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A CONTROLLABILITY MINIMUM PRINCIPLE

by

Walter Jervis Grantham

A Dissertation Submitted to the Faculty of the
AEROSPACE AND MECHANICAL ENGINEERING DEPARTMENT

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In the Graduate College
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GRADUATE COLLEGE

I hereby recommend that this dissertation prepared under my
direction by Walter Jervis Grantham
entitled A Controllability Minimum Principle

be accepted as fulfilling the dissertation requirement of the
degree of Doctor of Philosophy

Thomas L. Vincent
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April 5, 1973
Date

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SIGNED: *Jacky Grantham*

DEDICATION

To Kitty, Cindy, Brian, and Michelle.

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ABSTRACT

The basic problem considered in this work is that of determining the boundary of the set of points in state space from which a given dynamical system, governed by a system of ordinary differential equations subject to control, can be controlled to a prescribed target set along system paths which do not leave a specified state constraint set.

One possible method of determining the boundary of the controllable set would be to examine all of the system paths, corresponding to all possible control functions, which start from a point in the state constraint set and reach the target without leaving the state constraint set. This method has obvious limitations. Another approach, which has been investigated extensively for linear systems, would be to seek necessary and (or) sufficient conditions under which the entire state space is controllable to the target.

Rather than pursue any of the previously mentioned approaches, we assume that not all of the state constraint set is controllable to the target and then derive necessary conditions for a control function to generate a system path which lies entirely in the boundary of the controllable set. The basic necessary condition is in the form of a Controllability Minimum Principle analogous to the "abnormal" case of Pontryagin's Maximum Principle for optimal control problems. The condition consists of a system of adjoint differential equations that must be satisfied and a scalar function of the state, adjoint, and

control variables which must be minimized by the control function which generates the path.

A restricted Controllability Minimum Principle is derived for the case where the controllable set is the closure of an open set with a smooth manifold for a boundary. The derivation is simple and involves little more than the Mean Value Theorem of the Calculus coupled with continuity and differentiability results from the theory of ordinary differential equations. After this introductory derivation, these assumptions regarding the structure of the controllable set are dropped and a more general Controllability Minimum Principle is derived in a manner similar to Pontryagin's proof of the Maximum Principle for optimal control problems.

For target and state constraint sets which are the closures of open sets with smooth manifolds for boundaries, we derive boundary, or "transversality," conditions for the adjoint variables at points where system paths in the boundary of the controllable set reach the target or the boundary of the state constraint set. Since we are concerned only with determining the boundary of the controllable set and since the state constraint set is given, we do not follow the usual approach employed in dealing with state constrained optimal control problems. Specifically, we do not derive special adjoint equations and "jump" conditions which must be satisfied for a control to generate a path which lies in the intersection of the boundary of the controllable set with the boundary of the state constraint set. We merely derive transversality conditions for the usual adjoint variables at points where such a path may reach the boundary of the state constraint set.

The use of the Controllability Minimum Principle and of the Target and State Constraint Transversality Conditions is demonstrated through two simple problems from the field of optimal control.

CHAPTER 1

INTRODUCTION

Given a specified system subject to control, such as the national economy or an anti-ballistic missile, and a prescribed goal for the system, such as achieving a desired gross national product or intercepting an incoming missile, is it possible to control the system so that the goal is accomplished? These are examples of a general problem in systems and controls engineering known as controllability.

This dissertation is concerned with the controllability of a particular class of systems. Specifically, we shall consider a dynamical system whose state is governed by a system of ordinary differential equations subject to control and with the state constrained to lie in a prescribed subset of state space. For a given target set in state space we wish to determine the set of points in state space, called the controllable set, from which the system may be transferred to the target along paths which do not leave the state constraint set. We impose no constraint on the time required for such a transfer to occur other than the requirement that the transfer time be finite. In particular, we are not concerned with determining the set of points which are controllable to the target within some specified time.

The usual aim of controllability studies is to develop necessary and (or) sufficient conditions for a system without state constraints to be controllable to a point target from any other point in state space.

For example, Kalman, Ho, and Narendra (1962, pp. 189-213) show that a system of the form

$$\dot{x} = Ax + Bu,$$

where the state $x \in E^n$ and control $u \in E^m$ are unconstrained and where A and B are constant $n \times n$ and $n \times m$ matrices respectively, is controllable to any point in state space from any other point in state space if and only if the $n \times nm$ matrix

$$[B, AB, A^2B, \dots, A^{n-1}B]$$

is of rank n .

Rather than seek conditions under which the controllable set for a linear system is all of state space, we shall consider the case where the system, in general, is nonlinear and the controllable set is not all of state space. For such a situation, the controllable set has a boundary and it is this boundary that we wish to determine. Moreover, with a prescribed state constraint set (containing the controllable set), those points, if any, in the intersection of the boundary of the controllable set with the boundary of the state constraint set are considered known. Thus we will direct our attention to the problem of determining that portion of the boundary of the controllable set which lies in the interior of the state constraint set.

Under the assumption that the controllable set is closed, one can show (see Section 3.1) that in the absence of a state constraint there exist system paths, for the system we consider, from every point in the boundary of the controllable set to the target with the paths remaining entirely in the boundary of the controllable set. In the presence of a state constraint, the assumption that the controllable set is closed guarantees that from every point in the intersection of

the boundary of the controllable set with the interior of the state constraint set there exist system paths which remain in the boundary of the controllable set until they either reach the target or the boundary of the state constraint set.

Our approach to determining the boundary of the controllable set will be to develop necessary conditions governing a control function which generates a system path lying in the boundary of the controllable set. In addition, for a particular class of target and state constraint sets, we shall develop conditions governing the location of points where the boundary of the controllable set may intersect either the target or the boundary of the state constraint set.

The basic necessary condition is in the form of a Controllability Minimum Principle which is essentially the "abnormal" case (Blaquiere and Leitmann, 1967, p. 350) of Pontryagin's Maximum Principle (Pontryagin et al. 1962) for optimal control problems. Blaquiere and Leitmann (1967, pp. 336-350) give an explicit derivation of the abnormal case of Pontryagin's Maximum Principle for the same class of systems which we consider, but without state constraints. Their derivation, if closely examined, is seen to be totally independent of any optimization criterion and is, in effect, a statement of controllability although this was not their aim according to Dr. Leitmann (Leitmann 1972). Blaquiere and Leitmann's derivation assumes that the boundary of the controllable set, when projected orthogonally from state space into augmented (state-cost) space, possesses a unique tangent plane at every point. The more general abnormal case of the Maximum Principle, where the boundary of the controllable set may possess "corners," is not derived explicitly by Blaquiere and Leitmann, but is stated by

means of an analogy between the boundary of the controllable set, projected into augmented space, and another surface in augmented space, called a Σ -surface, which is used to establish the "normal" case of the optimal control Maximum Principle. This Σ -surface is defined in terms of the optimization criterion and its properties, which are employed in the analogy, are therefore dependent on the optimization criterion. Since Blaquiere and Leitmann's proof of the "normal" case of Pontryagin's Maximum Principle is long and involved, it is extremely difficult to verify that their proof through analogy is independent of the optimization criterion, as it must be if the theorem is to be interpreted as a statement of controllability.

In addition to the above difficulty, the conditions of optimality as stated by Pontryagin and his associates must hold for all times in the time interval of consideration for the case where the class of admissible control functions consists of all piecewise continuous functions of time. Blaquiere and Leitmann did not establish the conditions of optimality "everywhere" on the time interval of consideration for the general case where their Σ -surface may possess corners nor did they establish the weaker requirement that the conditions must hold "almost everywhere" in the time interval since they did not allow zero "velocity" vectors. They did, however, establish the requirement for the special case where their Σ -surface had a unique tangent plane at every point.

Using the techniques of Blaquiere and Leitmann, and Pontryagin and his associates, we will prove our Controllability Minimum Principle, for a slightly less restrictive case than the "general case" considered by Blaquiere and Leitmann, without any reference to an optimization criterion and we shall verify that the conditions of our theorem must,

indeed, hold everywhere in the time interval of consideration. In addition, we will allow the state of the system to be constrained to lie in a specified subset (or all) of state space. In this connection our objectives will differ slightly from those of other authors discussing optimal control problems with state constraints (Berkovitz 1962; Bryson, Denham, and Dreyfus 1963; Leitmann 1966; Litt 1971). Since we are only interested in determining that portion of the boundary of the controllable set which lies in the interior of the state constraint set, we will not derive special adjoint equations and "jump" conditions necessary to yield a control which will transfer the system along a portion of the boundary of the controllable set which coincides with the (known) boundary of the state constraint set. Rather, we shall consider system paths in the boundary of the controllable set which, with the possible exception of their endpoints, lie in the interior of the state constraint set. For such paths we will develop a Target Transversality Condition and a State Constraint Transversality Condition for the usual adjoint equations at points where the path may reach the target or the boundary of the state constraint set respectively.

In Chapter 2 the use of the Controllability Minimum Principle and the Target and State Constraint Transversality Conditions is demonstrated through two simple examples from the field of optimal control. In Chapter 3 we develop various properties of the controllable set to be used in the proof of the Controllability Minimum Principle. In Chapter 4 a restricted Controllability Minimum Principle is established for the case where the boundary of the controllable set is a smooth (C^1) $n-1$ dimensional manifold in Euclidian n -space. This proof follows that of Leitmann (1966) for the optimal control Maximum Principle with

a correspondingly smooth Σ -surface and is presented as an introduction to the proof in Chapter 5 of a more general Controllability Minimum Principle and as an indication of the geometric nature of controllability. In Chapter 5 the Controllability Minimum Principle is proven, using the techniques of Pontryagin and his associates, without any assumptions regarding the structure of the controllable set except for the requirement that the tangent cone to the controllable set (see Section 3.2) must possess a nonempty interior at the final point of the system path being considered. This requirement is essentially the non-degeneracy requirement of Blaquiere and Leitmann. In Chapter 6, for a class of target and state constraint sets, we derive Target and State Constraint Transversality Conditions governing the adjoint variables at points where a system path, in the intersection of the boundary of the controllable set with the interior of the state constraint set, may reach the target or the boundary of the state constraint set.

CHAPTER 2

INTRODUCTION TO THE CONTROLLABILITY MINIMUM PRINCIPLE AND TRANSVERSALITY CONDITIONS

In this chapter we present some necessary conditions for a system path, with the possible exception of its end points, to lie in the intersection of the boundary of the controllable set with the interior of the state constraint set. We do not derive these conditions at this time, but merely present them and demonstrate their usage by means of two simple examples.

2.1 The Dynamical System

We shall consider dynamical systems governed by state equations of the form

$$\dot{x} = f(x,u) \quad (2.1.1)$$

where $x = [x_1, \dots, x_n]^T \in X \subseteq E^n$ is the state of the system, which we require to lie in a given closed subset (possibly all) of Euclidian n -space (E^n) called the state constraint set X , and where the control is $u = [u_1, \dots, u_m]^T \in \Omega \subseteq E^m$ which we require to lie in a given subset (possibly all) of Euclidian m -space (E^m) called the control constraint set Ω . We denote differentiation with respect to time, t , by $(\dot{\cdot})$ and we denote the transpose of a matrix A by A^T . The function $f(x,u) = [f_1(x,u), \dots, f_n(x,u)]^T$ and its $n \times n$ matrix of partial derivatives with respect to x ,

$$\frac{\partial f(x,u)}{\partial x} \triangleq \begin{bmatrix} \frac{\partial f_1(x,u)}{\partial x_1} & \dots & \frac{\partial f_1(x,u)}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n(x,u)}{\partial x_1} & \dots & \frac{\partial f_n(x,u)}{\partial x_n} \end{bmatrix}, \quad (2.1.2)$$

are assumed to be continuous for all $x \in \chi$ and for all $u \in \Omega$ where χ is an open subset (or all) of E^n containing the closed state constraint set X .

After specifying the control $u = u(t)$ as a function of time on some time interval $[t_0, t_1]$, the state equations (2.1.1) become ordinary differential equations. We restrict our class of admissible controls to functions $u(\cdot): [t_0, t_1] \rightarrow E^m$ such that $u(\cdot)$ is defined and piecewise continuous on $[t_0, t_1]$ and $u(t) \in \Omega$ for all $t \in [t_0, t_1]$.

From the theory of ordinary differential equations (Coddington and Levinson 1955), the specification of an admissible (piecewise continuous) control $u(\cdot): [t_0, t_1] \rightarrow \Omega$, a time $\bar{t} \in [t_0, t_1]$, and a point $\bar{x} \in \chi$, determines a unique solution $x(t; \bar{x}, \bar{t}) \in \chi \subseteq E^n$ to the state equations defined on some open time interval containing $\bar{t}$. Henceforth we assume that the time interval $[t_0, t_1]$ is such that the solution $x(t; \bar{x}, \bar{t})$ is defined for all $t \in [t_0, t_1]$. In addition to the solution being unique (for a given control function) it is also continuous in t , $\bar{x}$, and $\bar{t}$, continuously differentiable in $\bar{x}$, and at worst piecewise differentiable in t . These properties will be useful later in the proof of the Controllability Minimum Principle. Except when it is necessary to display the explicit dependence of the solution $x(t; \bar{x}, \bar{t})$ on the boundary condition parameters $\bar{x}$ and $\bar{t}$, we will simply denote

a solution to the state equations by $x(\cdot): [t_0, t_1] \rightarrow \chi$. To indicate the dependence of the solution on the particular control function employed, we will say that $x(\cdot): [t_0, t_1] \rightarrow \chi$ is generated by $u(\cdot): [t_0, t_1] \rightarrow \Omega$.

2.2 Soft and Hard State Constraints

We have assumed that the function $f(x, u)$ and its partial derivatives with respect to the state variables are continuous for all $x \in \chi$ and all $u \in \Omega$ in order to guarantee the existence of solutions to the state equations in a neighborhood of any point in the closed state constraint set $X \subset \chi \subseteq E^n$ including points in ∂X (the boundary of X). Such an assumption, however, also guarantees the existence of solutions to the state equations from points in $\chi - X$ (the set of points in χ but not in X) since χ is an open set containing the closed set X . Thus we restrict ourselves to what we call "soft" state constraints, which the system can violate (mathematically), rather than to "hard" state constraints which the system cannot violate. As an example of a hard state constraint, to which the theory we present may not apply, consider the scalar system

$$\dot{x} = \sqrt{x} + u. \quad (2.2.1)$$

For the right-hand side of (2.2.1) to be defined and real valued for real x and u , we must impose the state constraint

$$x \in X \triangleq \{x: x \geq 0\}.$$

Since the right-hand side of (2.2.1) is not real valued for $x < 0$, no open set χ containing X exists such that $f(x, u) = \sqrt{x} + u$ is even defined for all $x \in \chi$, much less continuous in x .

2.3 The Controllable Set

We are concerned, basically, with the following problem: given that the system (2.1.1) is at some point $x^0 \in X$ at time t_0 , does there exist a time $t_1 \geq t_0$ and an admissible control $u(\cdot): [t_0, t_1] \rightarrow \Omega$ which will transfer the system to a given target set $\Theta \subset X$ along a system path which does not leave the state constraint set X ?

We define the controllable set C as the set of all points $x^0 \in X$ for which there exists a time interval $[t_0, t_1]$, an admissible control $u(\cdot): [t_0, t_1] \rightarrow \Omega$, and a corresponding solution $x(\cdot): [t_0, t_1] \rightarrow X$ to the state equations (2.1.1), with $x(t_0) = x^0$, such that

- (i) $x(t) \in X$ for all $t \in [t_0, t_1]$;
- (ii) $x(t_1) \in \Theta$.

