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NONLINEAR OSCILLATIONS AND STABILITY OF A NUCLEAR
REACTOR WITH TWO REACTIVITY FEEDBACKS

by

Theodore Reinold Schmidt Jr.

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1969

THE UNIVERSITY OF ARIZONA

GRADUATE COLLEGE

I hereby recommend that this dissertation prepared under my
direction by THEODORE REINOLD SCHMIDT, JR.
entitled NONLINEAR OSCILLATIONS AND STABILITY OF A NUCLEAR
REACTOR WITH TWO REACTIVITY FEEDBACKS.
be accepted as fulfilling the dissertation requirement of the
degree of DOCTOR OF PHILOSOPHY

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To Janice

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ABSTRACT

The autonomous nonlinear differential equations describing the dynamic behavior of nuclear reactors with two reactivity feedbacks are analyzed for stability and for the possibility of nonlinear oscillatory solutions. Fourier series expansions of the reactivity and power are used to search for limit cycles, successive approximations being generated by starting from the linear results. In the earlier work using this method, the second approximation for the amplitude of the reactivity was derived assuming that the feedback contained the fundamental frequency only. In this study, higher approximations including feedback from higher harmonic frequencies are derived for not only the amplitude of the reactivity but also the amplitude of the power.

The effective lifetime model and the prompt jump approximation with one group of delayed neutrons are the two models used for the neutron kinetics. This permits an analysis of the effect of delayed neutrons on oscillations since both models are the same for small amplitude low-frequency oscillations. The most general linear heat transfer equations for two regions are described and the linear transformation for decoupling the equations is derived. A special set of two-region heat transfer equations is discussed which describes the heat transfer in a nuclear reactor with in-core thermionic converters.

Linear stability is examined by applying the Routh criterion and plotting stability boundaries in a parameter space of reactivity

coefficients. Nonlinear stability is discussed in terms of Welton's criterion, boundedness, finite escape times, and limit cycles.

The results obtained with the Fourier series method are compared with exact solutions obtained from analog and digital computations. Accurate prediction of limit cycles by Fourier series is found when the operating power is near the critical power for linear stability, but only in the highest approximation used.

The system of equations with effective lifetime kinetics yields stable limit cycles only, while the system of equations with the prompt jump approximation yields both stable and unstable limit cycles. Also, solution trajectories between the shutdown and operating equilibrium points exhibit some differences with the two kinetics models.

The thermionic reactor system is restricted to physically meaningful parameters and then analyzed for stability. The exact solution indicates that the thermionic reactor system may be very stable.

CHAPTER I

INTRODUCTION

Purpose

The purpose of this study is to examine the autonomous nonlinear differential equations, which describe the dynamic behavior of nuclear reactors with two reactivity feedbacks, for stability and oscillatory solutions. This study finds application in the area of nuclear reactor operating safety. Historically, numerous nuclear power reactors have been built and operated safely for long periods of time under restrictive design and operating limits based on stringent safeguards. However, there is current interest, for economic and performance reasons, in the design of reactors that are safe but for which some of the highly restrictive safeguards may be relaxed.

For example, in present day power reactors, it is advantageous to increase the specific power levels so that the reactor core is smaller, and hence, the containment vessel and other structures are reduced in size. It becomes necessary to know the effects, if any, of increasing the operating power of a reactor and the resulting margin of safety.

Also, fast spectrum power reactors are being designed for construction early in the next decade. The possibility exists that these reactors will present more complex safety problems than the present thermal reactors. This does not mean that they can not be made and operated safely.

All of the temperature coefficients of highly enriched fast spectrum cores may not be negative. The effects of positive temperature feedbacks in the fuel or coolant need to be examined.

Usually, in reactor design the only type of stability permitted is asymptotic stability. This may be too restrictive in cases where there is sufficient time to arrest a slow excursion. It may not be restrictive enough in cases where a perturbation produces a large excursion before the disturbance dies out. Furthermore, a reactor can be designed to operate with stable oscillations if the oscillation amplitudes are sufficiently small. Also, a reactor with asymptotic stability may become unstable from sufficiently large disturbances.

In this study, the stability analysis is applied to a nuclear reactor with in-core thermionic converters which is an example of a fast spectrum reactor with possible positive temperature feedback in the fuel.

Background

The background includes an examination of the methods for analysis of oscillatory solutions or limit cycles, and a review of the early work in the Fourier series-transfer function method. In addition, the literature is surveyed for analysis of reactor systems with two reactivity feedbacks.

Methods of Limit Cycle Analysis

Several phase plane methods (Minorsky 1962) for finding limit cycles were examined. These methods are generally easy to use; however, they are only applicable to second order differential systems. This

characteristic limits the value of the method since only simple systems of equations can be solved.

Geometrical methods (Truxal 1955) are also restricted in order to third order systems of autonomous differential equations with constant coefficients. The results of the analysis for the two temperature reactor problem may be misleading.

Shotkin (1964) employed a method in which one coordinate of the system is eliminated, and hence, reduces the order of the system. He analyzed several third order reactor systems by reducing the system equations to second order and using phase plane methods.

Fourier series expansions and transfer function analyses were examined and appear to overcome most of the objections in the above methods. This method is not restricted by the order of the system, and yet is fairly straightforward in application.

Fourier Series Method

The early work was involved in modifying the system transfer functions to account for the nonlinearities. Later, as the effort progressed, it became known as describing function analysis which was used primarily with reactor oscillation experiments. The mathematical formulation for the natural, unforced, oscillations of the autonomous equations is similar to the results of the describing function.

Rumsey (1947) was probably the first to substitute Fourier series expansions of the neutron density, reactivity and delayed neutron precursor concentrations into the space-independent neutron kinetics equations.

He solved the resulting set of coupled algebraic equations for the Fourier coefficients of the neutron density assuming that the input reactivity coefficients were known. He iterated once and obtained a correction to the system transfer function.

Following the same general approach, Nelkin (1955) solved the neutron kinetics equations with an externally applied oscillating reactivity and linear feedback processes. He recognized that a negative reactivity bias was necessary for sustained power oscillations; however, he failed to consider in the solution the change in average neutron density from low power to high power operation.

Sandmeier (1959) contributed to the solution of the problem by including the change in average neutron density which Nelkin had omitted. Sandmeier then iterated for solutions of the power fundamental in terms of the applied reactivity.

Later, Wasserman (1962) provided a comprehensive review of the Fourier methods to 1962 when he extended and corrected the earlier works. He developed a perturbation method that appeared superior in some respects to the iteration scheme. His work was concerned with the applied oscillating reactivity problem and thus did not consider autonomous oscillations. He relaxed the usual requirements of describing functions (Weaver p. 147, 1963) and permitted feedback of all the harmonics generated in the nonlinearity.

Smets (1964) applied describing function analysis to natural oscillations in the reactor problem without delayed neutrons. Later, Smets (1965) included delayed neutrons in his analysis. Subsequently,

Akcasu and Shotkin (1967) modified the work of Smets to extend the validity of the results to large amplitudes. They analyzed the amplitude and stability of the limit cycle solutions of the neutron kinetics equations without delayed neutrons and concluded that only stable limit cycles exist for this model with linear feedback.

Torlin (1966) used Fourier series expansions in the autonomous reactor equations and iterated for the successive approximations of the reactivity amplitude. He applied his results to a homogeneous boiling water reactor with one group of delayed neutrons and found stable as well as unstable limit cycles, but his application of the method to this model seems incorrect.

Reactor Systems with Two-Path Feedback

The reactor kinetics equations with two-path reactivity feedback have been studied by several authors, of which only a few are mentioned. Bethe (1956) used a reactor model with delayed neutrons and feedback from two temperature regions in examining the stability of the fast reactor EBR-I. Gyftopoulos (1961) used a more general heat-transfer model than Bethe but neglected delayed neutrons. Shotkin (1964) performed a complete analysis of the two temperature problem without delayed neutrons using the canonical set of temperature equations. Miida and Suda (1961) included delayed neutrons by using the second order neutron kinetics equations. The resulting system was too complex, however, to permit detailed analysis of all the cases.

Summary

In this study the method of Torlin (1966), which used Fourier series expansions in solving for limit cycles, is simplified and extended.

In the derivation, Torlin used an average incremental power for defining the neutron density which required the introduction of a correction factor in the transfer function. This adjusted for the shift in the average power level produced by the nonlinearities. By defining the average incremental power differently and using the steady-state bias term in the Fourier series expansion, the correction factor is not needed and the derivation is more straightforward.

Torlin examined the second approximation for the fundamental amplitude of the reactivity which excluded second harmonics in the feedback. It is demonstrated in this study that the second approximation yields results which are partially incorrect. Therefore, in an effort to improve the results, the method is extended to the third, fourth, and infinite-order approximations in reactivity amplitude and is examined for inclusion of higher harmonics. In addition, the successive approximations are derived in terms of the fundamental amplitude in power which provides a check on the results of the reactivity amplitude. It is evident that in order for the presence of a limit cycle to be substantiated, it must be predicted by both the power and reactivity amplitudes.

Using series approximation methods, a solution of the system equations (Hetrick 1968) is presented which yields the same results as the infinite approximation. This method is an improvement over the

iterative method since the convergence of a series is not required in the derivation; however, both methods are restricted to describing limit cycles of small amplitude. In practical stability only small amplitude limit cycles are of interest since small stable limit cycles may permit reactor operation outside the region of linear stability whereas large stable limit cycles may not. On the other hand, small unstable limit cycles may restrict reactor operation to less than the region of linear stability whereas large unstable limit cycles may not.

An important result obtained with the prompt jump equations is the existence of unstable limit cycles. This result is predicted by the successive approximations and verified by solutions of the exact equations.

A digital computer program was written to solve the various approximations for the amplitudes and frequencies with different sets of parameters. The results of the various approximations for three different systems are given and compared to the exact solutions.

Integrated throughout this study is a comparison of the effective lifetime model and the prompt jump approximation for neutron kinetics. The differences between the linear stability regions of the two parameter spaces are emphasized. The prompt jump approximation is in close agreement with the standard second order neutron kinetics equations.

There are also significant differences in nonlinear stability. The effective lifetime model does not admit the possibility of unstable limit cycles while the prompt jump approximation yields both stable and unstable limit cycles. There are also important differences in the trajectories between shut down and operating conditions. It is concluded

that the prompt jump approximation provides a much more accurate description of neutron kinetics than the effective lifetime model.

The most general linear heat transfer equations for two regions are described. A linear transformation is derived for transforming the coupled equations into a general set of decoupled equations which are mathematically simpler and easier to use.

The stability of a special set of temperature equations (Hetrick, Schmidt and Davis 1967) is analyzed using the time constants and feedback coefficients which were calculated (Brehm 1968) for a nuclear reactor with in-core thermionic converters. A wide range of stability is demonstrated for the model. The form of the equations has application to systems other than the thermionic problem as Bethe (1956) used the same equations with different parameters for analyzing the kinetic behavior of the fast reactor EBR-I.

The iterative solution of the autonomous neutron kinetics equations is derived in Chapter II. The iterative solution is then used to find the first, second, third, and infinite successive approximations for the reactivity and power amplitudes. The solutions are compared to the results of a derivation using direct series methods to the same order of approximation. The method of analyzing the approximation for limit cycles is discussed.

In Chapter III the system equations are given. The two neutron kinetics models used in this study, the prompt jump approximation and effective lifetime model, are derived. The most general coupled temperature equations for two regions are given and a linear transformation

to the decoupled temperature equations is derived. A description is presented of the thermionic reactor concept, and the temperature equations, which describe the heat transfer in the reactor, are given.

Linear and nonlinear stability is discussed in Chapter IV. The transfer functions for the forward and feedback loops of the system are used to examine linear stability by applying the Routh stability criterion. The stability boundaries are displayed in a parameter space where the coordinates are functions of the feedback coefficients. Various aspects of nonlinear stability are discussed, and where boundaries exist, they are noted in the parameter space.

Chapter V presents the results of the Fourier method for three systems with different time constants using both neutron kinetics models. These results are compared with the exact solutions of the system equations obtained by analog and digital computer computation.

Chapter VI summarizes the conclusions.

CHAPTER II

FOURIER SERIES SOLUTION OF SYSTEM EQUATIONS

In this chapter, the iterative solution of the space-independent reactor kinetics equations using Fourier series expansions of reactivity and power will be derived. This solution describes analytically the frequency and the amplitude of the fundamentals in power and reactivity of certain steady oscillations in reactor systems. The derivation follows the derivation by Torlin (1966) closely except in the consideration of the incremental shift in mean power level between the linear analysis and the nonlinear analysis. The differences will be noted as they are included.

The iterative solution is used to calculate the next approximation with the linear solution as the starting equations or zero order approximation. Various approximations for the reactivity amplitude are then presented. The final approximation considers the inclusion of all higher harmonics and results from taking the limit of an infinite series of terms. This "infinite" iteration, however, includes only second harmonics in power and reactivity, as the higher harmonics would be expressed in terms of higher order than considered in this derivation.

The approximations corresponding to those developed for the reactivity amplitude are then calculated for the power amplitude. The differences in development and formulation are discussed.

The results of the iterative solution are compared to a derivation by Hetrick (1968) starting with the integro-differential formulation of the neutron kinetics equations. The results by Hetrick are identical to the results of the infinite iteration. In addition, Hetrick's solution does not require the convergence of a series which places a restriction on the validity of the iterative solution. The following section begins the derivation.

Derivation of Iterative Solution

The point reactor kinetics equations are used to describe the dynamic behavior of the mean neutron density and the mean concentrations of delayed neutron precursors over the reactor volume. Since the power of a nuclear reactor is proportional to the neutron density integrated over the volume, the power and neutron density are used interchangeably throughout the dissertation.

The validity of the point reactor kinetics equations, used in this analysis, is discussed by Weaver (1963).

The $n + 1$ order system of reactor kinetics equations are

$$\frac{dP(t)}{dt} = \frac{\beta}{\ell} [\rho(t) - 1]P(t) + \sum_{i=1}^n \lambda_i C_i(t) \quad (2-1)$$

$$\frac{dC_i(t)}{dt} = \frac{\beta_i}{\ell} P(t) - \lambda_i C_i(t) \quad i = 1, n \quad (2-2)$$

where; P is the mean neutron density or power, C_i is a mean delayed neutron precursor concentration, ρ is the reactivity of the system in dollars, ℓ is the neutron generation time, λ_i are the decay constants for the delayed neutron precursors, and β is the delayed neutron fraction,

$$\beta = \sum_{i=1}^n \beta_i.$$

Using (s) as the differential operator, d/dt, equations 2-1 and 2-2 may be rewritten as

$$s \left[\frac{\ell}{\beta} + \sum_{i=1}^n \frac{\beta_i/\beta}{s+\lambda_i} \right] P(t) = \rho(t)P(t) \quad (2-3)$$

An incremental neutron density is defined which gives the relative deviation of the neutron density about the equilibrium value, P_o .

$$p(t) = \frac{P(t)-P_o}{P_o} \quad (2-4)$$

Torlin (1966) used the nonlinear time-average value of the neutron density, $\bar{P}$, in this expression.

Replacing the linear differential operator by

$$P_o Z^{-1}(s) = s \left[\frac{\ell}{\beta} + \sum_{i=1}^n \frac{\beta_i/\beta}{s+\lambda_i} \right] \quad (2-5)$$

the reactor kinetics equation from equation 2-3 becomes

$$P_o Z^{-1}(s)p(t) = \rho(t)[p(t)+1] \quad (2-6)$$

which is considered the forward loop equation.

The feedback equation is written in similar form,

$$P_o H(s)p(t) = -\rho(t) \quad (2-7)$$

where $H(s)$ is the linear differential operator depending upon the state equations of the variables contained in the reactivity. Torlin (1966) included a factor of $\bar{P}/P_o$ in equation 2-7 to account for the deviation between $\bar{P}$ and P_o . This is necessary in his work because $H(s)$ is calculated for P_o while Torlin defined p in terms of $\bar{P}$. The factor is

not needed here since p is the incremental power about P_0 . The deviation between $\bar{P}$ and P_0 which is a shift in the mean equilibrium neutron density caused by the nonlinearities in the system is discussed by several authors (Sandmeier 1959, Smets 1964). For a linear system, the shift is not present; $\bar{P} = P_0$ and $\bar{P}/P_0 = 1$. This shift in power level is related to the negative reactivity bias required to keep the nonlinear oscillations from growing. This will be discussed later.

Equations 2-6 and 2-7 together completely describe the system. It is noted that the equations and the following method are not restricted by the order of the system. The equations are now in a form which will yield the simplest derivation of the iterative solution.

The neutron density and the reactivity are expanded in Fourier series expressed in exponential form. The reactivity is given by

$$\rho(\omega t) = \sum_{k=-\infty}^{\infty} a_k e^{ik\omega t} \quad (2-8)$$

and the power by

$$p(\omega t) = \sum_{\ell=-\infty}^{\infty} A_{\ell} e^{i\ell\omega t} \quad (2-9)$$

where a_k and A_{ℓ} are the complex amplitudes of the k^{th} and ℓ^{th} harmonic respectively. In this formulation, any shift in the equilibrium power level will be accounted for by A_0 .

Substitution of the Fourier series for power and reactivity into the forward loop equation, equation 2-6, gives

$$P_0 Z^{-1}(s) \sum_{\ell=-\infty}^{\infty} A_{\ell} e^{i\ell\omega t} = \sum_{k=-\infty}^{\infty} a_k e^{ik\omega t} \left(\sum_{\ell=-\infty}^{\infty} A_{\ell} e^{i\ell\omega t} + 1 \right) \quad (2-10)$$

which can be rewritten as

$$P_o \sum_{\ell=-\infty}^{\infty} Z^{-1}(i\ell\omega) A_{\ell} e^{i\ell\omega t} = \sum_{k=-\infty}^{\infty} a_k e^{ik\omega t} + \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} a_k A_{\ell} e^{i(\ell+k)\omega t} \quad (2-11)$$

The nonlinearity of the equation is the term with the double summation. Use of Cauchy's product theorem simplifies equation 2-11 to

$$P_o \sum_{\ell=-\infty}^{\infty} Z^{-1}(i\ell\omega) A_{\ell} e^{i\ell\omega t} = \sum_{\ell=-\infty}^{\infty} a_{\ell} e^{i\ell\omega t} + \sum_{k=-\infty}^{\infty} a_k A_{\ell-k} e^{i\ell\omega t} \quad (2-12)$$

Note that the summations in the equation now have different arguments. Equating the coefficients of the same order of exponential or frequency yields

$$P_o Z_{\ell}^{-1} A_{\ell} = a_{\ell} + \sum_{k=-\infty}^{\infty} a_k A_{\ell-k} \quad (2-13)$$

The convention is used that $Z(i\ell\omega) = Z_{\ell}$ and similarly for the feedback, $H(i\ell\omega) = H_{\ell}$.

Equation 2-13 is now used in the following form to iterate for the successive approximations of the reactor power harmonics.

$$A_{\ell}^{(v+1)} = \frac{Z_{\ell}}{P_o} \left[a_{\ell} + \sum_{k=-\infty}^{\infty} a_k A_{\ell-k}^{(v)} \right] \quad (2-14)$$

Performing the same manipulations with the feedback equation yields a second equation relating the power and reactivity harmonics. Substitution of the Fourier series of power and reactivity into equation 2-7 gives

$$P_o \sum_{\ell=-\infty}^{\infty} H_{\ell} A_{\ell} e^{i\ell\omega t} = - \sum_{k=-\infty}^{\infty} a_k e^{ik\omega t}. \quad (2-15)$$

Collecting the coefficients of the same frequencies reduces equation 2-15 to

$$A_{\ell} = - \frac{a_{\ell}}{P_o H_{\ell}} \quad (2-16)$$

Equations 2-14 and 2-16 constitute the system equations, and they must be satisfied simultaneously for any solution of the power and reactivity to be a valid one. In developing the successive approximations, both equations are used in order to eliminate either the power or the reactivity and solve for the other explicitly. Before going to the successive approximations, the steady state values of a_o and A_o are discussed.

Equation 2-13 can be used to determine the mean value or negative bias of the reactivity for steady oscillations to occur. Letting $\ell = 0$ and noting that $Z(0)^{-1} = 0$, the result is

$$a_o = - \sum_{k=-\infty}^{\infty} a_k A_{-k} \quad (2-17)$$

The limits of the sum are simplified by using the fact that a_{-k} is the complex conjugate of a_k and similarly for A_{-k} and A_k . Thus, $a_k + a_{-k} = 2\text{Re } a_k$ and

$$a_o = -2 \text{Re} \sum_{k=1}^{\infty} a_k A_{-k} \quad (2-18)$$

Equation 2-16 is then used to find the shift in the neutron density or power level

$$A_o = - \frac{a_o}{P_o H_o} \quad (2-19)$$

where A_o is related to the equilibrium power level by

$$A_o = \frac{\bar{P} - P_o}{P_o} \quad .$$

Note that the preceding equations are exact in that no approximations have been made to the point-reactor equations.

Successive Approximations

In this section the iterative solution of the system equations is used to solve for the various successive approximations. The steps will be outlined in some detail for the solution of the first approximation. For higher approximations only the additions will be discussed. Equations 2-14 and 2-16 with substitutions of equations 2-18 and 2-19 are used to find the fundamentals of power and reactivity. Eliminating one fundamental leads to an expression for the absolute magnitude squared of the other. The amplitude of the reactivity fundamental is derived in the following examples and the results for the corresponding amplitudes of the power fundamental are summarized afterwards. The linear results are used for the zero approximation.

First Approximation

The first approximation is developed in the following manner. For $\ell = 1$, equation 2-14 for the forward loop becomes

$$A_1^{(1)} = \frac{Z_1}{P_o} \left[a_1 + a_o A_1^{(o)} + a_1 A_o^{(o)} + a_{-1} A_2^{(o)} + \dots \right]. \quad (2-14)$$

As the zero approximation is the linear case in which all higher harmonics ($A_2^{(o)}$, $A_3^{(o)}$, ...) are zero, the expression for $A_1^{(1)}$ is immediately reduced to

$$A_1^{(1)} = \frac{Z_1}{P_o} \left[a_1 + a_o A_1^{(o)} + a_1 A_o^{(o)} \right]. \quad (2-20)$$

From the feedback, equation 2-16

$$A_1^{(1)} = - \frac{a_1}{P_o H_1}. \quad (2-21)$$

To proceed, the right hand side of $A_1^{(1)}$, must be expressed in terms of the appropriate fundamental which in this case is the fundamental of the reactivity. Thus, a_o , $A_1^{(o)}$ and $A_o^{(o)}$ in equation 2-20 must be expressed in terms of a_1 .

