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Stress singularities at crack corners

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STRESS SINGULARITIES AT CRACK CORNERS

by
Linlin Xu

A Thesis Submitted to the Faculty of the
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In the Graduate College

THE UNIVERSITY OF ARIZONA

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ABSTRACT

In this thesis the stress and displacement fields near an embedded crack corner in a linear elastic medium are analytically computed. The conical-spherical coordinate system is introduced to solve this problem. It is observed that the strength of the stress singularity is dependent on the angle of the crack corner. The singularity becomes weaker, varying from r^{-1} to r^0 , as the angle of the crack corner varies from 360° to 0° . Both symmetric and skew-symmetric loadings give the same variation of the behavior of the stress singularity. It is also observed that the order of the singularity is independent of the Poisson's ratio, unlike the corner cracks at a free surface where Poisson's ratio affects the results.

CHAPTER 1

INTRODUCTION

In linear elastic fracture mechanics it is well known that the stress field near an ideal crack tip has a square root ($r^{-\frac{1}{2}}$) singularity (Williams, 1957). This is true as long as the crack front is bounded by a smooth curve (Ouyang & Xu, 1987), *i.e.* a tangent can be drawn at the crack front as shown in Figure 1-1. For different crack geometries the order of the stress singularity does not change as long as the crack front is smooth but the strength of the stress field and hence the stress intensity factors (K_I , K_{II} , and K_{III}) vary. Stress intensity factors for some ideal crack geometries are available in the literature. [Tada *et. al.* (1973), Hellan (1984), and Sih (1973)].

Rice (1985), Gao & Rice (1986, 1987), Meade & Keer (1984) investigated the variation of the stress intensity factors as the crack geometry deviates from the ideal shapes (circular, elliptical, two dimensional cracks with straight fronts) as shown in Figure 1-2. Only the small deviations from the ideal shapes could be handled by these first order variation analyses presented by Rice (1985) and his co-workers. For large variations numerical methods such as finite element or boundary element techniques are necessary to obtain the appropriate stress intensity factors. However, all these approaches assume smooth crack fronts as shown in Figure 1-1 or 1-2 and does not work well near the crack corners as shown in Figure 1-3. Although a large volume of works has been devoted to the analytical and numerical stress analyses of cracks with smooth crack fronts very few investigators attempted to obtain analytically the stress field near a sharp corner of a crack where no tangent can be drawn.

Benthem (1977, 1980) and Bazant & Estensoro (1979) solved corner crack

Figure 1-1 Crack With Smooth Boundary

Circular

Elliptical

Flat

Figure 1-2 Ideal Crack Shapes (solid lines) and Variations
from Ideal Shapes (dotted lines)

Figure 1-3 Crack with a Sharp Corner at the Boundary

problems for the surface cracks. In these problems, which these investigators considered, one side of the crack corner coincides with the free surface of the body. Benthem (1977, 1980) solved the problem of quarter-infinite cracks ($2\alpha = 90^\circ$ in Figure 1-3); his results have been verified by Bažant & Estensoro (1979). Both Benthem and Bažant found that the strength of the stress singularity near the crack corner is different from $r^{-\frac{1}{2}}$. They also observed that the strength of the singularity is dependent on Poisson's ratio. For symmetric loading the stress field singularity becomes weaker as Poisson's ratio increases from 0 to 0.5 and for skew-symmetric loading the reverse is true. Folias (1975) also tried to solve the corner crack problems for surface cracks and concluded that the stress singularity at the corner should be dependent on the Poisson's ratio. However, the quantitative values of the order of the stress singularities predicted by Folias didn't match with those presented by Benthem or Bažant & Estensoro.

In this thesis the stress singularities near the crack corner of an embedded crack are considered under both symmetric and skew-symmetric conditions. The problems are reduced to potential theory with Trefftz's functions and then solved in the conical-spherical coordinate system.

Morrison & Lewis (1976) used conical coordinate system to compute charge singularity at the corner of a flat plate. Their interesting approach to compute the eigenvalues is to solve numerically the two sets of ordinary differential equations (shooting technique). However, in this thesis we use the eigenvalue equations for simply periodic Lamé functions (Bateman, 1955). Keer & Parihar (1978) used Green's function and arrived at singular integral equations for symmetric problems. They provided numerical results only for the angle near 0 and 2π .

CHAPTER 2

LITERATURE REVIEW

2.1 Brittle Fracture Theory

There is a broad range of phenomena of brittle failure which cannot in principle be explained from the standpoint of strength theory. Predicting the fracture strength of a solid should be based on the study of the actual process of fracture. The primary concept of fracture mechanics is founded on the work of Griffith (1920). The basic idea behind his theory is that a crack will begin to propagate if the elastic energy released by its growth is greater than the energy required to create the fractured surfaces. As a model, Griffith considered the problem of a crack of length $2l$ in a thin plate under tension p as in Figure 2-1. The length of the crack is assumed small in comparison to the dimension of the plate ($l \ll a$, $l \ll b$). Experiments show that, beginning at a certain p , the crack will grow, increasing its free surface. Therefore, Griffith introduced the surface energy of the brittle body and formulated the principle, according to which the existing crack begins to propagate in an avalanche-like manner as soon as the rate of liberation of elastic strain energy exceeds the increase in surface energy of the crack, *i.e.*, when

$$\frac{\partial(\delta U)}{\partial l} > 4\gamma \quad (2-1)$$

Here δU is the change in elastic potential of the plate due to the presence of the crack and γ is the free surface energy of a unit of free surface.

The elastic energy U of the plate with the crack is equal to $U_0 - \delta U$, where U_0 is the elastic potential of the plate without the crack. The value of δU is equal to the product of the mean area of the region of stress concentration (proportional

Figure 2-1. A Thin Cracked Plate Under Tension

to l^2) times the mean value of elastic potential density (proportional to p^2/E , where E is Young's modulus):

$$\delta U = \frac{\lambda_0 l^2 p^2}{E} \quad (2-2)$$

$$U = U_0 - \delta U$$

Here the factor λ_0 may depend only on Poisson's ratio. Griffith determined the value of constant λ_0 is 2π for a thin plate.

Since the quantity U_0 is independent of l , according to (2-1) and (2-2), we find in the critical state

$$2E\gamma = \lambda_0 p^2 l \quad (2-3)$$

Thus, from (2-3) we derive the following dependence of load on crack length:

$$p = \lambda_1 \sqrt{\frac{E\gamma}{l}} \quad (2-4)$$

Here λ_1 is factor of the order of unity. Formula (2-4), the expression for the failure load as a function of the length of the initial crack, is the primary achievement of the theory of Griffith.

