A Geostatistical Inverse Method for Variably Saturated Flow in the Vadose Zone

by

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ABSTRACT

A geostatistical inverse technique utilizing both primary and secondary information is developed to estimate conditional means of unsaturated hydraulic conductivity parameters (saturated hydraulic conductivity and pore-size distribution parameters) in the vadose zone. Measurements of saturated hydraulic conductivity and pore-size distribution parameters are considered as the primary information, while measurements of steady-state flow processes (soil-water pressure head and degree of saturation) are regarded as the secondary information. This inverse approach relies on the classical linear predictor (cokriging) theory and takes the advantage of the spatial cross-correlation between soil-water pressure head, degree of saturation, saturated hydraulic conductivity, and pore-size distribution parameter. Using an approximate perturbation solution for steady, variably saturated flow under general boundary conditions, the cross-covariances between the primary and secondary information are derived. The approximate solution is formulated based on a first-order Taylor series expansion of a discretized finite element equation. The sensitivity matrix in the solution is evaluated by an adjoint state sensitivity approach for flow in heterogeneous media under variably saturated conditions. Through several numerical examples, the inverse model demonstrates its ability to improve the estimates of the spatial distribution of saturated hydraulic conductivity and pore-size distribution parameters using the secondary information.

INTRODUCTION

The cokriging technique has been widely adopted to estimate transmissivity, head, velocity, and concentration of pollutants in aquifers (*Kitanidis and Vomvoris*, 1983; *Hoeksema and Kitanidis*, 1984 and 1989; *Rubin and Dagan*, 1987; *Gutjahr and Wilson*, 1989; *Harvey and Golerick*, 1995; *Yeh et al.*, 1995 and 1996). It has also been applied to estimate water content in the vadose zone, using water content, soil-water pressure head, soil surface temperature, and soil texture data sets (e.g., *Vauclin et al.*, 1983; *Yates and Warrick*, 1987; and *Mulla*, 1988). However, little attention has been directed toward the application of this method to the inverse problem in the vadose zone (i.e., estimation of unsaturated hydraulic conductivity parameters, using soil-water pressure head and water content data sets.)

Predicting water and solute movements in the vadose zone with a reasonable accuracy requires a large number of measurements of the unsaturated hydraulic conductivity (e.g., *Yeh and McCord*, 1994, and *Yeh*, 1995). Unlike saturated hydraulic conductivity, the hydraulic conductivity of unsaturated porous media is a nonlinear function of soil-water pressure head or moisture content. Due to this nonlinearity, measurements of the unsaturated hydraulic conductivity are thus a difficult, time-consuming and costly task. Subsequently, characterization of the vadose zone by direct measurements of the hydraulic conductivity at a large number of locations becomes a formidable task. On the other hand, information about flow processes such as soil-water pressure head and water content data sets can be collected with relative ease in most of shallow and unconsolidated vadose zones with inexpensive tools. Water content data sets may, however, be the only information that can be collected in large quantities in the thick vadose zone of the Western region of United States.

Poorly sorted alluvial deposits, conglomerates, and solid rock masses comprising the vadose zone in the region often prohibit the use of pressure measurement devices. As a result, it seems logical to take the advantage of the abundance of the information about flow processes to improve our estimates of unsaturated hydraulic properties in the field. In fact, *Harter and Yeh* (1995b) recently demonstrated that using cokriging and a numerical model that a large amount of soil-water pressure head measurements can greatly improve the prediction of movement of solutes in the vadose zone. While their finding is promising, the method is limited to unit mean gradient, uniform flow, and stationary conditions which may not be suitable for application to a wide range of field problems.

In this paper, a flexible geostatistical inverse technique that can utilize both soil-water pressure and water content data sets is developed to estimate the approximate conditional mean unsaturated hydraulic conductivity parameters in the vadose zone. The technique is flexible in the sense that it can be applied to nonuniform flow and nonstationary problems under field conditions. To demonstrate the ability of the proposed methodology and the benefit of using soil-water pressure head and water content to estimate the unsaturated hydraulic conductivity parameters, several numerical experiments were carried out. Explanations for the improvement on the estimates due to the use of both soil-water pressure and water content data sets are provided.

MATHEMATICAL FORMULATIONS

Governing Flow Equations

Steady-state flow in two-dimensional heterogeneous porous media under variably saturated conditions can generally be described by

$$\frac{\partial}{\partial x_i} \left[K(\psi) \frac{\partial (\psi + x_2)}{\partial x_i} \right] = 0 \quad i = 1, 2 \quad (1)$$

with specified boundary conditions:

$$\begin{aligned} \psi &= \psi^* & \text{on } \Gamma_1 \\ q_i \cdot n_i &= q^* & \text{on } \Gamma_2 \end{aligned} \quad (2)$$

where x_1 and x_2 are horizontal and vertical coordinates (positive upward), respectively. ψ is the soil-water pressure head and is positive for saturated flow and negative for unsaturated flow. ψ^* is prescribed head on boundary Γ_1 , and q^* is the prescribed flux normal to boundary Γ_2 . $K(\psi)$ is the unsaturated hydraulic conductivity (assumed locally isotropic), which varies with ψ under unsaturated conditions. In this study, the Gardner-Russo model (Russo, 1988) is used to describe the relationship between K and ψ . That is,

$$K(\psi, \mathbf{x}) = K_s(\mathbf{x}) \exp[\alpha(\mathbf{x}) \psi(\mathbf{x})] \quad (3)$$

where $\mathbf{x}$ is the position vector, $\{x_1, x_2\}$, K_s is the saturated hydraulic conductivity, and α is a pore-size distribution parameter. The relationship between water content and soil-water pressure head is described by the following function.

$$\Theta = \frac{\theta - \theta_r}{\theta_s - \theta_r} = e^{-0.5\alpha|\psi|} (1 + 0.5\alpha|\psi|)^{\frac{2}{m+2}} \quad (4)$$

where Θ is degree of saturation or effective saturation, θ is the moisture content, θ_s and θ_r are

saturated and residual moisture contents, respectively, and m is a soil parameter which accounts for the tortuosity of the flow path and the correlation between pores.