Using equation 2-14 for the linear case gives

$$A_1^{(o)} = \frac{Z_1}{P_o} a_1. \quad (2-22)$$

The expression for a_o from equation 2-18 is

$$a_o = -2 \operatorname{Re}[a_1 A_{-1}^{(o)} + a_2 A_{-2}^{(o)} + \dots], \quad (2-18)$$

and since in the linear case the higher harmonics are eliminated,

$$a_o = -2 \operatorname{Re}[a_1 A_{-1}^{(o)}] . \quad (2-23)$$

Using equation 2-22 for $A_{-1}^{(o)}$ with the property that $a_1 a_{-1} = a_1 a_1^* = |a_1|^2$, equation 2-23 becomes

$$a_1 = \frac{-2|a_1|^2}{P_o} \operatorname{Re} Z_1 \quad (2-24)$$

From equation 2-19 with the use of equation 2-24 it is seen that

$$A_o = \frac{2|a_1|^2}{P_o H_o} \operatorname{Re} Z_1 .$$

Substituting this together with equations 2-22 and 2-24 into equation 2-20 gives

$$A_1^{(1)} = \frac{Z_1 a_1}{P_o} \left[1 - \frac{2|a_1|^2}{P_o^2} (Z_1 - H_o^{-1}) \operatorname{Re} Z_1 \right] . \quad (2-25)$$

Eliminating the power fundamental with the feedback equation, equation 2-21, then yields the first approximation for the amplitude squared of the reactivity fundamental.

$$\frac{|a_1|^2}{P_o^2} = \frac{1 + Z_1 H_1}{2Z_1 H_1 [Z_1 - H_o^{-1}] \operatorname{Re} Z_1} \quad (2-26)$$

From this result, it is seen that the higher harmonic frequencies do not contribute in the first approximation. In the second approximation, the effects of higher harmonics are evident.

Second Approximation without a_2

The simplest expression for the second approximation which includes higher harmonics is determined by including A_2 only and letting all other harmonics in power and reactivity be zero. The forward loop equation, equation 2-14, is now written as

$$A_1^{(2)} = \frac{Z_1}{P_o} \left[a_1 + a_o A_1^{(1)} + a_1 A_o^{(1)} + a_{-1} A_2^{(1)} \right]. \quad (2-27)$$

This has the same three terms as equation 2-20 with an additional term, $a_{-1} A_2^{(1)}$. The approximations for A_o and a_o are the same as in the first approximation. However $A_1^{(1)}$, equation 2-25, now has two terms more than $A_1^{(o)}$.

To describe $A_2^{(1)}$ which is the last term of equation 2-27, the successive approximation, equation 2-14, is used with $l = 2$.

$$A_2^{(1)} = \frac{Z_2}{P_o} \left[a_2 + \sum_{k=-\infty}^{\infty} a_k A_{2-k}^{(o)} \right] \quad (2-28)$$

For this case, a_2 is not used, and only one term in the summation contributes.

$$A_2^{(1)} = \frac{Z_2}{P_o} a_1 A_1^{(o)} \quad (2-29)$$

Upon substitution of $A_1^{(o)}$ from equation 2-19 into equation 2-29, $A_2^{(1)}$ becomes

$$A_2^{(1)} = \frac{Z_1 Z_2}{P_o^2} (a_1)^2 \quad (2-30)$$

Substituting equations 2-19, 2-24, 2-25 and 2-30 into the second approximation for $A_1^{(2)}$ including A_2 , equation 2-27, yields

$$A_1^{(2)} = \frac{Z_1 a_1}{P_o} \left[1 + \frac{|a_1|^2}{P_o^2} [Z_1 Z_2 - 2(Z_1 - H_o^{-1}) \text{Re} Z_1] + O|a_1|^4 \right] \quad (2-31)$$

where the terms of order $|a_1|^4$ are omitted. The usual argument for validity of such a truncation is that $|a_1|^2 \ll 1$. This requirement may be relaxed by considering the right hand side of equation 2-31 as a polynomial in $|a_1|^2$. If the smallest root is sufficiently smaller than the other roots, the truncation is a fair approximation regardless of the size of the root or the order of the polynomial. However, it will be seen later that inclusion of higher harmonics affects only the higher order terms which tends to require that $|a_1|^2 \ll 1$ for sake of accuracy.

The feedback equation provides the second relationship for A_1 and a_1 .

$$A_1^{(2)} = - \frac{a_1}{P_o H_1} \quad (2-32)$$

Solving equations 2-31 and 2-32 for $|a_1|^2$ yields the second approximation.

$$\frac{|a_1|^2}{P_o^2} = \frac{1 + Z_1 H_1}{Z_1 H_1 [2(Z_1 - H_o^{-1}) \text{Re} Z_1 - Z_1 Z_2]} \quad (2-33)$$

This is identical to the equation used by Torlin (1966) in his analysis, although here it was derived in a different manner. This expression now includes terms having second harmonic frequencies. The next order of complexity is to include the second harmonic reactivity.

Second Approximation with a_2

The consideration of a_2 in the derivation adds one more term in the summation of the forward loop equation. Equation 2-14 is now

$$A_1^{(2)} = \frac{Z_1}{P_o} \left[a_1 + a_o A_1^{(1)} + a_1 A_o^{(1)} + a_{-1} A_2^{(1)} + a_2 A_{-1}^{(1)} \right]. \quad (2-14)$$

The contribution of a_2 to $A_1^{(2)}$ is determined by the use of the feedback equation for $\ell = 2$.

$$a_2 = - P_o H_2 A_2^{(1)} \quad (2-34)$$

where substitution of equation 2-30 for A_2 gives

$$a_2 = - \frac{(a_1)^2}{P_o} Z_1 Z_2 H_2 \quad (2-35)$$

The addition of this term in $A_1^{(2)}$ is subsequently reflected in $|a_1|^2$ in the following manner.

$$\frac{|a_1|^2}{P_o^2} = \frac{1 + Z_1 H_1}{Z_1 H_1 [2(Z_1 - H_o^{-1}) \text{Re} Z_1 - Z_1 Z_2 (1 - Z_{-1} H_2)]} \quad (2-36)$$

In this expression frequencies of the second harmonic appear in the feedback loop as well as in the forward loop.

Second Approximation with Higher Harmonics

The inclusion of third, fourth, or higher harmonics in power or reactivity in the above equations only yields terms of order $|a_1|^4$ and higher which are dropped in the derivation. Therefore, the equations developed above for the reactivity fundamental are unchanged and the effect of harmonics other than the second is not considered in this method.

Third Approximation

The third approximation with A_2 only yields the same expression as was derived for the second approximation with A_2 only. The third approximation including a_2 gives additional information. The effect is in $A_2^{(2)}$.

$$A_2^{(2)} = \frac{Z_2}{P_o} \left[a_2 + a_1 A_1^{(1)} \right] \quad (2-37)$$

Compare this expression with equation 2-29. With the substitution of equation 2-35 for a_2 into equation 2-37, $A_2^{(2)}$ becomes

$$A_2^{(2)} = \frac{(a_1)^2}{P_o^2} Z_1 Z_2 [1 - Z_2 H_2] \quad (2-38)$$

Putting this expression into the equation for the third approximation yields the following reactivity fundamental.

$$\frac{|a_1|^2}{P_o^2} = \frac{1 + Z_1 H_1}{Z_1 H_1 [2(Z_1 - H_o^{-1}) \text{Re} Z_1 - Z_1 Z_2 (1 - Z_1 H_2) (1 - Z_2 H_2)]} \quad (2-39)$$

This result has one term more than the second approximation. It will be shown in the next section that each new approximation adds another term.

Infinite Approximation

Upon examination of the system equations for the higher approximations, it is evident that the only change occurs in the expression for A_2 . Taking the n^{th} approximation,

$$A_2^{(n)} = \frac{(a_1)^2}{P_o^2} Z_1 Z_2 [1 - Z_2 H_2 + Z_2^2 H_2^2 - \dots (-)^n Z_2^n H_2^n] \quad (2-40)$$

This can be simplified by requiring that the power series be convergent and by taking an infinite number of terms.

$$A_2^{(\infty)} = \frac{(a_1)^2}{P_o^2} Z_1 Z_2 \frac{1}{1+Z_2 H_2} \quad (2-41)$$

Using this result, the infinite approximation for the reactivity fundamental becomes

$$\frac{|a_1|^2}{P_o^2} = \frac{1 + Z_1 H_1}{Z_1 H_1 [2(Z_1 H_1 - 1) \operatorname{Re} Z_1 - Z_1 Z_2 (1 - Z_1 H_2) (1 + Z_2 H_2)^{-1}]} \quad (2-42)$$

This is the approximation that is used to generate the most accurate results in the last chapter. The various approximations derived above for the reactivity fundamental may also be derived in terms of the power fundamental which will be discussed next.

Successive Approximation in Power Fundamental

In the same manner as the approximations for the reactivity amplitude are derived, expressions for the power amplitude may be derived. There are a few differences to be pointed out and then the results for the various approximations will be summarized.

The basic difference lies in describing the approximations in terms of the power fundamental, A_1 , instead of the reactivity fundamental. For example in the linear case,

$$a_1 = \frac{P_o A_1}{Z_1} \quad (2-43)$$

is used to eliminate a_1 in the approximation. Notice that in the linear

case, the feedback equation also provides a relationship between a_1 and A_1 (i.e., $a_1 = -P_o H_1 A_1$). It might appear that either relationship can be used in the development. The choice is optional but with one exception. To satisfy the requirement that $a_o = 0$ when $\text{Re}Z_1 = 0$, it is necessary that the forward loop expression, equation 2-43, be used. The problem arises because A_o is defined by the feedback equation instead of the forward loop equation. For other substitutions either expression can be used.

In the first approximation for describing the power fundamental, the reactivity bias in terms of A_1 is

$$a_o = -2P_o |A_1|^2 \text{Re}Z_1^{-1} \quad (2-44)$$

With the use of equation 2-43 and 2-44 and the system equations, the successive approximations for the power fundamental are developed.

In the first approximation, the power fundamental is given by

$$|A_1|^2 = \frac{1 + Z_1 H_1}{2Z_1 H_1 [-H_1^{-1} - H_o^{-1}] \text{Re}Z_1^{-1}} \quad (2-45)$$

Compare this expression with equation 2-26 for the reactivity fundamental.

Continuing on to higher approximations in the same manner as for the reactivity fundamental, the infinite iteration of the power fundamental is derived to be

$$|A_1|^2 = \frac{1 + Z_1 H_1}{Z_1 H_1 [-2(H_1^{-1} + H_o^{-1}) \text{Re}Z_1^{-1} + Z_2 H_{-1} (1 + H_{-1}^{-1} H_2) (1 + Z_2 H_2)^{-1}]} \quad (2-46)$$

This equations corresponds to equation 2-42 for the reactivity fundamental. The principal difference arises because the power is

described in terms of the feedback transfer function, H , rather than the forward loop transfer function, Z .

The various approximations of both the power and reactivity fundamental are summarized in Appendix A.

Integral Formulation

Starting with the integro-differential formulation for the forward loop of the reactor system, Hetrick (1968) has derived the same expression for the successive approximations as was derived above. Then by considering the expression a series expansion and keeping terms of second order, Hetrick was able to derive the same result as the infinite approximation but without taking the limit of an infinite series, and without iterating.

The method will be outlined so that comparison can be made.

The reactor kinetics equations may be written

$$\frac{\ell}{\beta} \frac{dP(t)}{dt} = \rho(t)P(t) + \int_0^{\infty} [P(t-u) - P(t)] D(u) du \quad (2-47)$$

where $D(u)$ is the delayed neutron yield function.

$$D(u) = \frac{1}{\beta} \sum_{i=1}^n \lambda_i \beta_i e^{-\lambda_i u} \quad (2-48)$$

and

$$\int_0^{\infty} D(u) du = 1$$

The other notation is standard, P is the mean neutron density, ℓ is the neutron generation time, β is the delayed neutron fraction and ρ is the reactivity in dollars.

The reactivity and incremental neutron density are expanded in Fourier series

$$\rho(\omega t) = \sum_{k=-\infty}^{\infty} a_k e^{ik\omega t} \quad (2-8)$$

$$p(\omega t) = \frac{P(\omega t) - P_0}{P_0} = \sum_{\ell=-\infty}^{\infty} A_{\ell} e^{i\ell\omega t} \quad (2-9)$$

Substitution of the Fourier expansions into equation 2-47, and after some manipulation, leads to the following results which is the same as equation 2-13.

$$P_0 Z_{\ell}^{-1} A_{\ell} = a_{\ell} + \sum_{k=-\infty}^{\infty} a_k A_{\ell-k} \quad (2-49)$$

The feedback equation is unchanged from equation 2-16.

Now considering the terms in equations 2-49 of the same order of a_1 to be an approximate solution, an expression may be found corresponding to the infinite approximation.

The amplitude of the second harmonic in power to order $(a_1)^2$ is given by

$$A_2 = \frac{Z_2}{P_0} [a_2 + a_1 A_1]. \quad (2-37)$$

Eliminating A_2 and solving for a_2 yields

$$a_2 = - \frac{(a_1)^2 Z_1 Z_2 H_2}{P_0 [1 + Z_2 H_2]} \quad (2-50)$$

Use of this expression for a_2 in equation 2-49 and elimination of either fundamental leads to expressions which are the same as the infinite approximation derived earlier in this section.

Method of Analysis

The expressions developed in the successive approximations for the fundamentals are composed of complex factors involving frequency. Since the magnitude squared of the fundamental is a real number, values of the frequency are sought which eliminate the imaginary portion of the expression. This is accomplished by solving a polynomial generated by requiring that the imaginary part of $|a_1|^2$ or $|A_1|^2$ be zero. Each root of the polynomial is a frequency squared, so that only the positive roots represent meaningful frequencies for possible oscillations or limit cycles. The positive frequency roots are used in the real part of the expression for $|a_1|^2$ or $|A_1|^2$ to calculate the oscillation amplitudes. Of the resulting magnitudes, only the positive magnitudes are meaningful for describing limit cycles. In the infinite approximation a subharmonic frequency gives rise to a positive real amplitude; however, this is not physically possible and represents a spurious result.

With the lower approximations, the equations were easily solved by slide rule, but for the higher approximations it was necessary to use machine calculations. To perform the analysis, a digital computer program, FREAMP, was written which describes the real and imaginary parts of an expression for the magnitude squared of an amplitude. The frequency squared roots are extracted from the imaginary part, and the fundamental amplitudes are calculated using the positive roots. The results of this analysis are given in Chapter V.

CHAPTER III

SYSTEM EQUATIONS

The system of equations analyzed in this dissertation are composed of the nuclear reactor kinetics equations and the two-region heat transfer equations. The $n + 1$ order system of neutron kinetics equations are discussed in Chapter II. In this chapter the second order neutron kinetics equations including one group of delayed neutron are given and then simplified to the prompt jump approximation. This approximation is a first order equation which nevertheless includes a time constant for delayed neutrons. The prompt jump equation is then simplified to the effective lifetime kinetics model in which the effect of delayed neutrons appears only in the weighting of the neutron lifetime. The prompt jump approximation and the effective lifetime kinetics are the two neutron kinetics models used in this analysis. The two models are identical for small amplitude low-frequency oscillations which facilitates analysis of the effect of delayed neutrons on oscillations. The effective lifetime model is the simplest neutron kinetics model and has been examined extensively (Gyftopoulos 1961, Shotkin 1964, Smets 1964, Akcasu and Noble (1966)). The prompt jump equation is slightly more complex than the effective lifetime model; however, it yields a much more accurate description of the neutron kinetics. The prompt jump approximation has been used by several authors (Bethe 1956, Meenan and Hetrick 1966) but has not enjoyed as wide a usage as the effective lifetime model. Both

reactor models are restricted in validity to some degree, but for limited disturbances about the operating conditions both equations provide valid results. Excursion analysis would require a different set of equations or parameters.

For a fast reactor where the mean neutron generation time is very short, solutions using the prompt jump approximation are very nearly the same as the one delayed group second-order kinetics equations provided the system does not approach prompt critical too closely. The differences will be discussed later.

Another important reason for use of the prompt jump approximation and the effective lifetime model is that the neutron generation time does not appear explicitly in the formulation. This permits much easier solution of the system equations by analog and digital computer. When a lifetime of approximately 10^{-7} seconds appears among the time constants of the system, it is difficult to scale the problem for analog solution or to solve the problem numerically because of the small time steps required in order to preserve accuracy.

The general two-region heat transfer equations are presented and the transformation to the canonical heat transfer equations is discussed.

The heat transfer model used in the analysis describes the heat transfer of an internally heated thermionic diode which is representative of a nuclear reactor with in-core thermionic converters. The model can be applied to other systems by merely adjusting the values of the time constants and heat capacities to conform to the thermal properties of the system.

Neutron Kinetics Equations

The time-dependent point reactor neutron kinetics equations repeated for convenience are

$$\frac{dP(t)}{dt} = \frac{\beta}{\ell} [\rho(t) - 1] P(t) + \sum_{i=1}^n \lambda_i C_i(t) \quad (3-1)$$

$$\frac{dC_i(t)}{dt} = \frac{\beta_i}{\ell} P(t) - \lambda_i C_i(t) \quad i=1, n. \quad (3-2)$$

The notation is the same as listed in Chapter II.

If the delayed neutron precursors are lumped into a single group with a mean decay constant,

$$\frac{1}{\bar{\lambda}} = \frac{1}{\beta} \sum_{i=1}^n \frac{\beta_i}{\lambda_i} \quad (3-3)$$

equations 3-1 and 3-2 are reduced to a second order set of equations with one group of delayed neutrons.

$$\frac{dP(t)}{dt} = \frac{\beta}{\ell} [\rho(t) - 1] P(t) + \bar{\lambda} C(t) \quad (3-4)$$

$$\frac{dC(t)}{dt} = \frac{\beta}{\ell} P(t) - \bar{\lambda} C(t) \quad (3-5)$$

C is the weighted neutron precursor group.

This set of equations can be reformulated as a single second order equation in neutron density or power. This is accomplished by eliminating C in equation 3-5 by use of equation 3-4.

$$\frac{dC(t)}{dt} = \frac{\beta}{\ell} \rho(t) P(t) - \frac{dP(t)}{dt} \quad (3-6)$$

Equation 3-4 is differentiated once and equation 3-6 is substituted for dC/dt which gives

$$\ell \left[\frac{d^2 P(t)}{dt^2} + \lambda \frac{dP(t)}{dt} \right] = - \beta \frac{dP(t)}{dt} + \beta \frac{d}{dt} \rho(t) P(t) + \rho(t) P(t) \quad (3-7)$$

The prompt jump approximation is an asymptotic form obtained by setting the right hand side of equation 3-7 equal to zero. When ℓ is small and $\rho < 1$, the transient term that is neglected will decay very rapidly. Since the order of the differential equation is reduced by one, an initial condition is lost, corresponding to the rapidly-decaying transient that is neglected.

Rearranging the remaining terms yields the prompt jump approximation in its usual form.

$$[1-\rho(t)] \frac{dP(t)}{dt} = \left[\frac{d\rho(t)}{dt} + \lambda\rho(t) \right] P(t) \quad (3-8)$$

Using the following assumption

$$\left| \frac{d}{dt} \rho(t)P(t) \right| < \lambda \left| \rho(t)P(t) \right| \quad (3-9)$$

in equation 3-8 the equation for the effective lifetime neutron kinetics results.

$$\frac{dP(t)}{dt} = \lambda\rho(t)P(t) \quad (3-10)$$

In this formulation β/λ plays the role of an effective neutron lifetime, ℓ' , that is determined essentially by the properties of the delayed neutrons.

$$\ell' = \ell + \sum_{i=1}^n \frac{\beta_i}{\lambda_i} \approx \sum_{i=1}^n \frac{\beta_i}{\lambda_i} \approx \frac{\beta}{\lambda} \quad (3-11)$$

A model useful for analyzing rapid excursions is obtained from equations 3-1 and 3-2 in the limit as $\beta_i \rightarrow 0$ and $\lambda_i \rightarrow 0$. Note that this formulation is the same as equation 3-10 except that the neutron generation time appears in place of the effective lifetime.

$$\frac{dP(t)}{dt} = \beta \frac{\rho(t)P(t)}{\ell} \quad (3-12)$$

The prompt jump approximation is a good approximation to second order neutron kinetics equations provided $\rho < 1$ and provided the following condition is satisfied (Hetrick 1968).

$$1 - \rho(t) \gg \sqrt{\frac{\ell}{\beta} \left| \lambda \rho(t) + \frac{d\rho(t)}{dt} \right|}$$

For small amplitude sinusoidal variation at angular frequency ω and amplitude ρ_o this requires

$$(\omega + \lambda) \rho_o \ll \frac{\beta}{\ell}$$

For effective lifetime neutron kinetics, the frequency limitation is

$$\frac{\omega}{\lambda} \rho_o \ll \rho_o \ll 1.$$

The prompt jump approximation, equation 3-8, and the effective lifetime model, equation 3-10, are the two reactor kinetics equations used in this study.

General Two-Region Temperature Equations

Nuclear reactor models with linear temperature feedback from two regions have been examined by several authors in the literature (Gyftopoulos 1961, Shotkin 1964, Miida and Suda 1961). In the majority of this previous work, the equations used are decoupled equations which are mathematically simpler and in general easier to analyze. If a decoupled system were to represent the physical system, it implies that the heat generated in each region is transferred directly to the heat sink or reservoir and that there is no heat transfer between regions. In most reactor models this is rather artificial.

However, if the decoupled equations do not directly represent the physics system, use of the equations is justified because of the

fact that the physically coupled heat transfer system is related to the decoupled system by a linear transformation. Thus, coupled temperature equations can be diagonalized into a decoupled system. In doing this one has to be careful in order to avoid certain pitfalls (Lellouche 1966).

In this section the most general heat transfer equations for two regions are given. The effects included are heat production in each region, heat transfer between the regions and heat loss from each region. The linear transformation for converting the coupled system into the diagonalized system is derived. Gyftopoulos' (1961) analysis of the two temperature problem neglected direct heat loss in one of the regions.

The general coupled temperature equations for two regions are

$$\frac{dT_1(t)}{dt} = b_1[P(t)-P_0] - g_1[T_1(t)-T_2(t)] - g_2T_1(t) \quad (3-13)$$

$$\frac{dT_2(t)}{dt} = b_2[P(t)-P_0] + g_3[T_1(t)-T_2(t)] - g_4T_2(t) \quad (3-14)$$

where

b_i = reciprocal heat capacity of i^{th} region,

g_i = reciprocal time constants for heat removal,

$T_i(t)$ = average incremental temperature in the i^{th} region.

