TRANSFORMATION OPTICS RELAY LENS DESIGN FOR IMAGING FROM A CURVED TO A
FLAT SURFACE

by

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Abstract

Monocentric lenses provide compact, broadband, high resolution, wide-field imaging. However, they produce a curved image surface and have found limited use. The use of an appropriately machined fiber bundle to relay the curved image plane onto a flat focal plane array (FPA) has recently emerged as a potential solution. Unfortunately the spatial sampling that is intrinsic to the fiber bundle relay can have a negative effect on image resolution, and vignetting has been identified as another potential shortcoming of this solution. This thesis describes a metamaterial lens yielding a high-performance image relay from a curved surface to a flat focal plane. Using quasi-conformal transformation optics, a Maxwell's fish-eye lens is transformed into a concave-plano shape. A design with a narrower range of constitutive parameters is deemed more likely to be manufacturable. Therefore, the way in which the particular shape of the concave-plano reimager influences the range of needed constitutive parameters is explored. Finally, image quality metrics, such as spot size and light efficiency, are quantified.

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Chapter 1 Problem Introduction

Spherically symmetric imaging systems are free from astigmatism and coma. Thus, spherically symmetric lenses provide better image resolution, for a given field of view, than lenses with planar focal surfaces. [1], for instance, compares the modulation transfer functions, or MTF's, of three lenses: a convex-plano lens, a Cooke triplet, and a ball lens. The three tested systems all have the same F/#, approximate size, and glass composition, but only the ball lens has spherical symmetry. The MTF is calculated over a range of spatial frequencies, over several optical wavelengths, for several field angles, for meridional and for sagittal rays. The ball lens' MTF's are the best overall. For instance, Table 1 lists the MTF's, as read from a plot in [1], corresponding to the spatial frequency of 68 cycles/mm. It includes the MTF's of on-axis rays, tangential rays at 40% of the instruments' field of view, tangential rays at 70% of the field of view, sagittal rays at 40% of the field of view, and sagittal rays at 70% of the field of view. The ball lens' MTF is superior at all tested field angles and in all tested propagation planes, except that the Cooke triplet performs as well for on-axis light.

MTF, 68 cycles/mm	Convex-Plano Lens	Cooke Triplet	Ball Lens
On-Axis	30%	60%	60%
Tangential, .4 FOV	0%	50%	60%
Tangential, .7 FOV	0%	58%	60%
Sagittal, .4 FOV	0%	40%	60%
Sagittal, .7 FOV	0%	25%	60%

Table 1: Comparison of MTF's of Different Lenses from [1]

The ball lens studied in [1] is fashioned out of a single kind of glass. A more advanced spherically symmetric lens, invented in 1859 [2], is the monocentric lens. It consists of

concentric glass shells, as diagrammed in Figure 1. The refractive index of the multiple layers can be chosen in order to minimize third-order geometric aberrations [3].

Figure 1: Monocentric Lens

Figure 1 diagrams a cross section of a monocentric lens. It is comprised of concentric, hemispherical surfaces. Ray bundles traced through the lens from field angles of 0° , 30° , and 60° are colored green, blue, and red respectively. The central physical stop is illustrated by the black rectangle. The dotted semicircle demarcates the image surface.

Despite the monocentric lens' image quality, its use in cameras has never been widespread. The problem with the monocentric lens, and spherically symmetric lenses in general, has been that there was not an economical way to record the hemispherical image. In the 1950's, for example, researchers tried to couple the curved focal surface to curved film [4]. This film, however, proved too difficult to produce and process. In the 1960's, scientists relayed the image to a detector via fiber bundles. Unfortunately, the relays suffered from too much cross talk [5]. Thus, for lack of a feasible method to capture the image, the use of monocentric lenses in cameras has been limited.

Recently, however, several promising ways to record the curved image have been proposed and prototyped. Since 1999, for instance, researchers have been investing in curved

digital sensor arrays [1] [6] [7] [8]. In 2013, researchers at Duke University implemented a camera with a monocentric lens coupled to micro camera relays [9], as illustrated in Figure 2. This camera achieves F/2.4 light collection, near-diffraction-limited resolution, and a full field of view of more than 120° [3]. Moreover, the stop is located in the micro cameras, eliminating the need for one in the monocentric lens itself, which causes vignetting. While large-scale manufacturers do not currently sell optics with this design, [8] anticipates that it may be used for military, commercial, and civilian applications in the future. Additionally, in 2013, at the University of California, San Diego, researchers demonstrated a camera with a monocentric lens that transmitted the image to focal plane arrays via multimode fiber bundles. This setup is illustrated in Figure 3. Unlike the ones that failed in the 1960's, the fiber bundles in this design adequately preserved the quality of the image because their core refractive index was much higher and thereby substantially reduced cross-talk. Because of the success of this prototype, it is conceivable that this design may also be used by wide-field camera manufacturers in the future [5].

Figure 2: Monocentric Lens Coupled to Micro Cameras

Micro cameras tiled near the focal surface capture the monocentric lens' image. The stop is located within the micro cameras.

Figure 3: Monocentric Lens Coupled to Fiber Bundles

Waveguides transport the hemispherical image to planar detector arrays.

Nevertheless, these promising reimager designs have drawbacks. The micro camera setup is complex and expensive [5]. Moreover, in order to ensure that the fields of view of the individual micro cameras overlap with each other, light from any particular field angle is

distributed between two or three micro cameras, which reduces the system's collection power [3]. In the fiber bundle design, each waveguide needs a diameter of at least $2\text{ }\mu\text{m}$ to reject cross talk [10]; thus, the optic's spot size cannot have a diameter finer than $2\text{ }\mu\text{m}$. Furthermore, in order to attain this maximum resolution, the pitch of the sensor pixels must be matched to the pitch of the fibers. In this case, the boundaries between fibers and the borders between sensor pixels coincide, and moiré patterns result [10]. On top of these problems, the coupling is inefficient and lossy, particularly at the points on the focal surface where the fiber's axis of symmetry is not aligned as well with the direction of propagation of the incident light. While image processing can improve some of these defects [11], some image quality is inevitably lost.

Because of these shortcomings, this thesis analyzes an alternative way to reimage the hemisphere: a lens with a concave object surface and a planar image surface, as illustrated in Figure 4. This lens is made with metamaterials, the constitutive parameters of which are determined by applying quasi-conformal transformation optics, or QCTO, to Maxwell's fish-eye lens. This metamaterial solution potentially preserves the image quality of the monocentric lens better than the fiber bundles and micro cameras. However, the lens would require anisotropic permeability, which is still beyond the scope of current metamaterial technology for optical wavelengths [12]. Still, the field of metamaterials is rapidly advancing; should such technology be developed, this solution could prove useful. This thesis also investigates the influence of the particular shape of the reimager on the range of values of required for constitutive parameters. The narrower the range of permittivity (ϵ) and permeability (μ), the more probable it is that the reimager can be built without resonant structures [13], which severely limit bandwidth and cause absorptive and dissipative losses. Then, the performance of the a reimager is evaluated.

Figure 4: Monocentric Lens Coupled to a Concave-Plano Reimager

The spherical image plane of the monocentric lens is imaged onto a flat detector plane.

The main contributions of this thesis are as follows. First, it puts forth the idea of a concave-plano metamaterial lens designed with transformation optics as a solution to the monocentric lens' reimaging problem. In support of this idea, it provides an example of a design of a particular concave-plano reimager for a monocentric lens with a 12-mm radius. In particular, it provides plots of the permittivity and permeability tensor values as functions of space that would be needed to implement it, as well as a ray trace of the meridional rays propagating through it to verify that it has sufficiently small spot sizes. This specific design was also detailed in [14]. Another contribution of this thesis is the idea that the specifics of the shapes of lenses designed with the quasi-conformal transformation optics design technique can be chosen to minimize the range of values of the permittivity and permeability prescriptions. In this paper, the effects of the particulars of the shape of the concave-plano reimager on the range of values of the constitutive parameters is studied, and it is shown that certain shapes yield a narrower range of constitutive parameter values than others. This result is potentially useful because the narrower the range of permittivity and permeability values, the more likely the device is to be implemented without resonant structures or to be able to be manufactured at all.

Chapter 2 Overview of Transformation Optics

Before describing the specific design, a basic summary of transformation optics (TO) and quasi-conformal transformation optics (QCTO) is presented. The reimager is designed based on this theory.

2.1 General Transformation Optics

Transformation optics is based on the following theorem:

Let there be a known solution to Maxwell's equations specified by: $\mathbf{E}(\mathbf{x}, t)$, the electric vector field; $\mathbf{H}(\mathbf{x}, t)$, the magnetic vector field; $\varepsilon(\mathbf{x})$, the relative permittivity tensor; and $\mu(\mathbf{x})$, the relative permeability tensor, where $\mathbf{x} \equiv (x, y, z)$, and all material is optically linear and temporally invariant. Let $\mathcal{T}_{x \rightarrow x'}(\mathbf{x})$ be a differentiable operator on $\mathbb{R}^3$, with Jacobian matrix $J_{x \rightarrow x'}(\mathbf{x})$. Let $\mathcal{T}_{x' \rightarrow x}(\mathbf{x}') \equiv \mathcal{T}_{x \rightarrow x'}^{-1}(\mathbf{x}')$, with Jacobian matrix $J_{x' \rightarrow x}(\mathbf{x}')$. Then, $\mathbf{E}'(\mathbf{x}', t)$, $\mathbf{H}'(\mathbf{x}', t)$, $\varepsilon'(\mathbf{x}')$, and $\mu'(\mathbf{x}')$ constitute another solution to Maxwell's equations, where:

$$\mathbf{E}'(\mathbf{x}', t) = J_{x \rightarrow x'}^{-1}(\mathbf{x}') \times \mathbf{E}(\mathcal{T}_{x' \rightarrow x}(\mathbf{x}'), t) \quad (1)$$

$$\mathbf{H}'(\mathbf{x}', t) = J_{x \rightarrow x'}^{-1}(\mathbf{x}') \times \mathbf{H}(\mathcal{T}_{x' \rightarrow x}(\mathbf{x}'), t) \quad (2)$$

$$\varepsilon'(\mathbf{x}') = \frac{J_{x \rightarrow x'}(\mathcal{T}_{x' \rightarrow x}(\mathbf{x}')) \times \varepsilon(\mathcal{T}_{x' \rightarrow x}(\mathbf{x}')) \times J_{x \rightarrow x'}(\mathcal{T}_{x' \rightarrow x}(\mathbf{x}'))^T}{|J_{x \rightarrow x'}(\mathcal{T}_{x' \rightarrow x}(\mathbf{x}'))|} \quad (3)$$

$$\mu'(\mathbf{x}') = \frac{J_{x \rightarrow x'}(\mathcal{T}_{x' \rightarrow x}(\mathbf{x}')) \times \mu(\mathcal{T}_{x' \rightarrow x}(\mathbf{x}')) \times J_{x \rightarrow x'}(\mathcal{T}_{x' \rightarrow x}(\mathbf{x}'))^T}{|J_{x \rightarrow x'}(\mathcal{T}_{x' \rightarrow x}(\mathbf{x}'))|} \quad (4)$$

Simply stated, if $\mathbf{E}(\mathbf{x}, t)$, $\mathbf{H}(\mathbf{x}, t)$, $\varepsilon(\mathbf{x})$, and $\mu(\mathbf{x})$ satisfies Maxwell's equations, (1)-(4) constitutes another solution to Maxwell's equations. A proof of this theorem appears in the appendix of [15].

In the TO algorithm, a material is designed using this theorem. (3) and (4) are used to determine the constitutive parameters of a material so that it realizes the desired field distributions of (1) and (2).

The most well-known TO design is the invisibility cloak, presented in [16]. It bends incident light around an object. Then it returns the light to the trajectory the light would be on in the object's absence. The cloak thus makes the object invisible; it is effectively a mirage of free space. The material prescription for this cloak is obtained with the TO algorithm, in the following steps. First, a known solution to Maxwell's equations is chosen. Since the cloak's purpose is to make the light behave as though free space were in place of the object and cloak, free space is the appropriate starting solution, populated with any $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{H}(\mathbf{x}, t)$ that could exist there. For instance, a monochromatic plane wave could exist in free space: $\mathbf{E}(\mathbf{x}, t) = A_0 \cos(\omega t - k_0 x) \hat{\mathbf{z}}$ and $\mathbf{H}(\mathbf{x}, t) = \frac{A_0}{Z_0} \cos(\omega t - k_0 x) (-\hat{\mathbf{y}})$, where A_0 , ω , and k_0 are positive constants, and Z_0 is the impedance of free space. Secondly, a transformation is specified: $(x', y', z') = \mathcal{T}_{x \rightarrow x'}(x, y, z)$. For the case of the invisibility cloak, the chosen transformation, described by (5) and illustrated in Figure 5, maps all points inside the sphere of radius R_2 onto the spherical shell between radii R_1 and R_2 , where $R_1 < R_2$. The Jacobian of this $\mathcal{T}_{x \rightarrow x'}(x, y, z)$ is substituted into (3) and (4) in order to convert the permittivity and permeability of free space into the constitutive parameters of the invisibility cloak. The resulting fields through the cloak, $\mathbf{E}'(\mathbf{x}', t)$ and $\mathbf{H}'(\mathbf{x}', t)$, are then described by (1) and (2), using the original $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{H}(\mathbf{x}, t)$ and the chosen $\mathcal{T}_{x \rightarrow x'}(x, y, z)$. Figure 6a illustrates the trajectories of incident light rays through the starting material (free space), while Figure 6b shows the ray paths through the final cloak. In the free-space case, in Figure 6a, rays are straight lines. If an object were in the center of the sphere of radius R_1 , it would scatter the light. In the final scenario, on the other hand, no incident light enters this sphere, as the sphere does not belong to the range of $\mathcal{T}_{x \rightarrow x'}(\mathbf{x})$. The object does not interact with the light. Moreover, outside the sphere of radius R_2 , $\mathcal{T}_{x \rightarrow x'}(\mathbf{x})$ is the identity mapping. Hence $\mathbf{E}'(\mathbf{x}', t) = \mathbf{E}(\mathbf{x}', t)$ and $\mathbf{H}'(\mathbf{x}', t) = \mathbf{H}(\mathbf{x}', t) \forall \mathbf{x}': \|\mathbf{x}'\| > R_2$. An

observer thus sees the same light as would be there if free space were in the place of the cloak and object.

$$R' = \begin{cases} R_1 + \left(\frac{R_2 - R_1}{R_2} \right) \times R, & R < R_2 \\ R, & R > R_2 \end{cases} \quad (5)$$

Figure 5: Invisibility Cloak Transformation

The mapping used to create the invisibility cloak. The independent axis is the radial coordinate in the original space. The dependent axis is the radial coordinate in the final space. The polar angle and azimuth angle of a point is invariant under the transformation.

Figure 6a: Invisibility Cloak, Original Solution

Figure 6b: Invisibility Cloak, Final Solution

The original solution, depicted in Figure 6a, consists of light rays traveling straight, through a vacuum. The final solution, in Figure 6b, illustrates the new trajectories of the light rays.

2.2 Quasi-Conformal Transformation Optics

While a powerful design tool in theory, the practical implementation of TO has limitations. The permittivity and permeability prescribed by the TO design algorithm are generally both anisotropic, and, furthermore, often span a large range of values. Such materials might require expensive or rare metamaterials, or they might not be feasible at all. Quasi-conformal transformation optics (QCTO) was therefore introduced by [17] to find TO prescriptions that were more likely to be manufacturable.

2.2.1 QCTO Basic Theory

According to the theory underlying QCTO, isotropy and non-magnetism in an optical material designed by the TO algorithm is achievable under the following stringent conditions. First, the materials in the starting solution must be isotropic and non-magnetic. Secondly, the system must be “2-D”: in one particular transverse direction, $\hat{\mathbf{u}}_T$, the constitutive parameters must not vary, and light propagating in this way must be inconsequential to the system. For

instance, in an imaging system, light that flows across the optical axis, in parallel with the detector plane never forms part of the image and is thus inconsequential. Moreover, the transverse dimension, $span(\hat{\mathbf{u}}_T)$, and its orthogonal complement, $U \equiv span(\hat{\mathbf{u}}_T)^\perp$, must both be invariant under $\mathcal{T}_{x \rightarrow x'}(\mathbf{x})$. Most critically, $\mathcal{T}_{x \rightarrow x'}|_U(\mathbf{u})$, or $\mathcal{T}_{x \rightarrow x'}(\mathbf{x})$ restricted to U , must be conformal. As a final requirement, the Eikonal approximation must be valid, and the way that energy is divided between the electric field and the magnetic fields must be inconsequential to the device's performance. When the aforementioned conditions are satisfied, the optic can be implemented with a non-magnetic dielectric.

