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THE STRUCTURE OF THE PLANETS JUPITER
AND SATURN.

The University of Arizona, Ph.D., 1976
Physics, astronomy and astrophysics

Xerox University Microfilms, Ann Arbor, Michigan 48106

THE STRUCTURE OF THE PLANETS JUPITER AND SATURN

by

Wayne Lewis Slattery

A Dissertation Submitted to the Faculty of the

DEPARTMENT OF PLANETARY SCIENCES

In Partial Fulfillment of the Requirements
For the Degree of

DOCTOR OF PHILOSOPHY

In the Graduate College

THE UNIVERSITY OF ARIZONA

1 9 7 6

THE UNIVERSITY OF ARIZONA

GRADUATE COLLEGE

I hereby recommend that this dissertation prepared under my
direction by Wayne Lewis Slattery
entitled The Structure of the Planets Jupiter and Saturn

be accepted as fulfilling the dissertation requirement of the
degree of Doctor of Philosophy

W. B. Hubbard
Dissertation Director

26 May 1976
Date

After inspection of the final copy of the dissertation, the
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Wayne Slotter

ACKNOWLEDGMENTS

I wish to express my gratitude to my supervising professor, Dr. W. B. Hubbard for suggesting this project and for his continued advice during the course of this work. I would also like to express my thanks to the other members of this committee, Dr. R. J. Greenberg, Dr. J. R. Jokipii, Professor R. G. Strom, and Dr. R. J. Weymann, for their helpful criticisms.

Acknowledgment is also made to the Department of Planetary Sciences of The University of Arizona for the research assistantships which I held during the years 1972-1976.

I wish also to thank my colleague Mr. Guy Consolmagno for suggestions of style and for proof reading this dissertation, Mrs. Beverly Epps and Miss Nancy Moore for typing it.

Finally, I wish to express my deepest gratitude to my wife, Phyllis, for her continued encouragement, love, and assistance.

This research was supported in part by NASA Grant NSG-7045.

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ABSTRACT

Planetary models for Jupiter and Saturn are computed using a fourth order theory and a new molecular equation of state. The equation of state for the molecular hydrogen and helium planetary envelopes is calculated using a Monte Carlo technique.

Models for Jupiter are found that have a small amount of heavy elements either mixed with hydrogen and helium throughout the interior of the planet or concentrated in a small dense core. Saturn is modeled with a solar composition hydrogen and helium envelope and a small dense core. I conclude that the molecular equation of state linked with suitable interior equations of state can produce Jovian models which satisfy the observational data. The planetary models show that the enrichment of heavy elements (relative to solar composition) is approximately three times for Jupiter and ten times for Saturn.

CHAPTER 1

INTRODUCTION

Space probes have greatly improved our knowledge of the physical parameters of the planet Jupiter, and within a few years we will have obtained equivalent data for Saturn. Since these planets compose most of the mass of the sun's satellite system, it is important to our understanding of the solar system to construct models of these planets consistent with these new data.

To deduce the structure of these planets I have used the new method of Hubbard, Slattery, and DeVito (1975) which solves the equation of hydrostatic equilibrium for a rotating planet to the fourth order in the ratio of rotational acceleration to gravitational acceleration, using an accurate perturbation expansion.

A new equation of state for the molecular hydrogen and helium atmospheres is also derived. In the past, due to lack of data for the molecular hydrogen and helium region, those who constructed models of the Jovian planets were forced to interpolate over a large gap between the perfect gas region and the metallic interior. Now with a new molecular equation of state this interpolation gap has been narrowed from somewhat less than three to less than one order of magnitude in the density. (There is some uncertainty in these data, and part of the purpose of this dissertation is to demonstrate that very credible models of the

planets Jupiter and Saturn can nevertheless be constructed using this new equation of state.)

In Chapter 2 a qualitative discussion of the models for the planets Jupiter and Saturn and an historical sketch of their development is presented. A new equation of state for molecular hydrogen and helium is derived in Chapter 3. Chapter 4 considers the metallic equation of state. The difficulties in using these numerical equations of state with the fourth order perturbation theory, and the provisions made to allow a small spherical core in the planets are discussed in Chapter 5. The resulting planetary models for Jupiter and Saturn are presented in Chapter 6.

CHAPTER 2

HISTORICAL DEVELOPMENT OF MODELS OF JUPITER AND SATURN, AND A QUALITATIVE SKETCH OF THEIR STRUCTURE

Historical Development of Jovian Planetary Models

The pioneering work on Jupiter models was done by Wildt (1947), Ramsey (1951), DeMarcus (1958), and Alfvén (1954). The first three authors postulated a condensed hydrogen and helium model planet, whereas Alfvén assumed a gaseous carbon, nitrogen, and oxygen model. Öpik (1962) evaluated both of the above types of composition and concluded that the condensed hydrogen and helium models fit the observational data better and were more satisfactory from a cosmogonical viewpoint. Building on DeMarcus' work, Peebles (1964) went on to show that an adiabatic, solar mixture of hydrogen and helium gave an even better fit to observational data.

The discovery of the infrared excess by Low (1966) prompted the investigation into plausible planetary energy generation methods. Smoluchowski (1967, 1971, 1973) suggested that the excess energy was due to the possible insolubility of helium in metallic hydrogen, resulting in helium settling toward the center of the planet, thus liberating gravitational energy. The interior of the planet according to Smoluchowski's ideas would be relatively cold in contrast to Hubbard's (1968, 1969, 1970) "hot" model. Hubbard constructed a model planet which had a hot convecting interior with well-mixed hydrogen and

helium. The luminosity of the planet would be derived from cooling and contracting of the planet. The hot convecting metallic interior would also nicely account for the existence of the planet's magnetic field.

Arguing that it is difficult to envisage a fractionation of hydrogen relative to helium, Podolak and Cameron (1974) constructed models in which they used an adiabatic solar composition envelope around a core of heavy elements. They obtained a fit to the observational data by varying the amount of heavy elements in and surrounding the core. They concluded that Jupiter (and Saturn) is enriched in heavy elements relative to the sun.

Zharkov, Makalkin, and Trubitsyn (1975) were the first to use third order results in constructing Jupiter models. This has increased in detail the knowledge of the composition of the high density core.

Now with the acquisition of data from Pioneers 10 and 11 it is necessary to use a fourth order theory to constrain the models of Jupiter.

In constructing more recent models, Hubbard and Slattery (1976) have concentrated on the distribution of the small admixture of heavy elements which still satisfies the gravity data. After constructing approximate models they conclude that either a relatively uniform distribution of heavy elements throughout the planet, or a distribution which has the heavy elements concentrated at the core would satisfy known constraints, and that Jupiter is enriched in heavy elements relative to the sun. (There is some doubt in their conclusion since their molecular equation of state has subsequently been found to be in error.)

The evolution of Saturn planetary models has been roughly parallel to the evolution of Jupiter models. However, since the mass of Saturn is approximately one-third that of Jupiter, much less of the planet is expected to be metallic hydrogen and helium. Instead Saturn has a much larger proportion of its mass in the molecular state, and thus its models are much more vulnerable to errors in our description of the dense molecular region. Until now there has been great uncertainty in the thermodynamic properties of molecular hydrogen and helium at high pressure, hence all previous models of Saturn should be regarded as rough approximations.

Qualitative Overview of Jovian Planetary Structure

Based on the best models constructed in the past we would expect the following regions to be encountered if we were to travel from the exterior of a Jovian planet to the core. The first significant mass we would encounter would be a very low density molecular region composed mostly of hydrogen and helium. The equation of state for this region, with a density up to 0.005 g-cm^{-3} , is treated very adequately by the perfect gas law. Going deeper, from a density of 0.005 to approximately 1.3 g-cm^{-3} , we would enter what is called the strongly interacting region, or the area in which the gas is definitely not perfect, but neither is it sufficiently pressure-ionized so that we may consider it to be metallic. For the equation of state for densities from 0.005 to 0.3 g-cm^{-3} the thermodynamic data derived in Chapter 3 will be used. For densities greater than 0.3 up to about

1.3 g-cm^{-3} there are not yet sufficient thermodynamic data, so it is necessary to interpolate from a density of 0.3 g-cm^{-3} to the metallic region. Beyond 1.3 g-cm^{-3} (the empirical phase transition from molecular to metallic hydrogen is close to this density [Grigoryev et al. 1972]) we would come to the metallic hydrogen region which has a well-known equation of state, which is considered to be accurate to within 5% (DeWitt and Hubbard 1976). So far we would not have expected to have encountered any solid layers since the planet is thought to be a convecting liquid and thus close to adiabatic (Hubbard, Trubitsyn, and Zharkov 1974).

