorem is one of several theorems which brings together the properties of the operators described above and also examines the different subspaces of the space of differential forms.

Theorem 2.6.2.1. *Let M be a compact, Riemannian manifold with boundary. Then*

$$\Omega^k(M) = B^k(M) \oplus \text{Im}d_k^* \oplus \mathcal{H}^k(M)$$

where $B^k(M)$, $\text{Im}d_k^*$ and $\mathcal{H}^k(M)$ are mutually orthogonal and

$$Z^k(M) = \mathcal{H}^k(M) \oplus B^k(M)$$

Proof. Let $\beta \in B^k(M)$ ($\beta = d_{k-1}\alpha$ for some $\alpha \in \Omega^{k-1}(M)$) and $\eta \in \mathcal{H}^k(M)$. Then the inner product of β and η is given by the following:

$$(\beta, \eta) = (d_{k-1}\alpha, \eta) = (\alpha, d_{k-1}^*\eta) \quad (2.6.2.2)$$

However, $\eta \in \mathcal{H}^k(M)$ implies that $\eta \in \ker d_{k-1}^*$. Thus, the inner product is zero and $B^k(M)$ and $\mathcal{H}^k$ are mutually orthogonal. Let $\beta \in B^k(M)$ ($\beta = d_{k-1}\alpha$ for some $\alpha \in \Omega^{k-1}(M)$) and let $\gamma \in \text{Im}d_k^*$ ($\gamma = d_k^*\tilde{\gamma}$ for some $\tilde{\gamma} \in \Omega^{k+1}(M)$). Then

$$(\beta, \gamma) = (d_{k-1}\alpha, d_k^*\tilde{\gamma}) = (d_k d_{k-1}\alpha, \tilde{\gamma}) \quad (2.6.2.3)$$

However, $d_k d_{k-1} = 0$. Thus, the inner product is zero and B^k and $\text{Im}d_k^*$ are mutually orthogonal. Now, let $\gamma \in \text{Im}d_k^*$ ($\gamma = d_k^*\tilde{\gamma}$ for some $\tilde{\gamma} \in \Omega^{k+1}(M)$) and let $\eta \in \mathcal{H}^k$. Then,

$$(\gamma, \eta) = (d_k^*\tilde{\gamma}, \eta) = (\tilde{\gamma}, d_k\eta) \quad (2.6.2.4)$$

However, $\eta \in \mathcal{H}^k$ implies that $\eta \in \ker d_k$, thus, the inner product is zero and $\text{Im}d_k^*$ and $\mathcal{H}^k$ are mutually orthogonal.

Now we want to show that every $\omega \in \Omega^k(M)$ can be decomposed into a direct sum of elements from $B^k(M)$, $\text{Im}d_k^*$ and $\mathcal{H}^k(M)$. Let $\omega \in \Omega^k(M)$. Because Δ_k is a self-adjoint operator and $\ker \Delta_k = \mathcal{H}^k(M)$, we can then write $\Omega^k(M) = \mathcal{H}^k(M) \oplus \mathcal{H}^{k\perp}(M)$. So, each $\omega \in \Omega^k(M)$ can be written uniquely as $\omega = \eta + \bar{\eta}$ where $\eta \in \mathcal{H}^k(M)$ and $\bar{\eta} \in \mathcal{H}^{k\perp}(M)$. Since Δ_k is self adjoint, then the restriction of Δ_k to a subspace orthogonal to the kernel of Δ_k , $(\mathcal{H}^{k\perp}(M))$, is one-to-one and $\mathcal{H}^{k\perp}(M)$ is invariant under $\Delta_k(M)$. Set $\tilde{\Delta}_k$ equal to Δ_k restricted to $\mathcal{H}^{k\perp}(M)$. Then $\tilde{\Delta}_k$ is invertible. So, for $\bar{\eta} \in \mathcal{H}^{k\perp}(M)$

$$\bar{\eta} = (\tilde{\Delta}_k)^{-1} \Delta_k \bar{\eta} \quad (2.6.2.5)$$

Note that the Laplacian commutes with $d_k^* d_k$ so the inverse Laplacian also commutes with $d_k^* d_k$. Therefore, the previous equation becomes,

$$\bar{\eta} = d_k^* d_k (\tilde{\Delta}_k)^{-1} \bar{\eta} + d_{k-1} d_{k-1}^* (\tilde{\Delta}_k)^{-1} \bar{\eta} \quad (2.6.2.6)$$

where the first term is an element of $\text{Im}d_k^*$ and the second term is an element of $B^k(M)$. Therefore, $\omega = \eta + \bar{\eta}$ where $\eta \in \mathcal{H}^k(M)$ and $\bar{\eta} \in \text{Im}d_k^* \oplus B^k(M)$ which implies that $\Omega^k(M) = \mathcal{H}^k(M) \oplus \text{Im}d_k^* \oplus B^k(M)$.

Similarly, we can show that $Z^k(M) = B^k(M) \oplus \mathcal{H}^k(M)$ where $Z^k(M) = \{\omega \in \Omega^k(M) \mid d_k \omega = 0\}$. Since $\mathcal{H}^k(M) \subset Z^k(M)$ and $B^k(M) \subset Z^k(M)$ then it follows that $\mathcal{H}^k(M) \oplus B^k(M) \subset Z^k(M)$. Now, let $\omega \in Z^k(M)$ ($d_k \omega = 0$) where $\omega = d_{k-1} \alpha + d_k^* \gamma + \eta$ and $\eta \in \mathcal{H}^k(M)$. Then applying d_k to ω , we get

$$d_k \omega = d_k d_{k-1} \alpha + d_k d_k^* \gamma + d_k \eta = 0 \quad (2.6.2.7)$$

The terms $d_k d_{k-1} \alpha$ and $d_k \eta$ are both equal to zero and we are left with the term, $d_k d_k^* \gamma = 0$. Therefore,

$$(d_k d_k^* \gamma, \gamma) = 0 \quad (2.6.2.8)$$

which, using the adjoint relation, gives $d_k^* \gamma = 0$. Therefore, any $\omega \in Z^k(M)$ can be written as $\omega = d_{k-1} \alpha + \eta$ and since $B^k(M) \perp \mathcal{H}^k(M)$ then $B^k(M) \cap \mathcal{H}^k(M) = \emptyset$. Therefore, $Z^k(M) = B^k(M) \oplus \mathcal{H}^k(M)$. $\square$

Chapter 3

CHAINS AND COCHAINS - THE DISCRETE

With the advance of computer technology, it is of importance to be able to represent a continuous space by an appropriate discrete space. A compact manifold equipped with differential forms, the exterior derivative and the adjoint of the exterior derivative can be naturally represented by a cellular decomposition of the manifold equipped with chains, cochains, the boundary and coboundary operators. In this chapter, the concepts of a cell complex, the group of k -chains which are the standard building blocks for the domains of integration on a cell complex, and the group of k -cochains which are related to differential forms are introduced. The ideas of the boundary operator, the transpose of the boundary operator, the coboundary operator, a metric, the star operator on cochains, and a combinatorial Laplacian on a cell complex will also be introduced.

3.1 Cellular Decomposition and Chains

A compact manifold can be represented discretely by the cellular decomposition of the manifold into a cell complex. A cell complex of a compact, differentiable manifold M , which is a Hausdorff space, is defined by the sequence of closed subspaces (topological)

$$M^0 \subset M^1 \subset \cdots \subset M^k \subset \cdots$$

such that the following properties hold:

1. $\bigcup M^k = M$ where M^k is called the k -skeleton.
2. M^0 is a discrete subspace, i.e. has the discrete topology, of M (the points of M^0 are called 0-cells or vertices).

3. For $k \geq 1$, $M^k - M^{k-1}$ is the disjoint union of open subsets, each of which is homeomorphic to $\mathbb{D}^k$ (the open k -dimensional disk). These subsets are called the k -dimensional cells of the decomposition and are denoted by s_k . The i -th k -dimensional cell will be denoted by s_k^i .
4. M is the disjoint union of the cells of various dimension and M^k is the union of all cells of dimension less than or equal to k .
5. $M^k = \{\emptyset\}$ for $k < 0$.

For the entirety of this dissertation, we will be examining only polyhedral cell complexes. These are complexes whose faces are polyhedrons.

There are many choices for the cellular decomposition of a manifold. One example of a cell complex on $\mathbb{R}^n$ is obtained by considering n -dimensional cubes whose vertices are the points of $\mathbb{R}^n$ with integer coordinates. The cells of $\mathbb{R}^n$ are the cubes, the faces of the cubes, the edges of the faces and the vertices of the edges. Each cell can be assigned one of two possible orientations. When an orientation is chosen then we say that the cell is oriented. For example, a 1-cell, s_1 , can be given by the directed line segment from s_0^1 to s_0^2 or by the directed line segment from s_0^2 to s_0^1 as in Figure 3.1.a and a 2-cell, s_2 , can be assigned a counterclockwise or clockwise orientation as in Figure 3.1.b. In general for a 3-dimensional manifold, a polyhedral cell complex consists of interconnected 0-cells denoted by vertices, 1-cells denoted by directed edges, 2-cells denoted by oriented faces or surface elements, and 3-cells denoted by oriented volume elements. By choosing the orientation, we can define the volume cell c_n as an n -dimensional cell in an n -dimensional cell complex.

A k -chain is a formal linear combination of cells of dimension k with coefficients in an arbitrary Abelian group, $\mathcal{G}$. If $\mathcal{G} = \mathbb{R}$, then $C_k(M, \mathbb{R})$ is the free module of k -chains with coefficients in $\mathbb{R}$ and the k -cells, s_k^i , are the basis elements. This means

FIGURE 3.1. a. Orientation of a 1-cell. b. Orientation of a 2-cell.

that every k -chain can be written as a linear combination of the s_k^i as follows:

$$C_k(M, \mathbb{R}) = \left\{ \sum_i h_i s_k^i : h_i \in \mathbb{R} \right\}$$

where the coefficients, h_i , are real numbers. Thus, a k -chain simply assigns a real number to the k -cells in a cell complex. For example, the length of a directed line segment may be the real number assigned to a 1-cell, or the area of a face may be the real number assigned to a 2-cell.

3.2 The Boundary Operator

The relationship between cells of dimension k and of dimension $k-1$ can be described by the boundary operator. When the boundary operator is applied to a cell-complex, it is sometimes referred to as the connection matrix because it describes how the cells of dimension k are connected to the cells of dimensions $k-1$ in the complex.

The boundary operator on the group of k -chains, $C_k(M, \mathbb{R})$, is the map

$$C_k(M, \mathbb{R}) \xrightarrow{\partial_k} C_{k-1}(M, \mathbb{R}).$$

On a k -cell, s_k^i , we define the boundary operator by

$$\partial_k(s_k^i) = \sum_j [s_k^i : s_{k-1}^j] s_{k-1}^j \quad (3.2.1)$$

where $[s_k^i : s_{k-1}^j]$ is called the incidence number of s_k^i and s_{k-1}^j . In general, the incidence number is an integer, however for polyhedral cells the incidence number takes on the values 0, +1, or -1 as follows:

$$[s_k^i : s_{k-1}^j] = \begin{cases} 0 & \text{if } s_{k-1}^j \text{ is not in the boundary of } s_k^i \\ +1 & \text{if } s_{k-1}^j \text{ is positively oriented on the boundary of } s_k^i \\ -1 & \text{if } s_{k-1}^j \text{ is negatively oriented on the boundary of } s_k^i \end{cases}$$

In particular, suppose the 1-cell, s_1 , is given by the directed line segment from s_0^1 to s_0^2 as shown in Figure 3.1.a. Then, $\partial_1(s_1) = s_0^2 - s_0^1$. If s_2 denotes the counterclockwise oriented 2-cell shown in Figure 3.1.b, then $\partial_2(s_2) = s_1^1 + s_1^2 - s_1^3 - s_1^4$. Since ∂_k is a linear operator, it can be extended to the space of k -chains such that

$$\partial_k\left(\sum_i h_i s_k^i\right) = \sum_i h_i \partial_k(s_k^i).$$

Also, due to the linearity of the boundary operator, we can express the boundary operator as a matrix (the connection matrix) in the following way: Suppose the cell complex of a compact orientable 3-dimensional manifold M consists of four 0-cells, six 1-cells, four 2-cells and one 3-cell. Then $\partial_2 : C_2(M, \mathbb{R}) \rightarrow C_1(M, \mathbb{R})$ is represented by a 6×4 matrix, (a_{ij}) . The entries of (a_{ij}) will be the incidence numbers $[s_2^j : s_1^i]$. Therefore, in general, the connection matrix, (a_{ij}) , representing the boundary operator, $\partial_k : C_k(M, \mathbb{R}) \rightarrow C_{k-1}(M, \mathbb{R})$, will have entries $a_{ij} = [s_k^j : s_{k-1}^i]$. The transpose of this matrix, $(a_{ij})^T$, will provide the connections between cells of dimension $(k-1)$ and the cells of dimension k . This matrix represents the operator

$$\partial_k^T : C_{k-1}(M, \mathbb{R}) \rightarrow C_k(M, \mathbb{R}). \quad (3.2.2)$$

The entries of $(a_{ij})^T$ will be given by $a_{ij} = [s_k^i : s_{k-1}^j]$. On a basis element, $s_{k-1}^i \in C_{k-1}(M, \mathbb{R})$, we define the transpose of the boundary operator by

$$\partial_k^T(s_{k-1}^i) = \sum_j [s_{k-1}^i : s_k^j] s_k^j \quad (3.2.3)$$

where the incidence number will take on the values 0, +1, and -1 as described for the boundary operator.

An example of the transpose of the boundary operator on a 0-cell, s_0 , with the 1-cells s_1^1 , s_1^2 , s_1^3 , and s_1^4 attached to s_0 is shown in Figure 3.2.a such that.

$$\partial_1^T(s_0) = s_1^1 + s_1^2 - s_1^3 - s_1^4.$$

FIGURE 3.2. a. The Transpose of the Boundary Operator Applied to a 0-cell. b. The Transpose of the Boundary Operator Applied to a 1-cell.

The orientations are chosen in this case such that if the directed line segment is directed towards the 0-cell, s_0 , then a positive sign is attached; and if the directed line segment is directed away from s_0 , then a negative sign is attached. The other choice would be to make the inward direction have a negative sign and the outward direction have a positive sign. Either choice is correct as long as consistency is upheld. If s_1 is

the 1-cell with the 2-cells s_2^1 , s_2^2 , s_2^3 , and s_2^4 connected to s_1 as in Figure 3.2.b, then the transpose of the boundary operator acting on s_1 gives.

$$\partial_2^T(s_1) = s_2^1 - s_2^2 - s_2^3 + s_2^4.$$

When the transpose of the boundary operator is applied to a cell complex, it is represented by the transpose of the connection matrix, $(a_{ij})^T$, which is generated by the boundary operator. The entries of $(a_{ij})^T$ will be the incidence numbers $[s_{k-1}^i : s_k^j]$.
-1z The boundary operators form a sequence of maps

$$0 \longrightarrow C_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_{k+1}} C_k \xrightarrow{\partial_k} C_{k-1} \xrightarrow{\partial_{k-1}} \cdots \longrightarrow 0$$

which satisfy the property $\partial_k \partial_{k+1} = 0$. where the proof of this property depends greatly on the choice of orientation for each cell. The group $Z_k(M, \mathbb{R}) = \{c \in C_k(M, \mathbb{R}) : \partial_k c = 0\}$ is called the group of k -cycles (those k -chains without boundary) and the group $B_k(M, \mathbb{R}) = \{c \in C_k(M, \mathbb{R}) : c = \partial_{k+1} b \text{ for some } b \in C_{k+1}\}$ is called the group of k -boundaries (those k -chains which are the boundary of a $(k+1)$ -chain). Since $\partial_k \partial_{k+1} = 0$, we know that if the boundary of a k -chain, c , is zero, then one possibility is that c is the boundary of a $(k+1)$ -chain, b . The other possibility occurs when the k -chain, c , is such that $\partial_k c = 0$ but $c \neq \partial_{k+1} b$. This is described by $H_k(M, \mathbb{R}) = Z_k(M, \mathbb{R})/B_k(M, \mathbb{R})$ where $H_k(M, \mathbb{R})$ is the k th homology group of the cell complex with coefficients in $\mathbb{R}$ (describes the number of holes in M). The k -th homology group $H_k(M, \mathbb{R})$ does not depend on the cell decomposition of M .

3.3 Cochains and the Metric

In order to describe an analogy to Stokes's theorem and integration on a cell complex, we must develop how differential forms are represented on a cell complex. The dual space of $C_k(M, \mathbb{R})$ will give such a representation. The dual space of $C_k(M, \mathbb{R})$, which will be denoted by $C^k(M, \mathbb{R})$, is called the space of cochains. A cochain, c^k , is

a linear functional on the space of chains, which means c^k takes a chain c_k into a real number. A basis s_k^i in $C_k(M, \mathbb{R})$ naturally gives rise to a basis s_j^k in $C^k(M, \mathbb{R})$ such that $(s_j^k, s_k^i) = \delta_{ij}$ where δ_{ij} is the Kronecker delta. In other words, the elementary k -dimensional cochains, s_j^k , are those cochains which assign the value one to the chain s_k^j and zero to all other chains. Since s_j^k is a basis in $C^k(M, \mathbb{R})$, we can then write any $c^k \in C^k(M, \mathbb{R})$ as a linear combination of the basis elements as follows:

$$C^k(M, \mathbb{R}) = \left\{ \sum_j d^j s_j^k : d^j \in \mathbb{R} \right\}.$$

Therefore, if $c_k \in C_k(M, \mathbb{R})$ is given by $\sum_i h_i s_k^i$, then

$$(c^k, c_k) = \left(\sum_j d^j s_j^k, \sum_i h_i s_k^i \right) = \sum_i d^i h_i$$

where $\sum_i d^i h_i \in \mathbb{R}$. This defines the metric on a cell-complex.

