NUMERICAL SIMULATION OF 2D ELECTROTHERMAL FLOW USING
BOUNDARY ELEMENT METHOD

by

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ABSTRACT

Microfluidics and its applications to Lab-on-a-Chip have attracted a lot of attention. Because of the small length scale, the flow is characterized by a low Re number. The governing equations become linear. Boundary element method (BEM) is a very good option for simulating the fluid flow with high accuracy. In this thesis, we present a 2D numerical simulation of the electrothermal flow using BEM.

In electrothermal flow the volumetric force is caused by electric field and temperature gradient. The physics is mathematically modeled by (i) Laplace equation for the electrical potential, (ii) Poisson equation for the heat conduction caused by Joule heating, (iii) continuity and Stokes equation for the low Reynolds number flow. We begin by solving the electrical potential and electrical field. The heat conduction is caused by the Joule heating as the heat generation term. Superposition principle is used to solve for the temperature field. The Coulomb and dielectric forces are generated by the electrical field and temperature gradient of the system. The buoyancy force is caused by the non-uniform temperature distribution inside the system. We analyze the Stokes flow problem by superposition of fundamental solution for free-space velocity caused by body force and BEM for the corresponding homogeneous Stokes equation. It is well known that a singularity integral arises when the source point approaches the field point. To overcome this problem, we solve the free-space velocity analytically. For the BEM part, we also calculate all the integrals analytically. With this effort, our solution is more accurate. In addition, we improve

the robustness of the matrix system by combining the velocity integral equation with the traction integral equation when we simulate the electrothermal pump.

One of our purpose is to design a pump for the microfluidics system. Since the system is a long channel, the flow is fully developed in the area far away from the electrodes. With this assumption, the velocity profile is parabolic at the inlet and outlet of the channel. So we can get appropriate boundary conditions for the BEM part of Stokes equation. Consequently, we can simulate the electrothermal flow in an open channel.

In this thesis, we will present the formulation and implementation of BEM to model electrothermal flow. Results of electrical potential, temperature field, Joule heating, electrothermal force, buoyancy force and velocity field will be presented.

CHAPTER 1: INTRODUCTION

The molecular diagnostics in a microchip is a fast developing technology which attracts the attention of researchers all over the world. The use of alternating current (AC) electrokinetics and the influence of nanotechnology have been fundamental components for this technology. AC electrokinetics has evolved to a commonly used technique to manipulate micro-and nano-scaled particles by forces generated by electrothermal system. AC electro-osmosis is an electro-hydrodynamic force that generates a microfluidic surface-driven flow at the electrode surface when an AC signal is applied to a fluid [1]. The travelling wave driven electrothermal flow is studied by Fuhr in 1994 [2]. AC electrothermal flow is analyzed as one of the phenomena in AC di-electrophoretic particle manipulating in microfluidics system [3-6]. A key component for this research area is the design of microelectrodes. Microelectrode consists of gold deposit contacts of varying design that are used to apply an electric current to create a voltage potential. The progressive development of the field has been accomplished by the improvements in the microelectronic methods used to generate microelectrode structures [7-9]. Based on this, it is obvious that research in mathematical modeling and computation method for AC electrokinetics is also necessary. Most of the simulation works are done according to Ramos's model [1]. P.Cheng stated that when the temperature rise caused by the Joule heating is small enough, this model is accurate [10]. In this modeling work, we analyze all the governing equations for electrothermal flow based on the model given by Ramos because the temperature rise is low in this case.

In this thesis, we will present a 2D numerical modeling of electrothermal flow driven by AC electrothermal forces and buoyancy force. We non-dimensionalize all the governing equations to illustrate the scaling. Boundary element method will be used to analyze the dimensionless governing equations: Laplace equation, Poisson equation, continuity equation and Stokes equation. The singular integral terms of Green's functions are computed by Cauchy principal value (CPV) and Hadamard finite part (HFP) [11-13]. All the integrals of the Green's functions are calculated analytically to provide an accurate numerical solution. We present all the formulations of these integral results in this thesis. All the details about the microfluidics system such as electrical field, temperature field, force field and velocity field will be presented in the thesis.

CHAPTER 2: MATHEMATICAL FORMULATION

Basic Equations

To analyze the phenomena for the electrothermal flow inside the microfluidics system, the first equation we need to solve is 2D Laplace equation for the electrical potential (Eq. 2.1). In this equation, φ is the electrical potential of the microfluidics system. The next equation is for the electrical field which is the negative gradient of the electrical potential (Eq. 2.2). $\mathbf{E}$ is the electrical field of the system. The Joule heating is caused by the electrical field (Eq. 2.3). In this equation, σ is the conductivity of the medium. The Joule heating is the source term for the energy equation. The next governing equation needs to be solved is the energy equation (Eq. 2.4). k , ρ , c are the thermal conductivity, density, specific heat of the fluid respectively. In this thesis, we choose water as the fluid inside the microfluidics. As the Peclet number is small for the microfluidics system, the convection term can be neglected in Eq. 2.4 [14]. Based on this, the energy equation can be simplified to Eq. 2.5. Navier-Stokes equations need to be solved to analyze the fluid flow inside the electrothermal system. For the electrothermal flow, we assume the fluid is incompressible. So we have the continuity equation for incompressible flow (Eq. 2.6). $\mathbf{u}$ is velocity vector. In the electrothermal system, because of the low velocity and small dimension scale, the Reynolds number for electrothermal flow is very small. The momentum equation can be simplified to Stokes equation (Eq. 2.7). P is pressure. $\mathbf{F}$ is body force. μ is the dynamic viscosity.

$$\nabla^2 \varphi = 0 \tag{2.1}$$

$$\mathbf{E} = -\nabla \varphi \tag{2.2}$$

$$JH = -\sigma(\mathbf{E} \cdot \mathbf{E}) \quad (2.3)$$

$$\rho c \left(\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T \right) = k \nabla^2 T + \sigma(\mathbf{E} \cdot \mathbf{E}) \quad (2.4)$$

$$k \nabla^2 T = JH \quad (2.5)$$

$$\nabla \cdot \mathbf{u} = 0 \quad (2.6)$$

$$0 = -\nabla P + \mu \nabla^2 \mathbf{u} + \mathbf{F} \quad (2.7)$$

In this thesis, we assume that the permittivity ε and conductivity σ of the medium are approximated by linear functions in temperature as shown [15]:

$$\varepsilon = \varepsilon_r(1 + \delta_\varepsilon \theta), \sigma = \sigma_r(1 + \delta_\sigma \theta) \quad (2.8)$$

Where

$\delta_\varepsilon = \frac{1}{\varepsilon_r} \frac{d\varepsilon}{dT} \Delta T$, $\delta_\sigma = \frac{1}{\sigma_r} \frac{d\sigma}{dT} \Delta T$, $\theta = \frac{T-T_r}{\Delta T}$, ε_r is reference permittivity, σ_r is the reference conductivity, T , T_r , and θ are dimensional, reference, and dimensionless temperature. ΔT is the characteristic temperature scale.

Because the current we apply is alternating current, we need to compute the time average electrothermal force given by Ramos [1]:

$$\mathbf{F} = \mathbf{F}_{electrothermal} + \mathbf{F}_{buoyancy},$$

$\mathbf{F}_{electrothermal} = \frac{1}{2} \left(\frac{\delta_\varepsilon(1-\alpha)}{\omega^2 \tau^2 + 1} \varepsilon_r (\nabla \theta \cdot \mathbf{E}) \right) \mathbf{E} - \frac{1}{4} \varepsilon_r \delta_\varepsilon \nabla \theta |\mathbf{E}|^2$, the first term of the electrothermal force is Coulomb force and the second part is dielectric force.

$\mathbf{F}_{buoyancy} = \rho_r \beta g (T - T_r)$, the direction for the buoyancy force is positive vertical direction. (2.9)

Where ω is frequency of the AC. ρ_r is reference density. β is thermal expansion coefficient. g is gravitational acceleration. $\alpha = \frac{\delta_\sigma}{\delta_\varepsilon}$ and $= \frac{\varepsilon}{\sigma}$.

Non-dimensionalization

To analyze this electrothermal system easily, we need to non-dimensionalize all the governing equations above. Then we can apply boundary element method to solve all the dimensionless equations. The non-dimensional variables are introduced in Table. 2.1.

Table 2.1 Non-dimensional variables

Non-dimensional variables		
Type	Expression	Notes
Length	$x^*, y^* = \frac{(x, y)}{L_c}$	L_c , characteristic length
Potential	$\varphi^* = \frac{\varphi}{\varphi_0}$	φ_0 , applied potential
Temperature	$\theta = \frac{T-T_r}{\Delta T}, \Delta T = \frac{\sigma \varphi_0^2}{k}$	T_r , reference temperature; ΔT , characteristic temperature scale
Permittivity	$\varepsilon^* = \frac{\varepsilon}{\varepsilon_r}$	ε_r , reference permittivity
Electrothermal force	$F_e^* = \frac{F_e}{\varepsilon \varphi_0^2 \delta_\sigma \Delta T / L_c^3}$	F_e , time average force; $\delta_\sigma = \frac{1}{\sigma} \frac{d\sigma}{dT}$
Buoyancy force	$F_b^* = \frac{F_b}{\rho_r \beta g \Delta T}$	F_b , buoyancy force
Velocity	$u^* = \frac{u}{V_c}; V_c = \frac{\varepsilon \varphi_0^2 \delta_\sigma \Delta T}{\mu L_c}$	V_c , characteristic velocity
Pressure	$P^* = \frac{P - P_r}{\mu V_c / L_c}$	P_r , reference pressure

After we apply all the non-dimensional variables, the 2D Laplace equation (Eq. 2.1) becomes:

$$\nabla^{*2} \varphi^* = 0 \quad (2.10)$$

Then the dimensionless electrical field can be calculated by:

$$\mathbf{E}^* = -\nabla^* \varphi^* \quad (2.11)$$

Now we can computed the non-dimensional Joule heating term as:

$$JH^* = -(\mathbf{E}^* \cdot \mathbf{E}^*) \quad (2.12)$$

The dimensionless energy equation is:

$$\nabla^{*2}\theta = JH^* \quad (2.13)$$

The continuity equation and Stokes equation after non-dimensionalization are shown by:

$$\nabla^* \cdot \mathbf{u}^* = 0 \quad (2.14)$$

$$0 = -\nabla^* P^* + \nabla^{*2}\mathbf{u}^* + \mathbf{F}_e^* + \frac{\rho_r \beta g L_g^3}{\varepsilon \varphi_0^2 \delta_\sigma} \mathbf{F}_b^* \quad (2.15)$$

With all these dimensionless governing equations, we can utilize BEM to solve all of them. All the details and calibration examples for BEM are shown in Chapter 3. Then we can multiply the dimensionless results by the non-dimensional variables in Table 2.1 to get the dimensional parameters of the electrothermal microfluidics system. All the variable values of the system we use in this modeling work are listed in Table 2.2. In this thesis, we will model the closed box electrothermal flow and electrothermal pump respectively.

