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SCATTERING FROM A THIN WIRE EXCITED BY A
PERPENDICULAR LINE CURRENT

by
Muntasir Mohammad Sheikh

A Dissertation Submitted to the Faculty of the
DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING
In Partial Fulfillment of the Requirements
For the Degree of
DOCTOR OF PHILOSOPHY
In the Graduate College
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ABSTRACT

In many applications it is necessary to determine coupling from a line current source to a nearby wire. Applications include current coupling in high-speed interconnects and wire interaction with a charged particle beam. A common physical configuration occurs when the source and wire are perpendicular to each other. In this dissertation, we investigate the scattered field and coupled current that result from such a configuration. We solve the problem for three different sources: a dipole, an array of dipoles, and a continuous line current. We detail the solution for the line current source where we obtain the scattered field by numerical integration, the far-zone approximation using steepest descents, and the excited current by numerical integration. We also show that the solution of an infinite number of phased dipoles approaches the continuous line source excitation. For the continuous line source case, we assume an infinite traveling wave line current. We also assume that the current magnitude and phase are not affected by the existence of the nearby wire. The current travels with a speed less than the speed of light in the surrounding medium. The wire is infinitely long and infinitesimally thin, and is located a distance d from the line source. We solve for the scattered field both numerically and approximately using steepest descents. We then add corrections to the saddle point approximation through two different approaches. We also solve numerically for the coupled current on the wire. Finally, we produce plots that allow us to compare the levels of the field with and

without the wire present. Our problem could serve as a prelude to investigation of a traveling wave of current and an array of parallel wires. However, such a problem is quite different since the physical configuration would then allow the presence of guided waves.

Chapter 1

INTRODUCTION

In many applications a situation arises when it is necessary to place conducting non-circuit elements close to other circuits. Examples of such elements are support cables and reinforcing steel rods. One common configuration is in the form of long wires perpendicular to current carrying conductors. The most notable effect of such a situation is that information that is to be bound to the relevant circuits may become coupled to these neighboring elements. This coupling can interfere with the operation of other circuits, cause cross-talk, or disturb shielding properties. Another physical situation that can be modelled using the same line of thought is lightning-induced current on horizontal wires.

The analysis of coupling to wires from dipoles or other wires is relatively recent. In a paper appearing in 1977 [1], the authors used full-wave scattering techniques to formulate the current induced on a horizontal wire above earth by a vertical electric dipole (VED), and obtained approximate solutions when the VED is electrically distant. In the same year, Hill and Wait [2] investigated coupling between an arbitrarily oriented electric or magnetic dipole and a horizontal cable over an ideal ground. They presented results for far field radiation and modal currents of a leaky coaxial cable excited by a VED. In a series of two papers, Wait investigated the excitation of a

coaxial cable above ground by a VED [3] and the excitation of an ensemble of parallel horizontal lines above a layered ground by a similar source [4]. No numerical results were presented in either paper. Later, the emphasis shifted to coupling between crossing conductors on microstrip and microwave circuits where one or both conductors may carry incident currents. Giri *et al.* [5] developed an equivalent circuit approximate model for the junction of two skewed transmission lines, where the inductive coupling was solved in close form and the capacitances obtained by solving coupled integral equations. In that paper the authors assumed TEM propagation on both conductors.

Different approaches were followed by other authors to develop similar coupling models: Kami and Sato [6] used transmission line theory and assumed that the power returning to the exciting line is negligible. Uwano *et al.* [7] analyzed crossing strip lines on a suspended substrate using a generalized transverse resonance technique. Papatheodorou *et al.* [8] used the method of moments to solve coupled integral equations derived for a microstrip crossover in a dielectric substrate backed by a ground plane. Young [9] developed a closed form approximation for TEM capacitive coupling between orthogonally crossing wires over a ground plane using full-wave scattering analysis. Finally, in a recent paper, Wait [10] presented a self-contained canonical solution for the currents excited on an overhead cable by a vertical line of current of finite height over a perfectly conducting plane, where low frequency approximations were incorporated to provide some insight. No numerical results

were presented.

We note that in all of the above mentioned papers there existed multiple conductors and/or ground planes, thus assuring the existence of TEM modes for most configurations. This facilitated developing approximate low frequency approximations if desired. In this dissertation we deal with the case of a single, infinitely-long wire close to a perpendicular infinite line-current source. As we shall show, the absence of a second conductor or ground plane makes results quite different from the above cases since no bound modes exist. The analysis presented here is primarily for the infinite current source, with the cases of single and multiple dipole sources presented for comparison purposes. The motivation for this research was to present a deeper understanding of the physics involved with scattering from a cross-wire.

In Chapter 2, we present the analytical formulation of the incident and scattered fields and currents excited on the wire (scatterer). We describe the geometry for different cases and derive representations for fields and currents. We also prove that the quantities for the continuous source case can be derived from those for the dipole array case as a limiting process.

In Chapter 3, we discuss features of the different complex planes involved in the problem. The presentation in this chapter focuses primarily on the continuous source case. We also demonstrate the importance of studying several complex planes for the same problem to better understand the relevant features. In the beginning of the chapter, we present a comprehensive explanation of how to determine the

proper branch cuts and Riemann sheets of a special form of the square root function. Although similar presentations can be found in electromagnetics and mathematics texts, in most of the cases the details are either confusing or not as comprehensive.

In Chapter 4, we present a standard asymptotic analysis for obtaining approximate expressions for the far-zone field. We also investigate the details of special cases that arise from certain values of parameters and develop improved and new expressions to account for the discrepancies between the numerically integrated fields and the standard asymptotic expressions. We also compare the asymptotic expressions for the incident and the scattered fields. At the end of the chapter, we present the asymptotic expressions for the single and multiple dipole cases, and demonstrate the travelling wave nature of the excited current away from the center of the scatterer.

In Chapter 5, we present the numerical results. First, we explain why fast Fourier transform techniques are not suitable for the integrals in hand. Next, we describe the methods used to integrate the field and current integrals, and compare the numerically integrated fields with standard and improved approximations. We also describe the features of numerically integrated fields and asymptotic expressions based upon mathematical features of the complex planes involved as well as physical characteristics of the excited currents. Finally, we present an example application that demonstrates the use of the results in a possible practical situation.

In Chapter 6, we summarize our results and recommend possible extensions to this problem as future work.

Chapter 2

FORMULATION

In this chapter we detail the analytical formulation of electric fields and currents for the continuous and discrete source cases. We begin by describing the geometry of the different cases. Then we present a detailed analysis of the continuous source case where the problem is split into incident and scattered parts. The procedure uses standard techniques for solving second order partial differential equations combined with coordinate transformations. Finally, we introduce the final expressions for the single and multiple dipole source cases without much detail as they are very similar to the former formulation, albeit simpler. Also, we analytically prove that the current excited on the scatterer from a dipole array indeed approaches that from a continuous source if enough properly phased elements are added to the array. This also shows that long enough arrays can be modeled by an infinite line source and vice versa.

2.1 Problem Geometry

The geometries of the different parts of the problem are shown in Figures 2.1, 2.2, and 2.3. Fig. 2.1 shows the geometry when the source is a single dipole, Fig. 2.2 shows that for the dipole array, and Fig. 2.3 for the continuous source. In all cases, the scatterer is an infinitely long wire placed parallel to the z -axis, passing through the

FIGURE 2.1. Geometry of the problem for the case of a single dipole source.

FIGURE 2.2. Geometry of the problem for the case of a dipole array source.

FIGURE 2.3. Geometry of the problem for the case of a continuous line current source.

point $(x, y) = (d, 0)$, and having a cross-sectional radius a . The radius a is assumed sufficiently small to satisfy thin-wire approximations. The conductivity of the wire is assumed to be infinite. We shall be referring mostly to Fig. 2.3, since we shall only detail the analysis of the continuous source case. The single dipole formulation is very similar, and that of a dipole array can be obtained by superposition. In the subsequent analysis of the continuous source case we shall be using local coordinate systems that will facilitate the derivations. These coordinate systems are shown in Fig. 2.3 as primed and double-primed Cartesian coordinates. Also, when expressing and plotting scattered fields only we shall use local spherical coordinates (r and θ) that pertain only to the double-primed system. This is because scattered fields have ϕ -symmetry in the double-primed system and, therefore, are easier to plot using those coordinates.

2.2 Continuous Line Current Source

In this section we detail the derivation of the incident and scattered fields and excited current when the source is a continuous line current. Although not mentioned explicitly, all quantities dealt with from now on are in the temporal frequency domain.

2.2.1 Current Source Model

In the continuous current source case, the current is assumed to have a constant phase behavior, i.e.

$$I(y) = I_0 e^{-i\beta y}. \quad (2.1)$$

where β is a constant. This current form has been used many times, for instance, by Wait [11], Dudley and Casey [12], and others, because the results can be extended to a more general form of the current using Fourier transform methods. We also assume that the traveling current speed is slower than that of light in the surrounding medium, i.e., $\beta > k$. We shall assume that the surrounding medium is free space, although the generalization to any lossless medium can be achieved by incorporating the proper permittivity, so long as β is kept larger than k . In Section 2.3 we shall use a discretized form of (2.1) as amplitudes for the dipole array source.

2.2.2 Incident Field

We split the fields into an incident field emanating from the source, and a scattered field caused by the adjacent wire, viz:

$$\mathbf{E}^{TOTAL} = \mathbf{E}^t = \mathbf{E}^i + \mathbf{E}^s \quad (2.2)$$

We begin by solving for the incident field. We define (Fig. 2.3) a local coordinate system (x', y', z') such that the line current source is aligned with the z' -axis, i.e.,

$x \rightarrow x'$, $y \rightarrow z'$, $z \rightarrow -y'$. Only the z' directed potential is needed. Indeed, if $\mathbf{A}^i$ is the vector potential of the incident field, then

$$\mathbf{A}^i = \mathbf{1}_{z'} \psi^i(\rho', z') \quad (2.3)$$

where [13, eq. 2-110]

$$(\nabla_{\rho', z'}^2 + k^2) \psi^i(\rho', z') = -I_0 e^{-i\beta z'} \frac{\delta(\rho')}{2\pi\rho'}, \quad (2.4)$$

ρ' and z' are cylindrical coordinates in the primed system, and $\mathbf{1}_{z'}$ is a unit vector in z' direction. Note that ψ^i is independent of ϕ' . We now define the following Fourier transform pair

$$\begin{aligned} \Psi(\lambda) &= \int_{-\infty}^{\infty} \psi(z') e^{-i\lambda z'} dz' \\ \psi(z') &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi(\lambda) e^{i\lambda z'} d\lambda \end{aligned} \quad (2.5)$$

and apply the forward transform to (2.4), viz:

$$\frac{1}{\rho'} \frac{\partial}{\partial \rho'} \left(\rho' \frac{\partial \Psi^i(\rho', \lambda)}{\partial \rho'} \right) + (k^2 - \lambda^2) \Psi^i(\rho', \lambda) = -I_0 \delta(\lambda + \beta) \frac{\delta(\rho')}{\rho'} \quad (2.6)$$

The solution to (2.6) is given by [14, eq. 4.116]

$$\Psi^i(\rho', \lambda) = \frac{2\pi I_0 \delta(\lambda + \beta)}{4i} H_0^{(2)} \left[(k^2 - \lambda^2)^{1/2} \rho' \right] \quad (2.7)$$

We now take the inverse Fourier transform of (2.7) to obtain the solution in z' , viz:

$$\psi^i(\rho', z') = \frac{I_0}{4i} e^{-i\beta z'} H_0^{(2)}(\lambda_0 \rho') \quad (2.8)$$

where

$$\lambda_0 = \sqrt{k^2 - \beta^2}, \text{Im}(\lambda_0) < 0. \quad (2.9)$$

In the primed coordinate system, the scatterer is parallel to y' , which means that the only electric field component to interact with the wire is $E_{y'}^i$; thus we require only that component, viz:

$$\begin{aligned} E_{y'}^i(\rho', z', \phi') &= \mathbf{1}_{y'} \cdot \mathbf{E}^i = (\mathbf{1}_{\rho'} \sin \phi' + \mathbf{1}_{\phi'} \cos \phi') \cdot (\mathbf{1}_{\rho'} E_{\rho'}^i + \mathbf{1}_{\phi'} E_{\phi'}^i + \mathbf{1}_z E_z^i) \\ &= \sin \phi' E_{\rho'}^i + \cos \phi' E_{\phi'}^i. \end{aligned}$$

The required cylindrical components of the field are given by [13, eq. 5-18]

$$\begin{aligned} E_{\rho'}^i &= \frac{1}{\hat{y}} \frac{\partial^2 \psi^i}{\partial \rho' \partial z'}, \text{ and} \\ E_{\phi'}^i &= \frac{1}{\hat{y} \rho'} \frac{\partial^2 \psi^i}{\partial \phi' \partial z'} = 0 \end{aligned}$$

where $\hat{y} = i\omega\epsilon$. Thus,

$$\begin{aligned} E_{y'}^i(\rho', z', \phi') &= \sin \phi' \frac{\partial^2 \psi^i}{\partial \rho' \partial z'} \\ &= \frac{I_0 \beta \lambda_0}{4\hat{y}} \sin \phi' e^{-i\beta z'} H_1^{(2)}(\lambda_0 \rho') \end{aligned} \quad (2.10)$$

2.2.3 Scattered Field

We consider now the scattered field, where we define (Fig. 2.3) another local coordinate system (x'', y'', z'') that is just a translated version of (x, y, z) , i.e., $x'' = x - d$, $y'' = y$, and $z'' = z$. We follow the same steps as with the incident field, but this time the current is unknown. Again

$$\Psi^s(\rho'', \lambda) = \frac{I(\lambda)}{4i} H_0^{(2)}(\gamma \rho'')$$

where $I(\lambda)$ is the Fourier transform of the unknown current, $\gamma = (k^2 - \lambda^2)^{1/2}$, and $\text{Im}(\gamma) < 0$. We require the scattered field component parallel to the wire, i.e. [13, eq. 5-18],

$$E_{z''}^s(\rho'', z'') = \frac{1}{\widehat{y}} \left(\frac{\partial^2}{\partial z''^2} + k^2 \right) \psi^s.$$

In the spatial frequency domain this is given by

$$\begin{aligned} E_{z''}^s(\rho'', \lambda) &= \frac{1}{\widehat{y}} \gamma^2 \Psi^s \\ &= \frac{\gamma^2 I(\lambda)}{4i\widehat{y}} H_0^{(2)}(\gamma \rho''). \end{aligned} \quad (2.11)$$

We now apply the inverse Fourier transform to (2.11) to obtain

$$E_{z''}^s(\rho'', z'') = \frac{1}{8\pi i \widehat{y}} \int_{-\infty}^{\infty} \gamma^2 I(\lambda) H_0^{(2)}(\gamma \rho'') e^{i\lambda z''} d\lambda. \quad (2.12)$$

In order to evaluate $I(\lambda)$ we apply the boundary condition on the surface of the scatterer, i.e.,

$$E^t|_{\text{tangential}} = [E^i + E^s]_{\text{tangential}} = 0. \quad (2.13)$$

To do this, we transform the coordinates (x'', y'', z'') to (x', y', z') , i.e., $x'' \rightarrow x' - d$, $y'' \rightarrow y'$, and $z'' \rightarrow -y'$. Thus,

$$\rho'' = (x''^2 + y''^2)^{1/2} \rightarrow [(x' - d)^2 + z'^2]^{1/2}$$

and, after changing λ to $-\lambda$, (2.12) becomes

$$E_{y'}^s(x', y', z') = \frac{1}{8\pi i \widehat{y}} \int_{-\infty}^{\infty} \gamma^2 I(-\lambda) H_0^{(2)} \left\{ \gamma [(x' - d)^2 + z'^2]^{1/2} \right\} e^{i\lambda y'} d\lambda. \quad (2.14)$$

