REPRESENTATIONS OF THE ALTERNATING GROUP WHICH ARE IRREDUCIBLE OVER SUBGROUPS. II

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ABSTRACT. We prove that non-trivial representations of the alternating group A_n are reducible over a primitive proper subgroup which is isomorphic to some alternating group A_m . A similar result is established for finite simple classical groups embedded in A_n via their standard rank 3 permutation representations.

1. INTRODUCTION

If Γ is a transitive permutation group with a point stabilizer X then Γ is primitive if and only if $X < \Gamma$ is a maximal subgroup. So studying primitive permutation groups is equivalent to studying maximal subgroups. In most problems involving a finite primitive group Γ , the Aschbacher-O’Nan-Scott theorem [AS] allows one to concentrate on the case where Γ is almost quasi-simple, i.e. $L \triangleleft \Gamma/Z(\Gamma) \leq \text{Aut}(L)$ for a non-abelian simple group L . The results of Liebeck-Praeger-Saxl [LPS] and Liebeck-Seitz [LS] then allow one to assume furthermore that Γ is a finite classical group.

In the latter case, the possible structure of maximal subgroups X is described by Aschbacher’s theorem [A]: if $X < \Gamma$ is maximal then X belongs to

$$(1.1) \quad \mathcal{S} \cup \bigcup_{i=1}^8 \mathcal{C}_i,$$

where $\mathcal{C}_1, \dots, \mathcal{C}_8$ are collections of certain explicit natural subgroups of Γ , and $\mathcal{S}$ is the collection of almost quasi-simple groups that act absolutely irreducibly on the natural module for the classical group Γ .

It is not true, however, that every subgroup X in (1.1) is actually maximal in Γ . For $X \in \bigcup_{i=1}^8 \mathcal{C}_i$, the maximality of X has been determined by Kleidman-Liebeck [KIL] (see also [BDR] for a complete classification of maximal subgroups in low-dimensional finite classical groups). So let $X \in \mathcal{S}$. If X is *not* maximal then $X < G < \Gamma$ for a certain maximal subgroup G in Γ . The most challenging case to handle is when $G \in \mathcal{S}$ as well. This motivates the following problem, where $\mathbb{F}$ is an algebraically closed field of characteristic $p \geq 0$:

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Problem 1.1. *Classify all triples (G, V, X) where G is an almost quasi-simple finite group, V is an $\mathbb{F}G$ -module of dimension greater than one, and X is a proper subgroup of G such that the restriction $\text{Res}_X^G V$ is irreducible.*

Many results have been obtained concerning various cases of Problem 1.1 — see for instance [KT2] and references therein. In this paper, we are mostly interested in the case G is the alternating group $G = A_n$ or the symmetric group S_n . In this case, under the assumption $p > 3$ (or $p = 0$), Problem 1.1 has been solved in [BK, KS2] — see also [KT1] for double-covers $\hat{A}_n$ and $\hat{S}_n$ and [S, KIW] for the characteristic zero case. A number of techniques employed in these papers unfortunately break down in the case $(G, X) = (A_n, A_m)$ and $p = 2, 3$ (and especially when X is a primitive subgroup of G). On the other hand, this case is of crucial importance in a number of applications. The purpose of this paper is to resolve this important case, and our main result is:

Theorem 1.2. *Let $X \cong A_m$ be a primitive subgroup of A_n with $n > m \geq 9$. Let $\mathbb{F}$ be an algebraically closed field of arbitrary characteristic and V be a non-trivial $\mathbb{F}A_n$ -module. Then V is reducible over X .*

The bound $m \geq 9$ is the best possible — see Remark 8.2 and Lemma 8.1.

We emphasize that our methods also apply to many other primitive subgroups of A_n . To illustrate this, in this paper we handle the simple classical groups X that embed in A_n via their standard rank 3 permutation representations:

Theorem 1.3. *Let $X = L/Z(L)$ be a finite simple classical group, where L is one of the following group: $SL_d(q)$, $SU_d(q)$, or $Sp_d(q)'$ with $d \geq 4$, and $\Omega_d^\pm(q)$ with $d \geq 5$. Let W denote the natural d -dimensional module for L , and let X be embedded in $\text{Sym}(\Omega) = S_n$ via its rank 3 permutation action on the set Ω of 2-dimensional subspaces of W in the case $L = SL_d(q)$, and of 1-dimensional singular subspaces of W otherwise. If V is any $\mathbb{F}A_n$ -module of dimension > 1 , then $\text{Res}_X^{A_n} V$ is reducible.*

We plan to extend this result to the remaining simple primitive subgroups of A_n in a sequel. Together with the results of [BK, KS2] and the current paper, this will completely solve Problem 1.1 for $G = A_n$ in the cases that are of most interest for the Aschbacher-Scott program.

The paper is organized as follows. Basic notions are recalled in §2. Theorem 3.2 in §3 compares the dimensions of the Hom-spaces of irreducible S_n -modules in characteristic 2 over certain Young subgroups of S_n when n is even. Then Propositions 4.1, 4.3, and 4.4 in §4 show in particular that the p -modular irreducible representations of A_n which do not extend to S_n must have large enough dimension (at least exponential in n). These results, which we believe are also of independent interest, allow us to discard non- S_n -extendible A_n -modules in the proof of Theorem 1.2. In §5 we describe the submodule structure of the permutation modules of S_n acting on subsets of $\{1, 2, \dots, n\}$ of cardinality 2 or 3 in characteristic 2, again in the case of even n . This description plays a key role in the proof of Theorem 6.5 in §6, which gives a criterion for a 2-modular irreducible S_n -representation to be reducible over certain subgroups of S_n . Theorem 6.5 is then used in §7 to show that non-trivial 2-modular irreducible A_n -representations are reducible over A_m , if A_m is embedded into A_n via its actions on subsets or set partitions of $\{1, 2, \dots, m\}$

— see Theorem 7.12. Theorem 1.2 is proved in §8, which also contains further results concerning non-primitive embeddings of A_m into A_n . The final §9 is devoted to the proof of Theorem 1.3.

2. PRELIMINARIES

Throughout the paper, unless otherwise stated, we assume that the ground field $\mathbb{F}$ is algebraically closed, and $p := \text{char}(\mathbb{F})$. For a group G , the trivial $\mathbb{F}G$ -module is denoted $\mathbf{1}_G$ or simply $\mathbf{1}$ if it is clear what G is. If V is an $\mathbb{F}G$ -module, we denote by $\text{soc}(V)$ the socle of V , and for $n = 1, 2, \dots$, define $\text{soc}^n(V)$ from $\text{soc}^1(V) = \text{soc}(V)$ and $\text{soc}^n(V)/\text{soc}^{n-1}(V) = \text{soc}(V/\text{soc}^{n-1}(V))$ for $n > 1$. We refer to the quotients $\text{soc}^n(V)/\text{soc}^{n-1}(V)$ as the *socle layers* of V and usually list them from bottom to top, i.e. first $\text{soc}(V)$, then $\text{soc}^2(V)/\text{soc}(V)$, etc.

For $n \in \mathbb{Z}_{>0}$, let

$$\Omega := \{1, 2, \dots, n\}.$$

For $r = 1, \dots, n$, denote by Ω_r the set of r -element subsets of Ω . The symmetric group S_n acts naturally on the sets $\Omega = \Omega_1, \Omega_2, \dots, \Omega_n$ and the stabilizer of an element of Ω_r is conjugate to the subgroup $S_{n-r,r} := S_{\{1,2,\dots,n-r\}} \times S_{\{n-r+1,\dots,n\}}$. We write $S_{n-1,1}$ simply as S_{n-1} .

We denote by

$$M_r = \mathbb{F}\Omega_r \cong \text{Ind}_{S_{n-r,r}}^{S_n} \mathbf{1}_{S_{n-r,r}} \quad (1 \leq r \leq n)$$

the permutation module for the action of S_n on Ω_r .

We recall some basic notions of representation theory of symmetric groups referring to [J2] for details. The irreducible $\mathbb{F}S_n$ -modules are labeled by p -regular partitions of n (if $p = 0$ then p -regular partitions are interpreted as all partitions). If λ is a p -regular partition of n , the corresponding irreducible module is denoted D^λ . The Specht modules over $\mathbb{F}S_n$ are labeled by partitions of n . If λ is such a partition, the corresponding Specht module is denoted S^λ .

Let $p = 2$. Consider the partition

$$\alpha_n = \begin{cases} (k+1, k-1) & \text{if } n = 2k \text{ is even,} \\ (k+1, k) & \text{if } n = 2k+1 \text{ is odd.} \end{cases}$$

The irreducible module D^{α_n} is called *the basic spin module* for S_n . It is known [W, Table III] that

$$\dim D^{\alpha_n} = 2^{\lfloor (n-1)/2 \rfloor}.$$

Let sgn_n be the sign module over $\mathbb{F}S_n$. For any p -regular partition λ , we have that $D^\lambda \otimes \text{sgn}_n$ is an irreducible $\mathbb{F}S_n$ -module, so we can write $D^\lambda \otimes \text{sgn}_n \cong D^{\lambda^M}$, where

$$\mathbf{M} : \lambda \mapsto \lambda^M$$

is the *Mullineux involution* on the set of p -regular partitions of n . To describe the Mullineux involution, we briefly recall the notion of the *Mullineux symbol* $G(\lambda)$ of λ , referring the reader to [FK] for details. Let h_1 be the number of nodes in the p -rim of λ ,

and let r_1 be the number of rows in λ . Delete the p -rim and repeat to obtain sequences $h_1, h_2, \dots$ and $r_1, r_2, \dots$. Let k be such that $h_{k+1} = r_{k+1} = 0$ but $h_k \neq 0 \neq r_k$. Then

$$G(\lambda) := \begin{pmatrix} h_1 & h_2 & \dots & h_k \\ r_1 & r_2 & \dots & r_k \end{pmatrix}.$$

It was proved in [Mu] that λ is uniquely determined by $G(\lambda)$. Moreover, we have

$$G(\lambda^{\mathbf{M}}) = \begin{pmatrix} h_1 & h_2 & \dots & h_k \\ h_1 - r_1 + \epsilon_1 & h_2 - r_2 + \epsilon_2 & \dots & h_k - r_k + \epsilon_k \end{pmatrix},$$

where $\epsilon_i := 0$ if $p|h_i$ and $\epsilon_i := 1$ otherwise. This description of $\mathbf{M}$ is the main result of [FK] (see also [BeO]), which was conjectured by Mullineux.

Given an irreducible representation D^λ , either the restriction $E^\lambda := \text{Res}_{\mathbb{A}_n}^{S_n} D^\lambda$ is irreducible or $\text{Res}_{\mathbb{A}_n}^{S_n} D^\lambda \cong E_+^\lambda \oplus E_-^\lambda$, a direct sum of two inequivalent irreducible representations. Moreover, every irreducible $\mathbb{F}\mathbb{A}_n$ -module is isomorphic to one of $E_{(\pm)}^\lambda$, and the only non-trivial isomorphism of the form $E_{(\pm)}^\lambda \cong E_{(\pm)}^\mu$ is $E^\lambda \cong E^{\lambda^{\mathbf{M}}}$.

If $p \neq 2$, then $\text{Res}_{\mathbb{A}_n}^{S_n} D^\lambda$ is reducible if and only if $\lambda = \lambda^{\mathbf{M}}$. If $p = 2$, then an explicit criterion for reducibility of $\text{Res}_{\mathbb{A}_n}^{S_n} D^\lambda$ is given in [Ben, Theorem 1.1].

3. COMPARING SOME HOM-SPACES

Throughout this section we assume that $p = 2$. In this section we get some results on the dimensions

$$(3.1) \quad d_r(V) := \dim \text{Hom}_{\mathbb{F}S_n}(M_r, \text{End}_{\mathbb{F}}(V)) = \dim \text{End}_{\mathbb{F}S_{n-r,r}}(\text{Res}_{S_{n-r,r}}^{S_n} V).$$

The last equality follows using $M_r = \text{Ind}_{S_{n-r,r}}^{S_n} \mathbf{1}_{S_{n-r,r}}$ and Frobenius reciprocity.

Lemma 3.1. *Let V be an irreducible $\mathbb{F}S_n$ -module and $1 \leq r \leq n$. Then $d_r(V) = 1$ if and only if $\text{Res}_{S_{n-r,r}}^{S_n} V$ is irreducible.*

Proof. The sufficiency of the condition is clear. Conversely, irreducible $\mathbb{F}S_n$ -modules are self-dual, so the restriction $\text{Res}_{S_{n-r,r}}^{S_n} V$ is self-dual. Since irreducible $\mathbb{F}S_{n-r,r}$ -modules are also self-dual, the head of $\text{Res}_{S_{n-r,r}}^{S_n} V$ is isomorphic to its socle. So if $\text{Res}_{S_{n-r,r}}^{S_n} V$ is reducible then $d_r(V) > 1$. $\square$

The goal of this section is to prove the following result:

Theorem 3.2. *Let V be a simple $\mathbb{F}S_n$ -module and $2|n \geq 6$. Then one of the following statements holds:*

- (i) $d_3(V) > d_1(V)$.
- (ii) $V \cong D^{\alpha_n}$ is the basic spin module or $V \cong \mathbf{1}$ is the trivial module, in which cases we have $d_3(V) = d_1(V)$.

Let $V = D^\lambda$ for a 2-regular partition

$$\lambda = (\lambda_1 > \dots > \lambda_s > 0)$$

of n . If $s = 1$, then V is the trivial module, and Theorem 3.2 holds trivially.

Lemma 3.3. *Theorem 3.2 holds if $d_1(V) = 1$.*

Proof. Suppose that $d_1(V) = d_3(V) = 1$ and $\dim V > 1$. By Lemma 3.1, V is irreducible over S_{n-1} and $S_{3,n-3}$, and so the lemma follows from [P, Theorem 10]. $\square$

Lemma 3.4. *Theorem 3.2 holds if $s = 2$.*

Proof. Since n is even, we have that the restriction $D^\lambda|_{S_{n-1}}$ is irreducible by [K1]. So $d_1(V) = 1$, and we may apply Lemma 3.3. $\square$

Lemma 3.5. *Let*

$$x := \sum_{g \in S_{\{1,2,3\}}, \sigma \in S_{\{1,4\}} \times S_{\{2,5\}} \times S_{\{3,6\}}} (\text{sign } \sigma) \sigma g \sigma^{-1} \in \mathbb{F}S_n.$$

If $s > 2$, then $x D^\lambda \neq 0$.

Proof. By [BaK, Lemma 4.7] with $(m, p) = (3, 2)$, we see that the restriction $V|_{S_6}$ has a composition factor isomorphic to $D^{(3,2,1)}$, which for $p = 2$ is isomorphic to the Specht module $S^{(3,2,1)}$. Since $x \in \mathbb{F}S_6$, it suffices to prove that $x S^{(3,2,1)} \neq 0$. We use the notation of [J2, §4]; in particular, e_t is the polytabloid and $\{t\}$ is the tabloid corresponding to a $(3, 2, 1)$ -tableau t . Let

$$t := \begin{array}{|c|c|c|} \hline 1 & 4 & 6 \\ \hline 2 & 5 & \\ \hline 3 & & \\ \hline \end{array}, \quad s := \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 & \\ \hline 6 & & \\ \hline \end{array}.$$

An explicit calculation shows that $\{s\}$ appears in $x e_t$ with coefficient 1. $\square$

We now complete the proof of Theorem 3.2. Recall that M_3 is the permutation module on all three element subsets $\{i, j, k\} \subseteq \Omega$, while M_1 is the permutation module on the one element subsets $\{i\} \subseteq \Omega$. Consider the $\mathbb{F}S_n$ -module homomorphism

$$f : M_3 \rightarrow M_1, \quad \{i, j, k\} \mapsto \{i\} + \{j\} + \{k\}.$$

It is easy to see that f is surjective. So it induces an injective linear map

$$f^* : \text{Hom}_{\mathbb{F}S_n}(M_1, \text{End}_{\mathbb{F}}(V)) \rightarrow \text{Hom}_{\mathbb{F}S_n}(M_3, \text{End}_{\mathbb{F}}(V)), \quad \psi \mapsto \psi \circ f.$$

It suffices to prove that f^* is not surjective.

We exhibit an element $\phi \in \text{Hom}_{\mathbb{F}S_n}(M_3, \text{End}_{\mathbb{F}}(V))$ which is not in the image of f^* . For $\Theta \subseteq \Omega$ let $S_\Theta \subseteq S_n = S_\Omega$ be the subgroup of all permutations which stabilize the elements of $\Omega \setminus \Theta$. Now, define ϕ as follows:

$$(3.2) \quad \phi(\{i, j, k\})(v) := \sum_{g \in S_{\{i, j, k\}}} g v, \quad (\{i, j, k\} \subseteq \Omega, v \in V).$$

If $\phi \in \text{Im } f^*$, then $\phi = \psi \circ f$ for some $\psi \in \text{Hom}_{\mathbb{F}S_n}(M_1, \text{End}_{\mathbb{F}}(V))$. Consider the element

$$E = \sum_{\sigma \in S_{\{1,4\}} \times S_{\{2,5\}} \times S_{\{3,6\}}} (\text{sign } \sigma) \sigma \{1, 2, 3\} \in M_3.$$

Note that $f(E) = 0$. So $\phi(E) = \psi(f(E)) = 0$. On the other hand, we compute $\phi(E)$ using (3.2):

$$\phi(E)(v) = \sum_{\sigma \in S_{\{1,4\}} \times S_{\{2,5\}} \times S_{\{3,6\}}} (\text{sign } \sigma) \sum_{g \in S_{\{\sigma(1), \sigma(2), \sigma(3)\}}} gv = xv,$$

where x is as in Lemma 3.5. Now Lemma 3.5 yields a contradiction.