Table 2.2 Parameter Values

symbol	Value	Units	Notes
σ, σ_r	0.05	$[S/m]$ $[A/Vm]$	Conductivity (<i>water at 20 °C</i>) [16]
$\varepsilon, \varepsilon_r$	7.08×10^{-10}	$[Farad/m]$	Permittivity
k	0.58	$[W/(m \cdot k)]$	Thermal Conductivity (<i>water at 25 °C</i>) [17]
ω	1,000,000	$[Hz]$	Frequency
ρ, ρ_r	998.23	$[kg/m^3]$	Density (<i>water at 25 °C</i>) [17]
β	2.07×10^{-4}	$1/K$	Thermal Expansion Coefficient (<i>water at 20 °C</i>)
μ	8.55×10^{-4}	$[N \cdot s/m^2]$	Dynamic Viscosity (<i>water at 25 °C</i>) [17]
φ_0	7	V	Applied Reference Voltage
c	4,181	$[J/kg \cdot K]$	Specific Heat Capacity (<i>water at 25 °C</i>)
L_c	0.001	$[m]$	Characteristic Length (1 mm)
ΔT	4.2241	$[K]$	
T_r	298	$[K]$	
τ	1.42×10^{-8}	$[s]$	Ratio between ε and σ
δ_σ	2.0%	$1/^\circ$	Used in [1]
δ_ε	-0.4%	$1/^\circ$	Used in [1]
g	9.832	m/s^2	Gravitational Acceleration

CHAPTER 3: BOUNDARY ELEMENT METHOD

Introduction

Boundary element method is a powerful approach to solve linear partial differential equations. In this part, we will use BEM to solve all the dimensionless governing equations for the electrothermal flow. The singular and near singular problem for the integrals of the Green's function is one of the difficult things to tackle when we use boundary element method or the method of fundamental solutions. Lots of researchers investigated this problem [18-21]. Our unique computation is that all the integrals for the Green's functions are calculated analytically. For the singular integral of the Green's functions, we use Cauchy principle value and Hadamard finite part. Furthermore, all the other integrals of Green's functions are also evaluated analytically. With all these efforts, our solutions become much more accurate. In this chapter, we will talk about how to use BEM to solve Laplace equation, Poisson equation, continuity equation and Stokes equation in two dimensional case. Some numerical examples will be presented to show the accuracy of our solvers.

Laplace Equation

We start with the Green's function of 2D Laplace equation. In this equation, (x^*, y^*) is the location for field point and (ζ^*, η^*) is the source point.

$$G(x^*, y^*, \zeta^*, \eta^*) = \frac{1}{4\pi} \ln[(x^* - \zeta^*)^2 + (y^* - \eta^*)^2] \quad (3.1)$$

With this, we can derive the integral equation (Eq. 3.2) [22] for 2D Laplace equation by using reciprocal relation and divergence theorem.

$$\lambda(\xi^*, \eta^*) \varphi^*(\xi^*, \eta^*) = \int \left(\varphi^*(x^*, y^*) \frac{\partial G(x^*, y^*, \xi^*, \eta^*)}{\partial n} - G(x^*, y^*, \xi^*, \eta^*) \frac{\partial \varphi^*(x^*, y^*)}{\partial n} \right) ds(x^*, y^*) \quad (3.2)$$

Where

$\lambda(\xi^*, \eta^*) = 0$, if (ξ^*, η^*) is outside of the domain;

$\lambda(\xi^*, \eta^*) = 1/2$, if (ξ^*, η^*) lies on a smooth boundary;

$\lambda(\xi^*, \eta^*) = 1$, if (ξ^*, η^*) is within the domain.

$\mathbf{n}$ is the outward normal vector of the boundary.

To calculate the $\varphi^*(x^*, y^*)$ or $\frac{\partial \varphi^*(x^*, y^*)}{\partial n}$ on the boundary, we need to divide our

boundary into N elements. On each element, $\varphi^*(x^*, y^*)$ and $\frac{\partial \varphi^*(x^*, y^*)}{\partial n}$ are constant.

For a well-posed problem, one of these two values or combination of them is known.

Based on this constant element approximation, Eq. 3.2 becomes:

$$\lambda(\xi^*, \eta^*) \varphi^*(\xi^*, \eta^*) = \sum_{k=1}^N \left(\varphi^{*(k)} F_2^{(k)} - \frac{\partial \varphi^{*(k)}}{\partial n} F_1^{(k)} \right) \quad (3.3)$$

Where

$$F_1^{(k)}(\xi^*, \eta^*) = \int_{c(k)} G(x^*, y^*, \xi^*, \eta^*) ds(x^*, y^*) \quad (3.4)$$

$$F_2^{(k)}(\xi^*, \eta^*) = \int_{c(k)} \frac{\partial G(x^*, y^*, \xi^*, \eta^*)}{\partial n} ds(x^*, y^*) \quad (3.5)$$

N is the number of the boundary elements.

$F_1^{(k)}$ and $F_2^{(k)}$ are computed analytically by Whye-Teong Ang [22]. To compute the gradient of $\varphi^*(\xi^*, \eta^*)$, we need to apply the gradient operator to the integral equations as:

$$\begin{aligned} \lambda(\xi^*, \eta^*) \frac{\partial \varphi^*(\xi^*, \eta^*)}{\partial \xi^*} &= \sum_{k=1}^N \int_{c(k)} \left[\begin{aligned} &\varphi^*(x^*, y^*)^k \frac{\partial}{\partial \xi^*} \frac{\partial G(x^*, y^*, \xi^*, \eta^*)^k}{\partial n} \\ & - \frac{\partial G(x^*, y^*, \xi^*, \eta^*)^k}{\partial \xi^*} \frac{\partial \varphi^*(x^*, y^*)^k}{\partial n} \end{aligned} \right] ds(x^*, y^*) \\ \lambda(\xi^*, \eta^*) \frac{\partial \varphi^*(\xi^*, \eta^*)}{\partial \eta^*} &= \sum_{k=1}^N \int_{c(k)} \left[\begin{aligned} &\varphi^*(x^*, y^*)^k \frac{\partial}{\partial \eta^*} \frac{\partial G(x^*, y^*, \xi^*, \eta^*)^k}{\partial n} \\ & - \frac{\partial G(x^*, y^*, \xi^*, \eta^*)^k}{\partial \eta^*} \frac{\partial \varphi^*(x^*, y^*)^k}{\partial n} \end{aligned} \right] ds(x^*, y^*) \end{aligned} \quad (3.6)$$

With all these we can compute the dimensionless electrical potential $\varphi^*(\xi^*, \eta^*)$ and dimensionless electrical field $\mathbf{E}^*$.

Now we will present one numerical example to test the accuracy of our BEM solver for 2D Laplace equation.

Example 3.1

To test the accuracy of our BEM solution for 2D Laplace equation, we give a square domain from (0,0) to (1,1). On the left boundary, we put $\varphi^*(x^*, y^*) = 1$. And for the right boundary, we have $\varphi^*(x^*, y^*) = 0$. For the top and bottom boundaries, we fix the boundary conditions as no flux ($\frac{\partial \varphi^*(x^*, y^*)}{\partial n} = 0$). Then this example is actually a

unidirectional heat conduction problem and we can get the analytical solution easily. The temperature field for this case is shown in Fig. 3.1. In the Fig. 3.2, we plot the temperature results at the line ($y = 0.5125$) by BEM and analytical solution. We can find that the BEM result matches the analytical result. We also plot the temperature gradient of this case in Fig. 3.3. Then we can see that our BEM solver for Laplace equation is accurate.

Fig. 3.1 Temperature field for Laplace equation testing case

Fig. 3.2 Comparison result for Laplace equation testing case

Fig. 3.3 Temperature gradient for Laplace equation testing case

Poisson Equation

The temperature field for the microfluidics system is governed by Eq. 2.13 which is actually Poisson equation. As Eq. 2.13 is linear, we apply the method of superposition,

$\theta = \theta_{free-space} + \theta_{BEM}$, to obtain the solution. Using the Green's function method, we solve

the free-space temperature which satisfies the inhomogeneous Joule heating term,

$$\theta_{free-space}(\zeta^*, \eta^*) = \sum_{k=1}^M JH(x^*, y^*)^k P(\zeta^*, \eta^*)^{(k)} \quad (3.7)$$

Where

$$P(\zeta^*, \eta^*)^{(k)} = \int_a^b \int_c^d G(x^*, y^*, \zeta^*, \eta^*) dx^{*(k)} dy^{*(k)} \quad (3.8)$$

The domain is sub-divided into M subdomains. In our case, each subdomain is a rectangle with $x^* \in [a, b]$, $y^* \in [c, d]$. We assume in each subdomain, the Joule heating term is a constant. The analytical expression for $P(\zeta^*, \eta^*)^{(k)}$ is computed as:

$$\begin{aligned} P_1(\zeta^*, \eta^*)^{(k)} &= 3d(a-b) + 2\eta^*(b-a) + ((a-\zeta^*)^2 - (d-\eta^*)^2) \arctan\left[\frac{(a-\zeta^*)}{(d-\eta^*)}\right] \\ &- ((b-\zeta^*)^2 - (d-\eta^*)^2) \arctan\left[\frac{(b-\zeta^*)}{(d-\eta^*)}\right] + (d\zeta^* + a\eta^* - ad - \zeta^*\eta^*) \ln[(a-\zeta^*)^2 + (d-\eta^*)^2] \\ &+ (bd + \zeta^*\eta^* - d\zeta^* - b\eta^*) \ln[(b-\zeta^*)^2 + (d-\eta^*)^2] \end{aligned}$$

$$\begin{aligned} P_2(\zeta^*, \eta^*)^{(k)} &= 3c(a-b) + 2\eta^*(b-a) + ((a-\zeta^*)^2 - (c-\eta^*)^2) \arctan\left[\frac{(a-\zeta^*)}{(c-\eta^*)}\right] \\ &- ((b-\zeta^*)^2 - (c-\eta^*)^2) \arctan\left[\frac{(b-\zeta^*)}{(c-\eta^*)}\right] + (c\zeta^* + a\eta^* - ac - \zeta^*\eta^*) \ln[(a-\zeta^*)^2 + (c-\eta^*)^2] \\ &+ (bc + \zeta^*\eta^* - c\zeta^* - b\eta^*) \ln[(b-\zeta^*)^2 + (c-\eta^*)^2] \end{aligned}$$

$$P(\zeta^*, \eta^*)^{(k)} = \frac{1}{4\pi} [P_1(\zeta^*, \eta^*)^{(k)} - P_2(\zeta^*, \eta^*)^{(k)}] \quad (3.9)$$