We assume that the controllable set is not all of X and, for notational reasons, we denote the boundary of the controllable set by Ξ . Except for this one instance, we adhere to the usual topological notations. If S is a set in E^n then

$$\begin{aligned}\bar{S} &\triangleq \text{closure of } S \\ \overset{\circ}{S} &\triangleq \text{interior of } S \\ \text{Comp } S &\triangleq \text{complement of } S \triangleq E^n - S \\ \partial S &\triangleq \text{boundary of } S = \bar{S} \cap \overline{\text{Comp } S}\end{aligned}$$

where $E^n - S$ is the set of points in E^n but not in S and where $\bar{S} \cap \overline{\text{Comp } S}$ denotes the intersection of $\bar{S}$ and $\overline{\text{Comp } S}$. We denote the union of two sets A and B by $A \cup B$ and we denote the empty set by ϕ .

2.4 The Controllability Minimum Principle

In Chapter 5 we derive the following necessary condition governing an admissible control that generates a path which, with the possible exception of its end points, lies in the intersection of the boundary of the controllable set with the interior of the state constraint set:

Theorem 2.1: (Controllability Minimum Principle)

Let $u^*(\cdot): [t_0, t_1] \rightarrow \Omega$ be an admissible control which generates a solution $x^*(\cdot): [t_0, t_1] \rightarrow \Xi$ to the state equations

$$\dot{x} = f(x, u) \quad (2.4.1)$$

such that $x^*(t) \in \Xi$ for all $t \in [t_0, t_1]$ and $x^*(t) \in \overset{\circ}{X}$ for all $t \in (t_0, t_1)$. Let δ be an arbitrarily small positive number if $x^*(t_1) \in \partial X$ and let δ be zero if $x^*(t_1) \in \overset{\circ}{X}$. Let $C_I[x^*(t_1 - \delta)]$ denote the interior of the tangent cone (see Section 3.2) to $\bar{C}$ at $x^*(t_1 - \delta)$.

If $\overset{\circ}{C}_I[x^*(t_1 - \delta)]$ is nonempty then, with $\lambda = [\lambda_1, \dots, \lambda_n]^T \in E^n$ and

$$H(\lambda, x, u) \triangleq \lambda^T f(x, u), \quad (2.4.2)$$

there exists a nonzero continuous solution $\lambda(\cdot): [t_0, t_1] \rightarrow E^n$ to the adjoint equations

$$\dot{\lambda}^T \triangleq - \frac{\partial H[\lambda, x^*(t), u^*(t)]}{\partial x} \triangleq - \lambda^T \frac{\partial f[x^*(t), u^*(t)]}{\partial x} \quad (2.4.3)$$

such that

$$H[\lambda(t), x^*(t), u^*(t)] = \min_{u \in \Omega} H[\lambda(t), x^*(t), u] = 0 \quad (2.4.4)$$

for all $t \in [t_0, t_1]$.

In examining Theorem 2.1 we see that the basic premise is that the control $u^*(\cdot)$ generates a solution $x^*(\cdot)$ to the state equations with the property that $x^*(t)$ is in the boundary of the controllable set for all $t \in [t_0, t_1]$ and in the interior of the state constraint set for all $t \in (t_0, t_1)$. Given this premise, the only assumption we make regarding the structure of the controllable set is that the controllable set has a tangent cone with a nonempty interior at the terminal point $x^*(t_1)$, if $x^*(t_1) \in \overset{\circ}{X}$, or that such a tangent cone exists "just before" the terminal point if $x^*(t_1) \in \partial X$. This is equivalent to requiring that the interior of the controllable set be nonempty at the terminal point (or just before it) and that the boundary of the controllable set not have a "cusp" at (or just before) the terminal point. Thus the assumption is analogous to the nondegeneracy assumption of Blaquiere and Leitmann (1967, p. 323) for optimal control problems and to the Kuhn-Tucker Constraint Qualification condition (Kuhn and Tucker 1951, p. 483) for nonlinear programming problems. In Chapter 4 we make the much stronger assumption that the controllable set is the closure of an open set with a smooth $n-1$ dimensional manifold as its boundary in order to present an uncomplicated proof of the Controllability Minimum Principle.

2.5 A Ship Navigation Problem

As an example of the application of the Controllability Minimum Principle, consider the problem of steering a ship in the middle of an ocean. We assume that the water has a constant speed $V_w(>0)$ in the negative x_1 direction relative to the stationary x_1 - x_2 coordinate system shown in Figure 2.1. If we take the speed of the ship relative to the water as a constant, $V_s(\geq 0)$, and denote the counterclockwise

Fig. 2.1 Coordinate System for the Ship Navigation Problem

angle from the positive x_1 -axis to the ship's relative velocity vector by u , then the state equations may be written as

$$\dot{x}_1 = V_S \cos(u) - V_W$$

$$\dot{x}_2 = V_S \sin(u).$$

We take the target as the origin and we impose no state constraint.

If $V_W > V_S$ then the ship cannot move in the positive x_1 direction and we may surmise that the controllable set to the origin is not all of E^2 .

From (2.4.2)

$$H \triangleq \lambda_1 [V_S \cos(u) - V_W] + \lambda_2 V_S \sin(u) \quad (2.5.3)$$

so that the adjoint equations (2.4.3) become

$$\dot{\lambda}_1 = - \frac{\partial H}{\partial x_1} = 0$$

$$\dot{\lambda}_2 = - \frac{\partial H}{\partial x_2} = 0$$

yielding the solutions

$$\lambda_1 = \text{constant} = \lambda_1(t_1)$$

$$\lambda_2 = \text{constant} = \lambda_2(t_1).$$

The requirement, from (2.4.4), that $u^*(t)$ minimize H to zero yields the matrix equation

$$\begin{bmatrix} H \\ \frac{\partial H}{\partial u} \end{bmatrix} = \begin{bmatrix} V_S \cos[u^*(t)] - V_W & V_S \sin[u^*(t)] \\ -V_S \sin[u^*(t)] & V_S \cos[u^*(t)] \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which has a nonzero solution $\lambda = [\lambda_1, \lambda_2]^T$ if and only if

$$V_S^2 - V_W V_S \cos[u^*(t)] = 0,$$

which implies that

$$\cos[u^*(t)] = V_s/V_w.$$

Here we have used the fact that $V_s = 0$ and $\cos [u^*(t)] = V_s/V_w = 0$ both yield the same state equations.

Using the constant control

$$u^*(t) = \text{constant} = \cos^{-1}(V_s/V_w),$$

the solutions to the state equations are straight lines which we require to pass through the origin. This boundary condition on the state equations is established in Section 3.1 where we demonstrate that if the controllable set is closed then the boundary of the controllable set must intersect the target if no state constraint is imposed.

Figure 2.2 shows the boundary of the controllable set for the speed ratios $V_s/V_w = 0, 0.5$, and 1.0 . For all speed ratios in the range $0 \leq V_s/V_w < 1$, the controllable set is closed and the boundary condition imposed on the state equations is justified. For $0 < V_s/V_w < 1$, the controllable set is the closure of an open set whose boundary (except for the point at the origin) is a smooth manifold (compare with Chapter 4). For $V_s/V_w = 0$, that is for $V_s = 0$, the controllable set is the non-negative portion of the x_1 -axis and every point in the controllable set is a boundary point.

The special case where $V_s/V_w = 1$ deserves closer examination. For this case the controllable set (excluding the origin) is the open half plane given by the relation

$$C = \{x \in E^2: x_1 > 0\}.$$

For this case the boundary of the controllable set (the x_2 -axis) still intersects the target, as it would if the controllable set were closed,

Fig. 2.2 Controllable Sets for the Ship Navigation Problem when the Target is the Origin

but an examination of the state equations for $\cos[u^*(t)] = V_S/V_W = 1$ indicates that the system is static; if the system is initially at a point it remains at the point. This condition does not violate the Controllability Minimum Principle since a "null path" in the boundary of the controllable set is a perfectly valid path. The interesting feature of this case is that while the boundary of the controllable set intersects the target, no path in the boundary of the controllable set exists from a point not in the target to the target. This situation, where no path to the target exists in the boundary of the controllable set, has been observed for other systems and it may well be the "typical" situation for the case where the target is closed and the controllable set (excluding points of intersection with the target) is open. In any event, the Controllability Minimum Principle indicated the (possible) existence of the phenomenon even though the resulting system paths were null paths.

Finally, we note that the case $V_S/V_W > 1$ (which the Controllability Minimum Principle eliminates as a possibility) corresponds to the entire x_1 - x_2 plane being controllable to the target so that the controllable set has no boundary. Again, application of the Controllability Minimum Principle indicated the (possible) occurrence of the phenomenon.

2.6 A Target Transversality Condition

In the previous example no boundary conditions were specified for the adjoint variables. These variables, as it turned out, were constants which could be eliminated from the control law that evolved from (2.4.4). The paths in the boundary of the controllable set were

then determined by requiring that they intersect the target. Since the adjoint variables are solutions to differential equations, they cannot, in general, be eliminated from the control law and the adjoint equations must be integrated, usually numerically, along with the state equations in order to furnish enough information to determine the control $u^*(t)$. To perform the integration, boundary conditions must be specified for the adjoint variables. From an examination of the Controllability Minimum Principle, we see that the theorem merely states that "a" solution to the adjoint equations exists such that the control $u^*(\cdot)$ minimizes H to zero; nothing is said about "which" solution to the adjoint equations we should choose.

Fortunately, for a particular class of target sets, boundary conditions for the adjoint variables exist which not only completely determine the solution to the adjoint equations, but also satisfy the nondegeneracy condition that the interior of the tangent cone to the boundary of the controllable set be nonempty at the terminal point of a path in the boundary of the controllable set.

In this section we consider target sets of the form

$$\Theta = \{x: h(x) \leq 0\}, \quad (2.6.1)$$

where $h(\cdot): E^n \rightarrow E^1$ is C^1 with a nonzero gradient, given by the (row) matrix

$$\nabla h(x) \triangleq \left[\frac{\partial h(x)}{\partial x_1}, \dots, \frac{\partial h(x)}{\partial x_n} \right] \neq 0, \quad (2.6.2)$$

for all x in a neighborhood of the target boundary

$$\partial\Theta = \{x: h(x) = 0\}. \quad (2.6.3)$$

For such target sets the nondegeneracy condition of the Controllability Minimum Principle is satisfied and we have

Theorem 2.2: (A Target Transversality Condition)

Let $u^*(\cdot): [t_0, t_1] \rightarrow \Omega$ be an admissible control which generates a solution $x^*(\cdot): [t_0, t_1] \rightarrow \Xi$ to the state equations with $x^*(t) \in \Xi \cap \overset{\circ}{X} - \emptyset$ for all $t \in (t_0, t_1)$ and with $x^*(t_1) \in \Xi \cap \partial\emptyset \cap \overset{\circ}{X}$.

If the target $\emptyset$ is of the form (2.6.1) - (2.6.3) then there exists a solution $\lambda(\cdot): [t_0, t_1] \rightarrow E^n$ to the adjoint equations such that $u^*(\cdot)$ minimizes H to zero and such that $\lambda(\cdot)$ satisfies the boundary (or transversality) condition

$$\lambda^T(t_1) = k \nabla h[x^*(t_1)] \quad (2.6.4)$$

where k is any positive number.

This theorem implies that $\lambda(t_1)$ is an "outward" normal to the boundary of the target set at points where the boundary of the controllable set intersects the boundary of the target set, providing such points occur in the interior of the state constraint set.

As an application of the Target Transversality Condition, we consider the example of Section 2.5 for the case where the target is a circular disk of radius R centered at the origin. That is,

$$\emptyset = \{x \in E^2: x_1^2 + x_2^2 - R^2 \leq 0\}$$

and

$$\partial\emptyset = \{x \in E^2: x_1^2 + x_2^2 - R^2 = 0\}.$$

Thus the boundary conditions on the solution $\lambda(\cdot): [t_0, t_1] \rightarrow E^2$ to the adjoint equations are taken as

$$\lambda_1(t_1) = 2kx_1^*(t_1)$$

$$\lambda_2(t_1) = 2kx_2^*(t_1)$$

where k is any positive number and $x^*(t_1)$ is any point where a path in

the boundary of the controllable set intersects the boundary of the target set.

As for the boundary conditions on the state variables, the specification, in the previous example, of a point target and the requirement that the boundary of the controllable set must intersect the target (for a closed controllable set), were sufficient to determine the state equation boundary conditions. Now, however, all we know is that the end point $x^*(t_1)$ of a path which reaches the boundary of the target set must satisfy the relation

$$x_1^{*2}(t_1) + x_2^{*2}(t_1) = R^2$$

or, in polar coordinates,

$$x_1^*(t_1) = R \cos \gamma$$

$$x_2^*(t_1) = R \sin \gamma$$

where γ is the counterclockwise angle from the positive x_1 - axis to the radius vector from the origin to the point $x^*(t_1)$.

From the requirement that

$$H[\lambda(t_1), x^*(t_1), u^*(t_1)] = 0,$$

we get

$$R \cos \gamma [V_s \cos[u^*(t_1)] - V_w] + R V_s \sin \gamma \sin[u^*(t_1)] = 0$$

or

$$\tan \gamma = [V_w - V_s \cos[u^*(t_1)]] / V_s \sin[u^*(t_1)].$$

Thus the control law

$$u^*(t_1) = \cos^{-1}(V_s/V_w)$$

yields

$$\tan \gamma = \pm \left[1 - \left(\frac{V_s}{V_w} \right)^2 \right]^{\frac{1}{2}}.$$

Figure 2.3 shows the boundary of the controllable set for the speed ratios $V_S/V_W = 0, 0.5$, and 1.0 . For all speed ratios in the range $0 \leq V_S/V_W < 1$, the controllable set is the closure of an open set. As before, the case $V_S/V_W = 1$ yields an open controllable set (excluding points where the boundary of the controllable set intersects the target).

2.7 A State Constraint Transversality Condition

In the presence of a state constraint, the boundary of the controllable set may intersect the boundary of the state constraint set. In this section we present a theorem which is useful for determining boundary conditions for the adjoint equations at points where a path, in the intersection of the boundary of the controllable set with the interior of the state constraint set, may reach the boundary of the state constraint set.

We consider state constraint sets of the form

$$X = \{x: m(x) \leq 0\} \quad (2.7.1)$$

where $m(\cdot): E^n \rightarrow E^1$ is C^1 with nonzero gradient, represented by the (row) matrix

$$\nabla m(x) = \left[\frac{\partial m(x)}{\partial x_1}, \dots, \frac{\partial m(x)}{\partial x_n} \right] \neq 0, \quad (2.7.2)$$

for all x in a neighborhood of the state constraint boundary

$$\partial X = \{x: m(x) = 0\}. \quad (2.7.3)$$

We assume that the controllable set has the same type of structure as that of the state constraint set in a neighborhood N of a point where the boundary of the controllable set intersects the boundary of the state constraint set. That is,

$$C \cap N = \{x \in N: g(x) \leq 0\} \quad (2.7.4)$$

Fig. 2.3 Controllable Sets for the Ship Navigation Problem when the Target is a Circular Disk at the Origin

and

$$\Xi \cap N = \{x \in N : g(x) = 0\} \quad (2.7.5)$$

where $g(\cdot): E^n \rightarrow E^1$ is C^1 with nonzero gradient, represented by the (row) matrix

$$\nabla g(x) = \left[\frac{\partial g(x)}{\partial x_1}, \dots, \frac{\partial g(x)}{\partial x_n} \right] \neq 0, \quad (2.7.6)$$

for all $x \in \Xi \cap N$.

With such an assumption concerning the structure of the controllable set, we have

Theorem 2.2: (A State Constraint Transversality Condition)

Let $u^*(\cdot): [t_0, t_1] \rightarrow \Omega$ be an admissible control which generates a solution $x^*(\cdot): [t_0, t_1] \rightarrow \Xi$ to the state equations with $x^*(t) \in \Xi$ for all $t \in [t_0, t_1]$, $x^*(t) \in \overset{\circ}{X}$ for all $t \in (t_0, t_1)$, and $x^*(t_1) \in \partial X$.