The temperature equations can be considered as heat balance equations where the heat transfer rate is proportional to the temperature difference between two regions. The temperatures are incremental temperature based on a reference reservoir sink temperature.

In the two temperature problem the reactivity or feedback in dollars is given as a linear combination of the temperatures,

$$\rho = -\frac{\alpha_1}{\beta} T_1 - \frac{\alpha_2}{\beta} T_2 \quad (3-15)$$

where α_i is the reactivity coefficient of temperature feedback from the i^{th} region. The convention is used that positive α_i denotes negative feedback.

The heat transfer equations written in matrix form are given by

$$\frac{dT}{dt} = b(P - P_0) + [A] T \quad (3-16)$$

where $\frac{dT}{dt}$, b , and T are column vectors and A is an $m \times m$ submatrix of the system matrix where m is the number of regions.

$$A = \begin{bmatrix} -g_1 - g_2 & g_1 \\ g_3 & -g_3 - g_4 \end{bmatrix} \quad (3-17)$$

Solving $|A - \lambda I| = 0$ for the eigenvalues yields (I is the unit matrix)

$$\lambda_1, \lambda_2 = -\frac{1}{2}(g_1 + g_2 + g_3 + g_4) \pm \sqrt{\frac{1}{4}(g_1 + g_2 + g_3 + g_4)^2 - (g_1 g_4 + g_2 g_3 + g_2 g_4)} \quad (3-18)$$

which can be written

$$\lambda_1, \lambda_2 = -\frac{1}{2}(g_1 + g_2 + g_3 + g_4) \pm \sqrt{\frac{1}{4}(g_1 + g_2 - g_3 - g_4)^2 + g_1 g_3} \quad (3-19)$$

From this expression for the eigenvalues, it is evident that the eigenvalues are always real since the time constants are inherently positive.

Using the eigenvalues it is possible by the following to solve for the eigenvectors, V , which are directly related to the linear transformation of the temperatures.

$$[A - \lambda I] V = 0 \quad (3-20)$$

The eigenvectors are

$$\underline{V}_1 = \begin{bmatrix} \frac{r_2}{r_2 - r_1} \\ \frac{-1}{r_2 - r_1} \end{bmatrix} \quad \underline{V}_2 = \begin{bmatrix} \frac{-r_1}{r_2 - r_1} \\ \frac{1}{r_2 - r_1} \end{bmatrix} \quad (3-21)$$

where

$$r_1, r_2 = \frac{-\frac{1}{2}(g_1 + g_2 - g_3 - g_4) \pm \sqrt{\frac{1}{4}(g_1 + g_2 - g_3 - g_4)^2 + g_1 g_3}}{g_1} \quad (3-22)$$

Since eigenvectors have arbitrary scale and sign, $\underline{V}_1$ and $\underline{V}_2$ were chosen such that the transformation would be simple.

The linear transformation is defined by

$$\underline{T} = [M] \underline{T}' \quad (3-23)$$

where M is the modal matrix whose columns are composed of the eigenvectors and $\underline{T}'$ is the column vector of the temperatures for the canonical system.

$$M = \begin{bmatrix} \frac{r_2}{r_2 - r_1} & \frac{-r_1}{r_2 - r_1} \\ \frac{-1}{r_2 - r_1} & \frac{1}{r_2 - r_1} \end{bmatrix} \quad (3-24)$$

By matrix algebra, the diagonalized matrix is found.

Substituting equation 3-23 into the heat transfer equation, equation 3-16, gives

$$[M] \frac{d\underline{T}'}{dt} = \underline{b}(P - P_o) + [A][M]\underline{T}' \quad (3-25)$$

Premultiplying by M^{-1} yields

$$\frac{d\underline{T}'}{dt} = [M]^{-1} \underline{b}(P - P_o) + [M]^{-1}[A][M]\underline{T}' \quad (3-26)$$

where

$$[M]^{-1}[A][M] = [\Lambda] = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad (3-27)$$

The relationship between the decoupled variables and the physical variables is given by

$$\tilde{T}' = [M]^{-1} \tilde{T} \quad (3-28)$$

where

$$[M]^{-1} = \begin{bmatrix} 1 & r_1 \\ 1 & r_2 \end{bmatrix} \quad (3-29)$$

so that the decoupled temperature are expressed as

$$T_i' = T_1 + r_i T_2 \quad (3-30)$$

The system equations with the decoupled heat transfer are then written

$$\frac{dT_1'}{dt} = b_1'(P - P_o) - \gamma_1 T_1' \quad (3-31)$$

$$\frac{dT_2'}{dt} = b_2'(P - P_o) - \gamma_2 T_2' \quad (3-32)$$

and

$$\rho = -\frac{\alpha_1'}{\beta} T_1' - \frac{\alpha_2'}{\beta} T_2' \quad (3-33)$$

The parameters of this system are related to the physical parameters by

$$\begin{aligned} b_i' &= b_1 + r_i b_2 \\ \gamma_i &= g_1 + g_3 - r_i g_3 \\ \alpha_1' &= \frac{\alpha_2 - \alpha_1 r_2}{r_1 - r_2} \\ \alpha_2' &= \frac{\alpha_2 - \alpha_1 r_1}{r_2 - r_1} \end{aligned} \quad (3-34)$$

The physical temperatures in terms of the decoupled temperatures are as follows. Using the modal matrix, the temperatures are related by

$$\begin{aligned} T_1 &= \frac{r_2}{r_2 - r_1} T_1' + \frac{r_1}{r_2 - r_1} T_2' \\ T_2 &= \frac{T_1' - T_2'}{r_1 - r_2} \end{aligned} \quad (3-35)$$

and the reactivity coefficients are

$$\begin{aligned} \alpha_1 &= \alpha_1' + \alpha_2' \\ \alpha_2 &= r_1 \alpha_1' + r_2 \alpha_2' \end{aligned} \quad (3-36)$$

Thermionic Heat Transfer Model

At the present time there is considerable interest in the stability of a nuclear reactor with in-core thermionic converters (direct conversion devices for changing heat into electricity). The impetus behind this interest stems from the fact that there are some unique problems associated with thermionic converters. There is a materials stability problem because of the very high temperatures required of the emitters and the thermal gradient caused by the relatively low temperatures of the collectors. Another aspect requiring investigation is the effect of changes in the load resistance. This may be a problem because as much as 60% of the heat transfer from the fuel is accomplished by electron cooling. This flow of electrons across the diode comprises part of the electrical circuit. Attendant with these problems, are the usual complex problems that are inherent in the design of a highly enriched fast spectrum reactor core. Highly enriched fissile fuels may have positive temperature feedback characteristics because of Doppler

broadening in the resonance region. This is a prompt reactivity effect. Changes in the coolant density may decrease the moderation and cause the flux spectrum to harden which could also introduce positive effects because the ratio of the cross section for capture to the cross section for fission decreases. This is a delayed reactivity effect. Doppler effects which arise in materials of the reactor other than the fuel and expansion effects resulting from the specific reactor design generally contribute negative feedback.

The basic concept for a thermionic converter is a fissile fuel pin, oxide or cermet, approximately an inch in diameter and an inch or two high, which is clad with an emitter of refractory material such as tungsten. Between the emitter and the collector is a spacing on the order of .010 inches. The collector may be molybdenum, nickel or other materials and is isolated electrically from the rest of the heat transfer system by an insulator sheath of aluminum oxide. The cladding material covers the insulator and must be compatible with the liquid metal coolants, which are necessary in this application because of the high temperatures and the high rates of heat transfer required in the core and in order to maintain the fast spectrum.

There are several mechanisms for heat transfer between the emitter and collector. They include electron flow across the gap, radiant heat transfer, heat conduction down the electrical lead connection, and other extraneous conduction losses. It has been shown that for constant current density the heat transfer is approximately linear over a significant range of operating parameters (Gronroos 1967). Thus, for

simplicity the model used in this analysis incorporates linear heat transfer across the spacing which is dependent upon the emitter temperature only.

Because of the much lower temperature of the collector, back emission from the collector is less than 5% of the total heat transfer and therefore is not included.

The two temperature regions considered are region one, the fuel-emitter region, and region two, the collector region. In this study region one is characterized by a prompt temperature feedback predominately due to Doppler effects and region two is characterized by a delayed temperature feedback which is caused by thermal expansion effects in the reactor structure.

The heat flow in the system is from the fuel-emitter region, where the heat is produced, to the collector region and subsequently to the coolant reservoir heat sink. A simplified diagram for a thermionic reactor system is shown in Figure 3.1.

The heat transfer equations (Hetrick, Schmidt and Davis 1967) for this system are

$$\begin{aligned}\frac{dT_1}{dt} &= b(P - P_o) - \eta_1 T_1 \\ \frac{dT_2}{dt} &= \eta_2 T_1 - \eta_3 T_2\end{aligned}\tag{3-37}$$

where b is the reciprocal heat capacity of the fuel-emitter region and the η 's are the reciprocal time constants for heat transfer.

η_1 is the reciprocal time constant for heat transfer out of region one, η_2 is the reciprocal time constant for heat transfer into region two, and η_3 is the reciprocal time constant for heat transfer out of region two. η_1 is in general different from η_2 because of the difference in the heat capacities of the two regions.

Brehm (Brehm, Hetrick and Schmidt 1969) has calculated time constants for a thermionic reactor design using uranium oxide as the fuel and sodium as the coolant. The values of the time constants are given in Table 4.1 in Chapter IV as Case III. The system equations using this set of time constants as well as two other sets of time constants will be analyzed in the results section.

Figure 3.1 Diagram for Thermionic Reactor System.

CHAPTER IV

SYSTEM STABILITY

In this chapter the stability of the system is analyzed. In order to understand the results of the stability analysis, the terms used in the description of the various solution trajectories are defined. It is assumed that the reader has a basic notion of the types of trajectories encountered in the solution of differential equations.

Linear stability analysis is then discussed and the results for the thermionic reactor model using three different sets of time constants are presented. A parameter space which is a function of the system parameters and the reactivity coefficients is used to show the stability boundaries for the linearized system. This parameter space is also valid for some aspects of the nonlinear analysis, particularly, since in a sufficiently small neighborhood about the equilibrium operating point, the solutions of the linear and nonlinear systems coincide. Linear stability analysis is valid only in a small region about the equilibrium. To extend the analysis beyond this small region, the nonlinearities in the system must be considered. An unstable linearized system may for all practical purposes be stable with the inclusion of the nonlinearities. The converse may also be true; a stable linearized system may become an unstable system when the nonlinearities are considered.

The section on nonlinear stability discusses these effects. Also, regions in the parameter space are defined where the solutions are

asymptotically stable in the large and where solutions are bounded in the large though not necessarily asymptotically stable. With linear systems, the size of the disturbance is unimportant since trajectories are stable or unstable in all of state space. In the nonlinear system, solution trajectories depend upon the magnitude of the disturbance and upon whether the disturbance is momentarily acting or continuously acting on the system. In this study only momentarily acting disturbances are considered. The primary concern is whether a disturbance of a certain magnitude results in a stable or an unstable solution trajectory.

The results of the linear and nonlinear stability analysis are summarized at the end of the chapter using a parameter space.

Stability Definitions

This discussion is not intended to be all inclusive; only those aspects of stability which are germane to the results of the analysis are presented. A comprehensive discussion is found elsewhere (Magiros 1967, Truxal 1955).

A trajectory is stable, if, with a small disturbance, the disturbed trajectory remains near to the undisturbed trajectory for all time. Disturbances are caused by perturbations which are reflected in the system as disturbing forces. A trajectory is unstable, if, with the disturbance, the trajectory does not remain near to the undisturbed trajectory.

A trajectory is asymptotically stable if the disturbed trajectory ultimately coincides with the undisturbed trajectory. A trajectory is asymptotically stable in the large if it is asymptotically stable for any size disturbance.

A trajectory is bounded if the solution is finite for all time. An unbounded trajectory is a solution which is not finite for all time.

A finite escape time is a solution which is unbounded and furthermore becomes infinite at some finite time. Such a trajectory is caused by a singularity in the solution or in the system differential equations. An example of a solution exhibiting a finite escape time is shown in Figure 4.1.

A center is a nonisolated periodic closed trajectory whose amplitude and frequency depend on initial conditions. An example of a center is shown in Figure 4.2.

A stable limit cycle is an isolated periodic closed trajectory whose amplitude and frequency are independent of initial conditions. Trajectories near the stable limit cycle converge to the limit cycle. This is shown in Figure 4.3.

An unstable limit cycle is an isolated periodic closed trajectory whose amplitude and frequency are independent of initial conditions. Trajectories near the unstable limit cycle diverge from the unstable limit cycle. An example of unstable limit cycle trajectories are shown in Figure 4.4.

The limit cycle and finite escape time trajectories are found only in the solution of nonlinear differential equations.

The other definitions apply to both linear and nonlinear systems. Limit cycles and centers are periodic solutions, or oscillations, which may be described by Fourier series expansions. The major effort in this dissertation is in the analysis of such periodic trajectories.

Figure 4.1 Example of a Solution Exhibiting Finite Escape Time.

Figure 4.2 Example of a Center.

Figure 4.3 Example of a Stable Limit Cycle.

Figure 4.4 Example of an Unstable Limit Cycle.

Linear Stability Analysis

Several well established and straightforward methods are available for use with linear systems. In this section the Routh criterion is used to find those parameters which produce a stable system. The results are displayed in a parameter space for each of the sets of time constants. The system equations are linearized and then the Laplace transform taken in order to find the transfer functions of the system.

For the prompt jump approximation the transfer function, using $\delta P = P - P_o$, is given by

$$Z(s) = \frac{\delta P(s)}{\delta \rho(s)} = P_o \frac{(s+\lambda)}{s} \quad (4-1)$$

and for the effective lifetime model

$$Z(s) = \frac{\delta P(s)}{\delta \rho(s)} = \frac{\lambda P_o}{s} \quad (4-2)$$

where the reactivity is in dollars.

The transfer function for the feedback is found by using the system diagram in Figure 4.5.

Figure 4.5 Thermionic System Block Diagram

From the temperature equations, equations 3-35,

$$T_1(s) = \frac{b\delta P(s)}{s+\eta_1} \quad (4-3)$$

and

$$T_2(s) = \frac{b\eta_2\delta P(s)}{(s+\eta_1)(s+\eta_3)} \quad (4-4)$$

Noting that the reactivity is given by $\rho = -\frac{\alpha_1}{\beta} T_1 - \frac{\alpha_2}{\beta} T_2$

and that the system is antonomous,

$$H(s) = -\frac{\delta\rho(s)}{\delta P(s)} = \frac{\alpha_1 T_1(s) + \alpha_2 T_2(s)}{\beta\delta P(s)}$$

which gives

$$H(s) = \frac{\alpha_1 b}{\beta} \frac{s+\eta_3+\eta_2(\alpha_2/\alpha_1)}{(s+\eta_1)(s+\eta_3)} \quad (4-5)$$

Now the system is in the standard form for analysis with composite forward and feedback loop transfer functions as shown in Figure 4.6.

Figure 4.6 Standard System Block Diagram

Considering the system with prompt jump approximation neutron kinetics, the open loop transfer function may be written as

$$Z(s)H(s) = \frac{K(s+\lambda)(s+\xi)}{s(s+\eta_1)(s+\eta_3)} \quad (4-6)$$

where

$$K = \frac{\alpha_1 b^P_o}{\beta} \quad (4-7)$$

and

$$\xi = \eta_3 + \eta_2 \alpha_2 / \alpha_1 \quad (4-8)$$

Stability is assured if the roots of the characteristic equation, $1 + GH = 0$, are in the left half plane.

The characteristic equation is

$$D(s) = s^3 + (K + \eta_1 + \eta_3)s^2 + \eta_1\eta_3 + K\lambda + K\xi s + K\lambda\xi = 0 \quad (4-9)$$

By the Routh criterion, the roots lie in the left half plane if the following three inequalities are satisfied.

$$K + \eta_1 + \eta_3 > 0 \quad (4-10)$$

$$\eta_1\eta_3 + K\lambda + K\xi - \frac{K\lambda\xi}{K + \eta_1 + \eta_3} > 0 \quad (4-11)$$

$$K\lambda\xi > 0 \quad (4-12)$$

The variables used for the coordinates of the parameter spaces are defined by letting

$$x = \frac{\alpha_1 b^P_o}{\beta} = K \quad (4-13)$$

and

$$y = \frac{\eta_2}{\eta_3} \frac{\alpha_2 b^P_o}{\beta} \quad (4-14)$$

The three Routh inequalities in terms of x and y are

$$x + \eta_1 + \eta_3 > 0 \quad (4-15)$$

$$\begin{aligned}
& (\lambda + \eta_3)x^2 + \eta_3xy + (\eta_3^2 + \eta_1\lambda + 2\eta_1\eta_3)x + \eta_3(\eta_1 + \eta_3 - \lambda)y + \\
& + \eta_1\eta_3(\eta_1 + \eta_3) > 0
\end{aligned}
\tag{4-16}$$

$$x + y > 0 \tag{4-17}$$

The system is stable in the region of the x, y plane for which the three inequalities are satisfied. If the inequality signs are replaced by equal signs, the equations denote the boundaries of stability. It is noted that the third condition is the requirement of a negative steady-state power coefficient where

$$-H(0) = \frac{-b}{\beta\eta_1\eta_3} (\eta_3\alpha_1 + \eta_2\alpha_2) \tag{4-18}$$

The second condition is a hyperbola. A part of the hyperbola is a resonance line where the characteristic equation has a pair of roots that are pure imaginary. When the other root of the characteristic equation is negative, that part of the hyperbola is the locus of sustained oscillations or centers.

The asymptotes of the hyperbola are given by

$$x + \eta_1 + \eta_3 - \lambda = 0 \tag{4-19}$$

$$(\lambda + \eta_3)x + \eta_3y + \lambda^2 + \eta_1\eta_3 = 0 \tag{4-20}$$

The power level on the resonance line is called the critical power level.

The coordinates of the parameter space, x and y , are defined such that Routh conditions are functions of only two parameters, η_1 and η_3 . λ is a nuclear constant and changes little. This is especially convenient in parametric studies since only η_1 and η_3 require changing.

The coupling parameter, η_2 , appears only in the definition of y so that changes in η_2 are merely a scale factor for y .

Considering now the system with effective lifetime neutron kinetics, equation 4-2, the open loop transfer function becomes

$$Z(s)H(s) = \frac{K(s+\xi)}{s(s+\eta_1)(s+\eta_3)} \quad (4-21)$$

The characteristic equation is

$$D(s) = s^3 + (\eta_1 + \eta_3)s^2 + (K\lambda + \eta_1\eta_3)s + K\lambda\xi = 0 \quad (4-22)$$

where K , ξ , x and y retain their original definitions in equations 4-7, 4-8, 4-13 and 4-14.

Application of the Routh criterion to this characteristic equation yields the following requirements

$$\eta_1 + \eta_3 > 0 \quad (4-23)$$

$$\eta_1\eta_3 + K\lambda - \frac{K\lambda\xi}{\eta_1 + \eta_3} > 0 \quad (4-24)$$

and

$$K\lambda\xi > 0 \quad (4-25)$$

The first condition is true in any case as the time constants are always positive. The second condition is the resonance line which corresponds to equation 4-11 in the prompt jump approximation. The third condition is the same as equation 4-12 which is the negative steady state power coefficient requirement. The two conditions using x and y are

$$y < \frac{\eta_1}{\eta_3} x + \frac{\eta_1(\eta_1 + \eta_3)}{\lambda} \quad (4-26)$$

and

$$x + y > 0 \quad (4-27)$$

If the kinetics equation for prompt neutrons only, equation 3-12, is used to find the Routh conditions, the resonance line is

$$y = \frac{\eta_1}{\eta_3} x + \frac{\lambda \eta_1 (\eta_1 + \eta_3)}{\beta}$$

Comparing the result to equation 4-26, it is seen that both equations have the same slope. However, the y intercept for the case of prompt neutrons only is much smaller. Thus, the effect of neglecting delayed neutrons is to severely restrict the region of linear stability; in fact, the resonance line for prompt neutrons nearly coincides with the nonlinear boundary for asymptotic stability which is discussed in the nonlinear stability section.

The parameters selected for the three cases are listed in Table 4.1. They are chosen to illustrate several aspects of stability as well as to compare the two kinetics models, prompt jump and effective lifetime. The parameter spaces are shown in Figure 4.7 for the three cases.

The stability boundaries in the parameter spaces are shown as heavy lines with the resonance lines for prompt jump and effective lifetime, equations 4-16 and 4-26, marked P. J. and E. L. respectively.

The first quadrant corresponds to both feedbacks negative the second quadrant is for a positive prompt and a negative delayed feedback, and the fourth quadrant is for a negative prompt and a positive delayed feedback. The stability boundaries do not enter the

Table 4.1
Parameters for Case Studies

Parameter	Case I	Case II	Case III
$\lambda (\text{Sec}^{-1})$	0.1	0.1	0.077
β	0.01	0.01	0.007
$b \text{ (}^\circ\text{C/Mw-Sec)}$	100	5	0.26 ($\ell^\circ\text{C/kwSec}$)
$P_o \text{ (Mw)}$	10	10	100 (kw/ ℓ)
$\eta_1 (\text{Sec}^{-1})$	0.5	0.025	0.031
$\eta_2 (\text{Sec}^{-1})$	0.25	0.0125	0.55
$\eta_3 (\text{Sec}^{-1})$	0.2	0.01	8.65
$\alpha_1 (/^\circ\text{C})$	$-\frac{1}{3} \times 10^{-6}$	$-\frac{1}{3} \times 10^{-6}$	$-\frac{1}{3} \times 10^{-6}$
$\alpha_2 (/^\circ\text{C})$	Various	Various	Various

third quadrant where both feedbacks are positive. In all the cases and for both kinetics models, one half of the fourth quadrant is unconditionally stable. Conditional stability exists when by virtue of changing the power level only, the system becomes unstable. An increase in power level for a given set of parameters moves the point out on a radial line from the origin.

In all cases the effective lifetime model is conditionally stable in both the first and second quadrants. Lellouche (1966) noted that with

Figure 4.7 Parameter Spaces with Linear Stability Boundaries for Prompt Jump and Effective Lifetime Neutron Kinetics.

the effective lifetime model, a prompt positive feedback may be more stabilizing than a large negative delayed feedback. Also, stability is not assured even with both feedbacks negative since the resonance line enters the first quadrant. These two effects occur in all of the effective lifetime cases using the coupled physical system. It should be noted that in the diagonalized system neither of these two effects is observed which can be very misleading. In the physical system, if instability occurs with either a negative or a positive delayed feedback, the system can be made stable by increasing the negative prompt feedback.