3.3.1 The Coboundary Operator

Since the boundary operator, $\partial_k : C_k(M, \mathbb{R}) \rightarrow C_{k-1}(M, \mathbb{R})$, is a linear operator, we can construct a dual mapping ∂_k^* such that

$$\partial_k^* : C^{k-1}(M, \mathbb{R}) \rightarrow C^k(M, \mathbb{R})$$

where

$$\langle \partial_k^* b, c \rangle = \langle b, \partial_k c \rangle \quad \forall b \in C^{k-1}(M, \mathbb{R}), c \in C_k(M, \mathbb{R}).$$

The dual operator is called the coboundary operator. An explicit formula for the coboundary operator, ∂_k^* , which is a linear operator, is defined for the basis of $C^k(M, \mathbb{R})$. Let $s_i^k \in C^k(M, \mathbb{R})$ be a basis element of the space of cochains and

$s_{k+1}^j \in C_{k+1}(M, \mathbb{R})$ be a basis element of the space of chains. Then.

$$\begin{aligned}
 (\partial_{k+1}^* s_i^k)(s_{k+1}^j) &= s_i^k(\partial_{k+1} s_{k+1}^j) \\
 &= s_i^k \sum_l [s_{k+1}^j : s_k^l] s_k^l \\
 &= \sum_l [s_{k+1}^j : s_k^l] s_i^k(s_k^l) \\
 &= \sum_l [s_{k+1}^j : s_k^l] \delta_i^l \\
 &= [s_{k+1}^j : s_k^i].
 \end{aligned}$$

3.4 Hodge Star on Cochains

Analogous to the Hodge star operator on differential forms, we can define a star operator, $\star_c$, on the space of cochains. The star operator, $\star_c$, is defined as the map

$$\star_c : C^k(M, \mathbb{R}) \rightarrow C^{n-k}(M, \mathbb{R})$$

such that

$$(c^k, b^k) c^n = c^k \star b^k \quad (3.4.1)$$

where c^n is the volume cochain and $c^k, b^k \in C^k(M, \mathbb{R})$. For example, in $\mathbb{R}^3$, the star operator on cochains, $\star_c$, will take a 1-cochain which is applied to a directed edge to a 2-cochain which is applied to an oriented face. The star operator on the space of cochains has the property that

$$\star_c^2 = (-1)^{k(n-k)+s}$$

where $(s, n-s)$ is the signature of the space.

3.5 Cohomology

The coboundary operators form a sequence of maps such that

$$0 \xrightarrow{\partial_1^*} C^1 \xrightarrow{\partial_2^*} \dots \xrightarrow{\partial_k^*} C^k \xrightarrow{\partial_{k+1}^*} C^{k+1} \xrightarrow{\partial_{k+2}^*} \dots \longrightarrow 0$$

where $\partial_k^* \partial_{k-1}^* = 0$. From this sequence of maps, we can define two subspaces of $C^k(M, \mathbb{R})$ which are of particular interest. The subspace $Z^k(M, \mathbb{R}) = \{c \in C^k(M, \mathbb{R}) : \partial_{k+1}^* c = 0\}$ is called the group of cocycles and the subspace $B^k(M, \mathbb{R}) = \{c \in C^k(M, \mathbb{R}) : c = \partial_k^* b \text{ for some } b \in C^{k-1}(M, \mathbb{R})\}$ is called the group of coboundaries. These two subspaces are analogous to the spaces of closed and exact differential forms, respectively. Since $\partial_k^* \partial_{k-1}^* = 0$, then one solution to the equation $\partial_k^* c = 0$ is $c = \partial_{k-1}^* b$ for some $b \in C^{k-2}(M, \mathbb{R})$. Another solution, would be those $c \in C^{k-1}(M, \mathbb{R})$ for which $c \neq \partial_{k-1}^* b$. To study these solutions, we study the group of k -cohomologies with coefficients in $\mathbb{R}$. $H^k(M, \mathbb{R}) = Z^k(M, \mathbb{R})/B^k(M, \mathbb{R})$.

3.6 Relationship Between Cochains and Differential Forms

The space of cochains is the discrete analogue to the space of differential forms. The relationship between a differential form and a cochain is given by integration such that $\int_k : \Omega^k(M) \rightarrow C^k(M, \mathbb{R})$. Therefore, $\int_k(\omega)$ for $\omega \in \Omega^k(M)$ will be a linear functional on the space of k -chains, $C_k(M, \mathbb{R})$, mapping a k -chain into the real numbers. We say that $\bar{c}_\omega$ is the cochain generated by ω if $\bar{c}_\omega = \int_k \omega$.

Since $\int_k$ is a linear functional of $C_k(M, \mathbb{R})$, we can define $\int_k(\omega)$ on the oriented k -cells, s_k^i . Let h be a smooth map from an open set, U , in the closure of the k -cell, s_k^i , to the manifold M . If $\omega \in \Omega^k(M)$, then $h^*(\omega)$ is the pullback of ω onto the open set U . Then, we can define the inner product of the cochain generated by $\omega \in \Omega^k(M)$ with $s_k^i \in C_k(M, \mathbb{R})$ as

$$(\bar{c}_\omega, s_k^i) = \int_k(\omega)(s_k^i) = \int_{s_k^i} h^*(\omega). \quad (3.6.1)$$

The following theorem, which was critical to the results of this dissertation, relies on the relationship between the space of k -forms and the space of k -cochains.

Theorem 3.6.2. *Let M be a compact, orientable manifold. Then,*

$$\int_k d_{k-1} = \partial_k^* \int_{k-1}. \quad (3.6.3)$$

Proof. Let $\alpha \in \Omega^{k-1}(M)$ where $d_{k-1}\alpha = \omega$ and let s_k^i be a k -cell. Then (3.6.1) yields

$$[\int_k d_{k-1}(\alpha)](s_k^i) = \int_{s_k^i} h^*(d_{k-1}\alpha).$$

Now, h^* and d_{k-1} commute and we have that

$$[\int_k d_{k-1}(\alpha)](s_k^i) = \int_{s_k^i} d_{k-1}(h^*(\alpha)).$$

By Stokes's theorem,

$$[\int_k d_{k-1}(\alpha)](s_k^i) = \int_{\partial_k s_k^i} h^*(\alpha).$$

Given how we defined the operator, $\int_k$, in (3.6.1), we then have

$$[\int_k d_{k-1}(\alpha)](s_k^i) = \int_{k-1} (\alpha)(\partial_k s_k^i).$$

Finally by the definition of the adjoint, we have

$$[\int_k d_{k-1}(\alpha)](s_k^i) = [\partial_k^* \int_{k-1} \alpha](s_k^i).$$

Therefore,

$$\int_k d_{k-1} = \partial_k^* \int_{k-1}.$$

□

Thus, if $\tilde{c}_\omega$ is generated by the form ω , then $\partial^* \tilde{c}_\omega$ is generated by the form $d\omega$.

Theorem 3.6.2 is equivalent to stating that the following diagram is commutative:

$$\begin{array}{ccc} \Omega^k(M) & \xrightarrow{d_k} & \Omega^{k+1}(M) \\ \downarrow & & \downarrow \\ C^k(M, \mathbb{R}) & \xrightarrow{\partial_{k+1}^*} & C^{k+1}(M, \mathbb{R}) \end{array}$$

where the map from $\Omega^k(M)$ to $C^k(M, \mathbb{R})$ is the integration described above. The integration maps closed forms to cocycles and exact forms to coboundaries. From the commutivity of the above diagram, we have that if $\omega \in \Omega^k(M)$ is a closed form such that $d_k \omega = 0$, then $\partial_{k+1}^*(\int_k(\omega)) = \int_{k+1}(d_k \omega) = 0$. This implies that $\int_k(Z^k(M)) \subset Z^k(M, \mathbb{R})$ where $Z^k(M)$ is the set of closed differential forms and $Z^k(M, \mathbb{R})$ is the set of cocycles. If $\omega \in \Omega^k(M)$ is an exact form such that $\omega = d_{k-1} \tau$ where $\tau \in \Omega^{k-1}(M)$, then $\int_k(\omega) = \int_k(d_{k-1} \tau) = \partial_k^*(\int_{k-1} \tau) \in \text{Im} \partial_k^*$. Therefore, $\int_k(B^k(M)) \subset B^k(M, \mathbb{R})$ where $B^k(M)$ is the set of exact differential forms and $B^k(M, \mathbb{R})$ is the set of coboundaries. Therefore, the relationship between differential forms and cochains takes closed forms to cocycles and the exact forms to coboundaries.

As an aside, the zero-dimensional cochains c^0 represent a discontinuous analogue of scalar functions in $\mathbb{R}$. The one-dimensional cochains c^1 represent a discontinuous analogue of a vector field in $\mathbb{R}$ by assigning a numerical value to a directed edge. The k -dimensional cochains c^k correspond to k -dimensional functions (k -dimensional differential forms).

3.6.1 Extension of Stokes's Theorem to a Cell Complex

Using the relationship between differential forms and cochains, we can define Stokes's theorem on a cell complex. By definition of the adjoint, we have that

$$\langle \partial_{k+1}^* \bar{c}, c \rangle = \langle \bar{c}, \partial_{k+1} c \rangle \quad \forall \bar{c} \in C^k(M, \mathbb{R}), c \in C_{k+1}(M, \mathbb{R}). \quad (3.6.1.1)$$

This is analogous to Stokes's theorem by the following argument. Suppose $\bar{c} \in C^k(M, \mathbb{R})$ is generated by the differential form η . Then

$$\langle \bar{c}, \partial b \rangle = \int_{\partial b} \eta$$

where $b \in C_{k+1}$. Then by Stokes's theorem in Section 2.3 we have that

$$\int_{\partial b} \eta = \int_b d\eta$$

which is also equal to $(\partial^* \tilde{c}, b)$ by (3.6.1.1). Therefore, $\partial^* \tilde{c}$ is generated by the differential form $d\eta$. Thus, (3.6.1.1) for a cell complex is analogous to Stokes's theorem for differential forms.

3.7 Combinatorial Laplacian

Just as we defined the Laplacian on the space of differential forms, we can also define the Laplacian on the space of chains and the space of cochains. The combinatorial Laplacian on the space of chains is the map, $\Delta_k : C_k(M, \mathbb{R}) \rightarrow C_k(M, \mathbb{R})$, where

$$\Delta_k = \partial_k^* \partial_k + \partial_{k+1} \partial_{k+1}^*.$$

The Laplacian is a linear and self-adjoint operator and non-negative when the metric is positive definite. Linearity follows from the linearity of both the boundary and coboundary operators. It is easy to show that Δ_k is self-adjoint by showing that $(\Delta_k)^* = \Delta_k$. Non-negativity for a positive definite metric then follows from the following:

$$(\Delta_k c, c) = (\partial_k^* \partial_k c + \partial_{k+1} \partial_{k+1}^* c, c) = (\partial_k c, \partial_k c) + (\partial_{k+1}^* c, \partial_{k+1}^* c)$$

where $c \in C_k(M, \mathbb{R})$ and the first and second terms are greater than or equal to zero so the sum of the two must be greater than or equal to zero. Thus, the Laplacian is non-negative for a manifold with a positive definite metric. Also,

$$(\Delta_k c, c) = 0 \Leftrightarrow (\partial_k c, \partial_k c) = 0 \text{ and } (\partial_{k+1}^* c, \partial_{k+1}^* c) = 0 \Leftrightarrow \partial_k c = 0 \text{ and } \partial_{k+1}^* c = 0$$

Let $\mathcal{H}_k = \{c \in C_k : \Delta_k c = 0\}$ be the set of harmonic chains. From the previous equation, it can be shown that $\ker \Delta_k = \ker \partial_k \cap \ker \partial_{k+1}^*$.

A similar argument follows for defining the combinatorial Laplacian on the space of cochains.

3.8 Hodge Theory for Chains and Cochains

The Hodge theorem for chains and cochains brings together the properties of the operators described above and also examines the different subspaces of the spaces of chains and cochains.

Theorem: $C^k = B^k \oplus \text{Im}\partial_{k+1} \oplus \mathcal{H}^k$ where B^k , $\text{Im}\partial_{k+1}$ and $\mathcal{H}^k$ are mutually orthogonal and $Z^k = \mathcal{H}^k \oplus B^k$. The same theorem holds for $C_k = B_k \oplus \text{Im}\partial_k^* \oplus \mathcal{H}_k$.

Proof. Let $b \in B^k$ ($b = \partial_k^* c$ for some $c \in C^{k-1}$) and $h \in \mathcal{H}^k$. Then the inner product of b with h is given by the following:

$$(b, h) = (\partial_k^* c, h) = (c, \partial_k h).$$

However, $h \in \mathcal{H}^k$ implies that $h \in \ker \partial_k$. Thus, the inner product is zero and B^k and $\mathcal{H}^k$ are mutually orthogonal. Let $b \in B^k$ ($b = \partial_k^* c$ for some $c \in C^{k-1}$) and let $d \in \text{Im}\partial_{k+1}$ ($d = \partial_{k+1} \tilde{d}$ for some $\tilde{d} \in C_{k+1}$). Then

$$(b, d) = (\partial_k^* c, \partial_{k+1} \tilde{d}) = (\partial_{k+1}^* \partial_k^* c, \tilde{d}).$$

However, $\partial_{k+1}^* \partial_k^* = 0$. Thus, the inner product is zero and B^k and $\text{Im}\partial_{k+1}$ are mutually orthogonal. Now, let $d \in \text{Im}\partial_{k+1}$ ($d = \partial_{k+1} \tilde{d}$ for some $\tilde{d} \in C_{k+1}$) and let $h \in \mathcal{H}^k$. Then,

$$(d, h) = (\partial_{k+1} \tilde{d}, h) = (\tilde{d}, \partial_{k+1}^* h).$$

However, $h \in \mathcal{H}^k$ implies that $h \in \ker \partial_{k+1}^*$, thus, the inner product is zero and $\text{Im}\partial_{k+1}$ and $\mathcal{H}^k$ are mutually orthogonal.

Now we want to show that every $c \in C^k$ can be decomposed into a direct sum of elements from B^k , $\text{Im}\partial_{k+1}$ and $\mathcal{H}^k$. Let $c \in C^k$. Since Δ^k is a self-adjoint operator and $\ker \Delta^k = \mathcal{H}^k$, we can then write $C^k = \mathcal{H}^k \oplus \mathcal{H}^{k\perp}$. So, each $c \in C^k$ can be written uniquely as $c = h + \tilde{h}$ where $h \in \mathcal{H}^k$ and $\tilde{h} \in \mathcal{H}^{k\perp}$. Since Δ^k is self adjoint, then the restriction of Δ^k to a subspace orthogonal to the kernel of Δ^k , $(\mathcal{H}^{k\perp})$, is one-to-one

and $\mathcal{H}^{k\perp}$ is invariant under Δ^k . Set $\bar{\Delta}^k$ equal to Δ^k restricted to $\mathcal{H}^{k\perp}$. Then $\bar{\Delta}^k$ is invertible. So, for $\tilde{h} \in \mathcal{H}^k$

$$\tilde{h} = (\bar{\Delta}^k)^{-1} \Delta^k \tilde{h}.$$

Note that the Laplacian commutes with $\partial^* \partial$ so the inverse Laplacian also commutes with $\partial^* \partial$. Therefore, the above equation becomes.

$$\tilde{h} = \partial_k^* \partial_k (\bar{\Delta}^k)^{-1} \tilde{h} + \partial_{k+1} \partial_{k+1}^* (\bar{\Delta}^k)^{-1} \tilde{h}.$$

where the first term is an element of B^k and the second term is an element of $Im \partial_{k+1}$. Therefore, $c = h + \tilde{h}$ where $h \in \mathcal{H}^k$ and $\tilde{h} \in Im \partial_{k+1} \oplus B^k$ which implies that $C^k = \mathcal{H}^k \oplus Im \partial_{k+1} \oplus B^k$.

Similarly, we can show that $Z^k = B^k \oplus \mathcal{H}^k$ where $Z^k = \{c \in C^k \mid \partial_{k+1}^* c = 0\}$. Since $\mathcal{H}^k \subset Z^k$ and $B^k \subset Z^k$ then it follows that $\mathcal{H}^k \oplus B^k \subset Z^k$. Now, let $c \in Z^k$ ($\partial_{k+1}^* c = 0$) where $c = \partial_k^* b + \partial_{k+1} d + h$ and $h \in \mathcal{H}^k$. Then applying ∂_{k+1}^* to c , we get

$$\partial_{k+1}^* c = \partial_{k+1}^* \partial_k^* b + \partial_{k+1}^* \partial_{k+1} d + \partial_{k+1}^* h = 0.$$

The terms $\partial_{k+1}^* \partial_k^* b$ and $\partial_{k+1}^* h$ are both equal zero. We are left with the term, $\partial_{k+1}^* \partial_{k+1} d = 0$. Therefore,

$$(\partial_{k+1}^* \partial_{k+1} d, d) = 0$$

which using the adjoint relation, gives $\partial_{k+1} d = 0$. Therefore, any $c \in Z^k$ can be written as $c = \partial_k^* b + h$ and since $B^k \perp \mathcal{H}^k$ then $B^k \cap \mathcal{H}^k = \emptyset$. Therefore, $Z^k = B^k \oplus \mathcal{H}^k$. $\square$

Chapter 4

THE DUAL OF THE CELL COMPLEX

Once we have obtained a polyhedral cellular decomposition for an n -dimensional manifold, it is possible to define the dual of the cell complex. The dual complex is important to our description of cochains, which represent physical quantities, on the complex. For example, as will be shown in Chapter 5, the cochain representing the electric field will be applied to chains on the initial (primary) complex and the cochain representing the magnetic field will be applied to chains on the dual complex.