If the state constraint set and the controllable set are of the form (2.7.1) - (2.7.3) and (2.7.4) - (2.7.6), respectively, then there exists a solution $\lambda(\cdot): [t_0, t_1] \rightarrow E^n$ to the adjoint equations such that $u^*(\cdot)$ minimizes H to zero and such that $\lambda(\cdot)$ satisfies the boundary (or transversality) condition

$$\lambda^T(t_1) = k \nabla g[x^*(t_1)]$$

where k is any positive number.

As with the Target Transversality Condition, Theorem 2.3 implies that $\lambda(t_1)$ is an "outward" normal to the boundary of the state constraint set at points where the boundary of the controllable set meets the boundary of the state constraint set.

2.8 A Regulator Problem

As an example of the application of the State Constraint Transversality Condition, consider the problem of controlling the system

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = u$$

to the origin, with $u \in E^1$ and $|u| \leq 1$, in the presence of a state constraint set of the form

$$X = \{(x_1, x_2): x_1 - a \leq 0; a > 0\}.$$

To determine if there exists a system path, in the intersection of the boundary of the controllable set with the interior of the state constraint set, which reaches the boundary of the state constraint set, we define

$$H \triangleq \lambda_1 x_2 + \lambda_2 u.$$

With this definition, the adjoint equations become

$$\dot{\lambda}_1 = - \frac{\partial H}{\partial x_1} = 0$$

$$\dot{\lambda}_2 = - \frac{\partial H}{\partial x_2} = - \lambda_1$$

yielding the solutions

$$\lambda_1 = \text{constant} = \lambda_1(t_1)$$

$$\lambda_2 = \lambda_1(t_1)[t_1 - t] + \lambda_2(t_1).$$

From the State Constraint Transversality Condition, with

$$m(x) \triangleq x_1 - a,$$

we get

$$\lambda_1(t_1) = k \frac{\partial m[x^*(t_1)]}{\partial x_1} = k > 0$$

$$\lambda_2(t_1) = k \frac{\partial m[x^*(t_1)]}{\partial x_2} = 0 .$$

Thus the solutions to the adjoint equations become

$$\lambda_1(t) = k > 0$$

$$\lambda_2(t) = k[t_1 - t] .$$

For $|u| \leq 1$, the control $u^*(t)$ necessary to minimize H to zero is

$$u^*(t) = -\operatorname{sgn}[\lambda_2(t)] .$$

Since $k > 0$ implies $\lambda_2(t) > 0$ for all $t < t_1$, the control necessary to generate a path, in the boundary of the controllable set, which reaches the boundary of the state constraint set is

$$u^*(t) = -1$$

for all $t < t_1$.

For the boundary conditions on the state variables, we require $x^*(t_1) \in \partial X$ which gives

$$x_1^*(t_1) = a .$$

To determine $x_2^*(t_1)$ we use $H[\lambda(t_1), x^*(t_1), u^*(t_1)] = 0$. For $\lambda_1(t_1) = k > 0$ and $\lambda_2(t_1) = 0$, we get

$$x_2^*(t_1) = 0 .$$

Figure 2.4 shows the resulting controllable set obtained by integrating the state equations "backward" from the point $x_1^*(t_1) = a$, $x_2^*(t_1) = 0$ with the control $u^*(t) = -1$, $t < t_1$. The controllable set is the closure of an open set with its boundary consisting of a half-parabola (generated by $x^*(t)$, $t < t_1$) for $x_2 \geq 0$ and the line $x_1 = a$

Fig. 2.4 Controllable Set for the Regulator Problem

for $x_2 \leq 0$. Figure 2.4 also shows the adjoint vector $\lambda(t_1)$ at the point where the two portions of the boundary of the controllable set meet.

This example illustrates the fact that the imposition of a state constraint, on a system whose (unconstrained) controllable set is all of state space, may induce a controllability boundary in the interior of the state constraint set. The half parabola in Figure 2.4 is such a boundary; points above the parabola are outside the controllable set not because their paths miss the target, but because the paths eventually leave the state constraint set.

CHAPTER 3

ANALYSIS OF THE CONTROLLABLE SET

In this chapter we establish some preliminary results which will be used in the subsequent proofs of the Controllability Minimum Principle and of the Target and State Constraint Transversality Conditions.

3.1 Some Properties of the Controllable Set

Since the basic assumption of this dissertation is that the boundary of the controllable set, in many cases, is composed of system paths, we first present results which substantiate this assumption. Because we are primarily, but not exclusively, concerned with system paths in the interior of the state constraint set (except possibly for their end points), many of our results will be similar to those obtained by Blaquiere and Leitmann (1967) for optimal control problems without state constraints.

Many interesting results, such as relations between the controllable sets obtained with and without state constraints, can be established. In the interest of brevity, however, we shall confine ourselves to establishing only those properties of the controllable set which have a direct bearing on the proofs of the Controllability Minimum Principle and the Target and State Constraint Transversality Conditions.

If a path starts outside the controllable set, and does not leave the state constraint set, it remains outside the controllable set:

Lemma 3.1 If $x(\cdot): [t_0, t_1] \rightarrow X$ is a solution to the state equations, generated by an admissible control $u(\cdot): [t_0, t_1] \rightarrow \Omega$ (see Section 2.1) with $x(t) \in X$ for all $t \in [t_0, t_1]$, then

$$x(t_0) \in \text{Comp } C$$

implies

$$x(t) \in \text{Comp } C$$

for all $t \in [t_0, t_1]$.

proof: Suppose $x(\bar{t}) \in C$ for some $\bar{t} \in (t_0, t_1]$. Then, from the definition of C , there exists a $\bar{t}_1 > \bar{t}$, an admissible control $\bar{u}(\cdot): [\bar{t}, t_1] \rightarrow \Omega$, and a corresponding solution $\bar{x}(\cdot): [\bar{t}, \bar{t}_1] \rightarrow X$ to the state equations, generated by $\bar{u}(\cdot)$, such that $\bar{x}(t) \in X$ for all $t \in [\bar{t}, \bar{t}_1]$, $\bar{x}(\bar{t}) = x(\bar{t})$ and $\bar{x}(\bar{t}_1) \in \Theta$. Thus there exists a solution

$$\tilde{x}(t) \triangleq \begin{cases} x(t) & t \in [t_0, \bar{t}] \\ \bar{x}(t) & t \in [\bar{t}, \bar{t}_1] \end{cases}$$

to the state equations, generated by the admissible control

$$\tilde{u}(t) \triangleq \begin{cases} u(t) & t \in [t_0, \bar{t}] \\ \bar{u}(t) & t \in (\bar{t}, \bar{t}_1] \end{cases},$$

such that $\tilde{x}(t) \in X$ for all $t \in [t_0, \bar{t}_1]$ and $\tilde{x}(\bar{t}_1) \in \Theta$. But this contradicts $\tilde{x}(t_0) = x(t_0) \in \text{Comp } C$.

The restriction, in Lemma 3.1, that $x(t)$ lie in X for all t in the interval $[t_0, t_1]$ is an essential one since it may be possible to

transfer the system from a point $x(t_0) \in \text{Comp } C$ to a point $x(t_1) \in \emptyset$ by doing so along a path which leaves the state constraint set. For example, this condition exists in the regulator problem of section 2.8 for any point above the parabolic portion of the boundary of the controllable set.

The following result, which essentially states that no path in $\overset{\circ}{X}$ may "enter" the interior of the controllable set, will play a fundamental role in the proof of the Controllability Minimum Principle:

Lemma 3.2 If $x(\cdot): [t_0, t_1] \rightarrow X$ is a solution to the state equations, generated by an admissible control $u(\cdot): [t_0, t_1] \rightarrow \Omega$, with $x(t) \in \overset{\circ}{X}$ for all $t \in [t_0, t_1]$, then

$$x(t_0) \in \overline{\text{Comp } C}$$

implies

$$x(t) \in \overline{\text{Comp } C}$$

for all $t \in [t_0, t_1]$.

proof: If $\overset{\circ}{C} = \emptyset$ then $\overline{\text{Comp } C} = E^n$ and the lemma is trivially true. Therefore we consider the case where $\overset{\circ}{C} \neq \emptyset$ and, by way of contradiction, we suppose that $x(\bar{t}) \in \overset{\circ}{C} = \text{Comp}(\overline{\text{Comp } C})$ for some $\bar{t} \in (t_0, t_1]$.

Since $x(t_0) \in \overset{\circ}{X} \cap \overline{\text{Comp } C}$, there exists a positive number α such that $B[x(t_0)] \cap \text{Comp } C \neq \emptyset$ and $B[x(t_0)] \subset \overset{\circ}{X}$ where $B[x(t_0)]$ is any open ball in E^n with center at $x(t_0)$ and radius $\rho \in (0, \alpha)$. Let $u(\cdot)$ generate a (neighboring) solution $\bar{x}(\cdot): [t_0, t_1] \rightarrow X$ to the state equations with $\bar{x}(t_0) \in B[x(t_0)] \cap \overset{\circ}{X} \cap \text{Comp } C$ (see Figure 3.1).

For $x(\bar{t}) \in \overset{\circ}{C}$ there exists an open ball $B'[x(\bar{t})]$ in E^n with center at $x(\bar{t})$ such that $B'[x(\bar{t})] \subset \overset{\circ}{C} \cap \overset{\circ}{X}$ (since $C \subset X$). Since the

Fig. 3.1 An Illustration of the Functions Employed in the Proof of Lemma 3.2

solutions to the state equations, corresponding to the same control $u(\cdot)$, are continuous functions of their boundary conditions, we have

$$\bar{x}(\bar{t}) \in B^{\circ}[x(\bar{t})] \subset \bar{C} \cap \bar{X}$$

for sufficient small $\rho \in (0, \alpha)$. Moreover, since $x(\bar{t}) \in \bar{C} \cap \bar{X}$ implies $x(t) \in \bar{X}$ for all $t \in [t_0, \bar{t}]$, the continuity of the solutions to the state equations with respect to their boundary conditions (for a given control function) yields

$$\bar{x}(t) \in \bar{X}$$

for all $t \in [t_0, \bar{t}]$ providing ρ (the radius of $B[x(t_0)]$) is sufficiently small. Thus there exists a solution $\bar{x}(\cdot): [t_0, \bar{t}] \rightarrow X$ to the state equations, generated by an admissible control $u(\cdot): [t_0, \bar{t}] \rightarrow \Omega$, with $\bar{x}(t_0) \in \text{Comp } C$, $\bar{x}(t) \in X$ for all $t \in [t_0, \bar{t}]$, and $\bar{x}(\bar{t}) \in C$ which contradicts Lemma 3.1.

Note that we exclude the case $x(t_0) \in \partial X$ in Lemma 3.2. This restriction is necessary since the ball $B[x(t_0)]$, if $x(t_0) \in \partial X$, may not contain any points in $\text{Comp } C$ from which there exists solutions $\bar{x}(t)$ which do not violate the state constraint for any $t \in [t_0, \bar{t}]$. The point $x(t_0) = (a, 0)$ in the regulator problem of Section 2.8 is an example of such a point.

We now present two lemmas concerning the existence of system paths in the intersection of the boundary of the controllable set with the interior of the state constraint set. Both of the results are for the case where the controllable set is closed, that is, the boundary, Ξ , of the controllable set belongs to the set.

Lemma 3.3 If the controllable set is closed ($C = \bar{C} = \Xi \cup \bar{C}^{\circ}$) and if no state constraint is imposed ($X = E^n$), then for every point $x^0 \in \Xi$

there exists a time interval $[t_0, t_1]$, an admissible control $u^*(\cdot): [t_0, t_1] \rightarrow \Omega$, and a corresponding solution $x^*(\cdot): [t_0, t_1] \rightarrow X$ to the state equations, generated by $u^*(\cdot)$, such that

- (i) $x^*(t_0) = x^0$;
- (ii) $x^*(t) \in \Xi$ for all $t \in [t_0, t_1]$;
- (iii) $x^*(t_1) \in \Theta$.

proof: From the definition of C , $x^0 \in \Xi \subseteq C$ implies the existence of a time interval $[t_0, t_1]$, an admissible control $u^*(\cdot): [t_0, t_1] \rightarrow \Omega$, and a corresponding solution $x^*(\cdot): [t_0, t_1] \rightarrow X$ to the state equations, generated by $u^*(\cdot)$, such that $x^*(t_0) = x^0$ and $x^*(t_1) \in \Theta$. The condition $x^*(t_1) \in \Theta$ implies $x^*(t) \in C$ for all $t \in [t_0, t_1]$ since $x^*(t_1) \in \Theta \subset C$ and $x^*(t) \in \text{Comp } C$, for some $t \in (t_0, t_1)$, would contradict Lemma 3.1. Similarly, from Lemma 3.2, $x^*(t_0) = x^0 \in \Xi \subset \overline{\text{Comp } C}$ implies $x^*(t) \in \overline{\text{Comp } C}$ for all $t \in [t_0, t_1]$. Therefore, since $\Xi = C \cap \overline{\text{Comp } C}$ (for C closed), $x^*(t) \in \Xi$ for all $t \in [t_0, t_1]$.

If the controllable set is closed and a state constraint is imposed, then we have

Lemma 3.4 If the controllable set is closed, then for every point $x^0 \in \Xi \cap \overset{\circ}{X}$ there exists a time interval $[t_0, t_1]$, an admissible control $u^*(\cdot): [t_0, t_1] \rightarrow \Omega$, and a corresponding solution $x^*(\cdot): [t_0, t_1] \rightarrow X$ to the state equations, generated by $u^*(\cdot)$, with $x^*(t_0) = x^0$ such that

$$x^*(t) \in \Xi \cap \overset{\circ}{X} \text{ for all } t \in [t_0, t_1],$$

and either $x^*(t_1) \in \Theta \cap \overset{\circ}{X}$ or $x^*(t_1) \in \partial X$.

proof: From the definition of C , $x^0 \in \Xi \subseteq C$ implies the existence of a time interval $[t_0, t_1]$, an admissible control $u^*(\cdot): [t_0, t_1] \rightarrow \Omega$, and a corresponding solution $x^*(\cdot): [t_0, t_1] \rightarrow X$ to the state equations with $x^*(t_0) = x^0$, $x^*(t_1) \in \Theta$, and $x^*(t) \in X$ for all $t \in [t_0, t_1]$. If $x^*(t) \in \overset{\circ}{X}$ for all $t \in [t_0, t_1]$, then, from Lemma 3.3, $x^*(t) \in \Xi$ for all $t \in [t_0, t_1]$ and one case of the lemma is established.

If $x^*(t) \notin \overset{\circ}{X}$ for all $t \in [t_0, t_1]$ then we let $\bar{t}_1 \in (t_0, t_1]$ denote the first time at which $x^*(t) \in \partial X$, that is, $x^*(t) \in \overset{\circ}{X}$ for all $t \in [t_0, \bar{t}_1)$ and $x^*(\bar{t}_1) \in \partial X$. From Lemma 3.2, with $x^*(t_0) = x^0 \in \Xi \subset \overline{\text{Comp } C}$, we have that $x^*(t) \in \overline{\text{Comp } C}$ for all $t \in [t_0, \bar{t}_1]$. From Lemma 3.1 we get $x^*(t) \in \bar{C}$ for all $t \in [t_0, \bar{t}_1]$ since $x^*(t_1) \in \Theta$ and $x^*(t) \in \text{Comp } \bar{C} = \text{Comp } C$, for some $t \in [t_0, \bar{t}_1]$, would contradict Lemma 3.1. Thus $x^*(t) \in \Xi = \bar{C} \cap \overline{\text{Comp } C}$ for all $t \in [t_0, \bar{t}_1]$ and the lemma is established.