Similarly, the temperature gradient in free-space can be obtained,

$$\begin{aligned}
\frac{\partial \theta_{free-space}(\zeta^*, \eta^*)}{\partial \zeta^*} &= \sum_{k=1}^M JH(x^*, y^*)^k B_1(\zeta^*, \eta^*)^k \\
\frac{\partial \theta_{free-space}(\zeta^*, \eta^*)}{\partial \eta^*} &= \sum_{k=1}^M JH(x^*, y^*)^k B_2(\zeta^*, \eta^*)^k
\end{aligned} \tag{3.10}$$

Where

$$\begin{aligned}
B_1(\zeta^*, \eta^*)^{(k)} &= \int_a^b \int_c^d \frac{\partial G(x^*, y^*, \zeta^*, \eta^*)}{\partial \zeta^*} dx^{*(k)} dy^{*(k)} \\
B_2(\zeta^*, \eta^*)^{(k)} &= \int_a^b \int_c^d \frac{\partial G(x^*, y^*, \zeta^*, \eta^*)}{\partial \eta^*} dx^{*(k)} dy^{*(k)}
\end{aligned} \tag{3.11}$$

The analytical expressions for these integrals are :

$$\begin{aligned}
B_{11}(\zeta^*, \eta^*)^{(k)} &= 2(a - \zeta^*) \arctan\left[\frac{(d - \eta^*)}{(a - \zeta^*)}\right] - 2(b - \zeta^*) \arctan\left[\frac{(d - \eta^*)}{(b - \zeta^*)}\right] \\
&+ (d - \eta^*) (\ln[(a - \zeta^*)^2 + (d - \eta^*)^2] - \ln[(b - \zeta^*)^2 + (d - \eta^*)^2]) \\
B_{12}(\zeta^*, \eta^*)^{(k)} &= 2(a - \zeta^*) \arctan\left[\frac{(c - \eta^*)}{(a - \zeta^*)}\right] - 2(b - \zeta^*) \arctan\left[\frac{(c - \eta^*)}{(b - \zeta^*)}\right] \\
&+ (c - \eta^*) (\ln[(a - \zeta^*)^2 + (c - \eta^*)^2] - \ln[(b - \zeta^*)^2 + (c - \eta^*)^2]) \\
B_1(\zeta^*, \eta^*)^{(k)} &= \frac{1}{4\pi} (B_{11}(\zeta^*, \eta^*)^{(k)} - B_{12}(\zeta^*, \eta^*)^{(k)}) \\
B_{21}(\zeta^*, \eta^*)^{(k)} &= 2(c - \eta^*) \arctan\left[\frac{(b - \zeta^*)}{(c - \eta^*)}\right] - 2(d - \eta^*) \arctan\left[\frac{(b - \zeta^*)}{(d - \eta^*)}\right] \\
&+ (b - \zeta^*) (\ln[(b - \zeta^*)^2 + (c - \eta^*)^2] - \ln[(b - \zeta^*)^2 + (d - \eta^*)^2])
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
B_{22}(\zeta^*, \eta^*)^{(k)} &= 2(c - \eta^*) \arctan\left[\frac{(a - \zeta^*)}{(c - \eta^*)}\right] - 2(d - \eta^*) \arctan\left[\frac{(a - \zeta^*)}{(d - \eta^*)}\right] \\
&+ (a - \zeta^*)(\ln[(a - \zeta^*)^2 + (c - \eta^*)^2] - \ln[(a - \zeta^*)^2 + (d - \eta^*)^2]) \\
B_2(\zeta^*, \eta^*)^{(k)} &= \frac{1}{4\pi}(B_{21}(\zeta^*, \eta^*)^{(k)} - B_{22}(\zeta^*, \eta^*)^{(k)})
\end{aligned} \tag{3.13}$$

We then apply the BEM method for the Laplace equation to solve for θ_{BEM} by matching the boundary conditions:

$$\theta_{BEM}(x^*, y^*) = \theta(x^*, y^*) - \theta_{free-space}(x^*, y^*) \tag{3.14}$$

$$\frac{\partial \theta_{BEM}(x^*, y^*)}{\partial n} = \frac{\partial \theta(x^*, y^*)}{\partial n} - \frac{\partial \theta_{free-space}(x^*, y^*)}{\partial n} \tag{3.15}$$

Now, we will give a testing example to show the accuracy of our Poisson equation solver.

Example 3.2

In this testing case, we have a square domain from (0,0) to (1,1). We assume the Joule heating is uniform throughout the whole domain as $JH(x^*, y^*) = 1.0$. We have temperature $\theta(x^*, y^*) = 0$ for all the boundary parts. We use the method of superposition discussed above. The Joule heating field for this example is displayed in Fig. 3.4. The free-space temperature field is shown in Fig. 3.5. Then after superposition with BEM part temperature result, the temperature field is plotted in Fig. 3.6. We also show the temperature gradient in Fig. 3.7. From all these figures, we can find that our results match the Joule heating field and boundary conditions.

Fig. 3.4 Joule heating field for Poisson equation testing case

Fig. 3.5 Free-space temperature field for Poisson equation testing case

Fig. 3.6 Temperature field for Poisson equation testing case

Fig. 3.7 Temperature gradient for Poisson equation testing case

Continuity Equation and Stokes Equation

The next step for modeling this microfluidics system is to solve the Stokes flow caused by the electrothermal force and buoyancy force. In this part, we also use the boundary

element method to solve the dimensionless continuity equation (Eq. 2.14) and dimensionless Stokes equation (Eq. 2.15). Once again, we solve for the velocity field by superposition of free-space velocity solution and pure BEM velocity solution.

In the case of two-dimensional Stokes flow, the free-space Green's functions are given by [23],

$$Q_{ij}(\vec{x} - \vec{x}_0) = -\delta_{ij} \ln r + \frac{\hat{x}_i \hat{x}_j}{r^2} \quad (3.16)$$

$$T_{ijl}(\vec{x} - \vec{x}_0) = -4 \frac{\hat{x}_i \hat{x}_j \hat{x}_l}{r^4} \quad (3.17)$$

$$\psi_{ijkl}(\vec{x} - \vec{x}_0) = \frac{4}{r^2} \delta_{ik} \delta_{jl} + \frac{4}{r^4} (\delta_{ij} \hat{x}_k \hat{x}_l + \delta_{il} \hat{x}_k \hat{x}_j + \delta_{kj} \hat{x}_i \hat{x}_l + \delta_{kl} \hat{x}_i \hat{x}_j) - 32 \frac{\hat{x}_i \hat{x}_j \hat{x}_k \hat{x}_l}{r^6} \quad (3.18)$$

Where

$$\hat{x} = \vec{x} - \vec{x}_0, r = |\hat{x}| \quad (3.19)$$

Dividing the domain into M subdomains similar as we compute the free-space temperature, the two-dimensional free-space velocity and stress tensor are,

$$u_i(\vec{x}) = \frac{1}{4\pi\mu} \sum_{k=1}^M F Q_{ij}^{(k)} f_j^{(k)} \quad (3.20)$$

$$\sigma_{ij}(\vec{x}) = \frac{1}{4\pi} \sum_{k=1}^M F T_{ijl}^{(k)} f_l^{(k)} \quad (3.21)$$

Where

$$F Q_{ij}^{(k)} = \int_a^b \int_c^d Q_{ij}(\vec{x}, \vec{x}_0) dx_0 dy_0 \quad (3.22)$$

$$FT_{ijl}^{(k)} = \int_a^b \int_c^d T_{ijl}(\vec{x}_0, \vec{x}) dx_0 dy_0 \quad (3.23)$$

For accuracy, all these integrals are computed analytically. The expressions for these analytical integrals are as:

$$FQ_{111}^{(k)} = 4d(b-a) + 2y(a-b) + 2(d-y)^2 \left(\arctan\left[\frac{(b-x)}{(y-d)}\right] - \arctan\left[\frac{(a-x)}{(y-d)}\right] \right) \\ + (ad - dx - ay + xy) \ln[(a-x)^2 + (d-y)^2] - (bd - dx - by + xy) \ln[(b-x)^2 + (d-y)^2]$$

$$FQ_{112}^{(k)} = 4c(b-a) + 2y(a-b) + 2(c-y)^2 \left(\arctan\left[\frac{(b-x)}{(y-c)}\right] - \arctan\left[\frac{(a-x)}{(y-c)}\right] \right) \\ + (ac - cx - ay + xy) \ln[(a-x)^2 + (c-y)^2] - (bc - cx - by + xy) \ln[(b-x)^2 + (c-y)^2]$$

$$FQ_{11}^{(k)} = 0.5(FQ_{111}^{(k)} - FQ_{112}^{(k)}) \quad (3.24)$$

$$FQ_{121}^{(k)} = -(d-y)^2 \ln[(a-x)^2 + (d-y)^2] + (d-y)^2 \ln[(b-x)^2 + (d-y)^2] \\ - (a-x)^2 \ln[(a-x)^2 + (d-y)^2] + (b-x)^2 \ln[(b-x)^2 + (d-y)^2]$$

$$FQ_{122}^{(k)} = -(c-y)^2 \ln[(a-x)^2 + (c-y)^2] + (c-y)^2 \ln[(b-x)^2 + (c-y)^2] \\ - (a-x)^2 \ln[(a-x)^2 + (c-y)^2] + (b-x)^2 \ln[(b-x)^2 + (c-y)^2]$$

$$FQ_{12}^{(k)} = 0.25(FQ_{121}^{(k)} - FQ_{122}^{(k)}) \quad (3.25)$$

$$FQ_{21}^{(k)} = FQ_{12}^{(k)} \quad (3.26)$$

$$FQ_{221}^{(k)} = 4b(d-c) + 2x(c-d) + 2(b-x)^2 \left(\arctan\left[\frac{(d-y)}{(x-b)}\right] - \arctan\left[\frac{(c-y)}{(x-b)}\right] \right) \\ + (bc - cx - by + xy) \ln[(b-x)^2 + (c-y)^2] - (bd - dx - by + xy) \ln[(b-x)^2 + (d-y)^2]$$