In this chapter, the dual of a polyhedral cell complex, the one-to-one correspondence between the k -cells on the primary complex and the $(n - k)$ -cells on the dual complex given by a star operator and the relationship between the boundary operators on the primary complex and the transpose of the boundary operators on the dual complex will be defined. Examples of the construction of the dual complexes from various polyhedral primary complexes and the action of the star operator on these complexes will also be discussed.

4.1 Obtaining the Dual Complex

As discussed in Chapter 3, an n -dimensional polyhedral cell complex consists of interconnected, oriented 0-cells, 1-cells, 2-cells, ..., n -cells. Every k -dimensional cell is connected to a set of $(k - 1)$ dimensional cells (or faces) by the boundary operator. Every k -dimensional cell in a cell complex is connected to a set of $(k + 1)$ dimensional cells by the transpose of the boundary operator. The dual of a polyhedral cell complex is defined by setting up a one-to-one correspondence between the k -cells of the (primary) complex with the $(n - k)$ -cells of the dual complex. (Note that the primary and dual complexes are sometimes referred to as the primary and dual grids, meshes,

or lattices). This correspondence is achieved by creating the dual cell by connecting the barycenters, or centers of mass, of each neighboring n -cell in the primary complex.

The one-to-one correspondence between the k -cells on the primary complex and the $(n - k)$ -cells on the dual complex is defined by a star operator as follows. To construct the dual complex, we first make a subdivision of the primary complex by connecting the barycenters of each n -cell with the 0-cells (vertices) of that cell and with the barycenters of adjacent n -cells. Note that this construction cannot always be done. In a highly skewed case, the cell which connects the barycenters of two adjacent n -cells may intersect a third n -cell. In this dissertation, we have not examined the possibility of a highly skewed grid. One attempt at solving this problem, may be to use a subdivision which connects the barycenters of all k -cells. In other words, the 0-cells are connected to the barycenters of the 1-cells, 2-cells, etc., to form a tetrahedral subdivision. In our case, for each vertex, s_0^i , in the original cellular decomposition, let $\star s_0^i$ be the union of all cells in the first subdivision which contain the vertex s_0^i . Then, for each k -cell, s_k^i , in the original decomposition, let

$$\star s_k^i = \bigcap_{s_0^j \in s_k^i} \star s_0^j$$

be the intersection of the n -cells, $\star s_0^j$, associated to the vertices of s_k^i . The cells, $\star s_k^i$, then give a decomposition of M called the dual cell decomposition. Therefore, $\star$ gives the 1-1 correspondence described above. This star operator takes k -cells on the primary complex to $(n - k)$ -cells on the dual complex. The cells in the dual complex have an induced orientation from the primary complex. That is, if a 1-cell in the primary complex intersects a 2-cell in the dual complex transversely, then the induced orientation on the 2-cell follows the right hand rule.

Since the star operator gives this one-to-one correspondence, then a duality also exists between the boundary operators on the primary complex and the transpose of the boundary operators on the dual complex. Suppose that $\partial_k : C_k(M, \mathbb{R}) \rightarrow$

$C_{k-1}(M, \mathbb{R})$. Now, the star operator maps k -dimensional chains on the primary complex to $(n-k)$ -dimensional chains on the dual complex, $\star : C_k(M, \mathbb{R}) \rightarrow \underline{C_{n-k}(M, \mathbb{R})}$, where the underline denotes the space of chains in the dual complex. We also have that the star operator maps $(k-1)$ -dimensional chains on the primary complex to $(n-(k-1))$ -dimensional chains on the dual complex, $\star : C_{k-1}(M, \mathbb{R}) \rightarrow \underline{C_{n-k+1}(M, \mathbb{R})}$. Therefore, the matrix representing the boundary operator, as given in (3.2.1), on k -dimensional chains, $\partial_k : C_k(M, \mathbb{R}) \rightarrow C_{k-1}(M, \mathbb{R})$, is equivalent to the matrix representing the transpose boundary operator, as given in (3.2.3), on $(n-k)$ -chains in the dual complex, $\partial_{n-k+1}^T : \underline{C_{n-k}(M, \mathbb{R})} \rightarrow \underline{C_{n-k+1}(M, \mathbb{R})}$, where the underline on the transpose of the boundary operator denotes the transpose of the boundary operator on the dual complex.

4.2 Example on an Orthogonal Complex

One example of a polyhedral cellular decomposition on $\mathbb{R}^3$ is obtained by considering three dimensional cubes with edges of unit length. Since the three edges from a vertex of the cell are orthogonal to each other, this complex is considered to be an orthogonal complex. For an orthogonal cell complex, the metric tensor (g_{ij}) will always be diagonal. The metric and its inverse for the cubical case are both given by the Euclidean metric, $(g_{ij}) = (\delta_{ij})$. This is obtained as follows. Let the three unit vectors from the vertex of a cell in the direction of the edges be given by $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, and $\frac{\partial}{\partial z}$. Then the metric, (g_{ij}) , is given by

$$(g_{ij}) = \begin{pmatrix} \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x} \right) & \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) & \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial z} \right) \\ \left(\frac{\partial}{\partial y}, \frac{\partial}{\partial x} \right) & \left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y} \right) & \left(\frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \\ \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial x} \right) & \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial y} \right) & \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right) \end{pmatrix}.$$

Using the property that $\cos \theta = (w, v) / \|w\| \|v\|$, where θ is the angle between w and v and $\| \cdot \|$ is the norm, and that for an orthogonal complex $\theta = \pi/2$, we obtain the

Euclidean metric

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The barycenters of adjacent cubes are connected to obtain the dual complex as shown in Figure 4.1. The one-to-one correspondence between the k -cells of the primary com-

FIGURE 4.1. An Orthogonal Cell Complex and Its Dual.

plex and the $(3 - k)$ -cells of the dual complex is easily seen in Figure 4.1. The 0-cells, 1-cells, 2-cells and 3-cells of the primary complex are in a one-to-one correspondence with the 3-cells, 2-cells, 1-cells, and 0-cells of the dual complex, respectively. Each vertex of the primary complex corresponds with the barycenter of a cube in the dual complex. Each edge of the primary complex corresponds to a face of the cube in the dual complex. Each face in the primary complex corresponds to an edge in the dual complex and each cube in the primary complex corresponds to the barycenter of cube in the dual complex.

The star operator applied to the cochains on the orthogonal cell complex can be

FIGURE 4.2. The Oriented Faces and Edges of a Cubical 3-Cell.

defined through the star operator applied to the differential forms,

$$\star : \begin{cases} dx & \longrightarrow dy \, dz \\ dy & \longrightarrow dz \, dx \\ dz & \longrightarrow dx \, dy \\ dx \, dy & \longrightarrow dz \\ dy \, dz & \longrightarrow dx \\ dz \, dx & \longrightarrow dy \\ dx \, dy \, dz & \longrightarrow 1 \end{cases} .$$

The connection matrix representing the boundary operator, $\partial_k : C_k \rightarrow C_{k-1}$ for $k = 1, \dots, 3$ on the primary complex is equivalent to the connection matrix representing the transpose of the boundary operator, $\partial_{3-k+1}^T : C_{3-k} \rightarrow C_{3-k+1}$, on the dual complex. For example, given the 3-cell, oriented as shown in Figure 4.2, the connection matrix, (a_{ij}) , representing the boundary operator, $\partial_2 : C_2(M, \mathbb{R}) \rightarrow$

$C_1(M, \mathbb{R})$, in the primary complex is given by

$$(a_{ij}) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 \end{pmatrix}. \quad (4.2.1)$$

As an example of how this matrix is constructed, we can look at face 1, the front face. This face corresponds to the first column, (a_{i1}) , in the matrix. Face 1 is oriented in the clockwise direction. Following the edges which surround face 1 in the clockwise direction yields the entries: $a_{11} = -1$ since s_1^1 has the opposite orientation from the face, $a_{31} = +1$ since s_1^3 has the same orientation as the face, $a_{51} = -1$ since s_1^5 has the opposite orientation as the face, and $a_{19} = +1$ since s_1^9 has the same orientation as the face. All other entries in the first column are equal to zero since the corresponding edges are not connected to face 1. It can be shown that the matrix given by (4.2.1) is equal to the connection matrix representing the coboundary operator acting on the 1-cells in the dual complex, $\partial_2^T : \underline{C}_1 \rightarrow \underline{C}_2$.

4.3 Examples on Non-Orthogonal Complexes

In the following examples, we will be examining non-orthogonal complexes which are structured and unstructured. A structured complex is one such that the basic n -cell is translated to generate the entire n -complex. Therefore, each n -cell has the same geometrical shape. In contrast, in an unstructured complex there is no one basic n -cell which, if translated, would generate the entire n -complex. The parallelepiped and tetrahedron complexes, given in the following examples, are structured complexes.

The trapezoid complex is unstructured. While structured complexes make computations simpler, unstructured complexes are useful for conforming to complex curved geometries.

4.3.1 Parallelepiped Complex

One example of a polyhedral cell complex on $\mathbb{R}^3$ is obtained by considering three-dimensional parallelepipeds as shown in Figure 4.3. In this case, the complex is

FIGURE 4.3. A Parallelepiped Complex.

structured and non-orthogonal. The metric on a parallelepiped is obtained as follows. Let the three unit vectors originating from a vertex of a cell in the direction of the edges be given by $\frac{\partial}{\partial \xi}$, $\frac{\partial}{\partial \eta}$, and $\frac{\partial}{\partial \alpha}$. Then the metric is given by

$$(g_{ij}) = \begin{pmatrix} \left(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \xi} \right) & \left(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \eta} \right) & \left(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \alpha} \right) \\ \left(\frac{\partial}{\partial \eta}, \frac{\partial}{\partial \xi} \right) & \left(\frac{\partial}{\partial \eta}, \frac{\partial}{\partial \eta} \right) & \left(\frac{\partial}{\partial \eta}, \frac{\partial}{\partial \alpha} \right) \\ \left(\frac{\partial}{\partial \alpha}, \frac{\partial}{\partial \xi} \right) & \left(\frac{\partial}{\partial \alpha}, \frac{\partial}{\partial \eta} \right) & \left(\frac{\partial}{\partial \alpha}, \frac{\partial}{\partial \alpha} \right) \end{pmatrix}.$$

where the entries are determined from the following: $\cos \theta_1 = (\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \eta}) / (\|\frac{\partial}{\partial \xi}\| \|\frac{\partial}{\partial \eta}\|)$, $\cos \theta_2 = (\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \alpha}) / (\|\frac{\partial}{\partial \xi}\| \|\frac{\partial}{\partial \alpha}\|)$, and $\cos \theta_3 = (\frac{\partial}{\partial \eta}, \frac{\partial}{\partial \alpha}) / (\|\frac{\partial}{\partial \eta}\| \|\frac{\partial}{\partial \alpha}\|)$. Thus, the metric for

the parallelepiped complex is given by

$$(g_{ij}) = \begin{pmatrix} 1 & \cos \theta_1 & \cos \theta_2 \\ \cos \theta_1 & 1 & \cos \theta_3 \\ \cos \theta_2 & \cos \theta_3 & 1 \end{pmatrix}.$$

The inverse of (g_{ij}) is given by

$$(g^{ij}) = \frac{1}{\det g} \begin{pmatrix} 1 - \cos^2 \theta_3 & \cos \theta_2 \cos \theta_3 - \cos \theta_1 & \cos \theta_1 \cos \theta_3 - \cos \theta_2 \\ \cos \theta_2 \cos \theta_3 - \cos \theta_1 & 1 - \cos^2 \theta_2 & \cos \theta_1 \cos \theta_2 - \cos \theta_3 \\ \cos \theta_1 \cos \theta_3 - \cos \theta_2 & \cos \theta_1 \cos \theta_2 - \cos \theta_3 & 1 - \cos^2 \theta_1 \end{pmatrix} \quad (4.3.1.1)$$

where $\det g = 1 - \cos^2 \theta_1 - \cos^2 \theta_2 - \cos^2 \theta_3 + 2 \cos \theta_1 \cos \theta_2 \cos \theta_3$.

The dual complex is found by joining the barycenters of adjacent parallelepipeds as shown in Figure 4.4.

FIGURE 4.4. A Parallelepiped Complex and Its Dual.

Using the inverse of the metric as given by (4.3.1.1) and the definition of the star operator, we can determine the action of the star operator for this parallelepiped complex. Recall, that the star operator on differential forms is defined to be the map

$$\star : \Omega^k(M) \rightarrow \Omega^{n-k}(M).$$

In terms of the cochains, this map will now take a k -dimensional cochain on the primary complex to an $(n-k)$ -dimensional cochain on the dual complex. For differential forms, the star operator is determined through the relationship

$$\langle \omega, \mu \rangle V = \omega \wedge \star \mu$$

where $\omega, \mu \in \Omega^k(M)$ and V is the volume form on M . The inner product shown in the previous equation is taken with respect to (g^{ij}) , which is the inverse of the metric (g_{ij}) . For example, the volume form for this example is given by

$$V = \sqrt{\det g} d\xi d\eta d\alpha. \quad (4.3.1.2)$$

Using the inverse metric given in (4.3.1.1), the volume form given by (4.3.1.2), and the action of the star operator on a 1-form, $\omega = \omega_\xi d\xi + \omega_\eta d\eta + \omega_\alpha d\alpha$, from (2.2.2.3), results in

$$\begin{aligned} \star \omega = & \frac{1}{\sqrt{\det g}} \left\{ [(1 - \cos^2 \theta_3)\omega_\xi + (\cos \theta_2 \cos \theta_3 - \cos \theta_1)\omega_\eta \right. \\ & + (\cos \theta_1 \cos \theta_3 - \cos \theta_2)\omega_\alpha] d\eta d\alpha + [(\cos \theta_2 \cos \theta_3 - \cos \theta_1)\omega_\xi \\ & + (1 - \cos^2 \theta_2)\omega_\eta + (\cos \theta_1 \cos \theta_2 - \cos \theta_3)\omega_\alpha] d\alpha d\xi \\ & + [(\cos \theta_1 \cos \theta_3 - \cos \theta_2)\omega_\xi + (\cos \theta_1 \cos \theta_2 - \cos \theta_3)\omega_\eta \\ & \left. + (1 - \cos^2 \theta_1)\omega_\alpha] d\xi d\eta \right\}. \end{aligned}$$

The connection matrices representing the boundary and coboundary operators are found in a similar manner to that of the cubical cellular decomposition. The entries of the matrices representing the boundary and the transpose of the boundary operators are 0, +1, or -1 depending on how the k -cells are connected to the $(k-1)$ -cells and $(k+1)$ -cells, respectively.

4.3.2 Trapezoid Complex

Another example of a polyhedral cell complex on $\mathbb{R}^3$, which is an unstructured complex, is obtained by considering the trapezoid shown in Figure 4.5. The dual complex

FIGURE 4.5. A Trapezoidal Complex.

is once again determined by joining the barycenters of the neighboring 3-D trapezoids as shown in Figure 4.6. The metric for this cell is determined by the edges of the dual complex originating at the barycenter of the primary complex. This is the same method used in the first two examples. The only difference in this case is that the n -cells in the primary complex no longer have the same geometrical shape as the n -cells in the dual complex. Therefore, the metric defined by the edges of the dual complex is different from the metric defined by the edges of the primary complex. Due to the angle θ , $0 < \theta < \pi$, between two of the edges which originate from the barycenter of the shaded cell in Figure 4.6 and an angle of $\pi/2$ between the other two edges, then the metric is given by

$$(g_{ij}) = \begin{pmatrix} 1 & \cos \theta & 0 \\ \cos \theta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

FIGURE 4.6. A Dual Trapezoid Complex.

The inverse of the metric is given by

$$(g^{ij}) = \begin{pmatrix} \frac{1}{\sin^2 \theta} & -\frac{\cos \theta}{\sin^2 \theta} & 0 \\ -\frac{\cos \theta}{\sin^2 \theta} & \frac{1}{\sin^2 \theta} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For this case, the volume form is given by

$$V = \sin \theta d\xi d\eta dz.$$

Using the inverse of the metric, the volume form, and the explicit formula for the star operator applied to forms, (2.2.2.3), we can determine how the star operator acts on differential forms of rank 0, 1, 2, and 3. For example, on a 1-form $\omega = \omega_\xi d\xi + \omega_\eta d\eta + \omega_z dz$, the star operator is given by

$$\star \omega = \left(\frac{1}{\sin \theta} \omega_\xi - \frac{\cos \theta}{\sin \theta} \omega_\eta \right) d\eta dz + \left(\frac{1}{\sin \theta} \omega_\eta - \frac{\cos \theta}{\sin \theta} \omega_\xi \right) dz d\xi + \sin \theta \omega_z d\xi d\eta.$$

Once again, the connection matrices representing the boundary and coboundary

operators have entries 0, +1, and -1 according to how the k -cells are connected to the $(k - 1)$ -cells and $(k + 1)$ -cells, respectively.