For the application of the boundary condition, we evaluate (2.10) at the center of the wire ($x' = d$, $z' = 0$), and (2.14) at the surface of the wire ($[(x' - d)^2 + z'^2]^{1/2} = a$). This is valid so long as $a \ll \lambda_m$, and $d \gg a$, where $\lambda_m = 2\pi/k$. Then from (2.13), (2.10), and (2.14) we get

$$\begin{aligned} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \gamma^2 \mathcal{I}(-\lambda) H_0^{(2)}(\gamma a) e^{i\lambda y'} d\lambda &= -I_0 \beta \lambda_0 \frac{y'}{(d^2 + y'^2)^{1/2}} H_1^{(2)} \left[\lambda_0 (d^2 + y'^2)^{1/2} \right] \\ &= I_0 \beta \frac{\partial}{\partial y'} H_0^{(2)} \left[\lambda_0 (d^2 + y'^2)^{1/2} \right]. \end{aligned} \quad (2.15)$$

We now apply the forward transform defined in (2.5) on both sides of (2.15) with respect to y' , viz:

$$\gamma^2 \mathcal{I}(-\lambda) H_0^{(2)}(\gamma a) = -I_0 \beta \lambda \mathcal{F} \left\{ H_0^{(2)} \left[\lambda_0 (d^2 + y'^2)^{1/2} \right] \right\} \quad (2.16)$$

where $\mathcal{F}\{\cdot\}$ denotes the Fourier transform operator. The transform of $H_0^{(2)}$ is obtained through a well-known integral identity [14, eq. 4.306]; thus (2.16) becomes

$$\gamma^2 \mathcal{I}(-\lambda) H_0^{(2)}(\gamma a) = -2I_0 \beta \lambda \frac{e^{-id(\lambda_0^2 - \lambda^2)^{1/2}}}{(\lambda_0^2 - \lambda^2)^{1/2}}$$

Rearranging yields

$$\mathcal{I}(\lambda) = \frac{2I_0 \beta \lambda e^{-id\hat{\gamma}}}{\gamma^2 \hat{\gamma} H_0^{(2)}(\gamma a)} \quad (2.17)$$

where

$$\hat{\gamma} = (\lambda_0^2 - \lambda^2)^{1/2}, \quad (2.18)$$

and $\text{Im}(\hat{\gamma}) < 0$. From (2.17) and (2.14) we obtain the complete expression for $E_{y'}^s$, viz:

$$E_{y'}^s(x', y', z') = -\frac{I_0\beta}{4\pi i\hat{y}} \int_{-\infty}^{\infty} \frac{\lambda H_0^{(2)} \left\{ \gamma [(x' - d)^2 + z'^2]^{1/2} \right\}}{H_0^{(2)}(\gamma a)} \frac{e^{-id\hat{\gamma}}}{\hat{\gamma}} e^{i\lambda y'} d\lambda. \quad (2.19)$$

Now that we have both the incident (2.10) and scattered (2.19) fields in the primed system, we transform both to the original unprimed system to cast the equations in their desired forms ($x' \rightarrow x$, $y' \rightarrow -z$, $z' \rightarrow y$). Note that because $y' \rightarrow -z$, we have to multiply the expressions by a minus sign to get the correct sign of the z -component, viz:

$$E_z^i(x, y, z) = -\frac{I_0\beta\lambda_0}{4\hat{y}} \frac{z}{(x^2 + z^2)^{1/2}} e^{-i\beta y} H_1^{(2)} \left(\lambda_0 (x^2 + z^2)^{1/2} \right) \quad (2.20)$$

and

$$E_z^s(x, y, z) = \frac{I_0\beta}{4\pi i\hat{y}} \int_{-\infty}^{\infty} \frac{\lambda H_0^{(2)} \left\{ \gamma [(x - d)^2 + y^2]^{1/2} \right\}}{H_0^{(2)}(\gamma a)} \frac{e^{-id\hat{\gamma}}}{\hat{\gamma}} e^{-i\lambda z} d\lambda. \quad (2.21)$$

In order to get the familiar form of the inverse Fourier transform, we change λ to $-\lambda$, viz:

$$E_z^s(x, y, z) = -\frac{I_0\beta}{4\pi i\hat{y}} \int_{-\infty}^{\infty} \frac{\lambda H_0^{(2)} \left\{ \gamma [(x - d)^2 + y^2]^{1/2} \right\}}{H_0^{(2)}(\gamma a)} \frac{e^{-id\hat{\gamma}}}{\hat{\gamma}} e^{i\lambda z} d\lambda. \quad (2.22)$$

To obtain the excited current as a function of z , we note that (2.17) was obtained by operating on y' . Thus we have to apply the same procedure followed in obtaining the fields regarding coordinate transformations, viz:

$$I(z) = -\frac{I_0\beta}{\pi} \int_{-\infty}^{\infty} \frac{\lambda e^{-id\hat{\gamma}}}{\gamma^2 \hat{\gamma} H_0^{(2)}(\gamma a)} e^{i\lambda z} d\lambda \quad (2.23)$$

2.3 Single Dipole and Dipole Array Sources

Here we simply mention the final formulas obtained if the excitation was a single dipole or a dipole array. The derivation is similar but simpler than that of the continuous source case: only one coordinate transformation is required because of the “point” nature of the dipole. If the dipole is placed as shown in Fig. 2.1, then the incident field is given by [13, eq. 2-113]

$$E_{Dz}^i(x, y, z) = \frac{\bar{I}_0 l}{2\pi} e^{-ik[x^2 + (y-y_0)^2 + z^2]^{1/2}} \frac{(y-y_0)z}{x^2 + (y-y_0)^2 + z^2} \\ \times \left[\frac{3}{2} \left(\frac{\eta}{x^2 + (y-y_0)^2 + z^2} + \frac{1}{i\omega\epsilon [x^2 + (y-y_0)^2 + z^2]^{3/2}} \right) \right. \\ \left. + \frac{i\omega\mu}{2 [x^2 + (y-y_0)^2 + z^2]^{1/2}} \right]$$

where the subscript D designates dipole quantities, l is the dipole length, and $\bar{I}_0 = I_0(\omega)f(y_0)$ is an arbitrary function of the operating frequency ω and dipole position at y_0 . An alternative form of the previous expression that is used to match boundary conditions can be obtained from an equivalent representation [3]:

$$E_{Dz}^i(x, y, z) = -\frac{\bar{I}_0 l}{8\pi\hat{y}} \frac{y-y_0}{[x^2 + (y-y_0)^2]^{1/2}} \int_{-\infty}^{\infty} \lambda \gamma H_1^{(2)} [\gamma (x^2 + (y-y_0)^2)]^{1/2} e^{i\lambda z} d\lambda \quad (2.24)$$

By matching boundary conditions on the surface of the wire, we obtain the excited current and scattered field, viz:

$$I_D(z) = \frac{\bar{I}_0 l}{i2\pi} \frac{y_0}{(d^2 + y_0^2)^{1/2}} \int_{-\infty}^{\infty} \lambda \frac{H_1^{(2)} [\gamma (d^2 + y_0^2)^{1/2}]}{\gamma H_0^{(2)}(\gamma a)} e^{i\lambda z} d\lambda, \text{ and} \quad (2.25)$$

$$E_{Dz}^s(x, y, z) = -\frac{\bar{I}_0 l}{8\pi\hat{y}} \frac{y_0}{(d^2 + y_0^2)^{1/2}} \times \int_{-\infty}^{\infty} \lambda \gamma \frac{H_1^{(2)} \left[\gamma (d^2 + y_0^2)^{1/2} \right]}{H_0^{(2)}(\gamma a)} H_0^{(2)} \left(\gamma [(x-d)^2 + y^2]^{1/2} \right) e^{i\lambda z} d\lambda \quad (2.26)$$

The current and field for the dipole array case are obtained by superimposing the above two equations. We demonstrate this for the current. Suppose that we want the current on the wire excited by an array of N dipoles placed tip to tail on the y -axis from $-Y_0$ to $+Y_0$, where we restrict N to be even (Fig. 2.2). Such a configuration will have $N/2$ dipoles on each side of the y -axis with no dipole placed at the origin. If all the dipoles are of equal lengths ($l_1 = l_2 = \dots = l_N = l$), then the spacing between each dipole and its neighbor is $\Delta y = l = 2Y_0/(N-1)$. Also, the position of each dipole will be given by

$$y_0(n) = -Y_0 \left(\frac{N-2n}{N-2} \right) - \Delta y \left(\frac{n-1}{N-2} \right), \quad n = 1, 2, \dots, N$$

In this development we make the current strength of each dipole a phased function of the position, i.e., $\bar{I}_0 = I_0 e^{-i\beta y_0(n)}$, where I_0 is a function of the operating frequency. The spatial-frequency current excited by the array can be expressed as the following Riemann sum

$$I_A(\lambda) = \frac{I_0 \lambda \Delta y}{i\gamma H_0^{(2)}(\gamma a)} \sum_{n=1}^N e^{-i\beta y_0(n)} \frac{y_0(n)}{(d^2 + y_0(n)^2)^{1/2}} H_1^{(2)} \left(\gamma [d^2 + y_0(n)^2]^{1/2} \right) \quad (2.27)$$

where A denotes the dipole array. If we take $N \rightarrow \infty$, then $\Delta y \rightarrow dy$, $y_0(n) \rightarrow y$, and the sum becomes an integral from $-Y_0$ to $+Y_0$. If we further take $Y_0 \rightarrow \infty$ then

we get

$$I_A(\lambda) = \frac{I_0 \lambda}{i \gamma H_0^{(2)}(\gamma a)} \int_{-\infty}^{\infty} e^{-i \beta y} H_1^{(2)} \left(\gamma [d^2 + y^2]^{1/2} \right) \frac{y}{(d^2 + y^2)^{1/2}} dy \quad (2.28)$$

$$= -\frac{I_0 \lambda}{i \gamma^2 H_0^{(2)}(\gamma a)} \int_{-\infty}^{\infty} e^{-i \beta y} \frac{\partial}{\partial y} \left[H_0^{(2)} \left(\gamma [d^2 + y^2]^{1/2} \right) \right] dy \quad (2.29)$$

$$= -\frac{I_0 \beta \lambda}{\gamma^2 H_0^{(2)}(\gamma a)} \int_{-\infty}^{\infty} H_0^{(2)} \left(\gamma [d^2 + y^2]^{1/2} \right) e^{-i \beta y} dy \quad (2.30)$$

The last equation was obtained using integration by parts where the conjunct evaluates to zero because of the choice of $\text{Im}(\gamma)$. Using a well known identity of Hankel functions [14, eq. 4.306] the above equation can be expressed as

$$I_A(\lambda) = -\frac{I_0}{i \gamma^2 H_0^{(2)}(\gamma a)} \lambda i \beta \left(\frac{2i e^{-d(\beta^2 - \gamma^2)^{1/2}}}{(\beta^2 - \gamma^2)^{1/2}} \right) \quad (2.31)$$

By letting $i \hat{\gamma} = (\beta^2 - \gamma^2)^{1/2}$ where $\text{Im}(\hat{\gamma}) < 0$ and taking the inverse Fourier transform we get

$$I_A(z) = -\frac{\beta I_0}{\pi} \int_{-\infty}^{\infty} \frac{\lambda e^{-i d \hat{\gamma}}}{\gamma^2 \hat{\gamma} H_0^{(2)}(\gamma a)} e^{i \lambda z} d\lambda. \quad (2.32)$$

This is the same current expression derived in the continuous source problem.

Chapter 3

FEATURES OF COMPLEX PLANES

Here we explain the details of the complex planes involved in solving the continuous source case; then we specialize to the dipole case. As an introduction, we review the method of choosing branch cuts for a frequently occurring form of the square root function. The detailed explanation presented here combines material from several sources and expands on it, providing a step-by-step procedure to choose the proper branch cuts and Riemann sheets. Next, we move on to the important topic of defining and discussing the different complex planes involved with the continuous source case. It is worth noting that going back and forth between different complex planes for the same problem helps in understanding its features, since some planes show those features better than others. The complex planes for the discrete source case (single and multiple dipoles) are a special case from those of the continuous source case, since they lack one branch cut. This simplifies the asymptotic analysis but introduces difficulties with numerical integration as we shall see in the following chapters.

3.1 Choice of Branch Cuts and Riemann Sheets for $\gamma = \sqrt{k^2 - \lambda^2}$

Before we begin discussing features of complex planes involved in this dissertation, it is illustrative to explain how branch cuts and Riemann sheets are chosen for an important form of the square root function, namely $\gamma = \sqrt{k^2 - \lambda^2}$, with k being real, and λ the independent complex variable. This section is based on the understanding of material in [15, Section 5.3b], [16, Section 11-3 and Appendix to Chapter 11], and [17]. Because the square root is a multi-valued function, we must restrict the domain to insure that we are working with one value at a time. This means we work on a specified Riemann sheet. To achieve this, we have to choose the configuration of branch cuts that would separate the different branches of the square root. The choice of branch cuts is in general arbitrary. However, since most of our applications stem from physical problems, our choice must comply with the physical characteristics of the original problem. For example, a common integral found in problems relating to scattering from cylindrical structures has the following general form

$$I = \int_C f(\lambda) H_n^{(2)}(\gamma\rho) e^{-i\lambda z} d\lambda$$

If we express the Hankel function in its large argument asymptotic form, the integral becomes

$$I \approx \int_C f(\lambda) \sqrt{\frac{2}{\pi\gamma\rho}} e^{i(2n+1)\pi/4 - i\gamma\rho - i\lambda z} d\lambda$$

By looking at the exponent we see that it represents a wave propagating in both

ρ and z . To satisfy radiation conditions for $z > 0$, we must have $\text{Re}(\lambda) > 0$ and $\text{Im}(\lambda) < 0$, and vice versa for $z < 0$. Similarly, since $\rho > 0$ we must choose $\text{Re}(\gamma) > 0$ and $\text{Im}(\gamma) < 0$. If we want to pick the branch cut according to the conditions on λ then a possible configuration is $\text{Re}(\lambda) = k$, $\text{Im}(\lambda) < 0$ and $\text{Re}(\lambda) = -k$, $\text{Im}(\lambda) > 0$ (Fig. 3.1a). The first set is used by closing in the lower half-plane when $z > 0$, where the second set is used by closing in the upper half-plane when $z < 0$. If we want to pick the branch cut according to the conditions on γ then we might pick $\text{Im}(\gamma) = 0$ (Fig. 3.2). The details of the second choice is the topic of this chapter.