4. DIMENSION AND EXTENDIBILITY TO S_n

First we prove the following statement, which relies on some results of [Ben] and [GLT]:

Proposition 4.1. *Let $p = 2$, $n \geq 5$, and let V be an irreducible $\mathbb{F}A_n$ -module. Suppose that V does not extend to S_n . Then $\dim V \geq 2^{(n-6)/4}$.*

Proof. By assumption, $W := \text{Ind}_{A_n}^{S_n}(V)$ is an irreducible $\mathbb{F}S_n$ -module, and $\dim W = 2 \dim V$. Let $\lambda = (\lambda_1 > \lambda_2 > \dots > \lambda_s > 0)$ be the partition of n into distinct parts corresponding to W . Since W is reducible over A_n , by [Ben, Theorem 1.1], we have $s > 1$ and $\lambda_1 - \lambda_2 \in \{1, 2\}$. In particular, $n \geq \lambda_1 + \lambda_2 \geq 2\lambda_1 - 2$, i.e. $\lambda_1 \leq (n+2)/2$. Now

$$\dim W \geq 2^{\frac{n-\lambda_1}{2}} \geq 2^{\frac{n-(n+2)/2}{2}} = 2^{\frac{n-2}{4}},$$

thanks to [GLT, Theorem 5.1]. $\square$

We will also need the following branching result which is of interest in its own right:

Proposition 4.2. *Let $p = 2$, and $\lambda = (\lambda_1 > \dots > \lambda_s > 0) \neq (n)$ be a non-trivial 2-regular partition of n . If $2\lambda_1 - n \geq k \geq 3$ then the restriction of D^λ to a natural subgroup S_k of S_n affords both $\mathbf{1} = D^{(k)}$ and $D^{(k-1,1)}$ as composition factors.*

Proof. We apply induction on $m := n - \lambda_1 \geq 1$. If $m = 1$, then $D^\lambda = D^{(n-1,1)}$ is the heart of the natural permutation module, and the statement follows easily. Let $m \geq 2$.

Case 1: $\lambda_1 - \lambda_2$ is odd. Then, in the terminology of [K2, Definition 0.3], 2 is a normal index. Let $j \geq 2$ be the largest normal index; in particular j is a good index in the sense of [K2, Definition 0.3] again. Then by [K2, Theorem 0.5], D^μ is a simple submodule of $D^\lambda|_{S_{n-1}}$, where

$$\mu = \lambda(j) := (\lambda_1, \dots, \lambda_{j-1}, \lambda_j - 1, \lambda_{j+1}, \dots, \lambda_s) \vdash (n-1).$$

Note that $2\lambda_1 - (n-1) \geq k+1$ and $(n-1) - \lambda_1 = m-1$. Hence we can apply the induction hypothesis to D^μ restricted to S_k .

Case 2: $\lambda_1 - \lambda_2$ is even. Now 1 is a normal index. Then by [K2, Theorem 0.4], D^ν is a composition factor of $D^\lambda|_{S_{n-1}}$, where

$$\nu = \lambda(1) := (\lambda_1 - 1, \lambda_2, \dots, \lambda_s) \vdash (n-1).$$

Since $(\lambda_1 - 1) - \lambda_2$ is odd, as in Case 1 we now see that 2 is a normal index for ν . Let $j \geq 2$ be the highest normal index of ν ; in particular j is a good index. Then again by [K2, Theorem 0.5], D^μ is a simple submodule of $D^\lambda|_{S_{n-2}}$, where

$$\mu = \nu(j) := (\lambda_1 - 1, \dots, \lambda_{j-1}, \lambda_j - 1, \lambda_{j+1}, \dots, \lambda_s) \vdash (n-2).$$

Note that $2(\lambda_1 - 1) - (n - 2) \geq k$ and $(n - 2) - (\lambda_1 - 1) = m - 1$. Hence we can apply the induction hypothesis to D^μ restricted to S_k . $\square$

Using the Mullineux involution, we prove an analogue of Proposition 4.1 for $p \neq 2$ (certainly, the most interesting case being $p \leq n$):

Proposition 4.3. *Let $n \geq 5$ and $p \neq 2$.*

- (i) *Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be a p -regular partition of n . Suppose that $\lambda_1 \geq (n + p + 2)/2$. Then D^λ is irreducible over A_n .*
- (ii) *Let V be an irreducible $\mathbb{F}A_n$ -module. Suppose that V does not extend to S_n . Then $\dim V \geq 2^{(n-p-5)/4}$.*

Proof. (i) Recalling the definition of the Mullineux map from Section 2, denote the partitions obtained from λ by successively removing p -rims as $\lambda^{(j)}$, $1 \leq j \leq k$. We prove the statement by induction on $n - \lambda_1$. Since $\lambda_1 \geq (n - \lambda_1) + p + 2$ by the assumption, the first p -segment of the p -rim of λ has length p . Assume for a contradiction that $\lambda = \lambda^{\mathbf{M}}$. Then

$$(4.1) \quad h_i + \epsilon_i = 2r_i$$

for $1 \leq i \leq k$.

Suppose first that the p -rim of $\lambda' := (\lambda_2, \lambda_3, \dots)$ has at most $p - 1$ nodes. Write $h_1 = p + x$ and $r_1 = 1 + y$, where $0 \leq x \leq p - 1$ and y is the number of rows of λ' . Then according to (4.1) we have

$$(4.2) \quad p + x \leq p + x + \epsilon_1 = h_1 + \epsilon_1 = 2r_1 = 2 + 2y.$$

Note that x is the length of the p -rim of λ' . Hence $y \leq x$, and so (4.2) yields $x \geq p - 2 > 0$. In turn, this implies that p/h_1 , whence $\epsilon_1 = 1$ and (4.2) yields $x = y = p - 1$ (as $x \leq p - 1$). Recall we are assuming that the p -rim of λ' has at most $p - 1$ nodes. It follows that the p -rim of λ' has exactly $p - 1$ nodes and λ' also has $p - 1$ rows. This can happen only when $\lambda' = (1^{p-1})$, a column of $p - 1$ nodes. In this case, $\lambda^{(1)} = (\lambda_1 - p)$ has one part, which is of length ≥ 2 . Hence the p -rim of $\lambda^{(1)}$ is of length p (if $\lambda_1 \geq 2p$), or $z \geq 2$ (where $2p - 1 \geq \lambda_1 = p + z \geq p + 2$). Correspondingly, $r_2 = 1$ and $(h_2, \epsilon_2) = (p, 0)$ or $(z, 1)$. In either case

$$h_2 + \epsilon_2 \geq z + 1 \geq 3 > 2r_2,$$

contrary to (4.1).

Assume now that the p -rim of λ' has at least p nodes. Then, aside from the first p -segment contained in the first row, the p -rim of λ contains at least p nodes of λ' . It follows that the condition $\lambda_1 \geq (n - \lambda_1) + p + 2$ also holds for $\lambda^{(1)}$. By the induction hypothesis, $\lambda^{(1)}$ is not equal to its Mullineux dual, i.e. $h_i - r_i + \epsilon_i \neq r_i$ for some $i \geq 2$, again contradicting (4.1).

(ii) By assumption, $W := \text{Ind}_{A_n}^{S_n}(V)$ is an irreducible $\mathbb{F}S_n$ -module and $\dim W = 2 \dim V$. Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be the p -regular partition of n corresponding to W . Since W is reducible over A_n , $\lambda_1 \leq (n + p + 1)/2$ by (i). It now follows by [GLT, Theorem 5.1] that

$$\dim W \geq 2^{\frac{n-\lambda_1}{2}} \geq 2^{\frac{n-(n+p+1)/2}{2}} = 2^{\frac{n-p-1}{4}},$$

which implies the result. $\square$

Here is another version of Proposition 4.3:

Proposition 4.4. *Let $p > 2$, $n \geq 5$, and $\lambda = (\lambda_1, \lambda_2, \dots)$ be a p -regular partition of n . Suppose that there is some $s \geq 1$ such that*

$$\lambda_1 - \lambda_2 \geq \lambda_2 - \lambda_3 \geq \dots \geq \lambda_s - \lambda_{s+1} \geq p$$

and

$$\sum_{i=1}^s \left\lfloor \frac{\lambda_i - \lambda_{i+1}}{p} \right\rfloor > \frac{n}{2p-1}.$$

Then D^λ is irreducible over A_n . In particular, if

$$\lambda_1 \geq \lambda_2 + p \left\lceil \frac{n+1}{2p-1} \right\rceil,$$

then D^λ is irreducible over A_n .

Proof. Assume that D^λ is reducible over A_n . Then $\lambda = \lambda^M$ and so

$$(4.3) \quad \sum_{i=1}^k (h_i - 2r_i + \epsilon_i) = 0.$$

We will estimate $h_1 - 2r_1 + \epsilon_1$ by going down the p -segments of the p -rim of λ . Since $\lambda_1 - \lambda_2 \geq p$, the first p -segment consists of p nodes of the first row and so contributes $p-2$ to $h_1 - 2r_1 + \epsilon_1$. More generally, any horizontal p -segment of length p contributes $p-2$ to $h_1 - 2r_1 + \epsilon_1$. On the other hand, since λ is p -regular, any non-horizontal p -segment of length p has height $\leq (p-1)$ and so it contributes at least $p - 2(p-1) = 2-p$ to $h_1 - 2r_1 + \epsilon_1$. Suppose the p -rim also has a p -segment of length j less than p . Then it must be the last p -segment, and $\epsilon_1 = 1$. So the contribution of this p -segment to $h_1 - 2r_1 + \epsilon_1$ is $\geq j - 2j + 1 = 1 - j$.

As the p -rims are removed in succession, let a be the total number of horizontal p -segments of length p , b_p be the total number of non-horizontal p -segments of length p , and b_j be the total number of p -segments of length $1 \leq j < p$, so that $n = pa + \sum_{j=1}^p j b_j$. Applying the above arguments to all successive p -rims of λ we have that

$$\begin{aligned} \sum_{i=1}^k (h_i - 2r_i + \epsilon_i) &\geq (p-2)a - (p-2)b_p - \sum_{j=1}^{p-1} (j-1)b_j \\ &\geq (2p-2)a + (2b_p + \sum_{j=1}^{p-1} b_j) - (pa + \sum_{j=1}^p j b_j) \\ &\geq (2p-2)a + (2b_p + \sum_{j=1}^{p-1} b_j) - n. \end{aligned}$$

Observe that

$$2b_p + \sum_{j=1}^{p-1} b_j \geq \frac{\sum_{j=1}^p j b_j}{p-1} = \frac{n - pa}{p-1}.$$

Under the hypothesis, we can find an integer $t \geq (n+1)/(2p-1)$ such that

$$\sum_{i=1}^s \lfloor \frac{\lambda_i - \lambda_{i+1}}{p} \rfloor \geq t.$$

Now observe that at least t horizontal p -segments from the first s rows of λ belong to these successive p -rims. Thus $a \geq t \geq (n+1)/(2p-1)$. Hence,

$$\sum_{i=1}^k (h_i - 2r_i + \epsilon_i) \geq (2p-2)a + \frac{n - pa}{p-1} - n \geq \frac{p-2}{p-1}((2p-1)a - n) > 0,$$

contradicting (4.3). $\square$

5. STRUCTURE OF PERMUTATION MODULES

Throughout the section $n \geq 6$ is an *even* integer and $p = 2$.

We will study permutation modules M_r , mainly for $r = 1, 2, 3$. For $n \geq 2r$, let $\mathcal{S}_r \subset M_r$ denote the Specht module $S^{(n-r, r)}$ and (assuming $n > 2r$) let $D_r = D^{(n-r, r)}$ be the unique simple quotient of $\mathcal{S}_r$. Let $T_r \in M_r$ be the sum of all r -element sets. Let

$$\eta_{r,s} : M_r \rightarrow M_s$$

denote the incidence homomorphism sending an r -set to the sum of s -sets incident with (i.e. containing or contained in) it. By [J2, Corollary 17.18],

$$(5.1) \quad \mathcal{S}_r = \bigcap_{t=0}^{r-1} \text{Ker } \eta_{r,t}.$$

We denote by M'_r the *augmentation module*, i.e. the submodule $\text{Ker } \eta_{r,0}$ of M_r (spanned by differences of pairs of basis elements).

The space M_r has a natural bilinear form $\langle \cdot, \cdot \rangle_r$, with respect to which the standard basis is orthonormal. If we identify M_r and M_s with their dual spaces, using the corresponding bilinear forms, then $\eta_{s,r}$ is the dual map of $\eta_{r,s}$. In particular, $\eta_{s,r}$ is injective iff $\eta_{r,s}$ is surjective and vice versa. Also $\text{Im } \eta_{s,r} \cong \text{Im } \eta_{r,s}^*$ as $\mathbb{F}\mathcal{S}_n$ -modules. We have

$$(5.2) \quad \langle x, \eta_{r,s}(y) \rangle_s = \langle \eta_{s,r}(x), y \rangle_r \quad (x \in M_s, y \in M_r).$$

The ranks of the maps $\eta_{r,s}$ are given in [Wil]. We state the special cases that we need of this general result.

Lemma 5.1. *For $r \leq \min\{s, n-s\}$ we have*

$$\text{rank}_{\mathbb{F}} \eta_{r,s} = \sum_{1 \leq i \leq r, \binom{s-i}{r-i} \text{ is odd}} \left(\binom{n}{i} - \binom{n}{i-1} \right).$$

In particular, $\eta_{1,3}$ is injective, and

$$\text{rank}_{\mathbb{F}} \eta_{1,2} = n - 1 = \dim \mathcal{S}_1, \quad \text{rank}_{\mathbb{F}} \eta_{2,3} = 1 + \frac{n(n-3)}{2} = 1 + \dim \mathcal{S}_2.$$

Lemma 5.2. M_1 is a uniserial $\mathbb{F}\mathcal{S}_n$ -module with socle layers $\mathbb{F} \cdot T_1 \cong \mathbf{1}, D_1, \mathbf{1}$.

Proof. This is well known, see e.g. [J2, Example 5.1]. $\square$

Let the $\mathbb{F}\mathcal{S}_n$ -module Q be defined by the short exact sequence

$$(5.3) \quad 0 \rightarrow \mathbb{F} \cdot T_1 \rightarrow M_1 \rightarrow Q \rightarrow 0.$$

In fact, $Q \cong \mathcal{S}_1^*$.

The following lemma can be deduced from [MO, Theorem (1.1)], but we give an independent proof for the reader's convenience:

Lemma 5.3. As $\mathbb{F}\mathcal{S}_n$ -modules, $\text{Im } \eta_{1,2} \cong Q$ and M_2 has the following structure:

- (i) If $n \equiv 0 \pmod{4}$ then the composition factors of M_2 are $\mathbf{1}$ (twice), D_1 (twice) and D_2 (once). Furthermore, $M_2/(\text{Im } \eta_{1,2} + \mathbb{F} \cdot T_2) \cong \mathcal{S}_2^*$, whose socle is D_2 .
- (ii) If $n \equiv 2 \pmod{4}$ then $M_2 \cong \mathbb{F} \cdot T_2 \oplus M'_2$, and M'_2 is uniserial with socle layers $D_1, \mathbf{1}, D_2, \mathbf{1}, D_1$.

Proof. The dimension of $\text{Im } \eta_{1,2}$ is given by Lemma 5.1, from which we see that it is isomorphic to Q . The composition factors are given by [J1].

(i) The second statement in (i) now follows using the facts that the dual Specht module $\mathcal{S}_2^*$ is a quotient of M_2 and $\mathcal{S}_2^*$ is uniserial with socle layers D_2, D_1 .

(ii) The submodule $\mathbb{F} \cdot T_2$ is a direct summand of M_2 because $\binom{n}{2}$ is odd. Then $M'_2 = \text{Ker } \eta_{2,0}$ is the complementary summand. The composition factors of M'_2 are $D_1, \mathbf{1}, D_2, \mathbf{1}, D_1$. Also we have $\mathcal{S}_2 \subseteq M'_2$. The composition factors of $\mathcal{S}_2$ are $D_2, D_1, \mathbf{1}$ and $\mathcal{S}_2$ has a simple head isomorphic to D_2 . Since $M'_2{}^{\mathcal{S}_n} = 0$, it follows that $\mathcal{S}_2$ must be uniserial with socle layers $D_1, \mathbf{1}, D_2$. The uniseriality of M'_2 and its socle layers now follow from the self-duality of M'_2 . $\square$

Lemma 5.4. The $\mathbb{F}\mathcal{S}_n$ -module M_3 has the following structure.