$$\begin{aligned}
FQ_{222}^{(k)} &= 4a(d-c) + 2x(c-d) + 2(a-x)^2 \left(\arctan\left[\frac{(d-y)}{(x-a)}\right] - \arctan\left[\frac{(c-y)}{(x-a)}\right] \right) \\
&+ (ac-cx-ay+xy) \ln[(a-x)^2 + (c-y)^2] - (ad-dx-ay+xy) \ln[(a-x)^2 + (d-y)^2]
\end{aligned}$$

$$FQ_{22}^{(k)} = 0.5(FQ_{221}^{(k)} - FQ_{222}^{(k)}) \quad (3.27)$$

$$\begin{aligned}
FT_{111}^{(k)} &= (a-x) \arctan\left[\frac{(-d+y)}{(a-x)}\right] - (b-x) \arctan\left[\frac{(-d+y)}{(b-x)}\right] \\
&+ (-d+y) (\ln[(a-x)^2 + (d-y)^2] - \ln[(b-x)^2 + (d-y)^2])
\end{aligned}$$

$$\begin{aligned}
FT_{112}^{(k)} &= (a-x) \arctan\left[\frac{(-c+y)}{(a-x)}\right] - (b-x) \arctan\left[\frac{(-c+y)}{(b-x)}\right] \\
&+ (-c+y) (\ln[(a-x)^2 + (c-y)^2] - \ln[(b-x)^2 + (c-y)^2])
\end{aligned}$$

$$FT_{111}^{(k)} = 2(FT_{1111}^{(k)} - FT_{1112}^{(k)}) \quad (3.28)$$

$$\begin{aligned}
FT_{121}^{(k)} &= 2((c-y) \left(\arctan\left[\frac{(a-x)}{(c-y)}\right] - \arctan\left[\frac{(b-x)}{(c-y)}\right] \right) \\
&- (d-y) \left(\arctan\left[\frac{(a-x)}{(d-y)}\right] - \arctan\left[\frac{(b-x)}{(d-y)}\right] \right))
\end{aligned} \quad (3.29)$$

$$FT_{112}^{(k)} = FT_{121}^{(k)} \quad (3.30)$$

$$\begin{aligned}
FT_{122}^{(k)} &= 2((a-x) \left(\arctan\left[\frac{(y-d)}{(a-x)}\right] - \arctan\left[\frac{(y-c)}{(a-x)}\right] \right) \\
&- (b-x) \left(\arctan\left[\frac{(y-d)}{(b-x)}\right] - \arctan\left[\frac{(y-c)}{(b-x)}\right] \right))
\end{aligned} \quad (3.31)$$

$$FT_{211}^{(k)} = FT_{121}^{(k)} \quad (3.32)$$

$$FT_{221}^{(k)} = FT_{122}^{(k)} \quad (3.33)$$

$$FT_{212}^{(k)} = FT_{122}^{(k)} \quad (3.34)$$

$$FT_{2221}^{(k)} = (c-y) \arctan\left[\frac{(-b+x)}{(c-y)}\right] - (d-y) \arctan\left[\frac{(-b+x)}{(d-y)}\right] \\ + (-b+x)(\ln[(b-x)^2 + (c-y)^2] - \ln[(b-x)^2 + (d-y)^2])$$

$$FT_{2222}^{(k)} = (c-y) \arctan\left[\frac{(-a+x)}{(c-y)}\right] - (d-y) \arctan\left[\frac{(-a+x)}{(d-y)}\right] \\ + (-a+x)(\ln[(a-x)^2 + (c-y)^2] - \ln[(a-x)^2 + (d-y)^2])$$

$$FT_{222}^{(k)} = 2(FT_{2221}^{(k)} - FT_{2222}^{(k)}) \quad (3.35)$$

To match the boundary condition, we need to get the free-space tractions,

$$t_1 = \sigma_{11}n_x + \sigma_{12}n_y \quad (3.36)$$

$$t_2 = \sigma_{21}n_x + \sigma_{22}n_y \quad (3.37)$$

Where

t_1 and t_2 are two component for traction, $\vec{t}$.

The boundary condition for BEM is then given by,

$$\vec{u}_{BEM} = \vec{u} - \vec{u}_{free-space} \quad (3.38)$$

$$\vec{t}_{BEM} = \vec{t} - \vec{t}_{free-space} \quad (3.39)$$

Now, we need to compute the BEM part Stokes flow. Dividing the boundary C in to N elements, the discretized integral equations for Stokes flow are,

$$\lambda(\vec{x}_0)u_j(\vec{x}_0) = -\frac{1}{4\pi\mu} \sum_{k=1}^N f_i^{(k)}(\vec{x}) LQ_{ji}^{(k)} + \frac{1}{4\pi} \sum_{k=1}^N u_i^{(k)}(\vec{x}) LT_{ijl}^{(k)} n_l^{(k)}(\vec{x}) \quad (3.40)$$

$$\lambda(\vec{x}_0)f_i(\vec{x}_0) = -\frac{1}{4\pi} \sum_{k=1}^N f_j^{(k)}(\vec{x}) LT_{ijl}^{(k)} n_l(\vec{x}_0) + \frac{\mu}{4\pi} \sum_{k=1}^N u_j^{(k)}(\vec{x}) LH_{ijk}^{(k)} n_l^{(k)}(\vec{x}) n_k(\vec{x}_0) \quad (3.41)$$

Where

$\lambda(\vec{x}_0) = 0$, if $\vec{x}_0$ is outside of the domain;

$\lambda(\vec{x}_0) = \frac{1}{2}$, if $\vec{x}_0$ lies on a smooth boundary;

$\lambda(\vec{x}_0) = 1$, if $\vec{x}_0$ is within the domain.

We define the matrix coefficients for all the integrals in the above equations:

$$LQ_{ji}^{(k)} = \int_{C(k)} Q_{ji}(\vec{x}_0, \vec{x}) dl^{(k)}(\vec{x}) \quad (3.42)$$

$$LT_{ijl}^{(k)} = \int_{C(k)} T_{ijl}(\vec{x}, \vec{x}_0) dl^{(k)}(\vec{x}) \quad (3.43)$$

$$LH_{ijk}^{(k)} = \int_{C(k)} \psi_{ijk}(\vec{x}, \vec{x}_0) dl^{(k)}(\vec{x}) \quad (3.44)$$

Because all these integrals in equations above are line integrals, we need to apply parametric function for line integral to do this.

For each element, we have:

$$\vec{x} = (x, y), x \in [x_1, x_2] \text{ and } y \in [y_1, y_2]. \quad (3.45)$$

So the parametric function for each element is:

$$x = (x_2 - x_1)\tau + x_1, y = (y_2 - y_1)\tau + y_1, \tau \in [0, 1]. \quad (3.46)$$

After applying the Jacobian for this integral, we have:

$$dl^{(k)} = J^{(k)} d\tau$$

$$J^{(k)} = \sqrt{(x_2^{(k)} - x_1^{(k)})^2 + (y_2^{(k)} - y_1^{(k)})^2} \quad (3.47)$$

When field point and source point are coincident, these line integrals become singular integrals. In this case, we should apply Cauchy principle value and Hadamard finite part to evaluate the integrals. With this effort, the results for all the Green's functions singular integrals are,

$$LQ_{11}^{(k)} = [-0.5 \ln[(x_2 - x_1)^2 + (y_2 - y_1)^2] + (1 - \ln[0.5]) + \frac{(x_2 - x_1)^2}{(x_2 - x_1)^2 + (y_2 - y_1)^2}] J^{(k)} \quad (3.48)$$

$$LQ_{12}^{(k)} = [\frac{(x_2 - x_1)(y_2 - y_1)}{(x_2 - x_1)^2 + (y_2 - y_1)^2}] J^{(k)} \quad (3.49)$$

$$LQ_{21}^{(k)} = [\frac{(x_2 - x_1)(y_2 - y_1)}{(x_2 - x_1)^2 + (y_2 - y_1)^2}] J^{(k)} \quad (3.50)$$

$$LQ_{22}^{(k)} = [-0.5 \ln[(x_2 - x_1)^2 + (y_2 - y_1)^2] + (1 - \ln[0.5]) + \frac{(y_2 - y_1)^2}{(x_2 - x_1)^2 + (y_2 - y_1)^2}] J^{(k)} \quad (3.51)$$

$$LT_{ijl}^{(k)} = 0 \quad (3.52)$$

$$LH_{1111}^{(k)} = \left\{ -\frac{4}{(x_2 - x_1)^2 + (y_2 - y_1)^2} - \frac{16(x_2 - x_1)^2}{[(x_2 - x_1)^2 + (y_2 - y_1)^2]^2} + \frac{32(x_2 - x_1)^4}{[(x_2 - x_1)^2 + (y_2 - y_1)^2]^3} \right\} J^{(k)} \quad (3.53)$$

$$LH_{1112}^{(k)} = \left\{ -\frac{8(x_2 - x_1)(y_2 - y_1)}{[(x_2 - x_1)^2 + (y_2 - y_1)^2]^2} + \frac{32(x_2 - x_1)^3(y_2 - y_1)}{[(x_2 - x_1)^2 + (y_2 - y_1)^2]^3} \right\} J^{(k)} \quad (3.54)$$

$$LH_{1121}^{(k)} = LH_{1112}^{(k)} \quad (3.55)$$

$$LH_{1122}^{(k)} = \left\{ -\frac{4}{(x_2 - x_1)^2 + (y_2 - y_1)^2} + \frac{32(x_2 - x_1)^2 (y_2 - y_1)^2}{[(x_2 - x_1)^2 + (y_2 - y_1)^2]^3} \right\} J^{(k)} \quad (3.56)$$

$$LH_{1211}^{(k)} = LH_{1112}^{(k)} \quad (3.57)$$

$$LH_{1212}^{(k)} = LH_{1122}^{(k)} \quad (3.58)$$

$$LH_{1221}^{(k)} = LH_{1122}^{(k)} \quad (3.59)$$

$$LH_{1222}^{(k)} = \left\{ -\frac{8(x_2 - x_1)(y_2 - y_1)}{[(x_2 - x_1)^2 + (y_2 - y_1)^2]^2} + \frac{32(x_2 - x_1)(y_2 - y_1)^3}{[(x_2 - x_1)^2 + (y_2 - y_1)^2]^3} \right\} J^{(k)} \quad (3.60)$$

$$LH_{2111}^{(k)} = LH_{1112}^{(k)} \quad (3.61)$$

$$LH_{2112}^{(k)} = LH_{1122}^{(k)} \quad (3.62)$$

$$LH_{2121}^{(k)} = LH_{1122}^{(k)} \quad (3.63)$$

$$LH_{2122}^{(k)} = LH_{1222}^{(k)} \quad (3.64)$$

$$LH_{2211}^{(k)} = LH_{1122}^{(k)} \quad (3.65)$$

$$LH_{2212}^{(k)} = LH_{1222}^{(k)} \quad (3.66)$$

$$LH_{2221}^{(k)} = LH_{1222}^{(k)} \quad (3.67)$$

$$LH_{2222}^{(k)} = \left\{ -\frac{4}{(x_2 - x_1)^2 + (y_2 - y_1)^2} - \frac{16(y_2 - y_1)^2}{[(x_2 - x_1)^2 + (y_2 - y_1)^2]^2} + \frac{32(y_2 - y_1)^4}{[(x_2 - x_1)^2 + (y_2 - y_1)^2]^3} \right\} J^{(k)} \quad (3.68)$$