According to equation (2-4), the process occurs as follows (Figure 2-2). At first, as load p increases, the length of the initial crack l_0 remains unchanged until a value of p is reached corresponding to equation (2-4); after this, a dynamic process of crack growth begins.

Figure 2-2. Failure Load As a Function of Crack Length

Until approximately the 1950s, it was considered that the theory of Griffith was applicable only for brittle materials such as glass; most structural materials exhibit ductile properties during failure. The experimental studies of Orowan (1950), Irwin (1957), and later of many other scientists led to the so-called concept of quasibrittle fracture, according to which equation (2-4) is also correct for most materials under conditions of quasibrittle fracture if the quantity γ is replaced by the irreversible energy dissipated in the thin layer of plastic strain near the surface of the crack per unit area of free surface. This last quantity was found to be hundreds and thousands of times larger than the theoretically calculated values of surface energy γ corresponding to an ideally brittle fracture.

The performance of analysis within the framework of the energy approach of Griffith for more complex configurations of bodies and cracks encountered computational difficulties. Therefore, until the late 1950s, the number of problems completely solved was very small.

These difficulties were overcome to a great extent by the force approach suggested by Irwin (1957), in principle equivalent to the method of Griffith. According to Irwin's force method, to solve the problem of the growth of cracks it is sufficient in the purely elastic (and, consequently, linear) problem to find certain stress intensity factors at the crack front, which fully determine the local distribution of stresses, displacements, and strains near the tip of the crack; further analysis is purely algebraic. This theory was therefore called "linear fracture mechanics".

Irwin proposed three modes of crack extension which are identified by their respective stress-intensity factors K_I , K_{II} and K_{III} .

Mode I: Opening

The opening mode, Figure 2-3a, is characterized by the motions of the crack

surfaces that tend to separate symmetrically with respect to the plane occupied by the crack prior to deformation.

Mode II: Sliding

The sliding mode, Figure 2-3b, concerns local deformation in which the crack surfaces glide over one another in opposite directions but in the same plane.

Mode III: Tearing

The movement of the crack surfaces, associated with the tearing mode of fracture, Figure 2-3c, can be related to the warping action of noncircular cylinders under torsion in which the material points, initially in the same plane, occupy different planes after deformation.

2.2 Stress Intensity Factor

One major contribution of fracture mechanics has been the study of crack extension behavior on the basis of the similarity of near crack front stress and deformation fields of different bodies. The degree of similarity is often assessed based solely on the strength of dominant asymptotic crack front fields emerging from various constitutive and kinematical idealizations. Familiar examples of such parameters are the stress intensity factors K_I , K_{II} , and K_{III} of linear elastic fracture mechanics (Irwin, 1957). Providing that such single parameter asymptotic characterizations actually dominate the complete near-crack-front fields (including the zone of operative microfracture processes) in two different bodies (e.g. a laboratory specimen and a structural component), then the respective loading can be scaled to give similar local crack front field, and similar macroscopic crack extension behavior may be expected in each body when crack front fields are similarly scaled. In particular, the stress intensity factor approach to the analysis of brittle

Figure 2-3. Three Basic Modes of Fracture

fracture has proven to be an effective means of correlating the beginning of crack propagation.

For isotropic linear elastic response, the asymptotic crack tip fields at any point s , as in Figure 2-4, along a three dimensional crack front have been characterized (Rice, 1968) by the variation with s of three stress intensity factors, $K_I(s)$, $K_{II}(s)$, and $K_{III}(s)$. Let the linear feature representing the crack front have a continuously turning tangent vector $t(s) = e_3$, local crack plane normal $n(s) = e_2$, and curve bi-normal $b(s) = e_1$. Let local Cartesian coordinates emanating from the crack front be x_i . For material points in the 1,2 plane located a distance $r = (x_1^2 + x_2^2)^{\frac{1}{2}}$ from the origin at angle $\theta = \tan^{-1}(x_2/x_1)$ counterclockwise from the e_1 axis, components of the asymptotic stress tensor, σ_{ij} , are given by

$$\sigma_{ij}(r, \theta, s) = \frac{K_I(s)f_{ij}(\theta) + K_{II}(s)g_{ij}(\theta) + K_{III}(s)h_{ij}(\theta)}{(2\pi r)^{\frac{1}{2}}}$$

Here the dimensionless tensor components f_{ij} , g_{ij} and h_{ij} are known functions of their argument. The task of engineering stress analysis in fracture mechanics is to evaluate numerically $K_I(s)$ [and $K_{II}(s)$, $K_{III}(s)$, if present] for a given body containing crack(s) of a given shape and size, subject to various loadings. There are many successful engineering applications where the stress intensity factors are used, and stress intensity factor solutions for a large number of crack geometries and boundary conditions now exist (Sih, 1973). While exact elastic solutions have been found for some configurations (Sneddon & Lowengrub, 1969), the complexity of three dimensional crack geometries has required the use of numerical methods such as singular integral equations, boundary integral equations, and finite element methods.

Figure 2-4. Stress State Near Crack Border

2.3 Surface Crack

An approximate formula for an elliptical surface crack in a thin elastic plate, Figure 2-5, was given by Irwin (1962). To get a complete understanding of the solutions to surface crack problem in elastic plates, further numerical analyses have been performed by numerous investigators. Smith (1972) and Shah & Kobayashi (1972) used the alternating method. Tracey (1973, 1974) and Newman & Raju (1979, 1983) adopted the finite element method with singularity elements. Atluri & Kathiresan (1976) developed a finite element method with displacement hybrid elements. Nishioka & Atluri (1983, 1986) combined the finite element with the alternating method.

The singular integral formulations (Erdogan & Gupta, 1972) have been recently extended to three dimensional crack configurations through use of a distribution of dipole singularities in conjunction with a finite element substructure (Keat et al., 1988, 1989). The method requires only a modestly refined finite element grid, sufficient to model the no-crack fields and external boundary conditions, while the singular integral equation models the crack. The integral equation is highly singular, and a special procedure for numerical evaluation should be devised. The surface crack problem was analyzed by using this method. For bonded plane-layered media, similar formulations have been developed based on approximate Green's functions (Fares, 1987).

Within all of the research mentioned above, the assumptions have been made that the square root stress singularity always exists along the three dimensional crack front and that crack corners have no influence on the asymptotic solutions. The corner singularity for a surface crack intersecting with a traction free surface has been studied by Benthem (1977, 1980) and Bazant & Estensoro (1979). For

Figure 2-5. Surface Crack In a Plate

mode-I loading of a crack making normal incidence with a free surface, it appears that the corner singularity is generally weaker than that of the square root. Thus K_I should vanish in a thin boundary layer. However, asymptotic corner solutions for skew-symmetric loading of perpendicularly intersecting cracks show that the corner singularity is stronger than the square root singularity, so the local strengths of K_{II} and K_{III} become unbounded. Consistent results with such an interpretation have been also found by intensive numerical analyses (Nakamura & Parks, 1988 and Yingzi, 1990).