- (i) If $n \equiv 0 \pmod{4}$, then $M_3 \cong M_1 \oplus U$, where U is uniserial with socle layers D_2, D_1, D_3, D_1, D_2 .
- (ii) If $n \equiv 2 \pmod{4}$ then $\text{Im } \eta_{2,3}$ is uniserial with socle layers $\mathbf{1}, D_2, \mathbf{1}, D_1$. The composition factors of M_3 are $\mathbf{1}$ (with multiplicity 4), D_1 (twice), D_2 (twice) and D_3 (once). Also, $\text{soc}(M_3) = \mathbb{F} \cdot T_3$.

Proof. The composition factors are given by [J1]. The dimensions of the images of the incidence maps $\eta_{r,s}$ are given by Lemma 5.1.

(i) A simple computation shows that $\eta_{3,1} \circ \eta_{1,3} = 1_{M_1}$, so $\eta_{1,3}$ is a split injection of M_1 into M_3 . In fact, by computing bilinear forms on the images of basis elements, $\eta_{1,3}$ is seen to be an isometry. So $M_3 = \text{Im } \eta_{1,3} \oplus U$, with $U = (\text{Im } \eta_{1,3})^\perp = \text{Ker } \eta_{3,1}$, where the last equality is by (5.2). The module U is a self-dual module and its composition factors are D_1 (twice), D_2 (twice) and D_3 , and U contains $\mathcal{S}_3$. The structure of U will follow from its self-duality if we prove that $\mathcal{S}_3$ is uniserial with socle layers D_2, D_1, D_3 . Since we know that the head of $\mathcal{S}_3$ is isomorphic to D_3 , it suffices to show that D_1 is not a submodule of $\mathcal{S}_3$. If it were, then we would have $\text{Hom}_{\mathbb{F}\mathcal{S}_n}(D_1, M_3) \neq 0$, whence $\text{Hom}_{\mathbb{F}\mathcal{S}_n}(M_3, D_1) \neq 0$, i.e. the fixed point subspace $D_1^{\mathcal{S}_n-3,3}$ is non-trivial, which is easily checked to be false.

(ii) We have $\text{Im } \eta_{1,2} \cong Q$. Also, it is easy to see that $\eta_{2,3} \circ \eta_{1,2} = 0$, whence by dimensions using Lemma 5.1, we conclude that $\text{Im } \eta_{1,2} = \text{Ker } \eta_{2,3}$. Moreover, $\text{Im } \eta_{1,2} \cong Q$ by Lemma 5.3. Since $\text{Im } \eta_{1,2} \not\subseteq M'_2$, the structure of Q implies that $\text{Ker } \eta_{2,3} \cap M'_2$ must be zero or isomorphic to D_1 . Since we know the rank of $\eta_{2,3}$, we see that the latter holds. Thus from the structure of M'_2 , we see that $\eta_{2,3}(M'_2) \cong M'_2 / \text{soc}(M'_2)$. By dimensions, we see that $\eta_{2,3}(M'_2) = \text{Im } \eta_{2,3}$, so $\text{Im } \eta_{2,3}$ is uniserial with the socle layers as stated.

It remains to show that $\text{soc}(M_3) = \mathbb{F} \cdot T_3$. For this, it suffices to prove that M_3 has no submodule isomorphic to D_1 , D_2 or D_3 . For D_1 we explicitly check as in (i) that $D_1^{S_{n-3,3}} = 0$. The unique D_3 composition factor of M_3 is the head of $\mathcal{S}_3$, so since $\mathcal{S}_3$ is not simple, it follows that M_3 has no submodule isomorphic to D_3 . Finally, we consider D_2 . By the first part of (ii), $\text{Im } \eta_{2,3}$ has one composition factor D_2 as its second socle layer, so it suffices to show that $M_3 / \text{Im } \eta_{2,3}$, which has a single D_2 composition factor, has no submodule isomorphic to D_2 . However, $M_3 / \text{Im } \eta_{2,3}$ has D_3 as a composition factor, so maps surjectively onto $\mathcal{S}_3^*$. By [J1], D_2 is a composition factor of $\mathcal{S}_3^*$, but $\mathcal{S}_3^*$ has a simple socle isomorphic to D_3 . So D_2 is not a submodule of M_3 . $\square$

Figures 1 and 2 below are given for the reader's convenience, but they will not be used in proofs. The pictures give *partial* information on submodule structure of the permutation modules M_2 and M_3 . The edges indicate the existence of uniserial subquotients.

 FIGURE 1. Submodule structures for $n \equiv 0 \pmod{4}$

 FIGURE 2. Submodule structures for $n \equiv 2 \pmod{4}$

Lemma 5.5. *We have:*

- (i) $\text{Im } \eta_{1,3}$ is the unique submodule of M_3 that is isomorphic to M_1 as $\mathbb{F}\mathcal{S}_n$ -modules.
- (ii) $\text{Ker } \eta_{3,1}$ is the unique submodule N of M_3 such that $M_3/N \cong M_1$ as $\mathbb{F}\mathcal{S}_n$ -modules.

Proof. Part (ii) follows from (i) by the duality of $\eta_{1,3}$ and $\eta_{3,1}$. For (i), note that $\dim \text{Hom}_{\mathbb{F}\mathcal{S}_n}(M_1, M_3) = 2$. The map $\eta_{1,3}$ and the map β sending each 1-set to T_3 form a basis of this Hom-space. Now $\eta_{1,3}$ is injective since $\text{soc}(M_1)$ is spanned by T_1 and $\eta_{1,3}(T_1) = 3T_3 = T_3 \neq 0$. Also we have $\text{Im } \beta \subset \text{Im } \eta_{1,3}$, so $\text{Im } \eta_{1,3}$ is the unique submodule of M_3 isomorphic to M_1 . $\square$

In the following two lemmas, N denotes the submodule of M_3 specified in Lemma 5.5(ii).

Lemma 5.6. *Let $n \equiv 0 \pmod{4}$. Then $\text{Ker } \eta_{3,2} \cap N = \text{soc}^3(N)$, and*

$$N/\text{soc}^3(N) \cong \eta_{3,2}(N) = \text{Im } \eta_{3,2} \cap M'_2 = \mathcal{S}_2.$$

Proof. By Lemma 5.4, $N/\text{soc}^3(N)$ is uniserial with socle D_1 and head D_2 . Since M_2 has no composition factor isomorphic to D_3 , we have $\text{soc}^3(N) \subseteq \text{Ker } \eta_{3,2}$.

We claim that the induced map $N/\text{soc}^3(N) \rightarrow M_2$ is injective. If not, its image is either zero or isomorphic to D_2 . The latter is impossible since M_2 has no submodule isomorphic to D_2 . The former is also impossible since it forces the rank of $\eta_{3,2}$ to be at most $\dim M_1 = n$, contrary to Lemma 5.1, which gives the actual rank as $1 + \dim \mathcal{S}_2$.

Thus the map $\eta_{3,2}$ induces an isomorphism of $N/\text{soc}^3(N)$ with a submodule $\eta_{3,2}(N) \subseteq M_2$. Since $N \subseteq M'_3$ and $\eta_{2,0} \circ \eta_{3,2} = \eta_{3,0}$, we have $\eta_{3,2}(N) \subseteq M'_2$. Comparing the dimensions, we see that $\eta_{3,2}(N) = \text{Im } \eta_{3,2} \cap M'_2$. This submodule of M_2 has the same dimension and the same composition factors as $\mathcal{S}_2$. Since any submodule of M_2 with D_2 as a composition factor must contain $\mathcal{S}_2$, we now conclude that $\eta_{3,2}(N) = \mathcal{S}_2$. $\square$

Lemma 5.7. *Let $n \equiv 2 \pmod{4}$, $W = \text{Im } \eta_{2,3} \subseteq M_3$, and $Y := M'_2 \subset M_2$. Then:*

- (i) $\text{soc}^2(W) \subseteq N$ and $\text{soc}^2(W)$ is uniserial with socle layers $\mathbf{1}, D_2$.
- (ii) $\text{Ker}(\eta_{2,3}|_Y) = \text{soc}(Y)$, and $Y/\text{soc}(Y) \cong \eta_{2,3}(Y) = W$.
- (iii) $N \cap W = N \cap \text{soc}^3(W) = \text{soc}^2(W)$, $M_3/(N + \text{soc}^3(W)) \cong Q$.
- (iv) We have $M_1 \cong \text{Im } \eta_{1,3} \subseteq N$, $\text{Im } \eta_{1,3} \cap W = \mathbb{F} \cdot T_3$, and the submodule

$$Q' := (\text{Im } \eta_{1,3} + \text{soc}^2(W))/\text{soc}^2(W) \subseteq N' := N/\text{soc}^2(W)$$

is isomorphic to Q . Moreover, $N'' := N'/Q'$ is uniserial with socle layers D_3, D_2 .

- (v) We have $\text{soc}^2(W) \subset \mathcal{S}_3 \subset N$, the submodule $D' := \mathcal{S}_3/\text{soc}^2(W) \subset N'$ is isomorphic to D_3 . Moreover, $\text{soc}(N') = D' \oplus \text{soc}(Q') \cong D_3 \oplus D_1$ and $N'/D' \cong N/\mathcal{S}_3 \cong \eta_{3,2}(N) \cong \mathcal{S}_2$.

Proof. (i) The structure of W is given in Lemma 5.4, which implies that $\text{soc}^2(W)$ is uniserial with socle layers $\mathbf{1}, D_2$. Any nonzero quotient of $\text{soc}^2(W)$ has D_2 as its head. But $M_3/N \cong M_1$ and D_2 is not a composition factor of M_1 . So we see that $\text{soc}^2(W) \subseteq N$.

(ii) Recall that $M_2 \cong Y \oplus \mathbf{1}$ and Y is uniserial with socle layers $D_1, \mathbf{1}, D_2, \mathbf{1}, D_1$ by Lemma 5.3. Next, $\eta_{2,3}(Y)$ has codimension ≤ 1 in $W = \text{Im } \eta_{2,3}$. Inspecting the submodule structures of W and Y given in Lemma 5.4(ii) and Lemma 5.3, we see that $\eta_{2,3}(Y) = W$ and $\text{Ker}(\eta_{2,3}|_Y) = \text{soc}(Y)$.

(iii) Note that $\eta_{3,1} \circ \eta_{2,3} \neq 0$, so $W \not\subseteq \text{Ker } \eta_{3,1} = N$. Moreover, $\text{soc}^3(W) \not\subseteq N$, since otherwise $W \cap N = \text{soc}^3(W)$, and $M_3/N \cong M_1$ contains $(W+N)/N \cong W/\text{soc}^3(W) \cong D_1$

as a submodule, which is a contradiction. As W is uniserial and $\text{soc}^2(W) \subseteq N$, it now follows that $N \cap W = N \cap \text{soc}^3(W) = \text{soc}^2(W)$. Now, $M_3/(N + \text{soc}^3(W))$ is a quotient of $M_3/N \cong M_1$ by

$$(N + \text{soc}^3(W))/N \cong \text{soc}^3(W)/(N \cap \text{soc}^3(W)) = \text{soc}^3(W)/\text{soc}^2(W) \cong \mathbf{1},$$

so this quotient must be isomorphic to Q .

(iv) We know that N' has composition factors D_3, D_2, D_1 , and $\mathbf{1}$. It is easy to check that $\eta_{3,1} \circ \eta_{1,3} = 0$, so $\text{Im } \eta_{1,3} \subseteq N = \text{Ker } \eta_{3,1}$. By Lemma 5.5(i), $\text{Im } \eta_{1,3} \cong M_1$. Using the submodule structure of W and $\text{Im } \eta_{1,3}$ and the fact that $\text{soc}(M_3) = \mathbb{F} \cdot T_3$, we conclude that $\text{Im } \eta_{1,3} \cap W = \mathbb{F} \cdot T_3$. Therefore the image Q' of $\text{Im } \eta_{1,3}$ in N' is isomorphic to Q .

Now we know that N'' has composition factors D_3 and D_2 . Note that $\mathcal{S}_3^*$ is a quotient of M_3 , so some submodule S' of $\mathcal{S}_3^*$ is a quotient of N . Also, D_3 is the head of $\mathcal{S}_3$ and the socle of $\mathcal{S}_3^*$. But D_3 is not a composition factor of $\text{soc}^2(W) + \text{Im } \eta_{1,3}$. It follows that $\mathcal{S}_3^*$ is a quotient of $M_3/(\text{soc}^2(W) + \text{Im } \eta_{1,3})$, whence S' is a quotient of $N/(\text{soc}^2(W) + \text{Im } \eta_{1,3}) = N''$ and also of N' . By [J1] the composition factors of $\mathcal{S}_3$ (and $\mathcal{S}_3^*$) are $D_3, D_2, \mathbf{1}$. Among these, only $\mathbf{1}$ is a composition factor of $M_1 \cong M_3/N$, so S' has both D_3 and D_2 as composition factors. We have shown that S' is a quotient of N'' which has exactly two composition factors D_3 and D_2 . It follows that in fact $N'' \cong S'$. In this case, $\text{soc}(N'') \cong D_3$ since $\text{soc}(\mathcal{S}_3^*) \cong D_3$ is simple.

(v) Recall that $\mathcal{S}_3 \subset M_3$ and $\text{head}(\mathcal{S}_3) \cong D_3$ is not a composition factor of $M_1 \cong M_3/N$. It follows that $\mathcal{S}_3 \subset N$ and, since D_1 is not a composition factor of $\mathcal{S}_3$, the image D' of $\mathcal{S}_3$ in N' intersects Q' trivially. The aforementioned structure of N'' implies that $\mathcal{S}_3$ has no quotient isomorphic to N'' . Therefore, under the natural projection $N' \rightarrow N''$, D' projects onto a module isomorphic to D_3 , or 0. The latter cannot happen since D_3 is not a composition factor of $\text{soc}^2(W) + \text{Im } \eta_{1,3}$. So D' projects onto a module isomorphic to D_3 . This implies that the composition factors of $D' + Q'$ are $D_3, D_1, \mathbf{1}$. Since $D' \cap Q' = 0$ and $Q' \cong Q$, it follows that $D' \cong D_3$. We have shown that $D_3 \oplus D_1 \cong D' \oplus \text{soc}(Q') \subseteq \text{soc}(N')$. On the other hand, $N'/Q' = N''$, $\text{soc}(N'') \cong D_3$, and $\text{soc}(Q') \cong D_1$. Together these imply that $\text{soc}(N')$ embeds into $D_3 \oplus D_1$. So $D' \oplus \text{soc}(Q') = \text{soc}(N')$.

Let π denote the natural projection $N \rightarrow N'$. Then we have shown that $\text{Ker}(\pi|_{\mathcal{S}_3})$ has two composition factors D_2 and $\mathbf{1}$. On the other hand, $\text{Ker } \pi = \text{soc}^2(W)$. It follows that $\text{Ker}(\pi|_{\mathcal{S}_3}) = \text{soc}^2(W)$ so that in fact $D' = \mathcal{S}_3/\text{soc}^2(W)$ and $N'/D' \cong N/\mathcal{S}_3$. By (5.1) we have $\mathcal{S}_3 = N \cap \text{Ker } \eta_{3,2}$. So $N/\mathcal{S}_3 = N/(\text{Ker } \eta_{3,2} \cap N) \cong \eta_{3,2}(N)$ is a submodule of M_2 with composition factors $D_1, \mathbf{1}, D_2$, which are precisely the composition factors of $\mathcal{S}_2$. Hence, $N'/D' \cong N/\mathcal{S}_3 = \eta_{3,2}(N) = \mathcal{S}_2$. $\square$

6. MAIN REDUCTION THEOREM

The following theorem is the main tool for proving reducibility of various restrictions $\text{Res}_X^{\mathbb{S}_n}$ in the key case $p = 2|n$. Note by Theorem 3.2 that the assumption $d_3(V) > d_1(V)$ is equivalent to the assumption that V is not trivial and not the basic spin module.

Theorem 6.1. *Let $p = 2|n \geq 6$, V be an irreducible $\mathbb{F}\mathbb{S}_n$ -module satisfying $d_3(V) > d_1(V)$, and X be a subgroup of $\mathbb{S}_n$. Let N be the $\mathbb{F}\mathbb{S}_n$ -submodule of M_3 specified in Lemma 5.5(ii). Suppose that for any nonzero $\mathbb{F}\mathbb{S}_n$ -quotient J of N we have $J^X \neq 0$ and if $J^{\mathbb{S}_n} \neq 0$ then $\dim J^X \geq 2$. Then the restriction $\text{Res}_X^{\mathbb{S}_n} V$ is reducible.*

Proof. Set $E := \text{End}_{\mathbb{F}}(V)$ so that $d_r(V) = \dim \text{Hom}_{\mathbb{F}\mathbb{S}_n}(M_r, E)$. By Schur's Lemma, $\dim E^{\mathbb{S}_n} = \dim \text{End}_{\mathbb{F}\mathbb{S}_n}(V) = 1$, so the $\mathbb{F}\mathbb{S}_n$ -module E contains a unique submodule $E_1 \cong \mathbf{1}$. Note that $E^X = \text{End}_{\mathbb{F}X}(\text{Res}_X^{\mathbb{S}_n} V)$, so it suffices to prove that $\dim E^X \geq 2$.