3.2 Tangent Vectors and Tangent Cones

In studying the geometry of Ξ (the boundary of the controllable set) and its relation to controls which generate paths in Ξ , we will need to define a (possibly one-sided) tangent vector to a subset of E^n . We will use a definition similar to the one employed by Blaquiére and Leitmann (1967 p. 282).

Let S be any nonempty, locally arcwise connected subset of E^n and let x be a point in $\bar{S}$. We will say that a vector $\eta \in E^n$ is tangent to S at x if and only if there exists a positive number α and a continuous function $\eta(\cdot): (0, \alpha) \rightarrow E^n$ such that

$$(i) \quad x + \delta \eta(\delta) \in S \text{ for all } \delta \in (0, \alpha);$$

(ii) $\eta(\delta) \rightarrow \eta$ as $\delta \rightarrow 0$.

Since this definition is not made in terms of a particular subset of E^n , it will serve to define tangent vectors to such sets as Ξ (the boundary of the controllable set), X (the state constraint set), and Θ (the target set), as well as to C (the controllable set) and $\text{Comp } C$. For the moment, however, we shall only be concerned with vectors tangent to Ξ , $\bar{C}$, and $\overline{\text{Comp } C}$.

We define the tangent cone to Ξ at $x \in \Xi$ as

$$C_{\Xi}(x) \triangleq \{x + \eta: \text{for all } \eta \text{ tangent to } \Xi \text{ at } x\}.$$

Similarly, we define the tangent cone to $\bar{C}$ at $x \in \bar{C}$ as

$$C_I(x) \triangleq \{x + \eta: \text{for all } \eta \text{ tangent to } \bar{C} \text{ at } x\},$$

and the tangent cone to $\overline{\text{Comp } C}$ at $x \in \overline{\text{Comp } C}$ as

$$C_O(x) \triangleq \{x + \eta: \text{for all } \eta \text{ tangent to } \overline{\text{Comp } C} \text{ at } x\}.$$

The subscripts "I" and "O" are mnemonics for "inside" and "outside" of the controllable set, respectively.

Since $\Xi = \bar{C} \cap \overline{\text{Comp } C}$, we have

$$C_{\Xi}(x) = C_I(x) \cap C_O(x). \quad (3.2.1)$$

From the relations $\Xi = \bar{C} \cap \overline{\text{Comp } C}$ and $\Xi \cup \overset{\circ}{C} \cup \overset{\circ}{\text{Comp } C} = E^n$, we have

$$\overset{\circ}{C}_I(x) = \text{Comp } C_O(x) = C_I(x) - C_{\Xi}(x), \quad (3.2.2)$$

and

$$\overset{\circ}{C}_O(x) = \text{Comp } C_I(x) = C_O(x) - C_{\Xi}(x). \quad (3.2.3)$$

For detailed proofs of (3.2.1) - (3.2.3), see Blaquiere and Leitmann (1967 pp. 276-277).

3.3 Perturbation and Adjoint Equations

If $x^*(\cdot): [t_0, t_1] \rightarrow X$ and $x(\cdot): [t_0, t_1] \rightarrow X$ are two different solutions to the state equations, generated by the same control function $u^*(\cdot): [t_0, t_1] \rightarrow \Omega$, with

$$x(\tau) = x^*(\tau) + \eta \quad (3.3.1)$$

for some $\tau \in [t_0, t_1]$ where $\eta = [\eta_1, \dots, \eta_n]^T \in E^n$, then (see Coddington and Levinson 1955, pp. 321-322; Pontryagin et al. 1962, pp. 83-85) for all $t \in [t_0, t_1]$,

$$x(t) = x^*(t) + \eta(t) + o(\eta) \quad (3.3.2)$$

where $o(\eta)/|\eta| \rightarrow 0$ uniformly as $|\eta| \rightarrow 0$ and where $\eta(\cdot): [t_0, t_1] \rightarrow E^n$ is the solution to the perturbation equations

$$\dot{\eta} = \frac{\partial f[x^*(t), u^*(t)]}{\partial x} \eta \quad (3.3.3)$$

satisfying the boundary condition

$$\eta(\tau) = \eta. \quad (3.3.4)$$

The perturbation equations define a nonsingular linear transformation (Coddington and Levinson 1955; Pontryagin et al. 1962, pp. 84-85) represented by an $n \times n$ perturbation matrix $\Phi(t, \tau)$ (also called the Fundamental Matrix) such that

$$\eta(t) = \Phi(t, \tau)\eta(\tau) \quad (3.3.5)$$

for all $t, \tau \in [t_0, t_1]$. The perturbation matrix is the solution of the (matrix) differential equation

$$\frac{\partial \Phi(t, \tau)}{\partial t} = \frac{\partial f[x^*(t), u^*(t)]}{\partial x} \Phi(t, \tau) \quad (3.3.6)$$

satisfying the boundary condition

$$\Phi(\tau, \tau) = I, \quad (3.3.7)$$

where I is the $n \times n$ identity matrix. The inverse of $\Phi(t, \tau)$ is given by

$$\Phi^{-1}(t, \tau) = \Phi(\tau, t) \quad (3.3.8)$$

for all $t, \tau \in [t_0, t_1]$ and, if t_1, t_2 , and t_3 are any times in $[t_0, t_1]$, then

$$\Phi(t_1, t_3) = \Phi(t_1, t_2)\Phi(t_2, t_3). \quad (3.3.9)$$

The perturbation equations (3.3.3) describe the behavior of the system when the state is (infinitesimally) perturbed at some point from the nominal path defined by $x^*(\cdot)$. Closely associated with the perturbation equations are the adjoint equations

$$\dot{\lambda}^T = -\lambda^T \frac{\partial f[x^*(t), u^*(t)]}{\partial x}, \quad (3.3.10)$$

where $\lambda = [\lambda_1, \dots, \lambda_n]^T \in E^n$. The adjoint equations derive their name from the following relation:

Lemma 3.5: Let $\eta(\cdot): [t_0, t_1] \rightarrow E^n$ and $\lambda(\cdot): [t_0, t_1] \rightarrow E^n$ be solutions to the perturbation and adjoint equations, respectively, corresponding to the admissible control $u^*(\cdot): [t_0, t_1] \rightarrow \Omega$ and the solution $x^*(\cdot): [t_0, t_1] \rightarrow X$ to the state equations. Then

$$\lambda^T(t)\eta(t) = \text{constant}$$

for all $t \in [t_0, t_1]$.

proof: From the forms of the perturbation and adjoint equations, we have

$$\begin{aligned} \frac{d}{dt}(\lambda^T \eta) &= \dot{\lambda}^T \eta + \lambda^T \dot{\eta} \\ &= -\lambda^T \frac{\partial f}{\partial x} \eta + \lambda^T \frac{\partial f}{\partial x} \eta \\ &= 0 \end{aligned}$$

where $d(\cdot)/dt$ is to be interpreted as a one-sided derivative at times when a jump discontinuity in $u^*(\cdot)$ induces a discontinuity in $\partial f[x, u]/\partial x$.

Just as the perturbation equations define a matrix $\phi(t, \tau)$ which transforms $\eta(\tau)$ into $\eta(t)$ according to (3.3.5), so also do the adjoint equations define a matrix $\phi_\lambda(t, \tau)$ which transforms $\lambda(\tau)$ into $\lambda(t)$ by the relation

$$\lambda^T(t) = \lambda^T(\tau) \phi_\lambda(t, \tau). \quad (3.3.11)$$

From Lemma 3.5 we have

$$\lambda^T(t) \eta(t) = \lambda^T(\tau) \eta(\tau) \quad (3.3.12)$$

for all $t, \tau \in [t_0, t_1]$. Combining (3.3.5), (3.3.11), and (3.3.12), we get

$$\lambda^T(\tau) \phi_\lambda(t, \tau) \phi(t, \tau) \eta(\tau) = \lambda^T(\tau) \eta(\tau)$$

for all $t, \tau \in [t_0, t_1]$, which implies that

$$\phi_\lambda(t, \tau) = \phi^{-1}(t, \tau) = \phi(\tau, t) \quad (3.3.13)$$

for all $t, \tau \in [t_0, t_1]$. Thus (3.3.11) becomes

$$\lambda^T(t) = \lambda^T(\tau) \phi(\tau, t) \quad (3.3.14)$$

for all $t, \tau \in [t_0, t_1]$.

We conclude this section with a final property of the perturbation matrix $\phi(t, \tau)$. Since it represents a nonsingular linear transformation, the matrix $\phi(t, \tau)$ transforms hyperplanes into hyperplanes of the same dimension. Specifically, if $T(\tau)$ is an $n-1$ dimensional hyperplane at $x^*(\tau)$, determining two open half spaces $T_+(\tau)$ and $T_-(\tau)$, then for all times $t, \tau \in [t_0, t_1]$,

$$T(t) \triangleq \Phi(t, \tau)T(\tau)$$

is an $n-1$ dimensional hyperplane, at $x^*(t)$, determining the two open half spaces

$$T_+(t) \triangleq \Phi(t, \tau)T_+(\tau)$$

and

$$T_-(t) \triangleq \Phi(t, \tau)T_-(\tau).$$

Moreover, from Lemma 3.5, if $\lambda(\tau)$ is a vector at $x^*(\tau)$ normal to $T(\tau)$, with $x^*(\tau) + \lambda(\tau) \in T_+(\tau)$, then the vector $\lambda(t)$ at $x^*(t)$, defined by

$$\lambda^T(t) = \lambda^T(\tau)\Phi(\tau, t),$$

is normal to $T(t)$ with $x^*(t) + \lambda(t) \in T_+(t)$.

CHAPTER 4

A RESTRICTED CONTROLLABILITY MINIMUM PRINCIPLE

In this chapter we will establish a Controllability Minimum Principle for the case where the controllable set C is the closure of an open subset (but not all) of E^n and where Ξ , the boundary of the controllable set, is an $n-1$ dimensional smooth (C^1) manifold.

4.1 The Assumed Structure of the Controllable Set

We assume that the controllable set is the closure of an open subset (but not all) of E^n with its interior, $\overset{\circ}{C}$, and boundary, Ξ , expressible in the form

$$\overset{\circ}{C} = \{x: g(x) < 0\} \quad (4.1.1)$$

$$\Xi = \{x: g(x) = 0\} \quad (4.1.2)$$

where $g(\cdot): E^n \rightarrow E^1$ is C^1 with non-zero gradient, represented by the (row) matrix

$$\nabla g(x) = \left[\frac{\partial g(x)}{\partial x_1}, \dots, \frac{\partial g(x)}{\partial x_n} \right] \neq 0 \quad (4.1.3)$$

for all x in a neighborhood of Ξ . From Fleming (1965, p. 122), Ξ is an $n-1$ dimensional (smooth) manifold with a continuously varying "outward" normal $\nabla g(x)$ at every point $x \in \Xi$. Using the Approximation (Mean Value) Theorem (Buck 1965, p. 243), one can show that the tangent cone $C_{\Xi}(x)$ to Ξ at x is the $n-1$ dimensional hyperplane, normal to $\nabla g(x)$, given by the relation

$$C_{\Xi}(x) = \{x + n: \nabla g(x) \cdot n = 0\}. \quad (4.1.4)$$

To verify (4.1.4) we let $\eta(\cdot): (0, \alpha) \rightarrow E^n$ generate a vector η tangent to Ξ at $x \in \Xi$. Then for $\delta \in (0, \alpha)$, the Approximation Theorem yields

$$g[x + \delta\eta(\delta)] = g(x) + \delta \nabla g(x) \eta(\delta) + o(\delta)$$

where $o(\delta)/\delta \rightarrow 0$ uniformly as $\delta \rightarrow 0$. For x and $x + \delta\eta(\delta)$ in Ξ , we get

$$\delta[\nabla g(x)\eta(\delta) + o(\delta)/\delta] = 0.$$

Dividing by $\delta > 0$ and taking the limit as $\delta \rightarrow 0$, we have

$$\nabla g(x)\eta = 0$$

for all η tangent to Ξ at x .

In a similar manner, one can verify that the tangent cones to $\overline{C}$ and $\overline{\text{Comp } C}$ at $x \in \Xi$ are the closed half spaces given by

$$C_1(x) = \{x + \eta: \nabla g(x)\eta \leq 0\} \quad (4.1.5)$$

and

$$C_0(x) = \{x + \eta: \nabla g(x)\eta \geq 0\} \quad (4.1.6)$$

respectively.

4.2 A Minimization Lemma

We have the following fundamental relation between the gradient $\nabla g(x)$ to the surface Ξ and all possible "velocity" vectors $f(x, u)$, $u \in \Omega$, at a point $x \in \Xi$:

Lemma 4.1: At every point $x \in \Xi \cap \overset{\circ}{X}$,

$$\nabla g(x)f(x, u) \geq 0$$

for all $u \in \Omega$.

proof: Let $x(\cdot): [t_0, t_1] \rightarrow X$ be the solution to the state equations, generated by the constant control $u(t) = u \in \Omega$ for all $t \in [t_0, t_1]$, with $x(t_0) = x$. Since $x \in \overset{\circ}{X}$ and since $x(\cdot)$ is continuous in t , there exists a positive number α such that

$$x(t_0 + \Delta t) \in \overset{\circ}{X}$$

for all $\Delta t \in [0, \alpha)$. From the definition of a (one-sided) derivative,

$$x(t_0 + \Delta t) = x + \Delta t f(x, u) + o(\Delta t)$$

where $o(\Delta t)/\Delta t \rightarrow 0$ uniformly as $\Delta t \rightarrow 0$. Employing the Approximation Theorem (Buck 1965, p. 243),

$$g[x(t_0 + \Delta t)] = g(x) + \Delta t [\nabla g(x) f(x, u) + o(\Delta t)/\Delta t]. \quad (4.2.1)$$

From Lemma 3.2, since $x(t_0) = x \in \Xi$ and $x(t) \in \overset{\circ}{X}$ for all $t \in [t_0, t_0 + \alpha)$, we have

$$x(t_0 + \Delta t) \in \overline{\text{Comp } C} = \{x: g(x) \geq 0\},$$

for all $\Delta t \in [0, \alpha)$. Thus, dividing (4.2.1) by $\Delta t > 0$, using $g(x) = 0$ for $x \in \Xi$, and then taking the limit as $\Delta t \rightarrow 0$, we get

$$\nabla g(x) f(x, u) \geq 0$$

for all $u \in \Omega$ and all $x \in \Xi \cap \overset{\circ}{X}$.

4.3 Transformation of Tangent Planes

As we mentioned at the close of Chapter 3, the perturbation matrix $\phi(t, \tau)$ transforms an $n-1$ dimensional plane into an $n-1$ dimensional plane. For a solution $x^*(\cdot): [t_0, t_1] \rightarrow \Xi$ to the state equations which (except possibly for its end points) lies in $\Xi \cap \overset{\circ}{X}$, we will demonstrate that the perturbation matrix transforms a plane tangent to Ξ into a plane tangent to Ξ .

Lemma 4.2: Let $x^*(\cdot): [t_0, t_1] \rightarrow \Xi$ be a solution to the state equations, generated by an admissible control $u^*(\cdot): [t_0, t_1] \rightarrow \Omega$, with $x^*(t) \in \Xi$ for all $t \in [t_0, t_1]$ and $x^*(t) \in \overset{\circ}{X}$ for all $t \in (t_0, t_1)$.