3.1.1 Argument Analysis

To begin, we require that the branch cuts in the λ -plane be defined along $\text{Im}(\gamma) = 0$, and that the top sheet is chosen so that $\text{Im}(\gamma) \leq 0$. Furthermore, since contour integration is usually involved, we require that the contour pass through regions where $\text{Re}(\gamma) \geq 0$. The above two conditions on γ are based upon the radiation condition. Let us consider a point Q in the λ -plane (Fig. 3.2). If λ_Q gives the coordinates of Q in the λ -plane, then

$$\lambda_Q - k = r_1 e^{i\theta_1}$$

$$\lambda_Q + k = r_2 e^{i\theta_2}$$

Now consider γ

$$\gamma = \sqrt{k^2 - \lambda_Q^2} = \sqrt{(k - \lambda_Q)(k + \lambda_Q)} = \sqrt{-r_1 r_2 e^{i(\theta_1 + \theta_2)}}$$

FIGURE 3.1. Branch choices for $\gamma = \sqrt{k^2 - \lambda^2}$: (a) $\text{Re}(\lambda) = k$, $\text{Im}(\lambda) < 0$ and $\text{Re}(\lambda) = -k$, $\text{Im}(\lambda) > 0$ for k real; (b) $\text{Im}(\gamma) = 0$ for k complex.

Because of the square root, we can have two values for γ :

$$\gamma_1 = \sqrt{r_1 r_2} e^{i(\theta_1 + \theta_2)/2 - i\pi/2}$$

$$\gamma_2 = \sqrt{r_1 r_2} e^{i(\theta_1 + \theta_2)/2 + i\pi/2}$$

To determine which one of these values (or branches) of γ is going to be used, we test the argument of γ at different locations in the λ -plane. In order for $\text{Im}(\gamma)$ to be nonpositive on the top sheet, we must have $-\pi \leq \text{Arg}(\gamma) \leq 0$, where $\text{Arg}(\cdot)$ represents the principal value of the argument. Similarly, for $\text{Re}(\gamma)$ to be nonnegative in the regions where the contour should pass, we must have $-\pi/2 \leq \text{Arg}(\gamma) \leq \pi/2$. Combining both ranges, we find that the permissible range for $\text{Arg}(\gamma)$ becomes $-\pi/2 \leq \text{Arg}(\gamma) \leq 0$. We now move Q in the λ -plane to determine the permissible ranges for θ_1 and θ_2 that will satisfy the condition on γ . Such analysis is necessary to determine the correct locations where contours of integration should pass, especially if contributions by other singularities may be included.

Quadrant I: For this quadrant, we begin by choosing the following ranges for θ_1 and θ_2 :

$$0 \leq \theta_1 \leq \pi \text{ and } 0 \leq \theta_2 \leq \pi/2$$

This choice can be clearly explained by looking at Fig. 3.2 and following Q as it traverses in the first quadrant, with the limiting positions being marked as P1, P2, and P3. Table 3.1 shows the values for the arguments of γ_1 and γ_2 . It is obvious that γ_1 is the right choice that satisfies the required conditions. *This also means that we*

Points	θ_1	θ_2	$\text{Arg}(\gamma_1)$	$\text{Arg}(\gamma_2)$
P1	π	0	0	π
P2	0	0	$-\frac{\pi}{2}$	$\frac{\pi}{2}$
P3	$\frac{\pi}{2}$	$\frac{\pi}{2}$	0	π

TABLE 3.1. Values of θ_1 and θ_2 and principal arguments of γ_1 and γ_2 corresponding to points in Quadrant I

Points	θ_1	θ_2	$\text{Arg}(\gamma_1)$	$\text{Arg}(\gamma_2)$
P4	π	0	0	π
P5	π	π	$\frac{\pi}{2}$	$\frac{3\pi}{2}$
P6	$\frac{\pi}{2}$	$\frac{\pi}{2}$	0	π

TABLE 3.2. Values of θ_1 and θ_2 and principal arguments of γ_1 and γ_2 corresponding to points in Quadrant II (improper branch)

have to make sure that γ_1 stays the right choice for the rest of the quadrants. What we have done here is to select the branch on which we allow the integration contour to pass.

Quadrant II: As the point Q moves to the second quadrant, we are tempted to make a simple continuation of the ranges for θ_1 and θ_2 , namely:

$$\pi/2 \leq \theta_1 \leq \pi \text{ and } 0 \leq \theta_2 \leq \pi$$

This range will lead to undesired values for $\text{Arg}(\gamma)$ on both branches, as can be seen from Table 3.2.

The reason is that the simple range continuation means that Q *crosses* the branch cut when traversing from the first quadrant to the second. This places Q on the improper (bottom) Riemann sheet, thus leading to the undesired values for $\text{Arg}(\gamma)$. Nevertheless, since the only way for Q to move from the first quadrant to the second

Points	θ_1	θ_2	$\text{Arg}(\gamma_1)$	$\text{Arg}(\gamma_2)$
P4	π	-2π	$-\pi$	0
P5	π	$-\pi$	$-\frac{\pi}{2}$	$\frac{\pi}{2}$
P6	$\frac{\pi}{2}$	$-\frac{3\pi}{2}$	$-\pi$	0

TABLE 3.3. Values of θ_1 and θ_2 and principal arguments of γ_1 and γ_2 corresponding to points in Quadrant II (proper branch)

is to cross the branch cut between them, we have to make Q cross *another* part of the branch cut to return to the proper (top) Riemann sheet and in the same time go to the second quadrant. We let Q move to the third quadrant (cross the negative real axis where $\text{Re}(\lambda) < -k$) then move again to the second quadrant by crossing the part of the branch cut located where $-k < \text{Re}(\lambda) < 0$ (see Q' in Fig. 3.2). This is achieved by shifting the range of θ_2 a complete period, i.e., $\pm 2\pi$ from the range shown immediately above. We choose to subtract 2π from both ends, viz.,

$$-2\pi \leq \theta_2 \leq -\pi,$$

while keeping the range of θ_1 the same ($\pi/2 \leq \theta_1 \leq \pi$). This leads to the values shown in Table 3.3. Note that the values for $\text{Arg}(\gamma_1)$ are not quite what we want. Although they satisfy the condition that $\text{Im}(\gamma) \leq 0$, they give the opposite sign for $\text{Re}(\gamma)$, i.e., $\text{Re}(\gamma) \leq 0$. This means that although we are on the correct sheet (branch), the contour of integration should not pass through this quadrant.

Quadrant III: The ranges of this quadrant should be a simple continuation of those in the second quadrant (point Q' crosses $\text{Re}(\lambda) < -k$ to the third quadrant;

Points	θ_1	θ_2	$\text{Arg}(\gamma_1)$	$\text{Arg}(\gamma_2)$
P7	π	0	0	π
P8	π	$-\pi$	$-\frac{\pi}{2}$	$\frac{\pi}{2}$
P9	$\frac{3\pi}{2}$	$-\frac{\pi}{2}$	0	π

TABLE 3.4. Values of θ_1 and θ_2 and principal arguments of γ_1 and γ_2 corresponding to points in Quadrant III

see Fig. 3.2¹). Hence the values:

$$\pi \leq \theta_1 \leq 3\pi/2 \text{ and } -\pi \leq \theta_2 \leq 0$$

The resulting values given in Table 3.4 show that $\text{Arg}(\gamma_1)$ now satisfies both required conditions, i.e., $\text{Im}(\gamma) \leq 0$ and $\text{Re}(\gamma) \geq 0$. Thus, the contour of integration is permitted to pass through this quadrant.

Quadrant IV: Based upon the previous analysis, the logical continuation of the ranges for θ_1 and θ_2 should be clockwise from the first quadrant, since Q can move from the first quadrant to the fourth without crossing a branch cut, i.e., through $\text{Re}(\lambda) > k$. Hence the ranges are chosen to be

$$-\pi \leq \theta_1 \leq 0 \text{ and } -\pi/2 \leq \theta_2 \leq 0$$

As expected, the resulting values for $\text{Arg}(\gamma_1)$ (Table 3.5) satisfy the condition on $\text{Im}(\gamma)$, but gives the wrong sign for $\text{Re}(\gamma)$, which means that the contour should not pass through this quadrant.

Another possible trajectory for Q is to go counter-clockwise from the first quadrant to the second across the branch cut, from the second quadrant to the third across the

¹Note that the path of Q' to the third quadrant is not shown.

Points	θ_1	θ_2	$\text{Arg}(\gamma_1)$	$\text{Arg}(\gamma_2)$
P10	$-\pi$	0	$-\pi$	0
P11	0	0	$-\frac{\pi}{2}$	$\frac{\pi}{2}$
P12	$-\frac{\pi}{2}$	$-\frac{\pi}{2}$	$-\pi$	0

TABLE 3.5. Values of θ_1 and θ_2 and principal arguments of γ_1 and γ_2 corresponding to points in Quadrant IV

real line where $\text{Re}(\lambda) < -k$, then to the fourth quadrant across the branch cut once more.

3.1.2 Practical Considerations

In the previous analysis we showed how to choose the ranges of θ_1 and θ_2 to satisfy the conditions on the top sheet and its different quadrants. In practice, instead of deforming the contour of integration to pass only through the third and first quadrants, we may choose to keep the contour on the real line and let k have a small negative imaginary part to move the branch points off the real axis, viz:

$$k = k_r + ik_i.$$

where $k_r > 0$, $k_i < 0$, and $|k_r| > |k_i|$. Since γ and λ must satisfy the relation

$$\gamma^2 + \lambda^2 = k^2$$

then we substitute the rectangular form expressions of γ and k and equate the real and imaginary parts to get

$$\gamma_r^2 - \gamma_i^2 + \lambda_r^2 - \lambda_i^2 = k_r^2 - k_i^2$$

$$\gamma_r \gamma_i + \lambda_r \lambda_i = k_r k_i$$

Now we employ the condition $\text{Im}(\gamma) = \gamma_i = 0$:

$$\gamma_r^2 + \lambda_r^2 - \lambda_i^2 = k_r^2 - k_i^2 \quad (3.1)$$

$$\lambda_r \lambda_i = k_r k_i \quad (3.2)$$

Equations (3.1) and (3.2) describe a hyperbola going through the points $\lambda = \pm k$ (Fig. 3.1b). Note that not the whole hyperbola curve represents the branch cuts. To determine which parts of the curve represent branch cuts, we must examine points that satisfy both equations. We note that for the part of the curve from $+k$ to A, $|\lambda_r| < |k_r|$ and $|\lambda_i| > |k_i|$, and this satisfies (3.1). However, for the part of the curve from $+k$ to B, $|\lambda_r| > |k_r|$ and $|\lambda_i| < |k_i|$, which does not satisfy (3.1). A similar analysis can be performed for the other branch of the hyperbola. This gives the branch cut represented by the solid curves shown in Fig. 3.1b.

One last note that deserves mentioning here is that in there are cases where it is necessary to keep k real while performing numerical integration along the real line. To satisfy the physical conditions without deforming the contour of integration we must be aware of two things: First, we must enforce the correct sign of both real and imaginary parts of the square root along the real line and not let the programming

language decide. Second, we have to know precisely the behavior of the integrand when approaching the branch points $\lambda = \pm k$. We have implemented the first point in all numerical integration routines performed for the problem in hand, and taken care of the second point by introducing suitable mappings. Details can be found in Chapter 4.

3.2 Continuous Line Source

We now discuss the features of complex planes pertaining to the continuous line source problem. The complex plane of (2.22) and (2.23) is shown in Fig. 3.3. We shall defer discussion of the contours C_I and C_{II} until Chapter 4. There are three branch cuts in this plane: The first is $\text{Im}(\gamma) = 0$ where $\gamma = (k^2 - \lambda^2)^{1/2}$. The branch points are $\lambda = \pm k$ and the branch cut covers the whole imaginary axis and the piece of real axis between branch points. The second branch cut is $\text{Im}(\hat{\gamma}) = 0$ where $\hat{\gamma} = (k^2 - \beta^2 - \lambda^2)^{1/2}$. The branch points are $\lambda = \pm \lambda_0$ (see (2.9)) and the branch cut $\in (-i\infty, \lambda_0] \cup [-\lambda_0, i\infty)$. Finally, the third branch cut is the one associated with the Hankel functions in (2.22) and (2.23), defined by $\text{Im}(\gamma) = 0$ and $\text{Re}(\gamma) < 0$. This branch cut lies on top of the first one but is of a different nature. The first branch cut is a square root represented by a two-sheet Riemann surface, while the third has a logarithmic nature near the origin represented by an infinitely sheeted Riemann surface. To insure convergence of (2.22) and (2.23), we define the top sheet of the complex λ -plane as $\text{Im}(\gamma) < 0$ and $\text{Im}(\hat{\gamma}) < 0$. Furthermore, to achieve proper phase

FIGURE 3.3. λ -plane, k real. Top sheet is defined by $\text{Im}(\gamma, \hat{\gamma}) < 0$.

we require that the contour of integration pass through the third and first quadrants of the λ -plane where $\text{Re}(\gamma) > 0$ and $\text{Re}(\hat{\gamma}) > 0$.

We can gain more insight into the structure of Riemann sheets and branch cuts of the problem by investigating the γ -plane (Fig. 3.4). In this view we assume a small negative imaginary part for k to separate the branch cuts. The square roots involved here are $\lambda = (k^2 - \gamma^2)^{1/2}$ and $\hat{\gamma} = (\gamma^2 - \beta^2)^{1/2}$. We define the square root branch cuts as $\text{Im}(\lambda) = 0$ and $\text{Im}(\hat{\gamma}) = 0$, and we also have the branch cut associated with the Hankel functions defined as $\text{Im}(\gamma) = 0$, $\text{Re}(\gamma) < 0$. We define the top Riemann sheet as $\text{Im}(\lambda, \hat{\gamma}) < 0$. If we are to draw the contour of integration, it will be situated in the lower half-plane where $\text{Im}(\gamma) < 0$ and may cross to a lower sheet so long as we have $\text{Im}(\gamma, \hat{\gamma}) < 0$. It is instructive to display the different sheets of the γ -plane to see how can we traverse from one sheet to another. To do that in a practical way we must remove Hankel functions from the picture, because the branch cut of the Hankel function has a logarithmic nature near the origin. Having done that, the Riemann sheets of the γ -plane, without Hankel functions, are shown in Fig. 3.5. If we are moving counter-clockwise on a circle of radius h , where $h > \max(\beta, |k|)$, we can move from one sheet to another and back to the starting point. Note that crossing a branch cut changes the sign of the imaginary part associated with the definition of *that* branch cut, but not the other. This is because of the way we defined those branch cuts. For example, crossing the lower part of the λ branch cut changes only the sign of $\text{Im } \lambda$, but not $\text{Im } \hat{\gamma}$ (see $2 \rightarrow 1$ in Fig. 3.5b).

$\text{Im}(\hat{\gamma} = \sqrt{\gamma^2 - \beta^2}) = 0$

Br. cut of $H_0^{(2)}(\bullet)$

$\text{Im}(\lambda = \sqrt{k^2 - \gamma^2}) = 0$

Complex zeros of $H_0^{(2)}(\bullet)$

I: $\text{Re}(\lambda) > 0$, II: $\text{Re}(\lambda) < 0$, I: $\text{Re}(\hat{\gamma}) < 0$, II: $\text{Re}(\hat{\gamma}) > 0$

Top sheet: $\text{Im}(\lambda, \hat{\gamma}) < 0$

FIGURE 3.4. γ -plane, k complex. Boundaries of regions I and II are along the two branches of the hyperbola (dotted line). Boundaries of I and II are the real and imaginary axes.

(a)

(b)

FIGURE 3.5. Riemann sheets of γ -plane without Hankel functions, k complex: (a) Isometric view of γ -plane showing the four sheets joined by branch cuts, (b) a plan view of γ -plane showing possible transitions from a sheet to another by crossing branch cuts. The definitions of sheets are: 1. $\text{Im } \lambda < 0$ and $\text{Im } \hat{\gamma} < 0$, 2. $\text{Im } \lambda > 0$ and $\text{Im } \hat{\gamma} < 0$, 3. $\text{Im } \lambda < 0$ and $\text{Im } \hat{\gamma} > 0$, 4. $\text{Im } \lambda > 0$ and $\text{Im } \hat{\gamma} > 0$.