By definition of N in Lemma 5.5(ii), we have an exact sequence

$$0 \rightarrow N \rightarrow M_3 \rightarrow M_1 \rightarrow 0.$$

Applying $\text{Hom}_{\mathbb{F}\mathbb{S}_n}(-, E)$ to this sequence and using the assumption $d_3(V) > d_1(V)$, we conclude that there is some $f \in \text{Hom}_{\mathbb{F}\mathbb{S}_n}(M_3, E)$ such that $J := f(N) \neq 0$.

If $J \cap E_1 = 0$, then E contains a submodule isomorphic to $J \oplus E_1$, whence $\dim E^X \geq 2$ as $J^X \neq 0$ by assumption. On the other hand, if $J \cap E_1 \neq 0$ then $J^{\mathbb{S}_n} \neq 0$. In this case $\dim E^X \geq \dim J^X \geq 2$ by assumption again. $\square$

Our main goal now will be to obtain permutation group theoretic conditions on the subgroup X which guarantee that the assumptions of Theorem 6.1 hold.

To bound dimensions of various fixed point subspaces, we will frequently use the following well-known estimates:

Lemma 6.2. *Let X be a group and $U \supseteq V$ be $\mathbb{F}X$ -modules. Then:*

- (i) $\dim(U/V)^X - \dim H^1(X, V) \leq \dim U^X - \dim V^X \leq \dim(U/V)^X$.
- (ii) *If in addition X acts trivially on V and $\text{Hom}(X, \mathbb{F}) = 0$, then*

$$\dim U^X = \dim V + \dim(U/V)^X.$$

Proof. (i) follows from the exact sequence $0 \rightarrow V^X \rightarrow U^X \rightarrow (U/V)^X \rightarrow H^1(X, V)$.

(ii) In this case $H^1(X, V) = \text{Hom}(X, \mathbb{F}) = 0$, whence the statement follows from (i). $\square$

Let $X \leq \mathbb{S}_n$ be any subgroup and $1 \leq r \leq n/2$. We set

$$(6.1) \quad f_r(X) := \dim(M_r)^X, \quad f_0(X) := 0, \quad e_r(X) := f_r(X) - f_{r-1}(X).$$

Note that $f_r(X)$ is the number of X -orbits on Ω_r . As in (5.3) let the $\mathbb{F}\mathbb{S}_n$ -module Q be defined by the short exact sequence

$$0 \rightarrow \mathbb{F} \cdot T_1 \rightarrow M_1 \rightarrow Q \rightarrow 0.$$

When $p = 2|n$ we also put

$$(6.2) \quad h(X) := \dim H^1(X, Q).$$

For any partition $\lambda \vdash n$, let χ^λ denote the irreducible ordinary $\mathbb{S}_n$ -character labeled by λ . It is well known (and follows for instance from the Littlewood-Richardson formula [J2, 16.4]) that the $\mathbb{S}_n$ -character afforded by the permutation module $\mathbb{C}\Omega_r$ is $\sum_{s=0}^r \chi^{(n-s, s)}$. Denote $\alpha := \chi^{(n-1, 1)}$, so that $\alpha + 1_{\mathbb{S}_n}$ is the permutation character of $\mathbb{S}_n$ acting on $\Omega := \{1, 2, \dots, n\}$. Applying [GT2, Lemma 3.3], we see that

$$\sum_{s=0}^r \chi^{(n-s, s)} = \begin{cases} S^2(\alpha) & \text{if } r = 2, \\ S^3(\alpha) - \wedge^2(\alpha) - \alpha & \text{if } r = 3, \end{cases}$$

where S^k denotes the k th symmetric power and $\wedge^k$ denotes the k th exterior power. If we know the restriction $\alpha_X := \text{Res}_X^{\mathbb{S}_n} \alpha$ explicitly, we can compute $f_2(X)$ and $f_3(X)$ by computing the scalar product of X -characters as follows:

$$(6.3) \quad f_2(X) = [S^2(\alpha_X), 1_X]_X, \quad f_3(X) = [(S^3(\alpha_X) - \wedge^2(\alpha_X) - \alpha_X), 1_X]_X.$$

Next we record some elementary observations:

Lemma 6.3. *Let $X \leq \mathbb{S}_n$ be a transitive subgroup. Then:*

- (i) $f_2(X) = 1$ if and only if X is 2-homogeneous.
- (ii) Suppose $|X|$ is even. Then $f_2(X) = 1$ if and only if X is 2-transitive.
- (iii) $f_2(X) \leq 2$ if X is a rank ≤ 3 subgroup of $\mathbb{S}_n$.

Proof. (i) is obvious: X is 2-homogeneous precisely when it acts transitively on Ω_2 . For (ii), observe that X contains an involution t , and so we can find $x, y \in \Omega$ interchanged by t . It follows that X is 2-homogeneous precisely when it is 2-transitive.

For (iii), note that if X is a rank 2 subgroup $f_2(X) = 1$ by (ii). If X is a rank 3 subgroup, then the point stabilizer of $x \in \Omega$ in X has two orbits on $\Omega \setminus \{x\}$, whence X has at most two orbits on Ω_2 . $\square$

Note that since $p = 2$ in the following proposition, the condition $X = O^2(X)$ is equivalent to the condition $\text{Hom}(X, \mathbb{F}) = 0$ from Lemma 6.2. In many applications X will be perfect, in which case this assumption of course holds.

Proposition 6.4. *Let $p = 2 | n \geq 6$, N the $\mathbb{F}\mathbb{S}_n$ -submodule of M_3 specified in Lemma 5.5(ii), and let J be any nonzero $\mathbb{F}\mathbb{S}_n$ -quotient of N . Suppose that $X = O^2(X) \leq \mathbb{S}_n$ is a subgroup such that*

$$f_1(X) = 1, \quad e_3(X) \geq h(X) + 1, \quad \text{and either } f_2(X) \geq 3 \text{ or } \mathcal{S}_2^X \neq 0.$$

Then $J^X \neq 0$. Moreover, if $J^{\mathbb{S}_n} \neq 0$, then $\dim J^X \geq 2$.

Proof. We write f_r for $f_r(X)$, e_r for $e_r(X)$, and h for $h(X)$. Note that $Q^X = 0$ and $D_1^X = 0$ since $f_1 = 1$ and $X = O^2(X)$. Combining this with the structure of M_2 given in Lemma 5.3 and applying Lemma 6.2, we see that

$$(6.4) \quad f_2 \geq 1 + \dim D_2^X - \dim H^1(X, Q) = 1 + \dim D_2^X - h.$$

Note that $D_2^X \neq 0$. Indeed, if $f_2 \geq 3$, this follows by considering composition factors of M_2 described in Lemma 5.3 and using $Q^X = D_1^X = 0$. On the other hand, if $\mathcal{S}_2^X \neq 0$, this follows by considering composition factors of $\mathcal{S}_2$ using $D_1^X = 0$.

Case 1: $n \equiv 0 \pmod{4}$. Then N is uniserial by Lemma 5.4, and we are going to check that $J^X \neq 0$ for each of its five non-trivial quotients J . This is all we have to do, since $\mathbf{1}$ is not a composition factor of N , and so we never have $J^{\mathbb{S}_n} \neq 0$.

Note that $\text{soc}(N) \cong D_2$, so $N^X \supseteq D_2^X \neq 0$. By assumption, we have $f_3 \geq f_2 + h + 1$, so (6.4) implies $f_3 \geq \dim D_2^X + 2$. Since $M_3 = M_1 \oplus N$ and $f_1 = 1$, it follows that $(N/\text{soc}(N))^X \neq 0$. Then since $D_1^X = 0$, we also get $(N/\text{soc}^2(N))^X \neq 0$. Next, $N/\text{soc}^3(N) \cong \mathcal{S}_2$ by Lemma 5.6. If $\mathcal{S}_2^X \neq 0$, we are done. Otherwise, the conditions $f_2 \geq 3$ and $D_1^X = 0$ imply that $(N/\text{soc}^3(N))^X \neq 0$. Finally, $N/\text{rad}(N) \cong D_2$ and we already have $D_2^X \neq 0$.

Case 2: $n \equiv 0 \pmod{4}$. We are going to use the notation of Lemma 5.7.

Step 1: we prove that $J^X \neq 0$ for any nonzero quotient $J = N/K$ of N .

By Lemma 5.7(i), we have the submodule $\text{soc}^2(W) \subset N$ which is uniserial with socle layers $\mathbf{1}, D_2$. So any nonzero quotient of $\text{soc}^2(W)$ either contains $\mathbf{1}$ or is isomorphic to D_2 , hence it contains nonzero X -fixed points. In particular, $(N/K)^X > 0$ if $\text{soc}^2(W) \not\subseteq K$, and we may now assume that $\text{soc}^2(W) \subseteq K$. In other words, we are reduced to showing that X has nonzero fixed points on every nonzero $\mathbb{F}\mathcal{S}_n$ -quotient of $N' = N/\text{soc}^2(W)$.

Recall that $M_2 \cong Y \oplus \mathbf{1}$, see Lemma 5.7. In particular, $\dim Y^X = f_2 - 1$, and so $\dim(\text{soc}^4(Y))^X = f_2 - 1$, since X has no fixed points on $U/\text{soc}^4(U) \cong D_1$. Applying Lemma 6.2(i) to the exact sequence

$$0 \rightarrow \text{soc}^2(Y) \rightarrow \text{soc}^4(Y) \rightarrow \text{soc}^4(Y)/\text{soc}^2(Y) \rightarrow 0$$

with $(\text{soc}^2(Y))^X \cong Q^X = 0$, we see that

$$(6.5) \quad \dim(\text{soc}^4(Y)/\text{soc}^2(Y))^X \leq f_2 + h - 1.$$

By Lemma 5.7(ii), we have $\eta_{2,3}(Y) = W \cong Y/\text{soc}(Y)$. So $\text{soc}^3(W) \cong \text{soc}^4(Y)/\text{soc}(Y)$ is an extension of $\text{soc}^4(Y)/\text{soc}^2(Y)$ by $\text{soc}^2(Y)/\text{soc}(Y) \cong \mathbf{1}$. Together with (6.5) and Lemma 6.2, this implies that

$$(6.6) \quad \dim(\text{soc}^3(W))^X \leq f_2 + h.$$

Since $Q^X = 0$, Lemma 5.7(iii) yields

$$(6.7) \quad \dim(N + \text{soc}^3(W))^X = f_3.$$

Moreover, by the same lemma, we have

$$N' = N/\text{soc}^2(W) = N/(N \cap \text{soc}^3(W)) \cong (N + \text{soc}^3(W))/\text{soc}^3(W).$$

Since $f_3 - f_2 = e_3 \geq h + 1$ we deduce from (6.6) and (6.7) that $(N')^X \neq 0$.

Now we apply Lemma 5.7(iv). Since $N'' = N'/Q'$ and $Q'^X \cong Q^X = 0$, we have that $(N')^X \neq 0$ implies $N''^X \neq 0$. Recalling that $D_2^X \neq 0$ and $\text{head}(N'') \cong D_2$, we have now shown that X has nonzero fixed points on every nonzero quotient of N'' .

It remains to consider quotients of N' by nonzero submodules R' which do not contain Q' . Since Q' has a simple socle and $Q'/\text{soc}(Q') \cong \mathbf{1}$, we only need to consider R' such that $R' \cap Q' = 0$. We have shown in Lemma 5.7(v) that $\text{soc}(N') = D' \oplus \text{soc}(Q')$. Hence the condition $R' \cap Q' = 0$ implies that $R' \supseteq D'$. So we must show that X has nonzero fixed points on every nonzero quotient of the $\mathbb{F}\mathcal{S}_n$ -module $N'/D' \cong \mathcal{S}_2$, see Lemma 5.7(v) again. By assumption, $\mathcal{S}_2^X \neq 0$ unless $f_2 \geq 3$, in which case $D_1^X = 0$ implies $\mathcal{S}_2^X \neq 0$. The only proper quotients of $\mathcal{S}_2$ have socles $\mathbf{1}$ and D_2 and we have seen already that X has non-trivial fixed points on these simple modules.

Step 2: we assume that $J = N/K$ contains a trivial $\mathcal{S}_n$ -submodule $E \cong \mathbf{1}$ and prove that $\dim J^X \geq 2$. We again use the notation of Lemma 5.7. Since we have shown that $D_2^X, (N'')^X \neq 0$, by Lemma 6.2(ii) it suffices to show that J contains an $\mathbb{F}\mathcal{S}_n$ -submodule L where L/E is isomorphic to D_2 or N'' . This is obvious if $K = 0$. So we may assume that $K \supseteq \text{soc}(W) = \text{soc}(N) \cong \mathbf{1}$.

If $K \cap \text{soc}^2(W) = \text{soc}(W)$, then J contains $\text{soc}^2(W)/\text{soc}(W) \cong D_2$, and we can take $L \cong D_2 \oplus E$. Thus we may assume that $K \supseteq \text{soc}^2(W)$ and so J can be regarded as

a quotient N'/K' of $N' = N/\text{soc}^2(W)$. Let P' be the preimage of E in N' so that $E \cong P'/K'$. Since $\mathbf{1}$ is not a composition factor of N'/Q' , we see that $Q' + K' \supseteq P'$, whence $Q'/(Q' \cap K') \cong (Q' + K')/K'$ contains $E = P'/K'$. But Q' has socle layers $D_1, \mathbf{1}$, so $Q' \cap K' = \text{soc}(Q')$ and $P' = Q' + K'$. Recall that $N'/Q' = N''$ is uniserial with socle layers D_3, D_2 . So if $P' \neq N'$ then $J \cong N'/K'$ is isomorphic to an extension of the nonzero quotient N'/P' of $N'/Q' \cong N''$ by $P'/K' = E$, and we may take $L = J$.

Assume $P' = N'$, so that $N'/K' \cong \mathbf{1}$. Since $D' \cong D_3$ is simple, it follows that $D' \subset K'$. Now we see that $N'/K' \cong \mathbf{1}$ is a quotient of N'/D' which is isomorphic to $\mathcal{S}_2$ by Lemma 5.7(v), a contradiction since $\text{head}(\mathcal{S}_2) \cong D_2$. $\square$

Now we can prove the main result of this section:

Theorem 6.5. *Let $p = 2|n| \geq 6$, V be an irreducible $\mathbb{F}\mathcal{S}_n$ -module satisfying $d_3(V) > d_1(V)$, and $X = O^2(X) \leq \mathcal{S}_n$ be a subgroup such that*

$$f_1(X) = 1, \quad e_3(X) \geq h(X) + 1, \quad \text{and either } f_2(X) \geq 3 \text{ or } \mathcal{S}_2^X \neq 0.$$

Then the restriction $\text{Res}_X^{\mathcal{S}_n} V$ is reducible.

Proof. Apply Theorem 6.1 and Proposition 6.4. $\square$

Next we show that the condition $\mathcal{S}_2^X \neq 0$ is a fairly mild condition which *always* holds for transitive permutation groups of even degree n of rank ≥ 3 :

Lemma 6.6. *Let $2|n| \geq 6$ and let $G \leq \mathcal{S}_n$ be a transitive permutation group of rank ≥ 3 . Then $\mathcal{S}_2^G \neq 0$.*

Proof. Since $2|n|$, $|G|$ is even and so G contains an involution j . Choose $x_0 \in \Omega = \{1, 2, \dots, n\}$ not fixed by j . By assumption, the stabilizer G_{x_0} of x_0 has an orbit $\Delta_1 \ni j(x_0)$ on $\Omega \setminus \{x_0\}$, and $\Delta_2 := (\Omega \setminus \{x_0\}) \setminus \Delta_1 \neq \emptyset$. Let Φ_1 denote the G -orbit of the 2-subset $\{x_0, j(x_0)\}$. Since the G -orbit of the ordered pair $(x_0, j(x_0))$ has length $n|\Delta_1|$ and j interchanges x_0 and $j(x_0)$, we see that $|\Phi_1| = n|\Delta_1|/2$. Let $\Phi_2 := \Omega_2 \setminus \Phi_1$, where Ω_2 denotes the set of all 2-subsets $\{x, y\}$ of Ω . Then

$$|\Phi_2| = |\Omega_2| - n|\Delta_1|/2 = n|\Delta_2|/2 > 0.$$

As $|\Delta_1| + |\Delta_2| = n - 1 \equiv 1 \pmod{2}$, there is (a unique) $k \in \{1, 2\}$ such that $|\Delta_k|$ is *even*.