If the controllable set is of the form (4.1.1) - (4.1.3), then for any two times t, τ , with $t_0 \leq \tau \leq t \leq t_1$, the tangent planes $C_{\Xi}[x^*(t)]$ and $C_{\Xi}[x^*(\tau)]$ to Ξ at $x^*(t)$ and $x^*(\tau)$, respectively, are related by

$$C_{\Xi}[x^*(t)] = \Phi(t, \tau) C_{\Xi}[x^*(\tau)].$$

That is, if $\eta(\tau)$ is tangent to Ξ at $x^*(\tau)$, then $\Phi(t, \tau)\eta(\tau)$ is tangent to Ξ at $x^*(t)$ where $\Phi(t, \tau)$ is the perturbation matrix corresponding to the solution $x^*(\cdot)$ to the state equations and the admissible control $u^*(\cdot)$.

proof: Initially, we restrict τ to the open time interval (t_0, t_1) in order to have $x^*(\tau) \in \Xi \cap \overset{\circ}{X}$.

Let $\eta(\cdot): (0, \alpha) \rightarrow E^n$ generate a vector η tangent to Ξ at $x^*(\tau)$.

That is,

$$x^*(\tau) + \delta\eta(\delta) \in \Xi$$

for all $\delta \in (0, \alpha)$ and $\eta(\delta) \rightarrow \eta$ as $\delta \rightarrow 0$. Let $x(\cdot): [t_0, t_1] \rightarrow X$ be the solution to the state equations (see Figure 4.1), generated by the same control $u^*(\cdot)$ that generates $x^*(\cdot)$, with

$$x(\tau) = x^*(\tau) + \delta\eta(\delta) \in \Xi.$$

From (3.3.2) and (3.3.5)

$$x(t) = x^*(t) + \delta[\Phi(t, \tau)\eta(\delta) + o(\delta)/\delta] \quad (4.3.1)$$

for all $t \in [t_0, t_1]$ where $o(\delta)/\delta \rightarrow 0$ uniformly as $\delta \rightarrow 0$ and where $\Phi(t, \tau)$ is the perturbation matrix corresponding to $x^*(\cdot)$ and $u^*(\cdot)$.

Since solutions to the state equations, for a given control, are continuous in their boundary conditions, and since $x^*(t) \in \overset{\circ}{X}$ for all $t \in (t_0, t_1)$, we have $x(t) \in \overset{\circ}{X}$ for all $t \in (t_0, t_1)$ and sufficiently small $\delta \in (0, \alpha)$. From Lemma 3.2, $x(\tau) \in \Xi$ implies

Fig. 4.1 An Illustration of the Functions
Employed in the Proof of Lemma 4.2

$$x(t) \in \overline{\text{Comp } C} \triangleq \{x: g(x) \geq 0\} \quad (4.3.2)$$

for all $t \in [\tau, t_1]$ and sufficiently small $\delta \in (0, \alpha)$. Combining (4.3.1) and (4.3.2), the Approximation Theorem yields

$$g[x(t)] = \delta \{\nabla g[x^*(t)]\phi(t, \tau)_\eta(\delta) + o(\delta)/\delta\} \geq 0 \quad (4.3.3)$$

for all $t \in [\tau, t_1]$ and sufficiently small $\delta \in (0, \alpha)$, where we have used $g[x^*(t)] = 0$ for $x^*(t) \in \Xi$. Dividing the right-hand side of (4.3.3) by $\delta > 0$ and then taking the limit as $\delta \rightarrow 0$, we get

$$\nabla g[x^*(t)]\phi(t, \tau)_\eta \geq 0 \quad (4.3.4)$$

for all $t \in [\tau, t_1]$, all $\tau \in (t_0, t_1)$, and all η tangent to Ξ at $x^*(\tau)$.

Now, suppose there exists an η tangent to Ξ at $x^*(\tau)$ and times $t \in [\tau, t_1)$ and $\tau \in (t_0, t_1)$ such that

$$\nabla g[x^*(t)]\phi(t, \tau)_\eta > 0. \quad (4.3.5)$$

Since the tangent cone, $C_\Xi[x^*(\tau)]$, to Ξ at $x^*(\tau)$ is an $n-1$ dimensional hyperplane, the vector

$$\overline{\eta} \triangleq -\eta \quad (4.3.6)$$

at $x^*(\tau)$ is also tangent to Ξ . Combining (4.3.5) and (4.3.6) yields

$$\nabla g[x^*(t)]\phi(t, \tau)_{\overline{\eta}} < 0$$

which contradicts (4.3.4). Thus we conclude that

$$\nabla g[x^*(t)]\phi(t, \tau)_\eta = 0 \quad (4.3.7)$$

for all $t \in [\tau, t_1)$, all $\tau \in (t_0, t_1)$, and all η tangent to Ξ at $x^*(\tau)$. From (4.1.4) and (4.3.7) we have

$$C_\Xi[x^*(t)] = \phi(t, \tau)C_\Xi[x^*(\tau)] \quad (4.3.8)$$

for all t and τ with $t_0 < \tau \leq t < t_1$ and the lemma is established, except for the case where $\tau = t_0$ or $\tau = t_1$. For $\tau = t_1$, which implies $\tau = t = t_1$, the lemma is trivially true since $\phi(t_1, t_1) = I$ (the identity matrix) by (3.3.7). The same result holds for

$t_0 = \tau = t$. Thus we are left with the case $t_0 = \tau < t \leq t_1$. If $x^*(t_0) \in \overset{\circ}{X}$, then the above proof also holds for $\tau = t_0$ and the lemma is established.

If $x^*(t_0) \in \partial X$, then the condition $x^*(t) \in \overset{\circ}{X}$ for all $t \in (t_0, t_1)$ implies that

$$x^*(t_0 + \Delta t) \in \overset{\circ}{X}$$

for all sufficiently small $\Delta t > 0$. From the continuity of $\nabla g(x)$ with respect to x and the continuity of $x^*(t)$ with respect to t , we have that

$$\nabla g[x^*(t_0 + \Delta t)] \rightarrow \nabla g[x^*(t_0)] \text{ as } \Delta t \rightarrow 0,$$

which implies, through (4.1.4), that

$$C_{\Xi}[x^*(t_0 + \Delta t)] \rightarrow C_{\Xi}[x^*(t_0)] \text{ as } \Delta t \rightarrow 0.$$

In addition we have that

$$\phi(t, t_0 + \Delta t) \rightarrow \phi(t, t_0) \text{ as } \Delta t \rightarrow 0$$

since the solution $\phi(t, \tau)$ to the differential equations (3.3.6) is a continuous function of the time τ at which the boundary condition (3.3.7) is applied (Coddington and Levinson 1955). Therefore, for $t \in [t_0 + \Delta t, t_1]$, (4.3.8) yields

$$C_{\Xi}[x^*(t)] = \phi(t, t_0 + \Delta t) C_{\Xi}[x^*(t_0 + \Delta t)],$$

which converges to

$$C_{\Xi}[x^*(t)] = \phi(t, t_0) C_{\Xi}[x^*(t_0)]$$

as $\Delta t \rightarrow 0$, and the lemma is established.

4.4 Proof of a Restricted Controllability Minimum Principle

As an immediate consequence of Lemmas 4.1 and 4.2, we have the following necessary condition for an admissible control $u^*(\cdot): [t_0, t_1] \rightarrow$

Ω to generate a solution $x^*(\cdot): [t_0, t_1] \rightarrow \Xi$ to the state equations which, with the possible exception of its end points, lies in the intersection of the boundary of the controllable set with the interior of the state constraint set, for the case where the controllable set is the closure of an open set with a smooth $n-1$ dimensional manifold as its boundary:

Theorem 4.1: (A Restricted Controllability Minimum Principle)

Let $u^*(\cdot): [t_0, t_1] \rightarrow \Omega$ be an admissible control which generates a solution $x^*(\cdot): [t_0, t_1] \rightarrow \Xi$ to the state equations with $x^*(t) \in \Xi$ for all $t \in [t_0, t_1]$ and $\dot{x}^*(t) \in \dot{\bar{X}}$ for all $t \in (t_0, t_1)$.

If the controllable set is of the form (4.1.1) - (4.1.3), then, with $\lambda = [\lambda_1, \dots, \lambda_n]^T \in E^n$ and

$$H(\lambda, x, u) \triangleq \lambda^T f(x, u), \quad (4.4.1)$$

there exists a nonzero continuous solution $\lambda(\cdot): [t_0, t_1] \rightarrow E^n$ to the adjoint equations

$$\dot{\lambda}^T \triangleq - \frac{\partial H[\lambda, x^*(t), u^*(t)]}{\partial x} \triangleq - \lambda^T \frac{\partial f[x^*(t), u^*(t)]}{\partial x} \quad (4.4.2)$$

such that

$$H[\lambda(t), x^*(t), u^*(t)] = \min_{u \in \Omega} H[\lambda(t), x^*(t), u] = 0 \quad (4.4.3)$$

for all $t \in [t_0, t_1]$. In addition, the vector $\lambda(t)$ is an "outward" normal to Ξ at $x^*(t)$. That is,

$$\lambda^T(t) = k(t) \nabla g[x^*(t)] \quad (4.4.4)$$

for all $t \in [t_0, t_1]$ where $k(t)$ is a positive number for each $t \in [t_0, t_1]$.

proof: Let $k(t_0)$ be any positive real number and let $\lambda(\cdot): [t_0, t_1] \rightarrow \mathbb{R}^n$ be the solution to the adjoint equations (4.4.2) with

$$\lambda^T(t_0) = k(t_0) \nabla g[x^*(t_0)]. \quad (4.4.5)$$

Since we have assumed that $\nabla g[x^*(t_0)] \neq 0$, for $x^*(t_0) \in \Xi$, we have that $\lambda(t_0) \neq 0$. Furthermore, since the adjoint equations are homogeneous, we know that $\lambda(t) \equiv 0$ for all $t \in [t_0, t_1]$ if and only if $\lambda(t) = 0$ for some $t \in [t_0, t_1]$. Thus $\lambda(t_0) \neq 0$ implies $\lambda(t) \neq 0$ for any $t \in [t_0, t_1]$. The continuity of $\lambda(t)$ with respect to t follows from $\lambda(t)$ being a solution to the adjoint differential equations, which are a system of homogeneous linear ordinary differential equations with time varying coefficients that are (at worst) piecewise continuous in time.

From (4.4.5), $\lambda(t_0)$ is an "outward" normal to Ξ at $x^*(t_0)$. That is, $\lambda(t_0)$ is normal to the $n-1$ dimensional hyperplane $C_{\Xi}[x^*(t_0)]$ tangent to Ξ at $x^*(t_0)$, with $x^*(t_0) + \lambda(t_0)$ in the open half space $\overset{\circ}{C}_0[x^*(t_0)]$ bounded by $C_{\Xi}[x^*(t_0)]$. As a result of Lemma 4.2 and the discussion at the end of Chapter 3, $\lambda(t)$ is normal to the $n-1$ dimensional hyperplane $C_{\Xi}[x^*(t)]$, tangent to Ξ at $x^*(t)$, with $x^*(t) + \lambda(t)$ in the open half space $\overset{\circ}{C}_0[x^*(t)]$, for all $t \in [t_0, t_1]$. Thus, from (4.1.1) and (4.1.6), we have

$$\lambda^T(t) = k(t) \nabla g[x^*(t)] \quad (4.4.6)$$

for all $t \in [t_0, t_1]$ where $k(t)$ is a scalar valued function which is positive for all $t \in [t_0, t_1]$.

Combining (4.4.1) and (4.4.6) with the conclusion of Lemma 4.1, we get, for all $t \in [t_0, t_1]$ and all $u \in \Omega$,

$$H[\lambda(t), x^*(t), u] \triangleq \lambda^T(t) f[x^*(t), u] \geq 0.$$

We now wish to show that the "velocity" vector $f[x^*(t), u^*(t)]$ is tangent to Ξ for all $t \in [t_0, t_1]$. Let δ be a small positive number. Since $u^*(\cdot)$ is piecewise continuous, we have that either

$$u^*(t) = u^*(t+0) \triangleq \lim_{\delta \downarrow 0} u^*(t + \delta) \quad (4.4.7)$$

or

$$u^*(t) = u^*(t-0) \triangleq \lim_{\delta \downarrow 0} u^*(t - \delta) \quad (4.4.8)$$

or both if $u^*(\cdot)$ is continuous at t . For convenience in what follows, we take $u^*(t_0) = u^*(t_0+0)$ and $u^*(t_1) = u^*(t_1-0)$.

If $t \in [t_0, t_1)$ is a time at which $u^*(t) = u^*(t+0)$, then the definition of a (one-sided) derivative yields

$$x^*(t + \delta) = x^*(t) + \delta f[x^*(t), u^*(t)] + o(\delta)$$

where $o(\delta)/\delta \rightarrow 0$ as $\delta \rightarrow 0$. Since $x^*(t) \in \Xi$ for all $t \in [t_0, t_1]$, we have $x^*(t + \delta) \in \Xi$ for sufficiently small $\delta > 0$. Defining

$$\eta(\delta) \triangleq f[x^*(t), u^*(t)] + o(\delta)/\delta,$$

we have

$$x^*(t) + \delta \eta(\delta) \in \Xi$$

for all sufficiently small $\delta > 0$ and

$$\eta(\delta) \rightarrow f[x^*(t), u^*(t)] \text{ as } \delta \rightarrow 0.$$

Thus $f[x^*(t), u^*(t)]$ is tangent to Ξ at $x^*(t)$.

In a similar manner, if $u^*(t) = u^*(t-0)$ then $-f[x^*(t), u^*(t)]$ is tangent to Ξ which implies that $f[x^*(t), u^*(t)]$ is tangent to Ξ since the tangent cone to Ξ is an $n-1$ dimensional hyperplane. Thus, from (4.1.4) and (4.4.6),

$$\lambda^T(t) f[x^*(t), u^*(t)] = 0$$

or, equivalently,

$$H[\lambda(t), x^*(t), u^*(t)] = 0$$

for all $t \in [t_0, t_1]$ and the proof is complete.

CHAPTER 5

A CONTROLLABILITY MINIMUM PRINCIPLE

In this chapter we derive the general Controllability Minimum Principle (Theorem 2.1) without assuming that the controllable set is the closure of an open set with a smooth $n-1$ dimensional manifold as its boundary. The only assumption we make is that $\overset{\circ}{C}_1[x^*(t_1)]$, the interior of the tangent cone to $\bar{C}$ at the terminal point $x^*(t_1)$ of a path in the intersection of the boundary of the controllable set with the interior of the state constraint set, is nonempty.

5.1 The Perturbation Cone

Let $x^*(\cdot): [t_0, t_1] \rightarrow \Xi$ be a solution to the state equations, generated by an admissible control $u^*(\cdot): [t_0, t_1] \rightarrow \Omega$, with $x^*(t) \in \Xi$ for all $t \in [t_0, t_1]$ and with $x^*(t) \in \overset{\circ}{X}$ for all $t \in (t_0, t_1)$. Let $\Phi(t, \tau)$ denote the perturbation matrix, for $t, \tau \in [t_0, t_1]$, corresponding to $x^*(\cdot)$ and $u^*(\cdot)$.

We define the perturbation cone $K[x^*(t)]$ at $x^*(t) \in \Xi$, $t \in (t_0, t_1]$, as the set of all points $x^*(t) + \eta$ where η is any finite non-negative linear combination of vectors at $x^*(t)$ of the forms

$$(i) \quad \pm \Phi(t, \tau) f[x^*(\tau), u^*(\tau)]; \quad \tau \in (t_0, t) \text{ and } u^*(\cdot) \text{ continuous at } \tau,$$

and

$$(ii) \quad \Phi(t, \tau) f[x^*(\tau), u]; \quad \tau \in (t_0, t) \text{ and } u \in \Omega.$$

Thus, $x^*(t) + \eta \in K[x^*(t)]$ if and only if η is of the form

$$\eta = \sum_{i=1}^s \phi(t, \tau_i) \left\{ a_i f[x^*(\tau_i), u^*(\tau_i)] + \sum_{j=1}^{r_i} b_{i,j} f[x^*(\tau_i), j^i, j] \right\} \quad (5.1.1)$$

where s and r_i are positive integers; a_i and $b_{i,j}$ are real numbers with $b_{i,j} \geq 0$; the $u^{i,j}$ are control vectors in Ω ; and the τ_i are distinct times in $(t_0, t) \subset (t_0, t_1)$ with $u^*(\cdot)$ continuous at τ_i if $a_i \neq 0$.