4.3.3 Tetrahedral Complex

In the fourth example, a cellular decomposition of $\mathbb{R}^2$ is obtained by considering triangular cells. The following argument can be extended to $\mathbb{R}^3$ using a tetrahedral decomposition. The importance of this type of a decomposition, which is called a simplicial decomposition, is that any compact manifold possesses a simplicial decomposition. The proof of this can be found in many standard algebraic topology text books [27].

Consider the equilateral triangular complex and its dual complex as shown in Figure 4.7. Although this example uses equilateral triangles, this is not necessary.

FIGURE 4.7. Triangular Complex and Its Dual.

It is only done here for simplicity. The dual complex as shown in Figure 4.7 is a hexagon. The metric on the dual complex is given by

$$(g_{ij}) = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$$

where $\cos(\pi/3) = 1/2$ and the inverse metric is given by

$$(g^{ij}) = \begin{pmatrix} 4/3 & -2/3 \\ -2/3 & 4/3 \end{pmatrix}.$$

For a triangular decomposition such that the triangles are not equilateral, the metric must be chosen according to which face of the dual complex that the update equations will be solved on. This occurs because there are now more than three edges originating from each vertex. The metric on the primary complex is determined similarly. Suppose the form ω does not have a z -component. Using the inverse metric and the volume form, $V = \sin\theta d\xi d\eta$, the star operator applied to the 1-form, $\omega = \omega_\xi d\xi + \omega_\eta d\eta$, is given by

$$\star\omega = \left(\frac{1}{\sin\theta}\omega_\xi - \frac{\cos\theta}{\sin\theta}\omega_\eta \right) d\eta - \left(\frac{1}{\sin\theta}\omega_\eta - \frac{\cos\theta}{\sin\theta}\omega_\xi \right) d\xi.$$

Once again, the boundary and the transpose of the boundary operators are determined in the same manner as discussed for the other examples.

Chapter 5

CHAINS, COCHAINS AND MAXWELL'S EQUATIONS

As was stated at the beginning of Chapter 4, cochains, which represent physical quantities, can be applied to chains on the primary complex or to chains on the dual complex. In particular, as we will show in this chapter, the cochain generated by the electric field 1-form is applied to 1-chains on the primary complex and the cochain generated by the magnetic field 1-form is applied to 1-chains on the dual complex. The cochains generated by the electric and magnetic flux 2-forms are related to the the cochains generated by the electric and magnetic field 1-forms through constitutive relationships involving the star operator and the medium properties.

In this chapter, the inner product of chains and cochains, its integral representation, and the applications of these constructs to Maxwell's equations will be discussed.

5.1 The Inner Product and Integral Relations

How cochains are applied to the cellular complex is described by the inner product of the cochain with a chain in the complex. The inner product of a cochain and a chain may be written in terms of an integral where the integrand is the differential form which generates the cochain and the domain of integration is the chain. For example, suppose $\tilde{c}_E \in C^1(M, \mathbb{R})$ is the cochain generated by the differential form E and $c_1 \in C_1(M, \mathbb{R})$ is a 1-chain in the complex. Then, we can write the inner product of this cochain and chain as

$$\langle \tilde{c}_E, c_1 \rangle = \int_{c_1} E.$$

If we apply the exterior derivative to E to obtain $dE \in \Omega^2(M)$, then the cochain generated by dE will be denoted by $\tilde{c}_{dE} \in C^2(M, \mathbb{R})$. Using (3.6.3), we can write

$$\langle \tilde{c}_{dE}, c_2 \rangle = \langle \partial^* \tilde{c}_E, c_2 \rangle = \int_{c_2} dE. \quad (5.1.1)$$

where $c_2 \in C_2(M, \mathbb{R})$ is a 2-chain. Recalling Stokes's theorem for chains, we can write (5.1.1) as

$$\langle \partial^* \tilde{c}_E, c_2 \rangle = \langle \tilde{c}_E, \partial c_2 \rangle = \int_{\partial c_2} E.$$

where $\partial c_2 \in C_1(M, \mathbb{R})$ is a 1-chain.

Now suppose $\tilde{c}_B \in C^2(M, \mathbb{R})$ is the cochain generated by the differential form. $B \in \Omega^2(M)$. The inner product of $\tilde{c}_B$ with the chain. $c_2 \in C^2(M, \mathbb{R})$ is written in terms of the integral equation as

$$\langle \tilde{c}_B, c_2 \rangle = \int_{c_2} B.$$

Once again, applying the exterior derivative to B results in the 3-form, $dB \in \Omega^3(M)$. The inner product of the cochain generated by dB , $\tilde{c}_{dB} \in C^3(M, \mathbb{R})$, with the 3-chain. $c_3 \in C_3(M, \mathbb{R})$, is given by the integral equation

$$\langle \tilde{c}_{dB}, c_3 \rangle = \langle \partial^* \tilde{c}_B, c_3 \rangle = \int_{c_3} dB. \quad (5.1.2)$$

Recalling Stokes's theorem for chains, we can write (5.1.2) as

$$\langle \partial^* \tilde{c}_B, c_3 \rangle = \langle \tilde{c}_B, \partial c_3 \rangle = \int_{\partial c_3} B.$$

5.2 Applications to Maxwell's Equations

5.2.1 Differential Forms and Maxwell's Equations

Before describing an application of chains and cochains on a cellular complex, it is important to understand the continuous representation of that application. Maxwell's

equations have a very natural representation in terms of differential forms, which are continuous objects. In vector notation, we can write Maxwell's equations as

$$\nabla \times \vec{E} = -\partial_t \vec{B} \quad (5.2.1.1)$$

$$\nabla \cdot \vec{B} = 0 \quad (5.2.1.2)$$

and

$$\nabla \times \vec{H} = \partial_t \vec{D} + \vec{J} \quad (5.2.1.3)$$

$$\nabla \cdot \vec{D} = \rho \quad (5.2.1.4)$$

where (5.2.1.1) and (5.2.1.2) are Faraday's law and its corresponding divergence relation, respectively, and (5.2.1.3) and (5.2.1.4) are Ampere's law and its corresponding divergence relation, respectively. The electric and magnetic fields are denoted by $\vec{E}$ and $\vec{H}$, respectively, and the electric and magnetic fluxes are denoted by $\vec{D}$ and $\vec{B}$, respectively. The current density is denoted by $\vec{J}$ and the charge is denoted by ρ .

Recalling that the exterior derivative of a 1-form is similar to the curl of a vector and that the exterior derivative of a 2-form is similar to the divergence of a vector, we can write Maxwell's equations in differential form notation as

$$dE = -\partial_t B \quad (5.2.1.5)$$

$$dB = 0 \quad (5.2.1.6)$$

and

$$dH = \partial_t D + J \quad (5.2.1.7)$$

$$dD = \rho \quad (5.2.1.8)$$

where $E \in \Omega^1(\mathbb{R}^3)$ and $H \in \Omega^1(\mathbb{R}^3)$ represent the electric and magnetic fields, respectively; $B \in \Omega^2(\mathbb{R}^3)$ and $D \in \Omega^2(\mathbb{R}^3)$ represent the electric and magnetic fluxes,

respectively; and $\rho \in \Omega^3(\mathbb{R}^3)$ represents the charge density and $J \in \Omega^2(\mathbb{R}^3)$ represents the current density. Another way to describe Maxwell's equations in terms of differential forms is obtained in space-time by defining the unified electromagnetic field, $F \in \Omega^2(\mathbb{R}^{3+1})$, as

$$F = B + E \wedge dt$$

where $\mathbb{R}^{3+1}$ refers to Minkowski space. By applying the exterior derivative to F and setting this equal to zero, we obtain Faraday's law and the corresponding divergence relation in differential form notation.

$$dF = 0 \implies \begin{cases} dE = -\partial_t B \\ dB = 0. \end{cases}$$

By applying the adjoint of the exterior derivative to F and setting this equal to ρ , we obtain Ampere's law and the corresponding divergence relation in differential form notation.

$$d^*F = \rho \implies \begin{cases} dH = \partial_t D + J \\ dD = \rho. \end{cases}$$

The magnetic field 1-form and the magnetic flux 2-form are related by the star operator through the constitutive relation

$$B = \mu \star H$$

where μ is the permeability of the medium. The electric field 1-form and the electric flux 2-form are related by the star operator through the constitutive relation

$$D = \epsilon \star E$$

where ϵ is the permittivity of the medium. The permeability and permittivity specify the characteristics of the medium in which the fields exist.

Using the constitutive relationships, we can write (5.2.1.5) through (5.2.1.8) as

$$dE = -\partial_t(\mu \star H) \quad (5.2.1.9)$$

$$d(\mu \star H) = 0 \quad (5.2.1.10)$$

and

$$dH = \partial_t(\varepsilon \star E) + J \quad (5.2.1.11)$$

$$d(\varepsilon \star E) = \rho. \quad (5.2.1.12)$$

5.2.2 Chains, Cochains and Maxwell's Equations

Now that Maxwell's equations have been written in terms of differential forms, the discrete equations involving chains and cochains and their integral forms can be determined. Using the relationship between differential forms and cochains, we can describe Maxwell's equations on a cellular complex as follows. The first observation which needs to be made is that the electric field components lie on the edges of the primary complex and the magnetic field components lie on the edges of the dual complex. Therefore, the 1-dimensional cochains representing $E \in \Omega^1(M)$ and $H \in \Omega^1(M)$ will be applied to 1-chains in the primary complex and dual complex, respectively.

Let $\tilde{c}_E$ be the cochain generated by the electric field 1-form E and $\tilde{c}_B$ be the cochain generated by the magnetic flux 2-form B . Then, from (5.2.1.5) and (5.2.1.6), we have for Faraday's law and the corresponding divergence relation on a complex that

$$\langle \tilde{c}_{dE}, c_2 \rangle = -\partial_t \langle \tilde{c}_B, c_2 \rangle \quad (5.2.2.1)$$

$$\langle \tilde{c}_{dB}, c_3 \rangle = 0 \quad (5.2.2.2)$$

where $c_2 \in C_2(M, \mathbb{R})$ is a 2-chain, or face, in the primary complex and $c_3 \in C_3(M, \mathbb{R})$ is a 3-chain, or volume element, in the primary complex. By (3.6.3), we have that

$\langle \bar{c}_{dE}, c_2 \rangle = \langle \partial^* \bar{c}_E, c_2 \rangle$. Using Stokes's theorem for chains, (5.2.2.1) and (5.2.2.2) become

$$\langle \bar{c}_E, \partial c_2 \rangle = -\partial_t \langle \bar{c}_B, c_2 \rangle \quad (5.2.2.3)$$

$$\langle \bar{c}_B, \partial c_3 \rangle = 0 \quad (5.2.2.4)$$

where $\partial c_2 \in C_1(M, \mathbb{R})$ is a 1-chain on the primary complex and $\partial c_3 \in C_2(M, \mathbb{R})$ is a 2-chain on the primary complex. Using the constitutive relationship, $B = \mu \star H$, then Faraday's law and the corresponding divergence relation on a complex become

$$\langle \bar{c}_E, \partial c_2 \rangle = -\partial_t \langle \bar{c}_{(\mu \star H)}, c_2 \rangle \quad (5.2.2.5)$$

$$\langle \bar{c}_{(\mu \star H)}, \partial c_3 \rangle = 0 \quad (5.2.2.6)$$

where $\bar{c}_{(\mu \star H)} \in C^2(M, \mathbb{R})$ is the cochain generated by the 2-form $\mu \star H$. Although the cochain generated by H is applied to chains on the dual complex, the star operator takes the cochain generated by H to the primary complex. Using (3.6.1), the integral forms of (5.2.2.5) and (5.2.2.6) are given by

$$\int_{\partial c_2} E = -\partial_t \int_{c_2} (\mu \star H) \quad (5.2.2.7)$$

$$\int_{\partial c_3} (\mu \star H) = 0. \quad (5.2.2.8)$$

In order to obtain Ampere's law and the corresponding divergence relation on a complex, we begin by letting $\bar{c}_H$, $\bar{c}_D$, $\bar{c}_J$ and $\bar{c}_\rho$ be the cochains generated by the magnetic field 1-form, H , the electric flux 2-form, D , the current density 2-form, J and the charge 3-form, ρ . Then, from (5.2.1.7) and (5.2.1.8), we have

$$\langle \bar{c}_{dH}, \underline{c}_2 \rangle = \partial_t \langle \bar{c}_D, \underline{c}_2 \rangle + \langle \bar{c}_J, \underline{c}_2 \rangle \quad (5.2.2.9)$$

$$\langle \bar{c}_{dD}, \underline{c}_3 \rangle = \langle \bar{c}_\rho, \underline{c}_3 \rangle \quad (5.2.2.10)$$

where $\underline{c}_2 \in C_2(M, \mathbb{R})$ is a 2-chain, or face, in the dual complex and $\underline{c}_3 \in C_3(M, \mathbb{R})$ is a 3-chain, or volume element, in the dual complex. Note that we are now working on the dual complex since the magnetic field cochain is applied to chains on the dual complex. By (3.6.3), we have that $\langle \bar{c}_{dH}, \underline{c}_2 \rangle = \langle \partial^* \bar{c}_H, \underline{c}_2 \rangle$. Using Stokes's theorem for chains, (5.2.2.9) and (5.2.2.10) become

$$\langle \bar{c}_H, \underline{\partial c}_2 \rangle = \partial_t \langle \bar{c}_D, \underline{c}_2 \rangle + \langle \bar{c}_J, \underline{c}_2 \rangle \quad (5.2.2.11)$$

$$\langle \bar{c}_D, \underline{\partial c}_3 \rangle = \langle \bar{c}_\rho, \underline{c}_3 \rangle \quad (5.2.2.12)$$

where $\underline{\partial c}_2 \in C_1(M, \mathbb{R})$ is a 1-chain on the dual complex and $\underline{\partial c}_3 \in C_2(M, \mathbb{R})$ is a 2-chain on the dual complex. Using the constitutive relationship, $D = \varepsilon \star E$, then Ampere's law and the corresponding divergence relation on the dual complex become

$$\langle \bar{c}_H, \underline{\partial c}_2 \rangle = \partial_t \langle \bar{c}_{(\varepsilon \star E)}, \underline{c}_2 \rangle + \langle \bar{c}_J, \underline{c}_2 \rangle \quad (5.2.2.13)$$

$$\langle \bar{c}_{(\varepsilon \star E)}, \underline{\partial c}_3 \rangle = \langle \bar{c}_\rho, \underline{c}_3 \rangle \quad (5.2.2.14)$$

where $\bar{c}_{(\varepsilon \star E)} \in C^2(M, \mathbb{R})$ is the cochain generated by the 2-form $\varepsilon \star E$. Now, whereas $\bar{c}_E$ is a cochain applied to 1-chains in the primary complex, $\bar{c}_{(\varepsilon \star E)}$ is a cochain applied to 2-chains in the dual complex. Using (3.6.1), the integral forms of (5.2.2.13) and (5.2.2.14) are given by

$$\int_{\underline{\partial c}_2} H = \partial_t \int_{\underline{c}_2} (\varepsilon \star E) + \int_{\underline{c}_2} J \quad (5.2.2.15)$$

$$\int_{\underline{\partial c}_3} (\varepsilon \star E) = \int_{\underline{c}_3} \rho. \quad (5.2.2.16)$$

5.2.3 Divergence Free Properties

It is important that the divergence of the electric and magnetic fields is preserved in Maxwell's equations. In the continuum, the divergence of the electric and magnetic

fields given by (5.2.1.6) and (5.2.1.8) is preserved if $dD = \rho$ and $dB = 0$ for every time t . In particular, if $\rho = 0$ at time $t = 0$, then $dD = 0$ for all time t . Showing $dD = 0$ and $dB = 0$ is equivalent to showing that $\varepsilon d \star E = 0$ and $\mu d \star H = 0$ or $d \star E = 0$ and $d \star H = 0$ in a simple medium. Applying the exterior derivative to Faraday's law, $-\mu \partial_t (\star H) = dE$, in differential form notation, results in the expression

$$-\mu \partial_t (d \star H) = ddE. \quad (5.2.3.1)$$

By the property that $dd = 0$, we know that the right hand side of (5.2.3.1) is identically equal to 0. Suppose the cochain generated by ddE is given by $\bar{c}_{ddE}$ and the cochain generated by $d \star H$ is given by $\bar{c}_{(d \star H)}$. Then, if we let $c_3 \in C_3(M, \mathbb{R})$ be a 3-chain on the primary complex, we can then write the inner product of c_3 with $\bar{c}_{ddE}$ as

$$-\mu \partial_t \langle \bar{c}_{(d \star H)}, c_3 \rangle = \langle \bar{c}_{ddE}, c_3 \rangle. \quad (5.2.3.2)$$