Since Φ_k is a non-empty union of G -orbits, it suffices to show that the orbit sum $\hat{\Phi}_k := \sum_{\{x,y\} \in \Phi_k} \{x, y\}$ belongs to $\mathcal{S}_2$. A standard fact about Specht modules following from (5.1) and (5.2) is that

$$\mathcal{S}_2 = \langle T_2 \rangle^\perp \cap \eta_{1,2}(M_1)^\perp,$$

where perpendicularity is with respect to the natural inner product $\langle \cdot, \cdot \rangle_2$ on M_2 . Now,

$$\langle \hat{\Phi}_k, T_2 \rangle_2 = |\Phi_k| = n|\Delta_k|/2 \equiv 0 \pmod{2}.$$

Next, for any $a \in \Omega \setminus \{x_0\}$, $\{x_0, a\} \in \Phi_1$ if and only if there is some $g \in G$ sending $\{x_0, j(x_0)\}$ to $\{x_0, a\}$. Replacing g by gj if necessary, we get $g \in G_{x_0}$ and $a \in \Delta_1$. It follows that $\{x_0, a\} \in \Phi_k$ if and only if $a \in \Delta_k$, whence

$$\langle \hat{\Phi}_k, \eta_{1,2}(x_0) \rangle_2 = |\Delta_k| \equiv 0 \pmod{2}.$$

By transitivity of G on Δ , $\langle \hat{\Phi}_k, \eta_{1,2}(x) \rangle_2 = 0$ for all $x \in \Omega$, and the claim follows. $\square$

7. SPECIAL EMBEDDINGS OF A_m INTO A_n

Let $X \cong A_m$ with $m \geq 5$. In this section, we consider two *special* kinds of embeddings of X into symmetric groups S_n . The first arises from the action of X on k -subsets of

$$\Delta := \{1, 2, \dots, m\}$$

with $2 \leq k < m/2$, giving rise to an embedding of X into S_n , where $n = \binom{m}{k}$. The second embedding comes from the action on set partitions of Δ into b subsets of size a , where $m = ab$. This gives rise to an embedding into S_n , where $n = (ab)!/((a!)^b \cdot b!)$.

Lemma 7.1. *Suppose that $m \geq 8$. Then for any of the two special embeddings, we have that $n \geq m(m-1)/2$.*

Proof. For the first embedding, observe that the sequence $\binom{m}{k}$ is increasing for $2 \leq k < m/2$, whence $\binom{m}{k} \geq \binom{m}{2}$. For the second embedding, denote $N_{a,b} := (ab)!/((a!)^b \cdot b!)$ and observe that

$$(7.1) \quad \frac{N_{a,b+1}}{N_{a,b}} = \frac{(ab+1)(ab+2) \dots (ab+a-1)}{(a-1)!} \geq ab+1 \geq 5$$

as long as $a, b \geq 2$. An induction on a shows that $N_{a,2} > \binom{2a}{2}$ for $a \geq 4$. Now another induction on $b \geq 2$ using (7.1) shows that $N_{a,b} > \binom{ab}{2}$ whenever $a \geq 4$ and $b \geq 2$. Next, an induction on a shows that $N_{a,3} \geq \binom{3a}{2}$ for $a \geq 2$, with equality only when $a = 2$. Now another induction on $b \geq 3$ using (7.1) shows that $N_{a,b} > \binom{ab}{2}$ if $a \geq 2$, $b \geq 3$, and $ab \geq 8$. $\square$

Recall that $h(X) := \dim H^1(X, Q)$, where the S_n -module Q is defined in (5.3).

Lemma 7.2. *Let $p = 2$. We have:*

- (i) $\dim H^2(X, \mathbb{F}) = 1$.
- (ii) *If n is even, then one of the following statements holds:*
 - (a) $\dim H^1(X, M_1) = 1$ and $h(X) \leq 2$.
 - (b) $4|m$, A_m embeds into A_n via its action on partitions $(m/2, m/2)$ of Δ , and $\dim H^1(X, M_1) = 2$, $h(X) \leq 3$.

Proof. (i) is a well-known fact about the Schur multiplier of A_m (see e.g. [KIL, Theorem 5.1.4(i)]).

(ii) By Frobenius reciprocity we have that

$$H^1(X, M_1) \cong H^1(X_1, \mathbb{F}) \cong \text{Hom}(X_1, (\mathbb{F}, +)),$$

where X_1 is the stabilizer in X of a point on the set Ω , and $(\mathbb{F}, +)$ is the additive group of the field $\mathbb{F}$. First we consider the case where X is acting on k -sets of Δ . Then

$$X_1 = (S_k \times S_{m-k}) \cap A_m \cong (A_k \times A_{m-k}) \cdot 2.$$

Since $p = 2$, we have that $\text{Hom}(A_s, (\mathbb{F}, +)) = 0$ for all $s \geq 1$. Denoting by C_2 the group of order 2, it follows that $\text{Hom}(X_1, (\mathbb{F}, +)) \cong \text{Hom}(C_2, (\mathbb{F}, +))$ is one-dimensional.

Next we consider the case X is acting on set partitions of Δ into $b \geq 2$ subsets of size $a = m/b \geq 2$. Then

$$X_1 = (S_a \wr S_b) \cap A_m.$$

We may assume that the transposition $(1, 2)$ fixes the set partition fixed by X_1 . Then $(1, 2)$ belongs to the base subgroup $B = S_a^b$, whence $[B : B \cap X_1] = 2$ and $X_1 \cong (B \cap X_1) \cdot S_b$. As mentioned above, $\text{Hom}(A_a, (\mathbb{F}, +)) = 0$. Hence

$$\text{Hom}(X_1, (\mathbb{F}, +)) \cong \text{Hom}(X_1/A_a^b, (\mathbb{F}, +)) = \text{Hom}(Y, (\mathbb{F}, +)),$$

where $Y = X_1/A_a^b \cong 2^{b-1} \cdot S_b$. If $b \geq 3$, then one can check that $[Y, Y]$ contains the normal subgroup 2^{b-1} , and so

$$\text{Hom}(Y, (\mathbb{F}, +)) \cong \text{Hom}(S_b, (\mathbb{F}, +)) \cong \text{Hom}(C_2, (\mathbb{F}, +)) \cong \mathbb{F}$$

as $\text{Hom}(A_b, (\mathbb{F}, +)) = 0$. Assume that $b = 2$, i.e. X_1 fixes the partition $\Delta = \{1, 2, \dots, a\} \cup \{a+1, \dots, m\}$. If $a \geq 3$ is odd, then the permutation

$$g : (1, a+1, 2, a+2)(3, a+3) \dots (a, 2a)$$

belongs to X_1 and $g^2 = (1, 2)(a+1, a+2)$ gives rise to an involution in Y . Thus Y is cyclic of order 4, and so again $\text{Hom}(Y, (\mathbb{F}, +)) \cong \mathbb{F}$. If a is even, then Y (of order 4) is generated by two involutions $(1, 2)(a+1, a+2)$ and $(1, a+1)(2, a+2) \dots (a, 2a)$, whence $\text{Hom}(Y, (\mathbb{F}, +)) \cong \mathbb{F}^2$. This proves the claims on $\dim H^1(X, M_1)$ in (ii).

Now the bounds on $h(X)$ in (ii) follow immediately from the portion

$$0 = H^1(X, \mathbb{F}) \rightarrow H^1(X, M_1) \rightarrow H^1(X, Q) \rightarrow H^2(X, \mathbb{F})$$

of the long exact sequence arising from (5.3). $\square$

Lemma 7.3. *Let X embed into S_n via its action on k -subsets of Δ for $2 \leq k < m/2$. Then $f_2(X) = k$.*

Proof. We claim that the orbits of X or of S_m on pairs $\{A, B\}$ of distinct k -subsets of Δ are labeled by $j := |A \cap B|$ for $0 \leq j \leq k-1$, hence $f_2(X) = k$. Indeed, the claim is obvious for S_m . Since $A \neq B$, we can find $i \in A \setminus B$ and $j \in B \setminus A$. Now the transposition (i, j) fixes the pair $\{A, B\}$, and so S_m and A_m have the same orbits on pairs $\{A, B\}$. $\square$

Next we handle the embedding of A_m into A_n via its action on 2-subsets:

Corollary 7.4. *Let $p = 2$ and $m \geq 6$ be such that $n := \binom{m}{2}$ is even. Let $X = A_m$ embed into A_n via its action on 2-subsets of $\{1, 2, \dots, m\}$. Suppose that an irreducible $\mathbb{F}S_n$ -module V satisfies the condition $d_3(V) > d_1(V)$. Then $\text{Res}_X^{S_n} V$ is reducible.*

Proof. By Lemma 7.2(ii), we have that $h(X) \leq 2$. On the other hand, $f_1(X) = 1$, and $f_2(X) = 2$ by Lemma 7.3. Also, $f_3(X) \geq 5$. Indeed, we can regard the $\mathbb{F}S_n$ -permutation module M_1 as having a basis consisting of all 2-subsets $\{i, j\}$ of $\Delta = \{1, 2, \dots, m\}$. Then the module M_3 has a basis consisting of unordered triples of distinct pairs, and S_m has 5 orbits on this set represented by the triples

$$\begin{aligned} & \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}, & \{\{1, 2\}, \{3, 4\}, \{4, 5\}\}, & \{\{1, 2\}, \{2, 4\}, \{3, 4\}\} \\ & \{\{1, 2\}, \{1, 3\}, \{1, 4\}\}, & \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}. \end{aligned}$$

In particular, $e_3(X) \geq 3 \geq h(X) + 1$; furthermore, $\mathcal{S}_2^X \neq 0$ by Lemma 6.6. Hence we are done by Theorem 6.5. $\square$

Lemma 7.5. *Suppose that $m \geq 11$. Then $e_2(X) \geq 2$, unless A_m embeds in A_n via its action on 2-subsets of Δ , in which case $e_2(X) = 1$.*

Proof. Recall that $f_1(X) = 1$ for the special embeddings of X into S_n in question. Now for the action of X on k -subsets of Δ the result follows from Lemma 7.3.

Now let X act on partitions $P = \{P_1, \dots, P_b\}$ of Δ into b a -subsets $P_1, \dots, P_b$. We will exhibit at least 3 orbits of X on pairs of partitions $\{P, Q\}$. Note that Δ admits two partitions with no common subset between them. It follows that, for each $j = 0, 1, \dots, b-2$, Δ admits a pair of partitions $\{P, Q\}$, where P and Q contain exactly j common subsets. Certainly, such pairs with different parameters j belong to different S_m -orbits. In particular, we are done if $b \geq 4$.

Suppose $b = 3$ and $a \geq 4$. Then we get at least one orbit with the above parameter $j = 0$. For $j = 1$, we get at least two orbits with representatives $\{P, Q\}$, where $P = \{P_1, P_2, R\}$, $Q = \{Q_1, Q_2, R\}$, and $|P_1 \cap Q_1| = 1$, respectively $|P_1 \cap Q_1| = 2$.

Suppose $b = 2$ and $a \geq 6$. Then for each $s = 1, 2, 3$ we get at least one orbit with representatives $\{P, Q\}$, where $P = \{P_1, P_2\}$, $Q = \{Q_1, Q_2\}$, and $|P_1 \cap Q_1| = s$. $\square$

Remark 7.6. It is easy to check that for the embedding via the action of X on b a -subsets, we have $e_2(X) = 0$ if $(a, b) = (3, 2)$, and $e_2 = 1$ if $(a, b) = (2, 3)$, $(4, 2)$, or $(5, 2)$. On the other hand, $e_2(X) = 3$ if $(a, b) = (3, 3)$, as one can compute using (6.3) above and [GAP].

Lemma 7.7. *Suppose $m \geq 6$ and $X = A_m$ embeds into S_n via its action on k -subsets of Δ for $2 \leq k < m/2$. Then either $e_3(X) \geq 4$, or $k = 2$ and $e_3(X) = 3$.*

Proof. By Lemma 7.3, we have $f_2(X) = k$. So we will try to exhibit at least $(k+4)$ S_m -orbits on triples $\{A, B, C\}$ of k -subsets. We may assume that

$$|A \cap B| \geq |A \cap C| \geq |B \cap C|$$

and call $(|A \cap B|, |A \cap C|, |B \cap C|)$ the *mark* of the triple $\{A, B, C\}$. Certainly, triples with different marks belong to different S_m -orbits.

Recall that $m \geq 2k+1$. First let $|A \cap B| = k-1$, so we may assume

$$A = \{1, 2, \dots, k-1, k\}, \quad B = \{1, 2, \dots, k-1, k+1\}.$$

For $0 \leq j \leq k-1$, by choosing

$$C = \{1, 2, \dots, j, k+2, k+3, \dots, 2k+1-j\}$$

we get a triple with the mark $(k-1, j, j)$. Similarly, for $1 \leq j \leq k-1$, by choosing

$$C = \{1, 2, \dots, j-1, k, k+2, k+3, \dots, 2k+1-j\}$$

we get a triple with the mark $(k-1, j, j-1)$.

Next we consider the case $|A \cap B| = k-2$, say

$$A = \{1, 2, \dots, k-2, k-1, k\}, \quad B = \{1, 2, \dots, k-2, k+1, k+2\}.$$

For $1 \leq j \leq k-2$, by choosing

$$C = \{1, 2, \dots, j, k+3, k+4, \dots, 2k+2-j\}$$

we get a triple with the mark $(k-2, j, j)$. Similarly, for $1 \leq j \leq k-2$, by choosing

$$C = \{1, 2, \dots, j-1, k, k+2, k+3, \dots, 2k+2-j\}$$

we get a triple with the mark $(k-2, j, j-1)$.

We have produced at least $4k-5$ different marks. So we have $e_3(X) \geq 3k-5 \geq 4$ if $k \geq 3$.

Finally, consider the case $k=2$. Then the triples

$$\{\{1, 2\}, \{1, 3\}, \{2, 4\}\}, \{\{1, 2\}, \{1, 3\}, \{4, 5\}\}, \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$$

have marks $(1, 1, 0)$, $(1, 0, 0)$, and $(0, 0, 0)$. Furthermore, the triples $\{\{1, 2\}, \{1, 3\}, \{1, 4\}\}$ and $\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ have the same mark $(1, 1, 1)$, but different cardinality of $A \cup B \cup C$, so they produce two more S_m -orbits. $\square$

To estimate $e_3(X)$ for the second special embedding of X into S_n , we need the following observation:

Lemma 7.8. *Let Y be any group and $\mathbb{K}$ any field. Suppose that A_1, A_2, B_1, B_2 are $\mathbb{K}Y$ -modules such that there is an injective $f \in \text{Hom}_{\mathbb{K}Y}(A, B)$ with $A = A_1 \oplus A_2$, $B = B_1 \oplus B_2$, and $f(A_2) \subseteq B_2$. Then*

$$\dim B^Y - \dim A^Y \geq \dim B_1^Y - \dim A_1^Y.$$

Proof. Clearly, $A^Y = A_1^Y \oplus A_2^Y$ and $B^Y = B_1^Y \oplus B_2^Y$. Now f embeds A_2^Y in B_2^Y , whence the claim. $\square$

The following statement is also well known and follows for example from the formula for $\text{rank}_{\mathbb{K}} \eta_{2,3}$ given in [Wil]:

Lemma 7.9. *Let $p \neq 2, 3$ and $n \geq 4$. Then $\eta_{2,3} : M_2 \rightarrow M_3$ is injective.* $\square$

Now we can prove a reduction lemma to help estimate $e_3(X)$ for the second special embedding.

Lemma 7.10. *Let $X = A_m$ and $Y := S_m$ embed into S_n via their actions on partitions of $\Delta = \{1, 2, \dots, m\}$ into b a -subsets, with $a, b \geq 2$. Suppose that $b > s \geq 2$. Set $n' := (sa)! / ((a!)^s \cdot s!)$ and let $Z := S_{sa}$ embed in $S_{n'}$ via its action on partitions of $\Delta' = \{1, 2, \dots, sa\}$ into s a -subsets. Also denote by N_r the permutation $S_{n'}$ -module corresponding to its action on r -subsets of $\{1, 2, \dots, n'\}$. Then*

$$\dim M_3^X - \dim M_2^X \geq \dim N_3^Z - \dim N_2^Z.$$

Proof. 1) Since M_r is a permutation module, $\dim M_r^X$ remains unchanged when we replace $\mathbb{F}$ by any other field. So throughout this proof we may assume $\mathbb{F} = \mathbb{C}$.

With respect to X and Y , any pair $\{P, Q\}$ in Ω_2 is a pair of two partitions

$$P = \{P_1, \dots, P_a\}, \quad Q = \{Q_1, \dots, Q_a\}.$$

Call $\{P, Q\}$ a *good pair* if at least $b-s$ subsets P_i occur among the Q_j . Next, we call a triple $\{P, Q, R\} \in \Omega_3$ with $R = \{R_1, \dots, R_a\}$ a *good triple* if all three pairs $\{P, Q\}$, $\{P, R\}$, $\{Q, R\}$ are good. We also call $\{P, Q, R\}$ a *very good triple* if there are at least $b-s$ subsets P_i which occur both among the Q_j and among the R_j . Then $Y > X$ acts

on the following sets: Ω_{21} of all good pairs, $\Omega_{22} := \Omega_2 \setminus \Omega_{21}$, Ω_{31} of all good triples, and $\Omega_{32} := \Omega_3 \setminus \Omega_{31}$. Thus

$$A_1 := \mathbb{C}\Omega_{21}, \quad A_2 := \mathbb{C}\Omega_{22}, \quad B_1 := \mathbb{C}\Omega_{31}, \quad B_2 := \mathbb{C}\Omega_{32}$$

are $\mathbb{C}Y$ -submodules of $M_2 = A_1 \oplus A_2$ and $M_3 = B_1 \oplus B_2$. Note that $Y > X$ also acts on the set Ω_{311} of *very good* triples, and so $B_{11} := \mathbb{C}\Omega_{311}$ is an X -submodule of B_1 . Certainly, $B_{11}^X \subseteq B_1^X$.