The perturbation cone has the following property with respect to the tangent cone $C_0[x^*(t)]$ to $\overline{\text{Comp } C}$ at $x^*(t)$:

Lemma 5.1: If $K[x^*(t)]$ is the perturbation cone at $x^*(t) \in \Xi$ and if

$C_0[x^*(t)]$ is the tangent cone to $\overline{\text{Comp } C}$ at $x^*(t)$, then

$$K[x^*(t)] \subseteq C_0[x^*(t)]$$

for all $t \in (t_0, t_1)$, if $x^*(t_1) \in \partial X$, or for all $t \in (t_0, t_1]$ if $x^*(t_1) \in \overset{\circ}{X}$.

proof: We begin by relabeling the τ_i in (5.1.1); if necessary, so that

$$t_0 < \tau_1 < \dots < \tau_s < t \leq t_1.$$

Let δ be a (small) positive number and define

$$\delta_i \triangleq a_i \delta; \quad (5.1.2)$$

$$\delta_{i,j} \triangleq b_{i,j} \delta; \quad (5.1.3)$$

$$\Delta_i \triangleq \sum_{j=1}^{r_i} \delta_{i,j} = \delta \sum_{j=1}^{r_i} b_{i,j}. \quad (5.1.4)$$

Thus $\delta, \delta_{i,j}$, and Δ_i are non-negative (δ_i will be negative if $a_i < 0$) and are such that

$$\delta_1, \delta_{i,j}, \Delta_i \rightarrow 0 \quad \text{as} \quad \delta \rightarrow 0.$$

Let $\Delta x^i(\cdot): [0, \Delta_i] \rightarrow X$ be the solution to the state equations (see Figure 5.1) generated by the piecewise constant (admissible) control

$$u(\bar{t}) \triangleq \begin{cases} u^{i,1} & \text{if } \bar{t} \in [0, \delta_{i,1}) \\ u^{i,2} & \text{if } \bar{t} \in [\delta_{i,1}, \delta_{i,1} + \delta_{i,2}) \\ \vdots & \vdots \\ u^{i,r_i} & \text{if } \bar{t} \in [\Delta_i - \delta_{i,r_i}, \Delta_i] \end{cases}$$

and satisfying the initial condition

$$\Delta x^i(0) = x^i$$

where x^i is to be specified later.

From (5.1.3) and the definition of a (one-sided) derivative, we have

$$\begin{aligned} \Delta x^i(\delta_{i,1}) &= x^i + \delta_{i,1} f[x^i, u^{i,1}] + o(\delta_{i,1}) \\ &= x^i + \delta b_{i,1} f[x^i, u^{i,1}] + o(\delta) \end{aligned} \quad (5.1.5)$$

where $o(\delta)/\delta \rightarrow 0$ as $\delta \rightarrow 0$. Applying the definition of a derivative again, we get

$$\begin{aligned} \Delta x^i(\delta_{i,1} + \delta_{i,2}) &= \Delta x^i(\delta_{i,1}) + \delta_{i,2} f[\Delta x^i(\delta_{i,1}), u^{i,2}] \\ &\quad + o(\delta_{i,2}) \end{aligned} \quad (5.1.6)$$

Fig. 5.1 An Illustration of the Functions Employed in the Proof of Lemma 5.1

where $o(\delta_{i,2})/\delta_{i,2} \rightarrow 0$ as $\delta_{i,2} \rightarrow 0$. Using (5.1.3) and (5.1.5) and applying the Approximation Theorem to $f[\cdot, u^{i,2}]$ in (5.1.6) yields

$$\begin{aligned} \Delta x^i(\delta_{i,1} + \delta_{i,2}) &= x^i + \delta b_{i,1} f[x^i, u^{i,1}] + \delta b_{i,2} f[x^i, u^{i,2}] \\ &\quad + \delta^2 b_{i,1} b_{i,2} \frac{\partial f[x^i, u^{i,1}]}{\partial x} f[x^i, u^{i,1}] \\ &\quad + o(\delta) + \delta o(\delta). \end{aligned} \quad (5.1.7)$$

Collecting all terms of higher order than δ into a single $o(\delta)$ term, (5.1.7) becomes

$$x^i(\delta_{i,1} + \delta_{i,2}) = x^i + \delta \sum_{j=1}^2 b_{i,j} f[x^i, u^{i,j}] + o(\delta).$$

At the end of the time interval $[0, \Delta_i]$ we get

$$\Delta x^i(\Delta_i) = x^i + \delta \sum_{j=1}^{r_i} b_{i,j} f[x^i, u^{i,j}] + o(\delta) \quad (5.1.8)$$

where $o(\delta)/\delta \rightarrow 0$ as $\delta \rightarrow 0$.

As seen from the above development, $\Delta x^i(\Delta_i)$ is a point which could be reached from the point x^i by applying the constant controls $u^{i,j}$ for small time intervals of length $\delta_{i,j}$. To establish Lemma 5.1, we perturb the state of the system by the (small) amount $\Delta x^i(\Delta_i) - x^i$ from a point x^i (near the "nominal" system state) at time τ_i (see Figure 5.1). After each perturbation, a new "nominal" system path evolves which we will perturb in a similar manner at time τ_{i+1} . By keeping each perturbation small we can make each of the resulting paths lie as near the unperturbed path $x^*(\cdot)$, as we like.

Let $x^i(\cdot): [\tau_i, t_1] \rightarrow X$ be the solution to the state equations, generated by the control $u^*(\cdot): [\tau_i, t_1] \rightarrow \Omega$, and satisfying the boundary condition

$$x^i(\tau_i) = \Delta x^i(\Delta_i). \quad (5.1.9)$$

We take the boundary condition on the solution $\Delta x^i(\cdot): [0, \Delta_i] \rightarrow X$ as

$$\Delta x^i(0) = x^i \triangleq x^{i-1}(\tau_i + \delta_i) \quad (5.1.10)$$

where $i = 1, \dots, s, \delta_i \triangleq \delta a_i$, and $x^0(\cdot) \triangleq x^*(\cdot)$. Thus x^i is a point on the "nominal" path near the "nominal" state $x^{i-1}(\tau_i)$.

From the definition of a derivative and from the continuity of $u^*(\cdot)$ at each τ_i for which $a_i \neq 0$, (5.1.10) can be written as

$$x^i = x^{i-1}(\tau_i) + \delta_i f[x^{i-1}(\tau_i), u^*(\tau_i)] + o(\delta_i) \quad (5.1.11)$$

regardless of the sign of $\delta_i = \delta a_i$. Combining (5.1.8) and (5.1.11), the boundary condition (5.1.9) on $x^i(\cdot)$ becomes

$$x^i(\tau_i) = x^{i-1}(\tau_i) + \delta \eta^i(\delta) + o(\delta) \quad (5.1.12)$$

where

$$\begin{aligned} \eta^i(\delta) &\triangleq a_i f[x^{i-1}(\tau_i), u^*(\tau_i)] \\ &+ \sum_{j=1}^{r_i} b_{i,j} f[x^{i-1}(\tau_i + \delta a_i), u^{i,j}] \end{aligned} \quad (5.1.13)$$

and where we have used the definition of δ_i and collected all terms of higher order than δ into a single $o(\delta)$ term.

To determine the location of the state of the perturbed system after the final ($i = s$) perturbation, we make use of the perturbation matrix $\Phi(t, \tau)$. With $x^0(\cdot) \triangleq x^*(\cdot)$, the boundary condition

(5.1.12) on $x^1(\cdot): [\tau_1, t_1] \rightarrow X$ is

$$x^1(\tau_1) = x^*(\tau_1) + \delta \eta^1(\delta) + o(\delta).$$

Then, from (3.3.1), (3.3.2) and (3.3.5),

$$x^1(\tau) = x^*(\tau) + \delta \Phi(\tau, \tau_1) \eta^1(\delta) + o(\delta) \quad (5.1.14)$$

for all $\tau \in [\tau_1, t_1]$ since $x^1(\cdot)$ and $x^*(\cdot)$ are generated by the same control $u^*(\cdot)$ on the time interval $[\tau_1, t_1]$.

Combining (5.1.12), for $i = 2$, with (5.1.14) evaluated at $\tau = \tau_2$, the boundary condition on $x^2(\cdot): [\tau_2, t_1] \rightarrow X$ becomes

$$x^2(\tau_2) = x^*(\tau_2) + \delta [\Phi(\tau_2, \tau_1) \eta^1(\delta) + \eta^2(\delta)] + o(\delta). \quad (5.1.15)$$

Thus, from (3.3.1), (3.3.2) and (3.3.5), we have

$$\begin{aligned} x^2(\tau) &= x^*(\tau) + \delta \Phi(\tau, \tau_2) \Phi(\tau_2, \tau_1) \eta^1(\delta) \\ &\quad + \delta \Phi(\tau, \tau_2) \eta^2(\delta) + o(\delta). \end{aligned} \quad (5.1.16)$$

for all $\tau \in [\tau_2, t_1]$. From (3.3.9) we have that

$$\Phi(\tau, \tau_2) \Phi(\tau_2, \tau_1) = \Phi(\tau, \tau_1)$$

so that (5.1.16) may be written as

$$x^2(\tau) = x^*(\tau) + \delta \sum_{i=1}^2 \Phi(\tau, \tau_i) \eta^i(\delta) + o(\delta)$$

for all $\tau \in [\tau_2, t_1]$.

Repeating the process, and using the fact that

$$\phi(t, \tau_i) \phi(\tau_i, \tau_{i-1}) = \phi(t, \tau_{i-1})$$

for all $i = 2, \dots, s$, we get

$$x^s(t) = x^*(t) + \delta \sum_{i=1}^s \phi(t, \tau_i) \eta^i(\delta) + o(\delta) \quad (5.1.17)$$

for all $t \in [\tau_s, t_1]$, where $o(\delta)/\delta \rightarrow 0$ as $\delta \rightarrow 0$.

Now that we have an expression for $x^s(t)$, the state of the system after the series of perturbations, we wish to show that $x^s(t) \in \overline{\text{Comp } C}$. To do this we will make use of Lemma 3.2, the fact that $x^*(t)$ is in $\overset{\circ}{X}$ for all $t \in (t_0, t_1)$ (or for all $t \in (t_0, t_1]$ if $x^*(t_1) \in \overset{\circ}{X}$), and the fact that we can make each $x^i(\tau)$, $\tau \in [\tau_i, t_1]$, lie as near $x^*(\tau)$ as we like by taking $\delta (> 0)$ sufficiently small.

Since $x^*(t) \in \Xi \cap \overset{\circ}{X}$ for all $t \in (t_0, t_1)$ and since $\tau_1 \in (t_0, t)$, we have, from (5.1.10), that

$$x^1 \triangleq x^*(\tau_1 + \delta a_1) \in \Xi \cap \overset{\circ}{X}$$

for sufficiently small $\delta > 0$. Lemma 3.2 and the fact, from (5.1.8), that $\Delta x^i(\Delta_1) \rightarrow x^i$ as $\delta \rightarrow 0$ imply that

$$\Delta x^1(\Delta_1) \in \overline{\text{Comp } C} \cap \overset{\circ}{X}$$

for sufficiently small $\delta > 0$. Since $\Delta x^1(\Delta_1)$ is the initial point for $x^1(\cdot)$ and since $x^*(\tau) \in \overset{\circ}{X}$ for all $\tau \in [\tau_1, t_1]$ (or for all $\tau \in [\tau_1, t_1]$ if $x^*(t_1) \in \overset{\circ}{X}$), the continuity of solutions to the state equations with respect to their boundary conditions (for a given control) implies that

$$x^1(\tau) \in \overset{\circ}{X}$$

for all $\tau \in [\tau_1, t_1)$ (or for all $\tau \in [\tau_1, t_1]$ if $x^*(t_1) \in \overset{\circ}{X}$) providing

δ , and therefore $|\Delta x^1(\Delta_1) - x^*(\tau_1)|$, is sufficiently small. Therefore Lemma 3.2 is applicable to $x^1(\cdot)$ and yields

$$x^1(\tau) \in \overline{\text{Comp } C} \cap \overset{\circ}{X}$$

for all $\tau \in [\tau_1, t_1)$ (or for all $\tau \in [\tau_1, t_1]$ if $x^*(t_1) \in \overset{\circ}{X}$) providing δ is sufficiently small. In particular, we have

$$x^2 \triangleq x^1(\tau_2 + \delta a_2) \in \overline{\text{Comp } C} \cap \overset{\circ}{X}$$

for sufficiently small δ since $\tau_2 \in (\tau_1, t_1)$.

Repeated application of the above argument yields

$$x^S(t) \in \overline{\text{Comp } C} \cap \overset{\circ}{X} \quad (5.1.18)$$

for all $t \in (t_0, t_1)$ (or for all $t \in (t_0, t_1]$ if $x^*(t_1) \in \overset{\circ}{X}$) providing δ is taken sufficiently small in (5.1.17). We have written this result in terms of the interval (t_0, t_1) rather than the intervals (τ_i, t_1) since $\tau_i \in (t_0, t)$ and we wish to consider arbitrary $t \in (t_0, t_1)$ (or $(t_0, t_1]$).

From the boundary conditions

$$\begin{aligned} \Delta x^1(0) &= x^*(\tau_1 + \delta a_1); & x^1(\tau_1) &= \Delta x^1(\Delta_1) \\ \vdots & & \vdots & \\ \Delta x^S(0) &= x^{S-1}(\tau_S + \delta a_S); & x^S(\tau_S) &= \Delta x^S(\Delta_S), \end{aligned}$$

equation (5.1.4), and the fact that the functions $\Delta x^i(\cdot)$ and $x^i(\cdot)$ are continuous functions of time and their boundary conditions, we have that for all $i = 1, \dots, s$

$$x^{i-1}(\tau_i + \delta a_i) \rightarrow x^*(\tau_i) \quad \text{as } \delta \rightarrow 0.$$

Thus from (5.1.13) and the continuity of $f(x, u)$ with respect to x , we have that

$$n^i(\delta) \rightarrow a_i f[x^*(\tau_i), u^*(\tau_i)] + \sum_{j=1}^{r_i} b_{i,j} f[x^*(\tau_i), u^{i,j}] \quad (5.1.19)$$

as $\delta \rightarrow 0$. Therefore, from (5.1.17) - (5.1.19),

$$n(\delta) \triangleq \sum_{i=1}^S \phi(t, \tau_i) n^i(\delta) + o(\delta)$$

is a (continuous) vector function such that

$$\begin{aligned} n(\delta) \rightarrow n \triangleq \sum_{i=1}^S \phi(t, \tau_i) \left\{ a_i f[x^*(\tau_i), u^*(\tau_i)] \right. \\ \left. + \sum_{j=1}^{r_i} b_{i,j} f[x^*(\tau_i), u^{i,j}] \right\} \end{aligned}$$

as $\delta \rightarrow 0$ with

$$x^*(t) + \delta n(\delta) \in \overline{\text{Comp } C}$$

for sufficiently small $\delta > 0$ and all $t \in (t_0, t_1)$ (or $(t_0, t_1]$ if $x^*(t_1) \in \overset{\circ}{X}$). Thus, from (5.1.1) and the definition (Section 3.2) of $C_0[x^*(t)]$, we have $K[x^*(t)] \subseteq C_0[x^*(t)]$ for all $t \in (t_0, t_1)$ if $x^*(t_1) \in \partial X$ and for all $t \in (t_0, t_1]$ if $x^*(t_1) \in \overset{\circ}{X}$, and the proof of Lemma 5.1 is complete.