Geostatistical Inverse Approach

Assume that the natural log of saturated hydraulic conductivity $\ln K_s$ and pore-size distribution coefficient $\ln \alpha$ are stochastic processes with means $E[\ln K_s] = F(x)$ and $E[\ln \alpha] = A(x)$ and perturbations $f(x)$ and $a(x)$, respectively. Similarly, the soil-water pressure ψ and effective saturation Θ are also considered as stochastic processes that can be expressed as $\psi(x) = H(x) + h(x)$ and $\Theta(x) = S(x) + s(x)$, where $H(x) = E[\psi(x)]$ and $S(x) = E[\Theta(x)]$ are the means and $h(x)$ and $s(x)$ are the perturbations, respectively. Suppose we have n_f observed saturated hydraulic conductivities $f(x_i)$, where $i = 1, 2, \dots, n_f$, n_a observed pore-size distribution coefficient $a(x_j)$, where $j = 1, 2, \dots, n_a$, n_h soil-water pressure measurements, $h(x_k)$, where $k = 1, 2, \dots, n_h$, and n_s sampled effective saturation $s(x_l)$, where $l = 1, 2, \dots, n_s$. We want to estimate f and a at locations where no samples are available. In the stochastic concept, one desirable estimate is the fields of f and a that represent the realizations conditioned on the measurements. The other is the expected value of all the possible realizations of f and a fields that are conditioned on the measurements of f , a , h , and s (conditional means). Theoretically, the former will have an infinite number of possibilities and the latter is unique. Our geostatistical inverse approach is intended to derive the conditional mean fields.

If we assume that the distributions of f , a , h , and s are jointly normal, the conditional mean estimates of $f(x)$ and $a(x)$ at unsampled locations x_o can be expressed by the linear combination of the weighted observed values of f , a , h , and s (assuming f and a are uncorrelated).

$$\begin{aligned}
f_{co}(\mathbf{x}_0) &= \sum_{i=1}^{n_f} P_{fi} f(\mathbf{x}_i) + \sum_{k=1}^{n_h} Q_{fk} h(\mathbf{x}_k) + \sum_{l=1}^{n_s} R_{fl} s(\mathbf{x}_l) \\
a_{co}(\mathbf{x}_0) &= \sum_{j=1}^{n_a} P_{aj} a(\mathbf{x}_j) + \sum_{k=1}^{n_h} Q_{ak} h(\mathbf{x}_k) + \sum_{l=1}^{n_s} R_{al} s(\mathbf{x}_l)
\end{aligned} \tag{5}$$

In (5), f_{co} and a_{co} are the cokriged values of f and a at the unsampled location, $\mathbf{x}_0$. P_{fi} , Q_{fk} , R_{fl} are cokriging coefficients for estimation of f with respect to the measurements of f , h , and s ; P_{aj} , Q_{ak} , R_{al} are cokriging coefficients for the estimate of a with respect to the samples of a , h , and s . The i , j , k , and l are the indices for observed f , a , h , and s , respectively. These cokriging coefficients are selected in such a way that the estimations expressed by (5) will have the minimal variances, or

$$\begin{aligned}
E[(f_{co} - f^*)^2] &= \text{Minimum} \\
E[(a_{co} - a^*)^2] &= \text{Minimum}
\end{aligned} \tag{6}$$

where f^* and a^* are the unknown true values of f and a at the unsampled location.

Substituting (5) into (6), taking the derivatives of (6) with respect to the coefficients, and setting the resultants to zero leads to two sets of the cokriging system of equations. The cokriging system of equations for calculating the cokriging coefficients related to saturated hydraulic conductivity, P_{fi} , Q_{fk} , and R_{fl} , are

$$\begin{aligned}
\sum_{mi=1}^{n_f} P_{fi} C_{ff}(\mathbf{x}_i, \mathbf{x}_{mi}) + \sum_{mk=1}^{n_h} Q_{fk} C_{fh}(\mathbf{x}_i, \mathbf{x}_{mk}) + \sum_{ml=1}^{n_s} R_{fl} C_{fs}(\mathbf{x}_i, \mathbf{x}_{ml}) &= C_{ff}(\mathbf{x}_i, \mathbf{x}_0) \\
\sum_{mi=1}^{n_f} P_{fi} C_{hf}(\mathbf{x}_k, \mathbf{x}_{mi}) + \sum_{mk=1}^{n_h} Q_{fk} C_{hh}(\mathbf{x}_k, \mathbf{x}_{mk}) + \sum_{ml=1}^{n_s} R_{fl} C_{hs}(\mathbf{x}_k, \mathbf{x}_{ml}) &= C_{hf}(\mathbf{x}_k, \mathbf{x}_0) \\
\sum_{mi=1}^{n_f} P_{fi} C_{sf}(\mathbf{x}_l, \mathbf{x}_{mi}) + \sum_{mk=1}^{n_h} Q_{fk} C_{sh}(\mathbf{x}_l, \mathbf{x}_{mk}) + \sum_{ml=1}^{n_s} R_{fl} C_{ss}(\mathbf{x}_l, \mathbf{x}_{ml}) &= C_{sf}(\mathbf{x}_l, \mathbf{x}_0)
\end{aligned} \tag{7}$$

The cokriging coefficients associated with pore-size distribution parameter, P_{al} , Q_{ak} and R_{al} , can be derived by solving the other set of the system of equations,

$$\begin{aligned}
 \sum_{mj=1}^{n_a} P_{ai} C_{aa}(\mathbf{x}_j, \mathbf{x}_{mj}) + \sum_{mk=1}^{n_h} Q_{ak} C_{ah}(\mathbf{x}_j, \mathbf{x}_{mk}) + \sum_{ml=1}^{n_s} R_{al} C_{as}(\mathbf{x}_j, \mathbf{x}_{ml}) &= C_{aa}(\mathbf{x}_j, \mathbf{x}_0) \\
 \sum_{mj=1}^{n_a} P_{ai} C_{ha}(\mathbf{x}_k, \mathbf{x}_{mj}) + \sum_{mk=1}^{n_h} Q_{ak} C_{hh}(\mathbf{x}_k, \mathbf{x}_{mk}) + \sum_{ml=1}^{n_s} R_{al} C_{hs}(\mathbf{x}_k, \mathbf{x}_{ml}) &= C_{ha}(\mathbf{x}_k, \mathbf{x}_0) \\
 \sum_{mj=1}^{n_a} P_{ai} C_{sa}(\mathbf{x}_l, \mathbf{x}_{mj}) + \sum_{mk=1}^{n_h} Q_{ak} C_{sh}(\mathbf{x}_l, \mathbf{x}_{mk}) + \sum_{ml=1}^{n_s} R_{al} C_{ss}(\mathbf{x}_l, \mathbf{x}_{ml}) &= C_{sa}(\mathbf{x}_l, \mathbf{x}_0)
 \end{aligned} \quad (8)$$

where C_{ff} , C_{aa} , C_{hh} , and C_{ss} are the auto-covariance functions of f , a , h , and s , respectively, and the others C_{pq} ($p, q = f, a, h, \text{ or } s$) are cross-covariance functions between p and q .