Because the composition of the planets is assumed to be solar, and if the silicates and iron were not mixed throughout the planet, at higher densities we would expect a small dense core of silicates and iron.

In the following chapters the term "solar mixture of hydrogen and helium" is defined as a mixture of hydrogen and helium only, with $X = 0.78$ and $Y = 0.22$, where X and Y are the hydrogen and helium mass fraction. These relative abundances are the current best estimate of solar composition (Cameron 1973). For future reference the solar mass fraction of elements heavier than helium is $Z = 0.016$ (Cameron 1973).

CHAPTER 3

EQUATION OF STATE FOR MIXTURES OF MOLECULAR HYDROGEN AND HELIUM AT HIGH PRESSURE

Introduction

A Monte Carlo technique similar to that which is described in Slattery and Hubbard (1973) (hereafter known as MC III) is used to calculate the pressure, the specific heat at constant volume, the adiabatic temperature gradient, and the internal energy for a mixture of molecular hydrogen and helium ($X = 0.73$), and for pure molecular hydrogen ($X = 1.0$). The densities which were investigated range from 0.005 g-cm^{-3} to 0.3 g-cm^{-3} , while the temperatures, which range from 500°K to 7600°K , roughly follow a Jovian adiabat.

This Monte Carlo technique is applicable in the following situation:

- (1) The temperature must be above the Debye temperature, i.e., where quantum effects become unimportant. To insure that we were above this boundary for all of the calculations reported below we also calculated the first order quantum correction to the internal energy (Landau and Lifshitz 1969); in all cases the corrections were on the order of a per cent or less of the total internal energy.
- (2) The temperatures must be below the H_2 dissociation curve; as in MC III we took the dissociative curve as calculated by Hayashi and Nakano (1963).

- (3) The density must be below the density where many body contributions to the total interaction potential become important. Ree and Bender (1974) have calculated the intermolecular distance at which the interactions of a third H_2 molecule become important relative to the two body potential, and from this we have calculated that the two body potential remains valid up to 0.3 g-cm^{-3} . However, as noted by Ross (1974), when a molecule is surrounded by many others the charge distortions are very likely to be less than in the three body case, and thus one can expect the Monte Carlo method to be valid beyond 0.3 g-cm^{-3} .

Method of Calculation

Boundary Conditions

In order to represent an infinite system with a finite one, a unit cubical cell is chosen with periodic boundary conditions, i.e., the unit cell is duplicated periodically on all sides of the basic cell, the interaction potential is calculated by centering a cube on each molecule in turn and then calculating the pairwise potential energy. This is illustrated in Figure 1.

Calculation of Thermodynamic Quantities

The total energy of the system can be expressed as:

$$E = K + \phi$$

Figure 1. Periodic Boundary Conditions

Basic cell is in the center. Dotted line shows how the potential "sees" the molecules around it when calculating the energy of the upper right-hand molecule (in basic cell).

where K is the total kinetic energy and ϕ is the total interaction potential energy.

Now the Helmholtz free energy of the molecules is given by the expression:

$$F = F_0 - kT \ln[V^{-N} \int d^{3N}q \exp(-\phi(q)/kT)] \quad (1)$$

where F_0 is the Helmholtz free energy of a perfect gas of diatomic molecules, N is the number of molecules, k is Boltzmann's constant, T is the temperature °K, V is the volume of the system, and $d^{3N}q$ represents all of the differentials over the coordinate space.

The Monte Carlo technique of Metropolis et al. (1953) was used to evaluate various moments (ensemble averages) of the above multidimensional integral to obtain the internal energy, the specific heat at constant volume, the pressure and the adiabatic temperature gradient. Using this method to calculate ensemble averages, the equation for the internal energy may be written as:

$$E = D_f NkT + NkT \langle \phi / NkT \rangle \quad (2)$$

where K has been replaced by its value $\frac{5}{2} D_f NkT$ where

$D_f = ((5/2)N_{H_2} + (3/2)N_{He})/N$, and N_{H_2} and N_{He} are the number of hydrogen and helium molecules respectively. Also ϕ has been replaced by its ensemble averaged value $NkT \langle \phi / NkT \rangle$.

Since only the intermolecular potential ϕ_{ij} is known, ϕ/kT is given by the expression:

$$\frac{\Phi}{kT} = \frac{1}{kT} \frac{1}{2} \sum_{i \neq j}^N \phi_{ij}$$

where the summation is over all pairs of molecules. The functional dependence of ϕ_{ij} is discussed below. To simplify the notation we next define

$$\frac{\langle u \rangle}{N} \equiv \left\langle \frac{\Phi}{NkT} \right\rangle ;$$

thus equation (2) transforms to

$$E = D_f NkT + kT \langle u \rangle \quad (3)$$

and hence $\langle u \rangle$ is the quantity evaluated by the Monte Carlo chain.

Following Hubbard and Slattery (1971) the expressions for the other thermodynamic quantities of interest are now derived.

To obtain the specific heat C_v it is noted that

$$\langle u \rangle = \frac{\int d^{3N}q \frac{\Phi(q)}{kT} \exp(-\Phi(q)/kT)}{\int d^{3N}q \exp(-\Phi(q)/kT)} \quad (4)$$

and from the first law of thermodynamics

$$C_v = T \left(\frac{\partial S}{\partial T} \right)_v = \left(\frac{\partial E}{\partial T} \right)_v . \quad (5)$$

Substituting for $\langle u \rangle$ from equation (4) into equation (5), substituting

this new expression for E into equation (5), and taking the indicated derivatives, the specific heat is given as:

$$\frac{C_v}{Nk} = D_f + (\langle u^2 \rangle - \langle u \rangle^2)/N .$$

Notice that once $\langle u \rangle$ is given it would be possible to derive all of the other thermodynamic variables of interest using numerical differentiation. However, since a Monte Carlo calculation is intrinsically noisy and numerical differentiation would accentuate this noise, it is best to evaluate the other variables by calculating the necessary ensemble averages.

The pressure is calculated by using the relation

$$P = - \left(\frac{\partial F}{\partial V} \right)_T .$$

Thus using equation (1) for F , taking the ensemble average, and changing from atomic units to c.g.s. units (since ϕ is given in atomic units*), the pressure is expressible as:

$$P = P_0 - \left\langle \frac{d\phi}{dV} \right\rangle \frac{e^2}{a_0 kT}$$

where $P_0 = \frac{NkT}{V}$, e is the charge on the electron, and a_0 is the Bohr radius. The second term is then a correction due to molecular interaction. Defining $\frac{N}{V} = \frac{1}{4/3\pi a^3}$ where a is then the volume per molecule

*In atomic units the unit of length is the Bohr radius a_0 , and the unit of energy is e^2/a_0 .

and letting $r_s = \frac{a}{a_0}$, then

$$\frac{(P-P_0)V}{NkT} = \frac{e^2}{a_0 kT} \frac{r_s}{3N} \left\langle \frac{\partial \phi}{\partial r_s} \right\rangle = \left\langle \frac{e^2}{a_0 kT} \frac{r_s}{3N} \frac{\partial \phi}{\partial r_s} \right\rangle = \langle v \rangle$$

where the quantity $\langle v \rangle$ is evaluated along with $\langle u \rangle$ in the Monte Carlo chains.