In all cases the prompt jump approximation is conditionally stable in the second quadrant and is unconditionally stable in the first quadrant, except for Case II. In Case II the resonance line enters the first quadrant and yields an unusual type of conditional stability discussed by Meenan and Hetrick (1966). The system is stable at low and high power levels while it is unstable at intermediate power levels. The necessary requirement for the resonance line to enter the first quadrant is determined from the asymptote of the hyperbola, equation 4-19, to be

$$\eta_1 + \eta_3 < \lambda \quad (4-28)$$

This case illustrates that both of Lellouche's comments on the stability of the coupled system with effective lifetime kinetics are applicable to the prompt jump approximation providing equation 4-28 is satisfied. If equation 4-28 is not satisfied, then neither of his observations apply. In the diagonalized system for any case, the resonance line does not enter the first quadrant which is similar to the effective lifetime results.

In comparing the stability regions of the two models, the effective lifetime is in general the more restrictive in that it does not overestimate the actual stability regions. This is desirable since the effective lifetime kinetics is an approximation to the prompt jump kinetics. However, the cross-hatched area in the parameter space for Case I is a region which is unstable with prompt jump kinetics and stable with effective lifetime kinetics. This illustrates an instance when the effective lifetime stability region is not a subset of the prompt jump stability region. The condition for this has been derived (Hetrick 1967) and is given by

$$\frac{1}{\lambda} > \frac{1}{\eta_1} + \frac{1}{\eta_3} - \frac{1}{\eta_1 + \eta_3} \quad (4-29)$$

As long as one of the η 's is nearly equal to or less than λ this condition is not satisfied.

When the condition is satisfied, care is warranted in the use of the effective lifetime model since it no longer represents a conservative system when compared to the prompt jump approximation.

In all three of the cases, except for the region just discussed in Case I, the effective lifetime model is overly restrictive with Case III being a prime example.

In order to check on the validity of the prompt jump approximation, the transfer function for the one delayed group, second order neutron kinetics equations was used in the system equations to delineate the stability boundaries in the parameter space. In the inequalities resulting from the Routh criterion, terms of order λ/β were present in addition to the terms given in equations 4-15, 4-16, and 4-17. For very

short neutron lifetimes characteristic of fast reactors, ℓ/β is approximately 10^{-5} . Thus, the agreement between the prompt jump approximation and the second order neutron kinetics is very close. Also, in the limit as $\ell/\beta \rightarrow 0$, the inequalities coincided with those found using prompt jump kinetics. The equations for the Routh criterion using the one delayed group neutron kinetics are given in Appendix B.

Nonlinear Stability

In this section nonlinear stability is analyzed for both reactor kinetics models. Nonlinear criteria are applied to the thermionic reactor system and the results are displayed in the parameter space.

Regions exhibiting finite escape times are described, and regions where bounded solutions exist are discussed. The possible regions where limit cycles may be found are also presented.

Welton's Criterion

Welton's criterion (Welton 1955) is a sufficient criterion for asymptotic stability in the large for nonlinear systems with linear feedback. The criterion is

$$\operatorname{Re}[H(j\omega)] > 0 \quad \text{for all } \omega, \quad (4-30)$$

where the feedback is summed negatively as shown in the block diagram, Figure 4.6. The criterion insures negative feedback at all frequencies and as such may be too restrictive. For instance, the band width of the system is not considered since the criterion is independent of $Z(s)$, the forward loop transfer function. This property is evident in this study. The system with either prompt jump or effective lifetime kinetics

has the same region of asymptotic stability derived by Welton's criterion because the feedback transfer function is identical in both models. Since a large variance exists in the regions of linear stability for the models, it seems reasonable to expect a variance in the regions of nonlinear asymptotic stability.

Applying Welton's criterion to the feedback transfer function for the thermionic reactor model, equation 4-6,

$$H(j\omega) = \frac{K(j\omega + \xi)}{(j\omega + \eta_1)(j\omega + \eta_3)} \quad (4-6)$$

yields two conditions

$$K\xi > 0 \quad (4-31)$$

and

$$\eta_1 + \eta_3 > \xi \quad (4-32)$$

Writing the conditions in terms of x and y gives the boundaries in the parameter space.

$$x + y = 0 \quad (4-17)$$

$$y = \frac{\eta_1}{\eta_3} x \quad (4-33)$$

The first condition is the same as equation 4-17 in the Routh array which is the requirement of a negative steady-state power coefficient.

The second condition is a line through the origin parallel to the resonance line of the effective lifetime model.

The regions of asymptotic stability in the large are shown in Figure 4.8 for the first case. The other cases are similar. Limit cycles are not present in this region of the parameter space.

Figure 4.8 Parameter Space for Case I Showing Boundaries for Various Aspects of Nonlinear Stability.

Using the second order neutron kinetics equations, Brehm (Brehm, et al 1967) has shown that Liapunov's method for asymptotic stability in the large yields the identical region resulting from Welton's criterion. Thus far, efforts to extend the region of asymptotic stability using Liapunov's theory have proved unsuccessful.

Boundedness

The region of bounded solutions includes the region defined by Welton's criterion as well as possible other regions. Akcasu and Noble (1966) derived a criterion for boundedness in reactor systems having arbitrary linear feedback with any size initial disturbance. It requires that $H(s)$ does not vanish on the positive real axis and $H(0) > 0$. This is only valid for solutions of the reactor kinetics equations that are Laplace transformable. This excludes solutions of the prompt jump approximation when ρ approaches 1 as were discussed in Chapter III.

$H(0) > 0$ is the requirement for a negative steady-state power coefficient, equations 4-17 and 4-18, which is necessary for a finite equilibrium power level. Equation 4-17, then, is one of the boundaries for bounded solutions and is repeated here.

$$x + y > 0 \quad (4-17)$$

The zero of $H(s)$ is real and given by

$$s = -\xi = -\eta_3 \frac{x+y}{x} \quad (4-34)$$

The numerator of equation 4-34, $x + y$, is positive from the condition $H(0) > 0$, and if $x > 0$, then $H(s)$ has no zeros on the positive real axis and the second requirement is satisfied.

Thus, the region in the parameter space for bounded solutions is defined by the Y axis and the $x + y = 0$ line. This region is shown in Figure 4.8.

The region of bounded solutions includes regions that are linearly unstable in both of the kinetics models. Solutions in these regions exhibit at least one stable limit cycle. In the linearly stable region not included by Welton's criterion, the solution is either asymptotically stable or exhibits limit cycles in pairs, one being stable and one being unstable. Boundedness does not imply that a reactor can be designed to operate in this region. Oscillations or stable limit cycles with large swings in power can not be tolerated in a reactor system. Reactor operation is feasible in the linearly unstable region if the amplitude of the stable limit cycle is on the order of the noise level present in the system.

Finite Escape Time

Finite escape time refer to trajectories that cease to exist at some finite time. Finite escape time trajectories are independent of linear stability. For example, in the linearly stable region, the nonlinear solution is stable for small disturbances; however, for a sufficiently large disturbance the trajectory may diverge and become unbounded at a finite time. The same effect may occur in the linearly unstable region for any size of disturbance. If stable limit cycles are present, a disturbance larger than the amplitude of the limit cycle is required for the trajectory to diverge.

The necessary condition (Akcasu and Noble 1966) for the existence of solutions with finite escape times is that $h(t) < 0$ for $t = 0$. This implies that a sudden increase in power causes an initial increase in reactivity.

Using the initial value theorem with the feedback equation

$$h(0) = \lim_{s \rightarrow \infty} \frac{K s (s+\xi)}{(s+\eta_1)(s+\eta_3)} = K = \frac{\alpha_1 b P_o}{\beta} \quad (4-35)$$

and the requirement for finite escape times becomes

$$\alpha_1 < 0 \quad (4-36)$$

Thus, $h(0)$ is less than zero for positive prompt feedback. In the parameter space, negative α corresponds to negative x . Thus with positive prompt feedback an initial disturbance of sufficient size can be selected so that the power diverges completely before the negative delayed feedback can become effective.

Solutions using the prompt jump approximation are excluded in this criteria because of the singularity in the differential equation. Because of the singularity, it is possible to obtain solutions with the prompt jump kinetics which diverge at a finite time even with $h(0) > 0$. The reactor model is no longer valid for such solutions.

The necessary condition holds for effective lifetime kinetics and the $n + 1$ group reactor kinetics equations.

The regions of finite escape times, restricted to reactors with negative power coefficients, are also shown in Figure 4.8 for Case I.

Limit Cycles

From the above analysis, the only regions excluding the possible existence of limit cycles are the region defined by Welton's criterion and the region of positive steady-state power coefficients. It is difficult to be more specific as to the location of limit cycles because there is no simple sufficiency test for regions where limit cycles exist. If a region could be defined for the existence of limit cycles which does not bound on the region of asymptotic stability, then a larger region of asymptotic stability may be inferred. In the first and second approximation of the successive approximations derived in Chapter II it is possible to define regions in the parameter space for existence of limit cycles which does effectively extend the region of asymptotic stability in the large.

In the results of the various cases using prompt jump kinetics limit cycles appear on either side of the resonance line and in either the first or second quadrant. In general, for a particular case the limit cycle appears on one side or the other of a point on the resonance line but not on both sides. This is because the effect of the nonlinearity does not change qualitatively when the resonance line is crossed. An unstable limit cycle is caused by a destabilizing force in a linearly stable region while a stable limit cycle is caused by a stabilizing force in a linearly unstable region.

In the effective lifetime model only stable limit cycles have been observed. These are found also in both the first and second quadrants. This is in agreement with the literature (Akcasu and Shotkin 1967). Shotkin (1964) has found a boundary for limit cycles in the

effective lifetime model. In the parameter space the boundary line is given by

$$x = - \frac{\eta_1 \eta_3}{\lambda} \quad (4-37)$$

This line is where the coefficient of the linear term in the characteristic equation vanishes. No limit cycles exist in the region to the left of this line, i.e., for large prompt positive feedbacks. The line has been verified on the computer as a boundary line for stable limit cycles with the effective lifetime model in Case I.

The regions of the parameter space where limit cycles are predicted by the Fourier method will be presented in the next chapter together with solutions of the exact equations for selected points in the parameter space.

CHAPTER V

APPLICATION TO THE TWO-REGION REACTOR MODEL

The results of the Fourier method with both neutron kinetics models are presented along with the exact solutions for three different cases. The exact solutions are derived from analog and digital computation.

Results of the Successive Approximations

The results of the various approximations of the Fourier method are given and compared. With the first approximation and the second approximation without a_2 for the transfer functions used, regions in the parameter space are determined analytically which predict the locations of limit cycles. This is possible with these approximations because the frequency can be expressed in terms of the system parameters. It is unfortunate that the two approximations do not yield more accurate results since direct analytical determination of regions in the parameter space is the most convenient and desirable approach.

In higher approximations the values of frequency result from the numerical solution of a polynomial for a given set of parameters. It is necessary to compute numerical values of the frequency for each point in the parameter space. However, with high speed computing facilities, regions of the parameter space can be scanned quite easily. To perform this operation, a digital program, FREAMP, was written which

calculates the limit cycle frequencies and amplitudes for the various approximations of the Fourier method. FREAMP is described in Appendix C.

The results of the analysis are presented in several ways. For the infinite approximation the amplitude squared of the reactivity and power are plotted versus the equilibrium power level. For the first and second approximations discussed above, the parameter space is used as well as plots of amplitude squared.

First Approximation

The first approximation is analyzed in this section using prompt jump kinetics. With the effective lifetime model, the first approximation yields no results because $\text{Re } Z_1$ which multiplies the denominator of the approximation, is zero. The first approximation in reactivity from Chapter II is

$$\frac{|a_1|^2}{P_o^2} = \frac{1 + Z_1 H_1}{2Z_1 H_1 [Z_1 - H_o^{-1}] \text{Re } Z_1} \quad (2-26)$$

Z_1 and H_1 are the system transfer functions for reactivity in dollars with the Laplace variable (s) replaced by (i ω).

$$Z_1 = \frac{P_o (i\omega + \lambda)}{i\omega} \quad (4-1)$$

$$H_1 = \frac{K(i\omega + \xi)}{P_o (i\omega + \eta_1)(i\omega + \eta_3)} \quad (4-6)$$

and

$$\text{Re } Z_1 = P_o \quad (5-1)$$

$$H_o^{-1} = \frac{P_o \eta_1 \eta_3}{K\xi} \quad (5-2)$$

Substitution of equations 4-1, 4-6, 5-1, and 5-2 into equation 2-26 gives

$$\begin{aligned}
|a_1|^2 &= K\xi \{ \omega^4 - [\eta_1 \eta_3 + K(\lambda + \xi)] \omega^2 \} + i\omega \{ -(\eta_1 + \eta_3 + K) \omega^2 + K\lambda \xi \} \cdot \\
& \{ \{-2K[2K\lambda\xi + K\xi^2 - \eta_1 \eta_3(\lambda + \xi)] \omega^2 + 2K^2 \lambda^2 \xi^2 \} + i\omega \{ 2K[\eta_1 \eta_3 - K\xi] \omega^2 + \\
& + 2K\lambda\xi[2K\xi + K\lambda - \eta_1 \eta_3] \} \}^{-1} \quad (5-3)
\end{aligned}$$

As $|a_1|^2$ is a real quantity, the complex numerator and denominator must be in phase so that $|a_1|^2$ can be written either as

$$|a_1|^2 = - \frac{K\xi\omega^2}{2K} \frac{[\omega^2 - \eta_1 \eta_3 - K(\lambda + \xi)]}{\{ [2K\lambda\xi + K\xi^2 - \eta_1 \eta_3(\lambda + \xi)] \omega^2 - K\lambda^2 \xi^2 \}} \quad (5-4)$$

or

$$|a_1|^2 = - \frac{K\xi}{2K} \frac{[(\eta_1 + \eta_3 + K) \omega^2 - K\lambda \xi]}{[(\eta_1 \eta_3 - K\xi) \omega^2 + \lambda \xi (2K\xi + K\lambda - \eta_1 \eta_3)]} \quad (5-5)$$

The square of the angular frequency, ω^2 , is found by setting the imaginary component of equation 5-3 to zero and solving for the roots.

$$\begin{aligned}
& \{ K\xi - \eta_1 \eta_3 \} \omega^6 + \{ (\eta_1 + \eta_3 + K) [K\lambda \xi + (\lambda + \xi) (K\xi - \eta_1 \eta_3)] - \\
& - (K\xi - \eta_1 \eta_3) (\eta_1 \eta_3 + K\lambda + K\xi) \} \omega^4 + K\lambda \xi \{ (\eta_1 \eta_3 + K\lambda + K\xi) (\lambda + 2\xi) - K\lambda \xi - \\
& - (\lambda + \xi) (K\xi - \eta_1 \eta_3) - \lambda \xi (\eta_1 + \eta_3 + K) \} \omega^2 + K^2 \lambda^3 \xi^3 = 0 \quad (5-6)
\end{aligned}$$

The regions in the parameter space where the approximation predicts limit cycles are found by examining equation 5-3 for the positive values of $|a_1|^2$. Boundaries may be sought by locating limiting values of $|a_1|^2$, that is, where $|a_1|^2$ becomes zero or infinite.

To find the locus of points where $|a_1|^2 = 0$, the numerator of equation 5-3 is set equal to zero which yields two conditions. One condition is derived by setting $K\xi$ equal to zero, which corresponds to the zero power coefficient line in parameter space, $x + y = 0$. The

second condition is found by examining the term in the brackets of the numerator. Since this term is complex, both the real and imaginary parts must be zero. Setting the real part equal to zero and solving for ω^2 ,

$$\omega^2 = \eta_1 \eta_3 + K(\lambda + \xi). \quad (5-7)$$

Substituting this frequency into the equation obtained by setting the imaginary part equal to zero gives

$$[\eta_1 \eta_3 + K(\lambda + \xi)][\eta_1 + \eta_3 + K] - K\lambda\xi = 0 \quad (4-11)$$

This is the equation in the parameter space for the hyperbolic resonance line corresponding to one of the Routh conditions for linear stability.

To find the locus of points where $|a_1|^2 \rightarrow \infty$, two methods may be used. First, the imaginary component of $|a_1|^2$ may be used to find infinite values of the frequency. This locates the infinite boundary for $|a_1|^2$ because as $\omega^2 \rightarrow \infty$, $|a_1|^2 \rightarrow \infty$.

The other method consists of setting the denominator of equation 5-3 equal to zero. This is used to provide one condition which is, $K = 0$ or $x = 0$. The other condition from the frequency polynomial is the vanishing of the coefficient of the first term in equation 5-6:

$$K\xi - \eta_1 \eta_3 = 0 \quad (5-8)$$

which corresponds to a line in the parameter space

$$y = -x + \eta_1 \quad (5-9)$$

The region outlined by the reactivity amplitude in the parameter space for the existence of the limit cycles for Case I is shown by a shaded area in Figure 5.1. To the right of the resonance line where linear

Figure 5.1 Parameter Space for First Approximation with Prompt Jump, Case I.

Figure 5.2 Representative Oscillation Amplitudes for First Approximation, Prompt Jump, Case I.

stability exists, the shaded area defines a region of unstable limit cycles, and to the left of the resonance line where linear instability exists, the shaded region is for the existence of stable limit cycles.

In the same manner the first approximation in power, $|A_1|^2$, is analyzed. The regions in the parameter space are solved for and compared to the results of $|a_1|^2$. Only the areas common to positive values of $|a_1|^2$ and $|A_1|^2$ are valid for predicting the existence of limit cycles.

The first approximation in power is given by

$$|A_1|^2 = \frac{1 + Z_1 H_1}{2Z_1 H_1 [-H_1^{-1} - H_0^{-1}] \operatorname{Re} Z_1^{-1}} \quad (2-45)$$

where

$$\operatorname{Re} Z_1^{-1} = \frac{\omega^2}{P_o(\omega^2 + \lambda^2)} \quad (5-10)$$

Substitution of equations 4-1, 4-6, 5-2 and 5-10 into the power amplitude, equation 2-45, gives

$$|A_1|^2 = K\xi \{ \omega^4 + [\lambda(\eta_1 + \eta_3) - \eta_1 \eta_3 - K\xi] \omega^2 - K\lambda^2 \xi \} + i\omega \{ [\lambda - K - (\eta_1 + \eta_3)] \omega^2 - [K\lambda^2 + \lambda\eta_1 \eta_3] \} \cdot \quad (5-11)$$

$$(2K\omega^2 \{ [-\xi\omega^2 + 2\xi\eta_1 \eta_3] + i\omega [\xi(\eta_1 + \eta_3) + \eta_1 \eta_3] \})^{-1}$$

Since $|A_1|^2$ is a real quantity, the amplitude is obtained from

either

$$|A_1|^2 = \frac{\omega^4 + [\lambda(\eta_1 + \eta_3) - \eta_1 \eta_3 - K\xi] \omega^2 - K\lambda^2 \xi}{2\omega^2 [-\omega^2 + 2\eta_1 \eta_3]} \quad (5-12)$$

or

$$A_1^2 = \frac{K\xi [(\lambda - K - \eta_1 - \eta_3) \omega^2 - (K\lambda^2 + \lambda\eta_1 \eta_3)]}{2K\omega^2 [\xi(\eta_1 + \eta_3) + \eta_1 \eta_3]} \quad (5-13)$$

The angular frequency squared is found by setting the imaginary component of equation 5-11 to zero and solving for the roots. The polynomial is quadratic in ω^2 .

$$\begin{aligned} & \{K^2 \xi - K \eta_1 \eta_3 - K \lambda \xi\} \omega^4 + \{K^2 \xi^2 (\eta_1 + \eta_3) + K \lambda \xi (\eta_1 \eta_3 + K \lambda - \eta_1^2 - \eta_3^2) + \\ & + K \eta_1 \eta_3 [\eta_1 \eta_3 - (\eta_1 + \eta_3)(\lambda + \xi) - K \xi]\} \omega^2 + \\ & + \{K \lambda \xi [K \lambda \xi (\eta_1 + \eta_3) + K \lambda \eta_1 \eta_3 - 2(\eta_1 \eta_3)^2]\} = 0 \end{aligned} \quad (5-14)$$

The boundaries for zero amplitude $|A_1|^2$ coincide with the boundaries for $|a_1|^2$ equal to zero. They are the hyperbolic resonance line and the zero power coefficient line. The boundaries for infinite $|A_1|^2$ are sought from the denominator since the expressions for $|A_1|^2$, equations 5-12 and 5-13, do not become infinite for infinite ω^2 . The denominator yields two conditions that must be satisfied simultaneously for $|A_1|^2$ to be infinite:

$$\omega^2 = 2\eta_1 \eta_3 \quad (5-15)$$

and

$$\xi = -\frac{\eta_1 \eta_3}{\eta_1 + \eta_3} \quad (5-16)$$

These two conditions satisfy the frequency polynomial, equation 5-14, which indicates that ω^2 has a constant value of $2\eta_1 \eta_3$ along the line of constant ξ given by equation 5-16. This line is the boundary of infinite $|A_1|^2$ and is shown in Figure 5.1 for Case I. Positive values of $|A_1|^2$ predicting unstable limit cycles occur to the right of the resonance line bounded by the zero power coefficient line, and the infinite $|A_1|^2$ boundary. Stable limit cycles are predicted to the left

of the resonance line between the resonance line and the infinite boundary. The regions common to both $|a_1|^2$ and $|A_1|^2$ are depicted by heavy lines and indicate that for this case unstable limit cycles exist for $\xi > -0.143$ with $P_o < P_c$ and stable limit cycles exist for $\xi < -0.16$ with $P_o > P_c$, while for $-0.143 > \xi > -0.16$ stable limit exist only for equilibrium power levels sufficiently greater than the critical power level.

In the exact solution, the only existing periodic solutions are unstable limit cycles which are located near the resonance line in and above the region predicted by the first approximation. Furthermore, the exact solution indicates that limit cycles are not found everywhere in the common region lying to the right of the resonance line. In part of the region, all of the eigenvalues are real and linear oscillations do not exist.

This approximation illustrates the importance of requiring both the power and reactivity to agree in frequency and amplitude since there is a wide variance in the resulting regions.

In Figure 5.2, representative amplitudes of the power and reactivity are plotted versus the equilibrium power level for several values of ξ as shown in Figure 5.1. Holding ξ constant fixes the ratio of the feedback coefficients, and increasing the power level corresponds to moving outward on a radial line in the parameter space.

The critical power level is the equilibrium power level on the resonance line and is denoted by a subscript c.

The power amplitude is shown as a broken line and the reactivity amplitude as a solid line. The common areas denoting limit cycles are shown by a heavy line on the equilibrium power axis.

Second Approximation without a_2

This approximation is more complex than the first approximation in that second harmonic frequencies are included in the forward loop transfer function. The approximation also yields results for both neutron kinetics models.