To establish the general Controllability Minimum Principle, we need one additional property of the perturbation cone embodied in

Lemma 5.2: The perturbation cone $K[x^*(t)]$ is convex for all $t \in (t_0, t_1]$.

proof: Let n^1 and n^2 be any two vectors at $x^*(t) \in \Xi$, $t \in (t_0, t_1]$, such that

$$x^*(t) + n^1 \in K[x^*(t)]$$

and

$$x^*(t) + n^2 \in K[x^*(t)]$$

and let c_1 and c_2 be any non-negative numbers. From (5.1.1), η^1 and η^2 are of the form

$$\eta^1 = \sum_{i=1}^{s_1} \phi(t, \tau_{i,1}) \left\{ a_{i,1} f[x^*(\tau_{i,1}), u^*(\tau_{i,1})] + \sum_{j=1}^{r_{i,1}} b_{i,j,1} f[x^*(\tau_{i,1}), u^{i,j,1}] \right\}$$

and

$$\eta^2 = \sum_{i=1}^{s_2} \phi(t, \tau_{i,2}) \left\{ a_{i,2} f[x^*(\tau_{i,2}), u^*(\tau_{i,2})] + \sum_{j=1}^{r_{i,2}} b_{i,j,2} f[x^*(\tau_{i,2}), u^{i,j,2}] \right\}$$

where s_1 , s_2 , $r_{i,1}$, and $r_{i,2}$ are positive integers; $a_{i,1}$, $a_{i,2}$, $b_{i,j,1}$, and $b_{i,j,2}$ are real numbers with $b_{i,j,1}$ and $b_{i,j,2}$ non-negative; the $u^{i,j,1}$ and $u^{i,j,2}$ are control vectors in Ω ; the $\tau_{i,1}$ are distinct times in (t_0, t) with $u^*(\cdot)$ continuous at $\tau_{i,1}$ if $a_{i,1} \neq 0$; and the $\tau_{i,2}$ are distinct times in (t_0, t) with $u^*(\cdot)$ continuous at $\tau_{i,2}$ if $a_{i,2} \neq 0$.

For convenience in what follows, we assume that the $\tau_{i,1}$ and the $\tau_{i,2}$ are collectively distinct times in (t_0, t) so that vectors of the form $f[x^*(\tau), u^*(\tau)]$ and $f[x^*(\tau), u]$, for a particular τ and u , occur in only one of the convex combinations η^1 or η^2 , if they occur at all. Although such an assumption is not essential, it will make the presentation of the proof simpler.

For $i = 1, \dots, s_1$, define

$$r_i \triangleq r_{i,1};$$

$$a_i \triangleq c_1 a_{i,1};$$

$$\tau_i \triangleq \tau_{i,1};$$

$$b_{i,j} \triangleq c_1 b_{i,j,1} \quad j = 1, \dots, r_i;$$

$$u^{i,j} \triangleq u^{i,j,1} \quad j = 1, \dots, r_i.$$

For $i = s_1+1, \dots, s_1+s_2 \triangleq s$, define

$$r_i \triangleq r_{i-s_1,2};$$

$$a_i \triangleq c_2 a_{i-s_1,2};$$

$$\tau_i \triangleq \tau_{i-s_1,2};$$

$$b_{i,j} \triangleq c_2 b_{i-s_1,j,2} \quad j = 1, \dots, r_i;$$

$$u^{i,j} \triangleq u^{i-s_1,j,2} \quad j = 1, \dots, r_i.$$

With these definitions

$$\begin{aligned} c_1 \eta^1 + c_2 \eta^2 = & \sum_{i=1}^s \phi(t, \tau_i) \left\{ a_i f[x^*(\tau_i), u^*(\tau_i)] \right. \\ & \left. + \sum_{j=1}^{r_i} b_{i,j} f[x^*(\tau_i), u^{i,j}] \right\} \end{aligned}$$

which, from (5.1.1), implies

$$x^*(t) + c_1 \eta^1 + c_2 \eta^2 \in K[x^*(t)].$$

Therefore $K[x^*(t)]$ is convex for all $t \in (t_0, t_1]$, that is, for all t at which it is defined.

5.2 Proof of the Controllability Minimum Principle

We are now ready to prove Theorem 2.1 which we repeat here for reference.

Theorem 2.1: (Controllability Minimum Principle)

Let $u^*(\cdot): [t_0, t_1] \rightarrow \Omega$ be an admissible control which generates a solution $x^*(\cdot): [t_0, t_1] \rightarrow \Xi$ to the state equations

$$\dot{x} = f(x, u)$$

such that $x^*(t) \in \Xi$ for all $t \in [t_0, t_1]$ and $x^*(t) \in \overset{\circ}{X}$ for all $t \in (t_0, t_1)$. Let δ be an arbitrarily small positive number if $x^*(t_1) \in \partial X$ and let δ be zero if $x^*(t_1) \in \overset{\circ}{X}$. Let $\overset{\circ}{C}_1[x^*(t_1-\delta)]$ denote the interior of the tangent cone (see Section 3.2) to $\bar{C}$ at $x^*(t_1-\delta)$.

If $\overset{\circ}{C}_1[x^*(t_1-\delta)]$ is nonempty then, with $\lambda = [\lambda_1, \dots, \lambda_n]^T \in E^n$ and

$$H(\lambda, x, u) \triangleq \lambda^T f(x, u),$$

there exists a nonzero continuous solution $\lambda(\cdot): [t_0, t_1] \rightarrow E^n$ to the adjoint equations

$$\dot{\lambda}^T \triangleq - \frac{\partial H[\lambda, x^*(t), u^*(t)]}{\partial x} \triangleq - \lambda^T \frac{\partial f[x^*(t), u^*(t)]}{\partial x}$$

such that

$$H[\lambda(t), x^*(t), u^*(t)] = \min_{u \in \Omega} H[\lambda(t), x^*(t), x^*(t), u] = 0$$

for all $t \in [t_0, t_1]$.

proof: From Lemmas 5.1 and 5.2, the perturbation cone $K[x^*(t_1-\delta)]$ is a convex subset of $\overset{\circ}{C}_0[x^*(t_1-\delta)]$, the tangent cone to $\overline{\text{Comp } C}$ at $x^*(t_1-\delta)$. From (3.2.2), the assumption that $\overset{\circ}{C}_1[x^*(t_1-\delta)] =$

Comp $C_0[x^*(t_1-\delta)]$ is not empty implies that $C_0[x^*(t_1-\delta)]$ is not all of E^n . Therefore, from the convexity of $K[x^*(t_1-\delta)] \subseteq C_0[x^*(t_1-\delta)]$, there exists an $n-1$ dimensional "separating hyperplane $T(t_1-\delta)$ at $x^*(t_1-\delta)$, which bounds a closed half space $\bar{T}_+(t_1-\delta)$, such that

$$K[x^*(t_1-\delta)] \subseteq \bar{T}_+(t_1-\delta).$$

In particular, from the definition of $K[x^*(t_1-\delta)]$, we have

$$x^*(t_1-\delta) + \phi(t_1-\delta, t)f[x^*(t), u] \in \bar{T}_+(t_1-\delta) \quad (5.2.1)$$

for all $t \in (t_0, t_1-\delta)$ and all $u \in \Omega$, and we have

$$x^*(t_1-\delta) \pm \phi(t_1-\delta, t)f[x^*(t), u^*(t)] \in \bar{T}_+(t_1-\delta) \quad (5.2.2)$$

for all $t \in (t_0, t_1-\delta)$ with $u^*(\cdot)$ continuous at t .

Since $\bar{T}_+(t_1-\delta)$ is a closed half space, the " $\pm$ " in (5.2.2) implies

$$x^*(t_1-\delta) \pm \phi(t_1-\delta, t)f[x^*(t), u^*(t)] \in T(t_1-\delta) \triangleq \partial \bar{T}_+(t_1-\delta) \quad (5.2.3)$$

for all $t \in (t_0, t_1-\delta)$ with $u^*(\cdot)$ continuous at t , where $T(t_1-\delta)$ is the $n-1$ dimensional hyperplane which forms the boundary of $\bar{T}_+(t_1-\delta)$.

Let $\lambda(\cdot): [t_0, t_1] \rightarrow E^n$ be the solution to the adjoint equations

$$\dot{\lambda}^T = -\lambda^T \frac{\partial f[x^*(t), u^*(t)]}{\partial x}$$

with $\lambda(t_1-\delta)$ nonzero and normal to $T(t_1-\delta)$ at $x^*(t_1-\delta)$ and with

$$x^*(t_1-\delta) + \lambda(t_1-\delta) \in T_+(t_1-\delta).$$

From the discussion in the first part of the proof of Theorem 4.1 (Section 4.4), $\lambda(t)$ is continuous and nonzero for all $t \in [t_0, t_1]$.

Since $\lambda(t_1 - \delta)$ is normal to $T(t_1 - \delta)$ and points into $T_+(t_1 - \delta)$, (5.2.1) becomes

$$\lambda^T(t_1 - \delta)\phi(t_1 - \delta, t)f[x^*(t), u] \geq 0 \quad (5.2.4)$$

for all $t \in (t_0, t_1 - \delta)$ and all $u \in \Omega$, and (5.2.3) becomes

$$\lambda^T(t_1 - \delta)\phi(t_1 - \delta, t)f[x^*(t), u^*(t)] = 0 \quad (5.2.5)$$

for all $t \in (t_0, t_1 - \delta)$ with $u^*(\cdot)$ continuous at t .

Since $\lambda(\cdot)$ and $\phi(\cdot, t)$ are continuous functions and δ is an arbitrarily small positive number (or zero if $x^*(t_1) \in \overset{\circ}{X}$), (5.2.4) implies that

$$\lambda^T(t_1)\phi(t_1, t)f[x^*(t), u] \geq 0 \quad (5.2.6)$$

for all $t \in (t_0, t_1)$, and (5.2.5) implies that

$$\lambda^T(t_1)\phi(t_1, t)f[x^*(t), u^*(t)] = 0 \quad (5.2.7)$$

for all $t \in (t_0, t_1)$ with $u^*(\cdot)$ continuous at t .

Postmultiplying both sides of (3.3.14), evaluated at $t = t_1$ and $\tau = t$, by $\phi(t_1, t) = \phi^{-1}(t, t_1)$ yields

$$\lambda^T(t_1)\phi(t_1, t) = \lambda^T(t). \quad (5.2.8)$$

Combining (5.2.6) - (5.2.8), we get

$$\lambda^T(t)f[x^*(t), u] \geq 0 \quad (5.2.9)$$

for all $t \in (t_0, t_1)$ and all $u \in \Omega$, and

$$\lambda^T(t)f[x^*(t), u^*(t)] = 0 \quad (5.2.10)$$

for all $t \in (t_0, t_1)$ with $u^*(\cdot)$ continuous at t .

Since $u^*(\cdot)$ is piecewise continuous, the theorem is established for "almost all" $t \in [t_0, t_1]$. To establish the theorem for all $t \in [t_0, t_1]$, that is, to show that (5.2.9) and (5.2.10) hold for all $t \in [t_0, t_1]$, we let δ (not the same as the previous δ) be a small positive number and we let $t \in [t_0, t_1]$ be either t_0, t_1 , or a time when $u^*(\cdot)$ is discontinuous. For simplicity, we suppose that $u^*(\cdot)$ is defined so that $u^*(t_0) = u^*(t_0+0)$ and $u^*(t_1) = u^*(t_1-0)$, where $u^*(t+0)$ and $u^*(t-0)$ are defined by (4.4.7) and (4.4.8) respectively.

If t is a time in $[t_0, t_1]$ at which $u^*(t) = u^*(t+0)$, then (5.2.10) implies

$$\lambda^T(t+\delta)f[x^*(t+\delta), u^*(t+\delta)] = 0 \quad (5.2.11)$$

for all sufficiently small $\delta > 0$. With $f(x, u)$ continuous in x and u , $\lambda(t)$ and $x^*(t)$ continuous in t , and $u^*(t)$ "one-sided" continuous in t , (5.2.11) implies, as $\delta \rightarrow 0$, that

$$\lambda^T(t)f[x^*(t), u^*(t)] = 0$$

for all $t \in [t_0, t_1]$ at which $u^*(t) = u^*(t+0)$. A similar argument, with the various functions evaluated at $t-\delta$, $\delta > 0$, shows that (5.2.10) holds for all $t \in [t_0, t_1]$. A repetition of the above procedure, first for $t = t_0$ then for $t = t_1$, shows that (5.2.9) also holds for all $t \in [t_0, t_1]$ and the proof is complete.

As we indicated at the close of Section 2.4, the requirement that $\overset{\circ}{C}_I[x^*(t_1)]$ (or $\overset{\circ}{C}_I[x^*(t_1-\delta)]$ if $x^*(t_1) \in \partial X$) be nonempty is analogous to the nondegeneracy condition of Blaquiére and Leitmann, for optimal control problems, and to the Kuhn-Tucker Constraint Qualification condition for nonlinear programming problems. As evidenced by its absence from the statement of Theorem 4.1, this nondegeneracy condition is satisfied if the controllable set is the closure (or interior) of an open set having a smooth $n-1$ dimensional manifold as its boundary.

CHAPTER 6

TRANSVERSALITY CONDITIONS

In this chapter we present the proofs of Theorems 2.2 and 2.3, the Target and State Constraint Transversality Conditions.

6.1 The Problem of Boundary Conditions

The Controllability Minimum Principle represents necessary conditions for a control to generate a system path which, with the possible exception of its end points, lies in the intersection of the boundary of the controllable set with the interior of the state constraint set. The control is determined by simultaneously solving a system of $2n$ ordinary differential equations, which in general are coupled and nonlinear, while imposing the two conditions

$$H[\lambda(t), x^*(t), u^*(t)] = \min_{u \in \Omega} H[\lambda(t), x^*(t), u]$$

and

$$H[\lambda(t), x^*(t), u^*(t)] = 0.$$

The most fundamental problem associated with this task, outside of the dimensionality resulting from large scale (large n) systems, is the problem of determining boundary conditions for the state and adjoint equations. How can we determine, for instance, whether or not a particular point x is in the boundary of the controllable set? Moreover, given that the point x is a valid point for applying boundary conditions to the state equations, what are the corresponding boundary conditions for the adjoint equations? The Controllability Minimum

Principle only guarantees the existence of "a" solution to the adjoint equations such that the resulting control minimizes H to zero. In general there could be many such solutions depending on the boundary conditions imposed on the state and adjoint equations. As of yet, the only connection we have between the two sets of boundary conditions is the single scalar relation $H = 0$.

From Lemmas 3.3 and 3.4, we know that if the controllable set is closed then the boundary of the controllable set either intersects the target at a point in the interior of the state constraint set, or it intersects the boundary of the state constraint set, or both. Since both the target and state constraint sets are given, we can greatly reduce the region of possible boundary condition points for the state equations simply by assuming that the controllable set is closed.

From the similarity between the Controllability Minimum Principle and the "abnormal" case of Pontryagin's Maximum Principle for optimal control problems, we might suspect the existence of "transversality" conditions, for the adjoint variables of the Controllability Minimum Principle, corresponding to the transversality conditions of optimal control theory. Indeed, such relations do exist for a class of target and state constraint sets.

In Section 6.2, under the assumption that the target set is the closure of an open set with a smooth $n-1$ dimensional manifold as its boundary, we will show that the adjoint vector $\lambda(t_1)$ is an "outward" normal to the boundary of the target set at a point $x^*(t_1)$, in the interior of the state constraint set, where a path in the boundary of the controllable set reaches the target set.