The adiabatic temperature gradient is calculated from the definition

$$\gamma \equiv - \left(\frac{\partial \ln T}{\partial \ln V} \right)_S = \frac{V}{T} \left(\frac{\partial T}{\partial V} \right)_S \quad (6)$$

to reduce equation (6) to a form suitable for calculation. The last factor may be written as:

$$\begin{aligned} \frac{\partial T}{\partial V}_S &= \frac{\partial(T,S)}{\partial(V,S)} = \frac{\partial(T,S)/\partial(V,T)}{\partial(V,S)/\partial(V,T)} \\ &= - \frac{(\partial S/\partial V)_T}{(\partial S/\partial T)_V} \end{aligned}$$

(Landau and Lifshitz 1969). Substituting for the denominator from equation (5) yields

$$\frac{\partial T}{\partial V} = - \frac{T}{C_V} \left(\frac{\partial S}{\partial V} \right)_T \quad (7)$$

Using one of Maxwell's identities $(\partial S/\partial V)_T = (\partial P/\partial T)_V$ equation (7) becomes $(\partial S/\partial V)_S = - T/C_V (\partial P/\partial T)_V$. Substituting this expression into equation (6) gives

$$\gamma = V(\partial P / \partial T)_V / C_V .$$

From the relation

$$P = P_0 + [\int d^{3N} q (d\phi/dV) \exp(-\phi/kT)] / [\int d^{3N} q \exp(-\phi/kT)]$$

and performing the indicated differentiation,

$$\gamma = \frac{Nk}{C_V} [1 + (\langle uv \rangle - \langle u \rangle \langle v \rangle) / N].$$

Potentials Used in the Monte Carlo Calculations

As mentioned earlier, this Monte Carlo method assumes that the total interaction potential, ϕ , can be represented as a pairwise sum, i.e.,

$$\phi = \frac{1}{2} \sum_{i \neq j} \phi(R_{ij})$$

where $\phi(R_{ij})$ is the two body potential between molecules i and j separated by a distance R_{ij} .

For a mixture of molecular hydrogen and helium, potentials are needed for the three cases; H_2-H_2 , H_2-He , and $He-He$. In all cases a spherically symmetric potential of the form

$$\phi(R) = C_1 e^{-C_2 R} - (C_3/R^6 + C_4/R^8) e^{-(C_5/R)^6} \quad (8)$$

where R is the molecular distance and the C_i 's are constants was used. For H_2-H_2 the potentials are given in the form of equation (8). For H_2-He and $He-He$ the sum of the squares of the relative errors between this potential and the given potential were minimized by varying the

C_i 's. The errors which result from the fitting for both H_2 -He and He-He average less than 3%. It is noted here that equation (8) fits the potentials of these molecules far better than even generalized versions of the $n(R) - 6$ potential of Maitland and Smith (1973). For all cases the choice of a spherical rather than a non-spherical potential was made for the following reasons:

- (1) The accuracy of the potentials at the current time does not warrant a non-spherical potential.
- (2) MC III used a harder non-spherical potential which was anisotropic by a factor of two, and the H_2 molecules still did not show a preferred orientation.
- (3) A spherical potential saves computer time.

The potentials which have been adopted here are now discussed. All parameters for the potentials are listed in Table I. For H_2 - H_2 the intermolecular potential as given by Hoover et al. (1972) was in the form of equation (1). This potential which was derived theoretically agrees with shock data up to 900kb (Ross 1974). The H_2 -He intermolecular potential was derived empirically from quantum scattering theory of rotational relaxation and spectral line shapes by Shafer and Gordon (1973). The He-He intermolecular potential, taken from Farrar and Lee (1972), is a combination of many theoretical and experimental results. Of these three intermolecular potentials only the H_2 - H_2 has been verified by high pressure data. Fortunately a solar mixture of hydrogen and helium is predominantly hydrogen, and thus the

Table I. Constants Used for the Potentials

All values are in atomic units. The unit of length is the Bohr radius a_0 and the unit of energy is e^2/a_0 where e is the charge on the electron. (See Chapter 3, equation 8.)

	C_1	C_2	C_3	C_4	C_5
H ₂ -H ₂	8.2	1.74	13.	116.	2.7144
H ₂ -He	15.644	2.0760	2.6139	189.54	5.3828
He-He	7.3875	2.1784	1.1996	38.356	4.1845

thermodynamic parameters of interest do not depend sensitively upon the other potentials.

For comparison the three potentials are plotted in Figure 2.

Results

Using 16 hydrogen molecules and 3 helium atoms, the above interaction potentials, and the standard periodic boundary conditions, an initial Monte Carlo chain was run for each temperature and density point to relax the system. This was followed by a longer chain to obtain the ensemble averages shown in Table II. For this data series the initial chains varied from 2000 to 30,000 configurations and the final chains varied from 15,000 to 180,000 configurations. The largest chains corresponded to the lower temperatures at the higher densities.

To justify using only 19 molecules I also did two computer runs with 108 particles (91 H_2 molecules and 17 He atoms). The first run had a density of 0.2 g-cm^{-3} and a temperature of 6100°K , the second run had a density of 0.3 g-cm^{-3} and a temperature of 7600°K . Comparison with the corresponding runs containing only 19 molecules showed that all errors were less than 3%. (A system with 108 particles has been shown to be adequate for longer range potentials [Brush, Shalin, and Teller 1966 and Hansen 1973]. The potentials used here approach zero much faster than the potentials used by Brush et al. and Hansen, therefore these results should be even less affected by the numbers of particles.)

Figure 2. Potentials Used for This Paper
Note change in scale for the ordinate.

Table II. Thermodynamic Quantities for a Solar Mixture of Hydrogen and Helium

U is the internal energy, C_v specific heat at constant volume and γ is the adiabatic temperature gradient, $\gamma = (\partial \ln T / \partial \ln \rho)_s$, N is the total number of molecules and k is Boltzmann's constant. ($N_{H_2} = 16$ and $N_{He} = 3$)

T (°K)	ρ (g/cm ³)	U/NkT	Pressure (Kilobars)	C_v/Nk	γ
500	0.005	2.34	0.0921	2.35	0.429
700		2.34	0.128	2.35	0.428
1300		2.34	0.237	2.35	0.428
700	0.01	2.34	0.263	2.36	0.431
900		2.35	0.339	2.36	0.431
2100		2.36	0.787	2.36	0.430
1400	0.05	2.41	3.24	2.42	0.462
1700		2.41	3.88	2.42	0.461
3300		2.41	7.19	2.40	0.450
2000	0.10	2.51	11.2	2.50	0.510
2400		2.52	13.7	2.49	0.501
3500		2.50	18.8	2.46	0.488
4400		2.50	22.8	2.46	0.484
3000	0.20	2.83	52.2	2.62	0.597
3500		2.80	53.0	2.61	0.591
4400		2.76	68.2	2.59	0.583
6100		2.70	85.4	2.53	0.544
3300	0.25	3.02	85.7	2.69	0.649
4400		2.94	102.	2.65	0.629
6100		2.81	117.	2.64	0.616
7000		2.82	139.	2.59	0.589
3500	0.30	3.25	130.	2.74	0.694
4400		3.14	147.	2.64	0.638
6100		3.02	179.	2.65	0.638
7600		2.94	204.	2.64	0.628

An immediate application of this data is the calculation of the adiabatic variation of temperature, density, and pressure. Noting that $\gamma = (\partial \ln T / \partial \ln \rho)_s$ is calculated at all temperature (T) and density points (ρ) and letting $y = \ln T$ and $x = \ln \rho - \ln \rho_0$ where ρ_0 is constant, (assuming a general quadratic model), then

$$\frac{dy}{dx} = b_1 + b_2 y + b_3 y^2 + b_4 xy + b_5 x + b_6 x^2. \quad (9)$$

Since dy/dx has been calculated as γ and the corresponding x and y are known, equation (9) is then fitted to the data by minimizing the square of the relative error using the $b_1, b_2 \dots b_6$ as parameters.

The adiabatic temperature variation with density is then obtained by solving equation (9) for y (see Appendix A). For application to the Jovian planets it is convenient to start an adiabat at 1 bar pressure and $T_1 = T(1 \text{ bar})$. We then follow the perfect gas adiabat with quantum mechanical corrections, using the method of Hubbard (1973) until $\rho_0 = 0.005$ and T_0 is the corresponding adiabatic temperature. We then continue to follow an adiabat using $y_0 = \ln T_0$ and $x_0 = \ln \rho_0 - \ln \rho_0 = 0$ as boundary conditions in equation (9).

The parameters $b_i (i = 1, 2 \dots 6)$ for $\rho_0 = 0.005$ are given in Table III.

Once the temperature is known for a given density it is then possible to interpolate in the data to get the corresponding adiabatic pressure. In Figure 3 we have plotted two pressure-density adiabats (solid lines). The numbers not in parentheses above each adiabat are the starting temperatures in degrees Kelvin for that adiabat ($Y = 0.27$).