The cokriging variances associated with these estimates are given by

$$\begin{aligned}
 \Sigma_f &= C_{ff}(\mathbf{x}_0, \mathbf{x}_0) - \sum_{i=1}^{n_f} P_{fi} C_{ff}(\mathbf{x}_i, \mathbf{x}_0) - \sum_{k=1}^{n_h} Q_{fk} C_{fh}(\mathbf{x}_k, \mathbf{x}_0) - \sum_{l=1}^{n_s} R_{fl} C_{fs}(\mathbf{x}_l, \mathbf{x}_0) \\
 \Sigma_a &= C_{aa}(\mathbf{x}_0, \mathbf{x}_0) - \sum_{j=1}^{n_a} P_{aj} C_{aa}(\mathbf{x}_j, \mathbf{x}_0) - \sum_{k=1}^{n_h} Q_{ak} C_{ah}(\mathbf{x}_k, \mathbf{x}_0) - \sum_{l=1}^{n_s} R_{al} C_{as}(\mathbf{x}_l, \mathbf{x}_0)
 \end{aligned} \quad (9)$$

where Σ_f and Σ_a denote the cokriging variance of $\ln K_s$ and α estimates, respectively.

Theoretically, if all the variables are jointly normal, the above approach should produce the conditional mean estimates and conditional variances. Using Monte Carlo simulation, *Yeh* (1989) showed that under one-dimensional steady-state infiltration h is normally distributed. *Harter and Yeh* (1995a) demonstrated that under two-dimensional flow conditions, h and f and a are likely to be jointly normal. However, their results were derived from numerical simulations of uniform flow in random porous media and the validity of these results remains to be tested in the field. To be conservative, we say our estimates are approximate conditional mean fields and variances.

Evaluation of Covariances and Cross-Covariances

In order to determine the cokriging coefficients in (7) and (8), auto- and cross-covariance functions of the random variables f , a , h , and s have to be known. The auto-covariance functions of f and a fields, in general, can be estimated from the field data set. The cross-covariance of f , a , h , and s , and the auto-covariance functions of h and s for unsaturated flow can be obtained either by: (1) statistical analysis of the observed data set or results from Monte Carlo simulations; (2) first-order perturbation approximation of stochastic flow governing equation via spectral/perturbation technique (e.g., *Yeh et al. 1985 a and b* ; and *Harter and Yeh, 1995a*); or (3) the combination of first-order Taylor series approximation of flow equation and sensitivity analysis (e.g., *Dettinger and Wilson, 1981*; *Hoeksema and Kitanidis, 1984*).

Statistical analysis of the sampled data requires a large number of samples of f , a , h , and s in order to estimate the cross-covariance and covariance functions. The estimated functions may, however, not be positive definite and this can cause numerical problems in solving (7) and (8). Functions derived from Monte Carlo simulations also suffer from the same mathematical difficulties. On the other hand, the covariance and cross-covariance functions derived from the spectral analysis are free from this problem but they are limited to infinite domains (without the effects of boundaries) as well as stationary random processes. To alleviate the problem associated with the spectral technique, an approach based on the first-order Taylor series approximation of the numerical model of (1) is used to evaluate these covariance functions. Similar to the spectral approach, this numerical approach is, however, limited to small variation in the hydraulic parameters due to the first-order approximation. The development of the numerical approach is given as follows.

If we choose a finite element method to discretize (1), it can be written in a matrix form

$$[G]\{\psi\} = \{B\} \quad (10)$$

where $[G]$ is the coefficient matrix, $\{B\}$ is the vector associated with boundary conditions, and $\{\psi\}$ is the solution vector. Further, $\{\psi\}$ can be written as

$$\{\psi\} = [G]^{-1}\{B\} \quad (11)$$

Assume $\{F\}$ and $\{A\}$ are the vectors of the mean values of log saturated hydraulic conductivity and pore-size distribution parameter, $\{F\} = E[\{\ln K_s\}]$ and $\{A\} = E[\{\ln \alpha\}]$, and $\{H\}$ is the vector of soil-water pressure evaluated at $\{F\}$ and $\{A\}$. Let $\{f\}$ and $\{a\}$ be the perturbation vectors of $\{\ln K_s\}$ and $\{\ln \alpha\}$, and $\{h\}$ be the deviation of soil-water pressure from $\{H\}$. Further, if we assume that the magnitudes of perturbation vectors $\{f\}$ and $\{a\}$ are small, we may approximate the soil-water pressure head $\{\psi\}$ with a multivariate first-order Taylor series approximation of (11) at $\{F\}$ and $\{A\}$:

$$\begin{aligned} \{\psi\} &= \{H\} + \{h\} = g(\{F\} + \{f\}, \{A\} + \{a\}) \\ &\approx g(\{F\}, \{A\}) + \left[\frac{\partial \psi}{\partial \ln K_s} \right]_{(\{F\}, \{A\})} \{f\} + \left[\frac{\partial \psi}{\partial \ln \alpha} \right]_{(\{F\}, \{A\})} \{a\} \end{aligned} \quad (12)$$

where g represents the matrix representation of (11). After subtracting the expected value of (12) from (12) itself, the first-order approximation of the perturbation of soil-water pressure can be written as

$$\{h\} \approx \left[\frac{\partial \psi}{\partial \ln K_s} \right]_{(\{F\}, \{A\})} \{f\} + \left[\frac{\partial \psi}{\partial \ln \alpha} \right]_{(\{F\}, \{A\})} \{a\} = [J^{(h,f)}] \{f\} + [J^{(h,a)}] \{a\} \quad (13)$$

where matrices $[J^{(hf)}]$ and $[J^{(ha)}]$ are the sensitivity matrices of $\{h\}$ with respect to $\{f\}$ and $\{a\}$ evaluated at the mean values $\{F\}$ and $\{A\}$, respectively.