If no state constraint is imposed and if the controllable set actually is closed, then the boundary of the controllable set can be completely determined by integrating the state and adjoint equations "backwards" from the target set using controls determined from the Controllability Minimum Principle and the Target Transversality Condition. In the presence of a state constraint, however, such a procedure may not determine all points in the boundary of the controllable set. In fact, the "backwards" integration procedure may not determine any of the points in the boundary of the controllable set since we have no guarantee that the boundary of the controllable set intersects the target set in the presence of a state constraint.

In Section 6.3 we will consider the problem of determining boundary conditions for the adjoint equations at a point where a system path, in the intersection of the boundary of the controllable set with the interior of the state constraint set, reaches the boundary of the state constraint set. We will restrict our discussion to a state constraint set which is the closure of an open set with a smooth $n-1$ dimensional manifold as its boundary. In addition we will assume, in a vicinity of a point where the system path reaches the boundary of the state constraint set, that the controllable set, locally, has the same type of structure as that of the state constraint set.

6.2 Proof of a Target Transversality Condition

In this section we consider targets of the form

$$\Theta = \{x: h(x) \leq 0\}, \quad (6.2.1)$$

where $h(\cdot): E^n \rightarrow E^1$ is C^1 with nonzero gradient, represented by the (row) matrix

$$\nabla h(x) = \left[\frac{\partial h(x)}{\partial x_1}, \dots, \frac{\partial h(x)}{\partial x_n} \right] \neq 0, \quad (6.2.2)$$

for all x in a neighborhood of the target boundary

$$\partial\Theta = \{x: h(x) = 0\}. \quad (6.2.3)$$

We now present the proof of Theorem 2.2. For reference, we repeat the statement of the theorem.

Theorem 2.2: (A Target Transversality Condition)

Let $u^*(\cdot): [t_0, t_1] \rightarrow \Omega$ be an admissible control which generates a solution $x^*(\cdot): [t_0, t_1] \rightarrow \Xi$ to the state equations with $x^*(t) \in \Xi \cap \overset{\circ}{X} - \Theta$ for all $t \in (t_0, t_1)$ and with $x^*(t_1) \in \Xi \cap \partial\Theta \cap \overset{\circ}{X}$.

If the target Θ is of the form (6.2.1) - (6.2.3) then there exists a solution $\lambda(\cdot): [t_0, t_1] \rightarrow E^n$ to the adjoint equations such that $u^*(\cdot)$ minimizes H to zero and such that $\lambda(\cdot)$ satisfies the boundary (or transversality) condition

$$\lambda^T(t_1) = k \nabla h[x^*(t_1)]$$

where k is any positive number.

proof: We define the tangent cone to the target set at $x^*(t_1) \in \partial\Theta$ as

$$C_\Theta[x^*(t_1)] \triangleq \{x^*(t_1) + n: \text{ for all } n \text{ tangent to } \Theta \text{ at } x^*(t_1)\}.$$

With the help of the Approximation Theorem, one can show (see Section 4.1) that

$$C_\Theta[x^*(t_1)] = \{x^*(t_1) + n: \nabla h[x^*(t_1)]_n \leq 0\}.$$

That is, the tangent cone to Θ at $x^*(t_1)$ is a closed half space.

Let $T(t_1)$ be the $n-1$ dimensional hyperplane bounding $C_\Theta[x^*(t_1)]$

and tangent to $\partial\theta$ at $x^*(t_1)$. Let $T_+(t_1)$ be the open half space $\text{Comp } C_0[x^*(t_1)]$. Finally, let $K[x^*(t_1)]$ be the perturbation cone at $x^*(t_1)$.

Since every point in the target set is (trivially) controllable to the target, we have $\theta \subset C$ which implies

$$C_\theta[x^*(t_1)] \subseteq C_I[x^*(t_1)].$$

Therefore, from (3.2.2), we have that

$$C_0[x^*(t_1)] = \bar{T}_+(t_1) = \{x^*(t_1) + \eta : \forall h[x^*(t_1)]_n \geq 0\}.$$

Since we have $x^*(t) \in \overset{\circ}{X}$ for all $t \in (t_0, t_1]$, Lemma 5.1 is applicable at time t_1 , that is, $K[x^*(t_1)]$ is a (convex) subset of $C_0[x^*(t_1)]$, which, in turn, is a subset of the closed half space $\bar{T}_+(t_1)$ bounded by the $n-1$ dimensional hyperplane $T(t_1)$. Let $\lambda(\cdot): [t_0, t_1] \rightarrow E^n$ be a solution to the adjoint equations with $\lambda(t_1)$ normal to $T(t_1)$ and pointing into $T_+(t_1)$, that is,

$$\lambda^T(t_1) = k \nabla h[x^*(t_1)]; \quad k > 0.$$

Then the proof that $u^*(\cdot)$ minimizes $H[\lambda(t), x^*(t), u]$ to zero follows the proof of Theorem 2.1 in Section 5.2.

Note that although we do not require $\theta \subset \overset{\circ}{X}$, we do require $x^*(t_1) \in \partial\theta \cap \overset{\circ}{X}$. Theorem 2.2 implies, through $H = 0$, that the path corresponding to $x^*(\cdot)$ must be tangent to $\partial\theta$ at $x^*(t_1)$. If $x^*(t_1) \in \partial\theta \cap \partial X$, this tangency requirement may not be a valid necessary condition. The critical importance of $x^*(t_1) \in \overset{\circ}{X}$ is that it guarantees, through Lemma 5.1, that the perturbation cone $K[x^*(t_1)]$ is a convex subset of the tangent cone $C_0[x^*(t_1)]$ to $\overline{\text{Comp } C}$ at $x^*(t_1)$. If $x^*(t_1) \in \partial X$ instead

of $\overset{\circ}{X}$, then a perturbed system path may reach the target by first violating the state constraint. In such a situation $K[x^*(t_1)]$ need not be a subset of $C_0[x^*(t_1)]$.

6.3 Proof of a State Constraint Transversality Condition

In this section we derive boundary conditions for the adjoint variables at points where system paths, in the intersection of the boundary of the controllable set with the interior of the state constraint set, reach the boundary of the state constraint set.

We consider state constraint sets of the form

$$X = \{x: m(x) \leq 0\}, \quad (6.3.1)$$

where $m(\cdot): E^n \rightarrow E^1$ is C^1 with nonzero gradient, represented by the (row) matrix

$$\nabla m(x) = \left[\frac{\partial m(x)}{\partial x_1}, \dots, \frac{\partial m(x)}{\partial x_n} \right] \neq 0, \quad (6.3.2)$$

for all x in a neighborhood of the state constraint boundary

$$\partial X = \{x: m(x) = 0\}. \quad (6.3.3)$$

We shall be concerned with solutions $x^*(\cdot): [t_0, t_1] \rightarrow \Xi$ to the state equations, generated by admissible controls $u^*(\cdot): [t_0, t_1] \rightarrow \Omega$, such that $x^*(t) \in \Xi$ for all $t \in [t_0, t_1]$, $x^*(t) \in \overset{\circ}{X}$ for all $t \in (t_0, t_1)$, and $x^*(t_1) \in \partial X$. In the absence of any a priori knowledge concerning the true structure of the controllable set C , we make the assumption that in the vicinity of the point $x^*(t_1) \in \Xi \cap \partial X$, the controllable set has the same type of structure as the state constraint set. Specifically, we assume that in a neighborhood N of the point $x^*(t_1) \in \Xi \cap \partial X$, the controllable set is of the form

$$C \cap N = \{x \in N: g(x) \leq 0\}, \quad (6.3.4)$$

with boundary

$$\Xi \cap N = \{x \in N : g(x) = 0\}, \quad (6.3.5)$$

where $g(\cdot): E^n \rightarrow E^1$ is C^1 with nonzero gradient, represented by the (row) matrix

$$\nabla g(x) = \left[\frac{\partial g(x)}{\partial x_1}, \dots, \frac{\partial g(x)}{\partial x_n} \right] \neq 0, \quad (6.3.6)$$

for all $x \in \Xi \cap N$.

With these assumptions regarding the structure of the controllable set, we may establish Theorem 2.3, which we repeat here for reference:

Theorem 2.3: (A State Constraint Transversality Condition)

Let $u^*(\cdot): [t_0, t_1] \rightarrow \Omega$ be an admissible control which generates a solution $x^*(\cdot): [t_0, t_1] \rightarrow \Xi$ to the state equations with $x^*(t) \in \Xi$ for all $t \in [t_0, t_1]$, $x^*(t) \in \overset{\circ}{X}$ for all $t \in (t_0, t_1)$, and $x^*(t_1) \in \partial X$. If the state constraint set and the controllable set are of the form (6.3.1) - (6.3.3) and (6.3.4) - (6.3.6), respectively, then there exists a solution $\lambda(\cdot): [t_0, t_1] \rightarrow E^n$ to the adjoint equations such that $u^*(\cdot)$ minimizes H to zero and such that $\lambda(\cdot)$ satisfies the boundary (or transversality) condition

$$\lambda^T(t_1) = k \nabla m[x^*(t_1)]$$

where k is any positive number.

proof: Since C is the closure of an open set (in a neighborhood of $x^*(t_1)$) with a smooth $n-1$ dimensional manifold as its boundary, the tangent cone $C_I[x^*(t_1)]$ to $\bar{C}$ at $x^*(t_1)$ is a closed half space. Thus the nondegeneracy condition, $\overset{\circ}{C}_I[x^*(t_1)] \neq \phi$, of the Controllability

Minimum Principle is satisfied. From the definition of the controllable set, we have

$$C \subseteq X$$

which implies that

$$C_0[x^*(t_1)] = \{x^*(t_1) + \eta: \nabla g[x^*(t_1)]\eta \geq 0\}$$

is also given by

$$C_0[x^*(t_1)] = \{x^*(t_1) + \eta: \nabla m[x^*(t_1)]\eta \geq 0\}.$$

That is, the tangent cones to $\overline{\text{Comp } C}$ and to $\overline{\text{Comp } X}$, which are both closed half spaces, are in fact the same closed half space.

Let $\lambda(\cdot): [t_0, t_1] \rightarrow E^n$ be a solution to the adjoint equations with $\lambda(t_1)$ normal to the tangent plane $C_{\Xi}[x^*(t_1)]$ bounding $C_0[x^*(t_1)]$ and with $\lambda(t_1)$ pointing into $C_0[x^*(t_1)]$. That is,

$$\lambda^T(t_1) = k \nabla m[x^*(t_1)]$$

where k is any positive constant.

As a result of Lemma 4.2, $\lambda(t_1 - \delta)$, where δ is an arbitrarily small positive number, is normal to the $n-1$ dimensional hyperplane

$$C_{\Xi}[x^*(t_1 - \delta)] = \Phi(t_1 - \delta, t_1) C_{\Xi}[x^*(t_1)]$$

and points into the closed half space

$$C_0[x^*(t_1 - \delta)] = \Phi(t_1 - \delta, t_1) C_0[x^*(t_1)],$$

where $\Phi(t, \tau)$ is the perturbation matrix corresponding to $x^*(\cdot)$ and $u^*(\cdot)$.

From Lemma 5.1, $K[x^*(t_1 - \delta)]$, the perturbation cone at $x^*(t_1 - \delta)$, is a convex subset of $C_0[x^*(t_1 - \delta)]$ and the remainder of the proof follows the proof of Theorem 2.1 in Section 5.2.

LIST OF SYMBOLS

$\triangleq$	equals by definition
$=$	equals
$\neq$	does not equal
$\equiv$	is identical to
$\leq$ ($\geq$)	is less (greater) than or equal to
$<$ ($>$)	is less (greater) than
$\rightarrow$	converges to
$\downarrow$	decreases to
$\in$	is an element of
$\notin$	is not an element of
$\subset$	is a subset of
$\subseteq$	is a subset or all of
$\cap$	intersection
$\cup$	union
α, δ, ρ	small numbers (usually positive)
k	a positive number
t, τ	time
$[t_0, t_1]$	the closed time interval $t_0 \leq t \leq t_1$
(t_0, t_1)	the open time interval $t_0 < t < t_1$
$[t_0, t_1)$	the semiopen time interval $t_0 \leq t < t_1$
E^n	euclidian space of dimension n ($n = 1$ implies real numbers)
$\text{sgn}( )$	signum function = +1 if its argument is positive and -1 if its argument is negative

$\Phi(t, \tau)$	the perturbation (or Fundamental) matrix ($n \times n$)
I	the identity matrix ($n \times n$)
A^T	the transpose of the matrix A
x^0, x^1	vectors
x_1, x_2	scalars
x	$= [x_1, \dots, x_n]^T$ state vector (column matrix) $\in E^n$
u	$= [u_1, \dots, u_m]^T$ control vector $\in E^n$
η	$= [\eta_1, \dots, \eta_n]^T$ a (perturbation or tangent) vector $\in E^n$
λ	$= [\lambda_1, \dots, \lambda_n]^T$ adjoint vector $\in E^n$
$f(x, u)$	$= [f_1(x, u), \dots, f_n(x, u)]^T$ velocity vector ($\dot{x} = f(x, u)$)
$H(\lambda, x, u)$	a scalar valued function defined by $H \triangleq \lambda^T f(x, u)$.
$\dot{x}$	time derivative of x
$ x $	norm (euclidian length) of x
$\{a: b\}$	the set of all "a" such that "b" is satisfied
$A - B$	the set of all elements which are in A but not in B
ϕ	the empty set
S	a general subset (or all) of E^n
$\bar{S}$	closure of S
$\overset{\circ}{S}$	interior of S
$\text{Comp } S$	compliment of S ($= E^n - S$)
∂S	boundary of S ($= \bar{S} \cap \overline{\text{Comp } S}$)
Ω	the control constraint set ($u \in \Omega \subseteq E^n$)
X	the state constraint set ($x \in X \subseteq E^n$)
χ	an open subset (or all) of E^n containing X
Θ	the target set ($\Theta \subset E^n$)
C	the controllable set ($C \subset X$)

Ξ	the boundary (∂C) of the controllable set
$C_{\Xi}(x)$	the tangent cone to Ξ at $x \in \Xi$
$C_I(x)$	the tangent cone to $\bar{C}$ at $x \in \bar{C}$
$C_0(x)$	the tangent cone to $\overline{\text{Comp } C}$ at $x \in \overline{\text{Comp } C}$
$C_{\Theta}(x)$	the tangent cone to Θ at $x \in \partial\Theta$
$K(x)$	the perturbation cone at $x \in \Xi$
$B(x)$	an open ball in E^n with center at x
T	an $n-1$ dimensional hyperplane in E^n
T_+, T_-	the open half spaces in E^n bounded by T
$h(\cdot): E^n \rightarrow E^1$	function $h(\cdot)$ with domain E^n and range E^1
$o(\delta)$	a quantity (scalar or vector) of higher order than the number δ $\{o(\delta)/\delta \rightarrow 0 \text{ as } \delta \rightarrow 0\}$
$o(n)$	a quantity of higher order than $ n $ $\{o(n)/ n \rightarrow 0 \text{ as } n \rightarrow 0\}$
C^1	class of continuously differentiable functions
$\nabla h(x)$	$= \left[\frac{\partial h(x)}{\partial x_1}, \dots, \frac{\partial h(x)}{\partial x_n} \right]$ row matrix representation of the gradient of the function $h(\cdot): E^n \rightarrow E^1$ at x
$\frac{\partial f(x,u)}{\partial x}$	the matrix $[\partial f_i(x,u)/\partial x_j]$ of partial derivatives of the function $f(x,u)$ with respect to the components $x_1, \dots, x_n$ of the vector x (i = row, j = column)

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