Table III. Parameters for Adiabats

See Chapter 3, equation 9, and Appendix A, equation 2.

i	b_i
1	1.8717
2	-0.43557
3	0.032982
4	-0.046732
5	0.25426
6	0.040438

Figure 3. Adiabats Computed in This Paper

The starting temperature at one bar is given in degrees Kelvin above each adiabat. Solid Curve: the starting temperature above the curves without parentheses are for a mixture of hydrogen and helium ($Y = 0.27$). The starting temperature in parentheses above the curves are for pure hydrogen. Dashed Curve: adiabat from the potential of Tapia, Bessis, and Bratoz (1971), $Y = 0$.

A series of Monte Carlo runs on pure molecular hydrogen indicates that the effect of reducing the helium abundance is to give essentially the same adiabat as a solar mixture, but at a lower starting temperature. The numbers in parentheses above the solid line adiabats in Figure 3 are the starting temperatures at 1 bar for pure hydrogen adiabats. Comparison of the pure hydrogen results with those for 73% hydrogen and 27% helium shows that to a good approximation adiabats for other compositions can be derived through linear interpolation, by noting that the pressure-density relation remains essentially constant when the hydrogen fraction increases from 73% to 100% and T_1 is simultaneously reduced by 30%. For example, to obtain adiabats for the composition $X = 0.78$ and $Y = 0.22$ (used for Jovian models here) merely involves relabeling the $X = 0.73$ adiabats 10° lower. (Since the interpolation interval is limited, linear interpolation is assumed to be an adequate approximation.)

A series of Monte Carlo computations were also made for molecular hydrogen and helium ($X = 0.73$) using for the H_2 - H_2 potential an empirical potential from Ross (1974). This potential is the same as the Hoover et al. potential, except that C_1 and C_2 are replaced by 7.5 and 1.69 respectively. The resulting changes in the $X = 0.73$ adiabats are small. The starting temperatures at 1 bar to obtain essentially the same pressure-density adiabats are reduced by about $10^\circ K$.

Remarks on the Thermodynamics
of Molecular Hydrogen and Helium Mixtures

Of the three types of potentials required for a mixture of molecular hydrogen and helium (H_2-H_2 , H_2-He , and $He-He$), only the H_2-H_2 potentials can be compared with high pressure data at this time. However, for the H_2-H_2 potential there still remains an uncertainty of about 40% in the value of the potentials at a given separation due to the uncertainty in the high pressure shock data. Since there are uncertainties in the shock data the potential of Hoover et al. (1972) is favored because of its theoretical background. However, it should be noted that the two H_2-H_2 potentials (used here), which appear to fit the shock data better than any others, arrive at essentially the same adiabats by changing the starting temperature at 1 bar by only 10°K.

One disadvantage of using only 19 molecules in a Monte Carlo calculation is that reliable information concerning the solubility of helium in molecular hydrogen cannot be obtained. It will be assumed here that helium is soluble in molecular hydrogen.

The Tapia potential (Tapia et al. 1971) used in MC III is much too hard to fit the shock data. For comparison, an adiabat which was computed from the potential of Tapia is plotted in Figure 3 (dashed curve $Y = 0$).

CHAPTER 4

METALLIC EQUATION OF STATE

For the metallic or pressure-ionized interiors of the planets a much greater choice of data is possible for the equation of state; however, the choice is essentially a matter of convenience since the available data agree within 5% (DeWitt and Hubbard 1976). DeWitt and Hubbard give the Helmholtz free energy, F , of a dense ionized hydrogen and helium plasma as a function of the density, ρ , the temperature, T , and the helium mass fraction, Y . Since the pressure is given as $P = \rho^2(\partial F/\partial \rho)_T$, the pressure for a given density and composition can be calculated once the temperature is known. Thermal effects are small, however, for a dense plasma, and knowledge of the adiabatic temperature gradient (Grüneisen parameter) for the molecular and metallic regions permits estimates of the temperature for a given density along an adiabat. We have used the equation

$$T = 120 T_1 (\rho/5)^{0.64} \quad (1)$$

where T_1 is the temperature at one bar pressure (Hubbard 1976). The errors which result when using this equation are less than five per cent in the pressure for a given density.

Another source of error is the inclusion of alpha particles for the helium atoms in the metallic equation of state, because helium

is not expected to be ionized until pressures of about 90 Mbars are reached. This is well above the pressures expected in the interior of Jupiter (Hubbard and Smoluchowski 1973). However, if $Y = 0.22$, helium only comprises 7% of the total number of particles, hence this problem is not expected to be significant. A related problem is whether or not the helium atoms are insoluble in the proton-electron plasma as suggested by Smoluchowski (1967, 1971, 1973). However, theoretical calculations by Stevenson and Salpeter (1976) suggest that for the temperature range of interest here, helium is soluble in a hydrogen plasma.

Because very little is known about the phase transition from molecular to metallic hydrogen, for convenience it is assumed here that the process is gradual or that the density changes smoothly with increasing pressure. Grigoryev et al. (1972) did in fact observe a small discontinuous density increase around 2.8 Mb., but the error introduced by ignoring this discontinuity is considered insignificant when compared with the error introduced by the required interpolation between the metallic and molecular regions.

CHAPTER 5

BOUNDARY CONDITIONS AND PLANETARY STRUCTURE COMPUTATION

Boundary Conditions

Before the numerical computation of the planetary structure is discussed, let us first consider the boundary conditions which must be satisfied by the models.

The basic physical data concerning the planets is given in Table IV. These values were derived from ground based observations and the space probes Pioneers 10 and 11 (Brouwer and Clemence 1961, Null, Anderson, and Wong 1975, and Anderson, Hubbard, and Slattery 1974), and they represent the best available values at this time. The J_2 's in Table IV are the standard gravitational moments (Hubbard and Smoluchowski 1973).

In addition to the parameters listed in Table I, the temperature in the envelopes of the planets at one bar pressure (T_1), is required as a boundary condition for the adiabats. A series of papers by Orton (1975a,b), Wallace, Prather, and Belton (1974) and Newburn and Gulkis (1973) give model atmospheres for Jupiter which arrive at a fairly narrow range for T_1 , $150^\circ\text{K} \leq T_1 \leq 190^\circ\text{K}$. With considerably less available data for Saturn, values for T_1 are much more speculative. Values given in the literature vary from 90°K to 150°K (Newburn and Gulkis

Table IV. Numerical Data on Jupiter and Saturn

Quantity	Jupiter	Saturn	Comments
mass (g)	1.90×10^{30}	$.569 \pm .055 \times 10^{30}$	
equatorial radius (km)	7,398	$60,400 \pm 500$	normalizing radius
	$71,400 \pm 100$		radius at one bar pressure
	$71,715 \pm 25$		radius atmospheric level where particle density $\sim 10^{14} \text{ cm}^{-3}$
$J_2 (\times 10^6)$	$14,750 \pm 50$	$16,670 \pm 20$	
$J_4 (\times 10^6)$	-580 ± 40	$1,020 \pm 70$	
$J_6 (\times 10^6)$	50 ± 60	assumed zero	

1973 and Zharkov et al. 1975). T_1 was left a free parameter for Saturn.

Another parameter which is critical in model building is the rotation rate of the planet. For Jupiter $9^h, 55^m, 29^s .5$ is used for the rotation rate. This value is a rough mean between the System III (1957.0) rotation rate and the rotation rates from decimeter and decimeter radiation (Newburn and Gulkis 1973). These values differ by about one part in 10^5 , and therefore any error introduced would be of higher order than the accuracy of this work. The radio rather than the spectroscopic or visual rotation rate is used here because of its assumed connection with the main body of the planets. The lack of any observable periodic radio emission has limited the observation of Saturn's rotation rate to visual and spectroscopic observations which have questionable validity in determining body rotation. Here the rotation rate of Saturn is considered a free parameter about the value $10^h, 30^m \pm 30^m$ (Hubbard and Smoluchowski 1973). This paper does not consider differential rotation for either Jupiter or Saturn.

Numerical Method of Calculating Planetary Structure

As mentioned earlier, the new fourth order perturbation theory of Hubbard, Slattery and DeVito (1975) is used here to solve the equation of hydrostatic equilibrium for a rapidly rotating planet. The small parameter for this expansion is defined as follows:

$$q = \frac{\omega^2 a^3}{GM} \quad (1)$$

where ω is the angular rotation rate, a is the equatorial radius, G is the gravitational constant, and M is the planet's mass; thus q is a dimensionless measure of the planet's rotational distortion. For Jupiter and Saturn q is on the order of 10^{-1} , so that a fourth order expansion can provide the precision of about 10^{-5} for Jupiter and somewhat less for Saturn. (If Saturn's rotation rate were known to the same accuracy as its gravitational moments, a fifth order theory would be required to constrain Saturn models properly.)