Performing some simple statistical operations using (13), the covariance function of $\{h\}$ and cross-covariance functions between $\{h\}$ and $\{f\}$ and $\{a\}$ can be approximated as

$$\begin{aligned} [C_{fh}] &\approx [J^{(hf)}] [C_{ff}] \\ [C_{ah}] &\approx [J^{(ha)}] [C_{aa}] \\ [C_{hh}] &\approx [J^{(hf)}] [C_{ff}] [J^{(hf)}]^T + [J^{(ha)}] [C_{aa}] [J^{(ha)}]^T \end{aligned} \quad (14)$$

where $[J]^T$ is the transpose of the sensitivity matrix $[J]$, and $[C_{ff}]$ and $[C_{aa}]$ are the specified auto-covariances of log saturated hydraulic conductivity $\ln K_s$ and log pore-size distribution coefficient $\ln \alpha$, respectively.

In the same way, the sensitivity matrices of effective saturation $\{\Theta\}$ with respect to $\{\ln K_s\}$ and $\{\ln \alpha\}$ are defined as

$$[J^{(sf)}] = \left[\frac{\partial \Theta}{\partial \ln K_s} \right] \quad [J^{(sa)}] = \left[\frac{\partial \Theta}{\partial \ln \alpha} \right] \quad (15)$$

Thus, the approximate cross-covariance functions between $\{f\}$, $\{a\}$, and $\{s\}$ are

$$\begin{aligned} [C_{fs}] &\approx [J^{(sf)}] [C_{ff}] \\ [C_{as}] &\approx [J^{(sa)}] [C_{aa}] \\ [C_{ss}] &\approx [J^{(sf)}] [C_{ff}] [J^{(sf)}]^T + [J^{(sa)}] [C_{aa}] [J^{(sa)}]^T \end{aligned} \quad (16)$$

and the cross-covariance function between $\{s\}$ and $\{h\}$ can be expressed as

$$[C_{hs}] \approx [J^{(sf)}] [C_{ff}] [J^{(hf)}]^T + [J^{(sa)}] [C_{aa}] [J^{(ha)}]^T \quad (17)$$

Notice that the above cross-covariance and covariance functions related to flow processes (soil-water pressure head h and effective saturation s) may vary with the locations and are not necessarily a function of separation distance only. Such a nonstationary behavior in the covariances can be attributed to the fact that the flow process is nonstationary since the flow domain considered in this study is finite and bounded.

Adjoint sensitivity analysis

To evaluate the sensitivity matrices of $[J^{(hf)}]$, $[J^{(ha)}]$, $[J^{(sf)}]$, and $[J^{(sa)}]$, the adjoint sensitivity theory is employed. Adjoint state approach has been widely applied to groundwater flow problems (e.g., *Neuman*, 1980; *Sykes et al.*, 1985; *Sun and Yeh*, 1992). Its application to unsaturated flow problems, however, has been limited. Here we followed the approach for saturated flow developed by *Sykes et al.* (1985).

In adjoint sensitivity theory, equation (1) with boundary conditions (2) is considered the “primary” problem. If the flow domain of the primary problem is discretized into M elements with L nodal points, $\ln K_s$ and $\ln \alpha$ are defined at every element, and ψ is the soil-water pressure at some locations x_n in the domain ($n=1,2,\dots,N$), two N by M sensitivities matrices (i.e., the corresponding derivatives) of ψ with respect to $\ln K_s$ and $\ln \alpha$ can be defined as

$$J^{(hf)}(n,m) = \frac{\partial \psi_n}{\partial (\ln K_s)_m} \quad J^{(ha)}(n,m) = \frac{\partial \psi_n}{\partial (\ln \alpha)_m} \quad (18)$$

To evaluate (18), we choose a performance measure:

$$P = \int_{\Omega} \mathcal{F}(\{\beta\}, \phi) d\Omega \quad (19)$$

where Ω is the flow domain bounded by boundary Γ , $\mathcal{F}(\{\beta\}, \phi)$ is a user-chosen function of system state ϕ and system parameters vector $\{\beta\}$. In this study, system state is the total hydraulic head $\phi = \psi + x_2$, and system parameters $\{\beta\}$ are $\ln K_s$, $\ln \alpha$, prescribed head boundary condition ϕ^* , and prescribed flux boundary conditions q^* .

The sensitivity of the performance measure P to any specific parameter β_k is defined as $dP/d\beta_k$ and can be expressed as:

$$\frac{dP}{d\beta_k} = \int_{\Omega} \left[\frac{\partial \mathcal{F}(\{\beta\}, \phi)}{\partial \beta_k} + \frac{\partial \mathcal{F}(\{\beta\}, \phi)}{\partial \phi} \gamma \right] d\Omega \quad (20)$$

where $\gamma = d\phi/d\beta_k$ is the sensitivity of the total head to parameter β_k and is called “state sensitivity”.

This state sensitivity can be evaluated by solving a new differential equation (sensitivity equation) derived by differentiating (1) with respect to a specific parameter β_k :

$$\frac{\partial}{\partial x_i} \left[\frac{\partial K}{\partial \beta_k} \frac{\partial \phi}{\partial x_i} + K \frac{\partial \gamma}{\partial x_i} \right] = 0 \quad i = 1, 2 \quad (21)$$

The boundary conditions associated with (21) are

$$\begin{aligned} \gamma &= \frac{\partial \phi^*}{\partial \beta_k} && \text{on } \Gamma_1 \\ \frac{\partial q^*}{\partial \beta_k} n_i + \frac{\partial q^*}{\partial h} \gamma n_i &= \frac{\partial q^*}{\partial \beta_k} && \text{on } \Gamma_2 \end{aligned} \quad (22)$$

Sensitivity equation (21) has a similar structure to its original equation (1). With boundary conditions (22), equation (21) can be solved numerically to obtain the state sensitivity γ . For each specific parameter β_k , sensitivity equation (21) has to be solved once to evaluate the sensitivity of performance measure P to the parameter β_k . If the system includes a large number of parameters β_k as in our analysis, this approach is very costly and time consuming (*Kabala and Milly, 1990*). Alternatively, we can avoid the direct evaluation of state sensitivity by formulating adjoint equations of the partial differential equations for γ .