Applying Stokes's theorem for chains twice to the right hand side of (5.2.3.2), we obtain

$$-\mu \partial_t \langle \bar{c}_{(d \star H)}, c_3 \rangle = \langle \bar{c}_E, \partial \partial c_3 \rangle = 0 \quad (5.2.3.3)$$

where the right-hand side is equal to zero from the property, $\partial \partial = 0$. Now, applying Stokes's theorem to the left hand side of (5.2.3.3), results in

$$-\mu \partial_t \langle \bar{c}_{(\star H)}, \partial c_3 \rangle = -\mu \partial_t \langle \bar{c}_{(d \star H)}, c_3 \rangle = 0 \quad (5.2.3.4)$$

where $\partial c_3 \in C_2(M, \mathbb{R})$ is the bounding 2-chain of the 3-chain given by c_3 . An important thing to notice is the following. Suppose c_2 is the 2-chain, a face, for which we want to determine the update equations for the magnetic field. Then by applying the transpose of the boundary operator to the face, c_2 , we obtain the 3-chain, the two volume cells, attached to this face. The boundary of c_3 , ∂c_3 , is then given by the twelve faces of those two 3-cells with assigned orientations. The face c_2 will be

counted twice but with opposite orientation. Therefore, it will not figure into the calculation of ∂c_3 . Thus, the divergence free properties will give the magnetic field on c_2 in terms of the field components on each of the surrounding ten faces. In integral form, we obtain

$$-\mu \partial_t \int_{\partial c_3} (\star H) = -\mu \partial_t \int_{c_3} (d \star H) = 0. \quad (5.2.3.5)$$

Similarly, applying the exterior derivative to Ampere's law, $\varepsilon \partial_t \star E = dH$, in differential form notation, we obtain

$$\varepsilon \partial_t (d \star E) = ddH. \quad (5.2.3.6)$$

Suppose the cochain generated by ddH is given by $\tilde{c}_{ddH}$ and the cochain generated by $d \star E$ is given by $\tilde{c}_{(d \star E)}$. Let $\underline{c}_3 \in \underline{C}_3(M, \mathbb{R})$ be a 3-chain in the dual complex attached to the 2-chain $\underline{c}_2 \in \underline{C}_2(\mathbb{R})$ by the coboundary operator, as in Ampere's laws on a complex. Then we can write the inner product of $\underline{c}_3$ with $\tilde{c}_{ddH}$ as

$$\varepsilon \partial_t \langle \tilde{c}_{(d \star E)}, \underline{c}_3 \rangle = \langle \tilde{c}_{ddH}, \underline{c}_3 \rangle. \quad (5.2.3.7)$$

Applying Stokes's theorem for chains to the right hand side of (5.2.3.7), we obtain

$$\varepsilon \partial_t \langle \tilde{c}_{(d \star E)}, \underline{c}_3 \rangle = \langle \tilde{c}_H, \partial \underline{c}_3 \rangle = 0 \quad (5.2.3.8)$$

since we know from Chapter 3 that $\partial \partial = 0$. Now applying Stokes's theorem to the left hand side of (5.2.3.8), results in

$$\varepsilon \partial_t \langle \tilde{c}_{(\star E)}, \underline{\partial c}_3 \rangle = \varepsilon \partial_t \langle \tilde{c}_{(d \star E)}, \underline{c}_3 \rangle = 0 \quad (5.2.3.9)$$

where $\underline{\partial c}_3 \in \underline{C}_2(M, \mathbb{R})$ is the bounding 2-chain of the 3-chain on the dual complex given by $\underline{c}_3$. In integral form, we obtain

$$\varepsilon \partial_t \int_{\underline{\partial c}_3} (\star E) = \varepsilon \partial_t \int_{\underline{c}_3} (d \star E) = 0. \quad (5.2.3.10)$$

Thus, both (5.2.3.5) and (5.2.3.10) show that if $d \star E = 0$ and $d \star H = 0$ at time $t = 0$, then they are so for all time: i.e., the divergence-free properties are preserved.

Chapter 6

ALGORITHMS

As was shown in Chapter 5, Maxwell's equations can be written in terms of chains and cochains on a primary complex and its dual complex. By introducing this notation, we are able to develop a numerical method for solving Maxwell's equations. When the geometry of the manifold under consideration is uncomplicated, Maxwell's equations can sometimes be solved analytically. However, as the geometry of the manifold becomes more complicated, numerical methods can provide a good alternative.

In this chapter, I will introduce the discrete differential form algorithm which follows from the mathematical background given in the preceding chapters. I will also introduce two other algorithms: the algorithm developed by K.S. Yee, and the Discrete Surface Integral (DSI) algorithm. Comparisons will be made between the three algorithms: Yee's, DSI, and discrete differential form. In the final section, I will provide examples of the update equations for Maxwell's equations using the discrete differential form algorithm for various cell decompositions. Throughout the development of this algorithm, a simple medium is assumed. Thus, the properties of the medium for which the electric and magnetic fields exist, is given by a constant permeability and permittivity.

6.1 Discrete Differential Form Algorithm

Solving Maxwell's equations numerically using chains and cochains is as natural as writing Maxwell's equations in terms of differential forms. Equations (5.2.2.5) and (5.2.2.13) in Chapter 5 express Faraday's and Ampere's laws on a complex, respec-

FIGURE 6.1. A 2-cell Model for the Discrete Differential Form Algorithm.

tively, as

$$-\partial_t \langle \tilde{c}_{(\mu \star H)}, c_2 \rangle = \langle \tilde{c}_E, \partial c_2 \rangle \implies -\partial_t \int_{c_2} (\mu \star H) = \int_{\partial c_2} E \quad (6.1.1)$$

and

$$\partial_t \langle \tilde{c}_{(\varepsilon \star E)}, \underline{c}_2 \rangle = \langle \tilde{c}_H, \partial \underline{c}_2 \rangle \implies \partial_t \int_{\underline{c}_2} (\varepsilon \star E) = \int_{\partial \underline{c}_2} H \quad (6.1.2)$$

where c_2 is a 2-chain in the primary complex and $\underline{c}_2$ is a 2-chain in the dual complex. A 2-cell model is shown in Figure 6.1. Using these equations and the divergence equations, the discrete differential form algorithm can be determined. For each step in the algorithm, an example of determining the update equation for the E_η component of a transverse electric wave will be given. The cell complex in the example will be a parallelepiped complex with one angle, $\theta \neq \pi/2$, skewed, as shown in Figure 6.2.

The first steps in the discrete differential form algorithm are to construct the dual complex, and to define the star operator on the primary and dual complexes. The construction of the dual complex by connecting the barycenters of the primary complex was given in Chapter 4. When applying the star operator to a differential 1-form that generates a cochain on the primary complex, such as E , we define the metric and its inverse at the barycenter of a cell on the dual complex. The barycenter of the

FIGURE 6.2. Parallelepiped Complex With One Angle Skewed.

dual cell is chosen since it corresponds to the vertex of a primary cell. For example, if (g_{ij}) is the metric defined at the barycenter of a dual cell, then

$$\begin{aligned}
 \star E &= (g^{11}E_\xi + g^{12}E_\eta + g^{13}E_\alpha)\sqrt{\det(g_{ij})}d\eta \wedge d\alpha \\
 &+ (g^{21}E_\xi + g^{22}E_\eta + g^{23}E_\alpha)\sqrt{\det(g_{ij})}d\alpha \wedge d\xi \\
 &+ (g^{31}E_\xi + g^{32}E_\eta + g^{33}E_\alpha)\sqrt{\det(g_{ij})}d\xi \wedge d\eta.
 \end{aligned} \tag{6.1.3}$$

Likewise, when applying the star operator to a differential 1-form that generates a cochain on the dual complex, such as H , we define the metric and its inverse at the barycenter of a cell on the primary complex. The barycenter of the primary cell is chosen since it corresponds to the vertex of a dual cell. Using this convention, we can then define the basis 1-chains as originating from that vertex. Using the star operator and metric as defined above, the star operator applied to the electric and magnetic field 1-forms is determined for the specific cell complex.

Thus, for the example of the skewed parallelepiped where $0 < \theta < \pi$, the metric

and its inverse are given by the matrices

$$(g_{ij}) = \begin{pmatrix} 1 & \cos \theta & 0 \\ \cos \theta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

and

$$(g^{ij}) = \begin{pmatrix} \frac{1}{\sin^2 \theta} & -\frac{\cos \theta}{\sin^2 \theta} & 0 \\ -\frac{\cos \theta}{\sin^2 \theta} & \frac{1}{\sin^2 \theta} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

respectively. The volume form, $V = \sqrt{\det(g_{ij})} d\xi d\eta dz$, for this complex is given by $V = \sin \theta d\xi d\eta dz$. With this volume form and the inverse metric, the star operator applied to the electric field 1-form $E = E_\xi d\xi + E_\eta d\eta$ ($E_z = 0$ since E is a transverse electric wave) yields

$$\varepsilon \star E = \varepsilon \left(\frac{1}{\sin \theta} E_\xi - \frac{\cos \theta}{\sin \theta} E_\eta \right) d\eta dz + \varepsilon \left(\frac{1}{\sin \theta} E_\eta - \frac{\cos \theta}{\sin \theta} E_\xi \right) dz d\xi. \quad (6.1.4)$$

The second step in determining the update equations, is to substitute $\mu \star H$ and $\varepsilon \star E$ into (6.1.1) and (6.1.2), respectively and simplify the equations. If you are interested in obtaining the update equations for the magnetic field components, then solve (6.1.1) for the magnetic field component of interest, i.e. H_ξ , H_η or H_α . If you are interested in obtaining the update equations for the electric field components, then solve (6.1.2) for the electric field component of interest, E_ξ , E_η or E_α . For example, substituting the magnetic field 1-form, $H = H_\xi d\xi + H_\eta d\eta + H_\alpha d\alpha$ and the star operator applied to the electric field 1-form, E , as in (6.1.3), into (6.1.2) results in

$$\begin{aligned} & \varepsilon \partial_t \int_{\underline{c_2}} (g^{11} E_\xi + g^{12} E_\eta + g^{13} E_\alpha) \sqrt{\det(g_{ij})} d\eta d\alpha \\ & + \varepsilon \partial_t \int_{\underline{c_2}} (g^{12} E_\xi + g^{22} E_\eta + g^{23} E_\alpha) \sqrt{\det(g_{ij})} d\alpha d\xi \\ & + \varepsilon \partial_t \int_{\underline{c_2}} (g^{13} E_\xi + g^{23} E_\eta + g^{33} E_\alpha) \sqrt{\det(g_{ij})} d\xi d\eta = \int_{\partial \underline{c_2}} (H_\xi d\xi + H_\eta d\eta + H_\alpha d\alpha). \end{aligned}$$

Thus, if we wanted to determine the E_η component on the 2-cell, $\underline{c}_2$, then

$$\begin{aligned}
 & \varepsilon \partial_t \int_{\underline{c}_2} \sqrt{\det(g_{ij})} E_\eta (g^{12} d\eta d\alpha + g^{22} d\alpha d\xi + g^{32} d\xi d\eta) \\
 &= \int_{\partial \underline{c}_2} (H_\xi d\xi + H_\eta d\eta + H_\alpha d\alpha) \\
 &- \varepsilon \partial_t \int_{\underline{c}_2} \left[(g^{11} E_\xi + g^{13} E_\alpha) \sqrt{\det(g_{ij})} d\eta d\alpha + (g^{21} E_\xi + g^{23} E_\alpha) \sqrt{\det(g_{ij})} d\alpha d\xi \right. \\
 &+ \left. (g^{31} E_\xi + g^{33} E_\alpha) \sqrt{\det(g_{ij})} d\xi d\eta \right].
 \end{aligned} \tag{6.1.5}$$

For example on the skewed grid, suppose we let $\underline{c}_2 \in C_2(M, \mathbb{R})$ be the face in the ξz -plane as shown in Figure 6.2. Then, substituting $\varepsilon \star E$ into (6.1.2) yields.

$$\varepsilon \partial_t \left[\int_{\underline{c}_2} \left(\frac{1}{\sin \theta} E_\xi - \frac{\cos \theta}{\sin \theta} E_\eta \right) d\eta dz + \int_{\underline{c}_2} \left(\frac{1}{\sin \theta} E_\eta - \frac{\cos \theta}{\sin \theta} E_\xi \right) dz d\xi \right] = \int_{\partial \underline{c}_2} H_z dz. \tag{6.1.6}$$

Since $\underline{c}_2$ is in the ξz -plane, we know that the first integral on the left-hand side of (6.1.6) is equal to zero. Therefore, (6.1.6) becomes

$$\varepsilon \partial_t \int_{\underline{c}_2} \left(\frac{1}{\sin \theta} E_\eta - \frac{\cos \theta}{\sin \theta} E_\xi \right) dz d\xi = \int_{\partial \underline{c}_2} H_z dz. \tag{6.1.7}$$

Suppose we are looking for the update equations for E_η on $\underline{c}_2$. Then, we can solve (6.1.7) for E_η in terms of E_ξ and H_z

$$\varepsilon \partial_t \int_{\underline{c}_2} \frac{1}{\sin \theta} E_\eta dz d\xi = \int_{\underline{c}_2} H_z dz + \varepsilon \int_{\underline{c}_2} \frac{\cos \theta}{\sin \theta} E_\xi dz d\xi.$$

Once we have solved for the field component of interest, we can determine the other field components on the right-hand side of (6.1.5) using the divergence preserving properties.

Thus, the third step is to use the divergence preserving properties, which were introduced in Section 5.2.3, to determine the field components on the right-hand side of (6.1.5). From (5.2.3.4) and (5.2.3.9), we know that

$$\langle \bar{c}_{(\star H)}, \partial c_3 \rangle = \langle \bar{c}_{(d \star H)}, c_3 \rangle \tag{6.1.8}$$

and

$$\langle \tilde{c}_{(\star E)}, \underline{\partial c_3} \rangle = \langle \tilde{c}_{(d \star E)}, \underline{c_3} \rangle \quad (6.1.9)$$

where $c_3 \in C_3(M, \mathbb{R})$ is the 3-chain on the primary complex connected through the transpose of the boundary operator to the 2-chain, $c_2 = \partial^T c_3$, in (6.1.1) and $\underline{c_3} \in \underline{C_3}(M, \mathbb{R})$ is the 3-chain on the dual complex connected through the transpose of the boundary operator to the 2-chain, $\underline{c_2} = \underline{\partial^T c_3}$, in (6.1.2). The integral relations for (6.1.8) and (6.1.9) are given by

$$\int_{c_3} (d \star H) = \int_{\partial c_3} (\star H) \quad (6.1.10)$$

and

$$\int_{\underline{c_3}} (d \star E) = \int_{\underline{\partial c_3}} (\star E), \quad (6.1.11)$$

respectively. Examining (6.1.10) and (6.1.11), we can see that the divergence of $\star H$ and $\star E$ on the volume chains c_3 and $\underline{c_3}$, respectively, reduces to the values of $\star E$ and $\star H$ on the bounding faces of those volume chains, respectively. This fact will help in determining the unknown field components.

For the example on the parallelepiped complex as shown in Figure 6.2, (6.1.11) is given as

$$\begin{aligned} &= \int_{\underline{c_3}} \left[\frac{1}{\sin \theta} (\partial_\xi E_\xi + \partial_\eta E_\eta) - \frac{\cos \theta}{\sin \theta} (\partial_\xi E_\eta + \partial_\eta E_\xi) \right] d\xi d\eta dz \\ &\int_{\underline{\partial c_3}} \left[\frac{1}{\sin \theta} E_\xi - \frac{\cos \theta}{\sin \theta} E_\eta \right] d\eta dz + \int_{\underline{\partial c_3}} \left[\frac{1}{\sin \theta} E_\eta - \frac{\cos \theta}{\sin \theta} E_\xi \right] dz d\xi = 0. \end{aligned} \quad (6.1.12)$$

where the right hand side is equal to zero because we are dealing with a source free region. Now, since $\underline{c_3}$ and $\underline{c_2}$ are related by the transpose of the boundary operator, we have that $\underline{c_3}$ is given by the two volume cells which share the face $\underline{c_2}$. Therefore, the boundary of $\underline{c_3}$ is given by the bounding faces of those two volume cells. If, for example, the faces of one of the volume cells are assigned the orientations as shown in

Figure 4.2, then we can determine the connection matrix based on the property that $\partial\partial = 0$. For example, for the chosen orientations of the faces bounding that cell, the connection matrix representing the boundary operator on the volume cell is given by

$$\partial_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

where the faces are ordered as top, bottom, front, back, left, and right. Therefore, the boundary of the cell shown in Figure 4.2 is given by $-(top\,face) + (bottom\,face) + (front\,face) - (back\,face) - (right\,face) + (left\,face)$.

In the fourth step, we make our numerical choices based on the divergence preserving properties. In order for the integral equations (6.1.10) and (6.1.11) to be zero to second order accuracy, we choose the unknown components to be the average of the values of the components on the bounding faces of the 3-cells attached to the face of interest. So, from (6.1.5), we would let

$$E_\xi = \frac{1}{4}(E_\xi^{L1} + E_\xi^{L2} + E_\xi^{R1} + E_\xi^{R2})$$

and

$$E_\alpha = \frac{1}{4}(E_\alpha^{B1} + E_\alpha^{B2} + E_\alpha^{T1} + E_\alpha^{T2})$$

where the superscripts L , R , B , and T denote the left, right, back and top faces, respectively, and the 1 and 2 denote cell 1 and cell 2, respectively, as shown in Figure 6.1. Then, solving for each E_ξ and E_α on the surrounding faces, we obtain E_η in terms of the twenty magnetic field components on the edges of the two 3-cells.