Effective Lifetime Model. The second approximation without a_2 yields results with the effective lifetime neutron kinetics since there are other terms in the denominator besides those multiplied by $\text{Re } Z_1$.

The second approximation without a_2 for the reactivity is given by

$$\frac{|a_1|^2}{P_o^2} = \frac{1 + Z_1 H_1}{Z_1 H_1 [2(Z_1 - H_o^{-1}) \text{Re } Z_1 - Z_1 Z_2]} \quad (2-33)$$

where for the effective lifetime model

$$Z_1 = \frac{\lambda P_o}{i\omega} \quad , \quad (4-2)$$

$$Z_2 = \frac{\lambda P_o}{2i\omega} \quad , \quad (5-17)$$

and

$$\text{Re } Z_1 = 0 \quad . \quad (5-18)$$

H_1 and H_o^{-1} are the same as in the first approximation, equations 4-6 and 5-2 respectively.

Substituting equations 4-2, 4-6, 5-2, 5-17 and 5-18 into the second approximation, equation 2-33, yields

$$|a_1|^2 = \frac{-2\omega^2 \{[(\eta_1 + \eta_3)\omega^2 - K\lambda\xi] + i\omega[\omega^2 - (\eta_1\eta_3 + K\lambda)]\}}{K\lambda^3\xi + i\omega K\lambda^3} \quad (5-19)$$

The amplitude is obtained from either expression

$$|a_1|^2 = \frac{-2\omega^2 [(\eta_1 + \eta_3)\omega^2 - K\lambda\xi]}{K\lambda^3\xi} \quad (5-20)$$

or

$$|a_1|^2 = \frac{-2\omega^2 [\omega^2 - (\eta_1\eta_3 + K\lambda)]}{K\lambda^3} \quad (5-21)$$

In finding the limiting values of $|a_1|^2$, the numerator is examined for zero values. From the imaginary component of the numerator,

$$\omega^2 = \eta_1\eta_3 + K\lambda$$

and substituting this into the real component yields

$$(\eta_1\eta_3 + K\lambda)(\eta_1 + \eta_3) - K\lambda\xi = 0 \quad (4-24)$$

which corresponds to the Routh condition for the resonance line in the parameter space. In examining $|a_1|^2$ for infinite values, the denominator yields no information and ω^2 is infinite on Welton's line, $y = \eta_1/\eta_3 x$. Checking the expressions for $|a_1|^2$, equations 5-20 and 5-21, infinite ω^2 leads to negative values of $|a_1|^2$ which provide no information concerning limit cycles.

In this model, however, the frequency can be solved for explicitly in terms of the system parameters. The values of frequency which are positive are then used to find positive reactivity amplitudes. The imaginary component of the second approximation, equation 5-19, gives the following value for the frequency.

$$\omega^2 = \frac{\xi \eta_1 \eta_3}{\xi - (\eta_1 + \eta_3)} \quad (5-22)$$

ω^2 is positive for all negative ξ and for positive ξ greater than $\eta_1 + \eta_3$. The region for positive ω^2 is bounded by the zero power coefficient line and Welton's line, $y = \eta_1 / \eta_3 x$. From equation 5-22, it is noted that ω^2 has a constant value on a line of constant ξ .

Either expression for $|a_1|^2$ may be used to check for positive values considering ω^2 to be positive. For equation 5-20, the requirement is

$$\omega^2 < \frac{K\lambda\xi}{\eta_1 + \eta_3} \quad (5-23)$$

which, using equation 5-22, leads to the resonance line in the parameter space. Equation 5-21 for the amplitude gives the same result.

The region predicted for limit cycles is between the resonance line and the zero power coefficient line in the linearly unstable portion of the parameter space which indicates stable limit cycles. The region is displayed in the Case I parameter space for the effective lifetime model in Figure 5.3. The results are in fair agreement with the exact solution in that stable limit cycles are produced with the effective lifetime model. However, the vertical line through the intersection of the resonance line and the zero power coefficient line is a boundary for limit cycles which is discussed in Chapter IV (Shotkin 1964). From the figure, it is seen that the second approximation predicts too large a region.

The second approximation without a_2 for the power is given by

$$|A_1|^2 = \frac{1 + Z_1 H_1}{Z_1 H_1 [-2(H_1^{-1} + H_0^{-1}) \operatorname{Re} Z_1^{-1} + H_{-1} Z_2]} \quad (5-24)$$

Figure 5.3 Parameter Space for Case I Showing Regions Predicted by the Second Approximation Without a_2 for Existence of Limit Cycles.

Substituting equations 4-2, 4-6, 5-17 and 5-18 into the second approximation, equation 5-24, yields

$$|A_1|^2 = \frac{-2\omega^2\{\omega^4 - [K\lambda - \eta_1^2 - \eta_3^2]\omega^2 + [\eta_1\eta_3(K\lambda + \eta_1\eta_3) - K\lambda\xi(\eta_1 + \eta_3)]\}}{K^2\lambda^2(\omega^2 + \xi^2)} - \frac{2i\omega\{K\lambda(\xi - \eta_1 - \eta_3)\omega^2 - K\lambda\xi\eta_1\eta_3\}}{K^2\lambda^2(\omega^2 + \xi^2)} \quad (5-25)$$

The amplitude is given by

$$|A_1|^2 = \frac{-2\omega^2\{\omega^4 - [K\lambda - \eta_1^2 - \eta_3^2]\omega^2 + [\eta_1\eta_3(K\lambda + \eta_1\eta_3) - K\lambda\xi(\eta_1 + \eta_3)]\}}{K^2\lambda^2(\omega^2 + \xi^2)} \quad (5-26)$$

and the frequency is

$$\omega^2 = \frac{\xi\eta_1\eta_3}{\xi - (\eta_1 + \eta_3)} \quad (5-22)$$

Using equation 5-25, the boundary for $|A_1|^2$ equal to zero is the resonance line in the parameter space, equation 4-24, and the boundary for infinite $|A_1|^2$ is Welton's line. The frequency expression is the same as was derived for $|a_1|^2$, equation 5-22. From equation 5-26 it is seen that $|A_1|^2$ is negative for infinite ω^2 and thus, Welton's line is not a valid boundary for limit cycles. The results of the second approximation without a_2 are the same for both $|a_1|^2$ and $|A_1|^2$ as shown in Figure 5.3.

Prompt Jump Approximation. The procedure for analyzing the second approximation without a_2 using the prompt jump approximation is similar to the method used in the first approximation.

For the reactivity amplitude, equations 4-1, 4-6, 5-1, 5-2 and 5-17 are substituted into equation 2-33 and the limiting values are

examined. In the parameter space $|a_1|^2$ is zero on the hyperbolic resonance line and the zero power coefficient line. Infinite values of $|a_1|^2$ are found on the lines

$$K\xi - 2\eta_1\eta_3 = 0 \quad (5-27)$$

and

$$K = 0 \quad (5-28)$$

The regions in the Case I parameter space for positive reactivity amplitudes are shown in Figure 5-3.

For the power amplitude, equation 5-24 is used. Substituting in the system transfer functions yields an expression for $|A_1|^2$ which is zero on the hyperbolic resonance line and the zero power coefficient line. The amplitude does not become infinite as ω^2 goes to infinity and zero roots of the denominator are difficult to obtain. The frequency polynomial obtained from the imaginary component of $|A_1|^2$ is of sixth order in ω^2 and so the frequency can not be solved for explicitly in terms of the system parameters. It is necessary to factor the frequency polynomial numerically for specific values of x and y and then using the positive frequency roots, solve for positive values of $|A_1|^2$. This procedure must be carried out for numerous points in the parameter space in order to define regions where limit cycles exist. To perform this calculation, a digital computer program, FREAMP, was written which is described in Appendix C.

In the prompt jump parameter space, Figure 5.3, two radial lines are shown for values of ξ equal to -0.2 and -0.6. FREAMP is used to calculate the oscillation amplitudes along these two lines for both

$|A_1|^2$ and $|a_1|^2$. The results of the computation are shown in Figures 5.4 and 5.5 for ξ equal to -0.2 and -0.6 respectively.

In both figures the locus for $|a_1|^2$ has two branches which is inferred by the lines of constant ξ in the parameter space, Figure 5.3. For ξ equal to -0.2 near the resonance line, $|a_1|^2$ indicates unstable limit cycles since $P_o < P_c$. This agrees with the exact solution. The other branch of $|a_1|^2$ does not satisfy the convergence criteria of $|a_1|^2 < 1$ even for very large P_o . The power amplitude also predicts unstable limit cycles near the resonance line. There is another branch of $|A_1|^2$ which indicates stable limit cycles for $P_o \gg P_c$, but since $|a_1|^2$ is not in agreement, no oscillation is predicted.

For ξ equal to -0.6 the results are completely different. Near the resonance line for $P_o > P_c$, both $|a_1|^2$ and $|A_1|^2$ indicate stable limit cycles. This does not agree with the exact solution. For $P_o < P_c$, $|a_1|^2$ has a branch which does not meet the convergence criteria and $|A_1|^2$ exhibits a positive amplitude near the resonance line. Neither of these two amplitudes exist in the same region; thus, the amplitudes are not meaningful. For this value of ξ , the second approximation without a_2 yields an incorrect solution for limit cycles.

Second Approximation with a_2

This approximation represents an extension of the approximation used by Torlin (1966) in his example. The values of ξ used in these examples are the same as used in the second approximation without a_2 as shown in Figure 5.3.

Effective Lifetime Model. The oscillation amplitudes for the effective lifetime model are shown in Figure 5.6. With ξ equal to -0.2

Figure 5.4 Oscillation Amplitudes for Second Approximation Without a_2 , Prompt Jump, $\xi = -0.2$, Case I.

Figure 5.5 Oscillation Amplitudes for Second Approximation Without a_2 , Prompt Jump, $\xi = -0.6$, Case I.

Figure 5.6 Oscillation Amplitudes for Second Approximation
With a_2 , Effective Lifetime, Case I, $\xi = -0.2, -0.6$.

the common amplitudes are for $P_o < P_c$ which represents unstable limit cycles, and with ξ equal to -0.6 the common amplitudes are for $P_o > P_c$ which represents stable limit cycles. Since the exact solution for this model yields only stable limit cycles, the results are partially incorrect. This indicates that the addition of the second harmonic in reactivity does not improve the results of the second approximation without a_2 . The approximation does reduce the region where limit cycles are predicted to regions in the vicinity of the resonance line.

Prompt Jump Approximation. The oscillation amplitudes for the prompt jump approximation are shown in Figure 5.7. For ξ equal to -0.2 and -0.6 , common amplitudes exist near the resonance line for $P_o < P_c$ which indicates unstable limit cycles. This agrees with the exact solution for this case. The results of the reactivity amplitude are conservative from a stability point of view while the results of the power amplitude are in better agreement with the exact solution since unstable limit cycles exist only near the resonance line.

Infinite Approximation

The intermediate approximations between the second approximation and the infinite approximation are not discussed since the infinite approximation is no more complex than the third approximation, and less complex than the fourth approximation in that the order of the frequency polynomial is less. The parameter spaces for Case I, Case II and Case III are repeated in Figures 5.8, 5.9, and 5.10 respectively in order to illustrate the lines of constant ξ and the points for which the exact solutions are shown. For Case I and III, three lines, and for Case II,

Figure 5.7 Oscillation Amplitudes for Second Approximation
With a_2 , Prompt Jump Approximation, Case I, $\xi = -0.2$,
-0.6.

Figure 5.8 Parameter Space for Case I Showing Values of ξ and Example Points for Exact Solutions.

Figure 5.9 Parameter Space for Case II Showing Values of ξ and Points for Exact Solutions.

Figure 5.10 Parameter Space for Case III Showing Values of ξ and Example Points for Exact Solutions.

two lines of constant ξ are examined for each of the neutron kinetics models. In Figure 5.9 for Case II, the region about the origin is expanded in the inset.

In the graphs for Case I and II all oscillation amplitudes resulting from positive frequencies, whether positive, negative, or resulting from subharmonic frequencies, are shown. In Case III, only the valid amplitudes are included. The amplitudes arising from subharmonic frequencies are considered non-physical. Smets (1965) has proven this rigorously for the effective lifetime model. When shown in the graphs, the amplitudes resulting from the subharmonic are given by dotted lines.

After the three cases are examined, the results for Case I of the second approximation without a_2 , the second approximation with a_2 , and the infinite approximation are plotted on the same graph for comparison. A complete index of the graphs for the oscillation amplitudes is given in Table 5.1.

Case I. Effective Lifetime. The oscillation amplitudes are calculated for ξ values of -0.1, -0.2 and -0.6 and are shown in Figures 5.11, 5.12 and 5.13 respectively. For the three examples near the resonance line, a common amplitude exists for $P_o > P_c$ which indicates the presence of stable limit cycles. The region of limit cycles yielded by the exact solution is described in the section on the second approximation and is larger than indicated by this approximation. For instance, in these cases limit cycles are found for $P_c < P_o < 300$ Mw in

Table 5.1
Listing of Successive Approximation Results

Figure	Approximation	Case	Neutron* Kinetics	ξ
5-2	1st	I	P.J.	Various
5-4	2 nd w.o. a_2	I	P.J.	-0.2
5-5	2 nd w.o. a_2	I	P.J.	-0.6
5-6	2 nd w. a_2	I	E.L.	-0.2, -0.6
5-7	2 nd w. a_2	I	P.J.	-0.2, -0.6
5-11	∞	I	E.L.	-0.1
5-12	∞	I	E.L.	-0.2
5-13	∞	I	E.L.	-0.6
5-15	∞	I	P.J.	-0.1
5-16	∞	I	P.J.	-0.2
5-17	∞	I	P.J.	-0.6
5-18	∞	II	E.L.	-0.1
5-19	∞	II	E.L.	0.2
5-20	∞	II	P.J.	-0.1
5-21	∞	II	P.J.	0.2
5-22	∞	III	E.L.	-0.2, -15, 11
5-23	∞	III	P.J.	-0.2, -4, -15
5-24	2 nd , ∞	I	E.L.	-0.2, -0.6
5-25	2 nd , ∞	I	P.J.	-0.2, -0.6

*E.L.: Effective Lifetime Kinetics

P.J.: Prompt Jump Approximation

Figure 5.11 Oscillation Amplitudes for Infinite Approximation, Effective Lifetime,
Case I, $\xi = -0.1$.

Figure 5.12 Oscillation Amplitudes for Infinite Approximation, Effective Lifetime,
Case I, $\xi = -0.2$.

Figure 5.13 Oscillation Amplitudes for Infinite Approximation, Effective Lifetime,
Case I, $\xi = -0.6$.

the exact solution. Individually, the region predicted by $|a_1|^2$ is much smaller, suggesting that the predictions based on $|A_1|^2$ are more realistic.

In these examples, the agreement between the oscillation amplitudes of the subharmonic frequencies and the calculated result is inconsistent. This is substantiated in other examples which indicates that the amplitudes of the subharmonic frequencies are generally unreliable for predicting oscillations.

As an illustration, in Figure 5.14 the angular frequencies of the power and reactivity amplitudes for Case I, effective lifetime model, with $\xi = -0.2$ are plotted versus the power level. On the plot are given the amplitudes with their algebraic signs as well as the resonant frequency and the critical power level. This frequency plot corresponds to amplitudes shown in Figure 5.12.

Case I. Prompt Jump. The oscillation amplitudes with the prompt jump approximation are calculated for the same values as for the effective lifetime model, ξ equal to -0.1 , -0.2 and -0.6 , and are shown in Figures 5.15, 5.16 and 5.17. In the three cases, a common amplitude exists for $P_0 < P_c$ which indicates the presence of unstable limit cycles. For ξ equal to -0.1 the approximation predicts limit cycles over a wide range of equilibrium power levels while for ξ equal to -0.2 and -0.6 the range becomes more restricted. In the exact solution the unstable limit cycles exist only for power levels very near the resonance line, so that the results of the infinite approximation with prompt jump kinetics overestimates the region for the existence of limit cycles.

Figure 5.14 Angular Frequencies for Power and Reactivity for Case I with Effective Lifetime Kinetics, $\xi = -0.2$.

Figure 5.15 Oscillation Amplitudes for Infinite Approximation, Prompt Jump, Case I,
 $\xi = -0.1$.

Figure 5.16 Oscillation Amplitudes for Infinite Approximation, Prompt Jump, Case I,
 $\xi = -0.2$.

Figure 5.17 Oscillation Amplitudes for Infinite Approximation, Prompt Jump, Case I,
 $\xi = -0.6$.

Individually, the predictions based on $|A_1|^2$ tend to be closer to the exact solution.

In comparing the two neutron kinetics models, for ξ equal to -0.1 and -0.2 the critical equilibrium power of the effective lifetime model is greater than the critical power of the prompt jump approximation which illustrates that the effective lifetime may be linearly stable when the prompt jump approximation is not. The situation is reversed for ξ equal to -0.6.

The stable limit cycles of the effective lifetime model tend to increase the stable operating region of the linear parameter space, whereas the unstable limit cycles of the prompt jump approximation tend to reduce the stable operating region. For example, if disturbance of 25% of equilibrium power were permitted and ξ was equal to -0.6, Figure 5.17, the prompt jump reactor could operate at power levels no higher than approximately 145 Mw which is well below the critical power level for linear stability of 183 Mw. The effective lifetime reactor could operate at approximately 170 Mw, Figure 5-13, if the oscillations could be tolerated. If not, reactor operation would be limited to 161 Mw, the critical power level.

Case II. Effective Lifetime. The oscillation amplitudes for this model are shown in Figures 5.18 and 5.19 for ξ values of -0.1 and 0.2 which are in the second and first quadrants respectively. In both graphs, the common amplitudes exist for $P_o > P_c$ and indicate stable limit cycles which agrees with the exact solution. The limit cycles are predicted for a region very near to the resonance line. In the exact solution limit cycles are found for $P_c < P_o < 15$ Mw with ξ equal

Figure 5.18 Oscillation Amplitudes for Infinite Approximation, Effective Lifetime,
Case II, $\xi = -0.1$.

Figure 5.19 Oscillation Amplitudes for Infinite Approximation, Effective Lifetime,
Case II, $\xi = -0.2$.

to -0.1 and for $P_o > P_c$ for ξ equal to 0.2 . From the graphs it is evident that $|A_1|^2$ yields a more accurate result than $|a_1|^2$ with $|a_1|^2$ being very restrictive. The results found in the first quadrant are similar to those found in the second quadrant for the values of ξ examined.

Case II. Prompt Jump. The examples for the prompt approximation, Case II, are shown in Figures 5.20 and 5.21 for ξ values of -0.1 and 0.2 respectively. In the second quadrant for ξ equals -0.1 , the common amplitude is for $P_o > P_c$ and indicates stable limit cycles. The region is restricted to the vicinity of the resonance line with the reactivity amplitude being the more conservative. In the first quadrant an unusual type of conditional stability is possible (Meenan and Hetrick 1966) in that a line of constant ξ may intersect the hyperbolic resonance line twice, yielding two critical power levels. The amplitudes shown in Figure 5.21 indicate that stable limit cycles exist near each critical power level. This is in agreement with the exact solution.

In this example, the power amplitude is more restrictive than the reactivity amplitude which is the opposite of the previous examples. At some point for a smaller positive value of ξ , the amplitude plots should connect the critical power levels.

Case III. Effective Lifetime. The oscillation amplitudes are calculated for ξ values of -0.2 , -15 and 11 and are shown in Figure 5.22. Only the positive amplitudes of the fundamental frequencies are included. All examples have common areas near the resonance line for $P_o > P_c$ which indicates stable limit cycles and agrees with the exact solution. The reactivity amplitude is again more restrictive than the power amplitude

Figure 5.20 Oscillation Amplitudes for Infinite Approximation, Prompt Jump, Case II, $\xi = -0.1$.

Figure 5.21 Oscillation Amplitudes for Infinite Approximation, Prompt Jump, Case II, $\xi = 0.2$.

Figure 5.22 Oscillation Amplitudes for Infinite Approximation, Effective Lifetime, Case III.

with the power amplitude being the more accurate solution. However, for $\xi = -0.2$ the power amplitude is not a restrictive solution since limit cycles are predicted to the left of Shotkin's line which corresponds to a P_0 greater than 2.8×10^5 kw/liter in the second quadrant.

Case III. Prompt Jump. The examples of this case are for ξ values of -0.2 , -4 and -15 and are shown in Figure 5.23 where only the positive amplitudes from the fundamental frequencies are included. In all cases the amplitudes describe unstable limit cycles near the resonance line since $P_0 < P_c$. The reactivity amplitude is more restrictive than the power amplitude in these cases.

Comparing the results of the two neutron kinetics models indicates that higher nonlinearly stable power levels are permitted with the prompt jump approximation than with the effective lifetime model. For example with ξ equal to -15 , the power level in the effective lifetime model, Figure 5.22, is restricted to 1.035×10^5 kw/liter while stable operation is permitted with the prompt jump approximation to 3.8×10^5 kw/liter. In a practical sense, these power levels do not represent constraints on the system as nuclear reactors generally operate at power levels below 10^3 kw/liter.

The physical restrictions for the thermionic reactor are discussed later.

Comparison of Approximations

In Figures 5.24 and 5.25, the results of the second approximation with and without a_2 and the infinite approximation are compared for Case I of the effective lifetime model and the prompt jump approximation.

Figure 5.23 Oscillation Amplitudes for Infinite Approximation, Prompt Jump, Case III.

Figure 5.24 Comparison of Oscillation Amplitudes for (1) Second Approximation without a_2 (2) Second Approximation with a_2 and (3) Infinite Approximation, Effective Lifetime, Case I.

Figure 5.25 Comparison of Oscillation Amplitudes for (1) Second Approximation without a_2 (2) Second Approximation with a_2 and (3) Infinite Approximation, Prompt Jump, Case I.

Figure 5.24 illustrates the comparison of power amplitudes of the approximations for the effective lifetime model. The second approximation without a_2 and the infinite approximation agree with the exact solution while the second approximation with a_2 for ξ equal to -0.2 is incorrect.

Figure 5.25 shows the comparison of reactivity amplitudes of the approximations for the prompt jump approximation. The second approximation with a_2 and the infinite approximation agree with the exact solution while for this case the second approximation without a_2 for ξ equal to -0.6 is incorrect.

These four examples are representative of the pattern of results yielded by the various approximations. The first approximation and the second approximation with and without a_2 produce valid results in some examples and invalid results in others. Also, the lower approximations, in general, tend to predict larger regions in the parameter space for the existence of limit cycles than exist in the exact solution.

The infinite approximation in all of the examples of the cases examined produces valid results in the neighborhood of the resonance line, and, as opposed to the lower approximations, yields results which are generally conservative from a stability view point. The predicted regions of stable limit cycles tend to be smaller than the exact solution while the regions of unstable limit cycles tend to be larger than the exact solution.