With all these equations above, we get the integrals of the Green's function when source point and field point are coincident. For the case when source points and field points are not on the same element, all the Green's function integrals can be computed analytically as follow. For simplicity, we transform the format of these two equations as:

$$x = a\tau + b, \quad y = d\tau + e, \quad \tau \in [0,1] \quad (3.69)$$

Where

$$a = x_2 - x_1, \quad b = x_1, \quad c = x_0, \quad (3.70)$$

$$d = y_2 - y_1, \quad e = y_1, \quad f = y_0 \quad (3.71)$$

$$J^{(k)} = \sqrt{a^2 + d^2} \quad (3.72)$$

$$\begin{aligned} LQ_{111}^{(k)} = & \left\{ 1 - \frac{(d(c-b) + a(e-f))}{(a^2 + d^2)} \arctan\left[\frac{(a^2 + a(b-c) + d^2 + d(e-f))}{(d(c-b) + a(e-f))}\right] \right. \\ & - 0.5 \ln[(a+b-c)^2 + (d+e-f)^2] \left[1 + \frac{(a(b-c) + d(e-f))}{(a^2 + d^2)} \right] + \frac{1}{(a^2 + d^2)^2} \\ & \{ a^2(a^2 + d^2) - (d(c-b) + a(e-f))[(a^2 - d^2) \arctan\left[\frac{(a^2 + a(b-c) + d(d+e-f))}{(d(c-b) + a(e-f))}\right] \} \\ & \left. + ad \ln[(a+b-c)^2 + (d+e-f)^2] \right\} \end{aligned}$$

$$\begin{aligned} LQ_{112}^{(k)} = & \left\{ -\frac{(d(c-b) + a(e-f))}{(a^2 + d^2)} \arctan\left[\frac{(a(b-c) + d(e-f))}{(d(c-b) + a(e-f))}\right] \right. \\ & - 0.5 \frac{(a(b-c) + d(e-f))}{(a^2 + d^2)} \ln[(b-c)^2 + (e-f)^2] - \frac{1}{(a^2 + d^2)^2} \\ & (d(c-b) + a(e-f)) \{ (a^2 - d^2) \arctan\left[\frac{(a(b-c) + d(e-f))}{(d(c-b) + a(e-f))}\right] \} \\ & \left. + ad \ln[(b-c)^2 + (e-f)^2] \right\} \end{aligned}$$

$$LQ_{11}^{(k)} = LQ_{111}^{(k)} - LQ_{112}^{(k)} \quad (3.73)$$

$$LQ_{121}^{(k)} = \frac{1}{2(a^2 + d^2)^2} \{2ad(a^2 + d^2) - (d(c-b) + a(e-f))[4ad \arctan$$

$$[\frac{(a^2 + a(b-c) + d(d+e-f))}{(d(c-b) + a(e-f))}] - (a^2 - d^2) \ln[(a+b-c)^2 + (d+e-f)^2]\}] \}$$

$$LQ_{122}^{(k)} = \frac{1}{2(a^2 + d^2)^2} \{-(d(c-b) + a(e-f))[4ad \arctan$$

$$[\frac{(a(b-c) + d(e-f))}{(d(c-b) + a(e-f))}] - (a^2 - d^2) \ln[(b-c)^2 + (e-f)^2]\}] \}$$

$$LQ_{12}^{(k)} = LQ_{121}^{(k)} - LQ_{122}^{(k)} \quad (3.74)$$

$$LQ_{21}^{(k)} = LQ_{12}^{(k)} \quad (3.75)$$

$$LQ_{221}^{(k)} = \{1 - \frac{(d(c-b) + a(e-f))}{(a^2 + d^2)} \arctan[\frac{(a^2 + a(b-c) + d^2 + d(e-f))}{(d(c-b) + a(e-f))}]\}$$

$$- 0.5 \ln[(a+b-c)^2 + (d+e-f)^2] [1 + \frac{(a(b-c) + d(e-f))}{(a^2 + d^2)}] \} + \frac{1}{(a^2 + d^2)^2}$$

$$\{d^2(a^2 + d^2) - (d(c-b) + a(e-f))[(a^2 - d^2) \arctan[\frac{(a^2 + a(b-c) + d(d+e-f))}{(d(b-c) + a(f-e))}]]$$

$$- ad \ln[(a+b-c)^2 + (d+e-f)^2]\}$$

$$LQ_{222}^{(k)} = \{-\frac{(d(c-b) + a(e-f))}{(a^2 + d^2)} \arctan[\frac{(a(b-c) + d(e-f))}{(d(c-b) + a(e-f))}]\}$$

$$- 0.5 \frac{(a(b-c) + d(e-f))}{(a^2 + d^2)} \ln[(b-c)^2 + (e-f)^2] \} - \frac{1}{(a^2 + d^2)^2}$$

$$(d(c-b) + a(e-f)) \{ (a^2 - d^2) \arctan[\frac{(a(b-c) + d(e-f))}{(d(b-c) + a(f-e))}]$$

$$- ad \ln[(b-c)^2 + (e-f)^2] \}$$

$$LQ_{22}^{(k)} = LQ_{221}^{(k)} - LQ_{222}^{(k)} \quad (3.76)$$

$$\begin{aligned}
LT_{1111}^{(k)} &= -\frac{2}{(a^2 + d^2)^2} \left\{ (d(c-b) + a(e-f)) \frac{(2a(b-c)d - d^2(d+e-f) + a^2(3d+e-f))}{((a+b-c)^2 + (d+e-f)^2)} \right. \\
&\quad \left. - (3a^2d + d^3) \arctan\left[\frac{(a^2 + a(b-c) + d(d+e-f))}{(d(c-b) + a(e-f))}\right] + a^3 \ln[(a+b-c)^2 + (d+e-f)^2] \right\} \\
LT_{1112}^{(k)} &= -\frac{2}{(a^2 + d^2)^2} \left\{ (d(c-b) + a(e-f)) \frac{(2a(b-c)d - d^2(e-f) + a^2(e-f))}{((b-c)^2 + (e-f)^2)} \right. \\
&\quad \left. - (3a^2d + d^3) \arctan\left[\frac{(a(b-c) + d(e-f))}{(d(c-b) + a(e-f))}\right] + a^3 \ln[(b-c)^2 + (e-f)^2] \right\}
\end{aligned}$$

$$LT_{111}^{(k)} = LT_{1111}^{(k)} - LT_{1112}^{(k)} \quad (3.77)$$

$$\begin{aligned}
LT_{1121}^{(k)} &= -\frac{2}{(a^2 + d^2)^2} \left\{ -(d(c-b) + a(e-f)) \frac{(a^3 + (a^2 - d^2)(b-c) + ad(-3d - 2e + 2f))}{((a+b-c)^2 + (d+e-f)^2)} \right. \\
&\quad \left. + (a^3 - ad^2) \arctan\left[\frac{(a^2 + a(b-c) + d(d+e-f))}{(d(c-b) + a(e-f))}\right] + a^2d \ln[(a+b-c)^2 + (d+e-f)^2] \right\} \\
LT_{1122}^{(k)} &= -\frac{2}{(a^2 + d^2)^2} \left\{ -(d(c-b) + a(e-f)) \frac{((a^2 - d^2)(b-c) + ad(-2e + 2f))}{((b-c)^2 + (e-f)^2)} \right. \\
&\quad \left. + (a^3 - ad^2) \arctan\left[\frac{(a(b-c) + d(e-f))}{(d(c-b) + a(e-f))}\right] + a^2d \ln[(b-c)^2 + (e-f)^2] \right\}
\end{aligned}$$

$$LT_{112}^{(k)} = LT_{1121}^{(k)} - LT_{1122}^{(k)} \quad (3.78)$$

$$LT_{121}^{(k)} = LT_{112}^{(k)} \quad (3.79)$$

$$\begin{aligned}
LT_{1221}^{(k)} &= -\frac{2}{(a^2 + d^2)^2} \left\{ (d(b-c) + a(f-e)) \frac{(2a(b-c)d - d^2(d+e-f) + a^2(3d+e-f))}{((a+b-c)^2 + (d+e-f)^2)} \right. \\
&\quad \left. + d(a^2 - d^2) \arctan\left[\frac{(a^2 + a(b-c) + d(d+e-f))}{(d(c-b) + a(e-f))}\right] + ad^2 \ln[(a+b-c)^2 + (d+e-f)^2] \right\}
\end{aligned}$$

$$LT_{1222}^{(k)} = -\frac{2}{(a^2 + d^2)^2} \{ (d(b-c) + a(f-e)) \frac{(2a(b-c)d - d^2(e-f) + a^2(e-f))}{((b-c)^2 + (e-f)^2)} \\ + d(a^2 - d^2) \arctan\left[\frac{(a(b-c) + d(e-f))}{(d(c-b) + a(e-f))}\right] + ad^2 \ln[(b-c)^2 + (e-f)^2] \}$$

$$LT_{122}^{(k)} = LT_{1221}^{(k)} - LT_{1222}^{(k)} \quad (3.80)$$

$$LT_{211}^{(k)} = LT_{112}^{(k)} \quad (3.81)$$

$$LT_{212}^{(k)} = LT_{122}^{(k)} \quad (3.82)$$

$$LT_{221}^{(k)} = LT_{122}^{(k)} \quad (3.83)$$

$$LT_{2221}^{(k)} = -\frac{2}{(a^2 + d^2)^2} \{ (d(c-b) + a(e-f)) \frac{(a^3 + (a^2 - d^2)(b-c) + ad(-3d - 2e + 2f))}{((a+b-c)^2 + (d+e-f)^2)} \\ - (a^3 + 3ad^2) \arctan\left[\frac{(a^2 + a(b-c) + d(d+e-f))}{(d(b-c) + a(f-e))}\right] + d^3 \ln[(a+b-c)^2 + (d+e-f)^2] \}$$