Multiplying (21) by an arbitrary differentiable function λ , and integrating over the flow domain Ω gives,

$$\int_{\Omega} \left[\lambda \frac{\partial}{\partial x_i} \left(K \frac{\partial \gamma}{\partial x_i} \right) + \lambda \frac{\partial}{\partial x_i} \left(\frac{\partial K}{\partial \beta_k} \frac{\partial \Phi}{\partial x_i} \right) \right] d\Omega = 0 \quad (23)$$

Applying Green's first identity to (23), then adding the resultant to the sensitivity of performance measure (20) yields,

$$\begin{aligned} \frac{dP}{d\beta_k} = & \int_{\Omega} \left\{ \frac{\partial \mathcal{F}(\{\beta\}, \Phi)}{\partial \beta_k} + \gamma \left[\frac{\partial \mathcal{F}(\{\beta\}, \Phi)}{\partial \Phi} + \frac{\partial}{\partial x_i} \left(K \frac{\partial \lambda}{\partial x_i} \right) \right] - \frac{\partial \lambda}{\partial x_i} \frac{\partial K}{\partial \beta_k} \frac{\partial \Phi}{\partial x_i} \right\} d\Omega \\ & + \int_{\Gamma} \left[\gamma K \frac{\partial \lambda}{\partial x_i} n_i + \lambda \frac{\partial q_1}{\partial \beta_k} \right] d\Gamma \end{aligned} \quad (24)$$

To eliminate the unknown state sensitivities γ in (24), an arbitrary function λ is chosen to satisfy the following equation and boundary conditions, known as the adjoint problem:

$$\frac{\partial \mathcal{F}(\{\beta\}, \psi)}{\partial \psi} + \frac{\partial}{\partial x_i} \left(K \frac{\partial \lambda}{\partial x_i} \right) = 0 \quad (25)$$

associated with boundary conditions,

$$\begin{aligned} \lambda &= 0 && \text{on } \Gamma_1 \\ K \frac{\partial \lambda}{\partial x_i} n_i &= 0 && \text{on } \Gamma_2 \end{aligned} \quad (26)$$

To evaluate the sensitivity matrices of total hydraulic head with respect to $\ln K_s$ and $\ln \alpha$, we define $\beta_k = \ln K_s$ and $\beta_k = \ln \alpha$, respectively, and choose the user-defined function of (19) as

$$\mathcal{F}(\{\beta\}, \phi) = \phi \delta(\mathbf{x} - \mathbf{x}_p) \quad (27)$$

Therefore, the adjoint sensitivity equation (25) becomes

$$\frac{\partial}{\partial x_i} \left(K \frac{\partial \lambda}{\partial x_i} \right) = -\delta(\mathbf{x} - \mathbf{x}_p) \quad (28)$$

where $\delta(\mathbf{x} - \mathbf{x}_p)$ is the Dirac delta function, and $\mathbf{x}_p$ is the location where the sensitivity of head to changes of parameters is to be calculated. As equation (28) is similar to primary problem (1) with the exception that K is known and independent of λ and the right hand side is the Dirac delta function instead of zero, (28) is linear and can be solved numerically just like saturated flow problem.

With the adjoint function (28) associated with boundary conditions (26), the sensitivity performance measure (20) can be determined by evaluating

$$\frac{\partial \psi_p}{\partial \beta_k} = \int_{\Omega} \left[-\frac{\partial \lambda}{\partial x_i} \frac{\partial K}{\partial \beta_k} \frac{\partial \phi}{\partial x_i} \right] d\Omega \quad (29)$$

Notice that $dP/d\beta_k = d\phi(\mathbf{x}_p)/d\beta_k = d\psi(\mathbf{x}_p)/d\beta_k$ because $P = \phi(\mathbf{x}_p) = \psi_l(\mathbf{x}_p) + x_2$. Compared with directly solving state sensitivity equation of (21), the major advantage of adjoint sensitivity analysis is that for different β_k , the adjoint equation needs to be solved only once for a specific user-chosen function $\mathcal{F}$.

If the system parameter β_k denotes the log saturated hydraulic conductivity in the element k , or $(\ln K_s)_k$, and the exponential model (3) is used, then

$$J^{(hf)}(p, k) = \frac{\partial \psi_p}{\partial (\ln K_s)_k} = - \sum_{l=1}^M u_l \int_{\Omega_l} \frac{\partial \lambda}{\partial x_i} \frac{\partial \phi}{\partial x_i} d\Omega \quad (30)$$

where l is the element index, M is the total number of elements, and Ω_l is the domain of element l .

In addition, u_l is

$$u_l = \frac{\partial K_l}{\partial (\ln K_s)_k} = K_l [\delta_{kl} + \alpha_l \frac{\partial \psi_l}{\partial (\ln K_s)_k}] \quad (31)$$

where K_l and α_l are the unsaturated conductivity and pore-size distribution coefficient of element l , respectively. δ is the Kronecker delta ($\delta_{kl}=1$ for $k=l$ and 0 otherwise).

If we define the integral in (30) as

$$S(p, l) = - \int_{\Omega_l} \frac{\partial \lambda}{\partial x_i} \frac{\partial \phi}{\partial x_i} d\Omega \quad (32)$$

in which the adjoint function λ is associated with the pressure head ψ_p located at $\mathbf{x}_p$, $S(p, l)$ can be evaluated numerically with the known λ and ϕ distributions in the flow domain. Therefore, (32) can

be rewritten as

$$\sum_{l=1}^M [\delta_{pl} - K_l \alpha_l S(p, l)] J^{(h_f)}(l, k) = -K_k S(p, k) \quad p = 1, 2, \dots, M \quad (33)$$

The sensitivity of soil-water pressure ψ_l with respect to $\ln K_s$ can then be obtained by solving matrix equation (33) which contains M unknowns $J(p, k)$, ($p=1, 2, \dots, M$, and k is fixed) with M equations. Notice that in (33), the matrix is independent of the element index k . Hence, the matrix has to be evaluated only once and the solution can be used directly to determine $J(p, k)$ for other elements.

Similarly, if the system parameter β_k is defined as log pore-size distribution coefficient on element k , that is, $\beta_k = (\ln \alpha)_k$, the matrix equation of the sensitivities of soil-water pressures in all elements with respect to $\ln \alpha$ of one element k , $J^{(h_a)}(p, k) = \partial \psi_l / \partial (\ln \alpha)_k$ can be obtained as

$$\sum_{l=1}^M [\delta_{pl} - K_l \alpha_l S(p, l)] J^{(h_a)}(l, k) = -K_k \alpha_k \psi_k S(p, k) \quad p = 1, 2, \dots, M \quad (34)$$

After sensitivity matrices of soil-water pressure with respect to $\ln K_s$ and $\ln \alpha$ are derived, sensitivity matrices of effective saturation Θ with respect to $\ln K_s$ and $\ln \alpha$ can be evaluated easily as follows.