We now investigate the steepest descents complex plane. Steepest descent analysis is performed first by applying the usual mapping to the scattering integral (2.22):

$$\lambda = k \sin w. \quad (3.3)$$

We then take $kr \gg 1$, where r is the distance from $(d, 0, 0)$ to the observer. The complex w -plane is shown in Fig. 3.6, where we note that γ becomes an analytic function ($\gamma = k \cos w$). However, the branch cut associated with the Hankel functions does not disappear. Note that since the branch point of $\hat{\gamma}$ depends on β , the branch point becomes very close to the origin for values of $\beta/k \approx 1$. In this situation the constant phase path intersects the branch cut for most values of θ (Fig. 3.7). In general, the constant phase path does intersect the branch cut near grazing ($\theta \rightarrow 0$) for all values of β/k .

3.3 Dipole Source

In the dipole case, the w plane has no $\hat{\gamma}$ branch cut, which means that there is no difficulty with the SD path intersecting it. All other features in the w and λ planes are the same as the continuous source case.

One final note here is that the SD path shown in Fig. 3.6 corresponds to $kr \gg 1$ and $r/d \gg 1$. For $r/d > 1$ but not very large, the situation is different. We no longer have a single saddle point on the real axis every 2π interval, and the constant phase path has a different shape. The details are in the next chapter.

FIGURE 3.6. Steepest descents in w -plane, k real. Shaded areas indicate proper regions on the top Riemann sheet. The top sheet is defined by $\text{Im} \hat{\gamma} < 0$. Roman numerals indicate γ quadrants, and Arabic numerals indicate λ quadrants.

FIGURE 3.7. w -plane where the SD path deforms around the $\hat{\gamma}$ branch cut. Dotted lines show possible locations of SD path as $\theta \rightarrow 0$. Note that both start and end of deformed contour is at iw_1 ; it appears otherwise because the drawing is not to scale.

Chapter 4

ASYMPTOTIC ANALYSIS

In this chapter we will discuss in detail the asymptotic analysis of the problem. Only field integrals are considered for continuous and dipole sources. We do not consider current integrals for asymptotic analysis for the reasons we discuss in Chapter 5. We begin by considering the field integrals for the continuous current source case. We show how to develop the standard steepest descents approximation for the scattered field and then compare it with the asymptotic form of the incident field. The purpose of this comparison is to show that, whereas the incident field is localized around the line source, the scattered field extends much further. We also show the limitations of the standard steepest descents approximation pertaining to this case and introduce improvements upon it. The improvements are directed towards approximating some of the phenomena appearing in numerically integrated field that cannot be predicted by standard steepest descents techniques. In particular, we introduce three approximations: the first is a branch cut integral expansion suggested by Ishimaru [16, Section 15-8], implemented here with a slight enhancement. The second is a new expansion of the same integral developed by the author. As we shall see, the new expansion performs better than the one suggested by Ishimaru for the cases considered. The third is an improved steepest descents analysis of the

scattering integral to incorporate the proximity of the branch point. We also point out how more improvements can be introduced based on more rigorous higher order steepest descents analysis. In the last section, we present the asymptotic analysis for single and multiple dipole source cases. We also prove that the current excited on the wire for any case assumes a slowly decaying travelling wave form away from the center of the wire. This holds for both sides of the wire but with odd symmetry about the center.

4.1 Continuous Line Current Source

4.1.1 Scattered Field

In this section we shall first consider the asymptotic analysis of (2.22). We shall assume that both kr and r/d are much greater than 1. Later on we shall relax the second condition. We use the method of steepest descents as described in [15] by applying (3.3) and the usual coordinate transformation, $z = r \cos \theta$ and $[(x - d)^2 + y^2]^{1/2} = r \sin \theta$, to (2.22), viz:

$$E_z^s = -\frac{I_0 \beta k^2}{4\pi i \hat{y}} \int_{COI} \frac{\sin w \cos w H_0^{(2)}(kr \cos w \sin \theta)}{H_0^{(2)}(ka \cos w)} \frac{e^{-i d \hat{\gamma}}}{\hat{\gamma}} e^{i k r \sin w \cos \theta} dw, \quad (4.1)$$

where

$$\hat{\gamma} = [(k \cos w)^2 - \beta^2]^{1/2}, \quad (4.2)$$

r and θ are as indicated in Fig. 2.3, and COI denotes the contour of integration shown in Fig. 3.6. This contour is basically the real line in the λ -plane mapped to the w -plane. We take $kr \gg 1$ and invoke the large argument approximation for the Hankel function [18, eq. 9.2.4] in the numerator, viz:

$$E_z^s \approx \int_{COI} f(w) e^{kr g(w)} dw, \quad (4.3)$$

where

$$g(w) = i \sin(w - \theta), \quad (4.4)$$

and

$$f(w) = -e^{i\pi/4} \frac{I_0 \beta k^2 \sin w \cos w e^{-id\tilde{\gamma}}}{4\pi i \tilde{\gamma} H_0^{(2)}(ka \cos w) \left(\frac{\pi kr \sin \theta \cos w}{2} \right)^{1/2}}. \quad (4.5)$$

Note that $g(w)$ does not include any terms in d because of the assumption $r/d \gg 1$.

We now attempt to find the saddle point (SP) and constant-phase path. The SP is found by equating $g'(w)$ to zero, viz:

$$\begin{aligned} g'(w) &= i \cos(w - \theta) \\ &= i \cos(w_r - \theta + i w_i) \\ &= i [\cos(w_r - \theta) \cosh w_i - i \sin(w_r - \theta) \sinh w_i] \\ &= 0 \end{aligned}$$

where $w_r = \text{Re}(w)$ and $w_i = \text{Im}(w)$. Both real and imaginary parts of $g'(w)$ must be equal to zero at SP, and this is satisfied by $w_r = m\pi/2 + \theta$, m odd, and $w_i = 0$. We

pick $m = -1$ to make the range of w be around the origin of the w -plane. Next the constant-phase path is found by equating $\text{Im}(g(w))$ to a constant, viz:

$$\begin{aligned}
 \text{constant} &= \text{Im}(g(w)) \\
 &= \text{Im} [i \sin(w - \theta)] \\
 &= \text{Im} [i (\sin(w_r - \theta) \cosh w_i + i \cos(w_r - \theta) \sinh w_i)] \\
 &= \sin(w_r - \theta) \cosh w_i
 \end{aligned} \tag{4.6}$$

Since the coordinates of the SP (i.e., $(w_r, w_i) = (-\pi/2 + \theta, 0)$) must lie on the constant-phase path, we find that the constant in (4.6) evaluates to -1 . Hence the constant phase path is given by

$$\cosh(\text{Im } w) = -\csc(\text{Re } w - \theta). \tag{4.7}$$

Since there are two curves (a steepest descent curve and a steepest ascent curve) that satisfy (4.7), we must determine which one to use. This is achieved by testing $\text{Re}(g(w))$ and choosing the curve along which it decays as we move away from SP. This curve is shown in Fig. 3.6.

Now that both SP and SD are determined, we deform COI to the SD path defined along the valleys of (4.7). In doing this, we do not encounter any singularities for most values of θ and β . For values of $\theta \rightarrow 0$ the SD path may have to wrap around the $\hat{\gamma}$ branch cut located on the positive imaginary w -axis in order to stay on the proper Riemann sheet (see Fig. 3.7). We shall consider this subsequently. By standard SD

techniques [15, chapter 4] we evaluate $f(w)$ at SP and perform the integral of the exponential term along the SD path to yield

$$E_{z,SP}^s = -I_0 \eta_m \frac{e^{-ikr}}{2\pi r} \frac{\beta}{k} \frac{\cos \theta}{H_0^{(2)}(ka \sin \theta)} \frac{e^{-kd \left[\left(\frac{\beta}{k} \right)^2 - \sin^2 \theta \right]^{1/2}}}{\left[\left(\frac{\beta}{k} \right)^2 - \sin^2 \theta \right]^{1/2}} \quad (4.8)$$

where η_m is the intrinsic impedance of the medium (free space here). The SP contribution gives an excellent approximation of the scattered field for

$$kr > 20\pi, r/d > 10, \beta/k \geq 1.11, \text{ and } 0.1 < \theta/\pi \leq 0.5. \quad (4.9)$$

This combination of values guarantees that the observer is well in the *far zone* and away from grazing angles, and that the source current velocity is not very close to the speed of light in free space. Mathematically this means that the SD path does not wrap around the $\hat{\gamma}$ branch cut and is not near the Hankel function branch cut. Strictly speaking, in this problem there is no *far zone region*, because the scatterer has infinite length. However, if the observer is not close to grazing, the SP approximation can be made very close to the actual field if kr is large enough. We will illustrate this further in Chapter 5.

4.1.2 Incident Field

To illustrate the important difference between the scattered field and the incident field of the continuous source problem, we rewrite the incident field here for convenience:

$$E_z^i = -\frac{I_0 \beta \lambda_0}{4\hat{y}} \frac{z}{(x^2 + z^2)^{1/2}} e^{-i\beta y} H_1^{(2)} \left(\lambda_0 (x^2 + z^2)^{1/2} \right). \quad (4.10)$$

We recall from (2.9) that λ_0 is negative imaginary. By letting $\lambda_0 = -i\tilde{\lambda}_0$ where $\tilde{\lambda}_0 = \sqrt{\beta^2 - k^2}$ is positive real, we can rewrite (4.10) in terms of modified Bessel functions using [18, eq. 9.6.4(2)], viz:

$$E_z^i = \frac{I_0 \beta \lambda_0}{2\pi \hat{y}} \frac{z}{(x^2 + z^2)^{1/2}} e^{-i\beta y} K_1 \left(\tilde{\lambda}_0 (x^2 + z^2)^{1/2} \right). \quad (4.11)$$

We obtain the asymptotic form of (4.11) by applying the large argument approximation of the modified Bessel function [18, eq. 9.7.2], viz:

$$E_z^i \approx -\frac{iI_0 \beta}{\hat{y}} \left(\frac{\tilde{\lambda}_0}{8\pi} \right)^{1/2} \frac{z}{(x^2 + z^2)^{3/4}} e^{-i\beta y - \tilde{\lambda}_0 (x^2 + z^2)^{1/2}} \quad (4.12)$$

Recall that (4.10) - and hence (4.12) - were calculated in the unprimed coordinate system, whereas (4.8) was calculated in the double-primed coordinate system (see Fig. 2.3). In order to make a comparison between the asymptotic forms, the two must be - strictly speaking - calculated in the same coordinate system. However, since the asymptotic forms are valid at distances much greater than d , the error in directly comparing (4.12) and (4.8) will be insignificant. Having neglected this error, let us use spherical coordinates in (4.12), i.e., $z = r \cos \theta$, $x = r \sin \theta \cos \phi$, and $y = r \sin \theta \sin \phi$, and choose $\phi = 0$:

$$E_{z,FZ}^i = -\frac{iI_0 \beta}{\hat{y}} \left(\frac{\tilde{\lambda}_0}{8\pi r} \right)^{1/2} \cos \theta e^{-\tilde{\lambda}_0 r} \quad (4.13)$$

where FZ denotes the far-zone approximation. We divide (4.13) by (4.8) and take the magnitude of the result, viz:

$$\left| \frac{E_{z,FZ}^i}{E_{z,SP}^s} \right| = \left(\frac{\tilde{\lambda}_0 \pi r}{2} \right)^{1/2} \frac{\left[\left(\frac{\beta}{k} \right)^2 - \sin^2 \theta \right]^{1/2} \left| H_0^{(2)}(ka \sin \theta) \right| e^{-\tilde{\lambda}_0 r}}{e^{-kd \left[\left(\frac{\beta}{k} \right)^2 - \sin^2 \theta \right]^{1/2}}}$$

It is clear from the above expression that as $r \rightarrow \infty$, the ratio of the incident to scattered field goes to 0. We conclude henceforth that the incident field is heavily attenuated in comparison with the scattered field. It is this very characteristic that proves that the introduction of a scattering wire may couple energy to otherwise unaffected neighboring circuits. One more thing to note here is that the localized behavior is not a characteristic of just the z -component of the incident field, but of all components. Indeed, the three rectangular components of the incident field are given as follows:

$$\mathbf{E}^i = \frac{I_0 \lambda_0}{4\hat{y}} e^{-i\beta y} \left[\mathbf{1}_x \beta \frac{x}{(x^2 + z^2)^{1/2}} H_1^{(2)} \left(\lambda_0 (x^2 + z^2)^{1/2} \right) - \mathbf{1}_y i \lambda H_0^{(2)} \left(\lambda_0 (x^2 + z^2)^{1/2} \right) - \mathbf{1}_z \beta \frac{z}{(x^2 + z^2)^{1/2}} H_1^{(2)} \left(\lambda_0 (x^2 + z^2)^{1/2} \right) \right]$$

4.2 Special Cases of SD Approximation of Scattered Field

It is instructive to modify the standard SD approximation for parameters other than ones mentioned in (4.9).

4.2.1 $1 < \beta/k < 1.11$

In this case, the $\hat{\gamma}$ branch point becomes very close to the origin of the w -plane. For example, if $\beta/k = 1.01$, the branch point is at $w = iw_i = i \cosh^{-1}(\beta/k) \approx i0.14$ (refer to the expression for $\hat{\gamma}$ given in (4.2)). This means that for most values of θ

the SD path will wrap around a portion of the $\hat{\gamma}$ branch cut (Fig. 3.7), producing a contribution that must be added to the SP approximation. Initially the SD path meets the branch cut at $iw_1 = i \cosh^{-1}(\csc \theta)$; then to stay on the proper sheet, the SD path goes down the branch cut, deforms around the branch point $iw_2 = i \cosh^{-1}(\beta/k)$, goes up along the other side of the branch cut to iw_1 , then continues on the SD path. Along the path around branch cut: $\text{Im}(\hat{\gamma}) = 0$, and $\text{Re}(\hat{\gamma})$ is negative on the downward portion and positive on the upward. From the above information, we use (4.3) to formulate the branch cut integral, viz:

$$E_{z,BC}^s = C_{BC} \int_{w_2}^{w_1} \frac{\sinh w_i (\cosh w_i)^{1/2} \cos(d\hat{\gamma}) e^{ikr \sin(iw_i - \theta)}}{\hat{\gamma} H_0^{(2)}(ka \cosh w_i)} dw_i \quad (4.14)$$

where

$$C_{BC} = \frac{e^{i\pi/4} I_0 \beta k^2}{i\pi \hat{\gamma} (2\pi k r \sin \theta)^{1/2}},$$

and $\hat{\gamma} = [k^2 \cosh^2 w_i - \beta^2]^{1/2}$, a real positive quantity along the integration path.