These special circumstances guarantee isotropy and non-magnetism in the TO design by way of the following logic. First, a transformation is conformal iff its Jacobian is, everywhere, a scaling and a rotation. Thus:

$$J_U(\mathbf{u}) = \sigma(\mathbf{u})\mathfrak{R}(\mathbf{u}) \forall \mathbf{u} \in U \quad (6)$$

where $J_U(\mathbf{u})$ is the Jacobian of $\mathcal{T}_{x \rightarrow x'}|_U(\mathbf{u})$; $\sigma: U \rightarrow \mathbb{R}^+$, and $\mathfrak{R}: U \rightarrow SO(2)$, where $SO(2)$ is the set of 2-D special orthogonal matrices. As a result, the full Jacobian of $\mathcal{T}_{x \rightarrow x'}(\mathbf{x})$ can be expressed in block form as:

$$J_{x \rightarrow x'}(\mathbf{x}) = \begin{bmatrix} J_U(P_U(\mathbf{x})) & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \sigma(P_U(\mathbf{x}))\mathfrak{R}(P_U(\mathbf{x})) & 0 \\ 0 & 1 \end{bmatrix} \quad (7)$$

in a basis in which the first two dimensions belong to U and the third is $span(\hat{\mathbf{u}}_T)$, and where $P_U(\mathbf{x})$ denotes the projection of $\mathbf{x}$ onto U . When this expression for the Jacobian is substituted into (3):

$$\varepsilon' = \varepsilon \frac{J_{x \rightarrow x'} J_{x \rightarrow x'}^T}{|J_{x \rightarrow x'}|} = \varepsilon \frac{\begin{pmatrix} \sigma^2 I_2 & 0 \\ 0 & 1 \end{pmatrix}}{\sigma^2} = \varepsilon \begin{pmatrix} I_2 & 0 \\ 0 & \frac{1}{\sigma^2} \end{pmatrix} = \begin{pmatrix} \varepsilon I_2 & 0 \\ 0 & \frac{\varepsilon}{|J_U|} \end{pmatrix} \quad (8)$$

where the following short-hand is used:

- $\varepsilon' \equiv \varepsilon'(\mathbf{x}')$
- $\varepsilon \equiv \varepsilon(\mathcal{T}_{x' \rightarrow x}(\mathbf{x}'))$
- $J_{x \rightarrow x'} \equiv J_{x \rightarrow x'}(\mathcal{T}_{x' \rightarrow x}(\mathbf{x}'))$
- $\sigma \equiv \sigma(P_U(\mathcal{T}_{x' \rightarrow x}(\mathbf{x}')))$
- $|J_U| \equiv |J_U(P_U(\mathcal{T}_{x' \rightarrow x}(\mathbf{x}')))|$
- $I_2 \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Hence, (3) reduces to (9); (4) is simplified to (10):

$$\varepsilon'(\mathbf{x}') = \begin{bmatrix} \varepsilon'_U(\mathbf{x}') I_2 & 0 \\ 0 & \varepsilon'_T(\mathbf{x}') \end{bmatrix} \quad (9)$$

$$\mu'(\mathbf{x}') = \begin{bmatrix} \mu'_U(\mathbf{x}') I_2 & 0 \\ 0 & \mu'_T(\mathbf{x}') \end{bmatrix} \quad (10)$$

where:

$$\varepsilon'_U(\mathbf{x}') = \varepsilon(\mathcal{T}_{x' \rightarrow x}(\mathbf{x}')) \quad (11)$$

$$\varepsilon'_T(\mathbf{x}') = \frac{\varepsilon(\mathcal{T}_{x' \rightarrow x}(\mathbf{x}'))}{|J_U(P_U(\mathcal{T}_{x' \rightarrow x}(\mathbf{x}')))|} \quad (12)$$

$$\mu'_T(\mathbf{x}') = \frac{1}{|J_U(P_U(\mathcal{T}_{x' \rightarrow x}(\mathbf{x}')))|} \quad (13)$$

$$\mu'_U(\mathbf{x}') = 1 \forall \mathbf{x}' \quad (14)$$

Furthermore, (9) and (10) can be modified in a way that eliminates their anisotropy and magnetism, without altering the paths of light flowing through the material. Specifically, $\varepsilon'_U(\mathbf{x}')$ and $\mu'_T(\mathbf{x}')$ can be recombined in the following way:

$$\varepsilon''_U(\mathbf{x}') = \varepsilon'_U(\mathbf{x}')\mu'_T(\mathbf{x}') = \frac{\varepsilon(\mathcal{T}_{x' \rightarrow x}(\mathbf{x}'))}{|J_U(P_U(\mathcal{T}_{x' \rightarrow x}(\mathbf{x}')))|} \quad (15)$$

$$\mu''_T(\mathbf{x}') = 1 \forall \mathbf{x}' \quad (16)$$

ε''_U replaces ε'_U in (9), and μ''_T takes the place of μ'_T in (10). The new, geometrical-optics-equivalent material therefore has the permittivity and permeability described by:

$$\varepsilon''(\mathbf{x}') = \begin{bmatrix} \varepsilon''_U(\mathbf{x}')I_2 & 0 \\ 0 & \varepsilon'_T(\mathbf{x}') \end{bmatrix} = \frac{\varepsilon(\mathcal{T}_{x' \rightarrow x}(\mathbf{x}'))}{|J_U(P_U(\mathcal{T}_{x' \rightarrow x}(\mathbf{x}')))|} I_3 \quad (17)$$

$$\mu''(\mathbf{x}') = \begin{bmatrix} \mu'_U(\mathbf{x}')I_2 & 0 \\ 0 & \mu''_T(\mathbf{x}') \end{bmatrix} = I_3 \forall \mathbf{x}' \quad (18)$$

where the tensor $I_3 \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Thus, the resulting optic is isotropic and non-magnetic. (19),

furthermore, gives this new optic's index of refraction, where n is the index in the starting solution:

$$n'(\mathbf{x}') = \frac{n(\mathcal{T}_{x' \rightarrow x}|_U(\mathbf{x}'))}{\sqrt{|J_U(\mathcal{T}_{x' \rightarrow x}|_U(\mathbf{x}'))|}} \quad (19)$$

Geometrical wave-fronts only depend on the index of refraction $n = \sqrt{\varepsilon\mu}$, not the permittivity ε or permeability μ individually. Therefore, this alteration does not change optical paths of light propagating in U in the Eikonal limit because it preserves the index of refraction experienced by all light propagating in U . Light propagating in U can be decomposed into a

transverse-electric (TE) component and a transverse-magnetic (TM) component. The TE light's electric field is oriented entirely in a $\pm \hat{\mathbf{u}}_T$ direction, while its magnetic field is oriented in U . The TM component's magnetic field is oriented in a $\pm \hat{\mathbf{u}}_T$ direction, while the vectors in its electric field are oriented in U . Hence, the index of refraction experienced by the TE component,

$n'_{TE}(\mathbf{x}') = \sqrt{\varepsilon'_T(\mathbf{x}')\mu'_U(\mathbf{x}')}$, is not affected by this alteration. Meanwhile, the index of refraction experienced by the TM component with the alteration, $n''_{TM}(\mathbf{x}')$, is the same as the index it would experience without it, $n'_{TM}(\mathbf{x}')$, since $n''_{TM}(\mathbf{x}') = \sqrt{\varepsilon''_U(\mathbf{x}')\mu''_T(\mathbf{x}')} = \sqrt{\varepsilon'_U(\mathbf{x}')\mu'_T(\mathbf{x}')} = n'_{TM}(\mathbf{x}')$.

On the other hand, the index of refraction experienced by light not propagating in U is not preserved. Its trajectory is in fact distorted. However, this light does not affect the “2-D” system.

2.2.2 Flattened Fish-eye

An example of such a 2-D QCTO design is the “flattened fish-eye” lens described in [18]. Illustrated in Figure 7, the starting point is a cylindrical Maxwell's fish-eye, an optic that images each surface point onto the diametrically opposite one, without any geometrical aberrations. Its index is described by (20), where $R = \sqrt{x^2 + y^2}$. In the flattened fish-eye design, a QCTO transformation straightens parts of the outer shell, as shown in Figure 8. The transformation is invariant in the z direction (out of the page). The result is an isotropic, non-magnetic lens that images its flat front surface onto its flat back surface. The flat front and back surfaces themselves have no optical power. This device might be employed to relay a planar image across some distance onto a flat detector.

Figure 7: Ray pencil traced through Maxwell's fish-eye lens

Figure 7 The black and white curves represent rays of light emitted from a point source on the surface. All rays launched into the cylinder and traveling in the x - y plane are focused onto the opposite side. The index is invariant in the z dimension (out of the page).

$$n_{\text{fish-eye}}(R) = \frac{n_0}{1 + \left(\frac{R}{R_0}\right)^2} \quad (20)$$

Figure 8: Fish-Eye Flattening

The transformation that flattens the cylindrical fish-eye. The circle represents the boundary of the original lens. The solid red and blue arcs are respectively mapped onto the dashed red and blue line segments.

2.2.3 3-D QCTO

While helpful for 2-D optics, QCTO unfortunately cannot be used to design dielectric-only 3-D systems, in which light propagates in all three dimensions. Nevertheless, in the special case of systems with axial symmetry, a QCTO method can still make the material much more manufacturable, albeit anisotropic. In this strategy, $\mathcal{T}_{x \rightarrow x'}(\mathbf{x})$ maps a point with cylindrical coordinates (ρ, ϕ, z) onto a point with cylindrical coordinates (ρ', ϕ', z') such that $(z', \rho') = \mathcal{T}_{x \rightarrow x'}|_U(z, \rho)$, where $\mathcal{T}_{x \rightarrow x'}|_U$ is a conformal transformation, and $\phi' = \phi$. Assuming the validity of the Eikonal approximation, the device can then be implemented with (21)-(24) [19]:

$$\varepsilon'_\rho(z', \rho') = \varepsilon'_z(z', \rho') = \varepsilon(\mathcal{T}_{x' \rightarrow x}|_U(z', \rho')) \times \left(\frac{\rho}{\rho'}\right)^2 \quad (21)$$

$$\varepsilon'_\phi(z', \rho') = \frac{\varepsilon(\mathcal{T}_{x' \rightarrow x}|_U(z', \rho'))}{|J_U(\mathcal{T}_{x' \rightarrow x}|_U(z', \rho'))|} \quad (22)$$

$$\mu'_\rho(z', \rho') = \mu'_z(z', \rho') = 1 \quad (23)$$

$$\mu'_\phi(z', \rho') = \frac{1}{|J_U(\mathcal{T}_{x' \rightarrow x}|_U(z', \rho'))|} \times \left(\frac{\rho'}{\rho}\right)^2 \quad (24)$$

In these equations, $\varepsilon'_\rho(z', \rho')$, $\varepsilon'_\phi(z', \rho')$, and $\varepsilon'_z(z', \rho')$ are the components of the permittivity tensor in the new optic, oriented in the radial, azimuthal, and longitudinal directions respectively; $\mu'_\rho(z', \rho')$, $\mu'_\phi(z', \rho')$, and $\mu'_z(z', \rho')$ are the corresponding components of the new optic's permeability tensor; $\varepsilon(z, \rho)$ is the isotropic permittivity of the original optic; ρ is the radial coordinate of the preimage of (z', ρ') under $\mathcal{T}_{x \rightarrow x'}|_U(z, \rho)$; and $|J_U(z, \rho)|$ denotes the determinant of the Jacobian of $\mathcal{T}_{x \rightarrow x'}|_U(z, \rho)$.

In the material described by (21)-(24), all light travelling in meridional planes, regardless of polarization, experiences a non-magnetic response with index of refraction described by (25):

$$n'(z', \rho') = \frac{n(\mathcal{T}_{x \rightarrow x'}|_U^{-1}(z', \rho'))}{\sqrt{|J_U(\mathcal{T}_{x \rightarrow x'}|_U^{-1}(z', \rho'))|}} \quad (25)$$

where $n(z, \rho)$ is the original index distribution. (25) can be compared with (19).

Unlike the case of a “2-D” optic, on the other hand, it is not acceptable for light propagating in the invariant $\pm \hat{\mathbf{u}}_r$ direction – the azimuth direction – to be distorted. Hence, the anisotropy and magnetism cannot be eliminated by an alteration to the constitutive parameters like the one implemented with (15) and (16). The “3-D” device therefore must be anisotropic and magnetic. Nevertheless, the 3-D QCTO prescription is still more likely to be manufacturable compared with general TO designs because only one component of the prescribed permeability tensor is not 1, and most magnetically-coupled metamaterials provide a magnetic response in only one direction [19].

An example of a prior application of 3-D QCTO is the flattened Luneburg lens. The Luneburg lens is a sphere. Its outer shell is an infinite conjugate. Far-off point sources in front of it image onto points on its back surface without geometrical aberration. In [19], its hemispherical back focal surface is flattened by 3-D QCTO, allowing it to be fit to a flat detector.

Even though the exact solutions to axially-symmetric designs from QCTO are anisotropic, a dielectric-only approximation may be acceptable in some cases. Such an approximation reportedly proved acceptable in the case of the flattened Luneburg lens of [20]. They neglected the magnetism in (24) and used the ϕ -component of the permittivity tensor in (

22) as an isotropic permittivity. While their lens happened to perform satisfactorily, this design strategy, in general, does not work because the skew rays do not follow the correct trajectories.

2.2.4 Using QCTO to Reshape an Optical System: Numerical Calculation of Index Using Grid Generation

Unlike general TO, QCTO cannot re-route light to arbitrary new trajectories – a conformal transformation does not generally exist under which a particular given set of new trajectories are images of the original ones. This inability, nevertheless, is not a problem in the particular application of reshaping optical systems, as in the flattening of Maxwell’s fish-eye, or the flattening of the back surface of the Luneburg lens, or the carpet cloak design of [17], which makes a region of free-space cover an object to be hidden. This application does not require any specific set of new trajectories – it merely requires that the ray bundles originating from points on the new object surface converge to points on the new image surface. This condition is indeed often attainable with a conformal mapping.

In order to reshape a lens with QCTO, a discrete set of input points and corresponding output points of the conformal function $(x', y') = \mathcal{T}_{x \rightarrow x'}|_U(x, y)$ are found numerically, by way of the following methodology. Initially, the boundary conditions are specified via two “curvilinear quadrilaterals” in U : Q and Q' . Curvilinear quadrilaterals are closed chains of four arbitrary arcs, such as the one in Figure 9. The left and right sides of Q define the object and image surfaces, respectively, of the original imager. Similarly, the left and right sides of Q' demarcate the object and image surfaces, respectively, of the new imager. Then, for each of these curvilinear quadrilaterals, and for any curvilinear quadrilateral in general, the arcs of which meet at right angles, there exists a unique *conformal module* $M \in \mathbb{R}^+$ such that the curvilinear quadrilateral is conformally equivalent to all rectangles of height-to-width ratio M . Moreover,

numerical grid generation techniques, as detailed in [21] and [22], can be used to calculate the preimage under such a mapping, from the curvilinear quadrilateral to a rectangle, of a set of points in a regularly-spaced grid. Figure 10a, for instance, illustrates an example of such grid points generated on and within a curvilinear quadrilateral Q . Figure 10b then shows the regularly-spaced grid that is the image of these grid points under the conformal transformation that maps Q onto the rectangle R . Thus, with these grid generation techniques, two $N_\xi \times N_\eta$ arrays of points, $(\hat{x}_1[i, j], \hat{y}_1[i, j])$ and $(\hat{x}'[i, j], \hat{y}'[i, j])$, are computed such that:

$$\mathcal{T}_{Q \rightarrow R}(\hat{x}_1[i, j], \hat{y}_1[i, j]) = ((i - 1) \cdot \Delta\xi, (j - 1) \cdot \Delta\eta) \quad (26)$$

$$\mathcal{T}_{Q' \rightarrow R'}(\hat{x}'[i, j], \hat{y}'[i, j]) = ((i - 1) \cdot \Delta\xi', (j - 1) \cdot \Delta\eta') \quad (27)$$

where $\mathcal{T}_{Q \rightarrow R}(x, y)$ and $\mathcal{T}_{Q' \rightarrow R'}(x', y')$ are conformal transformations. $\mathcal{T}_{Q \rightarrow R}(x, y)$ maps the sides of Q onto the corresponding sides of a rectangle R of unit-length base, the bottom left corner of which is the origin. $\mathcal{T}_{Q' \rightarrow R'}(x', y')$ maps the sides of Q' onto the corresponding sides of a rectangle R' of unit-length base, the bottom left corner of which is the origin. Then, from bilinear interpolation of these results, an array of preimage points and corresponding image points can be derived for the conformal transformation $\mathcal{T}_{x \rightarrow x'}|_U(x, y) = \{\mathcal{T}_{Q' \rightarrow R'}^{-1} \circ \mathcal{T}_{Q \rightarrow R}\}(x, y)$. For instance, the preimage under $\mathcal{T}_{x \rightarrow x'}|_U(x, y)$ of the point $(\hat{x}'[i, j], \hat{y}'[i, j])$ can be approximated by:

$$\begin{aligned} (\hat{x}[i, j], \hat{y}[i, j]) &\equiv \mathcal{T}_{x \rightarrow x'}|_U^{-1}(\hat{x}'[i, j], \hat{y}'[i, j]) = \mathcal{T}_{Q \rightarrow R}^{-1}((i - 1) \cdot \Delta\xi', (j - 1) \cdot \Delta\eta') \\ &\approx w_{11}(\hat{x}_1[i_a, j_b], \hat{y}_1[i_a, j_b]) + w_{12}(\hat{x}_1[i_a, j_{b+1}], \hat{y}_1[i_a, j_{b+1}]) \\ &\quad + w_{21}(\hat{x}_1[i_{a+1}, j_b], \hat{y}_1[i_{a+1}, j_b]) + w_{22}(\hat{x}_1[i_{a+1}, j_{b+1}], \hat{y}_1[i_{a+1}, j_{b+1}]) \end{aligned} \quad (28)$$

where:

- $i_a \equiv \text{floor}(a) + 1$
- $i_b \equiv \text{floor}(b) + 1$

- $w_{11} \equiv (a - (i_a - 1))(b - (i_b - 1))$
- $w_{12} \equiv (a - (i_a - 1))(i_b - b)$
- $w_{21} \equiv (i_a - a)(b - (i_b - 1))$
- $w_{22} \equiv (i_a - a)(i_b - b)$
- $a \equiv (i - 1) \frac{\Delta \xi'}{\Delta \xi}$
- $b \equiv (j - 1) \frac{\Delta \eta'}{\Delta \eta}$

In this way, discrete arrays of input and corresponding output points of $\mathcal{T}_{x \rightarrow x'}|_U(x, y) - (\hat{x}[i, j], \hat{y}[i, j])$ and $(\hat{x}'[i, j], \hat{y}'[i, j])$ – are computed.