The degree of saturation expressed by (4) can be simplified by taking m as zero (Russo, 1988), that is,

$$\Theta = \frac{\theta - \theta_r}{\theta_s - \theta_r} = e^{0.5 \alpha \psi} (1 - 0.5 \alpha \psi) \quad (35)$$

Then, taking derivatives of Θ with respect to $\ln K_s$ and $\ln \alpha$ yields

$$\begin{aligned}
\frac{\partial \Theta_i}{\partial (\ln K_s)_j} &= -0.25 \alpha_i^2 \psi_i e^{0.5 \alpha_i \psi_i} \frac{\partial \psi_i}{\partial (\ln K_s)_j} \\
\frac{\partial \Theta_i}{\partial (\ln \alpha)_j} &= -0.25 \alpha_i^2 \psi_i e^{0.5 \alpha_i \psi_i} \left[\psi_i \delta_{ij} + \frac{\partial \psi_i}{\partial (\ln \alpha)_j} \right]
\end{aligned} \tag{36}$$

Once the sensitivities $\partial \psi_i / \partial (\ln K_s)_j$ and $\partial \psi_i / \partial (\ln \alpha)_j$ are evaluated, the sensitivities $\partial \Theta_i / \partial (\ln K_s)_j$ and $\partial \Theta_i / \partial (\ln \alpha)_j$ can be calculated using (36). Subsequently, all of the sensitivity matrices of $[J^{(h)}]$, $[J^{(h)}$ ^a], $[J^{(sf)}]$, and $[J^{(sa)}]$ can be obtained.

NUMERICAL EXAMPLES

As mentioned previously, detailed characterization of a vadose zone is a difficult and expensive task. Consequently, few field experiments have been conducted in the past to provide sufficient data sets for testing any inverse models. As a result, we have to rely on numerical experiments to test our geostatistical inverse approach. The numerical experiments involved steady-state flows in a 7x7 meters vertical plane of a hypothetical vadose zone. The left and right boundaries of the flow domain were defined as impermeable, and the lower boundary was considered as a prescribed head boundary, while the upper boundary condition varied for different flow scenarios. The flow domain is discretized uniformly into a 35x35 finite elements with $dx=dy=20$ cm. Each element is assigned a saturated hydraulic conductivity value and a pore-size distribution coefficient value created from a random field generator (*Gutjahr*, 1989) with a modified form of the Whittle spectrum (spectrum A, anisotropic case, *Mizell et al*, 1982). The variances of $\ln K_s$ and variances of $\ln \alpha$ are taken as 1.0 and 0.1, respectively, while the mean value of $\ln K_s$ is given as -3.0 and the mean of $\ln \alpha$ is 0.0. The correlation scales for both $\ln K_s$ and $\ln \alpha$ are set to be 100 cm and 300 cm in the

vertical and horizontal directions, respectively. Figures 1a and b show the hypothetical $\ln K_s$ and $\ln \alpha$ fields, respectively. Once the hypothetical vadose zone is generated, a finite element model (Yeh et al., 1993) is used to solve the primary flow problem (1) to obtain the pressure head and saturation fields. These $\ln K_s$, $\ln \alpha$, ψ , and Θ fields are then regarded as the real-world analogues (true fields) where measurements of these parameters are taken. Sixteen $\ln K_s$ and $\ln \alpha$ values ($n_f = n_a = 16$) were sampled at a 4x4 uniform grid over the entire domain as our primary information. The secondary information, pressure head values, was then taken from a 10x10 uniform grid, resulting in a total of 100 pressure head measurements ($n_h = 100$). Similarly, a total of 81 sampled Θ values ($n_s = 81$) were obtained from a 9x9 uniform sampling grid.

Three flow cases, Case 1: nonuniform flow A, Case 2: nonuniform flow B, and Case 3: uniform flow, associated with different upper boundary conditions, are considered here to investigate the usefulness of the primary and secondary information on the estimation of $\ln K_s$ and $\ln \alpha$ fields. In Case 1, the lower boundary is set to be water table, the central ten nodes of the upper boundary are constant heads (-100, -300, and -500 cm) while the remaining parts are no flux boundary. The true soil-water pressure head and flow fields for this case with upper boundary of -300 cm pressure head are shown in Figure 1c. Since the mean soil-water pressure head varies, the mean gradient changes with location. As a result, the h , and s fields are nonstationary (Yeh et al., 1985a).

In Case 2, the boundary conditions are the same as Case 1 except that the entire upper boundary is set to be a prescribed constant head. The mean soil-water pressure head field varies with depth. It becomes less negative as it approaches the water table and the h and s fields are nonstationary. In this case, three different prescribed head values (-100 cm, -300 cm, and -500 cm) for the upper boundary condition are also considered.

The upper boundary conditions in Case 3 were set to the same prescribed head values as the lower boundary conditions. The same as the previous cases, three different prescribed head values (-100, -300, and -500 cm) were used. The mean flow in this case is uniform in the most part of the domain with the hydraulic gradient of one (unit gradient case) except near the left- and right-hand side boundaries (no flow). Under this condition, the h and s fields away from the boundaries are essentially stationary.

For the above three cases, four different geostatistical estimation approaches are employed to estimate $\ln K_s$ and $\ln \alpha$ fields. Approach I, an ordinary kriging method using the primary information is used. In Approach II, our cokriging technique using the soil-water pressure information in addition to the $\ln K_s$ and $\ln \alpha$ measurements is employed. The usefulness of secondary information about the effective saturation on the $\ln K_s$ and $\ln \alpha$ estimates is, then, examined in Approach III. Finally, cokriging using both soil-water pressure and effective saturation as auxiliary information is performed to evaluate the improvements in the estimates in Approach IV.

To assess the performance of each approach under different flow conditions and wetness of the flow field, two quantitative measures are used. The first is the mean square error,

$$MSE = \frac{1}{N} \sum_{i=1}^N (y_{oi} - y_{ei})^2 \quad (37)$$

where y_{oi} and y_{ei} are the observed and estimated parameter values at the i -th location, respectively. N is the total number of elements for f . The second is the (co)kriged variance (9) which evaluate the ability of each approach in reducing the prediction variance in the ensemble sense.