In particular for our example, in order for the integral equation given by (6.1.12) to be zero to second order accuracy, we choose E_ξ on the face c_2 to be the average of the values of the E_ξ 's on the surrounding faces. Particularly, for the example on the

parallelepiped complex, the value of E_ξ on face $\underline{c}_2$, which will be denoted by $E_\xi^{\frac{1}{2},0,0}$, referring to Figure 6.2 for the placement of the origin $(0,0,0)$, is then given by

$$E_\xi^{\frac{1}{2},0,0} = \frac{1}{4} \left(E_\xi^{0,-\frac{1}{2},0} + E_\xi^{1,-\frac{1}{2},0} + E_\xi^{0,\frac{1}{2},0} + E_\xi^{1,\frac{1}{2},0} \right), \quad (6.1.13)$$

where the four E_ξ 's in the parenthesis are positioned at the faces surrounding the face $\underline{c}_2$. Using this average of the bounding faces in our example, we make our numerical choices on $\underline{c}_3$ such that the derivative terms of E_ξ and E_η in (6.1.12) yield second order accurate numerical expressions. We thus choose the following central difference schemes for the partial derivatives in (6.1.12).

$$\begin{aligned} \partial_\xi E_\xi &= \frac{E_\xi^{1,\frac{1}{2},0} - E_\xi^{0,\frac{1}{2},0}}{\Delta\xi}, \\ \partial_\eta E_\eta &= \frac{E_\eta^{\frac{1}{2},1,0} - E_\eta^{\frac{1}{2},0,0}}{\Delta\eta}, \\ \partial_\xi E_\eta &= \frac{1}{2} \left[\frac{E_\eta^{\frac{3}{2},0,0} - E_\eta^{-\frac{1}{2},0,0}}{2\Delta\xi} + \frac{E_\eta^{\frac{3}{2},1,0} - E_\eta^{-\frac{1}{2},1,0}}{2\Delta\xi} \right], \\ \partial_\eta E_\xi &= \frac{1}{2} \left[\frac{E_\xi^{0,\frac{3}{2},0} - E_\xi^{0,-\frac{1}{2},0}}{2\Delta\eta} + \frac{E_\xi^{1,\frac{3}{2},0} - E_\xi^{1,-\frac{1}{2},0}}{2\Delta\eta} \right], \end{aligned} \quad (6.1.14)$$

where the superscripts denote the positions of the electric field components on the complex.

In step five, we substitute our choices for the unknown field components, E_ξ and E_α , given by the divergence free properties into the original update equation for E_η .

For the parallelepiped example, we obtain the update equation for E_η on the face $\underline{c}_2$,

$$E_\eta^{\frac{1}{2},0,0} = \frac{1}{\varepsilon \Delta\xi \sin \theta} [H_z^{0,0,0} - H_z^{1,0,0}] + \frac{\cos \theta}{4\varepsilon \Delta\eta \sin \theta} [H_z^{1,1,0} - H_z^{1,-1,0} + H_z^{0,1,0} - H_z^{0,-1,0}]. \quad (6.1.15)$$

which may seem to have miraculously appeared but actually terms simplify to this quite nicely. We then have the update equation for E_η in terms of the surrounding H_z 's. The same steps are followed for a face in the ηz -plane to determine the update equations for E_ξ .

6.2 Update Equations for Various Cell Complexes

6.2.1 Cubical Complex

Suppose we are solving for the update equations for the magnetic field components H_x , H_y , and H_z on the cubical complex as shown in Figure 6.3. The electric and

FIGURE 6.3. Cubical Complex for Solving Faraday's Law.

magnetic field 1-forms are given by

$$\begin{aligned} E &= E_x dx + E_y dy + E_z dz \\ H &= H_x dx + H_y dy + H_z dz, \end{aligned} \quad (6.2.1.1)$$

respectively. Applying the star operator to H , yields

$$\mu \star H = \mu H_x dy \, dz + \mu H_y dz \, dx + \mu H_z dx \, dy. \quad (6.2.1.2)$$

Substituting (6.2.1.1) and (6.2.1.2) into Faraday's law on the chain, c_2^F ,

$$-\partial_t \int_{c_2^F} \mu \star H = \int_{\partial c_2^F} E \quad (6.2.1.3)$$

where the superscript F denotes the front face, we obtain

$$-\partial_t \int_{c_2^F} (\mu H_x dy \, dz + \mu H_y dz \, dx + \mu H_z dx \, dy) = \int_{\partial c_2^F} E_x dx + E_y dy + E_z dz. \quad (6.2.1.4)$$

Since c_2^F is a face in the xz -plane, (6.2.1.4) simplifies to the equation,

$$-\partial_t \int_{c_2^F} \mu H_y dz dx = \int_{\partial c_2^F} E_x dx + E_z dz. \quad (6.2.1.5)$$

Thus, (6.2.1.5) results in the update equation

$$\begin{aligned} H_y^{n+\frac{1}{2}} \left(i + \frac{1}{2}, j, k + \frac{1}{2} \right) &= H_y^{n-\frac{1}{2}} \left(i + \frac{1}{2}, j, k + \frac{1}{2} \right) \\ &+ \frac{\Delta t}{\mu} \left[\left(\frac{E_x^n(i+1, j+\frac{1}{2}, k) - E_x^n(i, j+\frac{1}{2}, k)}{\Delta x} \right) \right. \\ &\quad \left. - \left(\frac{E_z^n(i+\frac{1}{2}, j+1, k) - E_z^n(i+\frac{1}{2}, j, k)}{\Delta z} \right) \right]. \end{aligned} \quad (6.2.1.6)$$

Now, substituting (6.2.1.1) and (6.2.1.2) into Faraday's law on the chain, c_2^L ,

$$-\partial_t \int_{c_2^L} \mu \star H = \int_{\partial c_2^L} E \quad (6.2.1.7)$$

where the superscript L denotes the left face, we obtain

$$-\partial_t \int_{c_2^L} \mu H_x dy dz = \int_{\partial c_2^L} E_y dy + E_z dz. \quad (6.2.1.8)$$

since c_2^L is a face in the yz -plane. Thus, (6.2.1.8) results in the update equation

$$\begin{aligned} H_x^{n+\frac{1}{2}} \left(i, j + \frac{1}{2}, k + \frac{1}{2} \right) &= H_x^{n-\frac{1}{2}} \left(i, j + \frac{1}{2}, k + \frac{1}{2} \right) \\ &+ \frac{\Delta t}{\mu} \left[\left(\frac{E_y^n(i, j+\frac{1}{2}, k+1) - E_y^n(i, j+\frac{1}{2}, k)}{\Delta z} \right) \right. \\ &\quad \left. - \left(\frac{E_z^n(i, j+1, k+\frac{1}{2}) - E_z^n(i, j, k+\frac{1}{2})}{\Delta y} \right) \right] \end{aligned} \quad (6.2.1.9)$$

To determine the update equations for H_z , we solve Faraday's law on the chain, c_2^B , where the superscript B denotes the bottom face, and simplify to obtain

$$-\partial_t \int_{c_2^B} \mu H_z dx dy = \int_{\partial c_2^B} E_x dx + E_y dy, \quad (6.2.1.10)$$

since c_2^B is in the xy -plane. Thus, (6.2.1.10) results in the update equation

$$H_z^{n+\frac{1}{2}} \left(i + \frac{1}{2}, j + \frac{1}{2}, k \right) = H_z^{n-\frac{1}{2}} \left(i + \frac{1}{2}, j + \frac{1}{2}, k \right) + \frac{\Delta t}{\mu} \left[\left(\frac{E_x^n \left(i + \frac{1}{2}, j + 1, k \right) - E_x^n \left(i + \frac{1}{2}, j, k \right)}{\Delta y} \right) - \left(\frac{E_y^n \left(i + 1, j + \frac{1}{2}, k \right) - E_y^n \left(i, j + \frac{1}{2}, k \right)}{\Delta x} \right) \right]. \quad (6.2.1.11)$$

Now, suppose we are solving for the update equations for the electric field components, E_x , E_y , and E_z , on the cubical dual complex as shown in Figure 6.4. Given

FIGURE 6.4. Dual Cubical Complex for Solving Ampere's Law.

the electric field 1-form in (6.2.1.1), and applying the star operator to E , we obtain

$$\varepsilon \star E = \varepsilon E_x dy dz + \varepsilon E_y dz dx + \varepsilon E_z dx dy. \quad (6.2.1.12)$$

Substituting (6.2.1.1) and (6.2.1.12) into Ampere's law on the chain, $\underline{c_2^F}$,

$$\partial_t \int_{\underline{c_2^F}} (\varepsilon \star E) = \int_{\underline{\partial c_2^F}} H \quad (6.2.1.13)$$

where the superscript F denotes the front face, we obtain

$$\partial_t \int_{\underline{c}_2^F} (\varepsilon E_x dy dz + \varepsilon E_y dz dx + \varepsilon E_z dx dy) = \int_{\underline{c}_2^F} H_x dx + H_y dy + H_z dz. \quad (6.2.1.14)$$

Since $\underline{c}_2^F$ is a face in the xy -plane, (6.2.1.14) simplifies to the equation,

$$\partial_t \int_{\underline{c}_2^F} \varepsilon E_z dx dy = \int_{\partial \underline{c}_2^F} H_x dx + H_y dy. \quad (6.2.1.15)$$

Thus, (6.2.1.15) results in the update equation

$$\begin{aligned} E_z^{n+1} \left(i, j, k + \frac{1}{2} \right) &= E_z^n \left(i, j, k + \frac{1}{2} \right) \\ &+ \frac{\Delta t}{\varepsilon} \left[\left(\frac{H_y^{n+\frac{1}{2}} \left(i + \frac{1}{2}, j, k + \frac{1}{2} \right) - H_y^{n+\frac{1}{2}} \left(i - \frac{1}{2}, j, k + \frac{1}{2} \right)}{\Delta x} \right) \right. \\ &\quad \left. - \left(\frac{H_x^{n+\frac{1}{2}} \left(i, j + \frac{1}{2}, k + \frac{1}{2} \right) - H_x^{n+\frac{1}{2}} \left(i, j - \frac{1}{2}, k + \frac{1}{2} \right)}{\Delta y} \right) \right] \end{aligned}$$

Similarly, the update equations for the electric field components, E_x and E_y , are given by

$$\begin{aligned} E_x^{n+1} \left(i + \frac{1}{2}, j, k \right) &= E_x^n \left(i + \frac{1}{2}, j, k \right) \\ &+ \frac{\Delta t}{\varepsilon} \left[\left(\frac{H_z^{n+\frac{1}{2}} \left(i + \frac{1}{2}, j + \frac{1}{2}, k \right) - H_z^{n+\frac{1}{2}} \left(i + \frac{1}{2}, j - \frac{1}{2}, k \right)}{\Delta y} \right) \right. \\ &\quad \left. - \left(\frac{H_y^{n+\frac{1}{2}} \left(i + \frac{1}{2}, j, k + \frac{1}{2} \right) - H_y^{n+\frac{1}{2}} \left(i + \frac{1}{2}, j, k - \frac{1}{2} \right)}{\Delta z} \right) \right] \end{aligned}$$

and

$$E_y^{n+1} \left(i, j + \frac{1}{2}, k \right) = E_y^n \left(i, j + \frac{1}{2}, k \right) + \frac{\Delta t}{\varepsilon} \left[\left(\frac{H_x^{n+\frac{1}{2}} \left(i, j + \frac{1}{2}, k + \frac{1}{2} \right) - H_x^{n+\frac{1}{2}} \left(i, j + \frac{1}{2}, k - \frac{1}{2} \right)}{\Delta z} \right) - \left(\frac{H_z^{n+\frac{1}{2}} \left(i + \frac{1}{2}, j + \frac{1}{2}, k \right) - H_z^{n+\frac{1}{2}} \left(i - \frac{1}{2}, j + \frac{1}{2}, k \right)}{\Delta x} \right) \right],$$

respectively.

6.2.2 Parallelepiped Complex with One Angle Skewed-TE Wave

Suppose we want to determine the update equations for a transverse electric wave on a parallelepiped complex with one angle, $\theta \neq \pi/2$, skewed as shown in Figure 6.5. Ampere's law on the dual complex is again given by

FIGURE 6.5. Neighboring Cells in a Parallelepiped Cell Complex with One Angle Skewed.

$$\varepsilon \partial_t \int_{\underline{c_2}} \star E = \int_{\underline{\partial c_2}} H. \quad (6.2.2.1)$$

The electric and magnetic field 1-forms are then given on this cell complex by

$$E = E_\xi d\xi + E_\eta d\eta \quad (6.2.2.2)$$

and

$$H = H_z dz, \quad (6.2.2.3)$$

respectively. Applying the star operator to (6.2.2.2), we obtain

$$\star E = \left(\frac{1}{\sin \theta} E_\xi - \frac{\cos \theta}{\sin \theta} E_\eta \right) d\eta dz + \left(\frac{1}{\sin \theta} E_\eta - \frac{\cos \theta}{\sin \theta} E_\xi \right) dz d\xi. \quad (6.2.2.4)$$

Substituting (6.2.2.4) and (6.2.2.3) into (6.2.2.1), results in

$$\begin{aligned} \varepsilon \partial_t \int_{\underline{c_2^{F2}}} \left(\frac{1}{\sin \theta} E_\xi - \frac{\cos \theta}{\sin \theta} E_\eta \right) d\eta dz + \varepsilon \partial_t \int_{\underline{c_2^{F2}}} \left(\frac{1}{\sin \theta} E_\eta - \frac{\cos \theta}{\sin \theta} E_\xi \right) dz d\xi \\ = \int_{\underline{\partial c_2^{F2}}} H_z dz. \end{aligned}$$

where the superscript $F2$ denotes the front face. Since, $\underline{c_2^{F2}}$ is in the ξz -plane, the above equation simplifies to

$$\varepsilon \partial_t \int_{\underline{c_2^{F2}}} \left(\frac{1}{\sin \theta} E_\eta - \frac{\cos \theta}{\sin \theta} E_\xi \right) dz d\xi = \int_{\underline{\partial c_2^{F2}}} H_z dz. \quad (6.2.2.5)$$

Solving (6.2.2.5) in terms of E_ξ and H_z results in

$$\begin{aligned} \varepsilon \left(\frac{1}{\sin \theta} \right) \partial_t E_\eta \left(i - \frac{1}{2}, j \right) \Delta z \Delta \xi = H_z^{n+\frac{1}{2}}(i-1, j) \Delta z - H_z^{n+\frac{1}{2}}(i, j) \Delta z \\ + \varepsilon \left(\frac{\cos \theta}{\sin \theta} \right) \partial_t E_\xi \left(i - \frac{1}{2}, j \right) \Delta z \Delta \xi. \end{aligned} \quad (6.2.2.6)$$

Let $E_\xi^n(i - \frac{1}{2}, j)$ be the average of the E_ξ 's on the four surrounding faces,

$$\begin{aligned} \partial_t E_\xi^n(i - \frac{1}{2}, j) &= \frac{1}{4} \left[\partial_t E_\xi^n(i, j + \frac{1}{2}) + \partial_t E_\xi^n(i, j - \frac{1}{2}) + \partial_t E_\xi^n(i - 1, j + \frac{1}{2}) \right. \\ &\quad \left. + \partial_t E_\xi^n(i - 1, j - \frac{1}{2}) \right]. \end{aligned} \quad (6.2.2.7)$$

Then, we can solve for each of those E_ξ 's on each of the surrounding faces. For example, we can solve for $\partial_t E_\xi^n(i, j + \frac{1}{2})$ on the face $\underline{c_2^{R2}}$, where the superscript $R2$ denotes the right face of cell 2 on the parallelepiped, such that

$$\partial_t E_\xi^n \left(i, j + \frac{1}{2} \right) = \frac{\sin \theta}{\varepsilon \Delta \eta} \left(H_z^{n+\frac{1}{2}}(i, j+1) - H_z^{n+\frac{1}{2}}(i, j) \right) + \cos \theta \left(\partial_t E_\eta^n \left(i - \frac{1}{2}, j \right) \right). \quad (6.2.2.8)$$

After determining the values for the $\partial_t E_\xi$'s on the surrounding faces, we substitute them back into (6.2.2.7). Performing a final substitution of (6.2.2.7) into (6.2.2.6), we obtain the update equation for E_η as

$$\begin{aligned} E_\eta^{n+1} \left(i - \frac{1}{2}, j \right) = E_\eta^n \left(i - \frac{1}{2}, j \right) &+ \frac{\Delta t}{\varepsilon \Delta \xi \sin \theta} \left[H_z^{n+\frac{1}{2}}(i-1, j) - H_z^{n+\frac{1}{2}}(i, j) \right] \\ &+ \frac{\cos \theta}{4\varepsilon \Delta \eta \sin \theta} \left[H_z^{n+\frac{1}{2}}(i, j+1) - H_z^{n+\frac{1}{2}}(i, j-1) \right. \\ &\quad \left. + H_z^{n+\frac{1}{2}}(i-1, j+1) - H_z^{n+\frac{1}{2}}(i-1, j-1) \right]. \end{aligned} \quad (6.2.2.9)$$