Exact Solutions

The exact solutions were determined by analog and digital computer computation. The analog facility in the Department of Nuclear Engineering

at The University of Arizona was used to simulate the system equations. The equations and the programs are given in Appendix D.

A digital computer program, DIFEQN, was written to check on the analog solutions and also to solve for solutions which were not within the range capability of the analog computer. DIFEQN is described in Appendix E. The CDC 6400 digital computer at The University of Arizona's Computer Center was used to perform the digital computation.

The exact solutions are presented in a power-temperature state space with time as a parameter and the temperature being that of region one. The plots represent the projection of a three dimensional trajectory onto the power region-one-temperature plane. The third dimension is the temperature of region two. The initial conditions for the temperatures and power were selected to illustrate typical trajectories in the state space. The initial conditions for the power and temperature of region one can be obtained from the plots. There are also two examples of finite escape time solutions which are given as the time response of the power. The examples are summarized in Table 5.2 and the points in the parameter spaces corresponding to the examples are shown in Figures 5.8, 5.9 and 5.10. Point K is shown in Figure 5.37.

The phase plane for the system of equations is characterized by equilibrium points at shutdown, i.e., zero power and zero temperatures, and at the operating point. In this study, for simplicity, the temperature axis was translated to the operating equilibrium point. In all of the examples the shutdown equilibrium point is a saddle point. The operating equilibrium point exhibits a variety of solutions.

Table 5.2

Summary of the Exact Solutions Presented for
the Various Cases

Figure	Case	Neutron [*] Kinetics	Parameter Space Location	Type Solution
5.26	I	E.L.	A	Stable Focus
5.27	I	E.L.	B	Stable Limit Cycle
5.28	I	E.L.	C	Finite Escape Time
5.29	I	P.J.	D	Stable Focus
5.30	I	P.J.	E	Unstable Limit Cycle
5.31	I	P.J.	E	"Finite Escape Time"
5.32	II	E.L.	F	Stable Focus
5.33	II	E.L.	G	Stable Limit Cycle
5.34	II	P.J.	H	Stable Limit Cycle
5.35	III	E.L.	I	Stable Focus
5.36	III	E.L.	J	Stable Limit Cycle
5.38	III	P.J.	K	Stable Node

* E.L.: Effective Lifetime Kinetics

P.J.: Prompt Jump Approximation

The examples illustrate that both stable and unstable limit cycles are possible with delayed neutrons while only stable limit cycles exist without delayed neutrons.

Case I

Two phase planes for Case I with effective lifetime kinetics are given in Figures 5.26 and 5.27. The figures are for the same value of ξ but with different equilibrium power levels. For $P_o < P_c$ the operating equilibrium point is a stable focus as shown in Figure 5.26. There is a separatrix which connects the shutdown and operating equilibrium points. For $P_o > P_c$, the operating equilibrium point is unstable and a stable limit cycle is present. The separatrix from the shutdown equilibrium point now converges to the limit cycle.

An example of a finite escape time solution for Case I is given in Figure 5.28. The figure illustrates that for a sufficiently large disturbance the power diverges at a finite time which is dependent upon the initial disturbance.

Two phase planes for Case I with prompt jump kinetics are given in Figure 5.29 and 5.30. Both figures are for the same ξ and for $P_o < P_c$. The operating equilibrium points are stable; however, close to the resonance line an unstable limit cycle exists which is shown in Figure 5.30. The separatrix from the shutdown equilibrium point diverges in both phase planes because of ρ approaching 1. However, for a sufficiently small operating equilibrium power level and appropriate initial conditions, the separatrix would connect the shutdown and operating equilibrium points. The time response of the power for three disturbances is given in Figure

Figure 5.26 Phase Plane for Case I with Effective Lifetime Kinetics,
 $\xi = -0.2$, $P_0 = 180$ Mw.

Figure 5.27 Phase Plane for Case I with Effective Lifetime Kinetics,
 $\xi = -0.2$, $P_0 = 270$ Mw.

Figure 5.28 Time Response for Various Power Disturbances for Case I with Effective Lifetime Kinetics; $\xi = -0.1$, $P_o = 300$ Mw.

Figure 5.29 Phase Plane for Case I with Prompt Jump Kinetics;
 $\xi = -0.2$, $P_0 = 150$ Mw.

Figure 5.30 Phase Plane for Case I with Prompt Jump Kinetics;
 $\xi = -0.2$, $P_0 = 183$ Mw.

5.31 for the same parameters used in Figure 5.30. The top two graphs are outside the unstable limit cycle while the bottom graph is inside the limit cycle.

Case II

Figures 5.32 and 5.33 are phase planes for Case II with effective lifetime kinetics for $P_o < P_c$ and $P_o > P_c$ respectively. The graphs are qualitatively the same as the corresponding examples in Case I.

The results for the prompt jump approximations as shown in Figure 5.34 are completely different than the prompt jump results in Case I. The operating equilibrium point for this power level is unstable and a stable limit cycle is present. Also the separatrix from the shutdown equilibrium point converges to the limit cycle.

Case III

Figures 5.35 and 5.36 are phase planes for Case III with effective lifetime kinetics for $P_o < P_c$ and $P_o > P_c$ respectively. The results are qualitatively the same as for corresponding examples in the other two cases. The shutdown equilibrium point is not included in the graphs.

The prompt jump approximation phase plane for this case is discussed in the next section.

Thermionic Reactor System Stability

The temperature equations describing the thermionic reactor system are discussed in Chapter III. The prompt jump approximation is used for the neutron kinetics in this analysis. The possible range of

Figure 5.31 Time Response for Various Power Disturbances, Case I with Prompt Jump Kinetics; $\xi = -0.2$, $P_0 = 183$ Mw.

Figure 5.32 Phase Plane for Case II with Effective Lifetime Kinetics;
 $\xi = -0.1$, $P_0 = 3.0$ Mw.

Figure 5.33 Phase Plane for Case II with Effective Lifetime Kinetics;
 $\xi = -0.1$, $P_0 = 4.5$ Mw.

Figure 5.34 Phase Plane for Case II with Prompt Jump Kinetics
 $\xi = 0.2$, $P_0 = 6.3$ Mw.

Figure 5.35 Phase Plane for Case III with Effective Lifetime Kinetics;
 $\xi = -15$, $P_0 = 5.25 \times 10^3$ kw/l.

Figure 5.36 Phase Plane for Case III with Effective Lifetime Kinetics, $\xi = -15$, $P_0 = 1.05 \times 10^5$ kw/l.

values for the time constants and heat capacities of the system have been calculated elsewhere (Brehm, Hetrick and Schmidt 1969). The limits on the thermal constants are specified by the dimensions of the converters and the thermal properties of the various materials. The nuclear constants, λ and β , are determined by the particular fissile fuel used as the heat source. The constants used in Case III correspond to a specific converter design and fissile fuel.

The parameters varied in this study are the equilibrium power level and the feedback coefficients. The equilibrium power level depends upon the material properties of the fuel and the rate of heat transfer out of the converter. The prompt feedback coefficient is largely determined by the fuel, its enrichment and the flux spectrum in the reactor. The delayed coefficient is in general determined by the structure and/or coolant of the reactor. The range in value of both the prompt and delayed feedbacks are from 10^{-5} to -10^{-5} /°C. By imposing these limits on the feedback coefficients for the reactor, the region in the parameter space is greatly reduced over that shown in Figure 5.10. The stability analysis of the thermionic reactor system is restricted to this physically meaningful region which is shown in Figure 5.37. The region is narrow because the ratio η_2/η_3 appears in the definition of y . The linear stability requirements are governed solely by the zero power coefficient line, the slant line in Figure 5.37, (Brehm, 1968) in that the resonance line does not enter the region. Hence, for the linear system the entire region is unconditionally stable. It is noted that because of the zero power coefficient line the limit for positive prompt feedback is approximately 0.63×10^{-6} /°C rather than 10^{-5} /°C.

Figure 5.37 Parameter Space for Thermionic Reactor Systems.

Figure 5.38 Phase Plane for Case III with Prompt Jump Kinetics; $\xi = -0.22$, $\alpha_1 = -0.62 \times 10^{-6}$, $P_0 = 100$ kw/liter.

The nonlinear system is asymptotically stable by Welton's criterion in part of the region in the first quadrant and in all of the region in the fourth quadrant. The remainder of the region in the first quadrant has bounded solutions which also may be asymptotically stable. The second quadrant admits the possibility of finite escape time solutions. To explore this possibility, the stability of a reactor system was analyzed having the limiting values of the parameters; that is, the largest positive prompt feedback with the corresponding smallest negative delayed coefficient. The location in the parameter space, Figure 5.37, for this example is identified by point K. The phase plane for the example is shown in Figure 5.38 with the power plotted as the relative power. The solution is a stable node with power disturbances of at least 100 percent of the equilibrium power level, illustrating that the thermionic reactor system may be very stable.

CHAPTER VI

CONCLUSIONS

In this study the Fourier series method of predicting the amplitudes of the power and reactivity oscillations in nuclear reactor systems is simplified and extended. Higher harmonics are considered in both the forward and feedback loops, and the infinite-order successive approximation is found. The highest approximation may also be derived by direct series methods.

The results of the Fourier series method are compared with exact solutions of the system equations. Accurate prediction of limit cycles by Fourier series is possible when the infinite-order approximation is used. Lower-order approximations yield incorrect results in some cases. The method is restricted to describing small amplitude limit cycles near the critical power level (the threshold of linear instability). It is concluded that small-amplitude limit cycles exist only when both the reactivity and power amplitude predict their presence. An important extension of this work would be to relax the restriction of the method to small amplitude limit cycles. This would permit complete description of the regions in the parameter space for nonlinear stability.

In general, direct analytical determination of oscillatory regions in the parameter space is not possible, because the oscillation frequency is obtained from the numerical solution of a high-degree polynomial corresponding to a single point in the parameter space. Each

real frequency must then be tested in formulas for reactivity and power amplitudes. Using this procedure, the Fourier series method is able to examine a large number of cases for oscillatory solutions much quicker than numerical integration schemes of the exact equations.

A linear transformation is derived for decoupling the most general coupled heat-transfer equations for two regions, and the relationship between the resulting canonical variables and the original physical variables is given. A special set of two-region equations for a nuclear reactor with in-core thermionic converters is discussed.

The prompt jump approximation is compared with the effective lifetime model for neutron kinetics. The effective lifetime model can be considered a simplification of the prompt jump approximation, yet it is not as restrictive in all cases for linear stability.

There are significant differences in the two models for nonlinear stability, because the effective lifetime model predicts stable limit cycles only, while the prompt jump approximation yields both stable and unstable limit cycles. These results are predicted by the successive approximations and verified by solution of the exact equations. There are also important differences in the trajectories between shutdown and operating equilibrium power levels. For the same parameters, the effective lifetime model may exhibit a separatrix connecting the two equilibrium points, while the prompt jump approximation may or may not exhibit such a trajectory. The power may diverge before reaching operating conditions, indicating that external control would be required during startup of the reactor. The prompt jump approximation is shown to be in close agreement with the standard second order neutron kinetics equations.

These results apply to any two-region lumped-parameter feedback model in the sense that all two-region models are equivalent under linear transformation. The parameter space used in this study is, in that sense, applicable to any two-region model provided the matrix of inverse time constants is non-singular and has real and distinct eigenvalues.

The thermionic reactor model which is an example of a two-region model is analyzed for stability using physically meaningful parameters. The exact solutions indicate that the thermionic reactor system may be very stable for a wide range of parameters.

APPENDIX A

FORMULAS FOR POWER AND REACTIVITY AMPLITUDES

This appendix contains a summary of the iterative solutions for power and reactivity amplitudes resulting from various successive approximations. The intermediate approximations between the third approximation and the infinite approximation are not listed.

First Approximation

Reactivity

$$\frac{|a_1|^2}{P_o^2} = \frac{1 + Z_1 H_1}{2 Z_1 H_1 [Z_1 - H_o^{-1}] \text{Re} Z_1}$$

Power

$$|A_1|^2 = \frac{1 + Z_1 H_1}{2 Z_1 H_1 [-H_1^{-1} - H_o^{-1}] \text{Re} Z_1^{-1}}$$

Second Approximation with A₂ Only

Reactivity

$$\frac{|a_1|^2}{P_o^2} = \frac{1 + Z_1 H_1}{Z_1 H_1 [2(Z_1 - H_o^{-1}) \text{Re} Z_1 - Z_1 Z_2]}$$

Power

$$|A_1|^2 = \frac{1+Z_1 H_1}{Z_1 H_1 [-2(H_1^{-1} + H_o^{-1}) \operatorname{Re} Z_1^{-1} + H_{-1} Z_2]}$$

Second Approximation with A_2 and a_2

Reactivity

$$\frac{|a_1|^2}{P_o^2} = \frac{1+Z_1 H_1}{Z_1 H_1 [2(Z_1^{-1} - H_o^{-1}) \operatorname{Re} Z_1 - Z_1 Z_2 (1 - Z_{-1} H_2)]}$$

Power

$$|A_1|^2 = \frac{1+Z_1 H_1}{Z_1 H_1 [-2(H_1^{-1} + H_o^{-1}) \operatorname{Re} Z_1^{-1} + H_{-1} Z_2 (1 + H_{-1}^{-1} H_2)]}$$

Third Approximation

Reactivity

$$\frac{|a_1|^2}{P_o^2} = \frac{1+Z_1 H_1}{Z_1 H_1 [2(Z_1^{-1} - H_o^{-1}) \operatorname{Re} Z_1 - Z_1 Z_2 (1 - Z_{-1} H_2) (1 - Z_2 H_2)]}$$

Power

$$|A_1|^2 = \frac{1+Z_1 H_1}{Z_1 H_1 [-2(H_1^{-1} + H_o^{-1}) \operatorname{Re} Z_1^{-1} + H_{-1} Z_2 (1 + H_{-1}^{-1} H_2) (1 - Z_2 H_2)]}$$

Infinite Approximation

Reactivity

$$\frac{|a_1|^2}{P_o^2} = \frac{1 + Z_1 H_1}{Z_1 H_1 [2(Z_1^{-1} H_o^{-1}) \operatorname{Re} Z_1^{-1} Z_1 Z_2 (1 - Z_1^{-1} H_2) (1 + Z_2 H_2)^{-1}]}$$

Power

$$|A_1|^2 = \frac{1 + Z_1 H_1}{Z_1 H_1 [-2(H_1^{-1} + H_o^{-1}) \operatorname{Re} Z_1^{-1} + H_1^{-1} Z_2 (1 + H_1^{-1} H_2) (1 + Z_2 H_2)^{-1}]}$$

APPENDIX B

LINEAR STABILITY COMPARISON OF PROMPT JUMP NEUTRON KINETICS WITH SECOND ORDER NEUTRON KINETICS

In this appendix, the Routh array for the second order neutron kinetics equations is calculated in order to examine the validity of the stability boundaries in the parameter space resulting from prompt jump neutron kinetics.

The transfer function for the second order kinetics equations, equations 3-4 and 3-5, is given by

$$Z(s) = \frac{P_o (s + \lambda)}{\ell s (s + \lambda + \frac{\beta}{\ell})} \quad (B-1)$$

which is simplified for the purposes of illustration to

$$Z(s) = \frac{P_o (s + \lambda)}{\ell s (s + \frac{\beta}{\ell})} \quad (B-2)$$

$\beta/\ell \gg \lambda$ is a good approximation in general.

Using $H(s)$ from equation 4-5, the characteristic equation becomes

$$\begin{aligned} D(s) = s^4 + (\eta_1 + \eta_3 + \frac{\beta}{\ell}) s^3 + [\eta_1 \eta_3 + \frac{\beta}{\ell} (K + \eta_1 + \eta_3)] s^2 + \\ + \frac{\beta}{\ell} [\eta_1 \eta_3 + K(\lambda + \xi)] s + \frac{\beta}{\ell} K \lambda \xi \end{aligned} \quad (B-3)$$

This approaches $D(s)$ for prompt jump kinetics, equation 4-9, for very large β/ℓ .

The Routh array yields four inequalities which must be satisfied for linear stability.

$$\eta_1 + \eta_3 + \frac{\beta}{\ell} > 0 \quad (\text{B-4})$$

$$\left(\frac{\beta}{\ell}\right)^2 [K + \eta_1 + \eta_3] + \frac{\beta}{\ell} [(K + \eta_1 + \eta_3)(\eta_1 + \eta_3) - K(\lambda + \xi)] + \eta_1 \eta_3 (\eta_1 + \eta_3) > 0 \quad (\text{B-5})$$

$$\begin{aligned} & \left(\frac{\beta}{\ell}\right)^2 [(K + \eta_1 + \eta_3)(\eta_1 \eta_3 + K\lambda + K\xi) - K\lambda\xi] + \left(\frac{\beta}{\ell}\right) \{[(\eta_1 + \eta_3)^2 + \\ & + K(\eta_1 + \eta_3 - \lambda - \xi)][\eta_1 \eta_3 + K(\lambda + \xi)] - 2K\lambda\xi(\eta_1 + \eta_3)\} + \\ & + (\eta_1 + \eta_3)[(\eta_1 \eta_3)^2 + K\eta_1 \eta_3(\lambda + \xi) - K\lambda\xi(\eta_1 + \eta_3)] > 0 \end{aligned} \quad (\text{B-6})$$

$$\frac{\beta}{\ell} (K\lambda\xi) > 0 \quad (\text{B-7})$$

The first condition is always true since all the quantities are inherently positive. When β/ℓ is very large the other three conditions approach the Routh conditions given for the prompt jump approximation in equations 4-10, 4-11, and 4-12. Had equation B-1 been used instead of B-2, the result as β/ℓ becomes very large would be the same.

The last condition is identical to the requirement for a negative steady-state power coefficient and corresponds directly to one of the prompt jump conditions.

The effect on the hyperbolic resonance line is readily seen by examining the asymptotes of the hyperbola. Using x and y , the asymptotes

are

$$y = \frac{\eta_1 - \lambda + \frac{\beta}{\ell}}{\eta_3} x - \eta_1 + \frac{\beta}{\ell} \eta_3 [\eta_1 \eta_3 + (\frac{\beta}{\ell} - \lambda)(\eta_1 + \eta_3 - \lambda)] \cdot [\eta_1 + \eta_3 + \frac{\beta}{\ell}] \quad (\text{B-8})$$

$$y = - \frac{\lambda + \eta_3}{\eta_3} x - \eta_1 - \frac{\ell \lambda^2}{\beta \eta_3} (\eta_1 + \eta_3 + \frac{\beta}{\ell}) \quad (\text{B-9})$$

Equation B-8 corresponds to the vertical asymptote of the prompt jump approximation, equation 4-19, and also admits the possibility of entering the first quadrant of the parameter space. The changes are very slight. The x coordinate intercepts are almost exact and the vertical asymptote now has a slope of approximately β/ℓ rather than an infinite slope.

The correspondence between the prompt jump approximation and the second order neutron kinetics equations is very close. By comparison, the correspondence between the prompt jump approximation and effective lifetime kinetics is very poor.

APPENDIX C

FREAMP PROGRAM

FREAMP is a digital computer program designed to analyze the various approximations for the frequencies and amplitudes of oscillations. The program is composed of the main program and subroutines CHECK and DPROOT.

The calculation of the frequency and amplitude for all of the cases examined involves the same basic steps which are outlined here as an example.

Substitution of the forward and feedback loop transfer functions into a given approximateion (as listed in Appendix A) results in an expression of the following form

$$|a_1|^2 = \frac{\frac{a + ib}{c + id}}{\frac{e + if}{g + ih}} \quad (C-1)$$

Where $a, b, \dots, g, h$ are frequency dependent polynomials. By algebra, equation C-1 is put into the form

$$|a_1|^2 = U(\omega^2) + i V(\omega^2) \quad (C-2)$$

For the amplitude to be real, the imaginary component, V , in equation C-2 must be zero.

V is a polynomial of the form

$$V = V_{n+1}\omega^{2n} + V_n\omega^{2(n-1)} + \dots + V_2\omega^2 + V_1 \quad (C-3)$$

which is factored numerically for the roots by subroutine DPR00T.

The positive real roots of equation C-3 are substituted into the real part of equation C-2,

$$|a_1|^2 = U(\omega^2). \quad (C-4)$$

Of the resulting amplitudes, only the positive values are meaningful.

Main Program

The main program is used for the input and output of the program and for calculating the coefficients of the imaginary frequency polynomial and the amplitude squared of the reactivity or power.

The input is read in using 8D10.3 format and is listed below. All variables in the program are double precision variables. Double precision is required to insure accuracy in the factoring of the frequency polynomial.

Card 1

N ~ order of polynomial in I2.

Card 2

XL - λ , delayed neutron decay constant
T1 - reciprocal time constant, heat out of region one
T2 - reciprocal time constant, heat into region two
T3 - reciprocal time constant, heat out of region two
BETA - delayed neutron fraction
B1 - reciprocal heat capacity of the fuel
XI - ξ , a constant
ALPHI - prompt reactivity coefficient

Card 3

PWRERF - reference equilibrium power
DELT - change in x for next calculation

The main program calculates the coordinates of the reference point in the parameter space and then prints out all of the parameters and the reference values to identify the data.

Constant values of ξ correspond to radial lines through the origin in the parameter space. For a given ξ , the program starts at some point on the line, and solves for the coefficients of the frequency polynomial. DPROOT is called to factor the polynomial and return the roots to the main program. The roots are used to find the amplitudes, denoted AA. The output consists of the location in the parameter space, the equilibrium power level, the frequency roots and the resulting amplitudes. Hollerith statements provide headings for the results. Then x is increased by DELT, y is increased accordingly,

and the procedure is repeated for a different power level.

There is a test in the program to check for the critical power level, PWRC, occurring when a radial line crosses a resonance line. The resonance frequency of the fundamental is calculated from GH and printed out to serve as a check on the factoring of the frequency polynomial and to identify the fundamental amplitude.

Subroutine CHECK

CHECK finds the intersections of ξ with the resonance hyperbola. The critical coordinates are XC(I) and YC(I). The resonance frequency, OMEGS(I), is calculated for each intersection.

Subroutine DPROOT

DPROOT is a double precision version of PROOT and was written by Shapiro (1965) using a method developed by Vordan at LASL. PROOT finds all the roots of a polynomial with real coefficients. The routine converges simultaneously towards a linear and a quadratic factor by Newton's and Bairstow's methods respectively.

The input required from the main program is

N - order of polynomial
 A(I) - coefficients of the polynomial
 NPLUS2 - dimension of arrays
 NR - check variable for zero values of A(N + 1)

The roots are returned to the main program in the form $U(I) + i V(I)$ where U and V are one dimensional arrays of N elements.