Since Jupiter and Saturn are expected to be enhanced in heavy elements relative to the sun, the procedure of Hubbard et al. (1975) was modified by allowing the introduction of a small high density core which is sufficiently small and dense so that it is not distorted by rotation. This change was straightforward in the sense that a small spherical core does not require the introduction of further oblate shells in the interior of the planet. This was done to speed up the calculations. The contribution from an oblate shell of the core to the gravitational moments would be given approximately by the expression: $J_{2\ell} \sim (r_c/a)^{5\ell+6} q^{\ell+1}$ where r_c is the radius of the core and a is the equatorial radius of the planet (Hubbard et al. 1975). For a core with a typical radius of 11,000 km the core contribution to J_2 for Jupiter would be on the order of 10^{-11} . Thus the error in using a spherical core is negligible.

Since numerical equations of state are only known at discrete points, the following modification must be introduced into the method of Hubbard et al. (1975). Hubbard et al. show that to solve the

perturbation hierarchy the fourth derivative of $z(\rho)$ must be continuous where

$$z(\rho) = \int_0^{\rho} \frac{1}{\rho'} \frac{dP}{d\rho'} d\rho' , \quad (2)$$

ρ is the density and $P = P(\rho)$ is obtained from the equation of state. (The modified James theory [James 1964] which is used here does not require the actual calculation of the derivatives of $z[\rho]$, but the derivatives appear implicitly in the calculation.) Since the fourth order derivative of $z(\rho)$ must be continuous, it is necessary that at least the third derivative of $P(\rho)/\rho$ also be continuous. For numerical equations of state this requires either interpolating between the data smoothly, or fitting the numerical data to a smooth function. After considerable experimentation with both approaches, the most accurate approach appears to be a fit of the data to the perturbed polytropic equation of state (see Appendix B).

$$P = \rho^{x_1} \exp(x_2 + x_3\rho + x_4\rho^2 + x_5\rho^3 + x_6\rho^4) \quad (3)$$

where the x_i 's are found by minimizing the sum of the square of the relative errors between the predicted pressure and the actual pressure for a given density. Fitted to physical equations of state this form has the required continuity of the function and derivatives. An additional important property of this form is that it gives a very reasonable equation of state within the interpolation region of $0.3 \text{ g-cm}^{-3} \leq \rho \leq 1.3 \text{ g-cm}^{-3}$.

The errors which result from fitting the perturbed polytropic equation of state to the numerical data average <3% for $\rho \geq 1.3 \text{ g-cm}^{-3}$ (the metallic region). For $0 \leq \rho \leq 0.3 \text{ g-cm}^{-3}$ (the molecular region) the average error is $\leq 10\%$. In both cases the fitting errors are less than the formal errors in the equation of state (see Chapter 2).

Before models of Jupiter and Saturn are presented three of the deduced variables which are not explicitly mentioned in Hubbard et al. (1975) are stated, and the method by which we found models to fit the observations is discussed.

The dynamical oblateness was determined to the third order using the formula (Zharkov and Trubitsyn 1970)

$$e = \frac{3}{2} J_2 + \frac{1}{2} q + \frac{9}{4} J_2^2 - \frac{1}{4} q^2 + \frac{5}{8} J_4 + \frac{81}{8} J_2^3 \\ + \frac{27}{8} J_2^2 q + \frac{3}{8} J_2 q^2 + \frac{1}{8} q^3 + \frac{15}{2} J_2 J_4 + \frac{15}{8} q J_4 \\ + \frac{21}{16} J_6 .$$

The optical oblateness is given by the usual formula:

$$f = (a - b)/a \quad (5)$$

where a and b are the equatorial and polar radii. Both are measured from the center of the planet to the point where the density goes to zero in a polytropic model.

The radius at one bar pressure, a_{1b} , is found by adding $H \ln(\rho_1/\rho_{1b})$ to the radius at the integration point nearest to the place where the pressure is one bar. Here H is the density scale

height which is approximately 30km and 50km for Jupiter and Saturn respectively, ρ_1 is the atmospheric density at the integration point, and ρ_{1b} which is obtained from the equation of state is the density at one bar. The uncertainty in a_{1b} is probably less than 150km.

To obtain models which would fit the observational data at least $2n + 2$ different models were computed (n is the number of independent variables) by varying the independent variables (quantities such as core mass, temperature of envelope at one bar, etc.) which would bracket the observed data. Each dependent variable (e.g., J_2 , J_4 , a , etc.) was then fitted by least squares to an equation which was quadratic in the independent variables. These equations were called the "observable predictor equations". The sum of the square of the relative errors between the correct value of the observable predictor equation were minimized using the independent variables as parameters. The relative error for each dependent variable was weighted in inverse proportion to its relative uncertainty. This procedure allowed us to quickly find a range of good models.

CHAPTER 6

RESULTS AND DISCUSSION

Jupiter Models

Two types of Jupiter models were constructed. The first had an envelope with a solar mixture of hydrogen and helium and a small dense core of heavy elements. For the second type the heavy elements were assumed to be convectively mixed throughout the metallic interior, while the molecular envelope retained a solar mixture of hydrogen and helium. In the following models all adiabats are parameterized by the temperature at one bar pressure.

Jupiter Models with Cores

The independent parameters for these models were (1) the core masses, (2) the core densities, and (3) the starting temperatures for the solar mixture hydrogen and helium adiabatic envelopes.

These models are very temperature dependent. A 190°K adiabat fits the observational data significantly better than adiabats at either 180°K or 200°K. Table V gives the coefficients for the perturbed polytropic equation of state for the 190°K adiabat.

Table VI summarizes two models which have cores of 13.91 M_{earth} and 16.3 M_{earth} . (M_{earth} = mass of earth = 5.976×10^{27} g.) Table VII presents one of the models in detail.

Table V. Coefficients for Perturbed Polytropic Equations of State

	x_1	x_2	x_3	x_4	x_5	x_6
190°K adiabatic solar mixture hydrogen and helium envelope	1.4916	-.72196	1.9654	-.80505	.17867	-.015596
160°K solar mixture molecular adiabatic envelope Polytropic interior $P(M \text{ bar}) = 1.95p + (1+1/0.93845)$	1.4767	-1.0458	3.2703	-1.9446	.51259	-.047538
160°K adiabatic solar mixture hydrogen and helium envelope	1.5030	-.87672	2.0800	-.85140	.18742	-.016240

Table VI. Summary of Three Jupiter Models

Both models J₁ and J₂ have a 190°K solar mixture hydrogen and helium ($\gamma = 0.22$) adiabatic envelope.

Model J₁ core mass = 13.91 M_{earth} ; core density = 22 g-cm⁻³

Model J₂ core mass = 16.3 M_{earth} ; core density = 13 g-cm⁻³

Model J₃ has a 160° solar mixture molecular adiabatic envelope.

The interior is the polytrope $P(M \text{ bar}) = 1.95_p + (1 + 1/0.93845)$

Quantity	J ₁	J ₂	J ₃
<u>Radius (10³Km)</u>			
Core	9.6222	12.139	-
Equatorial at zero pressure (a)	71.848	71.846	71.600
Equatorial at one bar pressure (a _{1b})	71.48	71.48	71.25
Equatorial at start of interpolation region = 0.3 g-cm ⁻³	67.33	67.32	67.47
Equatorial at start of metallic region = 0.3 g-cm ⁻³	54.18	54.18	53.01
<u>Density (g-cm⁻³)</u>			
Core-envelope boundary or center for J ₃	3.95	3.73	3.96
<u>Pressure (at M bar)</u>			
Core-envelope boundary or center for J ₃	42.33	37.52	36.10
<u>Temperature (°K)</u>			
Core-envelope boundary	19,600	18,900	-
<u>Oblateness</u>			
Optical at zero pressure (f)	.06527	.06527	.05494
Dynamical at normalizing radius 71,398km (e)	.06481	.06481	.06456
<u>Gravitational Moments (X 10⁶)</u>			
J ₂	14,779	14,778	14,732
J ₄	-605	-604	-606
J ₆	37	37	29
J ₈	-2	-2	-3

Table VII. Detailed Jupiter Model

Model J₁Envelope: 190°K solar mixture hydrogen and helium ($Y = 0.22$) adiabatic.Core: 13.91 M_{earth} density - 22 g-cm⁻³.