In order to obtain the update equation for E_ξ , we use Ampere's law

$$\varepsilon \partial_t \int_{\underline{c_2^{R1}}} \star E = \int_{\partial \underline{c_2^{R1}}} H. \quad (6.2.2.10)$$

where the superscript $R1$ denotes the right face on cell 1. So (6.2.2.10) simplifies to

$$\varepsilon \partial_t \int_{\underline{c_2^{R1}}} \left(\frac{1}{\sin \theta} E_\xi - \frac{\cos \theta}{\sin \theta} E_\eta \right) d\eta dz = \int_{\partial \underline{c_2^{R1}}} H_z dz, \quad (6.2.2.11)$$

since $\underline{c_2^{R1}}$ is in the ηz -plane. Solving (6.2.2.11) in terms of E_η and H_z results in

$$\begin{aligned} \varepsilon \left(\frac{1}{\sin \theta} \right) \partial_t E_\xi^n \left(i, j - \frac{1}{2} \right) \Delta \eta \Delta z &= H_z^{n+1/2}(i, j) \Delta z - H_z^{n+1/2}(i, j-1) \Delta z \\ &+ \varepsilon \frac{\cos \theta}{\sin \theta} \partial_t E_\eta^n \left(i, j - \frac{1}{2} \right) \Delta \eta \Delta z \end{aligned} \quad (6.2.2.12)$$

Let $E_\eta^n(i, j - \frac{1}{2})$ be the average of the E_η 's on the four surrounding faces,

$$\begin{aligned} \partial_t E_\eta^n \left(i, j - \frac{1}{2} \right) &= \frac{1}{4} \left[\partial_t E_\eta^n \left(i + \frac{1}{2}, j \right) + \partial_t E_\eta^n \left(i - \frac{1}{2}, j \right) \right. \\ &\quad \left. + \partial_t E_\eta^n \left(i + \frac{1}{2}, j - 1 \right) + \partial_t E_\eta^n \left(i - \frac{1}{2}, j - 1 \right) \right]. \end{aligned} \quad (6.2.2.13)$$

Now, solving for each of the four E_η 's in the previous equation and substituting (6.2.2.13) into (6.2.2.12) results in

$$E_\xi^{n+1}(i, j - \frac{1}{2}) = E_\xi^n(i, j - \frac{1}{2}) + \frac{\Delta t}{\varepsilon \Delta \eta \sin \theta} [H_z^{n+1/2}(i, j) - H_z^{n+1/2}(i, j - 1)] \\ + \frac{\cos \theta}{4\varepsilon \Delta \xi \sin \theta} [H_z(i - 1, j) - H_z(i + 1, j) + H_z(i - 1, j - 1) - H_z(i + 1, j - 1)].$$

Finally, to determine the update equation for H_z , we solve Faraday's law

$$\partial_t \int_{c_2^B} (\mu \star H) = \int_{\partial c_2^B} E \quad (6.2.2.14)$$

where the superscript B denotes the bottom face on the primary cell.

$$\star H = \sin \theta H_z d\xi d\eta. \quad (6.2.2.15)$$

Therefore, substituting (6.2.2.15) into (6.2.2.14), we obtain

$$H_z^{n+1/2}(i, j) = H_z^{n-1/2}(i, j) + \frac{\Delta t}{\mu \sin \theta} \left[\left(\frac{E_\eta(i + \frac{1}{2}, j) - E_\eta(i - \frac{1}{2}, j)}{\Delta \xi} \right) - \left(\frac{E_x(i, j + \frac{1}{2}) - E_x(i, j - \frac{1}{2})}{\Delta \eta} \right) \right].$$

6.2.3 Parallelepiped Cell Complex with Three Angles Skewed

For the parallelepiped cell complex with three angles skewed, I will show how to obtain the update equations for H_ξ . The update equations for the other components follow similarly. Suppose we want to solve for the magnetic field update equations on the three dimensional parallelepiped complex as shown in Figure 6.6, where the shaded face denotes the 2-chain c_2 . The electric and magnetic field 1-forms for this complex are given by

$$E = E_\xi + E_\eta + E_\alpha \quad (6.2.3.1)$$

and

$$H = H_\xi + H_\eta + H_\alpha. \quad (6.2.3.2)$$

FIGURE 6.6. Three-Dimensional Parallelepiped Complex.

Recalling that the determinant of the metric is given by $\det g = 1 - \cos^2 \theta_1 - \cos^2 \theta_2 - \cos^2 \theta_3 + 2 \cos \theta_1 \cos \theta_2 \cos \theta_3$, we can apply the star operator to H to obtain

$$\begin{aligned}
 \star H = & \frac{1}{\sqrt{\det g}} \{ [(1 - \cos^2 \theta_3)H_\xi + (\cos \theta_2 \cos \theta_3 - \cos \theta_1)H_\eta \\
 & + (\cos \theta_1 \cos \theta_3 - \cos \theta_2)H_\alpha] d\eta d\alpha + [(\cos \theta_2 \cos \theta_3 - \cos \theta_1)H_\xi \\
 & + (1 - \cos^2 \theta_2)H_\eta + (\cos \theta_1 \cos \theta_2 - \cos \theta_3)H_\alpha] d\alpha d\xi \\
 & + [(\cos \theta_1 \cos \theta_3 - \cos \theta_2)H_\xi + (\cos \theta_1 \cos \theta_2 - \cos \theta_3)H_\eta \\
 & + (1 - \cos^2 \theta_1)H_\alpha] d\xi d\eta \}. \quad (6.2.3.3)
 \end{aligned}$$

Once again Faraday's law on a complex is given by

$$-\partial_t \int_{c_2} \mu \star H = \int_{\partial c_2} E. \quad (6.2.3.4)$$

Substituting (6.2.3.3) into Faraday's law and simplifying the integral since c_2 is in the $\eta\alpha$ -plane, we obtain

$$\begin{aligned} & -\mu\partial_t \int_{c_2} \frac{1}{\sqrt{\det g}} [(1 - \cos^2 \theta_3)H_\xi + (\cos \theta_2 \cos \theta_3 - \cos \theta_1)H_\eta \\ & + (\cos \theta_1 \cos \theta_3 - \cos \theta_2)H_\alpha] d\eta d\alpha = \int_{\partial c_2} E_\xi d\xi + E_\eta d\eta + E_\alpha d\alpha. \end{aligned} \quad (6.2.3.5)$$

In this case we want to solve for H_ξ on c_2 . Integrating (6.2.3.5), we obtain as our first equation in a system of three equations

$$\begin{aligned} & -\frac{\mu\Delta\eta\Delta\alpha}{\sqrt{\det g}} \left[(1 - \cos^2 \theta_3)\partial_t H_\xi^{i+\frac{1}{2},j-\frac{1}{2},k} + (\cos \theta_2 \cos \theta_3 - \cos \theta_1)\partial_t H_\eta^{i+\frac{1}{2},j-\frac{1}{2},k} \right. \\ & \quad \left. + (\cos \theta_1 \cos \theta_3 - \cos \theta_2)\partial_t H_\alpha^{i+\frac{1}{2},j-\frac{1}{2},k} \right] \\ & = E_\eta^n(i + \frac{1}{2}, j, k - 1)\Delta\eta + E_\alpha^n(i + \frac{1}{2}, j + \frac{1}{2}, k - \frac{1}{2})\Delta\alpha - E_\eta^n(i + \frac{1}{2}, j, k)\Delta\eta \\ & \quad - E_\alpha^n(i + \frac{1}{2}, j - \frac{1}{2}, k - \frac{1}{2})\Delta\alpha. \end{aligned} \quad (6.2.3.6)$$

Letting H_η be the average of the H_η 's on the surrounding four faces,

$$\begin{aligned} \partial_t H_\eta(i + \frac{1}{2}, j - \frac{1}{2}, k) &= \frac{1}{4} \left[\partial_t H_\eta(i + 1, j + \frac{1}{2}, k - \frac{1}{2}) + \partial_t H_\eta(i + 1, j - \frac{1}{2}, k - \frac{1}{2}) \right. \\ & \quad \left. + \partial_t H_\eta(i, j - \frac{1}{2}, k - \frac{1}{2}) + \partial_t H_\eta(i, j + \frac{1}{2}, k - \frac{1}{2}) \right] \end{aligned}$$

and letting H_α be the average of the H_α 's on the surrounding four faces,

$$\begin{aligned} \partial_t H_\alpha(i + \frac{1}{2}, k) &= \frac{1}{4} [\partial_t H_\alpha(i + 1, k, k) + \partial_t H_\alpha(i, j, k - 1) \\ & \quad + \partial_t H_\alpha(i, j, k) + \partial_t H_\alpha(i, j, k - 1)] \end{aligned}$$

we obtain the other two equations in our system of three equations, namely,

$$\begin{aligned}
& \Delta\alpha\Delta\xi \left[\left(\frac{\cos\theta_2\cos\theta_3 - \cos\theta_1}{1 - \cos^2\theta_2} \right) \partial_t H_\xi(i + \frac{1}{2}, j - \frac{1}{2}, k) + \partial_t H_\eta(i + \frac{1}{2}, j - \frac{1}{2}, k) \right. \\
& \quad \left. + \left(\frac{\cos\theta_1\cos\theta_2 - \cos\theta_3}{1 - \cos^2\theta_2} \right) \partial_t H_\alpha(i + \frac{1}{2}, j - \frac{1}{2}, k) \right] \\
& = \frac{-\sqrt{\det g}}{4\mu(\sin^2\theta_2)} \left[\left\{ E_\xi^n(i, j + \frac{1}{2}, k) - E_\xi^n(i + 1, j + \frac{1}{2}, k - 1) + E_\xi^n(i + 1, j - \frac{1}{2}, k) \right. \right. \\
& \quad - E_\xi^n(i + 1, j - \frac{1}{2}, k - 1) + E_\xi^n(i, j - \frac{1}{2}, k) - E_\xi^n(i, j - \frac{1}{2}, k - 1) + E_\xi^n(i, j + \frac{1}{2}, k) \\
& \quad \left. \left. - E_\xi^n(i, j + \frac{1}{2}, k - 1) \right\} \Delta\xi - \left\{ E_\alpha^n(i + \frac{3}{2}, j + \frac{1}{2}, k - \frac{1}{2}) - E_\alpha^n(i - \frac{1}{2}, j + \frac{1}{2}, k - \frac{1}{2}) \right. \right. \\
& \quad \left. \left. + E_\alpha^n(i + \frac{3}{2}, j - \frac{1}{2}, k - \frac{1}{2}) - E_\alpha^n(i - \frac{1}{2}, j - \frac{1}{2}, k - \frac{1}{2}) \right\} \Delta\alpha \right]
\end{aligned}$$

and

$$\begin{aligned}
& \left[\left(\frac{\cos \theta_1 \cos \theta_3 - \cos \theta_2}{1 - \cos^2 \theta_1} \right) \partial_t H_\xi(i + \frac{1}{2}, j - \frac{1}{2}, k) + \right. \\
& \left. \left(\frac{\cos \theta_1 \cos \theta_2 - \cos \theta_3}{1 - \cos^2 \theta_1} \right) \partial_t H_\eta(i + \frac{1}{2}, j - \frac{1}{2}, k) + \partial_t H_\alpha(i + \frac{1}{2}, j - \frac{1}{2}, k) \right] \Delta \xi \Delta \eta \\
& = \frac{-\sqrt{\det g}}{4\mu(1 - \cos^2 \theta_1)} \left[\left\{ E_\xi^n(i + 1, j - \frac{1}{2}, k) - E_\xi^n(i + 1, j + \frac{1}{2}, k) \right. \right. \\
& \quad + E_\xi^n(i + 1, j - \frac{1}{2}, k - 1) - E_\xi^n(i + 1, j + \frac{1}{2}, k - 1) + E_\xi^n(i, j - \frac{1}{2}, k - 1) \\
& \quad \left. - E_\xi^n(i, j + \frac{1}{2}, k - 1) + E_\xi^n(i, j - \frac{1}{2}, k) - E_\xi^n(i, j + \frac{1}{2}, k) \right\} \Delta \xi \\
& \quad + \left\{ E_\eta^n(i + \frac{3}{2}, k) + E_\eta^n(i + \frac{3}{2}, j, k - 1) - E_\eta^n(i - \frac{1}{2}, j, k - 1) \right. \\
& \quad \left. - E_\eta^n(i + \frac{1}{2}, j, k) \right\} \Delta \eta \right].
\end{aligned}$$

Solving these three equations simultaneously, we obtain

$$\begin{aligned}
H_\xi^{n+\frac{1}{2}}(i+\frac{1}{2}, j-\frac{1}{2}, k) &= H_\xi^{n-\frac{1}{2}}(i+\frac{1}{2}, j-\frac{1}{2}, k) \\
&+ \frac{\Delta t}{4\mu\sqrt{\det g}} \left[\left\{ \left(\frac{-\cos\theta_1}{\Delta\alpha} + \frac{\cos\theta_2}{\Delta\eta} \right) \left(E_\xi^n(i+1, j+\frac{1}{2}, k) \right. \right. \right. \\
&+ E_\xi^n(i, j+\frac{1}{2}, k) - E_\xi^n(i, j+\frac{1}{2}, k-1) + E_\xi^n(i+1, j+\frac{1}{2}, k-1) \Big) \\
&+ \left(\frac{-\cos\theta_1}{\Delta\alpha} - \frac{\cos\theta_2}{\Delta\eta} \right) \left(E_\xi^n(i+1, j-\frac{1}{2}, k) \right. \\
&- E_\xi^n(i+1, j-\frac{1}{2}, k-1) + E_\xi^n(i, j-\frac{1}{2}, k) + E_\xi^n(i, j-\frac{1}{2}, k-1) \Big) \Big\} \\
&+ \frac{\cos\theta_1}{\Delta\xi} \left\{ E_\alpha^n(i+\frac{3}{2}, j+\frac{1}{2}, k-\frac{1}{2}) + E_\alpha^n(i+\frac{3}{2}, j-\frac{1}{2}, k-\frac{1}{2}) \right. \\
&- E_\alpha^n(i-\frac{1}{2}, j-\frac{1}{2}, k-\frac{1}{2}) - E_\alpha^n(i-\frac{1}{2}, j+\frac{1}{2}, k-\frac{1}{2}) \Big\} \\
&- \frac{\cos\theta_2}{\Delta\xi} \left\{ E_\eta^n(i+\frac{3}{2}, j, k) + E_\eta^n(i+\frac{3}{2}, j, k-1) - E_\eta^n(i-\frac{1}{2}, j, k-1) \right. \\
&- E_\eta^n(i-\frac{1}{2}, j, k) \Big\} \Big] + \frac{\Delta t}{\mu\sqrt{\det g}} \left[\frac{E_\eta^n(i+\frac{1}{2}, j, k) - E_\eta^n(i+\frac{1}{2}, j, k-1)}{\Delta\alpha} \right. \\
&- \left. \frac{E_\alpha^n(i+\frac{1}{2}, j+\frac{1}{2}, k-\frac{1}{2}) - E_\alpha^n(i+\frac{1}{2}, j-\frac{1}{2}, k-\frac{1}{2})}{\Delta\eta} \right]
\end{aligned} \tag{6.2.3.7}$$

as the update equation for H_ξ .

6.2.4 Trapezoid Complex

For the final example, suppose we want to solve for the update equations for the electric and magnetic field components on an arbitrary, trapezoid complex as shown in Figure 6.7. Assuming a transverse electric wave, the electric field and magnetic field 1-forms are given by

$$E = E_{\theta\xi}d\xi + E_{\theta\eta}d\eta \tag{6.2.4.1}$$

and

$$H = H_z dz, \tag{6.2.4.2}$$

FIGURE 6.7. Trapezoid Complex.

respectively, where θ denotes the angle at which the edges of the corresponding primary cell meet. To solve for the E_ξ component, we solve Ampere's law on the chain, c_2^{B1} where $B1$ denotes the back face of cell 1,

$$\varepsilon \partial_t \int_{\underline{c}_2^{B1}} \star E = \int_{\partial \underline{c}_2^{B1}} H. \quad (6.2.4.3)$$

Substituting (6.2.4.1) and (6.2.4.2) into Ampere's law, results in

$$\begin{aligned} \varepsilon \int_{\underline{c}_2^{B1}} \left[\left(\frac{1}{\sin \theta} \partial_t E_{\theta\xi} - \frac{\cos \theta}{\sin \theta} E_{\theta\eta} \right) d\eta dz + \int_{\underline{c}_2^{B1}} \left(\frac{1}{\sin \theta} \partial_t E_{\theta\eta} - \frac{\cos \theta}{\sin \theta} \partial_t E_{\theta\xi} \right) dz d\xi \right] \\ = H_z^\kappa \Delta z - H_z^\omega \Delta z. \end{aligned} \quad (6.2.4.4)$$

Solving the integrals in the above equation, yields

$$\begin{aligned} \varepsilon \left[\partial_t E_{\theta\eta} \left(\frac{\sin(\theta - \alpha) - \cos \theta \sin \alpha}{\sin^2 \theta} \right) \Delta \xi_\theta \Delta z \right. \\ \left. + \partial_t E_{\theta\xi} \left(\frac{\sin \alpha - \cos \theta \sin(\theta - \alpha)}{\sin^2 \theta} \right) \Delta \xi_\theta \Delta z \right] = H_z^\kappa \Delta z - H_z^\omega \Delta z. \end{aligned}$$

where $E_{\theta\eta}$ and $E_{\theta\xi}$ are located on the face $\underline{c}_2^{B1}$. Now, if we let $E_{\theta\xi}$ on $\underline{c}_2^{B1}$ be the

average of the four surrounding $E_{\theta\xi}$'s,

$$\partial_t E_{\theta\xi} = \frac{1}{4} (\partial_t E_{\phi\xi} + \partial_t E_{\tau\xi} + \partial_t E_{\omega\xi} + \partial_t E_{\kappa\xi}) \quad (6.2.4.5)$$

we can solve for each $\partial_t E_{\theta\xi}$, $\partial_t E_{\tau\xi}$, $\partial_t E_{\omega\xi}$, and $\partial_t E_{\kappa\xi}$ in terms of $\partial_t E_{\theta\eta}$ on $\underline{c_2}^{B1}$. For example, for $\partial_t E_{\tau\xi}$, we obtain

$$\begin{aligned} & \left(\frac{1}{\sin \theta} \partial_t E_{\tau\xi} - \frac{\cos \theta}{\sin \theta} \partial_t E_{\theta\eta} \right) \left(\frac{\sin(\tau + \alpha)}{\sin \theta} \Delta \eta_\tau \right) \Delta z \\ & + \left(\frac{1}{\sin \theta} \partial_t E_{\theta\eta} - \frac{\cos \theta}{\sin \theta} \partial_t E_{\tau\xi} \right) \left(\frac{-\sin(\tau + \alpha - \theta)}{\sin \theta} \Delta \eta_\tau \right) \Delta z = H_z^\kappa \Delta z - H_z^\tau \Delta z. \end{aligned}$$