Figure 9: "Curvilinear Quadrilateral"

While a quadrilateral is a closed chain of four line segments, a "curvilinear quadrilateral" is a closed chain of four arcs.

Figure 10a: Grid Generated within a Curvilinear Quadrilateral

A grid generated by the grid-generation algorithm. Each intersection of the red curves with the blue curves is a grid point. The cells between the grid points are nearly rectangles, all with the approximately the same height-to-width ratio.

Figure 10b: Regularly-Spaced Grid Points Resulting from Conformal Transformation of the Curvilinear One

The regularly-spaced grid that is the image of the grid points of Figure 10a under the conformal transformation that maps Q onto the rectangle R .

2.2.4.1 Theory Behind the Grid-Generation Algorithm

The grid generation algorithm, which produces $(\hat{x}_1[i, j], \hat{y}_1[i, j])$ and $(\hat{x}'[i, j], \hat{y}'[i, j])$, is based on the following two facts. First, $\mathcal{T}_{Q \rightarrow R}(x, y)$ must be a solution to the Laplace equation. Second, $\mathcal{T}_{Q \rightarrow R}(x, y)$ transforms Q onto a rectangle. These two facts provide sufficient information for the determination of $(\hat{x}_1[i, j], \hat{y}_1[i, j])$ and $(\hat{x}'[i, j], \hat{y}'[i, j])$.

$\mathcal{T}_{Q \rightarrow R}(x, y)$ must be a solution to the Laplace equation:

$$\nabla^2 \mathcal{T}_{Q \rightarrow R}(x, y) = \mathbf{0} \quad \forall (x, y) \in U \quad (29)$$

This relationship stems from the fact that $\mathcal{T}_{Q \rightarrow R}(x, y)$ is conformal. Since the Jacobian matrix function of the conformal $\mathcal{T}_{Q \rightarrow R}(x, y)$ is of the form $J_{Q \rightarrow R}(x, y) = \sigma(x, y)\mathfrak{R}(x, y)$, where $\sigma: U \rightarrow \mathbb{R}^+$ and $\mathfrak{R}: U \rightarrow SO(2)$, then it follows that the transpose of the Jacobian is:

$$J_{Q \rightarrow R}(x, y)^T = \sigma(x, y) \Re(x, y)^{-1} \quad (30)$$

Since

$$J_{Q \rightarrow R}(x, y)^T = \begin{bmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{bmatrix} \quad (31)$$

where $\xi_x \equiv \partial_x \mathcal{T}_{Q \rightarrow R}^{(\xi)}(x, y)$; $\xi_y \equiv \partial_y \mathcal{T}_{Q \rightarrow R}^{(\xi)}(x, y)$; $\eta_x \equiv \partial_x \mathcal{T}_{Q \rightarrow R}^{(\eta)}(x, y)$; $\eta_y \equiv \partial_y \mathcal{T}_{Q \rightarrow R}^{(\eta)}(x, y)$, and

$\mathcal{T}_{Q \rightarrow R}^{(\xi)}(x, y)$ and $\mathcal{T}_{Q \rightarrow R}^{(\eta)}(x, y)$ are the functions such that $\mathcal{T}_{Q \rightarrow R}(x, y) = \left(\mathcal{T}_{Q \rightarrow R}^{(\xi)}(x, y), \mathcal{T}_{Q \rightarrow R}^{(\eta)}(x, y) \right)$,

then:

$$J_{Q \rightarrow R}(x, y)^T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \nabla \mathcal{T}_{Q \rightarrow R}^{(\xi)}(x, y) \quad (32)$$

and

$$J_{Q \rightarrow R}(x, y)^T \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \nabla \mathcal{T}_{Q \rightarrow R}^{(\eta)}(x, y) \quad (33)$$

where ∇ is the gradient operator. The conformality of $\mathcal{T}_{Q \rightarrow R}(x, y)$ therefore dictates that:

$$\nabla \mathcal{T}_{Q \rightarrow R}^{(\eta)}(x, y) = \Re_{90^\circ} \nabla \mathcal{T}_{Q \rightarrow R}^{(\xi)}(x, y) \quad (34)$$

where $\Re_{90^\circ}$ is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, the matrix of a 90° counterclockwise rotation. Consequently, if

$\mathcal{T}_{Q \rightarrow R}(x, y)$ is conformal, then the divergence of $\nabla \mathcal{T}_{Q \rightarrow R}^{(\eta)}(x, y)$ is:

$$\nabla \cdot \nabla \mathcal{T}_{Q \rightarrow R}^{(\eta)}(x, y) = \nabla \cdot \Re_{90^\circ} \nabla \mathcal{T}_{Q \rightarrow R}^{(\xi)}(x, y) = -\nabla \times \nabla \mathcal{T}_{Q \rightarrow R}^{(\xi)}(x, y) = 0 \quad (35)$$

where $\nabla \times$ denotes the scalar curl operator. The equivalence between $\nabla \cdot \nabla \mathcal{T}_{Q \rightarrow R}^{(\eta)}(x, y)$ and

$-\nabla \times \nabla \mathcal{T}_{Q \rightarrow R}^{(\xi)}(x, y)$ is illustrated below in Figure 11. With some abuse in notation,

$$\nabla \cdot \Re_{90^\circ} = [\partial_x \quad \partial_y] \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = [\partial_y \quad -\partial_x] = -\nabla \times \quad (36)$$

Since the curl of a gradient is zero, moreover,

$$\nabla \cdot \nabla \mathcal{T}_{Q \rightarrow R}^{(\eta)}(x, y) = 0 \quad (37)$$

By a parallel argument,

$$\nabla \cdot \nabla \mathcal{T}_{Q \rightarrow R}^{(\xi)}(x, y) = 0 \quad (38)$$

as well. Since the divergence of the gradient is the Laplacian operator, (29) follows. The grid generation algorithm therefore attempts to solve (29) numerically.

Figure 11: Why a Conformal Mapping Must Satisfy the Laplace Equation

The equivalence between $\nabla \cdot \nabla \mathcal{T}_{Q \rightarrow R}^{(\eta)}(x, y)$ and $-\nabla \times \nabla \mathcal{T}_{Q \rightarrow R}^{(\xi)}(x, y)$. If $\nabla \mathcal{T}_{Q \rightarrow R}^{(\eta)}(x, y)$ were to diverge away from a point, as illustrated by the red arrows, $\nabla \mathcal{T}_{Q \rightarrow R}^{(\xi)}(x, y)$, designated by the blue arrows, would necessarily curl around it. Since the curl of a gradient is 0, $\nabla \cdot \nabla \mathcal{T}_{Q \rightarrow R}^{(\eta)}(x, y)$ must also be 0.

In addition to the requirement that a conformal mapping satisfy Laplace's equation, the second fact that grid generation algorithm exploits is that $\mathcal{T}_{Q \rightarrow R}(x, y)$ transforms Q onto a rectangle. Consequently, each side of Q is an isoline of either $\mathcal{T}_{Q \rightarrow R}^{(\xi)}(x, y)$ or $\mathcal{T}_{Q \rightarrow R}^{(\eta)}(x, y)$ – $\mathcal{T}_{Q \rightarrow R}^{(\xi)}(x, y)$ or $\mathcal{T}_{Q \rightarrow R}^{(\eta)}(x, y)$ has constant value along each side of Q . For instance, $\mathcal{T}_{Q \rightarrow R}^{(\xi)}(x, y) = 0$ for all (x, y) on the left side of Q . Meanwhile, from (26), the set of points in any column or row

of $(\hat{x}_1[i, j], \hat{y}_1[i, j])$ also belongs to an isoline of either $\mathcal{T}_{Q \rightarrow R}^{(\xi)}(x, y)$ or $\mathcal{T}_{Q \rightarrow R}^{(\eta)}(x, y)$. Therefore, the outermost rows and columns of $(\hat{x}_1[i, j], \hat{y}_1[i, j])$ can be chosen to lie on Q . Furthermore, since the isolines of $\mathcal{T}_{Q \rightarrow R}^{(\xi)}(x, y)$ intersect the isolines of $\mathcal{T}_{Q \rightarrow R}^{(\eta)}(x, y)$ at right angles, it follows that $(\hat{x}_1[i, j], \hat{y}_1[i, j])$ should be such that each quadrilateral with vertices $(\hat{x}_1[i, j], \hat{y}_1[i, j])$, $(\hat{x}_1[i + 1, j], \hat{y}_1[i + 1, j])$, $(\hat{x}_1[i + 1, j + 1], \hat{y}_1[i + 1, j + 1])$, and $(\hat{x}_1[i, j + 1], \hat{y}_1[i, j + 1])$ should be approximately rectangular for large N_ξ and N_η . In order to effect this perpendicularity, therefore, the grid generation algorithm enforces Neumann boundary conditions over each side of Q as it numerically solves the Laplace equation.

Based on these two facts – that $\mathcal{T}_{Q \rightarrow R}(x, y)$ satisfies the Laplace equation, and that $\mathcal{T}_{Q \rightarrow R}(x, y)$ transforms Q onto a rectangle – the grid generation algorithm can determine $(\hat{x}_1[i, j], \hat{y}_1[i, j])$. First, an initial guess of $(\hat{x}_1[i, j], \hat{y}_1[i, j])$ is made, perhaps using transfinite interpolation. The points in the outermost rows and columns are made to lie on Q . Additionally, an initial guess is made as to M_Q , the conformal module of Q . If the region bounded by Q is particularly tall and narrow, for instance, M_Q might be guessed to be substantially greater than 1. If the curvilinear quadrilateral is wide and short, on the other hand, M_Q is likelier to be less than 1. Secondly, using the points in the outermost rows and columns of $(\hat{x}_1[i, j], \hat{y}_1[i, j])$ as Dirichlet boundary conditions, the successive over-relaxation method is applied to the interior points of $(\hat{x}_1[i, j], \hat{y}_1[i, j])$, so that they satisfy a discrete approximation of (29) as closely as possible. After this update, the points in the outermost rows and columns of $(\hat{x}_1[i, j], \hat{y}_1[i, j])$ are repositioned, or “slid,” along Q . Each boundary point is moved to the position on Q at which the tangent vector to Q is perpendicular to the displacement vector from the point to the corresponding “interior” point. For the left and right boundaries, the “interior” point

corresponding to each boundary point is the one in the adjacent column and the same row; for the top and bottom boundaries, it is the one is the same column and adjacent row. For example, $(\hat{x}_1[1, j_0], \hat{y}_1[1, j_0])$ on the left side of Q is moved such that the tangent vector to the left arc of Q at the point $(\hat{x}_1[1, j_0], \hat{y}_1[1, j_0])$ is perpendicular to the vector $\begin{bmatrix} \hat{x}_1[1, j_0 + 1] \\ \hat{y}_1[1, j_0 + 1] \end{bmatrix} - \begin{bmatrix} \hat{x}_1[1, j_0] \\ \hat{y}_1[1, j_0] \end{bmatrix}$. This re-positioning is illustrated in Figure 12. The green curve illustrates an arc of Q , the yellow marker designates a boundary point, and the orange marker is an interior point. The displacement vector from the interior point to the boundary point is perpendicular to the tangent of the boundary curve at the boundary point.

Figure 12: Neumann Slipping Boundary Conditions

The grid generation algorithm enforces Neumann boundary conditions by repositioning the boundary points on Q each update. The boundary point, in yellow, is slid so that the displacement vector from the interior point to the boundary point is perpendicular to the tangent to Q .

2.2.4.2 Agreement of Conformal Modules

The resulting $\mathcal{T}_{x \rightarrow x'}|_U(x, y)$ may or may not reshape the imager's object and image surfaces in the desired manner, depending on how close the conformal modules of Q and Q' are. $\mathcal{T}_{Q \rightarrow R}(x, y)$ and $\mathcal{T}_{Q' \rightarrow R'}(x', y')$ are illustrated in Figure 13 below. As shown in Figure 13, since the bases of R and R' are the same, the lateral sides of R are either a subsets or supersets of the corresponding lateral sides of R' . As a result, $\mathcal{T}_{x \rightarrow x'}|_U(x, y)$ maps the left and right sides of Q onto either subsets or supersets of the left and right sides of Q' . The closer the conformal modules of Q and Q' , the closer in height R and R' are, the smaller the difference between the

lateral sides of R and R' , the more closely the lateral sides of Q are mapped onto the lateral sides of Q' , and thus the more closely the object and image surface of the original optic are mapped onto the desired object and image surfaces for the new imager.

Figure 13: Agreement of Conformal Modules

The closer the conformal modules of Q and Q' , the more closely the lateral sides of Q are mapped onto the lateral sides of Q' under $\mathcal{T}_{x \rightarrow x'}|_U(x, y) = \{\mathcal{T}_{Q' \rightarrow R'}^{-1} \circ \mathcal{T}_{Q \rightarrow R}\}(x, y)$.

2.2.4.3 Numerical Samples of Constitutive Parameters

If the conformal modules are indeed sufficiently close, and $\mathcal{T}_{x \rightarrow x'}|_U(x, y)$, or $\mathcal{T}_{x \rightarrow x'}|_U(z, \rho)$, in the 3-D case, does adequately reshape the optical system, then the two arrays in $U - (\hat{x}[i, j], \hat{y}[i, j])$ and $(\hat{x}'[i, j], \hat{y}'[i, j])$, or, in the 3-D case, $(\hat{z}[i, j], \hat{\rho}[i, j])$ and $(\hat{z}'[i, j], \hat{\rho}'[i, j])$

– can be used to compute numerical samples of the constitutive parameters in the new optic. The determinant of the Jacobian in the denominator of (19), (22), (24), and (25) can be estimated by numerical differentiation, as in (39), in the 2-D case, or (40), in the 3-D case:

$$|J_U(\mathcal{T}_{x' \rightarrow x}|_U(\hat{x}'[i, j], \hat{y}'[i, j]))| \approx |\hat{J}_U|_{i, j}$$

$$\equiv \left\| \begin{bmatrix} \frac{\hat{x}'[i+1, j] - \hat{x}'[i, j]}{\hat{x}[i+1, j] - \hat{x}[i, j]} & \frac{\hat{x}'[i+1, j] - \hat{x}'[i, j]}{\hat{y}[i+1, j] - \hat{y}[i, j]} \\ \frac{\hat{y}'[i+1, j] - \hat{y}'[i, j]}{\hat{x}[i+1, j] - \hat{x}[i, j]} & \frac{\hat{y}'[i+1, j] - \hat{y}'[i, j]}{\hat{y}[i+1, j] - \hat{y}[i, j]} \end{bmatrix} \right\| \quad (39)$$

$$|J_U(\mathcal{T}_{x' \rightarrow x}|_U(\hat{z}'[i, j], \hat{\rho}'[i, j]))| \approx |\hat{J}_U|_{i, j}$$

$$\equiv \left\| \begin{bmatrix} \frac{\hat{z}'[i+1, j] - \hat{z}'[i, j]}{\hat{z}[i+1, j] - \hat{z}[i, j]} & \frac{\hat{z}'[i+1, j] - \hat{z}'[i, j]}{\hat{\rho}[i+1, j] - \hat{\rho}[i, j]} \\ \frac{\hat{\rho}'[i+1, j] - \hat{\rho}'[i, j]}{\hat{z}[i+1, j] - \hat{z}[i, j]} & \frac{\hat{\rho}'[i+1, j] - \hat{\rho}'[i, j]}{\hat{\rho}[i+1, j] - \hat{\rho}[i, j]} \end{bmatrix} \right\| \quad (40)$$

In the 3-D case, furthermore, the $\frac{\rho}{\rho'}$ and $\frac{\rho'}{\rho}$ terms in (21) and (24) can be approximated by $\frac{\hat{\rho}[i, j]}{\hat{\rho}'[i, j]}$

and $\frac{\hat{\rho}'[i, j]}{\hat{\rho}[i, j]}$. The terms $n(\mathcal{T}_{x' \rightarrow x}|_U(\mathbf{x}'))$ in (19) and $\varepsilon(\mathcal{T}_{x' \rightarrow x}|_U(\mathbf{z}', \rho'))$ in (21) and (22) are

computed by evaluating the function $n(x, y)$ at $(\hat{x}[i, j], \hat{y}[i, j])$ or the function $\varepsilon(z, \rho)$ at

$(\hat{z}[i, j], \hat{\rho}[i, j])$, where the $n(x, y)$ or $\varepsilon(z, \rho)$ describe the index or permittivity of the initial optic,

and are known a priori. In this way, discrete spatial samples of the material parameters needed to implement the new imager are obtained.

2.3 Relationship to Freeform Optics

Quasi-conformal transformation optics may be compared with the new field of freeform optics. While traditional lenses have radially-symmetric surfaces the sag of which may be expressed with polynomials, freeform optics may have arbitrarily shaped, asymmetric lens surfaces. Such surface shapes can be chosen to reduce aberration, as well as to refract or reflect

the light in desired directions. [23] While freeform optics typically only reshape an optic's surface, without changing the index of the interior glass, it is similar to transformation optics in that it prescribes non-traditionally shaped optical systems.

Chapter 3 Reimager QCTO Design Space

The reimager, illustrated in Figure 4, is designed with the 3-D QCTO procedure described in the last chapter. The starting point is Maxwell's fish-eye lens, like the one described in Chapter 2, except it is spherical instead of cylindrical. Its index of refraction is described by (20), except that R is the spherical radius. Each point on its outer shell images onto the diametrically opposite one, without geometric aberration. The new imager has a concave-plano shape, in order to interface between the monocentric lens' concave image surface and a flat focal plane array.