RESULTS AND DISCUSSION

Figure 2 illustrates the estimated $\ln K_s$ and $\ln \alpha$ fields of Case 1 by the four different approaches. In this figure, rectangles represents the sample locations of $\ln K_s$ and $\ln \alpha$. Sample locations for h and s are denoted by circles and squares, respectively. As expected, the kriged $\ln K_s$ and $\ln \alpha$ fields based on the primary information only are much smoother than the true fields (Figures 1a and b) while the fields produced by the approaches using both primary and secondary information are more irregular and close to the real-world analog. While this figure provides a visual evaluation of the performance of the four approaches, Tables 1, 2, and 3 tabulate the quantitative assessment of their performances. The first rows of these tables list the values of the mean square errors and estimation variances for $\ln K_s$ and $\ln \alpha$ estimates based on kriging (Approach I). The percentage of the improvement due to the use of the geostatistical inverse technique along with different secondary information is listed in the other rows of the tables. This percentage represents the difference between the result of Approach I and the others (Approaches II, III, and IV) normalized by the result of Approach I. A positive value implies improvement and a negative value denotes deterioration. According to these tables, the geostatistical inverse approach, in general, provides better estimates than ordinary kriging in terms of the performance measures. The mean square errors of the $\ln K_s$ and $\ln \alpha$ fields by the inverse model are always smaller than those of ordinary kriging. Similarly, the estimation variances of the $\ln K_s$ and $\ln \alpha$ fields from the inverse approach are less than those of ordinary kriging.

More specifically, Approach II using the secondary information, h , improves the estimation of $\ln K_s$ in all cases but the improvement decreases as the soil becomes dry. Its impact on the estimation of $\ln \alpha$ is smaller than that on the $\ln K_s$ under wet conditions but becomes larger as the soil

becomes less saturated. Using the information about the effective saturation, s , Approach III dramatically enhances the estimation of $\ln \alpha$, especially in the relatively dry soils. Under wet conditions (e.g., $H = -100$ cm) the improvement due to the knowledge of s is not significant because some areas of the soil profile were saturated and the effective saturation no longer depends on the value of $\ln \alpha$. The effectiveness of Approach III on the estimation of $\ln K_s$ drops rapidly as the value of H becomes more negative. For all the cases, Approach IV using both the primary (f and a) and secondary information (h and s) performs much better than the others. In general, the flow field (uniform or nonuniform) does not significantly affect the geostatistical inverse approach.

The above numerical results are directly related to the cross-correlation between the primary and secondary information. To gain some insights into the behavior of the cross-correlation, analytical expressions of the cross-correlation functions between f and h , and a and h are derived. Since two-dimensional analytical results are difficult to obtain, the analysis assumes that the flow is one-dimensional and under steady-state and unit-mean-gradient condition, and the spectral/perturbation approach (Yeh et al., 1985) is used. The analytical expression for the cross-correlation function between f and h is

$$\rho_{fh}(\xi) = \frac{C_{fh}(\xi)}{[C_{ff}(0)C_{hh}(0)]^{1/2}} = \frac{\sigma_f}{\sqrt{(\sigma_f^2 + \sigma_a^2 H^2)}} e^{(\frac{\xi}{\lambda}) \left[\frac{A\lambda}{(A\lambda+1)} - \frac{\xi}{\lambda} \right]} \quad (38)$$

for $\xi > 0$. When $\xi < 0$,

$$\rho_{fh}(\xi) = \frac{\sigma_f}{\sqrt{(\sigma_f^2 + \sigma_a^2 H^2)}} \left\{ \left[\frac{A\lambda(A\lambda+1)}{(A\lambda-1)^2} - \frac{|\xi|(A\lambda+1)}{\lambda(A\lambda-1)} \right] e^{(-\frac{|\xi|}{\lambda})} - \frac{4A^2\lambda^2}{(A\lambda-1)^2(A\lambda+1)} e^{(-A|\xi|)} \right\} \quad (39)$$

Similarly, the cross-correlation function between a and h can be derived as

$$\rho_{ah}(\xi) = \frac{C_{ah}(\xi)}{[C_{aa}(0)C_{hh}(0)]^{1/2}} = \frac{-\sigma_a H}{\sqrt{(\sigma_f^2 + \sigma_a^2 H^2)}} e^{(-\frac{\xi}{\lambda})} \left[\frac{A\lambda}{(A\lambda+1)} - \frac{\xi}{\lambda} \right] \quad (40)$$

if $\xi > 0$, and

$$\rho_{ah}(\xi) = \frac{-\sigma_a H}{\sqrt{(\sigma_f^2 + \sigma_a^2 H^2)}} \left\{ \left[\frac{A\lambda(A\lambda+1)}{(A\lambda-1)^2} - \frac{|\xi|(A\lambda+1)}{\lambda(A\lambda-1)} \right] e^{(-\frac{|\xi|}{\lambda})} - \frac{4A^2\lambda^2}{(A\lambda-1)^2(A\lambda+1)} e^{(-A|\xi|)} \right\} \quad (41)$$

if $\xi < 0$. where ξ is the separation distance, λ is the correlation scale, σ_f and σ_a are the standard deviation of $\ln K_s$ and $\ln \alpha$, respectively.

If the soil is fully saturated (H and $A = 0$), the behavior of ρ_{fh} is then similar to the two-dimensional result by Mizell et al. (1982) which is independent of the mean pressure head and ρ_{ah} becomes zero. It is interesting that under unsaturated conditions, the absolute value of ρ_{fh} at any given separation distance decreases as the value of H becomes more negative (38 and 39). That is, the drier the soil is, the smaller the correlation between f and h is. Subsequently, measurements of h do not improve the estimate of $\ln K_s$ under dry conditions. On the other hand, the absolute value of ρ_{ah} increases as the soil becomes less saturated (40 and 41) and measurements of h improve the

estimate of $\ln \alpha$. As H becomes less negative (approaching zero) or the soil is near saturation, the cross-correlation between f and h increases, while the value of ρ_{ah} drops. As a result, measurements of h improve the estimate of $\ln K_s$ but not $\ln \alpha$. Formulas similar to (38-41) and (39) can also be derived for ρ_{fs} and ρ_{as} to explain the results of Approach III. However, they are far more complex and will not be presented in this paper. Plots of cross-correlation functions, ρ_{ah} , ρ_{fh} , ρ_{fs} and ρ_{as} , in x_1 and x_2 directions are shown in Figure 3 for the two-dimensional uniform flow case with $H=-100$ cm (solid line) and -500 cm (dashed line). In general, the cross-correlation value is higher in the direction of flow (x_2) than in the direction normal to the flow (x_1). It should be pointed out that the errors associated with the first-order approximation of these cross-covariance functions become large when the head perturbation grows as soil becomes less saturated (Yeh, 1989).