The convergence criterion is set at $5.D - 20$ and if the polynomial has not converged after a prescribed number of tries the convergence criterion is relaxed.

A simplified flow chart is shown in Figure C.1 and a listing of the program follows.

Figure C.1 Flow Chart for FREAMP.

```

PROGRAM FREAMP (INPUT,OUTPUT)
COMMON /A/ XC(2),YC(2),DMEGS(2),XM,XI
COMMON /B/ T1,T3,TS,TP,X,Y,XL,KK,XY,T
COMMON /C/ N,A(32),U(32),V(32),IPLUS2,NR
DOUBLE R,COM,C,GO,U,A,V,T,T2S,T3S,XX,TS,TP,TX,AN,BN,CN
,DN,EN,FN,GN
*,HN,PN,QN,AD,BD,CD,ED,FD,GO,HO,PD,QD,RD,A1,A3,AA,T1
,T3,XL,DELTX
*,DELT,Y,X,Y,P,Q,T4P,I2P,T3P,XK,XI,BP,CP,OP,EP,FP,GP,HP,
,PP,RP,SN,SD,
*,TPP,WP,XP,YP,ZP,HF,OF,PF,QF,RF,TF,UF,WF,XF,YF,ZF,VF,SF
,CP,XS,XY,A2
*,A4,AP,OP,SP,UP,VP,AF,BF,CF,DF,EF,FF,GF,XC,XSS,DMECS,X
DELT,S
*,T2,XM,DELT,PWRERF,PWRE,PWRC,ALPH1,ALPH2,XRF,YRF,YC,YS
S,BETA,XI,B1
C INFINITE APPROXIMATION INCLUDING THE SECOND HARMONIC
C TWO TEMPERATURE THERMIONIC HEAT TRANSFER MODEL
C PROMPT JUMP NEUTRON KINETICS
C POWER AMPLITUDE SQUARED
  READ 50,N
50 FORMAT (I2)
52 READ 51,XL,T1,T2,T3,BETA,B1,XI,ALPH1,PWRERF,DELT
51 FORMAT (6D10.3)
  IF(XL) 64,64,55
55 IPLUS2 = N+2
  X = 0.0
  Y = 0.0
  NC = 0
  I = 1
  TS = T1+T3
  TP = T1*T3
  XM = (XI-T3)/T3
  XRF = ALPH1*B1*PWRERF/BETA
  YRF = XM*XRF
  ALPH2 = YRF*T3*BETA/(T2*B1*PWRERF)
  PRINT 32,T1,T2,T3
32 FORMAT (1H1,2X18TIME CONSTANT 1 = D11.3,10X18TIME CO
NSTANT 2 = D
*11.3,10X18TIME CONSTANT 3 = D11.3)
  PRINT 34,ALPH1,ALPH2,PWRERF
34 FORMAT (/3X9HALPHA1 = D11.3,10X9HALPHA2 = D11.3,10X3CH
REFERENCE SG
*0118RIOP POWER = D11.3)
  PRINT 35,BETA,B1,XL,XI
33 FORMAT (/3X7BETA = D11.3,10X5B1 = D11.3,10X9HLA 18DA
= D11.3,10X5
*HXI = D11.3)
  PRINT 35,XRF,YRF
35 FORMAT (/3X13XREFERENCE = D11.3,10X13XREFERENCE = D1
1.3)
  CALL CHECK

```

```

DO 60 JJ = 1, 50
  Y = Y+DABS(XN)*DELTA
  X = X+XN*DELTA/DABS(XN)
  PWRE = X*BETA/(ALPH1*B1)
13 PRINT 30
30 FORMAT (///10X,31HLOCATION IN THE PARAMETER SPACE)
  IF(DABS(X-XC(1))-DELTA)44,44,43
44 IF(NC)45,45,46
45 XSS = X
  YSS = Y
  Y = YC(1)
  X = XC(1)
  NC = 1
  PWRC = XC(1)*BETA/(ALPH1*B1)
  PRINT 47,X,Y,OMECS(1),PWRC
47 FORMAT (15X4HX = D11.3,5X4HY = D11.3,10X19HROOT CHECK
  VALUE = D14.
  *7,10X17HCRITICAL POWER = D11.3)
  GO TO 62
46 Y = YSS+DABS(XN)*DELTA
  X = XSS+XN*DELTA/DABS(XN)
  PWRE = X*BETA/(ALPH1*B1)
  I = 2
  NC = 0
48 CONTINUE
  PRINT 61,X,Y,PWRE
61 FORMAT (15X4HX = D11.3,5X4HY = D11.3,10X20HEQUILIBRIUM
  POWER = D11
  *.3)
62 T2S = T1*T1+I3*I3
  T3S = T1**3+I3**3
  T = XL*T3*(X+Y)
  TX = X*XL+TP
  XX = X*XL
  XS = X*XL+I3*(X+Y)
  XY = I3*(X+Y)
  AP = TS+X
  BP = TP+XS
  CP = (-(XL+TS))
  DP = TP+XL*TS
  EP = (-(XL*TP))
  FP = 4.0*(X+TS)
  GP = 2.0*(XX+XY+TP)
  HP = AP+CP
  CP = DP+AP*CP+BP
  PP = EP+AP*DP+CP*CP+T
  GP = AP*EP+BP*CP+T*CP
  RP = EP*EP+T*CP
  SP = T*EP
  UP = XY
  VP = X*TP
  WP = 3.0*XY*TP-X*TP*TS-XY*TS*TS

```

```

XP = X*TP*IP-XY*IP*IS
YP = 2.0*XY*IP*IP
AF = 2.0*L*XY
BF = 4.0*XX*IS-5.0*XXY
CF = -(XY*IS+X*IP)
DF = -(2.0*XY*IP)
EF = 2.0*XX*XY
FI = XY*(XX+2.0*XY)
GI = XY*(I-2.0*XX*XL)
HI = -(XY*(XX*XL*XL+2.0*XL*I))
PI = -(XL*I*I)
AD = 2.0*(UP*FP+R.D*VP)
UD = 2.0*(UP*IP+VP*GP+AP*FP+9.0*XP)+AT*FF+LF*EE)
RD = -(2.0*(UP*IP+VP*GP+AP*FP+8.0*YP)+AT*GF+BF*F+CF*EF
HD = 2.0*(VP*IP+GP*AP+8.0*YP)+AT*GF+BF*F+CF*EF
PD = -(2.0*(XP*IP+YP*GP)+AF*FP+BF*HT+CF*GF+DE*FF))
RD = CF*FP+DE*FF
AH = FP+8.0*HP
BN = -(1+HP*GP+GP*FP+8.0*PP)
CN = UP*IP+PP*GP+GP*FP+8.0*RP
DN = -(CP*IP+RP*GP+SP*FP)
EN = 1*SP
FN = 8.0
GN = -(GP+HP*FP+8.0*QP)
HN = HP*IP+UP*GP+PP*FP+9.0*QP
PN = -(PP*IP+GP*CP+RP*FP+8.0*SP)
RN = RP*IP+SP*QP
A(1) = EV*RN-EA*RD
A(2) = DP*RN+EP*PI-DN*RD-EA*PD
A(3) = CL*RN+DP*PI+ED*HI-CM*RD-D.I*PD-BN*HD
A(4) = ED*RN+CE*PI+DP*HI+ED*CN-H.I*RD-CN*PD-D.I*HD-EA*GD
A(5) = AD*RN+ED*PI+CD*HI+GD*GN+ED*FI-A.N*RD-BN*PD-CN*HD
-CA*GD-EA*FD
A(6) = AD*PN+ED*HN+CD*GN+DD*FI-A.I*PD-BN*HD-C.I*GD-DN*FI
A(7) = AL*HN+ED*GN+CD*BN-A.I*HD-A.I*GD-CN*FD
A(8) = AB*CN+BD*FI-A.I*GD-9*PI
A(9) = AD*FN-AH*FD
PRINT 53
53 FORMAT (//20X,23HROOTS OF THE POLYNOMIAL)
CALL DP2001
PRINT 54,(U(1),V(1),I=1,4)
54 FORMAT(20X,014.7,2H+I,014.7)
PRINT 31
31 FORMAT (//20X,22HOSCILLATION AMPLITUDES)
DC 75 KN=1,4
IF(V(K))75,57,75
IF(U(K))75,57,59
57 IF(U(K))75,75,59
59 AI = 2*U(K)*(I+U(K))*(C1+U(K))*(F1+U(K)*(K)*A9)))

```

```

      AB = FD+U(KK)*(DD+U(KK)*(CD+U(KK)*(BD+U(KK)*AD)))
      AA = -(XY*A1/AB)
      PRINT 56, AA
56  FORMAT (25X,5HAA = ,D11.3)
75  CONTINUE
C   RESET ORDER OF POLYNOMIAL TO READ IN VALUE IF CHANGED
92  IF(NR)90,90,71
90  N = N+1
      NR = NR+1
      GO TO 92
91  CONTINUE
60  CONTINUE
      GO TO 52
64  CONTINUE
      END

```

```

SUBROUTINE CHECK
COMMON /A/ XC(2),YC(2),OMEGS(2),XV,XI
COMMON /G/ T1,T3,TS,TP,X,Y,XL,KK,XY,T
DOUBLE T1,T3,XL,X,Y,AP,BP,CP,ROOT,OMEGS,A,U,V
*,XC,YC,XSS,XM,TS,TP,XY,T
AP = XL+XI
BP = TP+TS*XL+TS*XI-XL*XI
CP = TP*TS
ROOT = BP**BP-4.D*AP*CP
IF(ROOT)10,12,12
12  XC(1) = (-BP+DSQRT(ROOT))/(2.D*AP)
      XC(2) = (-BP-DSQRT(ROOT))/(2.D*AP)
      IF(DABS(XC(1)+XC(2))-DABS(XC(2))) 14,13,13
14  XC(1) = XC(2)
13  DO 11 I=1,2
      YC(I) = XM*XC(I)
11  OMEGS(I) = XL*T3*(XC(I)+YC(I))/(1S+XC(I))
10  RETURN
      END

```

```

SUBROUTINE DPRCOT
DIMENSION H(32),B(32),C(32)
COMMON /G/ N,A(32),U(32),V(32),EPLUS2,NR
DOUBLE A,U,V,H,B,C,CNV,P,Q,R,F,L,CHAR,D,QP,PP
*,X,Y
CNV=1.E-35
NR = 1
NC=N+1
CSEND COEFFICIENTS TO REDUCED COEFFICIENT STORAGE
D011=1,NC
F(1) = A(1)
1  CONTINUE

```


COEPCCK ON ORDER OF POLYNOMIAL READ IN AND CHANGE IF NECESSAR

Y

82 IF(H(NC))60,81,80

81 NC=N

N=N-1

NR=NR-1

GO TO 82

80 CONTINUE

CINITIALIZE GUESSES AND SET REVERSAL INDICATOR NORMAL

P=0.0

Q=0.0

R=0.0

IREV=1

CSCALING TO BE DONE AT THIS POINT

CREMOVE ALL ZERO ROOTS

3 IF(H(1))4,2,4

2 NC=NC-1

V(NC)=0.0

U(NC)=0.0

DO1002I=1,NC

H(I) = H(I+1)

1002 CONTINUE

GOTO3

CTEST FOR VARIOUS DEGREES

4 IF(NC-1)5,100,5

5 IF(NC-2)7,6,7

6 R=-H(1)/H(2)

GOTO50

7 IF(NC-3)9,8,9

8 P=H(2)/H(3)

Q=H(1)/H(3)

GOTO70

CTEST TO REVERSE COEFFICIENTS AND DO SO IF TEST SUCCEEDS

9 IF(DABS(H(NC-1)/H(NC))-DABS(H(2)/H(1)))10,19,19

10 IREV=-IREV

M=NC/2

DO111I=1,M

NL=NC+1-I

F=H(NL)

H(NL)=H(1)

11 H(1)=F

IF(Q)13,12,13

12 P=0.0

GOTO15

13 P=P/Q

Q=1.0/Q

15 IF(R)16,19,16

16 R=1.0/R

CNEWTON, CALCULATE F(R) AND TEST FOR ROOT

19 E=5.0-20

B(NC)=H(NC)

C(NC)=H(NC)

```

0(NC+1)=0.0
C(NC+1)=0.0
NP=NC-1
DO 40J=1,500
  C02111=1,MP
  I=NC-1
  B(1)=H(1)+R*B(1+1)
  C(1)=B(1)+R*C(1+1)
  IF(DABS(B(1)/H(1))-(E)50,50,24
  24 IF(C(2))23,22,23
  22 R=R+1.0
  GOTO30
  23 R=R-B(1)/C(2)
  C MAKE A BAIRSTON REDUCTION AND CORRECT
  30 DC3711=1,MP
  I=NC-11
  B(1)=H(1)-P*B(1+1)-Q*B(1+2)
  C(1)=B(1)-P*C(1+1)-Q*C(1+2)
  C TEST FOR CONVERGENCE OF BAIRSTON PROCESS.
  IF(H(2))32,31,32
  31 IF(DABS(B(2)/H(1))-(E)33,33,34
  32 IF(DABS(C(2)/H(2))-(E)33,33,34
  33 IF(DABS(C(1)/H(1))-(E)70,70,34
  34 C0AR=C(2)-F(2)
  F=C(3)**2-C0AR*C(4)
  IF(D)35,35,36
  35 P=P-2.0
  Q=Q*(0+1.0)
  GOTO49
  36 P=P+(B(2)*C(3)-B(1)*C(4))/E
  Q=Q+(-B(2)*C0AR+B(1)*C(3))/E
  49 CONTINUE
  E=E*10.0
  IF(E-C0NV)20,20,40
  40 C0NV=E
  GOTO20
  CLINEAR, COMPUTE AND STORE LINEAR ROOTS
  50 NC=NC-1
  V(NC)=0.1
  IF(IREV)51,52,52
  51 U(NC)=1.0/8
  GOTO53
  52 U(NC)=8
  53 DC541=1,NC
  H(1)=L(1+1)
  54 CONTINUE
  GOTO4
  COUADRATIC, SOLVE QUADRATIC AND STORE ROOTS
  70 NC=NC-2
  IF(IREV)71,72,72
  71 C2=1.0/8
  PP=P/(1+8.0)

```

```

GCT073
72 GP=0
PP=P/2.0
73 F=(PP)**2-GP
IF(F) 74, 75, 75
CCASE GF IMAGINARY ROOTS.
74 U(NC+1)=-PP
U(NC)=-PP
V(NC+1)=DSQRT(-F)
V(NC)=-V(NC+1)
GCT076
CCASE GF REAL ROOTS.
75 U(NC+1)=(-DSIGN(DABS(PP)+DSQRT(F), PP))
V(NC+1)=0.0
U(NC)=GP/U(NC+1)
V(NC)=0.0
CHECK NEW REDUCED COEFFICIENTS
76 D0771=1, NC
H(1) = L(1+2)
77 CONTINUE
GCT09
100 RETURN
END

```

8

```

1.0000-01 5.0000-01 2.5000-01 2.0000-01 1.0000-02 1.0000 02
-1.0000-01-3.3330-07
1.0000 01 2.5000-02
-1.0000 00

```

APPENDIX D

ANALOG SIMULATION

All analog simulation was performed in the analog computer laboratory in the Department of Nuclear Engineering at The University of Arizona using the Dystac 5800 analog computer and associated equipment.

Equations

The kinetics equations which were programmed for the thermionic reactor system are listed below with the reactivity in dollars.

Power Equations:

Effective lifetime,

$$\frac{dP(t)}{dt} = \lambda \rho(t) P(t) \quad (D-1)$$

Linearized effective lifetime,

$$\frac{dP(t)}{dt} = \lambda \rho(t) P_o \quad (D-2)$$

Prompt jump,

$$\frac{dP(t)}{dt} = \frac{\frac{d\rho}{dt} + \lambda \rho(t)}{[1 - \rho(t)]} P(t) \quad (D-3)$$

Linearized prompt jump,

$$\frac{dP(t)}{dt} = \left[\frac{d\rho}{dt} + \lambda \rho(t) \right] P_o \quad (D-4)$$

Temperature Equations:

$$\begin{aligned}\frac{dT_1(t)}{dt} &= b[P(t) - P_o] - \eta_1 T_1(t) \\ \frac{dT_2(t)}{dt} &= \eta_2 T_1(t) - \eta_3 T_2(t)\end{aligned}\tag{D-5}$$

The definitions of the symbols are presented in Chapter III and are not repeated here. The two temperature equations and one of the power equations together with a linear reactivity feedback comprise a system of equations which describes the thermionic reactor model.

Scaling

In order to keep the values of the variables within the operating range of the analog computer, it is necessary to scale the variables.

The scale factors used are

$$\begin{aligned}T_1(t) &= x_1 T_1'(t) \quad ; \quad T_2(t) = x_2 T_2'(t) \\ P(t) &= x_3 P'(t) \quad ; \quad t = x_4 \tau\end{aligned}\tag{D-6}$$

Substituting the scaled variables into the system equations with prompt jump neutron kinetics and the reactivity, equation 3-15, yields

$$\frac{dP(\tau)}{d\tau} = \frac{-\alpha_1 x_1 \frac{dT_1(\tau)}{d\tau} - \alpha_2 x_2 \frac{dT_2(\tau)}{d\tau} - \lambda \alpha_1 x_1 x_4 T_1(\tau) - \lambda \alpha_2 x_2 x_4 T_2(\tau)}{\beta + \alpha_1 x_1 T_1(\tau) + \alpha_2 x_2 T_2(\tau)} P(\tau)\tag{D-7}$$

$$\frac{dT_1(\tau)}{d\tau} = \frac{bx_4 x_3}{x_1} [P(\tau) - P_o] - x_4 \eta_1 T_1(\tau)\tag{D-8}$$

$$\frac{dT_2(\tau)}{d\tau} = x_4 \left[\eta_2 \frac{x_1}{x_2} T_1(\tau) - \eta_3 T_2(\tau) \right] \quad (D-9)$$

Example Equations

The example problem described here is for Case I using prompt jump neutron kinetics. With Case I, scaling was required to reduce the values of the temperatures. The scale factors are

$$X_1 = 100, X_2 = 100, X_3 = 1, X_4 = 1$$

The values of the parameters used in Case I are listed in Table 4.1. Substitution of the appropriate scale factors and parameters into equations D-7, D-8 and D-9 with $\alpha_1 = -0.33 \times 10^{-6}$ and $\alpha_2 = 0.8 \times 10^{-5}$ gives the following result.

$$\frac{dP}{dt} = \frac{\left[0.33 \times 10^{-4} \frac{dT_1}{dt} - 0.8 \times 10^{-3} \frac{dT_2}{dt} + 0.33 \times 10^{-5} T_1 - 0.8 \times 10^{-4} T_2 \right] P}{(0.01 - 0.33 \times 10^{-4} T_1 + 0.88 \times 10^{-3} T_2)}$$

$$\frac{dT_1}{dt} = P - 10.0 - 0.5 T_1$$

$$\frac{dT_2}{dt} = 0.25 T_1 - 0.2 T_2$$

The nonlinear power equation is programmed by using a multiplier in a divide mode for the quotient with a subsequent multiplication by P. To improve the accuracy of this operation, the numerator and denominator were increased by a factor of 10^3 . This is necessary in order to have the multipliers operate in their linear range.

Analog Programs

Diagrams of the analog programs for the system of equations with the different power equations are shown in Figures D.1 through D.4. The diagrams are for positive prompt and negative delayed feedback corresponding to the second quadrant of the parameter space. To consider other quadrants of the parameter space, the sign or signs of the feedback coefficients must be changed in the programs. Several other modifications are necessary in the programs depending upon the range of variable values and the size of disturbances. For example, it was necessary in one case to translate the origin to the equilibrium power level in order to obtain accurate results for large disturbances.

The unscaled values of the potentiometers for the analog programs are listed in Table D.1.

Figure D.1 Analog Computer Diagram for Linear Effective Lifetime Neutron Kinetics.

Figure D.2 Analog Computer Diagram for Nonlinear Effective Lifetime Neutron Kinetics.

Figure D.3 Analog Computer Diagram for Linear Prompt Jump Neutron Kinetics.

Figure D.4 Analog Computer Diagram for Nonlinear Prompt Jump Neutron Kinetics.

Table D.1
Potentiometer Values for Analog Program Diagrams

Potentiometer Number	NL-EL* Figure D.1	NL-EL* Figure D.2	NL-PJ* Figure D.3	NL-PJ* Figure D.4
1	P_i	P_i	P_i	P_i
2	T_{1i}	T_{1i}	T_{1i}	T_{1i}
3	T_{2i}	T_{2i}	T_{2i}	T_{2i}
4	b	b	b	b
5	bP_o	bP_o	bP_o	bP_o
6	η_1	η_1	η_1	η_1
7	η_2	η_2	η_2	η_2
8	η_3	η_3	η_3	η_3
9	$\lambda P_o \alpha_1 / \beta$	$\lambda \alpha_1 / \beta$	$\lambda P_o \alpha_1 / \beta$	$\lambda \alpha_1$
10	$\lambda P_o \alpha_2 / \beta$	$\lambda \alpha_2 / \beta$	$\lambda P_o \alpha_2 / \beta$	$\lambda \alpha_2$
11			α_1 / β	α_1
12			α_1 / β	α_2
13				α_1
14				α_2
15				β

*L-EL: Linear, Effective Lifetime

NL-EL: Nonlinear, Effective Lifetime

L-PJ: Linear, Prompt Jump

NL-PJ: Nonlinear, Prompt Jump

APPENDIX E

DIFEQN PROGRAM

DIFEQN is a digital computer program which solves the differential system equations. The program was developed to verify analog computer solutions and to extend the work performed on the analog computer. The general coupled heat transfer equations are used in the program for the temperature equations. There are four subprograms in the program: DIFEQN, the main program; and subroutines, RHS, PARTIAL and DEQVAR. The subroutines required for the plotting routine are not supplied by the user and will not be discussed.

Main Program

The main program is used to control the input and output of the program. It consists of read and print statements and plotting subroutine calls.

The input is in format 8E10.3 and comprises the following:

Card 1

Y(1)	-	initial value of T_1
Y(2)	-	initial value of T_2
Y(3)	-	initial value of P
ALPH1	-	prompt feedback coefficient
ALPH2	-	delayed feedback coefficient

- B1 - inverse heat capacity of Region 1
- B2 - inverse heat capacity of Region 2
- BETA - delayed neutron fraction

Card 2

- YZERO - equilibrium power level
- ETA1 - reciprocal time constant for heat transfer out of
Region 1
- ETA2 - reciprocal time constant for heat transfer into
Region 1
- ETA3 - reciprocal time constant for heat transfer into
Region 2
- ETA4 - reciprocal time constant for heat transfer out of
Region 2
- XLAMB - decay constant for delayed neutrons
- ERROR - truncation error limit
- X - starting value of time

Card 3

- XEND - ending value of time
- PJ - control constant, 1.0 for prompt jump neutron
kinetics and 0.0 for effective lifetime neutron
kinetics

Stacked data decks are used for examining several cases on the same run or for selecting different initial conditions in the same case.