CHAPTER 3

CRACK CORNER PROBLEM

3.1 Crack Corner Geometry

In fracture mechanics, simple shapes of geometry have been used to approximate the complicated cracks. Characteristically, most investigators tried to model a crack as a smooth surface of displacement discontinuities bounded by a smooth curve. In this model, the well known $r^{-\frac{1}{2}}$ stress singularity dominates the stress field near the crack tip.

Here, the problem of more complicated crack geometries is considered, in which cracks have sharp corners as shown in Figure 1-3. Mathematically, there is no tangent vector at these corner points. Experience suggests that the corner points have strong influences on the fracture phenomenon.

For simplicity, the crack is modelled as one with an angle of 2α , extending to infinity (Figure 3-1).

3.2 Conical-Spherical Coordinates

Cartesian coordinates are introduced with the origin at the crack corner and xy plane coinciding with the crack surface, see Figure (3-1). To solve this problem it is convenient to use conical-spherical coordinates, (ρ, β, γ) , which are related to the Cartesian coordinates as (Bateman, 1955)

$$\begin{aligned} x &= \frac{1}{k'} \rho \operatorname{dn}(\beta) \operatorname{dn}(\gamma) \\ y &= k \rho \operatorname{sn}(\beta) \operatorname{sn}(\gamma) \\ z &= i \frac{k}{k'} \rho \operatorname{cn}(\beta) \operatorname{cn}(\gamma) \end{aligned} \tag{3.2 - 1}$$

(a) $2\alpha < \pi$

(b) $2\alpha > \pi$

Figure 3-1. An Angular Crack Extending to Infinity

where

$$k = \sin \alpha, \quad k' = |\cos \alpha| \quad (3.2 - 2)$$

and 2α is the angle of the crack corner, and $sn(x)$, $cn(x)$, $dn(x)$ are the Jacobian elliptical functions with modulus k .

Consider a half space in the Cartesian coordinate system defined by

$$\begin{aligned} z &\geq 0 \\ -\infty &< x < \infty \\ -\infty &< y < \infty \end{aligned} \quad (3.2 - 3)$$

After transformation into the conical-spherical coordinate system, this half space corresponds to

$$\begin{aligned} 0 &\leq \rho < \infty \\ \beta &= K + i\beta', \quad 0 \leq \beta' \leq 2K' \\ -K &\leq \gamma \leq K \end{aligned} \quad (3.2 - 4)$$

where K and K' are the complete elliptical integrals:

$$\begin{aligned} K &= \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \\ K' &= \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k'^2 \sin^2 \theta}} \end{aligned} \quad (3.2 - 5)$$

The xy plane can be divided into four regions (see Figure 3-2) which correspond to the four special cases as described below.

Case (1). Let us take $\beta' = 0$, then from Bateman (1955) one obtains

$$\begin{aligned} dn(\beta) &= dn(K) = k' \\ sn(\beta) &= sn(K) = 1 \\ cn(\beta) &= cn(K) = 0 \end{aligned} \tag{3.2 - 6}$$

So

$$\begin{aligned} x &= \rho \, dn(\gamma) \\ y &= k \, \rho \, sn(\gamma) \\ z &= 0 \end{aligned} \tag{3.2 - 7}$$

On the xy plane, this corresponds to *Region 1* in Figure 3-2.

$$x \geq 0, \quad -|\tan \alpha| \leq \frac{y}{x} \leq |\tan \alpha| \tag{3.2 - 8}$$

Case (2). Similarly, for $\beta' = 2K'$,

$$\begin{aligned} dn(\beta) &= dn(K + 2iK') = -k' \\ sn(\beta) &= sn(K + 2iK') = 1 \\ cn(\beta) &= cn(K + 2iK') = 0 \end{aligned} \tag{3.2 - 9}$$

Figure 3-2. Four Regions in the xy Plane

So

$$\begin{aligned}
 x &= -\rho \operatorname{dn}(\gamma) \\
 y &= k \rho \operatorname{sn}(\gamma) \\
 z &= 0
 \end{aligned}
 \tag{3.2 - 10}$$

On the xy plane, this corresponds to *Region 3*:

$$x \leq 0, \quad -|\tan \alpha| \leq \frac{y}{x} \leq |\tan \alpha| \tag{3.2 - 11}$$

Case (3). For $\gamma = -K$,

$$\begin{aligned}
 \operatorname{dn}(\gamma) &= \operatorname{dn}(-K) = k' \\
 \operatorname{sn}(\gamma) &= \operatorname{sn}(-K) = -1 \\
 \operatorname{cn}(\gamma) &= \operatorname{cn}(-K) = 0
 \end{aligned}
 \tag{3.2 - 12}$$

So

$$\begin{aligned}
 x &= \rho \operatorname{dn}(\beta) \\
 y &= -k \rho \operatorname{sn}(\beta) \\
 z &= 0
 \end{aligned}
 \tag{3.2 - 13}$$

On the xy plane, this corresponds to *Region 2*:

$$y \leq 0, \quad -\frac{1}{|\tan \alpha|} \leq \frac{x}{y} \leq \frac{1}{|\tan \alpha|} \tag{3.2 - 14}$$

Case (4). Finally, for $\gamma = K$,

$$\begin{aligned} dn(\gamma) &= dn(K) = k' \\ sn(\gamma) &= sn(K) = 1 \\ cn(\gamma) &= cn(K) = 0 \end{aligned} \tag{3.2 - 15}$$

So

$$\begin{aligned} x &= \rho \, dn(\beta) \\ y &= k \, \rho \, sn(\beta) \\ z &= 0 \end{aligned} \tag{3.2 - 16}$$

On the xy plane, this corresponds to *Region 4*:

$$y \geq 0, \quad -\frac{1}{|\tan \alpha|} \leq \frac{x}{y} \leq \frac{1}{|\tan \alpha|} \tag{3.2 - 17}$$