$$LT_{2222}^{(k)} = -\frac{2}{(a^2 + d^2)^2} \{ (d(c-b) + a(e-f)) \frac{((a^2 - d^2)(b-c) + ad(-2e + 2f))}{((b-c)^2 + (e-f)^2)} \\ - (a^3 + 3ad^2) \arctan\left[\frac{(a(b-c) + d(e-f))}{(d(b-c) + a(f-e))}\right] + d^3 \ln[(b-c)^2 + (e-f)^2] \}$$

$$LT_{222}^{(k)} = LT_{2221}^{(k)} - LT_{2222}^{(k)} \quad (3.84)$$

$$\begin{aligned}
LH_{1111}^{(k)} = & \frac{1}{(a^2 + d^2)^2} \left\{ \frac{1}{((b-c)^2 + (e-f)^2)^2} [-a(b-c)d^2((b-c)^2 - (e-f)^2) \right. \\
& - a^3(b-c)(3(b-c)^2 + (e-f)^2) + d^3(3(b-c)^2 + (e-f)^2)(e-f) + a^2d(5(b-c)^2 \\
& + 3(e-f)^2)(e-f)] + \frac{1}{((a+b-c)^2 + (d+e-f)^2)^2} [3a^6 + 9a^5(b-c) - d^3(3(b-c)^2 \\
& + (d+e-f)^2)(d+e-f) + a(b-c)d^2((b-c)^2 - 7d^2 - 8de + 8df - (e-f)^2) \\
& + a^4(9(b-c)^2 - 3d^2 - 3de + e^2 + 3df - 2ef + f^2) + a^3(b-c)(3(b-c)^2 - 6d^2 - 8de \\
& + 8df + (e-f)^2) - a^2d(7d^3 + 14d^2e + 10de^2 + 3e^3 + (b-c)^2(2d + 5e - 5f) - 14d^2f \\
& \left. - 20def - 9e^2f + 10df^2 + 9ef^2 - 3f^3)] \right\} \quad (3.85)
\end{aligned}$$

$$\begin{aligned}
LH_{1112}^{(k)} = & \frac{1}{(a^2 + d^2)^2} \left\{ \frac{1}{((b-c)^2 + (e-f)^2)^2} [-(b-c)d^3((b-c)^2 - (e-f)^2) \right. \\
& - a^2(b-c)d(3(b-c)^2 + (e-f)^2) - ad^2((b-c)^2 - (e-f)^2)(e-f) - a^3(3(b-c)^2 \\
& + (e-f)^2)(e-f)] + \frac{1}{((a+b-c)^2 + (d+e-f)^2)^2} [(b-c)d^3((b-c)^2 - (d+e-f)^2) \\
& + ad^2(-(d+e-f)^2(2d+e-f) + (b-c)^2(4d+e-f)) + 3a^4(b-c)(5d+2e-2f) \\
& + 3a^5(2d+e-f) + a^2(b-c)d(3(b-c)^2 + 6d^2 + 4de - 4df + (e-f)^2) + a^3(4d^3 + 6d^2e \\
& \left. + 4de^2 + e^3 + 3(b-c)^2(4d+e-f) - 6d^2f - 8def - 3e^2f + 4df^2 + 3ef^2 - f^3)] \right\} \quad (3.86)
\end{aligned}$$

$$LH_{1121}^{(k)} = LH_{1112}^{(k)} \quad (3.87)$$

$$\begin{aligned}
LH_{1122}^{(k)} = & \frac{1}{(a^2 + d^2)^2} \left\{ \frac{1}{((b-c)^2 + (e-f)^2)^2} [a^3(b-c)((b-c)^2 - (e-f)^2) \right. \\
& - a(b-c)d^2((b-c)^2 + 3(e-f)^2) - d^3((b-c)^2 - (e-f)^2)(e-f) - a^2d \\
& (3(b-c)^2 + (e-f)^2)(e-f)] + \frac{1}{((a+b-c)^2 + (d+e-f)^2)^2} [-a^6 + 3a^5(-b+c) \\
& - d^3(-(b-c)^2 + (d+e-f)^2)(d+e-f) - a^3(b-c)((b-c)^2 - 10d^2 - 8de + 8df \\
& - (e-f)^2) + a^4(-3(b-c)^2 + 5d^2 + 5de + e^2 - 5df - 2ef + f^2) + a(b-c)d^2 \\
& ((b-c)^2 + 5d^2 + 8de + 3e^2 - 8df - 6ef + 3f^2) + a^2d(5d^3 + 10d^2e + 6de^2 + e^3 \\
& \left. + 3(b-c)^2(2d+e-f) - 10d^2f - 12def - 3e^2f + 6df^2 + 3ef^2 - f^3)] \right\} \quad (3.88)
\end{aligned}$$

$$LH_{1211}^{(k)} = LH_{1112}^{(k)} \quad (3.89)$$

$$LH_{1212}^{(k)} = LH_{1122}^{(k)} \quad (3.90)$$

$$LH_{1221}^{(k)} = LH_{1122}^{(k)} \quad (3.91)$$

$$\begin{aligned} LH_{1222}^{(k)} = & \frac{1}{(a^2 + d^2)^2} \left\{ \frac{1}{((b-c)^2 + (e-f)^2)^2} [a^2(b-c)d((b-c)^2 - \right. \\ & (e-f)^2) - (b-c)d^3((b-c)^2 + 3(e-f)^2) + a^3((b-c)^2 - (e-f)^2) \\ & (e-f) - ad^2((b-c)^2 + 3(e-f)^2)(e-f)] + \frac{1}{((a+b-c)^2 + (d+e-f)^2)^2} \\ & [(b-c)d^3((b-c)^2 + 3(d+e-f)^2) + ad^2(3(d+e-f)^2(2d+e-f) + (b-c)^2 \\ & (4d+e-f)) - a^4(b-c)(5d+2e-2f) + a^5(-2d-e+f) - a^2(b-c)d((b-c)^2 \\ & - 6d^2 - 4de + 4df - (e-f)^2) + a^3(4d^3 + 6d^2e + 4de^2 + e^3 - (b-c)^2(4d+e-f) \\ & \left. - 6d^2f - 8def - 3e^2f + 4df^2 + 3ef^2 - f^3)] \right\} \end{aligned} \quad (3.92)$$

$$LH_{2111}^{(k)} = LH_{1112}^{(k)} \quad (3.93)$$

$$LH_{2112}^{(k)} = LH_{1122}^{(k)} \quad (3.94)$$

$$LH_{2121}^{(k)} = LH_{1122}^{(k)} \quad (3.95)$$

$$LH_{2122}^{(k)} = LH_{1222}^{(k)} \quad (3.96)$$

$$LH_{2211}^{(k)} = LH_{1122}^{(k)} \quad (3.97)$$

$$LH_{2212}^{(k)} = LH_{1222}^{(k)} \quad (3.98)$$

$$LH_{2221}^{(k)} = LH_{1222}^{(k)} \quad (3.99)$$

$$\begin{aligned}
LH_{2222}^{(k)} = & \frac{1}{(a^2 + d^2)^2} \left\{ \frac{1}{((b-c)^2 + (e-f)^2)^2} [a^3(b-c)((b-c)^2 + 3(e-f)^2) \right. \\
& + a(b-c)d^2(3(b-c)^2 + 5(e-f)^2) + a^2d((b-c)^2 - (e-f)^2)(e-f) - d^3 \\
& ((b-c)^2 + 3(e-f)^2)(e-f)] - \frac{1}{((a+b-c)^2 + (d+e-f)^2)^2} [a^6 + 3a^5(b-c) \\
& - d^3((b-c)^2 + 3(d+e-f)^2)(d+e-f) + a^3(b-c)((b-c)^2 + 14d^2 + 8de + 3e^2 \\
& - 8df - 6ef + 3f^2) + a^4(3(b-c)^2 + 7d^2 + 7de + 3e^2 - 7df - 6ef + 3f^2) + a(b-c) \\
& d^2(3(b-c)^2 + 3d^2 + 8de + 5e^2 - 8df - 10ef + 5f^2) + a^2d(3d^3 + 6d^2e + 2de^2 - e^3 \\
& \left. + (b-c)^2(10d + e - f) - 6d^2f - 4def + 3e^2f + 2df^2 - 3ef^2 + f^3) \right\} \quad (3.100)
\end{aligned}$$

With all these integrals for Green's function, we can apply Eq. 3.40 and Eq. 3.41 to solve the Stokes flow by BEM. To improve the condition of the final matrix equation, we combine these two integral equations, we can set two coefficients β_1 and β_2 for each equation respectively. For different problems, we will pick up different β_1 and β_2 for accurate results. According to Yijun Liu's conclusion [11], commonly a positive value for $\beta_1=1.0$ and $\beta_2=1.0$ works well for most problems. More discussions for choosing β_1 and β_2 can be found in references [24-25]. For a well-posed problem, on each element, either velocity or traction should be known for any direction. Based on this, we can set up system equations by combining Eq. 3.40 and Eq. 3.41. Now, some numerical examples will be given to show the accuracy of our Stokes flow solver.

Example 3.3

This model is a 2D square cavity filled with incompressible viscous flow. The Reynolds number is small in this case. We choose $\beta_1 = 1$ and $\beta_2 = 0.01$ to get the result because all the velocity is known on the boundary. The reason for picking up $\beta_2 = 0.01$ is to avoid the instability problem when solving the matrix. The fluid is moving at a constant unit

velocity on the top lid. On the other boundary parts, there are no slip and no penetration. We pick up a square domain from (0,0) to (1,1). We divide the boundary into 1000 elements and internal domain into 1600 subdomains. On each boundary element and subdomain, the velocity and traction are constant. The boundary configuration and internal domains are shown in Fig. 3.8. The velocity vectors are shown in Fig. 3.9. Then we compute the u velocity at $x=0.5$ and plot that in the Fig. 3.10. The v velocity at $y=0.5$ is shown in Fig 3.11. Our BEM result is compared with D.L. Young's work by multiquadrics method [26]. We can see that our 2D BEM result matches the D.L. Young's result for the 2D square cavity Stokes flow. Besides, we use lattice Boltzmann method with (100*100) lattice nodes to compute the square cavity problem for Stokes flow. As shown in the figures, our results of 2D BEM and 2D LBM match with each other.