Therefore,

$$\begin{aligned} \partial_t E_{\tau\xi} &= \frac{\sin^2 \theta}{\Delta \eta_\tau (\sin(\tau + \alpha) + \cos \theta \sin(\tau + \alpha - \theta))} (H_z^\kappa - H_z^\tau) \\ &+ \left(\frac{\cos \theta \sin(\tau + \alpha) + \sin(\tau + \alpha - \theta)}{\sin(\tau + \alpha) + \cos \theta \sin(\tau + \alpha - \theta)} \right) \partial_t E_{\theta\eta}. \end{aligned}$$

Substituting the values for the surrounding $\partial_t E_\xi$'s back into (6.2.4.5) yields the update equation for $E_{\theta\eta}$ on $\underline{c_2}^{B1}$

$$\begin{aligned} E_{\theta\eta}^{n+1} &= E_{\theta\eta}^n + \frac{\Delta t \sin^2 \theta}{a + \frac{b}{4} \left(\frac{A_\tau}{B_\tau} + \frac{A_\kappa}{B_\kappa} + \frac{A_\phi}{B_\phi} + \frac{A_\omega}{B_\omega} \right)} \left\{ \frac{1}{\varepsilon \Delta \xi_\theta} (H_z^\kappa - H_z^\omega) \right. \\ &\quad \left. - \frac{b}{4} \left[\frac{H_z^\kappa - H_z^\tau}{\Delta \eta_\tau B_\tau} + \frac{H_z^\mu - H_z^\kappa}{\Delta \eta_\kappa B_\kappa} + \frac{H_z^\omega - H_z^\phi}{\Delta \eta_\phi B_\phi} + \frac{H_z^\nu - H_z^\omega}{\Delta \eta_\omega B_\omega} \right] \right\} \quad (6.2.4.6) \end{aligned}$$

where

$$\begin{aligned}
A_\tau &= \cos \theta \sin(\tau + \alpha) + \sin(\tau + \alpha - \theta), \\
B_\tau &= \sin(\tau + \alpha) + \cos \theta \sin(\tau + \alpha - \theta), \\
A_\kappa &= \cos \theta \sin(\kappa + \alpha) + \sin(\kappa + \alpha - \theta), \\
B_\kappa &= \sin(\kappa + \alpha) + \cos \theta \sin(\kappa + \alpha - \theta), \\
A_\phi &= \cos \theta \sin(\phi + \alpha) + \sin(\phi + \alpha - \theta), \\
B_\phi &= \sin(\phi + \alpha) + \cos \theta \sin(\phi + \alpha - \theta), \\
A_\omega &= \cos \theta \sin(\omega + \alpha) + \sin(\omega + \alpha - \theta), \\
B_\omega &= \sin(\omega + \alpha) + \cos \theta \sin(\omega + \alpha - \theta), \\
a &= \sin(\theta - \alpha) - \cos \theta \sin \alpha, \\
b &= \sin \alpha - \cos \theta \sin(\theta - \alpha).
\end{aligned}$$

Now, to solve for the E_ξ field component on a trapezoid complex, we substitute $\star H$ and E into Ampere's law, we obtain

$$\begin{aligned}
&\varepsilon \partial_t \left[\int_{\underline{c_2^{R1}}} \left(\frac{1}{\sin \theta} E_{\theta\xi} - \frac{\cos \theta}{\sin \theta} E_{\theta\eta} \right) d\eta dz + \int_{\underline{c_2^{R1}}} \left(\frac{1}{\sin \theta} E_{\theta\eta} - \frac{\cos \theta}{\sin \theta} E_{\theta\xi} \right) dz d\xi \right] \\
&= H_z^\omega \Delta z - H_z^\phi \Delta z.
\end{aligned}$$

Solving the integrals in the above equation, results in

$$\begin{aligned}
&\varepsilon \left[\partial_t E_{\theta\xi} \left(\frac{\sin(\phi + \alpha) \Delta \eta_\phi + \cos \theta \sin(\phi + \alpha - \theta) \Delta \eta_\phi}{\sin^2 \theta} \right) \right. \\
&\quad \left. - \partial_t E_{\theta\eta} \left(\frac{\cos \theta \sin(\phi + \alpha) \Delta \eta_\phi - \sin(\phi + \alpha - \theta) \Delta \eta_\phi}{\sin^2 \theta} \right) \right] = H_z^\omega - H_z^\phi
\end{aligned} \tag{6.2.4.7}$$

where $\partial_t E_{\theta\xi}$ and $\partial_t E_{\theta\eta}$ are on the chain c_2^{R1} . Now, if we let $\partial_t E_{\theta\eta}$ on c_2^{R1} be the average of the four surrounding $E_{\theta\eta}$'s,

$$\partial_t E_{\theta\eta} = \frac{1}{4} (\partial_t E_{\omega\eta} + \partial_t E_{\kappa\eta} + \partial_t E_{\phi\eta} + \partial_t E_{\tau\eta}),$$

we can solve for each of the four E_η 's in terms of $\partial_t E_{\theta\xi}$ on c_2^{R1} . For example, solving for $\partial_t E_{\omega\eta}$, we obtain

$$\begin{aligned}\partial_t E_{\omega\eta} &= \frac{\sin^2 \theta}{\sin(\theta - \alpha) - \cos \theta \sin \alpha} \left(\frac{H_z^\gamma - H_z^\omega}{\varepsilon \Delta \xi_\omega} \right) \\ &\quad - \frac{\sin \alpha - \cos \theta \sin(\theta - \alpha)}{\Delta \xi_\omega (\sin(\theta - \alpha) - \cos \theta \sin \alpha)} (\partial_t E_{\theta\xi}).\end{aligned}$$

Therefore, we can write $\partial_t E_{\theta\eta}$ as

$$\begin{aligned}\partial_t E_{\theta\eta} &= \frac{1}{4} \left\{ \frac{\sin^2 \theta}{\varepsilon (\sin(\theta - \alpha) - \cos \theta \sin \alpha)} \left[\frac{H_z^\gamma - H_z^\omega}{\Delta \xi_\omega} + \frac{H_z^\omega - H_z^\kappa}{\Delta \xi_\kappa} \right. \right. \\ &\quad \left. \left. + \frac{H_z^\psi - H_z^\phi}{\Delta \xi_\phi} + \frac{H_z^\phi - H_z^\tau}{\Delta \xi_\tau} \right] \right. \\ &\quad \left. - \partial_t E_{\theta\xi} \left(\frac{\sin \alpha - \cos \theta \sin(\theta - \alpha)}{\sin(\theta - \alpha) - \cos \theta \sin \alpha} \right) \left(\frac{1}{\Delta \xi_\omega} + \frac{1}{\Delta \xi_\kappa} + \frac{1}{\Delta \xi_\phi} + \frac{1}{\Delta \xi_\tau} \right) \right\}.\end{aligned}$$

Substituting this into (6.2.4.7) results in the update equation for $E_{\theta\xi}$

$$\begin{aligned}E_{\theta\xi}^{n+1} &= E_{\theta\xi}^n + \frac{\Delta t \sin^2 \theta}{A_\phi + \frac{B_\phi b}{4a} \left(\frac{1}{\Delta \xi_\omega} + \frac{1}{\Delta \xi_\kappa} + \frac{1}{\Delta \xi_\phi} + \frac{1}{\Delta \xi_\tau} \right)} \left\{ \frac{1}{\varepsilon \Delta \eta_\phi} (H_z^\omega - H_z^\phi) \right. \\ &\quad \left. + \frac{B_\phi}{4a} \left[\frac{H_z^\gamma - H_z^\omega}{\Delta \xi_\omega} + \frac{H_z^\omega - H_z^\kappa}{\Delta \xi_\kappa} + \frac{H_z^\psi - H_z^\phi}{\Delta \xi_\phi} + \frac{H_z^\phi - H_z^\tau}{\Delta \xi_\tau} \right] \right\}.\end{aligned}$$

6.3 Yee's Algorithm

The finite-difference time-domain method introduced by K.S. Yee [32] in 1966 gives solutions to Maxwell's curl equations,

$$\begin{aligned}\nabla \times E &= -\mu \frac{\partial H}{\partial t} \\ \nabla \times H &= \varepsilon \frac{\partial E}{\partial t}.\end{aligned}\tag{6.3.1}$$

The algorithm developed by Yee uses an orthogonally gridded computational domain where the components of the electric and magnetic fields are positioned on the unit cell as in Figure 6.8. Yee's algorithm uses a staggered mesh approach as shown in Figure 6.8, where $\vec{E}$ and $\vec{H}$ are evaluated at half-integer separated spatial points and

FIGURE 6.8. Electric and Magnetic Field Components on a Unit Cell.

are time integrated in a leap-frog manner at separate half-time steps. The staggered mesh approach produces the same results obtained with the dual mesh approach. By examining Figure 6.8, one can clearly see that the positions of the electric field components are on the edges of the primary mesh and the positions of the magnetic field components are on the edges of the dual mesh. The first curl equation in (6.3.1) represents the three scalar equations in (x, y, z) coordinates:

$$\begin{aligned}\frac{\partial H_x}{\partial t} &= \frac{1}{\mu} \left(\frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial x} \right) \\ \frac{\partial H_y}{\partial t} &= \frac{1}{\mu} \left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} \right) \\ \frac{\partial H_z}{\partial t} &= \frac{1}{\mu} \left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} \right)\end{aligned}\quad (6.3.2)$$

Using the positions of electric and magnetic field components on the unit Yee cell, we obtain the magnetic field update equations

$$H_x^{n+\frac{1}{2}}(i, j + \frac{1}{2}, k + \frac{1}{2}) = H_x^{n-\frac{1}{2}}(i, j + \frac{1}{2}, k + \frac{1}{2}) + \quad (6.3.3)$$

$$\frac{\Delta t}{\mu} \left[\left(\frac{E_y^n(i, j + \frac{1}{2}, k + 1) - E_y^n(i, j + \frac{1}{2}, k)}{\Delta z} \right) \right] \quad (6.3.4)$$

$$+ \left(\frac{E_z^n(i, j, k + \frac{1}{2}) - E_z^n(i, j + 1, k + \frac{1}{2})}{\Delta x} \right) \Big]. \quad (6.3.5)$$

This update equation coincides with the results in Section 6.2.1, (6.2.1.6). Therefore, the staggered grid, Yee algorithm and the discrete differential form algorithm are

equivalent on cubical complexes. The other two update equations (for the magnetic field components and the update equations for the electric field components) are obtained similarly. With the central differencing above, Yee's algorithm is second order accurate. A more detailed explanation of Yee's algorithm is given in [29].

Yee's method has the advantage of being easy to implement and very stable yet it is inefficient and inaccurate for objects with complicated curved geometries.

6.4 Discrete Surface Integral Algorithm

With the Yee algorithm, in order to conform to curved geometries, one must use finer resolution to stair-step approximate a curved boundary. This method can be very costly in terms of computer memory and time. A non-orthogonally gridded computational domain can adapt much better to a curved surface without having to reduce the cell size as in the Yee algorithm. The Discrete Surface Integral (DSI) method is considered to be a generalized Yee algorithm for a non-orthogonal cell complex. For the DSI method, the cellular complex consists of convex polyhedral cells. Convex means that any two points in the interior of the cell can be joined by a straight line lying entirely within the cell. Two neighboring cells in a non-orthogonal complex are shown in Figure 6.9. Using Ampere's Law,

$$\epsilon \partial_t \vec{E} = \nabla \times \vec{H}$$

as an example, the DSI algorithm is determined using the following five steps:

1. Let $\epsilon \partial_t \vec{E} = \vec{J}_d$. Then the curl of $\vec{H}$ leads to

$$J_{d\alpha} = \vec{J}_d \cdot \vec{v}_\alpha = \frac{1}{Area_\alpha} \sum_{\beta=1}^4 H_{\alpha\beta} l_\beta$$

where $\vec{v}_\alpha$ is the normal to face α , the $H_{\alpha\beta}$'s are on the β -th edge of face α and $Area_\alpha$ is the area of face α . Thus, $J_{d\alpha}$ is the $\vec{J}_d$ value projected along the normal to face α , which may or may not coincide with the corresponding edge in the primary grid.

FIGURE 6.9. Neighboring Cells in a Non-Orthogonal Complex.

2. Face related scalar quantities, J_d , are then converted into vertex related scalar quantities. For example, referring to Figure 6.9, vertex A is associated with faces 0, 1, and 4. Find $\tilde{J}_{dAu}$ (A represents the vertex and u represents the upper cell) by solving the system of equations

$$\begin{aligned}\tilde{J}_{dAu} \cdot \vec{v}_0 &= J_{d0} \\ \tilde{J}_{dAu} \cdot \vec{v}_4 &= J_{d4} \\ \tilde{J}_{dAu} \cdot \vec{v}_1 &= J_{d1}\end{aligned}$$

Do this for each vertex in the upper and lower cell, which share a common face.

3. Average each vertex related $\tilde{J}_d$ to get face valued vectors using volume averaging from the triple product of the three normal vectors related to that vertex [16]. For example,

$$\begin{aligned}\tilde{J}_{d0} &= \left[\tilde{J}_{dAu}(\vec{v}_0 \cdot \vec{v}_4 \times \vec{v}_1) + \tilde{J}_{dBu}(\vec{v}_0 \cdot \vec{v}_1 \times \vec{v}_2) + \cdots + \tilde{J}_{dAl}(\vec{v}_0 \cdot \vec{v}_9 \times \vec{v}_6) \right. \\ &\quad \left. + \tilde{J}_{dBl}(\vec{v}_0 \cdot \vec{v}_6 \times \vec{v}_7) + \cdots \right] \div [(\vec{v}_0 \cdot \vec{v}_4 \times \vec{v}_1) + \cdots + (\vec{v}_0 \cdot \vec{v}_8 \times \vec{v}_9)]\end{aligned}$$

4. Project $\tilde{J}_{d0}$ onto the edge of the primary grid $\vec{v}_{p0}$, which intersects face 0.

$$J_{dl0} = \tilde{J}_{d0} \cdot \vec{v}_{p0}$$

5. Write the time derivative in an explicit centered difference form,

$$\epsilon \left[\frac{E_{\alpha}^{n+1} - E_{\alpha}^n}{\Delta t} \right] = J_{d\Omega}^{n+\frac{1}{2}}.$$

Note that the volume averaging used in the third step of the DSI algorithm uses the volumes of the parallelepipeds formed by the triple product and not the actual volume of the arbitrary polyhedron. Therefore, for the parallelepiped complex, the update equations found using the DSI algorithm coincide with the update equations found using the discrete differential form algorithm, given in (6.2.2.9) for the two-dimensional case and in (6.2.3.7) for the three dimensional case. For the unstructured trapezoid complex, the update equations found using the DSI algorithm differ from the update equations found using the discrete differential form algorithm in (6.2.4.6). This difference is due to the volume averaging done in the DSI algorithm. Detailed explanations of the Discrete Surface Integral Algorithm can be found in various papers and books [29, 16, 25].

Chapter 7

CONCLUSION

With the use of mathematical concepts from differential geometry and algebraic topology, a discrete differential form algorithm for solving Maxwell's equations on unstructured meshes was developed. Beginning with differential forms and differentiable manifolds, we were able to describe Maxwell's equations in the continuum. With the development of the cellular decomposition of a manifold and defining cochains to be the discrete analog to differential forms, we were then able to describe Maxwell's equations in the discrete domain. Using these discrete representations, the discrete boundary, coboundary and star operators were constructed. These operators were shown to maintain divergence free regions, therefore, not producing unwanted sources. We have also realized the corresponding numerical update equations for Maxwell's equations on unstructured meshes and shown that these equations reduce to the Yee algorithm for an orthogonal mesh and to the DSI algorithm for a parallelepiped mesh. The discrete differential form algorithm reveals potential defects in previously developed algorithms, such as the discrete surface integral algorithm.

7.1 Further Work

Further work on the discrete differential form algorithm includes numerical implementation of the algorithm and determining a permeability and permittivity based star operator. Numerical implementation of this algorithm is very important in determining its stability and accuracy in the modelling of Maxwell's equations. In the development of the discrete differential form algorithm presented in this dissertation, the medium in which the electric and magnetic fields exist was assumed to be a simple medium. It is believed that if the medium is not simple, then the star operator will

depend on the permeability and permittivity of each cell. This additional work will be helpful in generalizing the discrete differential form algorithm to media consisting of many different material properties so that the propagation of electromagnetic waves in such a media can be modelled accurately.

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