As detailed in Chapter 2, the QCTO procedure involves the specification of two curvilinear quadrilaterals: Q and Q' . For this design, the starting quadrilateral, Q is denoted Q_o , while the quadrilateral corresponding to the final device, Q' , is denoted Q_{cp} . As long as the quadrilaterals' lateral sides are aligned with the optics' object and image surfaces, arbitrary quadrilaterals accomplish the design goal. Different quadrilaterals Q_o and Q_{cp} , however, yield different constitutive parameters. In this chapter, therefore, the effect of several different parameters of the quadrilaterals on the range of the refractive index is assessed. The final design then uses the test results as a guide in its choice of quadrilaterals, with the aim of finding constitutive parameters with as narrow as possible a range of values. The narrower the range of ϵ and μ values, the more likely that the reimager can be built without resonant structures [13], which, as noted in Chapter 1, severely limit bandwidth and cause absorptive and dissipative losses.

3.1 Design Terminology

In this design, the following terms are used. The subscripts ‘o’ and ‘cp’ respectively distinguish the original space of Maxwell’s fish-eye lens (‘o’) from the new space of the concave-plano reimager (‘cp’).

3.1.1 2-D and 3-D Transformations

The transformation from the original space to the new space, denoted $\mathcal{T}_{x \rightarrow x'}(\mathbf{x})$ in Chapter 1 and Chapter 2, is designated $\mathcal{T}_{o \rightarrow cp}^{(3d)}: U_o^{(3d)} \rightarrow U_{cp}^{(3d)}$. $U_o^{(3d)}$ and $U_{cp}^{(3d)}$ are three-dimensional spaces described by the Cartesian coordinates (x_o, y_o, z_o) and (x_{cp}, y_{cp}, z_{cp}) respectively. The z_o - and z_{cp} - axes are the two optics’ axes of symmetry. This 3-D transformation is related to the 2-D conformal transformation $\mathcal{T}_{o \rightarrow cp}: U_o \rightarrow U_{cp}$ as follows:

$$(x_{cp}, y_{cp}, z_{cp}) = \mathcal{T}_{o \rightarrow cp}^{(3d)}(x_o, y_o, z_o) \Leftrightarrow \begin{cases} (z_{cp}, \rho_{cp}) = \mathcal{T}_{o \rightarrow cp}(\mathcal{T}_{\mathcal{P}o}(x_o, y_o, z_o)) \\ \phi_{cp} = \phi_o \end{cases} \quad (41)$$

where $\phi_{cp} \equiv \arg(x_{cp} + iy_{cp})$, $\phi_o \equiv \arg(x_o + iy_o)$, and $\mathcal{T}_{\mathcal{P}o}: U_o^{(3d)} \rightarrow U_o$ designates the projection from $U_o^{(3d)}$ onto U_o :

$$\mathcal{T}_{\mathcal{P}o}(x_o, y_o, z_o) = (z_o, \sqrt{x_o^2 + y_o^2}) = (z_o, \rho_o) \quad (42)$$

Similarly, $\mathcal{T}_{\mathcal{P}cp}: U_{cp}^{(3d)} \rightarrow U_{cp}$ designates the projection from $U_{cp}^{(3d)}$ onto U_{cp} :

$$\mathcal{T}_{\mathcal{P}cp}(x_{cp}, y_{cp}, z_{cp}) = (z_{cp}, \sqrt{x_{cp}^2 + y_{cp}^2}) = (z_{cp}, \rho_{cp}) \quad (43)$$

The inverse of $\mathcal{T}_{o \rightarrow cp}^{(3d)}(x_o, y_o, z_o)$ is $\mathcal{T}_{cp \rightarrow o}^{(3d)}(x_{cp}, y_{cp}, z_{cp})$:

$$(x_o, y_o, z_o) = \mathcal{T}_{cp \rightarrow o}^{(3d)}(x_{cp}, y_{cp}, z_{cp}) \Leftrightarrow \begin{cases} (z_o, \rho_o) = \mathcal{T}_{cp \rightarrow o}(\mathcal{T}_{\mathcal{P}cp}(x_{cp}, y_{cp}, z_{cp})) \\ \phi_o = \phi_{cp} \end{cases} \quad (44)$$

3.1.2 Refractive Index

Substituting these terms into (25), light traveling in meridional planes through the reimager experiences a non-magnetic response with index of refraction $n_{cp}(z_{cp}, \rho_{cp})$:

$$n_{cp}(z_{cp}, \rho_{cp}) = \frac{n_o(\mathcal{T}_{cp \rightarrow o}(z_{cp}, \rho_{cp}))}{\sqrt{|J_{o \rightarrow cp}(\mathcal{T}_{cp \rightarrow o}(z_{cp}, \rho_{cp}))|}} \quad (45)$$

where $J_{o \rightarrow cp}(z_o, \rho_o)$ is the Jacobian of $\mathcal{T}_{o \rightarrow cp}(z_o, \rho_o)$; $n_o(z_o, \rho_o) = n_{fish-eye}(\sqrt{z_o^2 + \rho_o^2})$, $n_{fish-eye}(R)$ given by (20); and $\mathcal{T}_{cp \rightarrow o} \equiv \mathcal{T}_{o \rightarrow cp}^{-1}$.

3.1.3 Quadrilaterals

The two quadrilaterals that determine the mapping, shown in Figure 14a and Figure 14b, are designated Q_o and Q_{cp} respectively, and belong to the spaces U_o and U_{cp} respectively. Q_o is specified in terms of the radius of Maxwell's fish-eye, R_o , which, arbitrarily, may be set to 1 mm. The way that these quadrilaterals are calculated is detailed in later sections.

Figure 14a: Q_o
The curvilinear quadrilateral Q_o in U_o

Figure 14b: Q_{cp}

The curvilinear quadrilateral Q_{cp} in U_{cp}

Figure 15 illustrates the relationship between the curvilinear quadrilaterals and the transformation. Q_o and Q_{cp} determine the conformal transformation $(z_{cp}, \rho_{cp}) = \mathcal{T}_{o \rightarrow cp}(z_o, \rho_o)$, which transforms the longitudinal and radial coordinates of the fish-eye onto the longitudinal and radial coordinates of the concave-plano reimager. Meanwhile, Q_o is the image under $\mathcal{T}_{\mathcal{P}o}(x_o, y_o, z_o)$ of a closed surface that mostly aligns with the outer shell of Maxwell's fish-eye. Q_{cp} sets the shape of the reimager: it is the image of the boundary surface of the reimager under $\mathcal{T}_{\mathcal{P}cp}(x_{cp}, y_{cp}, z_{cp})$. The left arc of Q_{cp} corresponds to the concave front surface. The right line segment corresponds to the flat back detector surface.

Figure 15: The relationship between the curvilinear quadrilaterals and the transformation

This figure illustrates the relationship between the curvilinear quadrilaterals and the transformation.

3.1.4 Intermediate “Logical” Space

Additionally, the 2-D “logical” space U_ℓ is the one containing the rectangles conformally equivalent to the quadrilaterals. It is described by coordinates (ξ, η) . $\mathcal{T}_{\ell \rightarrow o}: U_\ell \rightarrow U_o$ and $\mathcal{T}_{\ell \rightarrow cp}: U_\ell \rightarrow U_{cp}$ are conformal transformations such that $\mathcal{T}_{o \rightarrow cp} = \mathcal{T}_{\ell \rightarrow cp} \circ \mathcal{T}_{o \rightarrow \ell}$; $\mathcal{T}_{\ell \rightarrow o}(\xi, \eta)$ maps a rectangle with a base of unit length onto Q_o ; while $\mathcal{T}_{\ell \rightarrow cp}(\xi, \eta)$ maps a rectangle of unit-length base onto Q_{cp} ; and $\mathcal{T}_{o \rightarrow \ell} \equiv \mathcal{T}_{\ell \rightarrow o}^{-1}$. Figure 16 illustrates this decomposition of $\mathcal{T}_{o \rightarrow cp}(z_o, \rho_o)$ into $\mathcal{T}_{o \rightarrow \ell}(z_o, \rho_o)$ and $\mathcal{T}_{\ell \rightarrow cp}(\xi, \eta)$.

Figure 16: Decomposition of $T_{o \rightarrow cp}$

The conformal mapping $T_{o \rightarrow cp} = T_{\ell \rightarrow cp} \circ T_{o \rightarrow \ell}$, where $T_{\ell \rightarrow cp}$ and $T_{o \rightarrow \ell}$ are both conformal

3.1.5 Grid-Point Arrays

With the procedure described in section 2.2.4, grid-point arrays, $(\hat{z}_o[i, j], \hat{\rho}_o[i, j])$ and $(\hat{z}_{cp}[i, j], \hat{\rho}_{cp}[i, j])$, are computed on and inside Q_o and Q_{cp} respectively, such that:

$$T_{o \rightarrow \ell}(\hat{z}_o[i, j], \hat{\rho}_o[i, j]) = (i \cdot \Delta \xi, j \cdot \Delta \eta) \quad (46)$$

$$T_{\ell \rightarrow cp}(i \cdot \Delta \xi, j \cdot \Delta \eta) = (\hat{z}_{cp}[i, j], \hat{\rho}_{cp}[i, j]) \quad (47)$$

where $\Delta \xi$ and $\Delta \eta$ are constants, and, therefore:

$$T_{o \rightarrow cp}(\hat{z}_o[i, j], \hat{\rho}_o[i, j]) = (\hat{z}_{cp}[i, j], \hat{\rho}_{cp}[i, j]) \quad (48)$$

3.1.6 Index Arrays and Jacobian Arrays

Then, for each point $(\hat{z}_{cp}[i, j], \hat{\rho}_{cp}[i, j])$, an approximation of $n_{cp}(\hat{z}_{cp}[i, j], \hat{\rho}_{cp}[i, j])$ is computed with the equation:

$$\hat{n}_{cp}[i, j] = \frac{n_o}{1 + (\hat{z}_o[i, j]/R_o)^2 + (\hat{\rho}_o[i, j]/R_o)^2} \times \sqrt{\frac{|\hat{J}_{\ell \rightarrow o}|_{i, j}}{|J_{\ell \rightarrow cp}|_{i, j}}} \quad (49)$$

where:

$$|\hat{J}_{\ell \rightarrow o}|_{i,j} = \left\| \begin{bmatrix} \frac{\hat{z}_o[i+1,j] - \hat{z}_o[i,j]}{\Delta\xi} & \frac{\hat{z}_o[i+1,j] - \hat{z}_o[i,j]}{\Delta\eta} \\ \frac{\hat{\rho}_o[i+1,j] - \hat{\rho}_o[i,j]}{\Delta\xi} & \frac{\hat{\rho}_o[i+1,j] - \hat{\rho}_o[i,j]}{\Delta\eta} \end{bmatrix} \right\| \quad (50)$$

and

$$|\hat{J}_{\ell \rightarrow cp}|_{i,j} = \left\| \begin{bmatrix} \frac{\hat{z}_{cp}[i+1,j] - \hat{z}_{cp}[i,j]}{\Delta\xi} & \frac{\hat{z}_{cp}[i+1,j] - \hat{z}_{cp}[i,j]}{\Delta\eta} \\ \frac{\hat{\rho}_{cp}[i+1,j] - \hat{\rho}_{cp}[i,j]}{\Delta\xi} & \frac{\hat{\rho}_{cp}[i+1,j] - \hat{\rho}_{cp}[i,j]}{\Delta\eta} \end{bmatrix} \right\| \quad (51)$$

are numeric approximations of $|J_{\ell \rightarrow o}(\mathcal{T}_{o \rightarrow \ell}(\hat{z}_o[i,j], \hat{\rho}_o[i,j]))|$ and

$|J_{\ell \rightarrow cp}(\mathcal{T}_{cp \rightarrow \ell}(\hat{z}_{cp}[i,j], \hat{\rho}_{cp}[i,j]))|$ respectively.

3.1.7 Index Range Metric

The metric by which the narrowness of the range of $n_{cp}(z_{cp}, \rho_{cp})$ is assessed for different pairs of quadrilaterals is:

$$\Delta \hat{n}_{cp} \equiv \frac{\max_{(i,j)} \{\hat{n}_{cp}[i,j]\}}{\min_{(i,j)} \{\hat{n}_{cp}[i,j]\}} \quad (52)$$

$\Delta \hat{n}_{cp}$ is a numerical approximation of Δn_{cp} :

$$\Delta n_{cp} \equiv \frac{\max_{(z_{cp}, \rho_{cp}) \in \mathbb{Q}_{cp}} \{n_{cp}(z_{cp}, \rho_{cp})\}}{\min_{(z_{cp}, \rho_{cp}) \in \mathbb{Q}_{cp}} \{n_{cp}(z_{cp}, \rho_{cp})\}} \quad (53)$$

where $\mathbb{Q}_{cp}$ denotes the set of points on and interior to Q_{cp} . This quotient is independent of n_0 in (64), which can be chosen arbitrarily.

In addition to this ratio, another metric assessing the material range of $n_{cp}(z_{cp}, \rho_{cp})$ needed to only reimage a $\pm 60^\circ$ field is computed:

$$\Delta \hat{n}_{cp}^w \equiv \frac{\max_{(i,j) \in \mathbb{I}_\Omega} \{\hat{n}_{cp}[i,j]\}}{\min_{(i,j) \in \mathbb{I}_\Omega} \{\hat{n}_{cp}[i,j]\}} \quad (54)$$

$\mathbb{I}_\Omega$ in (54) denotes the set of points $\{(i,j): (\hat{z}_{cp}[i,j], \hat{\rho}_{cp}[i,j]) \in \Omega^{(60)}\}$. $\Omega^{(60)} \subset U_{cp}$ is the set of points on and interior to Q_{cp} to which $\mathcal{T}_{\mathcal{P}cp}(x_{cp}, y_{cp}, z_{cp})$, defined in (43), projects the points inside the reimager needed to reimage a $\pm 60^\circ$ field. $\Delta \hat{n}_{cp}^w$ is a numerical approximation of:

$$\Delta n_{cp}^w \equiv \frac{\max_{(z_{cp}, \rho_{cp}) \in \Omega^{(60)}} \{n_{cp}(z_{cp}, \rho_{cp})\}}{\min_{(z_{cp}, \rho_{cp}) \in \Omega^{(60)}} \{n_{cp}(z_{cp}, \rho_{cp})\}} \quad (55)$$

An example of $\Omega^{(60)}$ is illustrated below in Figure 17. $\Omega^{(60)}$ is generally a subset of $\mathbb{Q}_{cp}$. If only a $\pm 60^\circ$ is desired, only the points in the reimager corresponding to $\Omega^{(60)}$ need be populated with a metamaterial. $\Omega^{(60)}$ tends to exclude some of the higher values of ρ_{cp} , as apparent from Figure 17, and therefore it excludes some of the lowest values of $n_{cp}(z_{cp}, \rho_{cp})$. Δn_{cp}^w is thus generally smaller than Δn_{cp} and the more useful metric.

Figure 17: $\Omega^{(60)}$

$\Omega^{(60)} \subset U_{cp}$ is the set of points on and interior to Q_{cp} to which $\mathcal{T}_{p_{cp}}(x_{cp}, y_{cp}, z_{cp})$ projects the points inside the reimager needed to reimage a $\pm 60^\circ$ field. $\Omega^{(60)}$ generally excludes some of the uppermost points interior to Q_{cp} , which correspond to the lowest values of $n_{cp}(z_{cp}, \rho_{cp})$. The axes represent the coordinates z_{cp} and ρ_{cp} in terms of R_{cp} , the radius of the monocentric lens to which the reimager fits.

3.1.7.1 Computation of Index Range Metric

In order to compute $\Delta \hat{n}_{cp}^w$, it must be determined which points (i, j) belong to $\mathbb{I}_\Omega$. This determination is made based on the following logic. $(i, j) \in \mathbb{I}_\Omega \Leftrightarrow (\hat{z}_{cp}[i, j], \hat{\rho}_{cp}[i, j]) \in \Omega^{(60)} \Leftrightarrow (\hat{z}_{cp}[i, j], \pm \hat{\rho}_{cp}[i, j]) \in \Omega_{z_{cp}-x_{cp}}^{(60)} \Leftrightarrow (\hat{z}_o[i, j], \pm \hat{\rho}_o[i, j]) \in \Omega_{z_o-x_o}^{(60)}$, where $\Omega_{z_{cp}-x_{cp}}^{(60)}$ signifies the subset of points in the $z_{cp}-x_{cp}$ cross section of the reimager through which pass meridional rays from field angles less than 60° , and $\Omega_{z_o-x_o}^{(60)}$ denotes the image of $\Omega_{z_{cp}-x_{cp}}^{(60)}$ under $\mathcal{T}_{cp \rightarrow o}^{(3d)}(x_{cp}, y_{cp}, z_{cp})$. An example of $\Omega_{z_{cp}-x_{cp}}^{(60)}$ is shown in Figure 18, and Figure 17 illustrates the corresponding $\Omega^{(60)}$. Taking advantage of this equivalency, an expression for the outer boundaries of $\Omega_{z_o-x_o}^{(60)}$ is obtained, and this expression is used to determine whether each grid point $(\hat{z}_o[i, j], \hat{\rho}_o[i, j]) \in \Omega_{z_o-x_o}^{(60)}$ and hence whether each $(i, j) \in \mathbb{I}_\Omega$.