Enhanced Ishimaru Expansion: Since the branch point w_2 is our main concern

here, we follow Ishimaru's method [16, Section 15-8] in expanding relevant terms in the integrand around it, while evaluating other terms without expansion¹. In

¹The approach we follow here is slightly different from that of Ishimaru. Our approach does not produce a divergent behavior as $w_2 \rightarrow w_1$ because of the way we handle the branch cut integral.

particular, the following expansions are performed

$$\cos(d\hat{\gamma}) \approx 1 - \frac{\beta}{k} (kd)^2 \chi(w_i - w_2) \quad (4.15)$$

$$\hat{\gamma} \approx k \left[2 \frac{\beta}{k} \chi(w_i - w_2) \right]^{1/2} \quad (4.16)$$

$$\sin(iw_i - \theta) \approx -\alpha_1 + i\alpha_2 (w_i - w_2), \text{ where} \quad (4.17)$$

$$\alpha_1 = \frac{\beta}{k} \sin \theta - i\chi \cos \theta,$$

$$\alpha_2 = \frac{\beta}{k} \cos \theta + i\chi \sin \theta, \text{ and}$$

$$\chi = \left[\left(\frac{\beta}{k} \right)^2 - 1 \right]^{1/2}.$$

We shall call the result *Ishimaru's approximation* (IA), viz:

$$E_{IA} = C_{IA} \int_{w_2}^{w_1} \left[1 - \frac{\beta}{k} (kd)^2 \chi(w_i - w_2) \right] \frac{e^{-k\tau\alpha_2(w_i - w_2)}}{(w_i - w_2)^{1/2}} dw_i \quad (4.18)$$

where

$$C_{IA} = C_{BC} \frac{e^{-ik\tau\alpha_1}}{kH_0^{(2)}(\beta a)} \left(\frac{\chi}{2} \right)^{1/2}.$$

If w_1 is finite (the SD path wraps around a portion of the branch cut), then (4.18) can be expressed in terms of incomplete gamma functions, which we develop into a series for computational purposes. However, as $\theta \rightarrow 0$, $w_1 \rightarrow \infty$, and the SD path wraps around the whole branch cut. In this case we can express (4.18) in terms of $\Gamma(1/2)$ and $\Gamma(3/2)$ [18, eq. 6.1.1], viz:

$$E_{IA} = H(\epsilon) C_{IA} \times \begin{cases} 2e^{-\alpha\epsilon} \epsilon^{1/2} (1 + S\Omega), & \text{if } \theta > \theta_0 \text{ (or } w_2 < w_1 < \infty) \\ \left(\frac{\pi}{\alpha} \right)^{1/2} \Omega, & \text{if } \theta < \theta_0 \text{ (or } w_1 \rightarrow \infty) \end{cases} \quad (4.19)$$

where

$$\begin{aligned}
\alpha &= kr\alpha_2, \\
\Omega &= 1 - \frac{\beta}{2\alpha k} (kd)^2 \chi, \\
S &= \sum_{m=1}^{\infty} \frac{m!}{(2m+1)!} (4\alpha\epsilon)^m, \\
\epsilon &= w_1 - w_2, \\
\frac{\pi}{2} &> \theta_0 > \sin^{-1} \left[\operatorname{sech} \left(\frac{2}{\delta} + w_2 \right) \right], \\
\delta &= 2\alpha(\Omega - 1),
\end{aligned}$$

and $H(\cdot)$ is the Heaviside unit step function, which is inserted to draw the reader's attention to the fact that the branch cut contribution is computed only when $\csc(\theta) > \beta/k$. The lower limit on θ_0 is determined by limiting the second term of the cosine expansion in (4.15) to 2. For most combinations of parameters, this limit is too stringent (almost 0). A more computationally convenient value is 0.1π . Note that $E_{IA} \rightarrow 0$ as $w_2 \rightarrow w_1$ because when the SD path wraps around a finite length of branch cut (Fig. 4.1a), we properly handle the integral as a finite integral. In [16], Ishimaru extends the path to infinity, then crosses the branch cut to the proper sheet in order to complete the SD path (Fig. 4.1b). This causes the result to diverge when $w_2 \rightarrow w_1$.

New Approximation: The approximation obtained above is only valid for small values of d/r (< 0.2), because of the series in d^2 that results from (4.15). We shall

FIGURE 4.1. Deformation of SD path around branch cut: (a) Contour wraps around a portion of the branch cut, (b) contour wraps around the whole branch cut by crossing to and from the lower Riemann sheet.

demonstrate this numerically in Chapter 5. To expand the envelope of approximation, we must then add more terms of this series, complicating the resulting integral. We may obtain a better approximation of (4.14) by not expanding the cosine function. We begin by splitting (4.14) into two integrals, viz:

$$E_{BC} = \frac{C_{BC}}{2} (E^+ + E^-) \quad (4.20)$$

where

$$E^\pm = \int_{w_2}^{w_1} \frac{\sinh w_i (\cosh w_i)^{1/2} e^{\pm id\hat{\gamma}} e^{ikr \sin(iw_i - \theta)}}{\hat{\gamma} H_0^{(2)}(ka \cosh w_i)} dw_i. \quad (4.21)$$

Now we only expand the sine function and $\hat{\gamma}$ as in (4.16), (4.17) and evaluate everything else at w_2 , viz:

$$E^\pm \approx \bar{C} \int_{w_2}^{w_1} \frac{e^{\pm \bar{\alpha}(w_i - w_2)^{1/2}} e^{-\alpha(w_i - w_2)}}{(w_i - w_2)^{1/2}} dw_i \quad (4.22)$$

where

$$\bar{C} = \frac{e^{-ikr\alpha_1}}{kH_0^{(2)}(\beta a)} \left(\frac{\chi}{2}\right)^{1/2}, \text{ and}$$

$$\bar{\alpha} = ikd \left(2\chi \frac{\beta}{k}\right)^{1/2}.$$

The integral in (4.22) can be expressed in terms of error functions [18, eq. 7.1.1], viz:

$$E^\pm \approx \bar{C} \left(\frac{\pi}{\alpha}\right)^{1/2} e^{\bar{\alpha}^2/4\alpha} \left[\operatorname{erf} \left(\frac{\mp \bar{\alpha}}{2\alpha^{1/2}} + (\alpha\epsilon)^{1/2} \right) - \operatorname{erf} \left(\frac{\mp \bar{\alpha}}{2\alpha^{1/2}} \right) \right] \quad (4.23)$$

Substituting (4.23) in (4.20) yields the required improved branch cut contribution, viz:

$$E_{BC,imp} = C_{imp} \left[\operatorname{erf} \left((\alpha\epsilon)^{1/2} - \frac{\bar{\alpha}}{2\alpha^{1/2}} \right) + \operatorname{erf} \left((\alpha\epsilon)^{1/2} + \frac{\bar{\alpha}}{2\alpha^{1/2}} \right) \right] \quad (4.24)$$

where

$$C_{imp} = H(\epsilon) \frac{-e^{i\pi/4} I_0 \beta \eta}{4\pi k H_0^{(2)}(\beta a)} \left(\frac{\chi}{\alpha_2 \sin \theta} \right)^{1/2} \frac{e^{-ik\alpha_1 r}}{r}, \text{ and} \quad (4.25)$$

$$\operatorname{erf}(z) = \frac{2}{\pi^{1/2}} \int_0^z e^{-t^2} dt. \quad (4.26)$$

This approximation has the proper behavior when $kd \gg 1$; since as $\bar{\alpha} \rightarrow \infty$, we get $E_{BC,imp} \rightarrow 0$.

Physical Aspects of Low-Frequency Limit: We next investigate what the branch cut contribution physically produces in the scattered electric field. When $kd \rightarrow 0$, $\bar{\alpha} \rightarrow 0$, and we gain more insight by examining the branch point contribution in cylindrical coordinates. Letting $\rho = r \sin \theta$ and $z = r \cos \theta$ in (4.24) and applying the definitions of α, α_1 , and α_2 , we produce

$$E_{BC,imp} \rightarrow \bar{C}_{imp} \frac{e^{-i\beta\rho}}{\rho^{1/2}} \frac{e^{-kz\chi}}{\left(\frac{\beta}{k} z + i\rho\chi \right)^{1/2}} \operatorname{erf} \left[\left\{ \epsilon k \left(\frac{\beta}{k} z + i\rho\chi \right) \right\}^{1/2} \right] \quad (4.27)$$

where

$$\bar{C}_{imp} = H(\epsilon) \frac{-e^{i\pi/4} I_0 \beta \eta}{4\pi k} \frac{\chi^{1/2}}{H_0^{(2)}(\beta a)}.$$

Thus in the low frequency limit, the branch cut contribution manifests itself as a cylindrical wave modified by phase and amplitude factors. The role of the error

function can be made clearer for some limiting cases. For grazing angles, $\theta \rightarrow 0$ and $\epsilon \rightarrow \infty$; thus $\text{erf}(\cdot) \rightarrow 1$. Here the wave propagates in ρ with a speed slower than that of light in the surrounding medium, and attenuates in $|z|$. Although the derivation here was made for positive z , the behavior in negative z is the same because of symmetry. The spatial spread is not exactly that of a pure cylindrical wave.

For broadside angles ($\theta \rightarrow \pi/2$), the argument of the error function is very small (because the steepest descents path wraps around a very small portion of the branch cut). Thus $\text{erf}(z) \rightarrow (2z/\pi^{1/2}) \exp(-z^2)$, and

$$E_{BC} \rightarrow 2\bar{C}_{imp} \left(\frac{k\epsilon}{\pi} \right)^{1/2} \frac{e^{-i\rho(\beta+k\epsilon\chi)}}{\rho^{1/2}} e^{-z(k\chi+\epsilon\beta)} \quad (4.28)$$

which is a cylindrical wave localized to the plane normal to the scatterer.

Modified Saddle-Point Contribution: The above two approximations take care of the branch cut contribution when the SD path wraps around it, but they do nothing to the SP contribution itself when the SP approaches the $\hat{\gamma}$ branch point. The SD approximation assumes that the SP has the most contribution to the scattering integral; but when the SP is close to λ_0 , this is no longer valid, especially if d/r is not very small. Both the SP location and the SD path slope change as a result of proximity of the branch point. One way to include the effect of a branch point proximity to a SP is to apply a method first suggested by Bleistein [19] and explained later for this special case by Felsen and Marcuvitz [15, Section 4.4c]. This method

accommodates the effect of *algebraic* branch points near saddle points and assumes no change in SP location or SD slope. Although in our case the branch point is more than algebraic ($\hat{\gamma}$ is also in an exponent), we shall still use the method for illustrative purposes. We begin from (4.3), deform the contour of integration to the SD path, and evaluate everything but the second exponent and algebraic branch cut at w_0 , viz:

$$E_z^s \approx h(w_0) \int_{SD} \frac{e^{ikr \sin(w-\theta)}}{\left[\cos^2 w - \left(\frac{\beta}{k}\right)^2\right]^{1/2}} dw, \quad (4.29)$$

where

$$h(w_0) = e^{i\pi/4} \frac{I_0 \beta k \cos \theta e^{-id[(k \sin \theta)^2 - \beta^2]^{1/2}}}{4\pi i \hat{\gamma} H_0^{(2)}(ka \sin \theta) \left(\frac{\pi k r}{2}\right)^{1/2}}. \quad (4.30)$$

Since the above integral is significant only around $w_0 = \theta - \pi/2$, we employ the change of variables $u = w - \theta + \pi/2$ and expand around $u = 0$, viz:

$$E_z^s \approx h(w_0) \int_{\nearrow} \frac{e^{-ikr(1-u^2/2)}}{\left[\sin^2 \theta + u \sin 2\theta - \left(\frac{\beta}{k}\right)^2\right]^{1/2}} du \quad (4.31)$$

where the slanted arrow denotes the 45° local slope of the SD path at $u = 0$. In the vicinity of $u = 0$, the real part of the square root in the denominator of (4.31) changes sign from lower limit of integration to upper limit. This necessitates splitting the integral into two parts; we also use the same reasoning of asymptotic analysis to extend each part to infinity. The above integral can be then expressed in terms of parabolic cylinder functions as follows

$$E_{MSP} = C_{MSP} \left[D_{-\frac{1}{2}} \left(b(kr)^{1/2} \right) + i D_{-\frac{1}{2}} \left(-b(kr)^{1/2} \right) \right] \quad (4.32)$$

where

$$b = \frac{\left(\frac{\beta}{k}\right)^2 - \sin^2 \theta}{\sin 2\theta} e^{i3\pi/4}, \quad (4.33)$$

$$C_{MSP} = \frac{I_0 \beta \eta e^{-i\pi/8}}{4\pi (kr)^{3/4}} (\cot \theta)^{1/2} \frac{e^{-kd \left[\left(\frac{\beta}{k}\right)^2 - \sin^2 \theta \right]^{1/2}}}{H_0^{(2)}(ka \sin \theta)} e^{-ikr(1-|b|^2/4)},$$

and MSP denotes the modified saddle point contribution. The parabolic cylinder function is given by [15, eq. 4.5.36], viz:

$$D_{-\nu}(t) = \frac{2e^{t^2/4}}{\Gamma(\nu)} \int_0^\infty p^{2\nu-1} e^{-\frac{1}{2}(p^2+t)^2} dp, \quad \text{Re}(\nu) > 0.$$

The steps needed to proceed from (4.31) to (4.32) are elaborate and can be found in Appendix A. If we employ the large argument expansion of $D_{-\nu}(t)$ [20, Section 16.52], we obtain the standard SP contribution previously derived in (4.8).

Although there is some improvement introduced by (4.32) over the standard SP approximation, there still is deviation from numerically computed fields at certain angles and for some combinations of parameters. This is due to the assumption that neither the SP location nor the SD path change when approaching the $\hat{\gamma}$ branch point.

4.2.2 $\theta \rightarrow \pi/2$

Here we show why both standard and modified SD analyses fail to give the correct field for angles approaching scatterer broadside. The deviation is observed for $\beta/k \approx 1$

and $r/d < 10$. To show this, we go back to (4.3) where we include both exponents into our attempt to find a correct SD analysis, i.e.,

$$g(w) = ikr \sin(w - \theta) - ikd \left[\cos^2 w - \left(\frac{\beta}{k} \right)^2 \right]^{1/2}. \quad (4.34)$$

To find SP, we set $\partial g / \partial w = 0$, viz:

$$\cos(w_0 - \theta) = -\frac{d}{r} \frac{\cos w_0 \sin w_0}{\left[\cos^2 w_0 - \left(\frac{\beta}{k} \right)^2 \right]^{1/2}}. \quad (4.35)$$

Note that for $d/r \rightarrow 0$ we get the result obtained before. However, if this is not the case and d is comparable to r , then (4.35) is a transcendental equation that is not easily treated analytically. We may obtain a feeling for the situation by assuming a value for θ and examining w_0 . If $\theta = \pi/2$, then (4.35) can be simplified as

$$\left[\cos^2 w_0 \left\{ 1 - \left(\frac{d}{r} \right)^2 \right\} - \left(\frac{\beta}{k} \right)^2 \right] \sin^2 w_0 = 0$$

which has the solution

$$w_0 = m\pi, m\pi + iv, m\pi - iv,$$

where $v = \cosh^{-1} \left[s \frac{\beta}{k} \left(\frac{1}{1-(d/r)^2} \right)^{1/2} \right]$, $s = \pm(-1)^m$, $m = 0, \pm 1, \pm 2, \dots$. The upper (lower) sign in s is chosen when m is even (odd). This shows that for every value of m there are *three* saddle points instead of one. For example, if $m = 0$ we have the original SP at $w_0 = 0$ and two other SP's on the imaginary axis on either side of the origin. For other values of θ , a complex-plane root-search must be employed to find the two SP's off real axis. If these are found, one can use the approach in [15,

Section 4.5b] to find the effect of three neighboring SP's. We leave this for future work.