Radius ($\times 10^3 \text{ km}$)	Equatorial Density (g-cm ⁻³)	Pressure (M bar)	Polar	
			Density (g-cm ⁻³)	Pressure (M bar)
9.66	3.95	42.3	3.95	42.3
11.96	3.75	37.9	3.74	37.7
14.26	3.59	34.5	3.58	34.2
16.56	3.46	31.7	3.44	31.3
18.86	3.34	29.3	3.31	28.7
21.16	3.22	27.0	3.19	26.3
23.46	3.11	24.8	3.07	24.1
25.76	3.00	22.8	2.95	21.9
28.06	2.88	20.8	2.82	19.8
30.36	2.78	18.8	2.69	17.8
32.66	2.63	16.9	2.56	15.8
34.96	2.51	15.1	2.42	13.9
37.26	2.37	13.4	2.27	12.1
39.56	2.24	11.7	2.12	10.4
41.86	2.10	10.1	1.97	8.74
44.16	1.96	8.60	1.81	7.23
46.46	1.81	7.20	1.65	5.85
48.76	1.66	5.92	1.49	4.60
51.06	1.51	4.75	1.32	3.49
53.86	1.36	3.70	1.15	2.52
55.66	1.20	2.78	.971	1.70
57.96	1.04	1.99	.783	1.04
60.26	.871	1.33	.554	.539
62.56	.697	.800	.366	.201
64.86	.512	.406	.136	.320 E-1
67.16	.314	.149	.245 E-02	.624 E-4
67.63	.272	.113		
68.10	.230	.820 E-01		
68.56	.189	.570 E-01		
69.03	.149	.376 E-01		
69.50	.112	.228 E-01		
69.97	.766 E-01	.122 E-01		
70.44	.449 E-01	.518 E-02		
70.91	.197 E-01	.145 E-02		
71.38	.516 E-02	.190 E-03		
71.85	0.	0.		

Inspection of the data in Table VI shows that the observational data does not closely restrict the average densities of the cores. Comparing the core parameters with those of Podolak and Cameron (1974) and Zharkov et al. (1975) shows that although the core densities are in fair agreement, there are significant differences in the masses of the cores. The core masses of Podolak and Cameron's models are two to three times greater than masses predicted by the models presented here. Zharkov et al. use very different composition, so direct comparison is difficult. The differences between the results of Podolak and Cameron and those presented here can be attributed to the following reasons:

- (1) Podolak and Cameron use a second order theory which clearly has limited accuracy when compared with a fourth order theory.
- (2) Not having any data for the strongly interacting molecular region they use a van der Waals equation of state to interpolate between the perfect gas region and the metallic region. This interpolation spans somewhat less than three orders of magnitude in the density as compared with an interpolation spanning less than one order of magnitude in the density in the equations of state used here.

The temperatures at the core-envelope boundary, as computed from Chapter 4, equation (1) are on the order of $19,000^{\circ}\text{K}$ for these two models. This value is in reasonable agreement with the temperatures stated by both Podolak and Cameron (1974) and Zharkov et al. (1975).

Jupiter Models with Heavy Elements Distributed Throughout the Metallic Region

This type of model used the perturbed polytropic equation of state fitted to a solar mixture of molecular hydrogen and helium for the molecular envelope, and to a polytropic interior (see Appendix B). Thus the independent variables were (1) the starting temperatures of the adiabats, (2) the polytropic constants K , and (3) the polytropic indices n .

These models were only sensitive to the adiabatic starting temperatures in the broad sense of Hubbard's (1974) gravity inversion technique. Hubbard showed that the acceptable starting temperatures for adiabats in the molecular envelope can be limited because the gravitational moments higher than J_2 are mostly generated in the oblate shell of the planet. According to these criteria, the acceptable temperatures which were computed by Slattery and Hubbard (1976) range roughly from 110°K to 190°K.

Table VI summarizes the parameters for one of these models having an adiabatic envelope which started at 160°K. The coefficient for the perturbed polytropic equation of state is given in Table V.

Since this type of model assumes that the heavy elements are convectively mixed throughout the metallic interior, it is instructive to estimate the amount of heavy elements in Jupiter. Proceeding from the additive volume law (Hubbard and Slattery 1976),

$$\frac{1}{\rho_J(P)} = \frac{X + Y}{\rho_H(P)} + \frac{Z}{\rho_Z(P)} \quad (1)$$

where ρ_J , ρ_H , and ρ_Z are the densities (at the same pressure, P) for a Jupiter model, a hydrogen and helium adiabat, and the heavy elements respectively; and X , Y , and Z are the mass fractions of hydrogen, helium, and heavy elements respectively. Since $X + Y + Z = 1$, $Z \ll 1$ and $\rho_Z \gg 1$, then

$$Z = 1 - \rho_H/\rho_J . \quad (2)$$

Integrating equation (2) over the volume of the planet and using the theorem of the mean,

$$\bar{Z} = \int (\rho_J - \rho_H) dV / M_J = \Delta M_Z / M_J \quad (3)$$

where $\bar{Z}$ is the average mass fraction of the heavy elements in the planet, and M_J and ΔM_Z are the mass of Jupiter and the excess heavy elements respectively. For the well-mixed model given in Table VI $\bar{Z} = 0.048$ which is three times the solar abundance (Cameron 1973) for the heavy elements. Therefore, according to this model there are approximately $15 M_{\text{earth}}$ of heavy elements mixed throughout the metallic interior, whereas if Jupiter were of solar composition we would expect $5 M_{\text{earth}}$.

Comparing these new results with those of Hubbard and Slattery (1976) we see that the total composition of this type of model is very sensitive to the equation of state for the molecular region. Hubbard and Slattery found that approximately $30 M_{\text{earth}}$ were required to satisfy the gravity data. The differences in these results can be

traced directly to their use of an inappropriate molecular equation of state which admitted significantly higher temperature adiabats for the interior.

Saturn Models

The independent variables for Saturn models are (1) the core masses, (2) the core radii, (3) the starting temperatures for the solar mixture hydrogen and helium adiabatic envelopes, and (4) the rotation rates.

The rotation rate is a special parameter since in hydrostatic equilibrium

$$J_2 = \Lambda_2^0 q + \Lambda_2^1 q^2 + \dots \quad (4)$$

$$|J_4| = \Lambda_4^0 q^2 + \Lambda_4^1 q^3 + \dots \quad (5)$$

where the Λ_i 's are the response coefficients (Hubbard and Slattery 1976). Thus if two Saturn models are identical except for different q 's, the Λ_i 's from equations (4) and (5) can be calculated. Substituting the observed J_2 into equation (4) we may then solve for the interpolated q_I , which in turn can be substituted for q into equation (5) to see if the predicted J_4 is within the error range of the observed J_4 . If the predicted J_4 is satisfactory, linear interpolation between the two regional equatorial radii (using equation (4) to the first order) permits a prediction of the period that a model would have by the equation

$$P_I = P_m \left(\frac{q_m a_I^3}{q_I a_m^3} \right)^{1/2} \quad (6)$$

where the I subscript and the m subscript apply to the interpolated value and a model value respectively, the P's are the rotation periods, and the a's are the equatorial radii. Experimentally equation (6) is accurate to $\pm 2^m$, while the linear interpolation in the radii is accurate to within ± 20 km. J_6 and J_8 are found by equations analogous to equations (4) and (5).

As was the case in Jupiter models with dense cores, Saturn models are very sensitive to the starting temperature of the adiabatic envelope. A 160°K adiabat fits the observational data much better than 150°K adiabat, and the change between the 150°K and the 160°K models are sufficiently large that a 170°K adiabat would probably not fit the observed data either.

Using the 160°K adiabatic envelope, models which satisfy the observational data were constructed having core masses ranging from 15 M_{earth} to 17 M_{earth} and core densities ranging from 15 g-cm^{-3} to 9 g-cm^{-3} respectively. Two of these models are summarized in Table VIII (the 15 M_{earth} model was derived using equations [4], [5], and [6]). Table IX presents in detail the model with a 17 M_{earth} core. The coefficients for the perturbed polytropic equation of state for the 160°K adiabat are given in Table V. An 18 M_{earth} core can be ruled out since such a model requires a period of less than 10^h . Models with core masses less than 15 M_{earth} were not computed.