Since $\text{char}(\mathbb{C}) = 0$, $\eta_{2,3}$ is injective by Lemma 7.9. Next, if $\{P, Q\} \in \Omega_{22}$, then

$$\eta_{2,3}(\{P, Q\}) = \sum_{R \neq P, Q} \{P, Q, R\},$$

where all occurring triples $\{P, Q, R\}$ are not good (since $\{P, Q\}$ is not good). Thus $\eta_{2,3}(\{P, Q\}) \in B_2$. We have shown that $\eta_{2,3}(A_2) \subseteq B_2$. Applying Lemma 7.8 to the homomorphism $f = \eta_{2,3}$, we see that

$$\dim M_3^X - \dim M_2^X \geq \dim B_1^X - \dim A_1^X \geq \dim B_{11}^X - \dim A_1^X.$$

Recall that $b > s$. Hence, for any good pair $\{P, Q\}$, P and Q have at least one common a -subset P_1 , and since $a \geq 2$, some transposition $(i, j) \in Y \setminus X$ fixes both P and Q . It follows that X and Y have the same orbits on good pairs. Similarly, X and Y have the same orbits on *very good* triples. Thus

$$\dim B_{11}^X - \dim A_1^X = \dim B_{11}^Y - \dim A_1^Y.$$

2) It remains to prove that

$$(7.2) \quad \dim B_{11}^Y - \dim A_1^Y = \dim N_3^Z - \dim N_2^Z.$$

To do so, we will count the X -orbits on good pairs and very good triples.

Let $r \in \{2, 3\}$. Suppose the good pairs (if $r = 2$), respectively the very good triples (if $r = 3$), $\{P^1, \dots, P^r\}$ and $\{Q^1, \dots, Q^r\}$ belong to the same X -orbit. Then, without loss we may assume that

$$(7.3) \quad P^i = \{R_1, \dots, R_{b-s}, P_1^i, \dots, P_s^i\}, \quad Q^i = \{R_1, \dots, R_{b-s}, Q_1^i, \dots, Q_s^i\}$$

for $1 \leq i \leq r$. Denote

$$\bar{P}^i := \{P_1^i, \dots, P_s^i\}, \quad \bar{Q}^i := \{Q_1^i, \dots, Q_s^i\}$$

for $1 \leq i \leq r$.

By assumption, there is some $\sigma \in S_m$ sending $\{P^1, \dots, P^r\}$ to $\{Q^1, \dots, Q^r\}$. Let t be the number of common a -subsets that occur in all $P^1, \dots, P^r$. Then t is also the number of common a -subsets that occur in all $Q^1, \dots, Q^r$, and $t \geq b - s$.

Consider the case $t = b - s$. Then the t common a -subsets among all P^i are precisely $R_1, \dots, R_{b-s}$, and similarly, the t common a -subsets among all Q^i are precisely $R_1, \dots, R_{b-s}$. It follows that σ acts on the set $\{R_1, \dots, R_{b-s}\}$, and preserves the set

$$\cup_{j=1}^s P_j^i = \cup_{j=1}^s Q_j^i$$

which can be identified with Δ' . Now we can write $\sigma = \mu\tau$, where

$$\mu \in S_{m-sa} = \text{Sym}(\cup_{j=1}^{b-s} R_j)$$

acts on the set $\{R_1, \dots, R_{b-s}\}$, and $\tau \in Z = \text{Sym}(\Delta')$ sends $\{\bar{P}^1, \dots, \bar{P}^r\}$ to $\{\bar{Q}^1, \dots, \bar{Q}^r\}$. Thus, the two pairs, respectively triples, $\{\bar{P}^1, \dots, \bar{P}^r\}$ and $\{\bar{Q}^1, \dots, \bar{Q}^r\}$ of partitions of Δ' belong to the same Z -orbit.

Next assume that $t > b - s$ and set $v = t + s - b$. Then the t common a -subsets among all P^i are precisely $R_1, \dots, R_{b-s}, S_1, \dots, S_v$, and similarly, the t common a -subsets among all Q^i are precisely $R_1, \dots, R_{b-s}, T_1, \dots, T_v$, for some a -subsets S_j and T_j of Δ . It follows that σ sends $\{R_1, \dots, R_{b-s}, S_1, \dots, S_v\}$ to $\{R_1, \dots, R_{b-s}, T_1, \dots, T_v\}$, and Σ to Θ , where

$$\Sigma = \Delta \setminus \left(\bigcup_{j=1}^{b-s} R_j \cup \bigcup_{j=1}^v S_j \right), \quad \Theta = \Delta \setminus \left(\bigcup_{j=1}^{b-s} R_j \cup \bigcup_{j=1}^v T_j \right).$$

Set $R'_j := R_j$ for $1 \leq j \leq b - s$. Also set $R_{b-s+j} := S_j$ and $R'_{b-s+j} := T_j$ for $1 \leq j \leq v$. Then there is a permutation $\pi \in \mathbf{S}_t$ such that $\sigma(R_j) = R'_{\pi(j)}$. Now we can find $\gamma \in \mathbf{S}_m = \text{Sym}(\Delta)$ such that

$$\gamma_\Sigma = 1_\Sigma, \quad \gamma(R_j) = R_{\pi^{-1}(j)}.$$

Clearly, $\sigma\gamma$ sends R_j to R_j for $1 \leq j \leq b - s$ (and S_j to T_j), and sends $\{P^1, \dots, P^r\}$ to $\{Q^1, \dots, Q^r\}$. Now we can repeat the argument of the preceding case $t = b - s$ to show that the two pairs, respectively triples, $\{\bar{P}^1, \dots, \bar{P}^r\}$ and $\{\bar{Q}^1, \dots, \bar{Q}^r\}$ of partitions of Δ' belong to the same Z -orbit.

Conversely, it is obvious that if $\{\bar{P}^1, \dots, \bar{P}^r\}$ and $\{\bar{Q}^1, \dots, \bar{Q}^r\}$ belong to the same Z -orbit, then $\{P^1, \dots, P^r\}$ and $\{Q^1, \dots, Q^r\}$, defined as in (7.3), belong to the same Y -orbit. We have therefore proved that $\dim B_{11}^Y = \dim N_3^Z$ and $\dim A_1^Y = \dim N_2^Z$, whence (7.2) holds. $\square$

Theorem 7.11. *Let $p = 2$ and n be even. Suppose that $n > m \geq 11$ and $X \cong \mathbf{A}_m$ embeds into $\mathbf{A}_n$ via its actions on subsets or partitions of $\{1, 2, \dots, m\}$. Then $f_2(X) \geq 3$ and $e_3(X) \geq h(X) + 2$, unless $\mathbf{A}_m$ embeds into $\mathbf{A}_n$ via its action on 2-subsets of $\{1, 2, \dots, m\}$.*

Proof. 1) The inequality $e_2(X) \geq 2$, and hence $f_2(X) \geq 3$, is proved in Lemma 7.5.

If $\mathbf{A}_m$ embeds into $\mathbf{A}_n$ via its action of k -subsets of Δ with $2 < k < m/2$, then $h(X) \leq 2$ by Lemma 7.2, and $e_3(X) \geq 4$ by Lemma 7.7. Thus $e_3(X) \geq h(X) + 2$ as stated.

2) From now on, we assume that $\mathbf{A}_m$ embeds in $\mathbf{A}_n$ via its action on partitions of Δ into b a -subsets, $a, b \geq 2$. By Lemma 7.2, either $h(X) \leq 2$, or $h(X) = 3$ and $b = 2(m/2)$.

Here we consider the case $a \geq 5$ and show that $e_3(X) \geq 5 \geq h(X) + 2$ in this case. Applying Lemma 7.10 with $s = 2$, we are reduced to prove that $\dim M_3^Y - \dim M_2^Y \geq 5$ for $b = 2$ and $Y = \mathbf{S}_m$. In this base case $b = 2$, each partition is a pair

$$[A] := \{A, \Delta \setminus A\}$$

with $|A| = a = |\Delta|/2$. Hence the Y -orbits on pairs of partitions $\{[A], [B]\}$ are labeled by $1 \leq j \leq t := \lfloor a/2 \rfloor$, where $|A \cap B| = a - j$. In particular, $\dim M_2^Y = t$.

So we need to produce at least $(t + 5)$ Y -orbits on triples of partitions $\{[A], [B], [C]\} \in \Omega_3$. As in the proof of Lemma 7.7, we will label A, B, C so that

$$(7.4) \quad |A \cap B| \geq |A \cap C| \geq |B \cap C| \geq a/2$$

and call $(|A \cap B|, |A \cap C|, |B \cap C|)$ the *mark* of the triple $\{A, B, C\}$. Certainly, triples with different marks belong to different $\mathbf{S}_m$ -orbits.

First let $|A \cap B| = a - 1$, so we may assume

$$A = \{1, 2, \dots, a - 1, a\}, \quad B = \{1, 2, \dots, a - 1, a + 1\}.$$

For $1 \leq j \leq t$, by choosing

$$C = \{1, 2, \dots, a - j, a + 2, a + 3, \dots, a + j + 1\}$$

we get a triple with the mark $(a - 1, a - j, a - j)$. In fact, for $j = 1$, we have two choices for C :

$$\{1, 2, \dots, a - 1, a + 2\}, \text{ and } \{1, 2, \dots, a - 2, a, a + 1\}$$

which lead to two different S_m -orbits with mark $(a - 1, a - 1, a - 1)$ for any $a \geq 4$ (since $|A \cap B \cap C| = a - 1$ for the first choice and $|A \cap B \cap C| = a - 2$ for the second choice). Similarly, for $1 \leq j \leq t - 1$, by choosing

$$C = \{1, 2, \dots, a - j - 1, a, a + 2, a + 3, \dots, a + j + 1\}$$

we get a triple with the mark $(a - 1, a - j, a - j - 1)$.

Suppose in addition that $a \geq 6$. Then we choose

$$A = \{1, 2, \dots, a - 2, a - 1, a\}, \quad B = \{1, 2, \dots, a - 2, a + 1, a + 2\}.$$

Taking

$$C = \{1, 2, \dots, a - 2, a + 3, a + 4\} \text{ or } \{1, 2, \dots, a - 3, a + 3, a + 4, a + 5\}$$

we get triples with the mark $(a - 2, a - 2, a - 2)$ and $(a - 2, a - 3, a - 3)$. We have produced at least $(t + 1) + (t - 1) + 2 \geq t + 5$ orbits on triples, as desired.

Next assume that $a = 5$, and so $t = 2$. We have already produced 4 orbits with marks $(4, 4, 4)$ (two orbits), $(4, 4, 3)$, and $(4, 3, 3)$. We can also exhibit 3 more orbits with

$$\{A, B, C\} = \{12345, 12346, 12578\}, \quad \{12345, 12367, 12389\}, \quad \{12345, 12367, 12589\}$$

and mark $(4, 3, 2)$, $(3, 3, 3)$, and $(3, 3, 2)$, respectively. (Note that the last mark deviates from the convention (7.4), but it does not cause any problem since $a = 5$, as one can check.)

For the next part of the proof, we also consider the case $a = 4$, and so $s = 2$. We have already produced 4 orbits with marks $(3, 3, 3)$ (two orbits), $(3, 3, 2)$, and $(3, 2, 2)$. Next, the triples with

$$\{P, Q, R\} = \{1234, 1256, 3456\} \text{ and } \{1234, 1256, 1357\}$$

have the same mark $(2, 2, 2)$, but belong to different orbits. (Indeed, the former satisfies the identity $R = P + Q := (P \setminus Q) \cup (Q \setminus P)$, but for the latter no member of $[R]$ can be the sum of a member of $[P]$ with a member of $[Q]$.) Thus we get at least 6 orbits on triples of partitions. (One can show by [GAP] that the number of orbits on triples is indeed 6 for both A_m and S_m .)

3) It remains to consider the case $2 \leq a \leq 4$. Since $m \geq 11$, we must have that $b \geq 3$, and so $h \leq 2$ by Lemma 7.2. For $a = 2, 3$, or 4 , we set $s = 5, 3$, or 2 , respectively. Applying Lemma 7.10, we are reduced to prove that $e' := \dim M_3^Y - \dim M_2^Y \geq 4$ for $b = s$ and $Y = S_{sa}$. This has been done in 2) for $(a, s) = (4, 2)$. Using (6.3) and [GAP], one can check that $e' = 35$ for $(a, s) = (3, 3)$. Finally, assume that $(a, s) = (2, 5)$. Using

(6.3) and [GAP], we see that the number of A_{10} -orbits on 2-subsets, respectively on 3-subsets of $\Omega = \{1, 2, \dots, 945\}$ is 6, respectively 139. Since A_{10} has index 2 in $Y = S_{10}$, it follows that $\dim M_2^Y \leq 6$ and $\dim M_3^Y \geq 139/2$, whence $e' \geq 64$. $\square$

Now we can prove the main result concerning the special embeddings of A_m into A_n :

Theorem 7.12. *Let $X = A_m$ be embedded in A_n via its actions on partitions or on k -subsets of $\{1, 2, \dots, m\}$ with $2 \leq k < m/2$, and $11 \leq m < n$. Let $p = 2$ and let V be any $\mathbb{F}A_n$ -module of dimension greater than 1. Then $\text{Res}_X^{A_n} V$ is reducible.*

Proof. Assume for a contradiction that V is irreducible over X . By Lemma 7.1, $n \geq m(m-1)/2$. Since $m \geq 11$, we have

$$|X| = \frac{m!}{2} < 2^{\frac{m(m-1)-12}{4}} \leq (2^{\frac{n-6}{4}})^2.$$

The irreducibility of V on X forces that $\dim V < \sqrt{|X|} < 2^{(n-6)/4}$. This bound implies by Proposition 4.1 that V extends to S_n . Thus, without loss we may assume that V is an irreducible $\mathbb{F}S_n$ -module. Also, since $2^{(n-6)/4} < 2^{\lfloor (n-2)/2 \rfloor}$, V cannot be a basic spin module.

Now if n is odd, then the X -module V is reducible by [KS1, Theorem 3.10]. Hence $2|n$, and V satisfies the conclusion (i) of Theorem 3.2. By Corollary 7.4, X does *not* embed into A_n via its action on 2-subsets of $\{1, 2, \dots, m\}$. Hence, by Theorem 7.11, $f_2(X) \geq 3$ and $e_3(X) \geq h(X) + 2$. Clearly, X is a perfect subgroup and $f_1(X) = 1$. Therefore, the X -module V is reducible by Theorem 6.5, a contradiction. $\square$

8. GENERAL EMBEDDINGS OF A_m INTO A_n

Throughout this section, $X \cong A_m$ is a subgroup of A_n , with $n > m \geq 5$. Recall the notation $\Omega = \{1, 2, \dots, n\}$ and $\Delta = \{1, 2, \dots, m\}$.

First, we deal with the small cases $5 \leq m \leq 10$:

Lemma 8.1. *Let $X \cong A_m$ be a transitive subgroup of A_n with $n > m$ and $5 \leq m \leq 10$, V be an $\mathbb{F}A_n$ -module of dimension > 1 such that $\text{Res}_X^{A_n} V$ is irreducible. Then $(m, n) = (5, 6), (6, 10), (7, 15)$, or $(8, 15)$.*

Proof. 1) First we consider the case where V is not the heart of the natural permutation module of $Y := A_n$. Suppose for instance that $m = 10$. The assumptions that X is a transitive subgroup of Y and $n > m$ imply by [Atl] that $n \geq 45$. This in turn implies by [GT1, Lemma 6.1] that $\dim V \geq (n^2 - 5n + 2)/2 = 901 > 567 \geq b_{\mathbb{F}}(X)$, where $b_{\mathbb{F}}(X)$ denotes the largest degree of irreducible $\mathbb{F}X$ -representations. It follows that $\text{Res}_X^{A_n} V$ is reducible. The same arguments apply to the case $m = 9$, where we have $n \geq 36$ and $b_{\mathbb{F}}(X) \leq 216$.

Suppose now that $m = 8$. According to [Atl] and [JLPW], $n = 15$ or $n \geq 28$; furthermore, $b_{\mathbb{F}}(X) \leq 70$. As above, $\dim V > b_{\mathbb{F}}(X)$ if $n \geq 28$. It follows that $n = 15$. The same arguments apply to the case $5 \leq m \leq 7$. (In fact, one can show that we must have either $(m, n) = (5, 6)$ or $(m, n, p, \dim V) = (8, 15, 2, 64)$.)

2) Now we may assume that V is the heart of the natural permutation module $\mathbb{F}\Omega$ of Y . By the assumption, X acts transitively on Ω . If this action is not doubly transitive,

then the restriction of the permutation character of $\mathbb{C}\Omega$ is $1_X + \alpha + \beta$ where α, β are (not necessarily irreducible) X -characters of degree > 1 . Since the Brauer character of $\mathbb{F}\Omega$ is $\varphi + e \cdot 1_Y$ with $e \in \{1, 2\}$ and $\varphi \in \text{IBr}_p(Y)$, it follows that V is reducible over X . Thus X acts doubly transitively on Ω . Now we can read off the possible value of n using [Atl]. $\square$

Remark 8.2. The special cases (m, n) listed in the statement of Lemma 8.1 are indeed exceptional. For these pairs, we can embed A_m into A_n so that A_m acts doubly transitively on Ω . Take V to be the heart of the permutation module $\mathbb{C}\Omega$. It is well known (and follows for instance from the Mackey's formula) that any doubly transitive group is irreducible on V . This is also almost always true for $p > 0$ — see [Mo].