4.2.3 $\theta < 0.1\pi$

When θ becomes smaller and approaches grazing, another behavior appears in the numerically integrated scattered field that is not predicted by standard or modified SD approximations. This behavior exists in all combinations of β, d , and r but is sometimes masked by a stronger SP contribution. The behavior is manifested by oscillations on top of the scattered field envelope; the oscillations increase in magnitude as $\theta \rightarrow 0$. We show representative plots of this in Chapter 5. We may examine what is happening by looking at Figure 3.6. As $\theta \rightarrow 0$, the SD path moves towards the left and the SP approaches $w = -\pi/2$, which is the location of the Hankel function branch point. Also, the upper part of the SD path is now wrapped around more of the $\hat{\gamma}$ branch cut. The oscillations we observe are not accommodated by the branch cut contribution; so it must be the effect of the Hankel function branch point. We expect that because the branch point is logarithmically singular; it has a nontrivial effect on both the SP location and the constant phase path. For some combinations of parameters, the SP contribution is orders of magnitudes lower than the oscillations. The analysis of the situation is further complicated if we take into account the other two SP's we discussed above. However, by numerically evaluating the integral representation of the scattered field, we account for these oscillations and

all other phenomena that cannot be predicted by asymptotic methods. Physically, these oscillations appear because the observer is close to the scattering wire but far from its center. We shall show in Chapter 5 that along the wire there is always an excitation of traveling currents that will cause these oscillations when the observer is close to the wire.

4.3 Single Dipole and Dipole Array Sources

The asymptotic behavior of the scattered field excited by a single dipole is given by

$$E_{Dz,SD}^s = \frac{\bar{I}_0 l k^2}{4\pi\omega\epsilon} \frac{y_0}{(d^2 + y_0^2)^{1/2}} \frac{e^{-ikr}}{r} \cos\theta \sin\theta \frac{H_1^{(2)} \left[k(\sin\theta) (d^2 + y_0^2)^{1/2} \right]}{H_0^{(2)}(ka \sin\theta)}. \quad (4.36)$$

For dipole excitation, the situation is simpler than the continuous source case because we do not have a branch cut intercepting the SD path. Also, the Hankel function branch point has a much weaker effect because of the quotient of two Hankel functions. An observation worth mentioning is that the SP contribution begins to have a good agreement with numerical integration for $r/(d^2 + y_0^2)^{1/2} > 100$, a distance ten times that of the continuous source case. The reason is that the current excited on the wire from a single dipole has a non-localized behavior, unlike some cases of the dipole array and continuous source cases where we can achieve localized current behavior. We shall explain this further in Chapter 5.

Finally, the current excited by a single dipole has a logarithmic asymptotic be-

havior. We show this by casting (2.25) as

$$I_D(z) = -\frac{\bar{I}_0 l}{2\pi} \frac{y_0}{(d^2 + y_0^2)^{1/2}} \frac{\partial}{\partial z} \int_{-\infty}^{\infty} \frac{H_1^{(2)} \left[\gamma (d^2 + y_0^2)^{1/2} \right]}{\gamma H_0^{(2)}(\gamma a)} e^{i\lambda z} d\lambda \quad (4.37)$$

The integral in the above equation has been shown [21], [22] to have the following asymptotic behavior as $z \rightarrow \infty$

$$\int_{-\infty}^{\infty} \frac{H_1^{(2)} \left[\gamma (d^2 + y_0^2)^{1/2} \right]}{\gamma H_0^{(2)}(\gamma a)} e^{i\lambda z} d\lambda \rightarrow \frac{2\pi}{ik (d^2 + y_0^2)^{1/2}} \frac{e^{-ikz}}{\ln \left(-\frac{2iz}{\Gamma^2 k a^2} \right)} \quad (4.38)$$

where $\Gamma = 1.781 \dots$. Substituting (4.38) in (4.37) and performing the derivative produces a second term that decays faster in z , so in the limit we are left with

$$I_D(z) \rightarrow \frac{\bar{I}_0 l y_0}{d^2 + y_0^2} \frac{e^{-ikz}}{\ln \left(-\frac{2iz}{\Gamma^2 k a^2} \right)} \quad (4.39)$$

Chapter 5

NUMERICAL RESULTS

In this chapter we explain the methods used to numerically integrate the field and current integrals and compare the results with approximate methods. We also introduce an example application that uses the results presented in this chapter to assess a coupling situation. We begin by explaining the method used to integrate the field integrals. We explain why the direct use of the fast Fourier transform does not provide an efficient computational option; special numerical integration methods must be used. Next, we compare the numerical integration results with standard and improved asymptotic approximations developed in Chapter 4. We also explain the different characteristics of the scattered field patterns based upon the mathematical singularities of the integrals as well as physical characteristics of excited currents. We then calculate the current integrals and show why the FFT fails to produce correct results. (An interesting observation is that although the analytical contribution of current singularities is zero, most of the contribution to the current comes from integrating around the singularities and not from the integration on the real axis.) We verify the accuracy of the numerically integrated current by convolving it with the Green's function to produce the numerically integrated scattered field. Finally, we present an example application that shows how the results from this research can be

used in a practical environment. In brief, in the application we compare the levels of electric field with and without the existence of a scatterer. We use the results to determine the required location of the scatterer to achieve a specified level of the total electric field.

5.1 Field Calculation

We begin by examining the integrand of (2.22). We notice that $e^{i\lambda z}$ renders the integrand oscillatory for all λ . For $|\lambda| > k$ the integrand is orders of magnitude smaller than for $|\lambda| < k$ because γ becomes imaginary and the Hankel functions become modified Bessel functions. The integrand does not increase without limit at the singularities at $\lambda = \pm k$ because of the Hankel functions in both numerator and denominator. Indeed, if we designate the integrand of (2.22) as $\Xi(\lambda)$, then

$$\begin{aligned} \lim_{\lambda \rightarrow \pm k} |\Xi(\lambda)| &= \lim_{\lambda \rightarrow \pm k} \left| \frac{\lambda H_0^{(2)} \left\{ \gamma [(x-d)^2 + y^2]^{1/2} \right\}}{H_0^{(2)}(\gamma a)} \frac{e^{-id\gamma}}{\hat{\gamma}} e^{i\lambda z} \right| \\ &= \frac{ke^{-\beta d}}{\beta} \lim_{\lambda \rightarrow \pm k} \left| \frac{\ln \left\{ \gamma [(x-d)^2 + y^2]^{1/2} \right\}}{\ln(\gamma a)} \right| \\ &= \frac{ke^{-\beta d}}{\beta} < \infty. \end{aligned}$$

However, the vicinity of these singularities introduces well-known numerical difficulties [23]. Because of rapidly changing phase especially for large r , the frequency of oscillations rises as $\lambda \rightarrow \pm k$ from both sides, which makes numerical integration difficult (see Fig. 5.1). This is the main reason why (2.22) was not a candidate for

integration using fast Fourier transform methods. The necessity to properly resolve the rapid oscillations near singularities would have made the number of samples excessively large for the FFT to produce the correct result in a reasonable time. This problem is more acute in (2.23) because the singularities at $\lambda = \pm k$ cause the integrand to peak to ∞ , in which case the FFT cannot produce the correct result no matter how many sample points we include.

The first step in performing numerical integration of (2.22) is to combine both sides of the integrand into one from 0 to infinity to reduce the computation time. Figure 5.2 shows a representative plot of the integrand of the transformed integral. Note that computation of the scattering integral is referred to the origin of the double primed system (Fig. 2.3) in order to make proper comparison with asymptotic approximations. To circumvent the problem of rapid oscillations near the singularity, we introduce the following mapping [23]:

$$\lambda = \begin{cases} k \sin \alpha, & 0 \leq \lambda \leq k \\ k \cosh \left(\alpha - \frac{\pi}{2} \right), & \lambda > k \end{cases}$$

This mapping spreads the oscillations more evenly and facilitates numerical integration (Fig. 5.3). Since the integrand is on the real axis, it can be split into real and imaginary parts where zero crossings can be found analytically. We use a five-point Simpson's rule to individually integrate each of the real and imaginary parts on the real α axis between zero crossings. The same procedure is used for the scattering integral of the single dipole case (2.26). Because of the strong attenuation of the

FIGURE 5.1. Amplitude of the integrand of the scattering integral for the continuous source case: $\beta/k = 1.1$, $d = 0.1$, $r = 10.0$, $I_0 = 1$ Ampere, $\lambda_m = 1$ meter, and $\theta = 1.0$ radians. Parameters d and r are normalized with respect to λ_m ; the operating wavelength in free space, and θ is as defined in Fig.2.3.

FIGURE 5.2. Amplitude of the integrand of the semi-infinite scattering integral for the continuous source case: $\beta/k = 1.1$, $d = 0.1$, $r = 10.0$, $I_0 = 1$ Ampere, $\lambda_m = 1$ meter, and $\theta = 1.0$ radians (Independent variable is λ).

FIGURE 5.3. Amplitude of the integrand of the semi-infinite scattering integral for the continuous source case: $\beta/k = 1.1$, $d = 0.1$, $r = 10.0$, $I_0 = 1$ Ampere, $\lambda_m = 1$ meter, and $\theta = 1.0$ radians (Independent variable is α).

integrand for $\alpha > \pi/2$ (i.e., $\lambda > k$), it suffices to cut off the numerical integration at $\lambda = 3k$. Analysis of the asymptotic behavior of the integrands from the cutoff point to infinity has verified the validity of doing so. Figure 5.4 shows plots of standard and modified SP contributions with and without adding the contribution of the branch cut (BC). Note that in the mid-range of θ the standard SP contribution follows closely the numerically integrated (NI) field, but deviates from it closer to broadside. Adding the BC to the SP contribution adds the ripples missing in the SP result in the mid-range, and fixes part of the deviation from the NI result as can be seen just before $\theta = 0.4\pi$ (Fig. 5.5). The jump in the SP value and the modified SP (MSP) value around $\theta = 0.45\pi$ is a result of the assumption that the SP location and the SD path do not change when the SP approaches the $\hat{\gamma}$ branch point. The MSP result provides a better performance than the SP one for $0.4\pi < \theta < 0.5\pi$ because it accommodates some effect of the proximity of the $\hat{\gamma}$ branch point. The approximation for the BC contribution we are using is the improved one (4.24). Although for the parameters shown in Figure 5.4 both Ishimaru's and the improved approximation give the same result, IA starts to give erroneous results for $d > 1.5$ (Fig. 5.6). As can be seen in Figures 5.7 and 5.8, all of these effects near broadside disappear when β/k and/or kr increases. Note the excellent agreement according to (4.9) between the NI and the standard SP contribution in Figure 5.8. Near grazing, the asymptotic analysis cannot predict the existence of a singularity; however, this situation improves for larger r . Getting closer to grazing causes the observer to be closer to the scatterer,

FIGURE 5.4. Magnitude of scattered field for the continuous source case: $\beta/k = 1.01$, $d = 0.5$, $r = 10.0$, $I_0 = 1$ Ampere, and $\lambda_m = 1$ meter. SP: Saddle point contribution; BC: Branch cut contribution; MSP: Modified SP contribution using parabolic cylinder functions.

FIGURE 5.5. Magnitude of scattered field for the continuous source case: $\beta/k = 1.01$, $d = 0.5$, $r = 10.0$, $I_0 = 1$ Ampere, and $\lambda_m = 1$ meter.

FIGURE 5.6. Behavior of the $\hat{\gamma}$ branch cut integral approximation with increasing d : $\beta/k = 1.01$, $r = 10.0$, $I_0 = 1$ Ampere, and $\lambda_m = 1$ meter.

FIGURE 5.7. Magnitude of scattered field for the continuous source case: $\beta/k = 1.11$, $d = 1.9$, $r = 10.0$, $I_0 = 1$ Ampere, and $\lambda_m = 1$ meter.

FIGURE 5.8. Magnitude of scattered field for the continuous source case: $\beta/k = 1.11$, $d = 1.9$, $r = 30.0$, $I_0 = 1$ Ampere, and $\lambda_m = 1$ meter.

and makes the effect of the traveling current far down the wire to be stronger than it is near the center. This can be explained in Figure 5.9, where we show the fields produced from convolving current excited on the scatterer with the Green's function (Hertzian dipole field). The details of producing the field from current can be found in Appendix B. We note that the SP contribution does not predict any oscillations in the field, and that the scattered field generated by the excited current extending over $z \in [-8, 8]$ gives better approximation to the numerically integrated (NI) field than that generated by the excited current extending only over $z \in [-4, 4]$. By this we mean that the field resulting from convolving the Hertzian dipole field with the portion of the current excited on the scatterer from $z = -8$ to $z = 8$ is a better approximation to the numerically integrated field than that produced from the current on $z \in [-4, 4]$. However, the magnitude and frequency of the field oscillations near grazing are affected only slightly by the current on the negative side of z as can be seen from the plot indicating the field produced by convolving currents extending on $z \in [0, 8]$ (long-dash double-dot plot in Fig. 5.9). This can be made clearer by visualizing the observer grazing the positive z -axis at $r = 10$. At such a position the observer is much closer to the current on the positive side of the z -axis than on the negative side. Of course at other values of θ , both positive and negative side currents are needed to produce the correct field.

FIGURE 5.9. Comparison of scattered field produced from current convolution with that obtained numerically and by SD for the continuous source case: $\beta/k = 1.11$, $d = 1.9$, $r = 10.0$, $I_0 = 1$ Ampere, and $\lambda_m = 1$ meter.

5.2 Current Calculation

In producing plots as shown in Figure 5.9, we use numerically computed currents from (2.23). In computing the current integral we did not produce asymptotic results for two reasons. First, in most of the cases, the most important part of the current is the portion around the center of the scatterer. Second, as is shown in current plots (Fig. 5.10), the current quickly takes the form of traveling currents ($e^{-ik|z|}$) that are very slowly decaying. We expect this to happen because the current excited from a single dipole has the same behavior (4.39); indeed, we have shown that superimposing dipole contributions approach that of the continuous source (see (2.27) to (2.32)). Figure 5.11 shows a representative case of the excited current. Note that if we ignore the slight decay in the magnitude of the current, the $e^{-ik|z|}$ behavior away from $z = 0$ is evident from the phase difference between the real and imaginary parts. We can also see that the current wave number is k , because the current completes a whole period every $z = 1$ wavelength. Again, the current plots show only the positive side of the scatterer because currents on the negative side are odd symmetric.

We return to the numerical integration of (2.23) to point out that the technique used here is slightly different from that used for calculating the field integrals. At $\lambda = \pm k$ the integrand behaves as a logarithmic branch point (of $H_0^{(2)}(\gamma a)$) on top of a pole (from γ^2). This combination gives an overall singular behavior weaker than that of a pole. In addition, when performing the standard semi-circular contour

FIGURE 5.10. Magnitude of current excited on scatterer for the continuous source case: $I_0 = 1$ Ampere, and $\beta/k = 1.01$ except for bottom curve, where $\beta/k = 1.11$.