Table VIII. Summary of two Saturn Models

Both models S_1 and S_2 have a 160°K solar mixture hydrogen and helium ($\gamma = 0.22$) adiabatic envelope.

Model S_1 core mass = 15 M_{earth} ; core density = 15 g-cm^{-3}
 Model S_2 core mass = 17 M_{earth} ; core density = 10 g-cm^{-3}

Quantity	S_1	S_2
<u>Radius (10^3km)</u>		
Core	11.257	13.436
Equatorial at zero pressure (a)	60.27	60.089
Equatorial at one bar (a_{1b})	54.81	59.62
Equatorial at start of interpolation region	51.50	51.36
Equatorial at start of metallic region	28.08	28.15
<u>Density (g-cm^{-3})</u>		
Core-envelope boundary	2.19	2.03
<u>Pressure (M bar)</u>		
Core-envelope boundary	10.66	8.98
<u>Temperature ($^{\circ}\text{K}$)</u>		
Core-envelope boundary	11,300	10,700
<u>Oblateness</u>		
Optical, at zero pressure (f)	.1006	.10196
Dynamical, at normalizing radius	.1010	.10300
<u>Period</u>		
	$10^{\text{h}24^{\text{m}} \pm 2^{\text{m}}}$	$10^{\text{h}14^{\text{m}}}$
<u>Gravitational Moments ($\times 10^6$)</u>		
J_2	16,670	16,658
J_4	-953	-965
J_6	81	82
J_8	-10	-10
<u>q(at normalizing radius)</u>		
	.16419	.16890

Table IX. Detailed Saturn Model

Model S₂

Envelope: 160°K solar mixture hydrogen and helium adiabat.

Core: 17 M_{earth}, density = 10 g-cm⁻³.

Radius (X 10 ³ km)	Equatorial Density (g-cm ⁻³)	Pressure (M bar)	Polar	
			Density (g-cm ⁻³)	Pressure (M bar)
13.43	2.03	8.98	2.03	8.98
15.06	1.92	7.96	1.91	7.82
16.68	1.83	7.06	1.81	6.89
18.30	1.74	6.29	1.72	6.10
19.92	1.66	5.62	1.63	5.41
21.54	1.58	5.03	1.55	4.79
23.16	1.51	4.50	1.47	4.24
24.78	1.44	4.02	1.40	3.74
26.42	1.37	3.58	1.32	3.28
28.03	1.30	3.18	1.25	2.86
29.65	1.24	2.81	1.17	2.47
31.27	1.17	2.46	1.10	2.12
32.89	1.11	2.15	1.03	1.79
34.51	1.04	1.85	.951	1.50
36.14	.974	1.58	.875	1.23
37.76	.907	1.34	.797	.986
39.38	.840	1.12	.717	.771
41.00	.777	.914	.634	.584
42.62	.701	.733	.550	.423
44.24	.630	.574	.462	.289
45.86	.557	.436	.371	.182
47.49	.483	.318	.278	.101
49.17	.407	.221	.184	.465 E-01
50.73	.330	.143	.954 E-01	.147 E-01
52.35	.253	.845 E-01	.271 E-01	.194 E-02
53.97	.176	.432 E-01	.167 E-02	.278 E-04
54.58	.149	.319 E-01		
55.20	.122	.225 E-01		
55.81	.973 E-01	.152 E-01		
56.42	.743 E-01	.972 E-02		
57.03	.537 E-01	.573 E-02		
57.64	.355 E-01	.296 E-02		
58.25	.202 E-01	.123 E-02		
58.86	.918 E-02	.368 E-03		
59.48	.303 E-02	.686 E-04		
60.09	0.	0.		

The total masses of the cores and surrounding dense materials of Podolak and Cameron's (1974) Saturn models were in all cases greater than $18 M_{\text{earth}}$. These larger masses in their Saturn model were necessary for the same reasons that larger masses were required in their Jupiter models for the cores and surrounding dense material.

A solar composition Saturn model has $1.5 M_{\text{earth}}$ of heavy elements, hence the Saturn models proposed here would be significantly enriched in heavy elements relative to the Sun.

As given by equation (1), Chapter 4, the temperatures at the core-envelope boundaries for these two models are on the order of $11,000^{\circ}\text{K}$ which is essentially the same as that given by Podolak and Cameron (1974).

Due to computer time limitations no attempt was made to produce Saturn models which had the heavy elements distributed throughout the metallic interior.

CHAPTER 7

CONCLUSION

The equation of state derived in Chapter 3 for the molecular envelopes can be linked with the interior metallic equation of state described in Chapter 4 to produce models of Jupiter and Saturn which are in agreement with the observational data.

The enrichment in the heavy elements relative to solar abundances is about three times for Jupiter and about ten times for Saturn.

Since the interpolation region (the area where the gases are strongly interacting and about which we have very little thermodynamic data) spans approximately one-fourth the mass of Jupiter and one-half the mass of Saturn, the planetary models presented here are not definitive. More thermodynamic data are required in this region, especially for Saturn models. Other critical parameters needed for Saturn are an accurate determination of the rotation rate and the equatorial radius. Perhaps the data gathered by Pioneer 11 when it passes by Saturn in 1979 will satisfy these needs.

APPENDIX A

SOLUTION OF THE MOLECULAR ADIABATIC EQUATION

To solve equation (9), Chapter 3, we note that it is a Ricatti equation. We therefore let

$$y = \frac{dz}{dx} \frac{1}{b_3 z} \quad (1)$$

Substituting equation (1) for y into equation (9), Chapter 3, we obtain

$$\frac{d^2 z}{dx^2} - (b_2 + b_4 x) \frac{dz}{dx} + (b_1 + b_5 x + b_6 x^2) b_3 z = 0$$

which has a series solution

$$z = \sum_{n=0}^{\infty} a_n x^n \quad (2)$$

where $a_0 = 1$, $a_1 = -b_3 y_0$, $a_2 = (b_2 a_1 - b_1 b_3 a_0)/2$
 $a_3 = (2 \cdot b_2 a_2 - (b_1 b_3 - b_4) a_1 - b_5 b_3 a_0)/6$ and for $n \geq 4$
 $a_n = (b_2(n-1) a_{n-1} - (b_1 b_3 - b_4(n-2)) a_{n-2} - b_5 b_3 a_{n-3} - b_6 b_3 a_{n-4}) / (n(n-1))$. Consequently, given density ρ for which we want the corresponding adiabatic temperature, we let $x = \ln \rho - \ln \rho_0$, solve for z and dz/dx from equation (2) and then find $y = \ln T$ from equation (9), Chapter 3.

APPENDIX B

PERTURBED POLYTROPIC EQUATION OF STATE

A perturbed polytropic equation of state was initially chosen because Jupiter can be modeled satisfactorily using a polytrope of index $n = 0.95$ (Hubbard, Slattery, and DeVito 1975). The polytropic equation of state is defined as

$$P = K\rho^{1 + 1/n} \quad (1)$$

where K and n are the polytropic constant and index respectively (Chandrasekhar 1939).

Taking the logarithmic derivative of equation (1) we find

$$d \ln P / d \ln \rho = 1 + 1/n \quad (2)$$

which is constant if n is constant.

A generalization of equation (2) can be expressed as:

$$\frac{d \ln P}{d \ln \rho} = X_1 + X_3 \rho + 2X_4 \rho^2 + 3X_5 \rho^3 + \dots$$

where the X_K 's are parameters.

Integration yields

$$\ln P = X_1 \ln \rho + X_2 + X_3 \rho + X_4 \rho^2 + X_5 \rho^3 + \dots \quad (3)$$

where X_2 is a constant of integration.

Equation (3) may be written as

$$P = \rho^{X_1} \exp[X_2 + X_3 \rho + X_4 \rho^2 + X_5 \rho^3 + \dots]$$

whence we recover equation (3), Chapter 5.

We experimentally determined that 6 coefficients optimized the fit with a minimum number of parameters.