CHAPTER 4

ASYMPTOTIC SOLUTIONS

4.1 Trefftz's Potential Functions

Let a surface of discontinuity in an infinite solid be situated in the plane $z = 0$, where x, y, z are the rectangular Cartesian coordinates. In the absence of body forces, Navier's equations of equilibrium are

$$\begin{aligned}\frac{\partial e}{\partial x} + (1 - 2\nu)\nabla^2 u_x &= 0 \\ \frac{\partial e}{\partial y} + (1 - 2\nu)\nabla^2 u_y &= 0 \\ \frac{\partial e}{\partial z} + (1 - 2\nu)\nabla^2 u_z &= 0\end{aligned}\tag{4.1 - 1}$$

where e is the dilatation

$$e = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\tag{4.1 - 2}$$

ν is Poisson's ratio and ∇^2 is the Laplace operator

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Trefftz (1928) has shown that the solution of Eq. (4.1-1) for a plane of discontinuity in the xy plane can be expressed in terms of the functions ϕ_x, ϕ_y, ϕ_z , and ψ ,

$$\begin{aligned}u_x &= \phi_x + z \frac{\partial \psi}{\partial x} \\ u_y &= \phi_y + z \frac{\partial \psi}{\partial y} \\ u_z &= \phi_z + z \frac{\partial \psi}{\partial z}\end{aligned}\tag{4.1 - 3}$$

ϕ_x , ϕ_y , ϕ_z , and ψ are the harmonic functions, and should satisfy the following partial differential equation

$$\frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial y} + \frac{\partial \phi_z}{\partial z} + (3 - 4\nu) \frac{\partial \psi}{\partial z} = 0 \quad (4.1 - 4)$$

The stress-displacement expressions are, by Hooke's law,

$$\begin{aligned} \frac{\sigma_x}{2\mu} &= \frac{\partial u_x}{\partial x} + \frac{\nu}{1 - 2\nu} e, & \frac{\sigma_{yz}}{\mu} &= \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \\ \frac{\sigma_y}{2\mu} &= \frac{\partial u_y}{\partial y} + \frac{\nu}{1 - 2\nu} e, & \frac{\sigma_{zx}}{\mu} &= \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z} \\ \frac{\sigma_z}{2\mu} &= \frac{\partial u_z}{\partial z} + \frac{\nu}{1 - 2\nu} e, & \frac{\sigma_{xy}}{\mu} &= \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \end{aligned} \quad (4.1 - 5)$$

where μ is the shear modulus.

For problems which are symmetric about the plane $z = 0$, the displacement components u_x , u_y should be even functions of z , and u_z should be an odd function of z . In addition, the shear stresses σ_{xz} and σ_{yz} must vanish on the plane $z = 0$. On the other hand, in skew-symmetric problems, u_x , u_y are odd in z , and u_z is even in z . Hence, the normal stress σ_z vanishes at $z = 0$. In both of these problems, the displacements and stresses are required to be continuous across the plane $z = 0$ outside the surface of discontinuity. It follows that all those quantities, which are odd in z , must vanish on such a plane. To apply these symmetry conditions, it suffices to compute the stress components σ_z , σ_{yz} , σ_{zx} by substituting Eq. (4.1-3) into (4.1-5) and thus one gets

$$\begin{aligned}
\frac{\sigma_z}{2\mu} &= \frac{\nu}{1-2\nu} \left(\frac{\partial\phi_x}{\partial x} + \frac{\partial\phi_y}{\partial y} \right) + \frac{1-\nu}{1-2\nu} \frac{\partial(\phi_z + \psi)}{\partial z} + z \frac{\partial^2\psi}{\partial z^2} \\
\frac{\sigma_{yz}}{2\mu} &= \frac{\partial(\phi_z + \psi)}{\partial y} + \frac{\partial\phi_y}{\partial z} + 2z \frac{\partial^2\psi}{\partial y\partial z} \\
\frac{\sigma_{zx}}{2\mu} &= \frac{\partial(\phi_z + \psi)}{\partial x} + \frac{\partial\phi_x}{\partial z} + 2z \frac{\partial^2\psi}{\partial x\partial z}
\end{aligned} \tag{4.1-6}$$

4.2 Symmetric Solutions

Due to the symmetry of the stress system with respect to the plane containing the surface of discontinuity, the harmonic functions ϕ_x , ϕ_y , ϕ_z , and ψ , must be chosen so as to satisfy Eq. (4.1-4), the conditions

$$\sigma_{yz} = \sigma_{zx} = 0, \quad \text{for } z = 0 \tag{4.2-1}$$

and to yield bounded displacement components at large distances away from the plane of discontinuity. If both $\partial^2\psi/\partial x\partial z$ and $\partial^2\psi/\partial y\partial z$ are to remain finite as $z \rightarrow 0$, the aforementioned conditions may be expressed as

$$\begin{aligned}
\frac{\partial\phi_x}{\partial x} + \frac{\partial\phi_y}{\partial y} + \frac{\partial\phi_z}{\partial z} + (3-4\nu) \frac{\partial\psi}{\partial z} &= 0 \\
\frac{\partial(\phi_z + \psi)}{\partial y} + \frac{\partial\phi_y}{\partial z} &= 0, \quad \text{for } z = 0 \\
\frac{\partial(\phi_z + \psi)}{\partial x} + \frac{\partial\phi_x}{\partial z} &= 0, \quad \text{for } z = 0
\end{aligned} \tag{4.2-2}$$

To simplify the formulations, one harmonic function $f(x, y, z)$ is introduced such that

$$\phi_x = \omega \frac{\partial f}{\partial x}, \quad \phi_y = \omega \frac{\partial f}{\partial y}, \quad \phi_z = \omega' \frac{\partial f}{\partial z} \quad (4.2-3)$$

$$\psi = \frac{\partial f}{\partial z}$$

The substitutions of Eq. (4.2-3) into (4.2-2) yields

$$\omega = 1 - 2\nu, \quad \omega' = -2(1 - \nu) \quad (4.2 - 4)$$

After substitution of Eqs. (4.2-3) and (4.2-4) into (4.1-4), one gets the displacement field

$$\begin{aligned} u_x &= (1 - 2\nu) \frac{\partial f}{\partial x} + z \frac{\partial^2 f}{\partial x \partial z} \\ u_y &= (1 - 2\nu) \frac{\partial f}{\partial y} + z \frac{\partial^2 f}{\partial y \partial z} \\ u_z &= -2(1 - \nu) \frac{\partial f}{\partial z} + z \frac{\partial^2 f}{\partial z^2} \end{aligned} \quad (4.2 - 5)$$

from which the stresses are easily determined:

$$\begin{aligned} \frac{\sigma_x}{2\mu} &= \frac{\partial^2 f}{\partial x^2} + 2\nu \frac{\partial^2 f}{\partial y^2} + z \frac{\partial^3 f}{\partial z \partial x^2}, & \frac{\sigma_{yz}}{2\mu} &= z \frac{\partial^3 f}{\partial y \partial z^2} \\ \frac{\sigma_y}{2\mu} &= \frac{\partial^2 f}{\partial y^2} + 2\nu \frac{\partial^2 f}{\partial x^2} + z \frac{\partial^3 f}{\partial z \partial y^2}, & \frac{\sigma_{zx}}{2\mu} &= z \frac{\partial^3 f}{\partial x \partial z^2} \\ \frac{\sigma_z}{2\mu} &= -\frac{\partial^2 f}{\partial z^2} + z \frac{\partial^3 f}{\partial z^3}, & \frac{\sigma_{xy}}{2\mu} &= (1 - 2\nu) \frac{\partial^2 f}{\partial x \partial y} + z \frac{\partial^3 f}{\partial x \partial y \partial z} \end{aligned} \quad (4.2 - 6)$$