Figure 18: $\Omega_{z_{cp}-x_{cp}}^{(60)}$

$\Omega_{z_{cp}-x_{cp}}^{(60)}$ denotes the subset of points in the z_{cp} - x_{cp} cross section of the reimager through which pass meridional rays from field angles less than 60° . $\Omega^{(60)}$ is the image, under $\mathcal{T}_{\mathcal{P}_{cp}}(x_{cp}, y_{cp}, z_{cp})$, of $\Omega_{z_{cp}-x_{cp}}^{(60)}$. The axes represent the coordinates z_{cp} and ρ_{cp} in terms of R_{cp} , the radius of the monocentric lens to which the reimager fits

The expression for the outer boundaries of $\Omega_{z_o-x_o}^{(60)}$ is deduced as follows. First, the light incident on the reimager is the image formed by the F/1 monocentric lens. Therefore, ignoring surface refraction, from each point on the reimager's surface, there emanates a cone of light into the reimager, the cross section of which spans $\pm \tan^{-1}\left(\frac{1}{2}\right) = \pm 26.6^\circ$. Thus, the light ray paths that constitute $\Omega_{z_{cp}-x_{cp}}^{(60)}$ are the ones originating on the arc C_{cp} with initial headings between -26.6° and $+26.6^\circ$ with respect to the surface normal, where C_{cp} is:

$$C_{cp} = \left\{ (R_{cp} \sin \theta, 0, R_{cp} \cos \theta) \in U_{cp}^{(3d)} : -60^\circ \leq \theta \leq +60^\circ \right\} \quad (56)$$

Furthermore, because $\mathcal{T}_{o \rightarrow cp}^{(3d)}(x_o, y_o, z_o)$, when restricted to a meridional plane, is conformal, each meridional ray of which $\Omega_{z_{cp}-x_{cp}}^{(60)}$ is comprised is the image, under $\mathcal{T}_{o \rightarrow cp}^{(3d)}(x_o, y_o, z_o)$, of another meridional ray trajectory through the z_o - x_o cross section of Maxwell's fish-eye:

$\mathcal{T}_{cp \rightarrow o}^{(3d)}(x_{cp}, y_{cp}, z_{cp})$ conformally maps $\Omega_{z_{cp}-x_{cp}}^{(60)}$ onto $\Omega_{z_o-x_o}^{(60)}$, a subset of the z_o - x_o cross section of Maxwell's fish-eye. The light ray paths that constitute $\Omega_{z_o-x_o}^{(60)}$ are therefore the ones originating on the arc C_o with initial headings between -26.6° and $+26.6^\circ$ with respect to the surface normal, where C_o is defined as:

$$C_o \equiv \left\{ (R_o \sin \theta, 0, R_o \cos \theta) \in U_o^{(3d)} : \theta_o \leq \theta \leq 360^\circ - \theta_o \right\} \quad (57)$$

and where $90^\circ < \theta_o < 180^\circ$. θ_o is such that $\mathcal{T}_{o \rightarrow cp}^{(3d)}(R_o \sin \theta_o, 0, R_o \cos \theta_o) = (R_{cp} \sin(60^\circ), 0, R_{cp} \cos(60^\circ))$. It can be estimated numerically by bilinear interpolation of the arrays $(\hat{z}_{cp}[i, j], \hat{\rho}_{cp}[i, j])$ and $(\hat{z}_o[i, j], \hat{\rho}_o[i, j])$:

$$\theta_o \approx \tan^{-1} \left(\frac{\hat{\rho}_o[1, j_0]}{\hat{z}_o[1, j_0]} \right) + \left[\tan^{-1} \left(\frac{\hat{\rho}_o[1, j_0 + 1]}{\hat{z}_o[1, j_0 + 1]} \right) - \tan^{-1} \left(\frac{\hat{\rho}_o[1, j_0]}{\hat{z}_o[1, j_0]} \right) \right] \alpha \quad (58)$$

where j_0 is the row index such that:

$$\tan^{-1} \left(\frac{\hat{\rho}_{cp}[1, j_0]}{\hat{z}_{cp}[1, j_0]} \right) \leq 60^\circ \leq \tan^{-1} \left(\frac{\hat{\rho}_{cp}[1, j_0 + 1]}{\hat{z}_{cp}[1, j_0 + 1]} \right) \quad (59)$$

and α is the fraction such that:

$$60^\circ = \tan^{-1} \left(\frac{\hat{\rho}_{cp}[1, j_0]}{\hat{z}_{cp}[1, j_0]} \right) + \left[\tan^{-1} \left(\frac{\hat{\rho}_{cp}[1, j_0 + 1]}{\hat{z}_{cp}[1, j_0 + 1]} \right) - \tan^{-1} \left(\frac{\hat{\rho}_{cp}[1, j_0]}{\hat{z}_{cp}[1, j_0]} \right) \right] \alpha \quad (60)$$

Based on these initial conditions of the light ray paths of which $\Omega_{z_o-x_o}^{(60)}$ consists, the closed-form expression for each ray path in $\Omega_{z_o-x_o}^{(60)}$ is then deduced based on two properties of Maxwell's fish-eye. Firstly, a ray trajectory beginning on the surface of Maxwell's fish-eye emerges at the diametrically opposite point. Secondly, each ray trajectory through Maxwell's fish-eye is an arc on a circle [24]. If a ray, for instance, begins on C_o at coordinates $(R_o \sin \theta, 0, R_o \cos \theta)$, propagating meridionally in the z_o - x_o plane, with an initial heading that is displaced, counterclockwise, from the surface normal by ψ , then, as illustrated in Figure 19, its trajectory

through the fish-eye is an arc along the circle with radius R_p and center of curvature

$(z_o^{(cc)}, x_o^{(cc)})$, where:

$$R_p = \frac{R_o}{\sin(|\psi|)} \quad (61)$$

$$(z_o^{(cc)}, x_o^{(cc)}) = (R_p \cos(\psi) \cos(\theta \pm 90^\circ), R_p \cos(\psi) \sin(\theta \pm 90^\circ)) \quad (62)$$

The '+' sign in (62) corresponds to the case when $\psi > 0$; the '-' sign is used when $\psi < 0$.

Because the light rays traverse in this way, the boundaries of $\Omega_{z_o-x_o}^{(60)}$ are therefore defined by the pairs of outermost rays emanating from each of the endpoints of C_o , as illustrated in Figure 20.

Let $\hat{O}_+$ denote the circle on which lies the meridional ray entering Maxwell's fish-eye from the uppermost point in C_o with the upward-most initial heading; let $\check{O}_+$ denote the circle on which lies the meridional ray entering Maxwell's fish-eye from the uppermost point on C_o with the downward-most initial heading; let $\hat{O}_-$ denote the circle on which lies the meridional ray from the lowermost point in C_o with the upward-most bound path; and let $\check{O}_-$ denote circle on which lies the meridional ray from the lowermost point in C_o with the lowermost-bound path. Then, $\Omega_{z_o-x_o}$ consists of all the points in the z_o-x_o cross section of Maxwell's fish-eye except the ones that either both lie exterior to $\hat{O}_+$ and exterior to $\hat{O}_-$, or the ones that both lie exterior to $\check{O}_+$ and exterior to $\check{O}_-$. Hence, in the calculation of $\Delta \hat{n}_{cp}^w$, it is determined whether each coordinate pair $(i, j) \in \mathbb{I}_\Omega$ by assessing whether $(\hat{z}_o[i, j], \hat{\rho}_o[i, j])$ lies interior to the aforementioned circles. Secondly, for the set of $(i, j) \in \mathbb{I}_\Omega$, the maximum and minimum $\hat{n}_{cp}[i, j]$ are identified. $\Delta \hat{n}_{cp}^w$ is then computed as the ratio of these extrema.

Figure 19: Ray Trajectory through Maxwell's Fish-eye

The blue circle represents the boundary of Maxwell's fish-eye lens, and the solid red arc is the trajectory of a light ray through Maxwell's fish-eye. The dashed red arc is the remainder of the circle of which the ray path is an arc.

Figure 20: $\Omega_{z_o-x_o}^{(60)}$

This figure illustrates the set of points in $\Omega_{z_o-x_o}^{(60)}$. The black, solid circle is the outer boundary of Maxwell's fish-eye. The red, blue, green, and cyan dashed curves demarcate the circles $\hat{O}_+$, $\check{O}_-$, $\hat{O}_-$, and $\check{O}_+$, respectively. $\Omega_{z_o-x_o}^{(60)}$, indicated by the lightly shaded region, is the region inside Maxwell's fish-eye that is neither exterior to both $\hat{O}_+$ and $\check{O}_+$ nor exterior to both $\hat{O}_-$ and $\check{O}_-$.

3.2 Influence of the Shape of the Reimager on the Index Range

The choice of quadrilateral parameters, as well as the predictions and analyses of the test results, are based on the following mathematical decomposition of (45) into three different

terms. From (45), $n_{cp}(z_{cp}, \rho_{cp})$ depends on the two quantities $\sqrt{|J_{o \rightarrow cp}(z_o, \rho_o)|}$ and

$n_o(\mathcal{T}_{cp \rightarrow o}(z_{cp}, \rho_{cp}))$. The contributions from Q_o and Q_{cp} to $|J_{o \rightarrow cp}(z_o, \rho_o)|$, moreover, come from the denominator and numerator respectively in (63):

$$|J_{o \rightarrow cp}(z_o, \rho_o)| = \frac{|J_{\ell \rightarrow cp}(\mathcal{T}_{o \rightarrow \ell}(z_o, \rho_o))|}{|J_{\ell \rightarrow o}(\mathcal{T}_{o \rightarrow \ell}(z_o, \rho_o))|} \quad (63)$$

where $J_{\ell \rightarrow o}: U_\ell \rightarrow \mathbb{R}^{2 \times 2}$ and $J_{\ell \rightarrow cp}: U_\ell \rightarrow \mathbb{R}^{2 \times 2}$ are functions that return the Jacobians of $\mathcal{T}_{\ell \rightarrow o}(\xi, \eta)$ and $\mathcal{T}_{\ell \rightarrow cp}(\xi, \eta)$ respectively. Substitution of (20) and (63) into (45) yields:

$$n_{cp}(z_{cp}, \rho_{cp}) = \frac{n_o}{1 + (z_o/R_o)^2 + (\rho_o/R_o)^2} \times \sqrt{\frac{|J_{\ell \rightarrow o}(\xi, \eta)|}{|J_{\ell \rightarrow cp}(\xi, \eta)|}} \quad (64)$$

where $(\xi, \eta) = \mathcal{T}_{cp \rightarrow \ell}(z_{cp}, \rho_{cp})$ and $(z_o, \rho_o) = \mathcal{T}_{cp \rightarrow o}(z_{cp}, \rho_{cp})$. Therefore, three distinct influences contribute to $n_{cp}(z_{cp}, \rho_{cp})$:

- 1) the original index distribution of Maxwell's fish-eye: $\frac{n_o}{1 + (z_o/R_o)^2 + (\rho_o/R_o)^2}$
- 2) the flattening of the spherical fish-eye into a cylinder: $\sqrt{|J_{\ell \rightarrow o}(\xi, \eta)|}$
- 3) the unflattening of the cylinder into a concave-plano shape: $1/\sqrt{|J_{\ell \rightarrow cp}(\xi, \eta)|}$

The term $\sqrt{|J_{\ell \rightarrow o}(\xi, \eta)|}$ represents the factor by which the speed of light whose original propagation path belongs to a flat rectangle, of the size of the rectangle to which $\mathcal{T}_{o \rightarrow \ell}(z_o, \rho_o)$ maps the interior of Q_o , must be changed, at each point, in order for it to complete the

corresponding trajectory through Maxwell's fish-eye at the same rate. The $\sqrt{|J_{\ell \rightarrow cp}(\xi, \eta)|}$ term, likewise, represents the factor by which the speed of the rays propagating through a flat rectangle must be changed, at each point, in order for the corresponding ones in the reimager to flow at the same rate. Examples of the two Jacobian functions $|J_{\ell \rightarrow o}(\xi, \eta)|$ and $|J_{\ell \rightarrow cp}(\xi, \eta)|$ are illustrated in Figure 21a-d, in a normalized logarithmic scale. The function $|J_{\ell \rightarrow o}(\xi, \eta)|$, for instance, is largest at the bottom corners, because, in the reshaping of a rectangle to Q_o , the bottom corners are stretched outward more than the other points. Therefore, light rays through the bottom corners of

Q_o must move more quickly to deliver the same amount of energy in the same amount of time. $|J_{\ell \rightarrow cp}(\xi, \eta)|$, meanwhile, is greatest at the top left corner. When the rectangle is transformed into Q_{cp} , the top corner is stretched the most.

Figure 21a: $|J_{\ell \rightarrow o}(\mathcal{T}_{o \rightarrow \ell}(z_o, \rho_o))|$ Figure 21b: $|J_{\ell \rightarrow cp}(\mathcal{T}_{cp \rightarrow \ell}(z_{cp}, \rho_{cp}))|$ Figure 21c: $|J_{\ell \rightarrow o}(\xi, \eta)|$ Figure 21d: $|J_{\ell \rightarrow cp}(\xi, \eta)|$ Figure 21e: $n_{\text{fish-eye}}$

Examples of the different factors of which $n_{cp}(z_{cp}, \rho_{cp})$ is comprised. The two Jacobian functions $|J_{\ell \rightarrow o}(\xi, \eta)|$ and $|J_{\ell \rightarrow cp}(\xi, \eta)|$, in a normalized logarithmic scale are illustrated in Figure 21a-d.. $|J_{\ell \rightarrow o}(\xi, \eta)|$, for instance, is largest at the bottom corners, because, in the reshaping of a rectangle to Q_o , the bottom corners are stretched outward more than the other points. Therefore, light rays through the bottom corners of Q_o must move more quickly to deliver the same amount of energy in the same amount of time. $|J_{\ell \rightarrow cp}(\xi, \eta)|$, meanwhile, is greatest at the top left corner. When the rectangle is transformed into Q_{cp} , the top corner is stretched the most. Figure 21e shows the original index distribution of Maxwell's fish-eye.

Similarly, the ratios Δn_{cp} and Δn_{cp}^w , as defined in (52) and (53), can also be expressed in terms of three primary components:

- 1) $\Delta n_{fish-eye}$ and $\Delta n_{fish-eye}^w$, respectively
- 2) $\Delta |J_{\ell \rightarrow o}|$ and $\Delta |J_{\ell \rightarrow o}|^w$, respectively, and
- 3) $\Delta |J_{\ell \rightarrow cp}|$ and $\Delta |J_{\ell \rightarrow cp}|^w$, respectively

in the following manner:

$$\Delta n_{cp} = \Delta n_{fish-eye} \sqrt{\frac{\Delta |J_{\ell \rightarrow o}|}{\Delta |J_{\ell \rightarrow cp}|}} \quad (65)$$

$$\Delta n_{cp}^w = \Delta n_{fish-eye}^w \sqrt{\frac{\Delta |J_{\ell \rightarrow o}|^w}{\Delta |J_{\ell \rightarrow cp}|^w}} \quad (66)$$

where:

$$\Delta n_{fish-eye} \equiv \frac{n_{fish-eye}(z_{omax}, \rho_{omax})}{n_{fish-eye}(z_{omin}, \rho_{omin})} \quad (67)$$

$$\Delta n_{fish-eye}^w \equiv \frac{n_{fish-eye}(z_{omax}^w, \rho_{omax}^w)}{n_{fish-eye}(z_{omin}^w, \rho_{omin}^w)} \quad (68)$$

$$\Delta |J_{\ell \rightarrow o}| \equiv \frac{|J_{\ell \rightarrow o}(\xi_{max}, \eta_{max})|}{|J_{\ell \rightarrow o}(\xi_{min}, \eta_{min})|} \quad (69)$$

$$\Delta |J_{\ell \rightarrow o}|^w \equiv \frac{|J_{\ell \rightarrow o}^w(\xi_{max}^w, \eta_{max}^w)|}{|J_{\ell \rightarrow o}^w(\xi_{min}^w, \eta_{min}^w)|} \quad (70)$$

$$\Delta |J_{\ell \rightarrow cp}| \equiv \frac{|J_{\ell \rightarrow cp}(\xi_{max}, \eta_{max})|}{|J_{\ell \rightarrow cp}(\xi_{min}, \eta_{min})|} \quad (71)$$

$$\Delta |J_{\ell \rightarrow cp}|^w \equiv \frac{|J_{\ell \rightarrow cp}^w(\xi_{max}^w, \eta_{max}^w)|}{|J_{\ell \rightarrow cp}^w(\xi_{min}^w, \eta_{min}^w)|} \quad (72)$$

in which:

$$\begin{aligned} (z_{cpmax}, \rho_{cpmax}) &= \mathcal{T}_{\ell \rightarrow cp}(\xi_{max}, \eta_{max}) = \mathcal{T}_{o \rightarrow cp}(z_{omax}, \rho_{omax}) \\ &= \underset{(z_{cp}, \rho_{cp}) \in \mathbb{Q}_{cp}}{\operatorname{argmax}} \{n_{cp}(z_{cp}, \rho_{cp})\} \end{aligned} \quad (73)$$

$$\begin{aligned}
(z_{cpmin}, \rho_{cpmin}) &= \mathcal{T}_{\ell \rightarrow cp}(\xi_{min}, \eta_{min}) = \mathcal{T}_{o \rightarrow cp}(z_{omin}, \rho_{omin}) \\
&= \underset{(z_{cp}, \rho_{cp}) \in \mathbb{Q}_{cp}}{\operatorname{argmin}} \{n_{cp}(z_{cp}, \rho_{cp})\}
\end{aligned} \tag{74}$$

$$\begin{aligned}
(z_{cpmax}^w, \rho_{cpmax}^w) &= \mathcal{T}_{\ell \rightarrow cp}(\xi_{max}^w, \eta_{max}^w) = \mathcal{T}_{o \rightarrow cp}(z_{omax}^w, \rho_{omax}^w) \\
&= \underset{(z_{cp}, \rho_{cp}) \in \Omega^{(60)}}{\operatorname{argmax}} \{n_{cp}(z_{cp}, \rho_{cp})\}
\end{aligned} \tag{75}$$

$$\begin{aligned}
(z_{cpmin}^w, \rho_{cpmin}^w) &= \mathcal{T}_{\ell \rightarrow cp}(\xi_{min}^w, \eta_{min}^w) = \mathcal{T}_{o \rightarrow cp}(z_{omin}^w, \rho_{omin}^w) \\
&= \underset{(z_{cp}, \rho_{cp}) \in \Omega^{(60)}}{\operatorname{argmin}} \{n_{cp}(z_{cp}, \rho_{cp})\}
\end{aligned} \tag{76}$$