REFERENCES

- Alfvén, H., 1954, On the Origin of the Solar System, (Oxford: Clarendon Press), p. 194.
- Anderson, J. D., Hubbard, W. B., and Slattery, W. L., 1974, Structure of the Jovian Envelope from Pioneer 10 Gravity Data, Ap.J.Lett. 192, L 194 - L 150.
- Brouwer, D., and Clemence, G. M., 1961, in The Solar System Vol. 3: Planets and Satellites, ed. G. P. Kuiper and B. M. Middlehurst (Chicago: University of Chicago Press), p. 31.
- Brush, S. G., Shalin, H. L., and Teller, E., 1966, Monte Carlo Study of a One Component Plasma, J.Chem.Phys. 45, 2102-2118.
- Cameron, A. G. W., 1973, Abundances of the Elements in the Solar System, Spa.Sci.Rev. 15, 121-146.
- Chandrasekhar, S., 1939, An Introduction to the Study of Stellar Structure, (New York: Dover), p. 84ff.
- DeMarcus, W. C., 1958, The Constitution of Jupiter and Saturn, Ap.J. 63, 2-28.
- DeWitt, H. E., and Hubbard, W. B., 1976, Statistical Mechanics of Light Elements at High Pressure IV, A Model Free Energy for the Metallic Phase, Ap.J., in press.
- Farrar, J. M., and Lee, Y. T., 1972, Intermolecular Potentials from Cross Beam Differential Elastic Scattering Measurements. V. The Attractive Well, J.Chem.Phys. 56, 5801-5807.
- Grigoryev, F. V., Kormer, S. B., Miklailova, O. L., Tolchko, A. P., and Urlin, V. D., 1972, Experimental Determination of the Compressibility of Hydrogen Between 0.5 and 2g/cc. Metalization of Hydrogen, JETP Letters 16, 20k-204.
- Hansen, J. P., 1973, Statistical Mechanics of Dense Ionized Matter.I. Equilibrium Properties of the Classical One Component Plasma, Phys.Rev.A 8, 3096-3109.
- Hayashi, C., and Nakano, T., 1963, Evolution of Stars of Small Masses in Pre-main Sequence Stages, Prog.Theoret.Phys. 30, 460-474.
- Hoover, W. G., Ross, M., Bender, C. F., Rogers, F. J., and Olness, R. J., 1972, Correlation of Theory and Experiment for High Pressure Hydrogen, Phys.Earth Planet. Interiors 6, 60-64.

- Hubbard, W. B., 1968, Thermal Structure of Jupiter, Ap.J. 152, 745-754.
- _____ 1969, Thermal Models of Jupiter and Saturn, Ap.J. 155, 333-344.
- _____ 1970, Structure of Jupiter: Chemical Composition, Contraction, and Rotation, Ap.J. 162, 687-697.
- _____ 1973, Observational Constraints on the Structure of Hydrogen Planets, Ap.J.Lett. 182, L 35 - L 38.
- _____ 1974, Inversion of Gravity Data for Giant Planets, Icarus, 21, 157-165.
- _____ 1976, Private comm., Prof. of Planetary Sci., Univ. of Arizona.
- Hubbard, W. B., and Slattery, W. L., 1971, Statistical Mechanics of Light Elements at High Pressure I. Theory and Results for Metallic Hydrogen with Simple Screening, Ap.J. 168, 131-139.
- _____ 1976, Interior Structure of Jupiter: Theory of Gravity Sounding, in Jupiter, ed. T. Gehrels (Tucson: University of Arizona Press), p. 176-194.
- Hubbard, W. B., Slattery, W. L., and DeVito, C. L., 1975, High Zonal Harmonics of Rapidly Rotating Planets, Ap.J. 199, 504-516.
- Hubbard, W. B., and Smoluchowski, R., 1973, Structure of Jupiter and Saturn, Space Sci.Rev. 14, 599-662.
- Hubbard, W. B., Trubitsyn, V. P., and Zharkov, V. N., 1974, Significance of Gravitational Moments for Interior Structure of Jupiter and Saturn, Icarus, 21, 147-151.
- James, R. A., 1964, The Structure and Stability of Rotating Gas Masses, Ap.J. 140, 552-582.
- Landau, L. D., and Lifshitz, E. M., 1969; Statistical Physics (2d ed.; Menlo Park, Calif.: Pergamon Press) p. 52ff, 93ff.
- Low, F. J., 1966, Observations of Venus, Jupiter, and Saturn at $\lambda 20\mu$, A.J. 71, 391.
- Maitland, G. C., and Smith, E. B., 1973, A Simplified Representation of Intermolecular Potential Energy, Chem.Phys.Lett. 22, 443-446.
- Metropolis, N., Rosenbluth, A. W., Rosenbluth, M. N., Teller, A. H., and Teller, E., 1953, Equation of State Calculation by Fast Computing Machines, J.Chem.Phys. 21, 1087-1092.

- Newburn, R. L., and Gulkis, S., 1973, A Survey of the Outer Planets Jupiter, Saturn, Uranus, Neptune, Pluto, and their Satellites, Space Sci.Rev. 3, 179-271.
- Null, G. W., Anderson, J. D., and Wong, S. K., 1975, The Gravity Field of Jupiter from Pioneer 11 Tracking Data, Science, 188, 476-477.
- Öpik, E. J., 1962, Jupiter: Chemical Composition, Structure and Origin of a Giant Planet, Icarus, 1, 200-257.
- Orton, G. S., 1975a, The Thermal Structure of Jupiter I. Implications of Pioneer 10 Infrared Radiometer Data, Icarus 26, 125-141.
- Orton, G. S., 1976b, The Thermal Structure of Jupiter II. Observations and Analysis of 8-14 Micron Radiation, Icarus, 26, 142-158.
- Peebles, P. J. E., 1964, The Structure and Composition of Jupiter and Saturn, Ap.J. 140, 328-347.
- Podolak, M., and Cameron, A. G. W., 1974, Models of the Giant Planets, Icarus, 22, 123-148.
- Ramsey, W. H., 1951, On the Constitution of the Major Planets, M.N.R.A.S. 111, 427-447.
- Ree, F. H., and Bender, C. F., 1974, Nonadditive Interaction in Molecular Hydrogen at High Pressure, Phys.Rev.Lett. 32, 85-88.
- Ross, M., 1974, A Theoretical Analysis of Shock Compression Experiments of the Liquid Hydrogen Isotopes and a Prediction of Their Metallic Transition, J.Chem.Phys. 60, 3634-3644.
- Shafer, R., and Gordon, R. G., 1973, Quantum Scattering Theory of Rotational Relaxation and Spectral Line Shapes in H₂-He Gas Mixtures, J.Chem.Phys. 58, 5422-5443.
- Slattery, W. L., and Hubbard, W. B., (MC III), 1973, Statistical Mechanics of Light Elements at High Pressure. III. Molecular Hydrogen, Ap.J. 181, 1031-1038.
- 1976, Thermodynamics of a Solar Mixture of Molecular Hydrogen and Helium at High Pressure, Icarus, in press.
- Smoluchowski, R., 1967, Internal Structure and Energy Emission of Jupiter, Nature, 215, 691-692.
- 1971, Metallic Interiors and Magnetic Fields of Jupiter and Saturn, Ap.J. 166, 435-439.

- Smoluchowski, R., 1973, Dynamics of the Jovian Interior, Ap.J.Lett. 185, L 95 - L 99.
- Stevenson, D. J., and Salpeter, E. E., 1976, Interior Models of Jupiter, in Jupiter, ed. T. Gehrels, (Tucson: University of Arizona Press), p. 88-112.
- Tapia, O., Bessis, G., and Bratoz, S., 1971, Interactions Moleculaires a courte Portee dous le System H_2-H_2 , Int.J.Quantum Chem. 4, 289-297.
- Wallace, L., Prather, M., and Belton, M. J. S., 1974, Temperature and Structure of the Atmosphere of Jupiter, Ap.J. 193, 481-493.
- Wildt, R., 1947, The Constitution of the Planets, M.N.R.A.S. 107, 82-102.
- Zharkov, V. N., Makalkin, A. B., and Trubitsyn, V. P., 1975, Models of Jupiter and Saturn II. Structure and Composition. Sov.Astron.Eng. Transl. 18, 768-773.
- Zharkov, V. N., and Trubitsyn, V. P., 1970, Theory of the Figure of Rotating Planets in Hydrostatic Equilibrium--A Third Approximation, Sov.Phys.Astron.Eng.Transl. 13, 981-988.