To determine the unknown function $f(x, y, z)$, one needs to satisfy the traction free conditions of the crack surfaces,

$$\sigma_z = 0, \quad \text{for } z = 0 \text{ \& } (\pi - \alpha) < \arctan \frac{y}{x} \leq (\pi + \alpha) \quad (4.2 - 7)$$

and the continuity of the displacement component u_z across $z = 0$ plane outside the crack

$$u_z = 0, \quad \text{for } z = 0 \text{ \& } -(\pi - \alpha) \leq \arctan \frac{y}{x} \leq (\pi - \alpha) \quad (4.2 - 8)$$

Satisfaction of Eqs. (4.2-7) and (4.2-8) gives the following constraints on $f(x, y, z)$

$$\text{For } -(\pi - \alpha) \leq \arctan \frac{y}{x} \leq (\pi - \alpha) \text{ \& } z = 0$$

$$\frac{\partial f}{\partial z} = 0 \quad (4.2 - 9)$$

$$\text{and for } (\pi - \alpha) < \arctan \frac{y}{x} \leq (\pi + \alpha) \text{ \& } z = 0$$

$$\frac{\partial^2 f}{\partial z^2} = 0 \quad (4.2 - 10)$$

Let

$$\frac{\partial f}{\partial z} = \rho^n \phi(\beta) \psi(\gamma), \quad n > 0 \quad (4.2 - 11)$$

Since function $f(x, y, z)$ is harmonic and satisfies Eq. (4.2-11), one can show that

$\phi(\beta)$ and $\psi(\gamma)$ should satisfy the following Lamé equations (Bateman, 1955)

$$\phi''(\beta) + [h - n(n+1)k^2 \text{sn}^2(\beta)]\phi(\beta) = 0 \quad (4.2-12)$$

$$\psi''(\gamma) + [h - n(n+1)k^2 \text{sn}^2(\gamma)]\psi(\gamma) = 0$$

where n and h are two unknown constants and the primes indicate differentiation with respect to the argument in parentheses. The substitution of Eq. (4.2-11) into Eq. (4.2-9) and (4.2-10) yields

Case 1. $0 < 2\alpha < \pi$

$$\phi(K) = \phi'(K + 2iK') = 0 \quad (4.2-13)$$

$$\psi(-K) = \psi(K) = 0$$

Case 2. $\pi < 2\alpha < 2\pi$

$$\phi(K) = \phi'(K + 2iK') = 0 \quad (4.2-13')$$

$$\psi'(-K) = \psi'(K) = 0$$

The conditions that nontrivial solutions of Lamé equation (4.2-12) exist under boundary conditions (4.2-13) or (4.2-13') define the eigenvalues for parameters n and h (See Appendix). Numerical results for the parameter n are shown in Figure 4-1. Thus, the discontinuity δ of the crack surfaces can be obtained,

Case 1. $0 < 2\alpha < \pi$

$$\delta = \frac{4(1 - \nu^2)}{E} \rho^n \phi(K + 2iK') \psi(\gamma), \quad \text{for Region 1}$$

Case 2. $\pi < 2\alpha < 2\pi$

$$\delta = \frac{4(1 - \nu^2)}{E} \rho^n \phi(K + 2iK') \psi(\gamma), \quad \text{for Region 1}$$

and

$$\delta = \frac{4(1 - \nu^2)}{E} \rho^n \phi(\beta) \psi(\pm K), \quad \text{for Regions 2 \& 4}$$

4.3 Skew-Symmetric Solutions

In the skew-symmetric problem, the harmonic functions ϕ_x , ϕ_y , ϕ_z and ψ must be selected so as to satisfy Eq. (4.1-4), the condition $\sigma_z = 0$ for $z = 0$, and to provide finite displacements at large distances away from the plane of discontinuity. Assuming that $\partial^2 \psi / \partial z^2$ remains finite as $z \rightarrow 0$, the aforementioned conditions may be expressed as

$$\frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial y} + \frac{\partial \phi_z}{\partial z} + (3 - 4\nu) \frac{\partial \psi}{\partial z} = 0 \quad (4.3-1)$$

$$\frac{\nu}{1 - \nu} \left(\frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial y} \right) + \frac{\partial}{\partial z} (\phi_z + \psi) = 0, \quad \text{for } z = 0$$

To simplify the remainder of the work, let the harmonic functions $g(x, y, z)$ and $h(x, y, z)$ be introduced such that

Figure 4-1. Parameter n vs Angle α

$$\begin{aligned}\phi_x &= \omega \frac{\partial h}{\partial x} + 2 \frac{\partial g}{\partial y} \\ \phi_y &= \omega \frac{\partial h}{\partial y} - 2 \frac{\partial g}{\partial x}\end{aligned}\tag{4.3-2}$$

$$\phi_z = \omega' \frac{\partial h}{\partial z}$$

$$\psi = \frac{\partial h}{\partial z}$$

The substitution of Eq. (4.3-2) into (4.3-1) yields

$$\omega = 2(1 - \nu), \quad \omega' = -(1 - 2\nu)\tag{4.3-3}$$

Therefore, the displacement field in the skew-symmetric problem can be described by these two functions in the following form

$$\begin{aligned}u_x &= 2 \frac{\partial g}{\partial y} + 2(1 - \nu) \frac{\partial h}{\partial x} + z \frac{\partial^2 h}{\partial x \partial z} \\ u_y &= -2 \frac{\partial g}{\partial x} + 2(1 - \nu) \frac{\partial h}{\partial y} + z \frac{\partial^2 h}{\partial y \partial z} \\ u_z &= -(1 - 2\nu) \frac{\partial h}{\partial z} + z \frac{\partial^2 h}{\partial z^2}\end{aligned}\tag{4.3-4}$$