We will need the following group-theoretic result:

Lemma 8.3. *Let $X \cong A_m$ be a transitive subgroup of A_n with $n > m \geq 11$. Then $n \geq m(m-1)/2$ and X cannot act doubly transitively on Ω . If X is imprimitive, then $n \geq m(m-1)$. If X is primitive, then one of the following statements holds:*

- (i) X acts on Ω via its action on k -subsets of Δ with $2 \leq k < m/2$ or on partitions of Δ , and $n \geq m(m-1)/2$.
- (ii) $n > |X|^{3/10} > m(m-1)$.

Proof. From the classification of doubly transitive permutation representations of A_m [Ma] and the assumption $m \geq 11$, it follows that A_m has no doubly transitive permutation representation of degree $n > m$. Next, consider the transitive action of X on Ω and let X_1 be a point stabilizer of this action. Then $[X : X_1] = n > m$. We can find a maximal subgroup M of X containing X_1 .

Now we consider the action of M on Δ . If this action is intransitive, then by maximality of M , M is the stabilizer in X of a k -subset of Δ for some $1 \leq k < m/2$. On the other hand, if this action is transitive but imprimitive, then M is the stabilizer in X of a partition of Δ into b a -subsets, with $a, b \geq 2$ and $ab = m$. By Lemma 7.1, in either case we have $n \geq [X : M] \geq m(m-1)/2$ (whence $[X : X_1] \geq m(m-1)$ if X is imprimitive), unless M is intransitive and $k = 1$. In this exceptional case, $M = A_{m-1}$ and $[X : X_1] = n > m = [X : M]$. Thus X_1 is a proper subgroup of M , and so, as is well known (see e.g. [KIL, p. 175]), $[M : X_1] \geq m-1$ and $n \geq m(m-1)$.

Suppose now that M acts primitively on Δ . By Bochert's Theorem 14.2 of [Wie], $[S_m : M] \geq \lfloor (m+1)/2 \rfloor!$. Since $m \geq 11$, we have $\frac{\lfloor (m+1)/2 \rfloor!}{2} > \left(\frac{m!}{2}\right)^{3/10}$. Therefore $n \geq [X : M] > |X|^{3/10} > m(m-1)$ as $m \geq 11$.

We have proved the bound $n \geq m(m-1)/2$ and also that $n \geq m(m-1)$ if X is imprimitive. Assume now that X is primitive. Then $X_1 = M$, and the above analysis yields the final claim of the lemma. $\square$

Theorem 8.4. *Let $X \cong A_m$ be a primitive subgroup of A_n with $n > m \geq 9$. Let $p = 2$ and let V be an $\mathbb{F}A_n$ -module of dimension > 1 . Then the restriction $\text{Res}_X^{A_n} V$ is reducible.*

Proof. By Lemma 8.1, we may assume that $m \geq 11$. Assume for a contradiction that V is irreducible over X . We apply Lemma 8.3. In the case (i) of Lemma 8.3 we are done by Theorem 7.12.

Now suppose that the case (ii) of Lemma 8.3 holds. Then $|X| < n^{10/3}$ and $n \geq 155$ as $m \geq 11$. Since V is irreducible over X , we must have that

$$\dim V < n^{5/3} < \frac{1}{2}(n-1)(n-2).$$

In particular, $\dim V < 2^{(n-6)/4}$ (as $n \geq 155$), whence V extends to S_n by Proposition 4.1. We denote the (unique up to isomorphism) extension of V to S_n by the same letter V .

By Lemma 8.3, the action of $X = A_m$ on Ω is not doubly transitive. Hence, if n is odd then, we are done by appealing to [KS1, Theorem 3.10]. Assume that $2|n \geq 155$. Then the condition $1 < \dim V < (n-1)(n-2)/2$ for the irreducible $\mathbb{F}S_n$ -module V implies by [J3, Theorem 7] that $V \cong D^{(n-k,k)}$ with $k = 1$ or 2 . As before, let $1 + \alpha$ denote the (complex) permutation character of S_n on Ω .

Suppose $k = 1$. Then $V = D^{(n-1,1)}$ is irreducible over X . Denoting by α^0 the restriction of α to $2'$ -elements in S_n , we see that the Brauer character of V is $\alpha^0 - 1$. Furthermore, X is perfect and $[\text{Res}_X^{S_n} \alpha, 1_X]_X = 0$ by transitivity of X on Ω . Hence the irreducibility of V over X forces that α is also irreducible over X . In other words, X acts doubly transitively on Ω , a contradiction.

Assume now that $k = 2$. Then $\wedge^2(D^{(n-1,1)})$ has a composition series with composition factors V (once), and $\mathbf{1}$ (once if $4|n$ and twice if $n \equiv 2 \pmod{4}$). The same is true for $\wedge^2(D^{(n-1,1)})$ considered as an X -module. Suppose in addition that $D^{(n-1,1)}$ is reducible over X . Then we can write the Brauer character of the $\mathbb{F}X$ -module $\text{Res}_X^{S_n} D^{(n-1,1)}$ as $\beta + \gamma$, where β and γ are Brauer characters of X , $\beta(1) \geq 1$ and $\gamma(1) \geq 3$. Note that

$$\wedge^2(\beta + \gamma) = \wedge^2(\beta) + \wedge^2(\gamma) + \beta\gamma.$$

It follows that the Brauer character of the X -module $\wedge^2(D^{(n-1,1)})$ is the sum of three Brauer characters, with at least two of degree ≥ 3 . This contradicts the aforementioned composition structure of the X -module $\wedge^2(D^{(n-1,1)})$. Thus $D^{(n-1,1)}$ is irreducible over X . But then, arguing as in the previous paragraph, we again arrive at the contradiction that X acts doubly transitively on Ω . $\square$

The case $p \neq 2$ can be done using results of [KS1], [KS2] and Proposition 4.3:

Proposition 8.5. *Let $X \cong A_m$ be a transitive subgroup of A_n with $n > m \geq 9$. Let $p \neq 2$ and V be an $\mathbb{F}A_n$ -module of dimension > 1 . Then $\text{Res}_X^{A_n} V$ is reducible.*

Proof. By Lemma 8.1, we may assume that $m \geq 11$. Assume for a contradiction that $\text{Res}_X^{A_n} V$ is irreducible. By Lemma 8.3, X is not doubly transitive, and $n \geq m(m-1)/2$. If V is extendible to S_n , we can apply [KS1, Main Theorem] to get a contradiction. Suppose V is not extendible to S_n . If $p \neq 3$, we again arrive at a contradiction by using [KS2, Main Theorem]. So we may assume that $p = 3$. By Proposition 4.3(ii) we have that

$$\dim V \geq 2^{\frac{n-8}{4}} \geq 2^{\frac{m^2-m-16}{8}} =: c_m.$$

Now, if $m \geq 12$ then $c_m > \sqrt{m!/2} = \sqrt{|X|}$. If $m = 11$, then $c_m > 3444$ whereas the largest degree of complex irreducible representations of $X = A_{11}$ is 2310. In either case, X cannot be irreducible on V . $\square$

Now Theorem 1.2 follows immediately from Theorem 8.4 and Proposition 8.5.

Note that for $p \neq 2$, Proposition 8.5 shows that any proper transitive subgroup $X \cong A_m$ of A_n acts reducibly on all non-trivial modules over $\mathbb{F}A_n$ (provided $m \geq 9$). For $p = 2$, if X is primitive, the same result holds by Theorem 8.4. Now we handle imprimitive embeddings of A_m into A_n for $p = 2$. Here our result is a little weaker:

Proposition 8.6. *Let $X \cong A_m$ be a (transitive) imprimitive subgroup of A_n with $n > m \geq 9$. Let $p = 2$ and let V be an irreducible $\mathbb{F}A_n$ -module of dimension > 1 . Then the X -module $\text{Res}_X^{A_n} V$ cannot be primitive irreducible.*

Proof. By Lemma 8.1, we may assume that $m \geq 11$. Assume for a contradiction V is irreducible and primitive over X . Consider the action of X as a subgroup of A_n on Ω and let X_1 be a point stabilizer. Since X is imprimitive, there is a maximal subgroup $M > X_1$ of X . Now $b := [X : M] \geq m \geq 11$, and we may assume that $X < Y := (S_a \wr S_b) \cap A_n$ for $a := n/b > 1$. By Lemma 8.3, $n \geq m(m-1)$. If $a = 2$ or 4 , then $O_2(Y) > 1$, and so V cannot be irreducible over $Y > X$, a contradiction. So we have $a \geq 3$ and $a \neq 4$.

Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be a 2-regular partition of n such that V is a simple submodule of $\text{Res}_{A_n}^{S_n} D^\lambda$. Observe that $m! < (m/2)^m$ for $m \geq 6$. Since $\text{Res}_X^{A_n} V$ is irreducible, by [GLT, Theorem 5.1] we have

$$2^{\frac{n-\lambda_1}{2}-1} \leq \dim V \leq \sqrt{|A_m|} < \sqrt{\frac{m^m}{2^{m+1}}},$$

whence $n - \lambda_1 < m \log_2 m - m + 1$. This in turn implies that $2\lambda_1 - n \geq n/m$. Indeed, otherwise we would have $\lambda_1 < n/2 + n/2m$ and so

$$n - \lambda_1 > \frac{n}{2} - \frac{n}{2m} = n \cdot \frac{m-1}{2m} \geq \frac{(m-1)^2}{2} > m \log_2 m - m + 1,$$

a contradiction.

We have shown that $2\lambda_1 - n \geq n/m \geq n/b = a \geq 3$. Also, $n - \lambda_1 \geq 1$ since $\dim V > 1$. Applying Proposition 4.2, we see that the restriction of D^λ to a natural subgroup S_a of S_n affords both $\mathbf{1}$ and $D^{(a-1,1)}$ as composition factors. If $a \geq 5$, then $\dim D^{(a-1,1)} \geq 4$ and so any irreducible summand of the A_a -module $D^{(a-1,1)}$ has dimension ≥ 2 (in fact $D^{(a-1,1)}$ is irreducible over A_a). On the other hand, if $a = 3$, then $D^{(a-1,1)}$ splits into a direct sum of two non-trivial irreducible A_a -submodules. Thus $\text{Res}_{A_a}^{A_n} V$ affords both the trivial module and also another non-trivial irreducible module as composition factors. It follows that $\text{Res}_Z^{A_n} V$ cannot be homogeneous for $Z := A_a \times \dots \times A_a = (A_a)^b \triangleleft Y$. But this is a contradiction, as $Z \triangleleft Y$ and the Y -module V is primitive. $\square$

The following lemma deals with tensor indecomposable irreducible representations of A_m . These were studied extensively in [Z, BeK1, BeK2, GoK, GrJ, BeK3].

Lemma 8.7. *Let $X \cong A_m$ be a subgroup of A_n with $n \geq m \geq 5$. Let V be an irreducible $\mathbb{F}A_n$ -module of dimension > 1 . Assume that the X -module $\text{Res}_X^{A_n} V$ is irreducible and tensor indecomposable. Then one of the following holds:*

- (i) X is a transitive subgroup of A_n .

- (ii) *There is some $t \in \{1, 2\}$ such that X fixes t points and acts transitively on $n - t$ remaining points of Ω . Furthermore, V is irreducible over a natural subgroup A_{n-t} of A_n .*

Proof. Suppose that X acts intransitively on Ω . Each non-trivial orbit of X has at least $m \geq 5$ points. If X has at least two non-trivial orbits on X , then we may assume that there is some $5 \leq k \leq n/2$ such that X acts non-trivially on both $\Omega^{(1)} := \{1, 2, \dots, k\}$ and $\Omega^{(2)} := \{k+1, \dots, n\}$. These actions induce embeddings $\pi_i : X \rightarrow \text{Alt}(\Omega^{(i)})$, namely, $g \in X$ acts on $\Omega^{(i)}$ as $\pi_i(g)$. Now, $X < Y := A_k \times A_{n-k}$ and $\text{Res}_Y^{A_n} V$ is irreducible. Hence $V|_Y \cong V_1 \boxtimes V_2$ is an outer tensor product of irreducible modules V_1 over A_k and V_2 over A_{n-k} . Note that $\dim V_1 > 1$, as otherwise the 3-cycles in A_k would act trivially on V . Similarly $\dim V_2 > 1$. It follows that $\text{Res}_X^{A_n} V = U_1 \otimes U_2$, where $U_i \cong \text{Res}_{\pi_i(X)}^{A_{k_i}} V_i$ for $i = 1, 2$ and $(k_1, k_2) := (k, n - k)$. In particular, $\text{Res}_X^{A_n} V$ is tensor decomposable, contrary to the assumption.

Thus X has only one non-trivial orbit on Ω , say of length $n - t$ for some $t > 0$, and X fixes the remaining t points. Furthermore, $t \leq 2$, as otherwise X is centralized by a 3-cycle and so cannot act irreducibly on V . $\square$

Finally, in connection with Lemma 8.7, we bound the number of extensions of irreducible representations of A_{n-1} to A_n . For this type of question, it is convenient to use the following simple observation:

Lemma 8.8. *Let Y be a subgroup of X and U be a finite-dimensional vector space over $\mathbb{F}$. Let $\Psi : Y \rightarrow GL(U)$ be an irreducible representation. Suppose that $\Phi_i : X \rightarrow GL(U)$, $i = 1, 2$, are two isomorphic representations of X with $\text{Res}_Y^X(\Phi_i) = \Psi$. Then in fact $\Phi_1 = \Phi_2$.*

Proof. Fix a basis of the vector space U and write Ψ and Φ_i as matrix representations with respect to this basis. By the assumption, there is some invertible matrix A such that $\Phi_2(x) = A\Phi_1(x)A^{-1}$ for all $x \in X$. Since both Φ_i extend Ψ , we have that $\Psi(y) = A\Psi(y)A^{-1}$ for all $y \in Y$. By Schur's lemma, A is scalar, and so $\Phi_2 = \Phi_1$. $\square$

Now we deal with the aforementioned question for symmetric groups:

Lemma 8.9. *Assume $n \geq 3$ and fix a natural embedding of S_{n-1} into S_n . Then every irreducible $\mathbb{F}S_{n-1}$ -representation $S_{n-1} \rightarrow GL(U)$ has at most one extension to S_n .*

Proof. Consider U as an $\mathbb{F}S_{n-1}$ -module, and suppose that there are two distinct p -regular partitions $\lambda, \mu \vdash n$ such that

$$\text{Res}_{S_{n-1}}^{S_n}(D^\lambda) \cong \text{Res}_{S_{n-1}}^{S_n}(D^\mu) \cong U.$$

Since $\text{soc}(\text{Res}_{S_{n-1}}^{S_n}(D^\lambda)) \cong \text{soc}(\text{Res}_{S_{n-1}}^{S_n}(D^\mu))$, by [K3, Corollary 5.3], precisely one of the modules $\text{Res}_{S_{n-1}}^{S_n}(D^\lambda)$, $\text{Res}_{S_{n-1}}^{S_n}(D^\mu)$ is reducible, a contradiction. Now apply Lemma 8.8. $\square$

Proposition 8.10. *Assume $n \geq 5$ and fix a natural embedding of A_{n-1} into A_n . Then every irreducible $\mathbb{F}A_{n-1}$ -representation $A_{n-1} \rightarrow GL(U)$ has at most three distinct extensions to A_n .*

Proof. We assume that U extends to an $\mathbb{F}A_n$ -module V_1 and find all possible other extensions V_2 of U to A_n . Consider A_{n-1} as the derived subgroup of a natural subgroup S_{n-1} of S_n . By Lemma 8.8, it suffices to bound the number of extensions up to isomorphism. For each $i = 1, 2$, we can find an irreducible $\mathbb{F}S_n$ -module W_i such that $V_i \hookrightarrow \text{soc}(\text{Res}_{A_n}^{S_n}(W_i))$, and an irreducible $\mathbb{F}S_{n-1}$ -module T such that $U \hookrightarrow \text{soc}(\text{Res}_{A_{n-1}}^{S_{n-1}}(T))$. We also fix a transposition $g \in S_{n-1} \setminus A_{n-1}$, and distinguish the following cases.

Case I: U is not S_{n-1} -invariant. Then for $i = 1, 2$ we have that

$$\text{Res}_{A_{n-1}}^{A_n}(V_i^g) = U^g \not\cong U = \text{Res}_{A_{n-1}}^{A_n}(V_i),$$

and so V_i is not S_n -invariant. Hence $\text{Res}_{A_n}^{S_n}(W_i) \cong V_i \oplus V_i^g$, and similarly $\text{Res}_{A_{n-1}}^{S_{n-1}}(T) \cong U \oplus U^g$. Now $\text{Res}_{A_{n-1}}^{S_n}(W_i) \cong U \oplus U^g \cong \text{Res}_{A_{n-1}}^{S_{n-1}}(T)$. It follows that $\text{Res}_{S_{n-1}}^{S_n}(W_1) \cong T \cong \text{Res}_{S_{n-1}}^{S_n}(W_2)$. By Lemma 8.9, $W_1 \cong W_2$, whence V_2 is isomorphic to V_1 or V_1^g . Since $\text{Res}_{A_{n-1}}^{A_n}(V_1^g) = U^g \not\cong U$, we must have that $V_2 \cong V_1$.