The geostatistical inverse model has its limitations. It may suffer numerical stability problems similar to those in the classical inverse models for saturated flow (see Yeh, 1986). Dietrich and Newsam (1989) pointed out that as the amount of available data increases and the discretization of the system is refined, both a numerically ill-conditioned parameter estimation problem and ill-conditioned cokriging equation may appear. Subsequently, they demonstrated that the cokriged transmissivity field in aquifers may contain some anomalies. To avoid this problem, addition of an error term to the cokriging equation is suggested for stabilizing the numerical solution in the method. But an addition of such an error term may result in the loss of information. In our analysis, the amount of primary and secondary information ($16 \times 2 + 100 + 81 = 213$) is relatively small compared to the total number of parameters ($1225 - 16 \times 2 = 1193$) to be estimated. The conditional number of the cokriging matrices seems large but no anomalies were observed in the estimated $\ln K_s$ and $\ln \alpha$ fields for all the cases examined. However, the stability problem may occur as the amount of

information increases.

As mentioned previously, the geostatistical inverse approach assumes a linear relationship between the primary and secondary information. Under fully saturated conditions, the linear assumption is valid as long as the variability of the primary variable is small (see *Yeh et al.*, 1995). However, the validity of the assumption may be further limited in the case of variably saturated flow due to the additional nonlinearity in the unsaturated hydraulic conductivity and water release curves. Theoretically, we do not know how to quantify the error associated with this assumption. It is, however, our conjecture that if the nonlinearity is incorporated in our estimation technique, more detailed spatial distributions of $\ln K_s$ and $\ln \alpha$ fields can be revealed. The linear assumption merely reduces the effectiveness of the secondary information. For depicting the general spatial patterns of $\ln K_s$ and $\ln \alpha$ fields, such a linear predictor serves as a useful and practical tool as demonstrated in our numerical examples.

Finally, while results of the numerical experiments appear interesting, application of this geostatistical inverse approach to field situations still remains to be explored. Differences in the sample volume of various sampling devices (e.g., tensiometers and neutron probes), methods for in-situ measurements of effective saturation, ... etc. are issues to be addressed. In addition, evaluation of the covariance functions of f and a may require a sufficient number of measurements and it will be affected by errors in the measurements (see *Russo and Jury*, 1987a and b). To alleviate this problem, a maximum likelihood approach proposed by *Kitanidis and Vomvoris* (1983) can be incorporated in our approach. In addition, unsaturated hydraulic property models other than the Gardner and Russo model (*Russo*, 1988) should be considered. Although other models such as the van Genuchten model for the unsaturated hydraulic properties can be implemented in our geostatistical approach

without difficulties, the computation burden will increase significantly and is beyond our capacity at this moment. Rapid advances in computational tools may alleviate this difficulty in the near future.

CONCLUSIONS

Based on the results of our studies, it appears that the geostatistical inverse approach using both the primary and secondary information is a promising tool for delineating detailed spatial distribution of unsaturated hydraulic heterogeneities. The estimates from the inverse approach are better than those derived from kriging based on the primary information only. Further, our results indicate that the degree of improvement on the estimates using either h or s will depend upon the wetness of the soil. This characteristic is different from that of the geostatistical inverse problem in the saturated aquifer. Finally, the usefulness of this approach under field conditions remains to be investigated. Carefully designed field experiments are needed.

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Table 1, Cokriging Results of Nonuniform Flow A (Case 1)

Head(cm)	Approach	MSE $\ln K_s$	MSE $\ln \alpha$	$\Sigma \ln K_s$	$\Sigma \ln \alpha$
	I (f,a)	0.3570	0.0454	0.4063	0.0406
H= -100	II (f,a,h)	45%	35%	41%	10%
	III (f,a,s)	36%	46%	15%	36%
	IV (f,a,h,s)	65%	71%	50%	47%
H= -300	II (f,a,h)	19%	46%	31%	18%
	III (f,a,s)	4%	72%	3%	46%
	IV (f,a,h,s)	34%	83%	47%	50%
H= -500	II (f,a,h)	10%	41%	26%	21%
	III (f,a,s)	1%	78%	0.5%	49%
	IV (f,a,h,s)	12%	80%	45%	50%

*** H is the constant soil-water pressure on the central part of the upper boundary

Table 2, Cokriging Results of Nonuniform Flow B (Case 2)

Head(cm)	Approach	MSE $\ln K_s$	MSE $\ln \alpha$	$\Sigma \ln K_s$	$\Sigma \ln \alpha$
	I (f,a)	0.3570	0.0454	0.4063	0.0406
H= -100	II (f,a,h)	55%	17%	48%	3%
	III (f,a,s)	62%	28%	30%	23%
	IV (f,a,h,s)	71%	33%	52%	42%
H= -300	II (f,a,h)	11%	53%	34%	15%
	III (f,a,s)	9%	73%	5%	45%
	IV (f,a,h,s)	49%	85%	49%	50%
H= -500	II (f,a,h)	5%	40%	27%	22%
	III (f,a,s)	1%	78%	1%	48%
	IV (f,a,h,s)	22%	83%	47%	50%

Table 3, Cokriging Results of Uniform Flow (Case 3)

Head(cm)	Approach	MSE $\ln K_s$	MSE $\ln \alpha$	$\Sigma \ln K_s$	$\Sigma \ln \alpha$
	I (f,a)	0.3570	0.0454	0.4063	0.0406
H= -100	II (f,a,h)	57%	10%	46%	4%
	III (f,a,s)	61%	30%	30%	26%
	IV (f,a,h,s)	77%	41%	52%	43%
H= -300	II (f,a,h)	6%	50%	26%	23%
	III (f,a,s)	34%	74%	10%	41%
	IV (f,a,h,s)	62%	85%	47%	50%
H= -500	II (f,a,h)	-17%	53%	14%	35%
	III (f,a,s)	19%	74%	4%	45%
	IV (f,a,h,s)	35%	78%	43%	52%

Fig. 1, True $\ln K_s$, $\ln \alpha$, and ψ fields and streamlines

Fig. 2 Estimated $\text{Ln}K_s$ and $\text{Ln}\alpha$ fields ($H=-300$ cm, case 1)

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Fig. 3, Cross-correlation functions for Case III