The stress field is obtained by substituting Eq. (4.3-4) into (4.1-7),

$$\begin{aligned}
 \frac{\sigma_x}{2\mu} &= 2\frac{\partial^2 g}{\partial x \partial y} + 2\frac{\partial^2 h}{\partial x^2} + 2\nu\frac{\partial^2 h}{\partial y^2} + z\frac{\partial^3 h}{\partial z \partial x^2} \\
 \frac{\sigma_{yz}}{2\mu} &= -\frac{\partial^2 g}{\partial x \partial z} + \frac{\partial^2 h}{\partial y \partial z} + z\frac{\partial^3 h}{\partial y \partial z^2} \\
 \frac{\sigma_y}{2\mu} &= -2\frac{\partial^2 g}{\partial x \partial y} + 2\frac{\partial^2 h}{\partial y^2} + 2\nu\frac{\partial^2 h}{\partial x^2} + z\frac{\partial^3 h}{\partial z \partial y^2} \\
 \frac{\sigma_{xz}}{2\mu} &= \frac{\partial^2 g}{\partial y \partial z} + \frac{\partial^2 h}{\partial x \partial z} + z\frac{\partial^3 h}{\partial x \partial z^2} \\
 \frac{\sigma_z}{2\mu} &= z\frac{\partial^3 h}{\partial z^3} \\
 \frac{\sigma_{xy}}{2\mu} &= \frac{\partial^2 g}{\partial y^2} - \frac{\partial^2 g}{\partial x^2} + 2(1-\nu)\frac{\partial^2 h}{\partial x \partial y} + z\frac{\partial^3 h}{\partial x \partial y \partial z}
 \end{aligned} \tag{4.3-5}$$

The traction free conditions of the crack surfaces

$$\sigma_{xz} = 0, \quad \sigma_{yz} = 0, \quad \text{for } z = 0 \text{ \& } (\pi - \alpha) < \arctan \frac{y}{x} \leq (\pi + \alpha) \tag{4.3-6}$$

and the continuity of the displacement components u_x and u_y across $z = 0$ plane outside the crack

$$u_x = 0, \quad u_y = 0 \quad \text{for } z = 0 \text{ \& } -(\pi - \alpha) \leq \arctan \frac{y}{x} \leq (\pi - \alpha) \tag{4.3-7}$$

can be rewritten in terms of the displacement potential functions $g(x, y, z)$ and $h(x, y, z)$,

For $-(\pi - \alpha) \leq \arctan \frac{y}{x} \leq (\pi - \alpha)$ & $z = 0$

$$\frac{\partial g}{\partial y} + (1 - \nu) \frac{\partial h}{\partial x} = 0 \quad (4.3-8)$$

$$-\frac{\partial g}{\partial x} + (1 - \nu) \frac{\partial h}{\partial y} = 0$$

and for $(\pi - \alpha) < \arctan \frac{y}{x} \leq (\pi + \alpha)$ & $z = 0$

$$\frac{\partial^2 g}{\partial y \partial z} + \frac{\partial^2 h}{\partial x \partial z} = 0 \quad (4.3-9)$$

$$-\frac{\partial^2 g}{\partial x \partial z} + \frac{\partial^2 h}{\partial y \partial z} = 0$$

Let

$$\frac{\partial g}{\partial y} + (1 - \nu) \frac{\partial h}{\partial x} + \nu z \frac{\partial^2 h}{\partial x \partial z} = \rho^n \phi_1(\beta) \psi_1(\gamma), \quad n > 0 \quad (4.3-10)$$

$$-\frac{\partial g}{\partial x} + (1 - \nu) \frac{\partial h}{\partial y} + \nu z \frac{\partial^2 h}{\partial y \partial z} = \rho^n \phi_2(\beta) \psi_2(\gamma), \quad n > 0$$

Since functions $g(x, y, z)$ and $h(x, y, z)$ are harmonic and satisfy Eq. (4.3-10), one can derive the following Lamé equations

$$\begin{aligned}
\phi_1''(\beta) + [h_1 - n(n+1)k^2 sn^2(\beta)]\phi_1(\beta) &= 0 \\
\psi_1''(\gamma) + [h_1 - n(n+1)k^2 sn^2(\gamma)]\psi_1(\gamma) &= 0
\end{aligned}
\tag{4.3-11}$$

$$\begin{aligned}
\phi_2''(\beta) + [h_2 - n(n+1)k^2 sn^2(\beta)]\phi_2(\beta) &= 0 \\
\psi_2''(\gamma) + [h_2 - n(n+1)k^2 sn^2(\gamma)]\psi_2(\gamma) &= 0
\end{aligned}$$

where n , h_1 and h_2 are unknown constants and the primes indicate differentiation with respect to the argument in parentheses. The substitutions of Eq. (4.3-10) into Eq. (4.3-8) and (4.3-9) yields

Case 1. $0 < 2\alpha < \pi$

$$\begin{aligned}
\phi_1(K) = \phi_1'(K + 2iK') &= 0 \\
\psi_1(-K) = \psi_1(K) &= 0
\end{aligned}
\tag{4.3-12}$$

$$\begin{aligned}
\phi_2(K) = \phi_2'(K + 2iK') &= 0 \\
\psi_2(-K) = \psi_2(K) &= 0
\end{aligned}$$

Case 2. $\pi < 2\alpha < 2\pi$

$$\begin{aligned}
\phi_1(K) = \phi_1'(K + 2iK') &= 0 \\
\psi_1'(-K) = \psi_1'(K) &= 0
\end{aligned}
\tag{4.3-12'}$$

$$\begin{aligned}
\phi_2(K) = \phi_2'(K + 2iK') &= 0 \\
\psi_2'(-K) = \psi_2'(K) &= 0
\end{aligned}$$

Following the similar procedure as for the symmetric problem, we can arrive at the same eigenvalues for parameter n .