Case IIa: U is S_{n-1} -invariant and both V_1, V_2 are S_n -invariant. In particular, we have $\text{Res}_{A_n}^{S_n}(W_i) = V_i$ for $i = 1, 2$ and similarly $\text{Res}_{A_{n-1}}^{S_{n-1}}(T) = U$. Now if $p = 2$, then the Brauer character of any extension of U to S_{n-1} is uniquely determined by its restriction to A_{n-1} which is the Brauer character of U . Thus U has a unique extension to S_{n-1} , and so $\text{Res}_{S_{n-1}}^{S_n}(W_1) \cong \text{Res}_{S_{n-1}}^{S_n}(W_2)$. It follows by Lemma 8.9 that $W_1 \cong W_2$ and so $V_1 \cong V_2$.

Next suppose that $p \neq 2$. Then U has two distinct extensions T and T^g to S_{n-1} and V_i has two distinct extensions W_i and W_i^g to S_n . It follows that $\text{Res}_{S_{n-1}}^{S_n}(W_i) \in \{T, T^g\}$ and so $\text{Res}_{S_{n-1}}^{S_n}(W_2) \in \{\text{Res}_{S_{n-1}}^{S_n}(W_1), \text{Res}_{S_{n-1}}^{S_n}(W_1^g)\}$. By Lemma 8.9, $W_2 \cong W_1$ or W_1^g , and so $V_2 \cong V_1$.

Thus Case IIa shows that among the extensions of U to A_n , at most one of them is S_n -invariant.

Case IIb: U is S_{n-1} -invariant, but neither V_1 nor V_2 is S_n -invariant. In this case for $i = 1, 2$ we have that $W_i = \text{Ind}_{A_n}^{S_n}(V_i)$, and so

$$\text{Res}_{S_{n-1}}^{S_n}(W_i) \cong \text{Ind}_{A_{n-1}}^{S_{n-1}}(\text{Res}_{A_{n-1}}^{A_n}(V_i)) \cong \text{Ind}_{A_{n-1}}^{S_{n-1}}(U)$$

is reducible. Also, $\text{soc}(\text{Res}_{S_{n-1}}^{S_n}(W_1)) \cong \text{soc}(\text{Res}_{S_{n-1}}^{S_n}(W_2))$. If moreover $W_1 \not\cong W_2$, then the latter implies by [K3, Corollary 5.3] that (precisely) one of the modules $\text{Res}_{S_{n-1}}^{S_n}(W_i)$, $i = 1, 2$, is irreducible, a contradiction. It follows that $W_1 \cong W_2$ and so $V_2 \in \{V_1, V_1^g\}$ since $\text{Res}_{A_n}^{S_n}(W_1) = V_1 \oplus V_1^g$.

Thus Case IIb shows that among the extensions of U to A_n , at most two of them can be non- S_n -invariant. Hence in the case where U is S_{n-1} -invariant, there are at most three extensions to A_n . $\square$

9. RANK 3 PERMUTATION GROUPS

To illustrate applicability of Theorem 6.5 to other primitive subgroups of S_n , in this section we consider finite simple classical groups X acting as rank 3 permutation groups on Ω , where (Ω, X) is one of the following:

- (i) Ω is the set of 2-dimensional subspaces of $W = \mathbb{F}_q^d$ and $X = PSL(W) = PSL_d(q)$ with $d \geq 4$;
- (ii) Ω is the set of singular 1-dimensional subspaces of $W = \mathbb{F}_q^d$ and $X = PSp(W)' = PSp_d(q)'$ with $2|d \geq 4$ (note that $Sp_d(q)' \neq Sp_d(q)$ only when $(d, q) = (4, 2)$);
- (iii) Ω is the set of singular 1-dimensional subspaces of $W = \mathbb{F}_q^d$ and $X = P\Omega(W) = P\Omega_d^\pm(q)$ with $d \geq 5$;
- (iv) Ω is the set of singular 1-dimensional subspaces of $W = \mathbb{F}_{q^2}^d$ and $X = PSU(W) = PSU_d(q)$ with $d \geq 4$.

According to the main result of [KaL], these families account for all the standard rank 3 permutation representations of finite simple classical groups.

Lemma 9.1. *Let $X < \text{Sym}(\Omega) = S_n$, where (Ω, X) is as listed above, and $2|n$.*

- (i) *Assume furthermore that $d \geq 6$ when $X = PSp_d(q)$ or $X = PSU_d(q)$, and $d \geq 7$ when $X = P\Omega_d^\pm(q)$. Then $f_3(X) \geq 6$.*
- (ii) *Also, $f_3(X) \geq 5$ if $X = PSp_4(q)$, $PSU_4(q)$, $PSU_5(q)$, $\Omega_5(q)$, or $P\Omega_6^\pm(q)$.*

Proof. (a) First we consider the case $X = PSL(W)$ with $W = \langle e_1, \dots, e_d \rangle_{\mathbb{F}_q}$ of dimension $d \geq 4$. Then X has at least six orbits on Ω_3 , with the following representatives:

- $A = \langle e_1, e_2 \rangle_{\mathbb{F}_q}$, $B = \langle e_1, e_3 \rangle_{\mathbb{F}_q}$, $C = \langle e_2, e_3 \rangle_{\mathbb{F}_q}$ (note that $\dim_{\mathbb{F}_q}(A + B + C) = 3$ and $A \cap B \cap C = 0$ here);
- $A = \langle e_1, e_2 \rangle_{\mathbb{F}_q}$, $B = \langle e_1, e_3 \rangle_{\mathbb{F}_q}$, $C = \langle e_1, e_2 + e_3 \rangle_{\mathbb{F}_q}$ (here, $\dim_{\mathbb{F}_q}(A + B + C) = 3$ and $A \cap B \cap C \neq 0$);
- $A = \langle e_1, e_2 \rangle_{\mathbb{F}_q}$, $B = \langle e_3, e_4 \rangle_{\mathbb{F}_q}$, $C = \langle e_1 + e_3, e_2 + e_4 \rangle_{\mathbb{F}_q}$ (here, $\dim_{\mathbb{F}_q}(A + B + C) = 4$ and $\{\dim_{\mathbb{F}_q}(A \cap B), \dim_{\mathbb{F}_q}(A \cap C), \dim_{\mathbb{F}_q}(B \cap C)\} = \{0, 0, 0\}$);
- $A = \langle e_1, e_2 \rangle_{\mathbb{F}_q}$, $B = \langle e_1, e_3 \rangle_{\mathbb{F}_q}$, $C = \langle e_2 + e_3, e_4 \rangle_{\mathbb{F}_q}$ (here, $\dim_{\mathbb{F}_q}(A + B + C) = 4$ and $\{\dim_{\mathbb{F}_q}(A \cap B), \dim_{\mathbb{F}_q}(A \cap C), \dim_{\mathbb{F}_q}(B \cap C)\} = \{1, 0, 0\}$);
- $A = \langle e_1, e_2 \rangle_{\mathbb{F}_q}$, $B = \langle e_1, e_3 \rangle_{\mathbb{F}_q}$, $C = \langle e_2, e_4 \rangle_{\mathbb{F}_q}$ (here, $\dim_{\mathbb{F}_q}(A + B + C) = 4$ and $\{\dim_{\mathbb{F}_q}(A \cap B), \dim_{\mathbb{F}_q}(A \cap C), \dim_{\mathbb{F}_q}(B \cap C)\} = \{1, 1, 0\}$);
- $A = \langle e_1, e_2 \rangle_{\mathbb{F}_q}$, $B = \langle e_1, e_3 \rangle_{\mathbb{F}_q}$, $C = \langle e_1, e_4 \rangle_{\mathbb{F}_q}$ (here, $\dim_{\mathbb{F}_q}(A + B + C) = 4$ and $\dim_{\mathbb{F}_q}(A \cap B \cap C) = 1$).

(b) In the remaining cases of (i), the assumption on d implies that W contains a non-degenerate 6-dimensional subspace with hyperbolic (Witt) basis $(e_1, e_2, e_3, f_1, f_2, f_3)$. Also, since $n = |\Omega|$ is even, q is odd. We will write any element in Ω as $\langle a \rangle$, the 1-space generated by a vector $a \in W$. In the Ω -case, for any unordered triple $\pi = \{A, B, C\}$, with $A = \langle a \rangle$, $B = \langle b \rangle$, $C = \langle c \rangle \in \Omega$ and $\dim(A + B + C) = 3$, we can associate to it the Gram matrix Γ of the bilinear form $(\cdot, \cdot)$ written in the basis (a, b, c) . Since changing to another basis of $A + B + C$ changes $\det(\Gamma)$ by a factor which belongs to the subgroup $F_0 := \mathbb{F}_q^{\times 2}$ of $F := \mathbb{F}_q^\times$, we can associate to such π a canonical element $\delta := \det(\Gamma)F_0 \in F/F_0$. Then the following unordered triples $\{A, B, C\} \in \Omega_3$ belong to disjoint X -orbits:

- $A = \langle e_1 \rangle$, $B = \langle e_2 \rangle$, $C = \langle e_1 + e_2 \rangle$ (note that $A + B + C$ is a 2-dimensional totally singular subspace here);
- In the Sp/SU -case: $A = \langle e_1 \rangle$, $B = \langle f_1 \rangle$, $C = \langle e_1 + \lambda f_1 \rangle$, where $\lambda = 1$ in the Sp -case and $\lambda^{q-1} = -1$ in the SU -case. Note that $A + B + C$ is a 2-dimensional non-degenerate subspace here;

- $A = \langle e_1 \rangle$, $B = \langle e_2 \rangle$, $C = \langle e_3 \rangle$ (here, $A + B + C$ is a 3-dimensional totally singular subspace);

- $A = \langle e_1 \rangle$, $B = \langle f_1 \rangle$, $C = \langle e_2 \rangle$. Note that $U := A + B + C$ is a 3-dimensional subspace with $C = \text{rad}(U)$;

- $A = \langle e_1 \rangle$, $B = \langle f_1 \rangle$, $C = \langle e_1 + e_2 \rangle$. Note that $U := A + B + C$ is a 3-dimensional subspace with $\dim \text{rad}(U) = 1$ but $A, B, C \neq \text{rad}(U)$. In fact, in the Sp -case, we get one more triple with the same A, B , but with $C' = \langle e_1 + f_1 + e_2 \rangle$ – note that $A \perp C$, but no two of A, B, C' are orthogonal to each other;

- In the Ω/SU -case: $A = \langle e_1 \rangle$, $B = \langle f_1 \rangle$, $C = \langle e_2 + \lambda f_2 + e_1 - \lambda f_1 \rangle$, where $\lambda \in F$ in the Ω -case and $\lambda = 1$ in the SU -case. Here, $W := A + B + C$ is a 3-dimensional non-degenerate subspace. Furthermore, in the Ω -case, we have $\delta = -2\lambda F_0$, whence we can choose λ so that $\delta = F_0$, respectively $\delta \neq F_0$.

(c) Ignoring the vectors e_3, f_3 , the arguments in (b) also show that $f_2(X) \geq 5$ if $X = PSp_4(q)$, $PSU_4(q)$, $PSU_5(q)$, $\Omega_5(q)$, or $\Omega_6^\pm(q)$. $\square$

Lemma 9.2. *Let $X < \text{Sym}(\Omega) = S_n$, where (Ω, X) is as listed above, that is, $d \geq 4$ when $X = PSL_d(q)$, $PSU_d(q)$, or $PSp_d(q)$, and $d \geq 5$ when $X = P\Omega_d^\pm(q)$. Suppose that $p = 2|n$. Then $h(X) \leq 3$ if $X = P\Omega_d^+(q)$ with $4|d \geq 8$, and $h(X) \leq 2$ otherwise.*

Proof. As mentioned in the proof of Lemma 9.1, $2|n$ implies that q is odd. We follow the proof of Lemma 7.2 and its notation. First, $\dim H^2(X, \mathbb{F})$, which is the 2-rank of the Schur multiplier of X , is ≤ 2 if $X = P\Omega_d^+(q)$ with $4|d \geq 8$, and ≤ 1 otherwise by [KL, Theorem 5.1.4]. Hence it suffices to show that

$$(9.1) \quad \dim \text{Hom}(X_1, (\mathbb{F}, +)) \leq 1,$$

where X_1 is the point stabilizer in X of a point in Ω . Without loss we may replace X by its central cover $SL_d(q)$, $SU_d(q)$, $Sp_d(q)$, or $\Omega_d^\pm(q)$. We also fix a basis $(e_1, \dots, e_d)$ of the natural module W of X .

Consider the case $X = SL_d(q)$. Then $X_1 = \text{Stab}_X(\langle e_1, e_2 \rangle) = Q \rtimes Y$, where $|Q|$ is a q -power and $Y = (SL_2(q) \times SL_{d-2}(q)) \rtimes C_{q-1}$. Since q is odd, we have that $O^2(Q) = Q$ and $O^2(SL_e(q)) = SL_e(q)$ for any $e \geq 2$. Hence (9.1) follows.

From now on we may assume $X \neq SL_d(q)$. We can then choose e_1 to be singular and let $X_1 = \text{Stab}_X(\langle e_1 \rangle)$.

Let $X = SU_d(q)$. Then $X_1 = Q \rtimes Y$, where $|Q|$ is a q -power and $Y = SU_{d-2}(q) \rtimes C_{q^2-1}$. As $d \geq 4$ and q is odd, we see that $O^2(SU_{d-2}(q)) = SU_{d-2}(q)$, and so (9.1) follows.

Suppose $X = Sp_d(q)$. Then $X_1 = Q \rtimes Y$, where $|Q|$ is a q -power and $Y = Sp_{d-2}(q) \rtimes C_{q-1}$. As $d \geq 4$ and q is odd, we see that $O^2(Sp_{d-2}(q)) = Sp_{d-2}(q)$, yielding (9.1).

Suppose $X = \Omega_d^\epsilon(q)$. Then $X_1 = Q \rtimes Y$, where $|Q|$ is a q -power and $Y = \Omega_{d-2}^\epsilon(q) \rtimes C_{q-1}$. As $d \geq 5$ and q is odd, we see that $O^2(\Omega_{d-2}^\epsilon(q)) = \Omega_{d-2}^\epsilon(q)$, and so we are done. $\square$

Proof of Theorem 1.3. Assume the contrary: $\text{Res}_X^{\Lambda^n} V$ is irreducible; in particular, V is irreducible. Note that X is not 2-transitive on Ω . Hence, by the main results of [KS1, KS2] we must have that $p = 2$ or 3.

(i) In the case $X = PSp_4(2)' \cong A_6$, we have $n = 15$, and so $\dim V \geq 13$ (see [Dec]), whereas the largest dimension $\mathfrak{b}(X)$ of irreducible $\mathbb{F}X$ -modules is at most 10, cf. [JLPW].

Thus V is reducible over X . Next, in the cases $X = SU_4(2) \cong PSp_4(3)$, respectively $SL_4(2)$, $Sp_6(2)$, we have $n \geq 27, 35, 63$, and $\mathfrak{b}(X) \leq 81, 70, 512$, respectively, according to [JLPW]. Certainly, $\text{Res}_X^{\mathbb{A}_n} V$ is reducible if $\dim V > \mathfrak{b}(X)$. So we must have that $\dim V \leq \mathfrak{b}(X)$. Since $\mathfrak{b}(X) < (n^2 - 5n + 2)/2$, by [GT1, Lemma 6.1] we see that $V \cong D^{(n-1,1)}$ is isomorphic to the heart of the natural permutation module $\mathbb{F}\Omega$. As in the proof of Lemma 8.1, we conclude that X is 2-transitive on Ω , a contradiction.

(ii) We may now assume that X is not isomorphic to any of the groups considered in (i). Direct computation shows that $2^{(n-8)/4} > |X|^{1/2}$. Since $\text{Res}_X^{\mathbb{A}_n} V$ is irreducible, we must have that

$$(9.2) \quad \dim(V) < 2^{(n-8)/4},$$

which implies by Propositions 4.1 and 4.3 that V extends to S_n . Applying the Main Theorem and Theorem 3.10 of [KS1], we again arrive at the contradiction that X is 2-transitive in the case $p = 3$, as well as in the case $p = 2 \nmid n$.

Thus we have shown that $p = 2|n$. The upper bound (9.2) implies by Theorem 3.2 that $d_3(V) > d_1(V)$. Also, $f_1(X) = 1$, $f_2(X) = 2$ by Lemma 6.3, and $e_3(X) \geq h(X) + 1$ by Lemmas 9.1 and 9.2. Now we can apply Theorem 6.5. $\square$

Non-standard rank 3 permutation representations of finite classical groups, as well as other primitive subgroups of $\mathbb{A}_n$, will be considered elsewhere.

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