Fig. 3.8 Square cavity flow domain

Fig. 3.9 Square cavity flow velocity vectors

Fig. 3.10 Square cavity flow u velocity at $x=0.5$

Fig. 3.11 Square cavity flow v velocity at y=0.5

Example 3.4

This model is unidirectional flow driven by uniform body force. In this case, we choose $\beta_1 = 1$ and $\beta_2 = 1$. We put body force $\vec{F} = 1.0$ throughout the whole domain. The domain we compute is from (0,0) to (5,1). We have 600 boundary elements. The domain and boundary configuration are shown in Fig. 3.12. Force field is displayed in Fig. 3.13. The free-space velocity caused by the body force is shown in Fig. 3.14. The boundary condition for the unidirectional viscous flow in a rectangular channel is: no-slip and no penetration for channel surface. $\frac{\partial u}{\partial x} = 0$, $v = 0$ and $p = p_\infty$ at inlet and outlet of the channel. The velocity vectors for this unidirectional flow are shown in Fig. 3.15. The comparison between the analytical result and BEM result is shown in Fig. 3.16. We find that the BEM solution matches the analytical solution.

Fig. 3.12 Unidirectional flow domain

Fig. 3.13 Unidirectional flow force field

Fig. 3.14 Unidirectional flow free-space velocity

Fig. 3.15 Unidirectional flow velocity vectors

Fig. 3.16 Unidirectional flow u velocity comparison

Convergence Discussion

To test the number of the grids for convergence, we present a calibration case with linear body force in the center region of the lower half domain as seen in Fig. 3.17. The velocity vectors for this calibration case with 720 internal subdomains and 804 boundary elements are plotted in Fig. 3.18. We can find the velocity field is consistent with the force field. We plot the free-space u velocity at $x = 0$ with 720 internal subdomains and 1280 internal subdomains in Fig. 3.19. From this, we can see the results are converged when we use 720 internal subdomains. Then we plot the final u velocity at $x = 0.3625$ for the case with 804 boundary elements and 979 boundary elements in Fig. 3.20. In this case, we use 720 internal subdomains. We can get the conclusion that when we use 804 boundary elements, the results are converged.

Fig. 3.17 Body force for convergence testing case

Fig. 3.18 Velocity vectors for convergence testing case

Fig. 3.19 Free-space u velocity at x=0 for convergence testing case

Fig. 3.20 The u velocity at x=0.3625 for coverage testing case

Conclusion

With all these equations and examples above, I can get the conclusion that our BEM solvers can solve all the governing equations for electrothermal system with high accuracy. In the following chapters, we will present the modeling results for different electrothermal systems. In chapter 4, we will talk about the closed box electrothermal system with two symmetric electrodes. In chapter 5, we will talk about the closed box electrothermal system with three electrodes. We will show the results of an electrothermal pump with two symmetric electrodes in chapter 6. Finally, in chapter 7, we will present an electrothermal pump system with four electrodes.

CHAPTER 4: TWO SYMMETRIC ELECTRODE CONFIGURATION

System Setup

The microfluidics system domain is a 1mm square box which is shown in Fig. 4.1. The two electrodes are 150 μm in width and 150 nm in thickness. The distance between the electrodes is 100 μm . The location of the two electrodes can also be seen in Fig. 4.1. In this modeling work, we use 804 boundary elements and 2000 internal subdomains. For the boundary condition, we apply negative 7 V electrical potential on the left electrode and positive 7 V on the right one. We have the electrical potential gradient $\left(\frac{\partial\phi}{\partial n} = 0\right)$ on the other part of the boundary. For thermal boundary condition, we fix the temperature of the electrodes at 298 K. On the other boundary part, we apply no heat flux $\left(\frac{\partial T}{\partial n} = 0\right)$ as the boundary condition. As this electrothermal system is closed, we apply no-slip and no-penetration for the velocity on the boundary. In this modeling, we have $\beta_1 = 1$ and $\beta_2 = 0$ because it is not necessary to apply the traction integral equations. The reason is that all the velocity on the boundary is known in this case.

Fig. 4.1 Microfluidics system domain with two symmetric electrodes

BEM Solution

After solving the dimensionless Laplace equation, we multiply the result by the common non-dimensional variable for electrical potential. All the parameter values we use in this thesis can be found in Table 2.2. Then we get the electrical potential for the electrothermal system in Fig. 4.2. We can see the electrical potential is anti-symmetric at symmetric x locations. The magnitude of the electrical potential is a little smaller than the boundary value because we plot the electrical potential for the internal domain. The next result we get is electrical field in Fig. 4.3. The electrical field has a source and sink flow behavior where the electrical field flows from the negative electrode to the positive electrode.

Fig. 4.2 Electrical potential with two symmetric electrodes

Fig. 4.3 Electrical field with two symmetric electrodes

The Joule heating is generated by the electrical field inside the fluid flow domain. The Joule heating field is shown in Fig. 4.4. As we know, Joule heating is the source term for the energy equation. Then the temperature field corresponding to Joule heating and

thermal boundary condition is plotted in Fig. 4.5. The temperature near the electrodes is lower because the electrode temperature is 298 K. And in the other region, the temperature is higher because of Joule heating. Then we can have the conclusion that the temperature field matches our BEM boundary condition.

Fig. 4.4 Joule heating with two symmetric electrodes

Fig. 4.5 Temperature field with two symmetric electrodes

Now we can talk about the force field inside the electrothermal system. In this modeling work, we have electrothermal force and buoyancy force. There are two kinds of electrothermal force. One is Coulomb force and the other one is dielectric force. Both of these two forces are generated by the electrical field and temperature gradient. The buoyancy force is directly proportional to the temperature. The Coulomb force is shown in Fig. 4.6. The dielectric force is plotted in Fig. 4.7. For the electrothermal force, the dominant region is close to the electrodes because in this region the electrical field and temperature gradient are both large. We show the buoyancy force in Fig. 4.8. Around the two electrodes, the buoyancy force is smaller because of lower temperature in that region. According to the parameter values we use in this thesis, the Coulomb force is dominant. So the total force field is displayed in Fig. 4.9.

Fig. 4.6 Coulomb force with two symmetric electrodes

Fig. 4.7 Dielectric force with two symmetric electrodes

Fig. 4.8 Buoyancy force with two symmetric electrodes

Fig. 4.9 Force field with two symmetric electrodes

Then we can apply the BEM solver for Stokes flow to get the fluid flow velocity inside the microfluidics system. The velocity vectors are shown in Fig. 4.9. There is circulation around the electrodes which is consistent with Ramos's work [1]. To get a better

description for the velocity near the electrodes, we plot the velocity vectors close to the electrodes in Fig. 4.10. The u velocity at some symmetric x locations is shown in Fig. 4.11. We can find that u velocity is anti-symmetric at symmetric x locations because we apply symmetric boundary conditions. Finally, the v velocity at different y locations is displayed in Fig. 4.12. The v velocity is symmetric along x direction. From the figure, we can see that the magnitude of the v velocity decreases when y increases. This result is consistent with our force field because the force is weak in the region far away from the electrodes. Also the order of the magnitude for the velocity is about $100 \text{ } \mu\text{m/s}$.

Fig. 4.10 Velocity vectors with two symmetric electrodes

Fig. 4.11 Velocity vectors around two symmetric electrodes

Fig. 4.12 The u velocity with two symmetric electrodes

Fig. 4.13 The v velocity with two symmetric electrodes

CHAPTER 5: THREE-ELECTRODE CONFIGURATION

System Setup

The microfluidics system with three-electrode configuration is 1 mm square closed box shown in Fig. 5.1. The left and right electrodes are 150 μm in width. The width of the middle electrode is 200 μm . All the electrodes are 150 nm in thickness. The distance between electrodes is 50 μm . The location of the electrodes can also be seen in Fig. 5.1. In this modeling work, we use 804 boundary elements and 2000 internal subdomains. We apply positive 7 V electrical potential on the middle electrode and negative 7 V electrical potential for the other two electrodes. The electrical potential gradient is zero ($\frac{\partial \phi}{\partial n} = 0$) on the other boundary parts. For the thermal boundary condition, we fix the electrodes at 298 K. We have no heat flux ($\frac{\partial T}{\partial n} = 0$) for the other boundary region. Similar to the two symmetric electrodes case, this electrothermal system is closed. Then we have both no-slip and no-penetration for velocity on the boundary. In this work, we choose $\beta_1 = 1$ and $\beta_2 = 0$ because we do not need to apply the traction integral equation for the case when all the velocity is known on the boundary.

Fig. 5.1 Microfluidics system domain with three electrodes

BEM Solution

The first equation we analyze is Laplace equation for electrical potential and electrical field. The electrical potential is plotted in Fig. 5.2. The result is consistent with the boundary condition we have for electrical potential. Then we have the electrical field in Fig. 5.3. We can see the electrical field flows from the negative electrical potential electrodes to the middle positive electrical potential electrode. And the electrical field is also symmetric along the x direction.

Fig. 5.2 Electrical potential with three electrodes

Fig. 5.3 Electrical field with three electrodes

The Joule heating is caused by the electrical field in the electrothermal domain. We plot the Joule heating field in Fig. 5.4. The Joule heating serves as the source term for the conduction equation. By solving the dimensionless Poisson equation and multiplying the

non-dimensional variables, we can get the temperature field in Fig. 5.5. The temperature around the the electrodes is lower because of the 298 K boundary condition on the electrodes. For the region far away from the electrodes, we have higher temperature which is caused by the Joule heating and no heat flux boundary condition.

Fig. 5.4 Joule heating with three electrodes

Fig. 5.5 Temperature field with three electrodes

With the electrical field and temperature field above, we can analyze the force inside the electrothermal system. Similarly, we have electrothermal force and buoyancy force. The electrothermal force includes Coulomb force and dielectric force. Both of these two forces are computed by the electrical field and temperature gradient of the system. The buoyancy force is generated by the temperature difference along the vertical direction. The Coulomb force is shown in Fig. 5.6. The dielectric force is displayed in Fig. 5.7. We plot the buoyancy force in Fig. 5.8. According to the parameter values we use in this modeling work, the Coulomb force is dominant. The total force field is shown in Fig. 5.9. We can also find that the force is symmetric in the x direction.

Fig. 5.6 Coulomb force with three electrodes

Fig. 5.7 Dielectric force with three electrodes

Fig. 5.8 Buoyancy force with three electrodes

Fig. 5.9 Force field with three electrodes

The next step is to apply the BEM Stokes solver to analyze the fluid flow inside the system. The velocity vectors are plotted in Fig. 5.10. We see big circulation above the left and right electrodes going down in the center. This is consistent with the work done by S

Loire in [14]. It is interesting to see there is small circulation going up above the middle electrode. To get a clear figure for the circulation, we plot the velocity vectors around the electrodes in Fig. 5.11. For some details about the velocity, the u velocity at symmetric x locations is plotted in Fig. 5.12. We can get the conclusion that the u velocity is anti-symmetric at symmetric x locations. Then we show the v velocity at different y locations in Fig. 5.13. The magnitude of the v velocity decreases when y increases because the force in the region away from the electrodes is small.