Thus, the discontinuity $\delta_x = 2u_x$ and $\delta_y = 2u_y$ at the crack surfaces can be obtained,

$$\delta_x = \frac{4(1+\mu)}{E} \rho^n \phi_1(\beta) \psi_1(\gamma) \quad (4.3-13)$$

$$\delta_y = \frac{4(1+\mu)}{E} \rho^n \phi_2(\beta) \psi_2(\gamma)$$

CHAPTER 5

RESULTS and CONCLUSIONS

5.1 Results

From the asymptotic solutions presented in the previous section one can easily compute the stress components near the crack corner on the plane $z = 0$. For brevity, only symmetric problem has been analyzed. Substituting Eq. (4.2-11) into (4.2-6) with $z = 0$, we have

Case 1. $0 < 2\alpha < \pi$:

$$\sigma_z = \frac{1 + \nu}{E} \frac{\phi'(K)}{ik} \rho^{n-1} \frac{\psi(\gamma)}{cn(\gamma)} \quad (5.1 - 1)$$

In *Region 1*:

$$x = \rho \, dn(\gamma)$$

$$y = k\rho \, sn(\gamma) \quad (5.1 - 2)$$

$$z = 0$$

and

$$\sigma_z = \frac{1 + \nu}{E} \frac{\pm \psi'(\pm K)}{ik} \rho^{n-1} \frac{\phi(\beta)}{cn(\beta)} \quad (5.1 - 3)$$

In *Regions 2 & 4*:

$$x = \rho \, dn(\beta)$$

$$y = \pm k\rho \, sn(\beta) \quad (5.1 - 4)$$

$$z = 0$$

It can be seen that the stress component σ_z tends to infinity as $cn(\beta)$ goes to zero at the crack boundary $\beta = K + 2iK'$ & $\gamma = \pm K$. Noticing that

$$\frac{1}{cn(\beta)} = \frac{1}{\sqrt{1 - sn^2(\beta)}} = \frac{k\sqrt{x^2 + y^2}}{\sqrt{|kx - k'y|}\sqrt{|kx + k'y|}} \quad (5.1 - 5)$$

we can see that the stress singularity is still of the form $r^{-\frac{1}{2}}$. However, at the corner point the stress becomes unbounded as $(x^2 + y^2)^{-\frac{1-n}{2}}$. Because the parameter n varies between 0 and 1 (See Figure 4-1), the stress singularity will be stronger and/or weaker than the $r^{-\frac{1}{2}}$ singularity as α changes.

Case 2. $\pi < 2\alpha < 2\pi$:

$$\sigma_z = \frac{1 + \nu}{E} \frac{\phi'(K)}{ik} \rho^{n-1} \frac{\psi(\gamma)}{cn(\gamma)} \quad (5.1 - 6)$$

In *Region 1*:

$$x = \rho \, dn(\gamma)$$

$$y = k\rho \, sn(\gamma) \quad (5.1 - 7)$$

$$z = 0$$

Obviously, results are characteristically the same as those of Case 1.

5.2 Conclusions

In this thesis, we computed the stress field near a crack corner. The crack corner is a point on the crack boundary where no tangent can be drawn.

Conical-spherical coordinates were adopted, and the displacement potential functions were separated from their arguments in these coordinates. The asymptotic solutions were derived at the corner point around the crack front, and it has been found that although the $r^{-\frac{1}{2}}$ stress singularity still dominates crack front fields along the smooth part of the crack boundary, different singularities exist at the corner point. If the angle at the corner point is greater than π , it is stronger than the $r^{-\frac{1}{2}}$ stress singularity; while if the angle is smaller than π , it is weaker. The results agree with the interpretations of Benthem (1977 & 1980) and Bazant & Estensoro (1979) for surface crack problems. However, unlike surface cracks for an embedded crack the stress singularity is found to be independent of the Poisson's ratio of the material.

APPENDIX A

Lamé Equation

As discussed in Bateman (1955), the nontrivial solutions of Lamé equation

$$F''(x) + [h - n(n+1)k^2 \operatorname{sn}^2(x)]F(x) = 0 \quad (A-1)$$

can be expanded in trigonometric series.

Case 1.

$$F_1(x) = \frac{A_0}{2} + \sum_{r=1}^{\infty} A_{2r} \cos(2r\theta) \quad (A-2)$$

where $\cos(\theta)$ is equal to the Jacobian elliptical function $\operatorname{sn}(x)$ with modulus k . The coefficients A_{2r} are determined by the following iteration procedure

$$\begin{aligned} \frac{1}{2}(n-2r+2)(n+2r-1)k^2 A_{2r-2} - [2h - n(n+1)k^2 - 4r^2(2-k^2)]A_{2r} \\ (A-3) \end{aligned}$$

$$+ \frac{1}{2}(n-2r-1)(n+2r+2)k^2 A_{2r+2} = 0$$

If the eigenvalues n and h are properly selected, such that

$$(n-1)(n+2)k^2 A_2 - [2h - n(n+1)k^2]A_0 = 0 \quad (A-4)$$

$$\lim_{r \rightarrow \infty} \frac{A_{2r+2}}{A_{2r}} = \frac{1-k'}{1+k'} \quad (A-5)$$

then

$$F_1'(-K) = F_1'(K) = 0 \quad (A-6)$$

Case 2.

$$F_2(x) = \sum_{r=0}^{\infty} B_{2r+1} \sin[(2r+1)\theta] \quad (A-7)$$

where

$$\frac{1}{2}(n-2r+1)(n+2r)k^2 B_{2r-1} - [2h - n(n+1)k^2 - (2r+1)^2(2-k^2)]B_{2r+1} \quad (\text{A-8})$$

$$+\frac{1}{2}(n-2r-2)(n+2r+3)k^2 B_{2r+3} = 0$$

If

$$\frac{1}{2}(n-2)(n+3)k^2 B_3 - [2h - n(n+1)k^2 - 2 + k^2 + \frac{1}{2}n(n+1)k^2]B_1 = 0 \quad (\text{A-9})$$

$$\lim_{r \rightarrow \infty} \frac{B_{2r+1}}{B_{2r-1}} = \frac{1-k'}{1+k'} \quad (\text{A-10})$$

then

$$F_2(-K) = F_2(K) = 0 \quad (\text{A-11})$$

Case 3.

$$F_3(x) = \sum_{r=0}^{\infty} C_r \sin[(r + \frac{1}{2})\phi] \quad (\text{A-12})$$

where $\cos(\phi)$ is equal to Jacobian elliptical function $sn(ix - iK + K')$ with modulus k' .

$$\frac{k'^2}{2}(n-r+\frac{3}{2})(n+r-\frac{1}{2})C_{r-2} + \frac{k'^2}{2}(n+r+\frac{5}{2})(n-r-\frac{3}{2})C_{r+2} \quad (\text{A-13})$$

$$-[n(n+1)(2-k'^2) - 2h - (2-k'^2)(r+\frac{1}{2})^2]C_r = 0$$

If

$$\begin{aligned} & \frac{k'^2}{8}(2n-3)(2n+5)C_2 - \frac{k'^2}{8}(2n-1)(2n+3)C_1 \\ & - [n(n+1)(2-k'^2) - 2h - \frac{1}{4}(2-k'^2)]C_0 = 0 \end{aligned} \quad (\text{A-14})$$

$$\begin{aligned} & \frac{k'^2}{8}(2n-5)(2n+7)C_3 - \frac{k'^2}{8}(2n+1)^2C_0 \\ & - [n(n+1)(2-k'^2) - 2h - \frac{9}{4}(2-k'^2)]C_1 = 0 \end{aligned}$$