FIGURE 5.11. Current excited on scatterer for $I_0 = 1$ Ampere, $\beta/k = 1.01$, and $d = 0.5$.

integration around the singularity we get a residue of zero. Indeed, (2.23) can be written as

$$I(z) = -\frac{2iI_0\beta}{\pi} \int_0^\infty \frac{\lambda e^{-i\lambda\hat{\gamma}}}{\gamma^2 \hat{\gamma} H_0^{(2)}(\gamma a)} \sin(\lambda z) d\lambda \quad (5.1)$$

$$= -\frac{2iI_0\beta}{\pi} \lim_{\delta \rightarrow 0} \left[\int_0^{k-\delta} \dots d\lambda + \int_{C_\delta} \dots d\lambda + \int_{k+\delta}^\infty \dots d\lambda \right] \quad (5.2)$$

where the first and last integrals are on the real axis and the middle integral is on a semi-circular contour C_δ centered at $\lambda = k$. For this integral we introduce the change of variables $\lambda = k + \delta e^{i\theta}$ where $-\pi \leq \theta \leq 0$. Under this change of variables and with δ being understood to be a small positive number, we have

$$\begin{aligned} \hat{\gamma} &= \left(k^2 - \beta^2 - (k + \delta e^{i\theta})^2 \right)^{1/2} \\ &\approx (-\beta^2 - 2k\delta e^{i\theta})^{1/2} \\ &= -i(\beta^2 + 2k\delta e^{i\theta})^{1/2} \\ &\approx -i \left(\beta + \frac{k\delta e^{i\theta}}{\beta} \right), \end{aligned} \quad (5.3)$$

and

$$\begin{aligned} \gamma &= \left(k^2 - (k + \delta e^{i\theta})^2 \right)^{1/2} \\ &\approx (-2k\delta e^{i\theta})^{1/2} \\ &= -i(2k\delta e^{i\theta})^{1/2}. \end{aligned} \quad (5.4)$$

We show now that taking the limit as $\delta \rightarrow 0$ produces zero for the semi-circular

contour integral, viz:

$$\begin{aligned}
\lim_{\delta \rightarrow 0} \left| \int_{C_\delta} \dots d\lambda \right| &= \lim_{\delta \rightarrow 0} \left| \int_{-\pi}^0 \frac{(k + \delta e^{i\theta}) e^{-d(\beta + \frac{k\delta e^{i\theta}}{\beta})} \sin [z (k + \delta e^{i\theta})]}{-2ik\delta e^{i\theta} \left(\beta + \frac{k\delta e^{i\theta}}{\beta} \right) H_0^{(2)} [-ia (2k\delta e^{i\theta})^{1/2}]} i\delta e^{i\theta} d\theta \right| \\
&\leq \frac{e^{-d\beta}}{2k} \lim_{\delta \rightarrow 0} \int_{-\pi}^0 e^{-(dk\delta \cos \theta)/\beta} \left| \frac{(k + \delta e^{i\theta}) \sin [z (k + \delta e^{i\theta})]}{\left(\beta + \frac{k\delta e^{i\theta}}{\beta} \right) H_0^{(2)} [-ia (2k\delta e^{i\theta})^{1/2}]} \right| d\theta \\
&\leq \frac{\pi e^{-d\beta}}{k} \lim_{\delta \rightarrow 0} \int_{-\pi}^0 e^{-(dk\delta \cos \theta)/\beta} \left| \frac{(k + \delta e^{i\theta}) \sin [z (k + \delta e^{i\theta})]}{\left(\beta + \frac{k\delta e^{i\theta}}{\beta} \right) \ln [a (2k\delta e^{i\theta})^{1/2}]} \right| d\theta. \\
&\leq \frac{\pi^2 e^{-d\beta}}{k} \lim_{\delta \rightarrow 0} M(\delta, \theta_0)
\end{aligned} \tag{5.5}$$

where $M(\delta, \theta_0)$ is an upper bound for the above integrand and $\theta_0 \in [-\pi, 0]$. We now attempt to manipulate $M(\delta, \theta_0)$, which can be given by:

$$M(\delta, \theta_0) = e^{-(dk\delta \cos \theta_0)/\beta} \left| \frac{(k + \delta e^{i\theta_0}) \sin [z (k + \delta e^{i\theta_0})]}{\left(\beta + \frac{k\delta e^{i\theta_0}}{\beta} \right) \ln [a (2k\delta e^{i\theta_0})^{1/2}]} \right|. \tag{5.6}$$

In the numerator of the above expression, we have the following:

$$e^{-(dk\delta \cos \theta_0)/\beta} \leq e^{dk\delta/\beta} \tag{5.7}$$

$$|k + \delta e^{i\theta_0}| \leq k + \delta \tag{5.8}$$

$$\begin{aligned}
|\sin [z (k + \delta e^{i\theta_0})]| &= \frac{1}{2} \left| e^{iz(k + \delta e^{i\theta_0})} + e^{-iz(k + \delta e^{i\theta_0}) + i\pi} \right| \\
&\leq \frac{1}{2} \left[\left| e^{iz(k + \delta e^{i\theta_0})} \right| + \left| e^{-iz(k + \delta e^{i\theta_0})} \right| \right] \\
&= \frac{1}{2} [e^{-z\delta \sin \theta_0} + e^{z\delta \sin \theta_0}] \\
&= \cosh (z\delta \sin \theta_0) \\
&\leq \cosh (z\delta).
\end{aligned} \tag{5.9}$$

In the denominator of (5.6), we have the following:

$$\left| \beta + \frac{k\delta e^{i\theta_0}}{\beta} \right| \geq \beta - \frac{k\delta}{\beta} \quad (5.10)$$

$$\begin{aligned} \left| \ln \left[a (2k\delta e^{i\theta_0})^{1/2} \right] \right| &= \left| \ln a + \frac{1}{2} (\ln 2 + \ln k + \ln \delta + i\theta_0) \right| \\ &= \left[\left(\ln a + \frac{\ln 2 + \ln k + \ln \delta}{2} \right)^2 + \frac{\theta_0^2}{4} \right]^{1/2} \\ &> \frac{\ln \delta}{2}. \end{aligned} \quad (5.11)$$

Substituting from (5.7)-(5.11) into (5.6) yields:

$$M(\delta, \theta_0) < \frac{2e^{dk\delta/\beta} (k + \delta) \cosh(z\delta)}{\left(\beta - \frac{k\delta}{\beta} \right) \ln \delta}, \quad (5.12)$$

which means that the limit in (5.5) evaluates to zero. Therefore, in order to account for the contribution of the singularity, we have to perform the integration numerically in the complex plane on a contour that deforms around it.

To do so, we use a three-segment contour around $\lambda = \pm k$ (see C_I in Fig. 3.3). The figure shows the contour deforming around both singularities. However, we use symmetry to transform the integral into a semi-infinite one. This means that we only integrate around $\lambda = +k$. The first segment of C_I starts from the origin at a 45° slope up to $\lambda = k + i\lambda_i$, where

$$\lambda_i = \begin{cases} 1, & |z| < 1 \\ 1/z, & |z| \geq 1 \end{cases}.$$

This is necessary to avoid growing exponentials in the integrand of (2.23) when $|z| \geq 1$.

The second segment extends horizontally from $\lambda = k + i\lambda_i$ to $\lambda = \lambda_1 + i\lambda_i$, where λ_1

is the location of the first zero crossing of the integrand¹. The third segment goes vertically down from $\lambda = \lambda_1 + i\lambda_i$ to $\lambda = \lambda_1$. On these segments the integrand is computed numerically, split into real and imaginary parts, then integrated using the quadrature routine DQAGS from the public domain SLATEC package [24]. For the rest of the real line integral ($\lambda > \lambda_1$) we use a combination of integration between zero crossings and asymptotic evaluation. By using asymptotic evaluation we save computation time. Details of evaluating the asymptotic integrand can be found in Appendix C. In order to verify the results obtained from the singularity contribution, we use another contour to go around it. The contour used here is a sine-shaped contour (see C_{II} in Fig. 3.3) described parametrically as $\lambda = \sigma + i\lambda_i \sin(\pi\sigma/\lambda_1)$, $\sigma \in [0, \lambda_1]$, where λ_1 and λ_i are as defined above. The reason for using such a contour instead of say, a semi-circle, is that we still can transform the integral into a semi-infinite one. Both contours produce the same result, but the three-segment contour is computationally faster by a factor of 1.5. Convolution of the resulting current with the field from a single dipole (Green's function) to produce the numerically integrated field is another means to verify the accuracy of the computed current.

In Figure 5.10 we show representative plots for the current on the scatterer for different distances from the source. Note that the current quickly takes the traveling form with the localized behavior becoming more apparent for $d/\lambda_m > 1$, and that increasing β/k accentuates the localization. By localized we mean that current far

¹The transformation to a semi-infinite integral results in a purely imaginary integrand for $\lambda > k$, which means that the only zero crossings are for this imaginary part.

from the center of the scatterer is one or more orders of magnitude smaller than the current around the center. It is in these cases that the oscillations appearing near grazing cannot be predicted by SD analysis, because the SP contribution is manifested physically from the current “bulge” around the center of the scatterer. Since at grazing the observer is closer to the traveling current on the scatterer than the “bulge”, the field produced from traveling currents is more dominant than that produced from the central bulge.

In Figure 5.12 we show representative plots for the excited scatterer current when the source is a single dipole. We note that as we move along the scatterer and away from the center the excited current exhibits a traveling-wave nature as predicted by (4.39). In Figure 5.13 we show that superimposing contributions from dipoles in a linear array approaches that of a continuous source. In order to show the converging behavior, it was very important to keep the distance between dipoles less or equal $0.01\lambda_m$.

5.3 An Example Application

Here we present an example application from the results of this research. We begin by showing a relief plot of the magnitude of the incident field from the continuous source for $\beta/k = 1.01$ (2.20) taken at the $y = 0$ plane (Fig. 5.14). If this were a three dimensional plot of xyz coordinates, we would see the current source running

FIGURE 5.12. Magnitude of the current excited on scatterer by a single dipole source: $I_0 = 1$ Ampere, $\beta/k = 1.11$ and $d = 0.01$.

FIGURE 5.13. Magnitude of scattered electric field (SD approximation) for the cases of continuous and dipole array sources: $\beta/k = 1.11$, $d = 0.01$, $r = 50.0$, $I_0 = 1$ Ampere, $\lambda_m = 1$ meter, and $\Delta y_0 = 0.01\lambda_m$.

FIGURE 5.14. Magnitude of the incident electric field for the continuous source case measured at the $y = 0$ plane, $I_0 = 1$ Ampere, and $\lambda_m = 1$ meter.

vertically through the origin between the two peaks. We find two peaks because the incident field has a null at $z = 0$. Note that for a spherical coordinate system with the origin at $(x, y, z) = (0, 0, 0)$, the incident field has no symmetry with respect to any coordinate of $(r, \phi, \theta)^2$. Note also that all the previously shown plots of the scattered field were taken at a shifted coordinate system centered at $(x, y, z) = (d, 0, 0)$. This means that although the scattered field is ϕ -independent in the shifted system, it is not in the original system. Our goal is to show the scattering effects in the original coordinate system of the incident field.

Figure 5.15 contains plots of the incident field with $\beta/k = 1.01$ and at four different values of ϕ : $\phi = 0^\circ, 30^\circ, 60^\circ$, and 90° . The reason for choosing these values is that they also represent the field at other values. For example, the incident field has the same magnitude for the following groups of ϕ : $(0^\circ, 180^\circ)$, $(90^\circ, 270^\circ)$, $(30^\circ, 150^\circ, 210^\circ, 330^\circ)$, and $(60^\circ, 120^\circ, 240^\circ, 300^\circ)$. The upper and lower halves of the vertical axis of the figure represent the $+z$ axis, while the horizontal axis represents the xy plane. Therefore θ can be read the usual way, from the $+z$ axis down to the xy plane. The numbers $(0, 18, 36, 54, 72, \text{ and } 90)$ are values of θ in degrees. We do not take θ further because the fields for $\theta \in [90^\circ, 180^\circ]$ are a mirror image of those for $\theta \in [0^\circ, 90^\circ]$. On each quadrant, there are four plots of the incident field computed at different values of r measured in wavelengths. The magnitude of the incident field is shown in dB referred to 1 volt/meter (dBV). The concentric dotted circles represent different

²It does have partial symmetry in θ . The field in $\theta \in [0, \pi/2]$ is a mirror image to that in $\theta \in [\pi/2, \pi]$ but opposite in sign.

FIGURE 5.15. An example chart used to determine the minimum values of d for placing the scattering wire at selected angles if the total E_z field is not to exceed -40 dBV: $\beta/k = 1.01$, $I_0 = 1$ Ampere, and $\lambda_m = 1$ meter.

levels of dBV from -20 to -120 , while the boundary of the chart is at 0 dBV. We can immediately see the strong attenuation of the incident field. For example, at $\phi = 0^\circ$, increasing r by five wavelengths produces a magnitude dip of 40 dBV. Now suppose that we have a source that can be modeled by our continuous current source in free space. Suppose further that we need to place a perpendicular line conductor somewhere near the source such that at any given distance from $(x, y, z) = (0, 0, 0)$ we should have no more than -40 dBV total field. We orient the coordinate system so that the source is aligned on the y -axis, and the scatterer on the line described by $(x, y) = (d, 0)$. Assume that we are interested in measuring the level of the electric field at $\phi = 0$. We examine the first quadrant of the figure. We discard measuring the field at $r = 5$, because without the scatterer the incident field is more than -20 dBV. At $r = 10$, however, the incident field is below -40 dBV. Note that at this value of ϕ , the scatterer is between the observer and the line current source. The numbers written on the $r = 10$ curve indicate the *minimum* distance (d values) at which a scatterer can be placed without producing a total field exceeding -40 dBV at the indicated θ sector. For example, if we are interested in levels of the total field for the sector $\theta \in [0^\circ, 36^\circ]$, then we may place the scatterer at $d \geq 1$. However, if we are interested in the sector $\theta \in [0^\circ, 54^\circ]$ then we must have the scatterer placed at $d \geq 2$. The rest of the figure can be read in the same manner. The most important thing to remember is that although the observer's position changes for different (r, ϕ, θ) , the position of the scatterer is always on the positive x -axis, d wavelengths from the

origin. Sometimes at a certain distance only part of the curve meets our criterion of being below -40 dBV, in which case we limit our attention to only that part. For example, at $\phi = 60^\circ$ and at $r = 10$ we cannot look beyond $\theta = 18^\circ$ because the incident field itself exceeds -40 dBV there. The figure serves as a concise and convenient means to examine many parameters of the incident and scattered fields simultaneously without the need to consult many plots.

Chapter 6

CONCLUSIONS AND RECOMMENDATIONS FOR FUTURE WORK

In this dissertation we have presented a full-wave analysis of the problem of scattering from an infinite thin wire excited by a perpendicular traveling current line source. We have presented results from standard and modified asymptotic analyses as well as numerical integration of the electric field and current integrals. We have shown that although the geometry and structure are simple, the characteristics of the fields and currents are subtle. In particular, we have shown that although standard steepest descents techniques give excellent agreement with numerically computed fields when both are measured according to some combinations of parameters, this is not the case for other combinations. We have shown that by careful consideration of some features in the complex planes, we may greatly improve upon standard approximate methods. We have introduced an enhancements on an approximations suggested in the literature, developed a new one, and applied improved SD techniques.

Similar to verifying the asymptotic expressions by comparing them to numerically integrated fields, we have verified the accuracy of numerically integrated currents by using them to produce scattered fields through convolution. We have also presented a useful chart that can be used in place of many plots to determine the level of the fields with and without the presence of the scatterer.

As future work, we recommend a rigorous analysis of the steepest descents problem for values of d comparable to r . As we have shown, this produces multiple saddle points instead of just one. The next logical step is to add another scatterer parallel to the first and on the other side of the line source. This structure will - in general - excite guided modes. We also recommend extending to an array of scatterers on both sides, in which case Floquet's method might be used. Another extension might be to add dielectric cladding to the single or multiple scatterer cases, where guided modes will exist, even for the single scatterer. For all of the above cases, we recommend incorporating coupling effects between the source and the scatterer(s) to accurately determine the value of β . Finally, allowing β to be smaller than k will introduce totally different behaviors of both the incident and scattered fields. In such a case, the incident field will no longer be localized to the source and the excited currents may not have the simple travelling wave behavior.