Fig. 5.10 Velocity vectors with three electrodes

Fig. 5.11 Velocity vectors around three electrodes

Fig. 5.12 The u velocity with three electrodes

Fig. 5.13 The v velocity with three electrodes

CHAPTER 6: ELECTROTHERMAL PUMP WITH TWO ELECTRODES

System Setup

As one of the application for AC electrothermal flow, the electrothermal pump is an interesting topic for both simulation and experimental works [10]. To design a pump, we need to make the total force asymmetric so that the mass flow rate will not be zero. According to the electrothermal force equation (Eq. 2.9), it is clear that if we fix a temperature difference between two electrodes, the total electrothermal force will drive the fluid flow inside the channel as a pump. In this chapter, we will show the results of designing a pump with two electrodes in electrothermal system. The microfluidics system is a long channel from (0,0) to (5,1) as shown in Fig. 6.1. In this case, we have two electrodes on the bottom surface. The width of the electrodes is 100 μm . The distance between the electrodes is 100 μm . We neglect the thickness of the electrodes. In this modeling work, we apply 600 boundary elements and 2000 internal subdomains. We apply positive 7 V electrical potential on the left electrode and negative 7 V electrical potential on the right one. On the other part boundary, we have electrical gradient equals to zero ($\frac{\partial \phi}{\partial n} = 0$). For thermal boundary condition, we fix the left electrode at 299 K and right electrode at 297 K to create a temperature difference along x direction. The heat flux is zero ($\frac{\partial T}{\partial n} = 0$) on other boundary parts. On the top and bottom surface, we have no-slip and no-penetration boundary conditions for velocity. As Reynolds number is small in this case, we assume the fluid flow is fully developed at the inlet and outlet. Based on this, we have $\frac{\partial u}{\partial x} = 0$, $v = 0$ and $P = P_\infty$ for the inlet and outlet of the channel.

In this electrothermal pump modeling, we have $\beta_1 = 1$ and $\beta_2 = 1$ to combine the velocity integral equation and traction integral equation.

Fig. 6.1 Electrothermal pump domain with two electrodes

BEM Solution

The first result we present is the electrical potential for the two electrode pump in Fig. 6.2. The electrical potential is consistent with our boundary condition. Then we can get the electrical field for this case in Fig. 6.3. The electrical field flows from the right electrode to the left electrode. The electrical field is symmetric in the x direction.

Fig. 6.2 Electrical potential in two electrode pump

Fig. 6.3 Electrical field in two electrode pump

Joule heating field which is created by electrical field is shown in Fig. 6.4. The Joule heating is the source term for the heat conduction equation. Then the temperature field is

plotted in Fig. 6.5. We can see that there is a temperature drop between two electrodes corresponding to the boundary conditions.

Fig. 6.4 Joule heating in two electrode pump

Fig. 6.5 Temperature field in two electrode pump

In this pump, we only consider the electrothermal force. The electrothermal force is created by electrical field and temperature gradient inside the system. The electrothermal force includes Coulomb force and dielectric force. The Coulomb force is displayed in Fig. 6.6. While the dielectric force is shown in Fig. 6.7. According to the parameter values we use in this electrothermal pump modeling, the Coulomb force is dominant. The total force field is shown in Fig. 6.8.

Fig. 6.6 Coulomb force in two electrode pump

Fig. 6.7 Dielectric force in two electrode pump

Fig. 6.8 Force field in two electrode pump

The next step is to solve the velocity for the electrothermal pump. The velocity vectors are shown in Fig. 6.9. We can see clockwise circulation above the left electrode and counterclockwise circulation around the right electrode. The velocity vectors are

consistent with the force field. To get more details about the velocity field, we plot u velocity at different x locations in Fig. 6.10. At $x=0.05$ mm and $x=4.95$ mm, the u velocity is parabolic and match with each other because the fluid flow is fully developed at the inlet and outlet. The order of the maximum velocity in this case is about 100 $\mu\text{m/s}$.

Fig. 6.9 Velocity vectors in two electrode pump

Fig. 6.10 The u velocity at different x locations in two electrode pump

CHAPTER 7: ELECTROTHERMAL PUMP WITH FOUR ELECTRODES

System Setup

The microfluidics system domain for this modeling case is shown in Fig. 7.1. The four electrodes can be seen in the figure. The width of the electrodes is 100 μm . And the distance between the electrodes is also 100 μm . We apply positive 7 V electrical potential on the left two electrodes and negative 7 V electrical potential on the right two electrodes. Similar as previous chapters, we have electrical potential gradient equals to zero ($\frac{\partial \phi}{\partial n} = 0$) on the other boundary parts. We fix the temperature of two left electrodes at 299 K and the two right electrodes at 297 K. For the other boundary parts, there is no heat flux ($\frac{\partial T}{\partial n} = 0$). For the velocity boundary condition, we assume the fluid flow is fully developed at the inlet and outlet of the channel. Under this assumption, we have $\frac{\partial u}{\partial x} = 0$, $v = 0$ and $P = P_\infty$ for the inlet and outlet boundary parts. We apply no-slip and no-penetration for velocity on the top and bottom surface. In this modeling work, we use 600 boundary elements and 2000 internal subdomains. We need to combine the velocity integral equation and traction integral equation by choosing $\beta_1 = 1$ and $\beta_2 = 1$ to model this electrothermal pump.

Fig. 7.1 Electrothermal pump domain with four electrodes

BEM Solution

The electrical potential inside the electrothermal system is shown in Fig. 7.2. There is an electrical potential drop between the left electrodes and right electrodes. The result is consistent with the boundary conditions for electrical potential. The electrical field is plotted in Fig. 7.3. The electrical field flows from the right electrodes to the left electrodes. In this case, the electrical field is symmetric in both x and y directions because of the symmetric boundary conditions.

Fig. 7.2 Electrical potential in four electrode pump

Fig. 7.3 Electrical field in four electrode pump

The Joule heating is generated by the electrical field inside the fluid flow domain. The Joule heating, the source term for the energy equation, is plotted in Fig. 7.4. We can see that the Joule heating is also symmetric in both x and y directions. Then we show the

temperature field in Fig. 7.5. Corresponding to the boundary condition, we have a temperature drop between the left electrodes and right electrodes.

Fig. 7.4 Joule heating in four electrode pump

Fig. 7.5 Temperature field in four electrode pump

Then we consider the electrothermal force inside the system. There are two kinds of electrothermal force for the electrothermal pump. One is Coulomb force and the other one is dielectric force. The Coulomb force is shown in Fig. 7.6. In Fig. 7.7, we plot the dielectric force. Using all the parameter values in Table 2.2, the Coulomb force is dominant. The total force field is displayed in Fig. 7.8.

Fig. 7.6 Coulomb force in four electrode pump

Fig. 7.7 Dielectric force in four electrode pump

Fig. 7.8 Force field in four electrode pump

The velocity vectors for the electrothermal pump with four electrodes are shown in Fig. 7.9. We can find that there are circulations around the two left electrodes. To get more information about the velocity field, we plot the u velocity at different x locations in Fig. 7.10. At $x=0.05$ mm and $x=4.95$ mm, the u velocity is parabolic and match with each other. It is consistent with our boundary condition for fully developed fluid flow. By comparing with the velocity for a two electrode pump, we can see that the mass flow rate increases because we have two more electrodes and body forces become larger.

Fig. 7.9 Velocity vectors in four electrode pump

Fig. 7.10 The u velocity at different x locations in four electrode pump

Electrode Configuration Analysis

To get more details about the electrothermal pump system design, we want to compare the volumetric flow rate between two electrodes case and four electrodes case. Besides, we will vary the electrode width or the distance between electrodes. First, we keep the right terminal of the left electrode at $x=2$ mm and the left terminal of the right electrode at $x=3$ mm for both two and four electrodes case. So the distance between the electrodes is fixed at 1 mm. Then we vary all the electrode width simultaneously. We plot the volumetric flow rate versus the electrode width in both two electrodes and four electrodes cases in Fig. 7.11. From this figure, the volumetric flow rate increases with increasing electrode width. However, when the electrode width is large enough, the volumetric flow rate will not have obvious increase by using longer electrode. The volumetric flow rate with four electrodes is almost two times of that with two electrodes.

Fig. 7.11 The electrothermal pump volumetric flow rate with different electrode width

We plot the volumetric flow rate in the AC electrothermal pump with different distance between electrodes in Fig. 7.12. In this case, we fix all the electrode width as 1mm. All the electrodes' locations are symmetric along the centerline ($x=2.5$ mm). The volumetric flow rate decreases when we increase the electrode distance. The decrease in volumetric flow rate is more quickly when the distance between electrodes is small. Similarly, it is clear that the volumetric flow with four electrodes is almost twice of two electrodes case.

Fig. 7.12 The electrothermal pump volumetric flow rate with different electrode distance

CHAPTER 8: DISCUSSION AND CONCLUSION

Conclusion

In this thesis, we apply the boundary element method (BEM) to model the 2D electrothermal flow. As well known, BEM is a powerful approach to solve the partial differential equations with arbitrary boundary shape. However, the Green's function integrals become singular or near singular when the source points approach the field points. To overcome this problem, our unique computation is to solve all these integrals analytically. By doing this, we can get accurate BEM solvers for all the governing equations of electrothermal flow. Some numerical examples are presented to demonstrate the accuracy of our BEM solutions. Then we design the electrothermal box with two electrodes and three electrodes. All the physical parameters for the electrothermal system are shown. And these results match well with our boundary conditions and previous work. It is useful and interesting to design a electrothermal pump. By combining the velocity integral equation and the traction integral equation in BEM for Stokes flow, we can solve the velocity field for an open channel. So two interesting electrothermal pump systems are designed and presented in this thesis. Based on all the above, we can get the conclusion that our numerical simulation work for 2D electrothermal system is accurate and useful.

Future Work

As the fluid flow in microfluidics system is a very interesting research topic attracting the attention of researchers from all over the world, the modeling work for designing the electrothermal system is necessary. The 2D numerical modeling of the electrothermal

system is limited in some cases when the microfluidics system can not be assumed as 2D case. So we should complete the work for 3D numerical simulation of the electrothermal microfluidics system. Then we can model and understand more interesting cases in this research topic.

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