The parameters and the initial conditions read in are printed out at the beginning of each set of data in order to identify the run. The printed output consists of the time, power, temperature in region one, temperature in region two, the step size, the number of the step and the reactivity. In the case of prompt jump kinetics, $(1-\rho)$ is printed instead of the reactivity. The data is printed in the DEQVAR subroutine. A variable, KKK, is included to control the number of time steps between printing.

Power-temperature plots are also produced using the Calcomp plotter. This permits rapid comparison with analog results. The procedure for using the plotting routine is described in the UA Computer Systems Calcomp Instructions. For producing multiple plots on the same graph using the scaling subroutines, the largest trajectory must be plotted first and the values of the scaling parameters stored for subsequent use with the other trajectories. A variable, K00, is used to control the storage of data to be plotted. The program limits the arrays of plotter data to 2500 words for each variable.

The program is listed at the end of this appendix.

Subroutine RHS

Subroutine RHS contains the system equations where the differentials are defined by the one dimensional array, $F(I)$. In this program $F(1) = \frac{dT_1}{dt}$, $F(2) = \frac{dT_2}{dt}$ and $F(3) = \frac{dP}{dt}$. $F(3)$ in this subroutine is changed automatically as the different neutron kinetics equations are used.

Subroutine PARTIAL

Subroutine PARTIAL defines the Jacobian which is used by DEQVAR to adjust the time step in order to insure numerical stability. The elements in the Jacobian are defined by the two dimensional array DF(I,J). The changes occurring in this subroutine with use of different neutron kinetics are taken care of automatically by the value of PJ.

Subroutine DEQVAR

DEQVAR integrates a system of first order ordinary differential equations

$$\frac{dy_i}{dx} = f_i(x, y, \dots y_n) \quad i = 1, n$$

from x_1 to x_2 with given initial conditions. DEQVAR was written by Lewis and Stovall (1965) at LASL and is briefly summarized here.

The method is equivalent to finding the fifth degree polynomial approximation to the desired solution of the differential equations. Provision is made for automatic increase or decrease of the interval size during the course of the integration. Tests are included to insure that the truncation error is bounded and that the integration scheme remains numerically stable.

In addition to the variables discussed above, the main program also provides to DEQVAR the following values.

- NE - number of equations
- H - starting value step size
- HMAX - maximum value step size
- JUMP - set equal to - 1 in main program
- ERROR - related to truncation error

ERROR is an approximate bound on the sum of truncation errors incurred at each integration step. As ERROR is decreased past a certain level, the accuracy may start to decrease because of round off error.

```

PROGRAM DIFFER.(INPUT,OUTPUT,TAPE6)
COMMON/A/ALPH1,ALPH2,B1,B2,BETA,YZERO,ETA1,ETA2,ETA3,ETA4,XLAMB
COMMON/B/Y(20),F(20),DF(20,20),RHO
COMMON/C/X,XFND,ERROR,NE,H,HMAX,JUMP,XN(2504),XT1(2504),XT2(2504),
*KC
CALL INITIAL(3E4100083E,6,0.6,0,0)
NC = 0
1 READ 2,Y(1),Y(2),Y(3),ALPH1,ALPH2,B1,B2,BETA,YZERO,ETA1,ETA2,ETA3,
*ETA4,XLAMB,ERROR,X,XFND,PJ
2 FORMAT(8F10.3)
IF(BETA)4,5,5
5 KC = 0
JUMP = -1
HMAX = 1.0
H = 0.03125
NE = 3
PRINT 6
6 FORMAT(1H19X13HINITIAL POWER3X13HINITIAL TEMP13X13HINITIAL
TEMP26X
*6H1ALPH112X6H1ALPH211X4H5H1A11X6H1LAMBDA)
PRINT 7,Y(3),Y(1),Y(2),ALPH1,ALPH2,BETA,XLAMB
7 FORMAT( 4X,7(3XE13.3))
PRINT 8
8 FORMAT(//7X17HEQUILIBRIUM POWER5X4HETA112X4HETA212X4HETA312X4HETA4
*13X2HB114X2HB2)
PRINT 9,YZERO,ETA1,ETA2,ETA3,ETA4,B1,B2
9 FORMAT( 4X,7(3XE13.3))
PRINT 3
3 FORMAT(///9X4HT1H5E15X5HPOWER13X5HT ONE13X5HT TWO11X9HS
TEMP SIZE7X5H
*KSTEP6X5HBETA-RHO//)
CALL DEQVAR
IF(NC) 10,10,11
10 CALL PLOT(2.0,-11.,-3)
CALL PLOT(5.0,2.0,-3)
CALL SCALE(X1,7.,KO,1,10.0)
CALL SCALE(XT1,5.0,KO,1,10.)
XN1 = XN(KO+1)
XA2 = XL(KO+2)
XT11 = XT1(KO+1)
XT12 = XT1(KO+2)
PRINT 13,XV1,XN2,XT11,XT12
13 FORMAT (2X,4(5XE13.3))
IF(10)12,12,11
11 XN(KO+1) = XN1
XN(KO+2) = XA2
XT1(KO+1) = XT11
XT1(KO+2) = XT12

```

```

12 CONTINUE
   CALL LIND(XT1,XN,K0,1,0,1)
   NC = 1
   GO TO 1
4  CONTINUE
   CALL PLOT(0.,0.,999)
   END

```

```

SUBROUTINE RHS
COMMON/A/ALPH1,ALPH2,B1,B2,BETA,YZERO,ETA1,ETA2,ETA3,ETA4,XLAMB
COMMON/B/Y(20),F(20),DF(20,20),RHO
RHO = (-ALPH1)*Y(1)-ALPH2*Y(2)
V = BETA-PJ*RHO
U1 = ALPH1*(PJ*ETA1-XLAMB)-PJ*ALPH2*ETA3
U2 = ALPH2*(PJ*ETA4-XLAMB)-PJ*ALPH1*ETA2
U3 = PJ*(-ALPH1*B1-ALPH2*B2)
U = U3*(Y(3)-YZERO)+U1*Y(1)+U2*Y(2)
F(1) = B1*(Y(3)-YZERO)-ETA1*Y(1)+ETA2*Y(2)
F(2) = B2*(Y(3)-YZERO)+ETA3*Y(1)-ETA4*Y(2)
F(3) = U*Y(3)/V
RETURN
END

```

```

SUBROUTINE PARTIAL
COMMON/A/ALPH1,ALPH2,B1,B2,BETA,YZERO,ETA1,ETA2,ETA3,ETA4,XLAMB
COMMON/B/Y(20),F(20),DF(20,20),RHO
RHO = (-ALPH1)*Y(1)-ALPH2*Y(2)
V = BETA-PJ*RHO
U1 = ALPH1*(PJ*ETA1-XLAMB)-PJ*ALPH2*ETA3
U2 = ALPH2*(PJ*ETA4-XLAMB)-PJ*ALPH1*ETA2
U3 = PJ*(-ALPH1*B1-ALPH2*B2)
U = U3*(Y(3)-YZERO)+U1*Y(1)+U2*Y(2)
DUDY1 = U1
DV DY1 = PJ*ALPH1
DUDY2 = U2
DV DY2 = PJ*ALPH2
DF(1,1) = -(ETA1)
DF(1,2) = ETA2
DF(1,3) = B1
DF(2,1) = ETA3
DF(2,2) = -(ETA4)
DF(2,3) = B2
DF(3,1) = (V*DUDY1-U*DV DY1)*Y(3)/V**2
DF(3,2) = (V*DUDY2-U*DV DY2)*Y(3)/V**2
DF(3,3) = (U+Y(3)*U3)/V
RETURN

```

END

```

SUBROUTINE DEGVAP
COMMON/A/ALPH1,ALPH2,B1,B2,BETA,YZERO,ETA1,ETA2,ETA3,ETA4,XLANE
COMMON/L/Y(20),F(20),DF(20,20),RHO
COMMON/C/X,XEND,ERROR,NE,H,HMAX,JUMP,XN(2504),XT1(2504),XT2(2504),
*KO
DIMENSION STARTY(20),SY(20),SAVEY(20),FP(20),DELTA(20),
,DELTA(20),A
*(20),B(20),C(20),H(20),AA(20),BB(20),CC(20),DD(20),TOP
(2),SF(20)
KK = 0
KCO = 0
KKK = 0
2004 IF(KSTEP.LT.281474976710655) GO TO 993
KSTEP=28
C TEST FOR TYPE OF ENTRY
993 IF(JUMP)1,996,999
998 GO TO 1A,(1000,11,21,302,003)
C JUMP POS. RESTORE VALUES
999 X=SAVLX
JUMP=0
DO 901 I=1,NL
F(I)=SF(I)
901 Y(I)=SAVEY(I)
GO TO 102
C JUMP NEG. INITIALIZE
1 DO 5 I=1,NF
STARTY(I)=Y(I)
A(I)=B(I)=C(I)=D(I)=.001
5 CONTINUE
KSTEP=KDELAY=KCON=0
T=95./((283.*64.))
U=863./((12.*5040.))
V=95./280.
P=25./24.
Q=35./72.
R=5./48.
S=1./120.
ASSIGN 1000 TO 1A
JUMP=0
GO TO 1111
C BEGIN INTEGRATION STEP
1000 DO 1111 I=1,NL
SF(I)=F(I)
1111 SAVEY(I)=Y(I)
C IF TOO SMALL=RETURN WITH JUMP NEG.
IF(ABS(A+H).GE.ABS(X)) GO TO 875

```

```

JUMP=-1
GO TO 1101
X=X+H
DO 10 I=1,NE
Y(I)=Y(I)+H*(F(I)+A(I)+B(I)+C(I)+D(I))
10 FP(I)=F(I)+2.0*A(I)+3.0*B(I)+4.0*C(I)+5.0*D(I)
ASSIGN 11 TO 1A
GO TO 1101
DO 12 I=1,NE
SY(I)=Y(I)
12 CONTINUE
DO 20 I=1,NE
DELTA(I)=F(I)-FP(I)
20 Y(I)=Y(I)+V*DELTA(I)*H
ASSIGN 21 TO 1A
KCON=1
GO TO 1101
21 KCON=0
DO 30 I=1,NE
DELTA(I)=F(I)-FP(I)
Y(I)=SY(I)+V*DELTA(I)*H
30 CONTINUE
TEST FOR STAIRING SEQUENCE
IF(KSTEP,30.28) GO TO 40
APPLY TEST 2 ON ZEROTH STEP
IF(KSTEP)50,50,60
HALVING TESTS
DO 45 I=1,NE
IF(ABS(DALTA(I)).GT.7302/ABS(H)) GO TO 55
45 CONTINUE
50 IF(V*H*CLIFFLE,0.125) GO TO 60
X=X-H
55 FAIL TESTS, HALVE H
H=0.5*H
KDELAY=0
DO 56 I=1,NE
A(I)=0.5*A(I)
B(I)=0.25*B(I)
C(I)=0.125*C(I)
F(I)=SF(I)
Y(I)=SAV(Y(I))
D(I)=0.0625*D(I)
GO TO 1000
60 PASS TESTS, CORRECT A,B,C,D
KSTEP=KSTEP+1
DO 65 I=1,NE
A(I)=A(I)+3.0*B(I)+6.0*C(I)+10.0*D(I)+P*DELTA(I)
F(I)=B(I)+4.0*C(I)+10.0*D(I)+2*DELTA(I)
C(I)=C(I)+5.0*D(I)+1*DELTA(I)
D(I)=D(I)+5*DELTA(I)
65 CONTINUE
IF I, STAIRING SEQUENCE, REACH

```

```

IF (KSTEP-24) 76,90,100
70 GO TO (1000,1000,1000,1000,74,1000,1000,78,1000,1000,1
000,74,1000,
1000,1000,66,1000,1000,1000,74,1000,1000,1000),KSTEP
4TH,12TH,20TH,STEP,60 BACK
74 H=-H
DC 75 I=1,NE
A(I)=-A(I)
75 C(I)=-C(I)
GO TO 1000
3TH STEP, GO FORWARD
78 H=-H
DC 79 I=1,NE
Y(I)=STAY(I)
A(I)=-A(I)
79 C(I)=-C(I)
GO TO 1000
16TH STEP, HALVE H, APPLY TEST 1
86 H=0.5*H
DC 87 I=1,NE
A(I)=0.5*A(I)
B(I)=0.25*B(I)
C(I)=0.125*C(I)
D(I)=0.0625*D(I)
DC 88 I=1,NE
IF (ABS (CALTA(I)).GT.EMERG/ABS (I)) GO TO 89
88 CONTINUE
PASS TEST GO FORWARD WITH HALVED H
GO TO 76
FAIL TEST BEGIN AGAIN WITH HALVED H
89 H=-H
DC 92 I=1,NE
Y(I)=STAY(I)
92 CONTINUE
GO TO 1
24TH STEP, DOUBLE H, STARTING SEQUENCE ENDS
90 H=2.*H
DC 91 I=1,NE
A(I)=A(I)*2.0
B(I)=B(I)*4.0
C(I)=C(I)*8.0
D(I)=D(I)*16.0
GO TO 76
100 KDELAY=KDELAY+1
C
WILL NEXT STEP BE A FAIL
102 IF (ABS (ZEND-X).GT.400 (I)) GO TO 110
YES.....SAVE X AND Y, INTEGRATE TO XEND, RETURN.
FNEH=XEND-X
DC 103 I=1,NE
A(I)=ENDH*A(I)/
B(I)=ENDH*B(I)/H**2
C(I)=ENDH*C(I)/H**3

```

```

105 DD(I)=ENLH**4*D(I)/H**4
    SAVEX=X
    DO 800 I=1,NE
    SF(I)=F(I)
800 SAVEY(I)=Y(I)
    X=XEND
    DO 106 I=1,NE
    Y(I)=Y(I)+ENDH*(F(I)+AA(I)+BE(I)+CC(I)+DD(I))
106 FP(I)=F(I)+2.0*AA(I)+3.0*BE(I)+4.0*CC(I)+5.0*DD(I)
    ASSIGN 802 TO 1A
    GO TO 1101
802 DO805 I=1,NE
    SY(I)=Y(I)
805 CONTINUE
    DO 107 I=1,NE
    DELTA (I)=F(I)-FP(I)
107 Y(I)=Y(I)+V*DELTA (I)*ENDH
    ASSIGN 803 TO 1A
    GO TO 1101
803 DO 108 I=1,NE
    DALTA (I)=F(I)-FP(I)
108 Y(I)=SY(I)+V*DALTA (I)*ENDH
    JUMP=1
    GO TO 1101
C    KC.....TEST FOR DOUBLING. IF OK, BEGIN NEXT STEP AFTER
    R DOUBLING
110 IF(ABS (XEND-X).LE.ABS (2.*H)) GO TO 1000
    IF(KDELAY.LT.4) GO TO 1000
    IF(ABS (2*H).GT.ABS (HMAX)) GO TO 1000
    DO 125 I=1,NE
    IF(ABS (DALTA(I)).GT.ERROR/(128*ABS (H))) GO TO 1000
125 CONTINUE
    IF(V*P*CLIF.GE.0.3625) GO TO 1000
    H=2.0*H
    DO 135 I=1,NE
    A(I)=2.0*A(I)
    B(I)=4.0*B(I)
    C(I)=8.0*C(I)
135 D(I)=16.0*D(I)
    KDELAY=0
    GO TO 1000
1101 IF(JUMP) 2000,2003,2002
2003 CALL RHS
2010 IF(KCEN.EQ.0) GO TO 2004
    IF(KSTEP-23)200,201,201
201 KCC = KCC+1
    IF(KC-2500)202,202,2002
202 IF(KC-KK)200,203,203
203 KE = KCC+2
    KC = KC+1
    XN(KC) = Y(3)
    XT1(KC) = Y(1)

```

```

      XT2(KD) = Y(2)
      IF(KDD-KKK)200,204,204
2004 KKK = KDD+20
      VV = BETA-RHO
      PRINT 9997,X,Y(3),Y(1),Y(2),D,KSIEP,VV
9997 FORMAT(5X'E13.5,5X'E13.5,5X'E13.5,5X'E13.5,5X'E13.5,5X
F10.6)
      200 CONTINUE
9998 CONTINUE
      CLIF = 1.0
      GO TO 2004
2000 PRINT 2001
2001 FORMAT(52X'31EINTEGRATION: STEP SIZE TOO SMALL ')
2002 RETURN
      END

```

```

1.500E 01 0.000E 00 1.000E-02-5.100E-02 5.000E-02 1.000E 00
      0.000E 00 1.000E 00
1.000E 01 5.000E-01 0.000E 00 2.500E-01 2.000E-01 1.000E-01
      1.000E-04 0.000E 00
2.000E 02 1.000E 00
      -1.000E 00

```


LIST OF REFERENCES

- Akcasu, A. Ziya, and Noble, Larry D., "A Nonlinear Study of Reactor Systems with a Linear Feedback," Nucl. Sci. and Eng., 25, 47 (1966).
- Akcasu, A. Ziya, and Shotkin, Louis M., "Power Oscillations and the Describing Function in Reactors with Linear Feedback," Nucl. Sci. and Eng., 28, 72-81 (1967).
- Bethe, Hans A., "Reactor Safety and Oscillator Tests," APDA-117, Oct. (1956).
- Brehm, Richard L., Private Communication (1968).
- Brehm, Richard L., Hetrick, David L., and Schmidt, Theodore R., "Stability Study of Thermionic In-core Reactor Concepts," Nuclear Applications, (To be Published) (1969).
- Brehm, Richard L., Davis, Monte V., Hetrick, David L., Guppy, James G., and Schmidt, Theodore R., "Dynamics of a Fast Reactor with In-core Cesium Thermionic Converters," Engineering Experiment Station, University of Arizona, Nov. (1969).
- Brehm, Richard L., Hetrick, David L., Guppy, James G., Kearns, Kenneth D., and Schmidt, Theodore R., "The Second Annual Report on the Dynamics of a Fast Reactor with In-core Thermionic Converters," Engineering Experiment Station, University of Arizona, March (1969).
- Chernick, Jack, "Evolution of the Finite-Escape-Time Concept in Reactor Kinetics," Neutron Dynamics and Control, U.S.A.E.C., Symposium Series No. 7, 38 (1966).
- Gronroos, H., "Analog Studies of Thermionic Reactor Dynamics," JPL SPS 37-45, Vol. IV, pp. 151-162 (1967).
- Gyftopoulos, Elias P., "Applications of Geometric Theory to Nonlinear Reactor Dynamics," Nucl. Sci. and Eng., 10, 254 (1961).
- Hetrick, David L., "An Illustration of the Destabilizing Effects of Delayed Neutrons in the Linear Two-Temperature Reactor Model," Trans. Am. Nucl. Soc., 10, 248 (1967).

LIST OF REFERENCES--Continued

- Hetrick, David L., Unpublished Notes (1968).
- Hetrick, David L., and Schmidt, Theodore R., "An Example of Two-Region Reactor Stability with Delayed Neutrons," Engineering Experiment Station. University of Arizona, Sept. (1967).
- Hetrick, David L., Schmidt, Theodore R., and Davis, Monte V. "Stability of a Fast Reactor with In-core Cesium Thermionic Converters," Trans. Thermionic Conversion Specialist Conf., Palo Alto, California, Nov. (1967)
- Lellouche, Gerald S., "Reactor Kinetics Stability Criteria," Nucl. Sci. and Eng., 24, 72 (1966).
- Lewis, H. R., and Stovall, E. J., "Fortran Version of Nordsieck's Scheme for Numerical Integration of Differential Equations", Los Alamos Scientific Laboratory, LA-3292, March (1965).
- Magiros, Demetrios G., "Unification and Classification of Stability Concepts of Dynamical Systems," Symposium of Space Technology and Science, Japan, May (1967).
- Meenan, Peter, and Hetrick, David L., "Stability of the Two-Temperature Reactor Model with Delayed Neutrons," Trans. Am. Nucl. Soc., 9, 458 (1966).
- Miida, J., and Suda, N., "General Stability Criteria for Nuclear Reactor with Two Feedback Paths of Single Time Constant," Nucl. Sci. and Eng., 11, 55 (1961).
- Minorsky, Nicholas, Nonlinear Oscillations, D. Van Nostrand Co. Inc., New York (1962).
- Nelkin, M., Appendix in "The Effect of Temperature Coefficients on Reactor Stability and Reactor Transfer Function," Siegel, R., and Hurwitz, H. Jr., KAPL-1138, 53-57, March (1955).
- Rumsey V. H., "The Time Variation of Pile Power Due to Periodic Modulation of the Effective Multiplication Constant," CRP-333, PD-217 (1947).
- Sandmeier, Henry A., "Nonlinear Treatment of Large Perturbation in Power Reactor Stability," Nucl. Sci. and Eng., 6, 85 (1959).

LIST OF REFERENCES--Continued

- Shapiro, Miriam, AEC Computing and Applied Mathematics Center, New York University, New York, Nov. (1965).
- Shotkin, Louis M., "A Nonlinear Analysis of a Reactor with Two Temperature Coefficients," Nucl. Sci. and Eng., 18, 271 (1964).
- Smets, Henri B., "Power Oscillation in Nuclear Reactors," Revue A, Belgian Society for Automatic Control, 6, 137 (1964).
- Smets, Henri B., "Derivation of the Describing Function for a Reactor at Power Level," Nukleonik 1, 399 (1965).
- Smets, Henri B., "On the Effect of Delayed Neutrons in Reactor Dynamics" Nucl. Sci. and Eng., 25, 236 (1966).
- Torlin, H. D., "Calculations of Weak Self-Oscillation Regions in Nuclear Reactors," J. Nucl. Energy, 20, 485 (1966).
- Truxal, J. C., Control System Synthesis, McGraw-Hill Book Company, Inc., New York (1955).
- Wasserman, A. A., "Contributions to Two Problems in Space-Independent, Nuclear-Reactor Dynamics", AEC Research and Development Report Reactor Technology, TID-4500, March (1962).
- Weaver, Lynn E., System Analysis of Nuclear Reactor Dynamics, Rowman and Littlefield, Inc., New York (1963).
- Welton, T. A., "Kinetics of Stationary Reactor Systems," Proc. First International Conference on Peaceful Uses of Atomic Energy, 5, 377 (1955).