Appendix A

DERIVATION OF THE MODIFIED SADDLE-POINT CONTRIBUTION

Here we detail the derivation of (4.32). We begin by rewriting (4.31) for convenience, viz:

$$E_z^s \approx h(w_0) \int_{\gamma} \frac{e^{-ikr(1-u^2/2)}}{\left[\sin^2 \theta + u \sin 2\theta - \left(\frac{\rho}{k}\right)^2\right]^{1/2}} du \quad (\text{A.1})$$

where $h(w_0)$ is given by (4.30). In this development, we keep track of the signs for the real and imaginary parts of the square root in the integrand. Since the above equation was obtained from (4.3) by expanding the relevant quantities around the SP and evaluating everything else at it, there has been no change in the definition of the square root. Hence the above square root (originally $\hat{\gamma}$) retains the same definition, i.e., a negative imaginary part. Regarding the real part, it changes sign around $u = 0$ (Fig. A.1). We did not encounter this problem in the standard SD analysis because $\hat{\gamma}$ is evaluated at the SP and removed outside the integral. We handle this situation by splitting the integral around $u = 0$, viz:

$$E_z^s \approx h(w_0) \frac{e^{-ikr}}{(\sin 2\theta)^{1/2}} (I_1 + I_2), \quad (\text{A.2})$$

FIGURE A.1. Change of sign of $\text{Re}(\hat{\gamma})$ along SD path

where

$$I_1 = \int_{\epsilon e^{-i3\pi/4}}^0 \frac{e^{ikru^2/2}}{\left[u - \frac{(\beta/k)^2 - \sin^2 \theta}{\sin 2\theta} \right]^{1/2}} du, \quad (\text{A.3})$$

and

$$I_2 = \int_0^{\epsilon e^{i\pi/4}} \frac{e^{ikru^2/2}}{\left[u - \frac{(\beta/k)^2 - \sin^2 \theta}{\sin 2\theta} \right]^{1/2}} du, \quad (\text{A.4})$$

where ϵ is a small positive number not to be confused with the permittivity. While both integrals have a negative imaginary part, the square root in I_1 has a positive real part, while that in I_2 has a negative real part. It is instructive to process both integrals together. Let us introduce the change of variables $u = \rho e^{-i3\pi/4}$ in I_1 and $u = \rho e^{i\pi/4}$ in I_2 . We also make the usual argument that the integrals are only significant around $\rho = 0$ and hence we can extend the nonzero limit to infinity without introducing much error, viz:

$$I_1 = e^{-i3\pi/4} \int_{\infty}^0 \frac{e^{-k\rho^2/2}}{\left[\rho e^{-i3\pi/4} - \frac{(\beta/k)^2 - \sin^2 \theta}{\sin 2\theta} \right]^{1/2}} d\rho, \quad (\text{A.5})$$

and

$$I_2 = e^{i\pi/4} \int_0^{\infty} \frac{e^{-k\rho^2/2}}{\left[\rho e^{i\pi/4} - \frac{(\beta/k)^2 - \sin^2 \theta}{\sin 2\theta} \right]^{1/2}} d\rho. \quad (\text{A.6})$$

Before we proceed, we draw attention to the fact that the arguments of the square roots above are mostly controlled by the first terms under the square roots. This is because, for most combinations of the parameters, the second terms are much less in magnitude than the first. For example, the argument of the square root in I_1 is very

close to $-3\pi/8$, which produces a negative imaginary part and a real positive part. A very important observation is that the argument is actually slightly $< -3\pi/8$; that is because of subtracting the second positive term under the square root. In I_2 , the situation is more subtle. From the argument of the first term, we are tempted to say that the argument of the square root is $\pi/8$, whereas the right answer is $\pi/8 \pm \pi$, thus producing negative real and imaginary parts. We choose the lower sign to make the argument $-7\pi/8$. By the same previous argument, we state that the argument of the square root is actually slightly $> -7\pi/8$. We now proceed by factoring out $e^{-i3\pi/4}$ from both terms under the square root in I_1 , viz:

$$I_1 = e^{-i3\pi/8} \int_{-\infty}^0 \frac{e^{-k\tau\rho^2/2}}{\left[\rho - \frac{(\beta/k)^2 - \sin^2\theta}{\sin 2\theta} e^{i3\pi/4}\right]^{1/2}} d\rho \quad (\text{A.7})$$

$$= e^{i5\pi/8} \int_0^{\infty} \frac{e^{-k\tau\rho^2/2}}{\left[\rho - \frac{(\beta/k)^2 - \sin^2\theta}{\sin 2\theta} e^{i3\pi/4}\right]^{1/2}} d\rho. \quad (\text{A.8})$$

Extracting the exponential is equivalent to rotating the square root by $3\pi/8$ counter-clockwise. But since the argument of the square root was less than $-3\pi/8$, the rotation did not change the signs of the real and imaginary parts. Likewise, extracting $e^{i\pi/4}$ from the square root in I_2 does not change the signs of the real and imaginary parts of the associated square root, viz:

$$I_2 = e^{i\pi/8} \int_0^{\infty} \frac{e^{-k\tau\rho^2/2}}{\left[\rho - \frac{(\beta/k)^2 - \sin^2\theta}{\sin 2\theta} e^{-i\pi/4}\right]^{1/2}} d\rho. \quad (\text{A.9})$$

From (A.2), (A.8), and (A.9) we get

$$E_z^s \approx -2h(w_0) \frac{e^{-i(kr-\pi/8)}}{(\sin 2\theta)^{1/2}} \frac{\partial}{\partial b} \left[\int_0^\infty (\rho+b)^{1/2} e^{-k\tau\rho^2/2} d\rho + i \int_0^\infty (\rho-b)^{1/2} e^{-k\tau\rho^2/2} d\rho \right] \quad (\text{A.10})$$

where

$$b = \frac{(\beta/k)^2 - \sin^2 \theta}{\sin 2\theta} e^{i3\pi/4} \quad (\text{A.11})$$

So far, the integrals in (A.10) are on the real line. However, to put the integrals in the form of parabolic cylinder functions, we must make the change of variable $(\rho-b)^{1/2} = u$ in the first integral, and $(\rho+b)^{1/2} = u$ in the second. This maps the contour of integration to be on a part of a hyperbola. Figure A.2 shows the relevant contours of integration. Note that there are two possible contours for each integral. It does not matter which one is chosen so long as we are consistent with the definitions of square roots. Our choice is shown in the figure as Γ_1 and Γ_2 . Therefore, (A.10) becomes

$$E_z^s \approx -4h(w_0) \frac{e^{-i(kr-\pi/8)}}{(\sin 2\theta)^{1/2}} \frac{\partial}{\partial b} \left[\int_{\Gamma_1} u^2 e^{-k\tau(u^2-b)^2/2} du + i \int_{\Gamma_2} u^2 e^{-k\tau(u^2+b)^2/2} du \right] \quad (\text{A.12})$$

Since there are no singularities in the u -plane, we can deform Γ_1 to Λ_1 and the positive real line, and Γ_2 to Λ_2 and the positive real line (Fig. A.2), viz:

$$\begin{aligned} E_z^s \approx & -4h(w_0) \frac{e^{-i(kr-\pi/8)}}{(\sin 2\theta)^{1/2}} \\ & \times \frac{\partial}{\partial b} \left[\int_0^\infty u^2 e^{-k\tau(u^2-b)^2/2} du \right. \\ & \left. + i \int_0^\infty u^2 e^{-k\tau(u^2+b)^2/2} du + I_{\Lambda_1} + i I_{\Lambda_2} \right]. \end{aligned} \quad (\text{A.13})$$

FIGURE A.2. Contours of integration for integrals involved in the derivation of the modified saddle point contribution.

The upper limit on I_{Λ_1} is $b^{1/2} = \sigma e^{i3\pi/8}$ and the lower limit is 0, whereas the upper limit on I_{Λ_2} is $(-b)^{1/2} = \sigma e^{-i\pi/8}$ and the lower limit is zero. If we make the change of variables $u = \xi e^{i3\pi/8}$ in I_{Λ_1} and $u = \xi e^{-i\pi/8}$ in I_{Λ_2} , $I_{\Lambda_1} + iI_{\Lambda_2}$ becomes

$$\begin{aligned}
 I_{\Lambda_1} + iI_{\Lambda_2} &= e^{i9\pi/8} \int_{\sigma}^0 \xi^2 \exp \left[-\frac{k\tau (\xi^2 e^{i3\pi/4} - \sigma^2 e^{i3\pi/4})^2}{2} \right] d\xi \\
 &\quad + ie^{-i3\pi/8} \int_{\sigma}^0 \xi^2 \exp \left[-\frac{k\tau (\xi^2 e^{-i\pi/4} - \sigma^2 e^{-i\pi/4})^2}{2} \right] d\xi \\
 &= e^{i9\pi/8} \int_{\sigma}^0 \xi^2 \exp \left[\frac{ik\tau (\xi^2 - \sigma^2)^2}{2} \right] d\xi \\
 &\quad + e^{i\pi/8} \int_{\sigma}^0 \xi^2 \exp \left[\frac{ik\tau (\xi^2 - \sigma^2)^2}{2} \right] d\xi \\
 &= 0.
 \end{aligned}$$

Hence we end up with (A.13) being

$$E_z^s \approx -4h(w_0) \frac{e^{-i(kr-\pi/8)}}{(\sin 2\theta)^{1/2}} \frac{\partial}{\partial b} \left[\int_0^\infty u^2 e^{-k\tau(u^2-b)^2/2} du + i \int_0^\infty u^2 e^{-k\tau(u^2+b)^2/2} du \right]. \quad (\text{A.14})$$

which is cast in the form of parabolic cylinder functions [15, eq. 4.5.36]. In particular

$$D_{-\frac{3}{2}}(\pm t) = \frac{2e^{t^2/4}}{\Gamma(\frac{3}{2})} \int_0^\infty p^2 e^{(p^2 \pm t)^2/2} dp.$$

Therefore, (A.14) can be expressed as

$$E_{MSP} = -\pi^{1/2} h(w_0) \frac{e^{-i(kr-\pi/8)}}{(\sin 2\theta)^{1/2}} (k\tau)^{-3/4} \quad (\text{A.15})$$

$$\times \frac{\partial}{\partial b} \left\{ e^{-k\tau b^2/4} \left[D_{-\frac{3}{2}}(-b(k\tau)^{1/2}) + i D_{-\frac{3}{2}}(b(k\tau)^{1/2}) \right] \right\}. \quad (\text{A.16})$$

Performing the derivative on b and employing the recurrence relations in [20, section 16.61] produces (4.32).

Appendix B

PRODUCING THE SCATTERED FIELD FROM THE EXCITED CURRENT

Here we show how to produce the scattered electric field from currents excited on the wire. First, we cast the scattered field as a convolution integral. Using operator notation, (2.22) can be written as

$$E_z^s = \mathcal{F}^{-1} [I(\lambda) G(\lambda)]$$

where

$$\begin{aligned} I(\lambda) &= 2I_0\beta \frac{\lambda e^{-id\hat{\gamma}}}{\gamma^2 \hat{\gamma} H_0^{(2)}(\gamma a)}, \\ G(\lambda) &= -\frac{1}{4i\hat{y}} \gamma^2 H_0^{(2)} \left\{ \gamma [(x-d)^2 + y^2]^{1/2} \right\}, \text{ and} \\ \mathcal{F}^{-1}[\cdot] &= \frac{1}{2\pi} \int_{-\infty}^{\infty} (\cdot) e^{i\lambda z} d\lambda. \end{aligned}$$

This means that

$$\begin{aligned} E_z^s &= I(z) \otimes G(z) \\ &= \int_{-\infty}^{\infty} I(\xi) G(z - \xi) d\xi \end{aligned}$$

where $I(z)$ and $G(z)$ are the inverse Fourier transforms of $I(\lambda)$ and $G(\lambda)$ respectively. A careful examination of $G(z)$ reveals that it is the field generated by a Hertzian dipole (see, for example, [13, eq. 2-113]). To arrive at the familiar form for the radiation

of a Hertzian dipole that involves exponentials, we use a well-known Hankel function identity [14, eq. 4.306]. Now that we have established the field as a convolution, we have to use numerical techniques since $I(z)$ is only available as data around the center of the scatterer. We use a very simple algorithm that represents the following formula

$$E_z^s(x, y, z) = \Delta_{z'} \sum_{j=1}^N I(z'_j) G(x, y, z - z'_j), \text{ where}$$

$$\Delta_{z'} = z'_2 - z'_1,$$

and N is the number of data points in $I(z)$. Note that in the above formula, each set of N points of G has the correct (x, y, z) dependence.

Appendix C

ASYMPTOTIC EXPRESSION OF THE CURRENT INTEGRAL

Here we show how to develop the expressions for the current integral for large $|\lambda|$. Referring to (2.23), we use symmetry and the fact that $\hat{\gamma}$ is pure imaginary to reduce the domain of integration, viz:

$$I(z) = -\frac{2I_0\beta}{\pi} \int_0^\infty \frac{\lambda e^{-d\tilde{\gamma}}}{\gamma^2 \tilde{\gamma} H_0^{(2)}(\gamma a)} \sin(\lambda z) d\lambda,$$

where $\tilde{\gamma} = i\hat{\gamma}$ is real. We now focus our attention on evaluating the integral for $|\lambda| \gg k$ where γ becomes imaginary. We split the above integral into two parts, one evaluated numerically from 0 to λ_c and the other evaluated asymptotically from λ_c to ∞ , where λ_c is a cutoff value after which asymptotic approximations are valid. In this region we assume that $\tilde{\gamma} \rightarrow \lambda$, $\gamma \rightarrow -i\lambda$, and $H_0^{(2)}(\gamma a) \rightarrow (2i/\pi)K_0(\lambda a) \rightarrow (2i/\pi) \exp(-\lambda a)/(2\lambda a/\pi)^{1/2}$. Hence the asymptotic integral becomes

$$I_c(z) = i \left(\frac{\pi a}{2} \right)^{1/2} \text{Im} \left(\int_{\lambda_c}^\infty \lambda^{-3/2} e^{-\lambda(d-a)} e^{i\lambda z} d\lambda \right). \quad (\text{C.1})$$

Let us denote the integral inside the parentheses above by $\bar{I}_c$ and let us perform the change of variables $t = \lambda/\lambda_c$, viz:

$$\bar{I}_c = \lambda_c^{-1/2} \int_1^\infty t^{-3/2} e^{-\zeta t} dt, \text{ where}$$

$$\zeta = \lambda_c (d - a - iz).$$

The above integral can be expressed in terms of the confluent hypergeometric function defined in [18, eq. 13.2.6] as follows

$$U(a, b; \zeta) = \frac{1}{\Gamma(a)} e^{\zeta} \int_1^{\infty} e^{-\zeta t} (t-1)^{a-1} t^{b-a-1} dt$$

where $\text{Re}(a) > 0$, $\text{Re}(\zeta) > 0$. Thus $\bar{I}_c$ becomes

$$\bar{I}_c = \lambda_c^{-1/2} e^{-\zeta} U(1, \frac{1}{2}; \zeta).$$

We have developed an efficient algorithm to compute U in much shorter time compared to numerical integration.

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