As evidenced in Figure 21c and d, the minimum values of $|J_{\ell \rightarrow o}(\xi, \eta)|$ occur near the maximum values of $|J_{\ell \rightarrow cp}(\xi, \eta)|$ and vice versa. In general, $|J_{\ell \rightarrow o}(\xi, \eta)|$ is greater near the base of the rectangle and smaller near the top. In particular, because the base of Q_o is wider than the top of Q_o , the maximum value of $|J_{\ell \rightarrow o}(\mathcal{T}_{o \rightarrow \ell}(z_o, \rho_o))|$ occurs twice on the optical axis: at $z_o = \pm R_o$. Meanwhile, $|J_{\ell \rightarrow cp}(\xi, \eta)|$ is generally larger at larger η and smaller at smaller η . The minimum of $|J_{\ell \rightarrow cp}(\mathcal{T}_{cp \rightarrow \ell}(z_{cp}, \rho_{cp}))|$ occurs on the optical axis at $z_{cp} = R_{cp}$. As a result, the ratio $\frac{|J_{\ell \rightarrow o}(\xi, \eta)|}{|J_{\ell \rightarrow cp}(\xi, \eta)|}$ in (64) has a maximum at $(\xi, \eta) = (0, 0)$, corresponding to $(z_o, \rho_o) = (-R_o, 0)$ and $(z_{cp}, \rho_{cp}) = (R_{cp}, 0)$. Moreover, the factor $\frac{|J_{\ell \rightarrow o}(\xi, \eta)|}{|J_{\ell \rightarrow cp}(\xi, \eta)|}$ is generally maximum at:

$$(z_{cpmax}, \rho_{cpmax}) = (z_{cpmax}^w, \rho_{cpmax}^w) = (R_{cp}, 0) \tag{77}$$

Meanwhile, $\frac{|J_{\ell \rightarrow o}(\xi, \eta)|}{|J_{\ell \rightarrow cp}(\xi, \eta)|}$ is minimal at large η , corresponding to large ρ_{cp} and large ρ_o . In the transformation from the sphere to the concave-plano shape, meridional rays maximally increase in speed near the optical axis and minimally at large radial distances. As a consequence of this way in which the distributions of $|J_{\ell \rightarrow o}(\xi, \eta)|$ and $|J_{\ell \rightarrow cp}(\xi, \eta)|$ align, or, equivalently, the way in

which the speed of the rays is changed, $\Delta|J_{\ell \rightarrow o}|$ and $\Delta|J_{\ell \rightarrow o}|^w$ in (65) and (66) are likely to be larger, the greater the range of values in $|J_{\ell \rightarrow o}(\xi, \eta)|$. Similarly, the greater the range of values in $|J_{\ell \rightarrow cp}(\xi, \eta)|$, the smaller $\Delta|J_{\ell \rightarrow cp}|$ and $\Delta|J_{\ell \rightarrow cp}|^w$ are likely to be. Therefore, the larger the range of values of either $|J_{\ell \rightarrow o}(\xi, \eta)|$ or $|J_{\ell \rightarrow cp}(\xi, \eta)|$, the more likely, it is predicted in this analysis, that Δn_{cp} and Δn_{cp}^w are to be large. More erratic Q_o boundaries, or more erratic Q_{cp} boundaries, are predicted to yield a wider range of index.

More specifically, the extents of the ranges of $|J_{\ell \rightarrow o}(\xi, \eta)|$ and $|J_{\ell \rightarrow cp}(\xi, \eta)|$ depend on the degree to which the isolines of $\mathcal{T}_{o \rightarrow \ell}(z_o, \rho_o)$ and $\mathcal{T}_{cp \rightarrow \ell}(z_{cp}, \rho_{cp})$ respectively change in length and curvature. For example, Figure 22 illustrates isolines in a conformal $(\xi, \eta) = \mathcal{T}_{x \rightarrow \ell}(z_x, \rho_x)$. The i^{th} red curve from the left is an isoline over which ξ is constant $\xi_0 + \Delta\xi(i - 1)$. The j^{th} blue curve from the bottom is an isoline over which $\eta = \eta_0 + \Delta\eta(j - 1)$. The red isolines differ substantially in curvature. The blue ones differ substantially in length and curvature. As a result of these differences in curvature and length, the upper rectangular cells bounded by the crossing isolines are smaller than the lower ones. As $\Delta\xi$ and $\Delta\eta \rightarrow 0$, moreover, the areas of these rectangular cells approach $|J_{\ell \rightarrow x}(\mathcal{T}_{x \rightarrow \ell}(\rho_x, z_x))| \times \Delta\xi \times \Delta\eta$. Thus, change in isoline curvature and length over space leads to change in the determinant of the Jacobian over space.

Figure 22: Change in Curvature and Length in Isolines Leads to Change in Jacobian Determinant

Isolines in a conformal mapping $(\xi, \eta) = \mathcal{T}_{x \rightarrow \ell}(z_x, \rho_x)$, which may be either $\mathcal{T}_{o \rightarrow \ell}(z_o, \rho_o)$ or $\mathcal{T}_{cp \rightarrow \ell}(z_{cp}, \rho_{cp})$. The i^{th} red curve from the left is an isline over which ξ is constant $\xi_0 + \Delta\xi(i - 1)$. The j^{th} blue curve from the bottom is an isline over which $\eta = \eta_0 + \Delta\eta(j - 1)$. The red isolines differ substantially in curvature. The blue ones differ substantially in length and curvature. As a result of these differences in curvature and length, the upper rectangular cells bounded by the crossing isolines are smaller than the lower ones. As $\Delta\xi, \Delta\eta \rightarrow 0$, moreover, the areas of these rectangular cells approach $|J_{\ell \rightarrow x}(\mathcal{T}_{x \rightarrow \ell}(\rho_x, z_x))| \times \Delta\xi \times \Delta\eta$. Thus, change in isline curvature and length over space leads to change in the Jacobian determinant over space.

The sides of Q_o and Q_{cp} , moreover, are isolines of $\mathcal{T}_{o \rightarrow \ell}$ and $\mathcal{T}_{cp \rightarrow \ell} \equiv \mathcal{T}_{\ell \rightarrow cp}^{-1}$

respectively. Hence, when opposite sides of Q_o and Q_{cp} have greater difference in length, or when one side curves in ways uncoordinated with its opposite, $|J_{\ell \rightarrow o}(\xi, \eta)|$ and $|J_{\ell \rightarrow cp}(\xi, \eta)|$, respectively, generally take on a greater range of values, and Δn_{cp} and Δn_{cp}^w , it is hypothesized, are more likely to be large. For each of the tests, reported below, this “isoline principle” is used to predict and interpret the results.

3.3 Test Procedure

The analysis is conducted with the following procedure. First, a particular Q_o is chosen. Then Q_{cp} is defined by a parameterized formula. The value of each Q_{cp} parameter is varied over several test trials, while the others are held fixed. For each test case, $\Delta \hat{n}_{cp}$ and $\Delta \hat{n}_{cp}^w$ are calculated. Finally, these data points are used to assess the effects of the quadrilateral parameters on index range and determine the final design.

3.3.1 The Structure of Q_o

The chosen structure of Q_o is diagrammed in Figure 23. The bottom side of Q_o is the diameter of Maxwell's fish-eye lens that is on the optical axis: $\{(z_o, 0) \in U_o : |z_o| \leq R_o\}$. R_o is the radius of Maxwell's fish-eye. R_o simply scales $|J_{\ell \rightarrow o}(\xi, \eta)|$ uniformly; hence it has no effect on $\Delta|J_{\ell \rightarrow o}|$ or $\Delta|J_{\ell \rightarrow o}|^w$ and is irrelevant to $\Delta\hat{n}_{cp}$ and $\Delta\hat{n}_{cp}^w$. Arbitrarily, it might be set to 1 mm. Meanwhile, the left and right sides are: $\{(\mp R_o \cos \theta, R_o \sin \theta) \in U_o : 0 \leq \theta \leq \varphi_o\}$, where $\varphi_o = 80^\circ$; where the negative sign corresponds to the left side; and where the positive sign to the right side. The top side is symmetric in z_o . Each half of this symmetric top side is defined by the cubic parametric equations in (78) and (79) for $0 \leq s \leq 1$, where the top signs in the ' $\mp$ ' and ' $\pm$ ' correspond to the left part; and the bottom signs to the right segment. (78) specifies $z_{oT}(s)/R_o$, the ratio of the z_o -coordinate of the top side to R_o , as a function of the parameter s . (79) similarly specifies $\rho_{oT}(s)/R_o$, the ratio of the ρ_o -coordinate to R_o as a function of s . The top is configured such that it forms a 90° angle with the side, such that it is level at the midpoint, at height $h_o = .8R_o$, and, arbitrarily, such that $|z_{oT}'(0)| = \frac{1}{4} \times R_o \cos \varphi_o$.

$$z_{oT}(s)/R_o = \mp \cos \varphi_o \pm \frac{1}{4} \cos \varphi_o s + \left\{ \pm \frac{5}{2} \cos \varphi_o - \frac{1}{4} \right\} s^2 + \left\{ \mp 1.75 \cos \varphi_o + \frac{1}{4} \right\} s^3 \quad (78)$$

$$\rho_{oT}(s)/R_o = \sin \varphi_o - \frac{1}{4} \sin \varphi_o s + \left\{ -\frac{5}{2} \sin \varphi_o + 3h_o \right\} s^2 + \{1.75 \sin \varphi_o - 2h_o\} s^3 \quad (79)$$

Figure 23: Q_o
The chosen structure of Q_o

While it might be useful in general to study variations in Q_o , in addition to Q_{cp} , this analysis only varies Q_{cp} parameters. For all tests, $\varphi_o = 80^\circ$ and $h_o = .8R_o$.

3.3.2 Parameterization of Q_{cp}

For this analysis, Q_{cp} is defined with the parameters R_{cp} , W_0 , ϕ , H_D , and a depicted in Figure 23. R_{cp} is the radius of the monocentric imager. Since this reimager is designed based on the monocentric of [3], $R_{cp} = 11.9885 \text{ mm}$. W_0 , meanwhile, is the length of the reimager along the optical axis. As shown in Figure 24, the bottom side of Q_{cp} is a horizontal line segment on the z_{cp} -axis: $\{(z_{cp}, 0) \in U_{cp} : R_{cp} \leq z_{cp} \leq R_{cp} + W_0\}$. The left side, additionally, is a subset of the image, under $\mathcal{T}_{\mathcal{P}_{cp}}(x_{cp}, y_{cp}, z_{cp})$, of the monocentric's image surface. It subtends angle $\frac{\phi}{2}$.

The monocentric lens' center of curvature corresponds to the origin of U_{cp} ; hence the left side is:

$\{(R_{cp} \cos \theta, R_{cp} \sin \theta) \in U_{cp} : 0 \leq \theta \leq \frac{\phi}{2}\}$. $\frac{\phi}{2}$ must be at least the desired half field of view. By

contrast, the right side is a straight vertical line segment of length $\frac{H_D}{2}$, where H_D is the full

diameter of the flat back surface, where the detector can be placed. It is the set $\{(R_{cp} + W_0, \rho_{cp}) \in U_{cp} : 0 \leq \rho_{cp} \leq \frac{H_D}{2}\}$.

Figure 24: Q_{cp}

Q_{cp} is parameterized by R_{cp} , W_0 , ϕ , $\frac{H_D}{2}$ and a (a is directly related to H)

The top side of Q_{cp} , shown in red in Figure 24, is defined by parametric equations,

$(z_{cp}^{(top)}(s), \rho_{cp}^{(top)}(s))$, which are polynomial functions of s for $0 \leq s \leq 1$. For smoothness, the curve is chosen to have third-order geometric (G^3) continuity – i.e. if s represented time, the path's linear and angular acceleration would be continuous. According to [25], the set of polynomials of the lowest order that can be parametric equations of a G^3 curve, given any particular set of starting and ending points; starting and ending tangential angles; starting and ending curvatures; and starting and ending angular accelerations, are the seventh-order ones that [25] describes, in which the coefficients are functions of these eight boundary conditions along with six other parameters: $\eta_1, \eta_2, \eta_3, \eta_4, \eta_5$, and η_6 . The top side of Q_{cp} is thus defined by such a pair of polynomials. The starting point $(z_{cp}^{(i)}, \rho_{cp}^{(i)})$ and ending point $(z_{cp}^{(f)}, \rho_{cp}^{(f)})$ used are the uppermost points of the left and right sides respectively:

$$\left(z_{cp}^{(i)}, \rho_{cp}^{(i)}\right) = \left(R_{cp} \cos \frac{\phi}{2}, R_{cp} \sin \frac{\phi}{2}\right) \quad (80)$$

$$\left(z_{cp}^{(f)}, \rho_{cp}^{(f)}\right) = \left(R_{cp} + W_0, \frac{H_D}{2}\right) \quad (81)$$

Because the boundaries of Q_{cp} are isolines of the conformal $\mathcal{T}_{cp \rightarrow \ell}(z_{cp}, \rho_{cp})$, moreover, the right and left sides of Q_{cp} necessarily meet the top and bottom sides of Q_{cp} at right angles. Therefore, the starting and ending tangential angles of the top curve are the ones that form right angles with the adjacent sides. The starting tangential angle is:

$$\psi_i = \frac{\phi}{2} \quad (82)$$

and the ending one is:

$$\psi_f = 0^\circ \quad (83)$$

In all test cases, the initial and final angular “acceleration” (second-order derivative with respect to s) is arbitrarily set to 0.

$$\dot{\kappa}_i = 0 \quad (84)$$

$$\dot{\kappa}_f = 0 \quad (85)$$

Additionally, the initial curvature is set to $-\frac{1}{R_{cp}}$, while the final curvature is set to 0:

$$\kappa_i = -1/R_{cp} \quad (86)$$

$$\kappa_f = 0 \quad (87)$$

"G3-Splines for the Path Planning of Wheeled Mobile Robots" ([25]) furthermore states that, given the boundary conditions, an approximately optimally smooth pair of polynomials is, in many “typical” cases, the one in which η_1 and η_2 are both set to the distance between the starting and ending points, and the other η ’s are set to 0:

$$\eta_1 = \eta_2 = \left\| \left(z_{cp}^{(f)}, \rho_{cp}^{(f)}\right) - \left(z_{cp}^{(i)}, \rho_{cp}^{(i)}\right) \right\| \quad (88)$$

$$\eta_3 = \eta_4 = \eta_5 = \eta_6 = 0 \quad (89)$$

Based on experimentation, we find that if

$$\eta_1 = \eta_2 = a \left\| \left(z_{cp}^{(f)}, \rho_{cp}^{(f)} \right) - \left(z_{cp}^{(i)}, \rho_{cp}^{(i)} \right) \right\| \quad (90)$$

then variation of the parameter a linearly affects the overall height H of the reimager. ($\frac{H}{2}$ is

illustrated in Figure 24.) Thus, $z_{cp}^{(top)}(s)$ and $\rho_{cp}^{(top)}(s)$ are provided by (91) and (92). The a and b coefficients in these formulas are obtained by substitution of (80)-(87), (89) and (90) into (93)-(108).

$$z_{cp}^{(top)}(s) = a_0 + a_1 s + a_2 s^2 + a_3 s^3 + a_4 s^4 + a_5 s^5 + a_6 s^6 + a_7 s^7 \quad (91)$$

$$\rho_{cp}^{(top)}(s) = b_0 + b_1 s + b_2 s^2 + b_3 s^3 + b_4 s^4 + b_5 s^5 + b_6 s^6 + b_7 s^7 \quad (92)$$

$$a_0 = z_{cp}^{(i)} \quad (93)$$

$$a_1 = \eta_1 \cos \psi_i \quad (94)$$

$$a_2 = -\frac{1}{2} \eta_1^2 \kappa_i \sin \psi_i \quad (95)$$

$$a_3 = 0 \quad (96)$$

$$a_4 = 35 \left(z_{cp}^{(f)} - z_{cp}^{(i)} \right) - CI \cos \psi_i + SI \sin \psi_i - CF \cos \psi_f - SF \sin \psi_f \quad (97)$$

$$\text{where } CI = 20\eta_1; SI = 5\eta_1^2 \kappa_i; CF = 15\eta_2; SF = \frac{5}{2} \eta_2^2 \kappa_f.$$

$$a_5 = -84 \left(z_{cp}^{(f)} - z_{cp}^{(i)} \right) + CI \cos \psi_i - SI \sin \psi_i + CF \cos \psi_f + SF \sin \psi_f \quad (98)$$

$$\text{where } CI = 45\eta_1; SI = 10\eta_1^2 \kappa_i; CF = 39\eta_2; SF = 7\eta_2^2 \kappa_f$$

$$a_6 = 70 \left(z_{cp}^{(f)} - z_{cp}^{(i)} \right) - CI \cos \psi_i + SI \sin \psi_i - CF \cos \psi_f - SF \sin \psi_f \quad (99)$$

$$\text{where } CI = 36\eta_1; SI = \frac{15}{2} \eta_1^2 \kappa_i; CF = 34\eta_2; SF = \frac{13}{2} \eta_2^2 \kappa_f$$

$$a_7 = -20 \left(z_{cp}^{(f)} - z_{cp}^{(i)} \right) + CI \cos \psi_i - SI \sin \psi_i + CF \cos \psi_f + SF \sin \psi_f \quad (100)$$

$$\text{where } CI = 10\eta_1; SI = 2\eta_1^2 \kappa_i; CF = 10\eta_2; SF = 2\eta_2^2 \kappa_f$$

$$b_0 = \rho_{cp}^{(i)} \quad (101)$$

$$b_1 = \eta_1 \sin \psi_i \quad (102)$$

$$b_2 = \frac{1}{2} \eta_1^2 \kappa_i \cos \psi_i \quad (103)$$

$$b_3 = 0 \quad (104)$$

$$b_4 = 35 \left(\rho_{cp}^{(f)} - \rho_{cp}^{(i)} \right) - CI \cos \psi_i - SI \sin \psi_i + CF \cos \psi_f - SF \sin \psi_f \quad (105)$$

where $CI = 5\eta_1^2 \kappa_i$; $SI = 20\eta_1$; $CF = \frac{5}{2}\eta_2^2 \kappa_f$; $SF = 15\eta_2$

$$b_5 = -84 \left(\rho_{cp}^{(f)} - \rho_{cp}^{(i)} \right) + CI \cos \psi_i + SI \sin \psi_i - CF \cos \psi_f + SF \sin \psi_f \quad (106)$$

where $CI = 10\eta_1^2 \kappa_i$; $SI = 45\eta_1$; $CF = 7\eta_2^2 \kappa_f$; $SF = 39\eta_2$

$$b_6 = 70 \left(\rho_{cp}^{(f)} - \rho_{cp}^{(i)} \right) - CI \cos \psi_i - SI \sin \psi_i + CF \cos \psi_f - SF \sin \psi_f \quad (107)$$

where $CI = \frac{15}{2}\eta_1^2 \kappa_i$; $SI = 36\eta_1$; $CF = \frac{13}{2}\eta_2^2 \kappa_f$; $SF = 34\eta_2$

$$b_7 = -20 \left(\rho_{cp}^{(f)} - \rho_{cp}^{(i)} \right) + CI \cos \psi_i + SI \sin \psi_i - CF \cos \psi_f + SF \sin \psi_f \quad (108)$$

where $CI = 2\eta_1^2 \kappa_i$; $SI = 10\eta_1$; $CF = 2\eta_2^2 \kappa_f$; $SF = 10\eta_2$

3.3.3 List of Tests

Table 2 lists the tests performed. The first test is intended to determine the effect on index range of W_0 , the depth of the reimager. Test 2 varies H_D , in order to assess the effect of the height of the back detector plane on the index range. The third test varies ϕ and gauges the relationship between the index range and the angle subtended by the front surface of the reimager. Finally, the fourth test varies the parameter a defined in (90). This test effectively varies the overall height H of the reimager with each trial, and thus assesses the relationship between H and index range.