$$\lim_{r \rightarrow \infty} \frac{C_{r+2}}{C_r} = \frac{1-k}{1+k} \quad (\text{A-15})$$

then

$$F_3(K) = F'_3(K + 2iK') = 0 \quad (\text{A-16})$$

APPENDIX B

FORTRAN Program for Eigenvalue Calculation

```

real function qaa(sita,rn,h,n)
tan1=tan(sita/180.0*3.1415926)
rk=(tan1**2/(1.0+tan1**2))**0.5
gh=h+h-rn*(rn+1)*rk*rk
rk1=(1.0-rk*rk)**0.5
s1=2*(2.0-rk*rk)/rk/rk
s=(1.0-rk1)/(1.0+rk1)
do 10 i=n,1,-1
    alfa=((rn+2)/2/i-1.0)*((rn-1.0)/2/i+1.0)
    beta=s1-gh/2/rk/rk/i/i
    gama=((rn-1.0)/2/i-1.0)*((rn+2.0)/2/i+1.0)
10  s=-alfa/(beta+gama*s)
    qaa=s*(rn-1.0)*(rn+2.0)*rk*rk-gh
    return
end

real function qa(sita,rn,h,n)
tan1=tan(sita/180.0*3.1415926)
rk=(tan1**2/(1.0+tan1**2))**0.5
gh=h+h-rn*(rn+1)*rk*rk
rk1=(1.0-rk*rk)**0.5
s1=2*(2.0-rk*rk)/rk/rk
s=(1.0-rk1)/(1.0+rk1)
do 10 i=n,1,-1
    alfa=((rn+1.0)/2.0/i-1.0)*(rn/2/i+1.0)
    beta=s1*(1.0+1.0/2/i)**2-gh/2/rk/rk/i/i
    gama=((rn-2)/2/i-1)*(rn+3)/2/i+1)
10  s=-alfa/(beta+gama*s)
    if (sita.lt.90) then
        sign=1.0
    else
        sign=-1.0
    end if
    qa=s*(rk*rk/2*(rn-2)*(rn+3))
    &    -(gh-2*rk*rk*(1+sign/2*rn*(rn+1)))
    return
end

```

```

real function f(sita,rn,h,n)
if (sita.lt.90.0) then
    f=(qa(sita,rn,h,n)**2+qxy(sita,rn,h,n)**2)**0.5
else
    f=(qaa(sita,rn,h,n)**2+qxy(sita,rn,h,n)**2)**0.5
end if
return
end

subroutine goal(sita,rn_in,h_in,n,
&               delt_n,delt_h,epislon,rn_out,h_out)
delt_n=0.000001
delt_h=0.000001
check=10.0
rn_out=rn_in
h_out =h_in
do while (abs(check).gt.epislon)
    fcheck=f(sita,rn_out,h_out,n)
    gradient_n=(f(sita,rn_out+delt_n,h_out,n)-fcheck)/delt_n
    gradient_h=(f(sita,rn_out,h_out+delt_h,n)-fcheck)/delt_h
    delt=f(sita,rn_out,h_out,n)/(gradient_n**2+
&               gradient_h**2)
    rn_out=rn_out-delt*gradient_n
    h_out =h_out -delt*gradient_h
    check=f(sita,rn_out,h_out,n)
end do
return
end

```

```

real function qxy(sita,rn,h,n)
tan1=tan(sita/180.0*3.1415926)
rk=(tan1**2/(1.0+tan1**2))**0.5
qx=(1.-rk)/(1.+rk)
qy=qx
rk1=(1.-rk*rk)**0.5
s0=2*(rk1*rk1-2)/rk1/rk1
h1=rn*(rn+1.)-h
gh1=h1+h1-rn*(rn+1.)*rk1*rk1
do 10 i=n,1,-1
    alfa1=(1.-3/4./i)*(1.-1/4./i)-rn*(rn+1)/4/i/i
    betal=s0*(1.+1./4/i)**2+gh1/2/rk1/rk1/i/i
    gamal=(1.+5/4./i)*(1.+3/4./i)-rn*(rn+1)/4/i/i
    alfa2=(1.-1/4./i)*(1.+1/4./i)-rn*(rn+1)/4/i/i
    beta2=s0*(1.+3/4./i)**2+gh1/2/rk1/rk1/i/i
    gama2=(1.+7/4./i)*(1.+5/4./i)-rn*(rn+1.)/4/i/i
    qx=-alfa1/(betal+gamal*qx)
10    qy=-alfa2/(beta2+gama2*qy)
    q1=(rk1*rk1-2)/4+gh1+qx*(15/8.*rk1*rk1-rn*(rn+1)/2*rk1*rk1)
    q2=3/8.*rk1*rk1-rn*(rn+1)/2*rk1*rk1
    q3=rn*(rn+1)/2*rk1*rk1+rk1*rk1/8
    q4=(rk1*rk1-2)*9/4.+gh1+qy*(35/8.*rk1*rk1-rn*(rn+1)/2*rk1*rk1)
    qxy=q1*q4+q2*q3
return
end

```



```

program linlin
open(88,file='fr.dat',status='old')
open(66,file='chech.dat',status='new')
* SSS= QA(0.3,0.9746637,0.9982792,60)
* TTT=QXY(0.3,0.9746637,0.9982792,60)
* WRITE(6,*)SSS,TTT
* deltn=0.0000001
* delth=0.0000001
* epislon=0.00001
* read(5,*)rn_in,h_in,epislon
* read(6,*)h_in,h_ed,h_inc
epislon=0.0001
do sita=0.05,179.07,1.0
if (sita.lt.90.0) then
    rn_in=0.7
    h_in=0.7
else
    rn_in=0.3
    h_in=0.3
end if
* sita=120.0
    call goal(sita,rn_in,h_in,60,delt_n,delt_h,
&         epislon, rn_out,h_out)
* do rn_out=rn_in,rn_ed,rn_inc
*     write(6,*)'rn_out=',rn_out
*     do h_out=h_in,h_ed,h_inc
* read(88,*)sita,rn_out,h_out
* rn_out=1.22606
* h_out=2.18096
* rn=0.35635
* h=0.20438
if (sita.lt.90.0) then
    check1=qa(sita,rn_out,h_out,60)
else
    check1=qaa(sita,rn_out,h_out,60)
end if
check2=qxy(sita,rn_out,h_out,60)
* check=f(sita,rn_out,h_out,60)
* write(6,*)sita
* write(66,*)'          h_out=',h_out,check1,check2
* write(88,*)sita,rn_out,h_out,check1,check2
* end do
end do
stop
end

```

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