Test #	Q_{cp} Parameter			
	W_0/R_{cp}	ϕ	H_D/R_{cp}	a
1	{1.75,2, ...,5.5}	180°	2.5	1
2	3.5	180°	{.75,1, ...,5.5}	1
3	3	{130°,131°, ...,180°}	3.5	1
4	2.5	180°	2.5	{0.5,0.6, ...,1.5}

Table 2: List of Tests

3.4 Test Results

3.4.1 Test 1: Effect of W_0 on $\Delta\hat{n}_{cp}$ and $\Delta\hat{n}_{cp}^w$

Using the parameters listed in the first row of Table 2, this test investigates the effect of W_0 on index range. W_0 is varied from $1.75R_{cp}$ to $5.5R_{cp}$ in increments of $.25R_{cp}$. The resulting $\Delta\hat{n}_{cp}$ and $\Delta\hat{n}_{cp}^w$ measurements are plotted in Figure 25a and Figure 25b.

Figure 25a: $\Delta\hat{n}_{cp} \text{ v. } W_0$
 $\Delta\hat{n}_{cp}$ decreases with W_0 .

Figure 25b: $\Delta\hat{n}_{cp}^w \text{ v. } W_0$
 $\Delta\hat{n}_{cp}^w$ decreases with W_0 .

Both $\Delta\hat{n}_{cp}$ and $\Delta\hat{n}_{cp}^w$ decrease with W_0 . This decrease is interpreted to be a result of the following factors.

Firstly, the isolines corresponding to constant η -value of the function $(\xi, \eta) = \mathcal{T}_{cp \rightarrow \ell}(z_{cp}, \rho_{cp})$, as described in section 3.2, become smoother as W_0 is increased. For instance, Figure 26 compares curvatures of the top side of Q_{cp} for different trials. The curvature is measured at the point where $\xi = \xi_{min}$ (the blue curve) as well as where $\xi = \xi_{min}^w$ (the red curve). Additionally, for each trial case in this test, the curvature of the isoline $\eta = \eta_{min}^w$ of $\mathcal{T}_{cp \rightarrow \ell}(z_{cp}, \rho_{cp})$ is computed at the point where $\xi = \xi_{min}^w$ (the green curve). These isolines and points are depicted in Figure 27. As W_0 increases, these curvatures diminish in magnitude. This smoothing accounts for the decrease in $1/\sqrt{\Delta|J_{\ell \rightarrow cp}|}$ and $1/\sqrt{\Delta|J_{\ell \rightarrow cp}|^w}$ with W_0 , which is

shown in Figure 28a and b. The decrease in $1/\sqrt{|\Delta J_{\ell \rightarrow cp}|}$ and $1/\sqrt{|\Delta J_{\ell \rightarrow cp}|^w}$ is, thus, one contributing factor to the decrease in $\Delta \hat{n}_{cp}$ and $\Delta \hat{n}_{cp}^w$ with W_0 .

Figure 26: Isoline Curvatures

This figure shows the effect of W_0 on the curvatures of η -isolines of the function $(\xi, \eta) = \mathcal{T}_{cp \rightarrow \ell}(z_{cp}, \rho_{cp})$. The dependent axis is the curvature, κ , in terms of R_{cp}^{-1} . The blue curve is the curvature of the top side of Q_{cp} at the point where $\xi = \xi_{min}$. This point is marked in Figure 27 by the pentagram. The red curve, meanwhile, is the computed curvature of the top side of Q_{cp} at the point where $\xi = \xi_{min}^w$, demarcated in Figure 27 by the downward-pointing triangle. The green curve is the computed curvature of the η -isoline $\eta = \eta_{min}^w$ of $\mathcal{T}_{cp \rightarrow \ell}(z_{cp}, \rho_{cp})$ at the point where $\xi = \xi_{min}^w$. This point is indicated in Figure 27 by the hexagram.

Figure 27: Isolines and Points at Which Their Curvature Is Measured

This diagram illustrates the isolines and points at which their curvatures are measured in Figure 26. For all tested W_0 , the maximum index is consistently located at $(z_{cpmax}, \rho_{cpmax}) = (z_{cpmax}^w, \rho_{cpmax}^w) = (R_{cp}, 0)$, as indicated by the circular marker. At this position, the η -isoline is the perfectly straight bottom side of Q_{cp} . The lower multi-colored arc is the isoline $\eta = \eta_{min}^w$. The upper multi-colored arc is the top side of Q_{cp} . It is also the isoline $\eta = M_{cp} = \eta_{min}$, where M_{cp} is the conformal module of Q_{cp} . These isolines are colored according to their curvatures. The hexagram marks the point on the isoline $\eta = \eta_{min}^w$ at which $\xi = \xi_{min}^w$. The pentagram marks the point on the upper isoline at which $\xi = \xi_{min}$. The down-ward pointed triangle represents the point on the upper isoline at which $\xi = \xi_{min}$.

Figure 28a: Three Components of $\Delta \hat{n}_{cp}$ v. W_0

Figure 28b: Three Components of $\Delta \hat{n}_{cp}^w$ v. W_0

In a decibel scale, $\Delta\hat{n}_{cp}$ is plotted alongside its 3 contributing factors: $\sqrt{\Delta|J_{\ell \rightarrow o}|}$, $1/\sqrt{\Delta|J_{\ell \rightarrow cp}|}$, and $\Delta n_{fish-eye}$ in (65). The 3 factors, in decibel form, sum up to $\Delta\hat{n}_{cp}$ in decibel form. For example, the decibel form of a quantity x is $10\log_{10}(x)$.

In a decibel scale, $\Delta\hat{n}_{cp}^w$ is plotted alongside the 3 factors $\sqrt{\Delta|J_{\ell \rightarrow o}^w|}$, $1/\sqrt{\Delta|J_{\ell \rightarrow cp}^w|}$, and $\Delta n_{fish-eye}$ in (66). The 3 factors, in decibel form, sum up to $\Delta\hat{n}_{cp}^w$ in decibel form.

An additional reason for the decrease in index ratio with W_0 is the decrease in the conformal module of Q_{cp} . In Figure 29, measurements of the conformal module of the tested Q_{cp} are plotted as a function of W_0 . The conformal module goes down with W_0 . Consequently, the preimage of the interior of Q_{cp} under $\mathcal{T}_{o \rightarrow cp}(z_o, \rho_o)$ occupies a shorter and shorter subset of the interior of Q_o . The shorter this subset, the less its top η -isolines of $\mathcal{T}_{o \rightarrow \ell}(z_o, \rho_o)$ differ from the straight base of Q_o , and thus the less $\Delta|J_{\ell \rightarrow o}|$ and $\Delta|J_{\ell \rightarrow o}|^w$ are. This decrease in $\Delta|J_{\ell \rightarrow o}|$ and $\Delta|J_{\ell \rightarrow o}|^w$ with W_0 is seen from the plots of calculated $\Delta|J_{\ell \rightarrow o}|$ and $\Delta|J_{\ell \rightarrow o}|^w$ for the different W_0 in Figure 28a and b.

Figure 29: Conformal Module of Q_{cp} v. W_0

The conformal module of Q_{cp} decreases with W_0 .

Figure 30a and b illustrate the paths of ray bundles through the reimager for the two cases in which $W_0 = 1.75R_{cp}$ and $W_0 = 5.5R_{cp}$ respectively. The ray bundles in the longer reimager

are bent more gradually, which necessitates a narrower index range, in agreement with the finding that a longer reimager corresponds to a lower $\Delta\hat{n}_{cp}$ and $\Delta\hat{n}_{cp}^w$.

Figure 30a: Ray Bundles through Reimager with $W_0 = 1.75R_{cp}$

The ray bundles propagating through the shorter reimager are bent more severely.

Figure 30b: Ray Bundles through Reimager with $W_0 = 5.5R_{cp}$

The ray bundles propagating through the longer reimager are bent more gradually.

Based on these results, it is concluded that a longer reimager is more likely to be manufacturable. The larger W_0 , the narrower the range of index values is required.

3.4.2 Test 2: Effect of H_D on $\Delta\hat{n}_{cp}$ and $\Delta\hat{n}_{cp}^w$

In this test, the effect of the height of the back surface of the reimager, or image height, on the index range is gauged. Q_{cp} is generated with the parameters in the second row of Table 2. H_D , the full height of the back plane of the reimager, is varied from $.75R_{cp}$ to $5.5R_{cp}$ in increments of $.25R_{cp}$. The different Q_{cp} 's corresponding to each test case are shown in Figure 31. The test is created with the hypothesis that the index range is likely to be narrower when H_D is close to values that correspond to an average unit magnification of the image: $H_D \approx \pi R_{cp}$.

Figure 31: How the Top Side of Q_{cp} varies with H_D

The top side of Q_{cp} terminates at the coordinate $\rho_{cp} = \frac{H_D}{2}$. As H_D is increased, the top side of Q_{cp} becomes smoother. It also becomes taller, which increases the conformal module. As $H_D \rightarrow 0$, the back side of Q_{cp} becomes increasingly compressed. The case where $H_D = 2R_{cp}$ is emboldened, since it has the narrowest index range.

The computed $\Delta \hat{n}_{cp}$ and $\Delta \hat{n}_{cp}^w$ for each value of H_D are plotted in Figure 32a and b. In both cases, for $H_D < 2R_{cp}$, $\Delta \hat{n}_{cp}$ and $\Delta \hat{n}_{cp}^w$ decrease with R_{cp} . The narrowest index range occurs for $H_D = 2R_{cp}$. Then, as H_D is increased beyond $2R_{cp}$, the index range widens.

Figure 32a: $\Delta \hat{n}_{cp}$ v. H_D

$\Delta \hat{n}_{cp}$ decreases with H_D for $H_D < 2R_{cp}$ and increases with H_D for $H_D > 2R_{cp}$.

Figure 32b: $\Delta \hat{n}_{cp}^w$ v. H_D

$\Delta \hat{n}_{cp}^w$ decreases with H_D for $H_D < 2R_{cp}$ and increases with H_D for $H_D > 2R_{cp}$. (The points with the red x's represent cases in which Q_{cp} does not encompass the area needed for all the rays paths originating within 60° of the axis of symmetry.)

The widening beyond $H_D = 2R_{cp}$ happens despite the fact that larger H_D values yield smoother top sides of Q_{cp} . This increased smoothness is evidenced by Figure 33, which plots the curvature of the top side of Q_{cp} at the point where $\xi = \xi_{min}$ as a function of H_D . (This point is illustrated by the downward-pointing triangle in Figure 27.) What does account for the increase in index ranges is the rise in the conformal module of Q_{cp} with H_D . This rise is depicted in Figure 34, which plots the conformal module for each H_D . The greater the conformal module of Q_{cp} , the larger the subset of Q_o that is transformed by $\mathcal{T}_{o \rightarrow cp}(z_o, \rho_o)$ onto the area encompassed by Q_{cp} . Consequently, the more severe the reshaping effected by $\mathcal{T}_{o \rightarrow \ell}(z_o, \rho_o)$. In accordance with this speculation, $\sqrt{\Delta|J_{\ell \rightarrow o}|}$ and $\sqrt{\Delta|J_{\ell \rightarrow o}|^w}$ increase with H_D beyond $H_D = 2R_{cp}$, as shown in Figure 35a and b.

Figure 33: Curvature v. H_D

The curvature of the top side of Q_{cp} at the point where $\xi = \xi_{\min}$, as a function of H_D .

Figure 34: Conformal Module v. H_D

The conformal module increases with H_D .

Figure 35a: Three Components of $\Delta \hat{n}_{cp}$ v. H_D

$\Delta \hat{n}_{cp}$ is plotted alongside the three factors that contribute to it: $\sqrt{\Delta |J_{\ell \rightarrow o}|}$, $1/\sqrt{\Delta |J_{\ell \rightarrow cp}|}$, and $\Delta n_{fish-eye}$ in (65). The factor $\Delta n_{fish-eye}$ is mostly insignificant. The factor $1/\sqrt{\Delta |J_{\ell \rightarrow cp}|}$ decreases, since the increase in H_D results in smoother isolines of $\mathcal{T}_{cp \rightarrow \ell}(z_{cp}, \rho_{cp})$. Since the conformal module of Q_{cp} goes up, however, $\sqrt{\Delta |J_{\ell \rightarrow o}|}$ increases with H_D at a rate that, for $H_D > 2R_{cp}$, is greater than the rate at which the $1/\sqrt{\Delta |J_{\ell \rightarrow cp}|}$ decreases. For $H_D < 2R_{cp}$, on the other hand, the $1/\sqrt{\Delta |J_{\ell \rightarrow cp}|}$ decreases at a much faster rate with H_D . This behavior is because for $H_D < 2R_{cp}$, the back end of the reimager is compressed into such a short space that the minimum value of $|J_{\ell \rightarrow cp}(\mathcal{T}_{cp \rightarrow \ell}(z_{cp}, \rho_{cp}))|$ is changed in location to the back of the reimager, and as $H_D \rightarrow 0$, the Jacobian $|J_{\ell \rightarrow cp}(\mathcal{T}_{cp \rightarrow \ell}(z_{cp}, \rho_{cp}))|$ at the back of the reimager increases without bound.

Figure 35b: Three Components of $\Delta\hat{n}_{cp}^w$

$\Delta\hat{n}_{cp}^w$ is plotted alongside its three contributing factors $\sqrt{\Delta|J_{\ell \rightarrow o}^w|}$, $1/\sqrt{\Delta|J_{\ell \rightarrow cp}^w|}$, and $\Delta n_{fish-eye}$ in (66). The trends are similar to the ones corresponding to $\Delta\hat{n}_{cp}$ v. H_D .

When H_D is less than $2R_{cp}$, on the other hand, $\Delta\hat{n}_{cp}$ and $\Delta\hat{n}_{cp}^w$ decrease with R_{cp} . This decline happens because the back end of the reimager is compressed into such a short space that the minimum value of $|J_{\ell \rightarrow cp}(\mathcal{T}_{cp \rightarrow \ell}(z_{cp}, \rho_{cp}))|$ is no longer at its usual $(z_{cp}, \rho_{cp}) = (R_{cp}, 0)$, but rather at the back end, with a larger z_{cp} -coordinate. This change in location is supported by Figure 36, a plot of the computed location of z_{cpmax} as a function of H_D ($z_{cpmax} = z_{cpmax}^w$). For trials with $H_D < 2R_{cp}$, $z_{cpmax} \approx 4R_{cp}$. As H_D is decreased, the back of Maxwell's fish-eye lens is more and more severely compressed, accounting for the increase in the factors $1/\sqrt{\Delta|J_{\ell \rightarrow cp}|}$ and $1/\sqrt{\Delta|J_{\ell \rightarrow cp}|^w}$ as $H_D \rightarrow 0$ evident in Figure 35a and b. Hence the negative slope in the $\Delta\hat{n}_{cp}$ v. H_D and $\Delta\hat{n}_{cp}^w$ v. H_D profiles.

Figure 36: z_{cpmax} v. H_D

z_{cpmax} as a function of H_D . In the cases where $H_D < 2R_{cp}$, the back of the reimager is so short that the compression implemented by $\mathcal{T}_{\ell \rightarrow cp}(\xi, \eta)$ results in particularly small values of $|J_{\ell \rightarrow cp}(\mathcal{T}_{cp \rightarrow \ell}(z_{cp}, \rho_{cp}))|$

Figure 37a, b, and c show traces of ray bundles, from field angles of 0° and 60° , through the reimager for the cases in which $H_D = 0.75R_{cp}$, $H_D = 2R_{cp}$, and $H_D = 5.5R_{cp}$ respectively. (The 60° bundle in the $H_D = 0.75R_{cp}$ case is not shown because this Q_{cp} does not encompass enough points to image a 60° field of view.) On the one hand, in the case where $H_D = 0.75R_{cp}$, the rays must be abruptly straightened in order to fit inside the compressed back of the reimager. This abrupt change in direction necessitates a wider index range. On the other hand, in the case where $H_D = 5.5R_{cp}$, the 60° ray bundle is elongated, compared to the one in the case where $H_D = 2R_{cp}$. A wider index range is also needed for this stretching. In this way, these ray bundle diagrams in Figure 37a-c provide a physical interpretation of the trends in Figure 32a and b.

Figure 37a: Ray Bundles for the Case in Which $H_D = 0.75 R_{cp}$

Figure 37b: Ray Bundles for the Case in Which $H_D = 2 R_{cp}$

Figure 37c: Ray Bundles for the Case in Which $H_D = 5.5 R_{cp}$

Traces of ray bundles, from field angles of 0° and 60° . (The 60° bundle in the $H_D = 0.75 R_{cp}$ case is not shown because this Q_{cp} does not encompass enough points to image a 60° field of view.) In the case where $H_D = 0.75 R_{cp}$, the rays must be abruptly straightened in order to fit inside the compressed back of the reimager. In the case where $H_D = 5.5 R_{cp}$, the 60° ray bundle is elongated, compared to the one in the case where $H_D = 2 R_{cp}$. Thus the $H_D = 0.75 R_{cp}$ and $H_D = 5.5 R_{cp}$ cases correspond to a higher $\Delta \hat{n}_{cp}$ and $\Delta \hat{n}_{cp}^w$ than the case in which $H_D = 2 R_{cp}$.

This test suggests that there is an optimal H_D for the reimager. In this test, the optimal value is $H_D = 2 R_{cp}$. It is anticipated in general, however, that the optimum might depend on the other Q_{cp} parameters, since the value $2 R_{cp}$ corresponds to the point at which the set of Q_{cp} repositions the minimum $|J_{\ell \rightarrow cp}(\mathcal{T}_{cp \rightarrow \ell}(z_{cp}, \rho_{cp}))|$.

3.4.3 Test 3: Effect of ϕ on $\Delta \hat{n}_{cp}$ and $\Delta \hat{n}_{cp}^w$

In this test, the parameter ϕ is varied from 130° to 180° in increments of 1° , in order to ascertain its effect on the index range. The higher ϕ , the more the top side of Q_{cp} must turn. The more the top side of Q_{cp} turns, the more the light ray trajectories are distorted by the transformation $\mathcal{T}_{\ell \rightarrow cp}(\xi, \eta)$. The test is therefore undertaken with the hypothesis that lower ϕ produces narrower index range.

Figure 38a and b plot the calculated $\Delta \hat{n}_{cp}$ and $\Delta \hat{n}_{cp}^w$ for each value of ϕ . In Figure 38a, the index range widens with ϕ , as predicted. This increase, moreover, as hypothesized, is the

result of the increase in the factor $1/\sqrt{\Delta|J_{\ell \rightarrow cp}|}$ with ϕ , as demonstrated by Figure 39a. In Figure 38b, on the other hand, the change in $\Delta\hat{n}_{cp}^w$ is small. The measurements plotted with red x's represent cases in which Q_{cp} does not encompass the area needed for all the rays paths originating within 60° of the axis of symmetry – ϕ must be at least 135° in order to image the $\pm 60^\circ$ field of view. As for the values of ϕ that do successfully image the $\pm 60^\circ$ field of view, plotted in blue, $\Delta\hat{n}_{cp}^w$ increases very slightly with ϕ between 135° and around 160° , then decreases slightly as ϕ is increased beyond 160° . Figure 39b decomposes the $\Delta\hat{n}_{cp}^w$ measurements into the three contributing factors $\sqrt{\Delta|J_{\ell \rightarrow o}^w|}$, $1/\sqrt{\Delta|J_{\ell \rightarrow cp}^w|}$, and $\Delta n_{\text{fish-eye}}$. As predicted, the factor $1/\sqrt{\Delta|J_{\ell \rightarrow cp}^w|}$ does increase with ϕ , but less dramatically than $1/\sqrt{\Delta|J_{\ell \rightarrow cp}|}$ in the $\Delta\hat{n}_{cp}$ case. The factor $\sqrt{\Delta|J_{\ell \rightarrow o}^w|}$, meanwhile, decreases with ϕ , more than in the $\Delta\hat{n}_{cp}$ case.

Figure 38a: $\Delta\hat{n}_{cp}$ v. ϕ

$\Delta\hat{n}_{cp}$ increases with ϕ .

Figure 38b: $\Delta\hat{n}_{cp}^w$ v. ϕ

$\Delta\hat{n}_{cp}^w$ increases with ϕ up to a certain point and then levels off. (The points with the red x's represent cases in which Q_{cp} does not encompass the area needed for all the rays paths originating within 60° of the axis of symmetry. These reimagers would not successfully image the $\pm 60^\circ$ field of view.)

Figure 39a: Three Components of $\Delta\hat{n}_{cp}$ v. ϕ

$\Delta\hat{n}_{cp}$ is plotted alongside the three factors components $\sqrt{\Delta|J_{\ell \rightarrow o}|}$, $1/\sqrt{\Delta|J_{\ell \rightarrow cp}|}$, and $\Delta n_{fish-eye}$ in (65), in a decibel scale. The increase in $\Delta\hat{n}_{cp}$ is the result of the increase in the factor $1/\sqrt{\Delta|J_{\ell \rightarrow cp}|}$.

Figure 39b: Three Components of $\Delta\hat{n}_{cp}$ v. ϕ

$\Delta\hat{n}_{cp}^w$ is plotted alongside the three factors $\sqrt{\Delta|J_{\ell \rightarrow o}^w|}$, $1/\sqrt{\Delta|J_{\ell \rightarrow cp}^w|}$, and $\Delta n_{fish-eye}$ in (66), in a decibel scale.

Based on these test results, it is necessary that ϕ be sufficiently large in order to image the desired field of view. Furthermore, within the acceptable range of ϕ , the index range needed for the $\pm 60^\circ$ field of view is narrowest when $\phi = 180^\circ$. Hence this test suggests that a full hemispherical concavity be used for the surface of the reimager.

3.4.4 Test 4: Effect of Overall Reimager Height on $\Delta\hat{n}_{cp}$ and $\Delta\hat{n}_{cp}^w$

This test seeks to determine the effect of the overall height of the reimager on index range. It uses the Q_{cp} parameters listed in the last row of Table 2. In order to make the overall height H vary, the parameter a in (90) is changed from .5 to 1.5 in increments of .1. The overall height of Q_{cp} increases linearly with a . This relationship is demonstrated in Figure 40, which illustrates the Q_{cp} corresponding to each tested a , as well as by the measurements of overall height H as a function of a plotted in Figure 41. The test is undertaken with the expectation that the Q_{cp} 's with values of a nearest 1 yield the narrowest index ranges, since their top sides are likely to be the smoothest, according to [25].

Figure 40: Q_{cp} for Different Values of a

The parameter a is used to vary the overall height of the reimager.

Figure 41: H v. a

For each test case, the overall height H of the reimager is measured. (The parameter H is illustrated in Figure 24.) H varies linearly with a .

The calculated $\Delta\hat{n}_{cp}$ and $\Delta\hat{n}_{cp}^w$ for each test are plotted in Figure 42a and b respectively.

As expected, $\Delta\hat{n}_{cp}$ is minimal at values of a between 1 and 1.1. $\Delta\hat{n}_{cp}^w$, on the other hand, is minimal around $a = .7$.

Figure 42a: $\Delta\hat{n}_{cp}$ v. a

$\Delta\hat{n}_{cp}$ is minimal at values of a close to 1 or 1.1.

Figure 42b: $\Delta \hat{n}_{cp}^w$ v. a
The minimum $\Delta \hat{n}_{cp}^w$ is at $a \approx .7$.

Figure 43a plots the factors $\sqrt{\Delta |J_{\ell \rightarrow o}|}$, $1/\sqrt{\Delta |J_{\ell \rightarrow cp}|}$, $\Delta n_{fish-eye}$; Figure 43b plots

$\sqrt{\Delta |J_{\ell \rightarrow o}^w|}$, $1/\sqrt{\Delta |J_{\ell \rightarrow cp}^w|}$, and $\Delta n_{fish-eye}^w$. From Figure 43a and b, $1/\sqrt{\Delta |J_{\ell \rightarrow cp}|}$ and $1/\sqrt{\Delta |J_{\ell \rightarrow cp}^w|}$,

and therefore the transformation $\mathcal{T}_{\ell \rightarrow cp}(\xi, \eta)$, are the predominant factors that influence the index

range. While the trend $\Delta \hat{n}_{cp}$ plausibly is highly correlated to the smoothness of the top side of

Q_{cp} , $\Delta \hat{n}_{cp}^w$, nevertheless, is more likely affected by the smoothness of the η -isolines of

$\mathcal{T}_{cp \rightarrow \ell}(z_{cp}, \rho_{cp})$ at the upper boundary of $\Omega^{(60)}$. While it is reasonable that smoothness in the top

side of Q_{cp} is linked with smoothness in the η -isolines of $\mathcal{T}_{cp \rightarrow \ell}(z_{cp}, \rho_{cp})$ at the top of $\Omega^{(60)}$, the

two factors are not perfectly correlated. Some looseness in this correlation, therefore, may

account for why the minimum $\Delta \hat{n}_{cp}^w$ occurs at $a = .7$, a little off from $a = 1$.

Figure 43a: Three Components of $\Delta \hat{n}_{cp}$ v. a

$\Delta \hat{n}_{cp}$ is plotted alongside the 3 contributing factors $\sqrt{\Delta|J_{\ell \rightarrow o}|}$, $1/\sqrt{\Delta|J_{\ell \rightarrow cp}|}$, and $\Delta n_{fish-eye}$ in (65). $1/\sqrt{\Delta|J_{\ell \rightarrow cp}|}$ is the predominant factor that influences $\Delta \hat{n}_{cp}$.

Figure 43b: Contributions of Three Components to $\Delta\hat{n}_{cp}^w$ v. a

$\Delta\hat{n}_{cp}^w$ is plotted alongside the three factors $\sqrt{\Delta|J_{\ell \rightarrow o}^w|}$, $1/\sqrt{\Delta|J_{\ell \rightarrow cp}^w|}$, and $\Delta n_{fish-eye}$ in (66). $1/\sqrt{\Delta|J_{\ell \rightarrow cp}^w|}$ is the predominant factor that influences the shape of $\Delta\hat{n}_{cp}^w$ v. a .

Figure 44a, b, and c show traces of ray bundles, from field angles of 0° and 60° , through the reimager for the cases in which $H = 2.9R_{cp}$, $H = 3.2R_{cp}$, and $H = 4.17R_{cp}$ respectively. The difference between the ray bundle trajectories in the $H = 2.9R_{cp}$ case and the $H = 3.2R_{cp}$ case does not appear significant. On the one hand, in the case where $H = 4.17R_{cp}$, the rays must arch higher. This longer, more arched path explains the necessity for the wider index range associated with the greater heights. In this way, these ray bundle diagrams in Figure 44a-c provide a physical interpretation of the trends in Figure 42a and b.

Figure 44a: Ray Bundles for the Case in Which $H = 2.9R_{cp}$

Figure 44b: Ray Bundles for the Case in Which $H = 3.2R_{cp}$

Figure 44c: Ray Bundles for the Case in Which $H = 4.17R_{cp}$

Traces of ray bundles, from field angles of 0° and 60° , through the reimager for the cases in which (a) $H = 2.9R_{cp}$, (b) $H = 3.2R_{cp}$, and (c) $H = 4.17R_{cp}$. In the case where $H = 4.17R_{cp}$, the rays must arch higher. This longer, more arched path explains the necessity for the wider index range.

The test results reveal that a and thus H have an optimal value. More vertical room for the reimager beyond the height associated with this optimum is not seen to improve the manufacturability – in fact requiring the light trajectories to take paths that are too high necessitates greater index variation.

Chapter 4 Design of a Particular Reimager

In this chapter, the design procedure documented in Chapter 3 is used to find the material prescriptions for a reimager that might be coupled to the particular monocentric lens specified by [3]. This monocentric lens has a radius of 11.9885 mm, an F/# of 1.71, and a full field of view of greater than 120° . The reimager is designed using a Q_{cp} with $W_0 = 4R_{cp}$, $\phi = 180^\circ$, $H_D = 3.5R_{cp}$, and $H = 4.549R_{cp}$. This Q_{cp} is depicted in Figure 14b.

Constitutive parameters

The calculated constitutive parameters for the reimager needed to reimage a $\pm 60^\circ$ field are plotted in Figure 45a-c. The permittivity is scaled so that the minimum value of $\varepsilon_{\phi_{cp}}$, the azimuthal component of the permittivity tensor, is 1. As a result, $\varepsilon_{\phi_{cp}}$ ranges in value from 1 to 25.6. It then works out that the radial and longitudinal components of the relative permittivity tensor, $\varepsilon_{\rho_{cp}}$ and $\varepsilon_{z_{cp}}$, shown in Figure 45b, range in value from 2.6 to 25.6. The azimuthal component of the permeability tensor, μ_ϕ , ranges from 0.2 to 1.0. From (23), the radial and longitudinal components of the permeability tensor are unity.

Figure 45a: Azimuthal Component of Permittivity Tensor

The permittivity is scaled so that the minimum value of $\epsilon_{\phi_{cp}}$, the radial component of the permittivity tensor, is 1. As a result, $\epsilon_{\phi_{cp}}$ ranges in value from 1 to 25.6.

Figure 45b: Radial and Longitudinal Components of the Permittivity Tensor

The radial and longitudinal components of the relative permittivity tensor, $\epsilon_{\rho_{cp}}$ and $\epsilon_{z_{cp}}$, range in value from 2.6 to 25.6.

Figure 45c: Azimuthal Component of Permeability Tensor

The azimuthal component of the permeability tensor, μ_ϕ , ranges from 0.2 to 1.0.

As seen from Figure 45c, the device requires a strong magnetic response in the azimuth direction: the relative permeability tensor component varies from approximately .2 to 1. As previously stated, such a response cannot be manufactured for optical frequencies using current metamaterial technology. This reimager design, therefore, is not currently manufacturable for visible wavelengths, although it might be built for IR light.

Spot sizes

Using the constitutive parameter profiles computed and displayed in Figure 45a-c, bundles of meridional rays are traced through a cross-section of the reimager. Each bundle spans $\pm 25^\circ$, approximately the angular extent of the cones of light entering the reimager from the F/1 monocentric lens. (The bundles at larger field angles are expected to have a narrower span, due to surface refraction.) The ray trace is shown in Figure 46, and the spot sizes are shown in Figure 47. The spot sizes of the 30° and 60° field angles at the back surface are less than 2.5 microns –

comparable to the size of the fibers bundles. The on-axis rays have a spot size of less than 6 microns. According to transformation optics theory, since the points imaged by Maxwell's fish-eye lens have no finite extent, at least in the geometric limit, then these spot sizes should also be 0. It is therefore not unreasonable to suppose that these small measured spot sizes might be a precision error, either in the discrete model of the constitutive parameters, or in the ray-tracer.

Meridional Rays through z -x Cross Section

Figure 46: Ray Trace

Bundles of meridional rays, from field angles of 0°, 30°, and 60°, are traced through a cross-section of the reimager. Each bundle spans $\pm 25^\circ$.

Figure 47: Spot Sizes of Meridional Rays

Light efficiency

Since n_0 in (20) can be chosen arbitrarily, the permittivity values in the reimager can be arbitrarily scaled. Moreover, the permittivity and permeability at the front face of the reimager determine the amount of light lost to Fresnel reflection. Therefore, different permittivity scales yield different light efficiencies. In order to gauge the effect of different n_0 values on light efficiency, the transmissivity, as a function of field angle, is calculated for different scales of the permittivity. Figure 48 plots the average transmissivity as a function of field angle for various n_0 . Figure 49a plots the transmissivity, averaged over all incident angles and field angles, as a function of n_0 . Figure 49b, meanwhile, plots the minimum, maximum, and average index of refraction experienced by meridionally-propagating light rays at the surface of the optic. The highest overall transmissivity corresponds to the case in which $n_0 \approx 1.4$, and the surface values

of the index of refraction are relatively close to 1. It is therefore concluded that the permittivity values should be scaled so that their average is close to 1 in order to maximize the reimager's efficiency.

Figure 48: Transmissivity for Various Scales of Permittivity

Lower permittivity ranges correspond to less light loss.

Figure 49a: Average Transmissivity v. n_0

Figure 49b: Surface Index of Refraction v. n_0

The overall light transmissivity is highest for $n_0 \approx 1.4$, at which point the index of refraction at the front surface is relatively close to 1.

Chapter 5 Other Applications

This thesis provides the material prescription for a concave-plano reimager for the monocentric lens. It has an anisotropic relative permittivity, with values ranging from 2.6 to 25.6. It has an anisotropic relative permeability as well. Specifically, the azimuth component of the permeability tensor ranges from .2 to 1. Because of this required magnetic response, the

device is not manufacturable at optical frequencies, although it could be built for the infrared region.

Based on the studies in Chapter 3, moreover, this thesis has provided an example of how a designer might adjust the shape of a desired object of a quasi-conformal transformation optics design in order to increase the likelihood of its manufacturability. It is concluded that building a longer reimager yields a narrower range of constitutive parameters. The tradeoff of increasing the length, however, would be the increased cost of the extra volume. Additionally, this thesis finds that there is an optimal height for the reimager. This strategy of choosing the optic's shape in order to make the range of constitutive parameters as narrow as possible might potentially prove useful to TO designs in future applications, given that, in general, it is very difficult to realize the metamaterial prescriptions, particularly for optical wavelengths.

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