OPTICAL SCIENCES

THE UNIVERSITY OF ARIZONA
TUCSON, ARIZONA

TECHNICAL REPORT NO. 13

OPTICAL TRANSFER OF THE
THREE-DIMENSIONAL OBJECT*

B. ROY FRIEDEN

January, 1967

*This paper has been accepted for publication in
the Journal of the Optical Society of America.

OPTICAL SCIENCES

THE UNIVERSITY OF ARIZONA
TUCSON, ARIZONA

TECHNICAL REPORT NO. 13

OPTICAL TRANSFER OF THE

THREE-DIMENSIONAL OBJECT*

B. ROY FRIEDEN

January, 1967

*This paper has been accepted for publication in
the Journal of the Optical Society of America.

ABSTRACT

This paper presents a transfer theory for determination of the image space and image spectrum of a three-dimensional object. The theory assumes the existence of volumes of stationarity, called "isotomes," into which the object must be divided. Isotomicity is approximated, over sufficiently small volumes, in the diffraction-limited case.

The main development assumes the object to be radiating incoherently; the results are as follows:

The image (irradiance) distribution $i(x,y,z)$ is the three-dimensional convolution of the point spread function $s(\alpha,\beta,\gamma)$ with the object distribution $o(x',y',z')$. The image spectrum $I(\omega_1,\omega_2,\omega_3)$ is defined as the three-dimensional Fourier transform of $i(x,y,z)$. It is found that I obeys a transfer theorem, $I = F \cdot O$, where $F(\omega_1,\omega_2,\omega_3)$ is the three-dimensional Fourier transform of $s(\alpha,\beta,\gamma)$ and $O(\omega_1,\omega_2,\omega_3)$ is an integral transform of $o(x',y',z')$. This transfer theorem establishes the value of using F as a criterion of optical design. In the Fraunhofer approximation, F may be represented as a line integral across the pupil U . This shows that F contains a simple pole at $\omega_1 = \omega_2 = 0$. Nevertheless, all integrals involving F are convergent. The pupil representation for F also shows that F is zero outside a restricted volume Σ of $(\omega_1,\omega_2,\omega_3)$ -space. Because F is bandwidth-limited, F , I , s , i and τ (the optical transfer function) individually obey sampling theorems. These theorems imply that if each point in image space is regarded as an independent degree of freedom, there can be no more than $1/\lambda^3 f_{\#}^4$ degrees of freedom/volume in image space.

For coherent object radiation, analogous theorems of convolution, transfer, and sampling can be constructed. In addition, the "amplitude"

transfer function $W(\omega_1, \omega_2, \omega_3)$, defined as the Fourier transform of the point amplitude distribution $u(\alpha, \beta, \gamma)$, is proportional to the pupil function and to $\delta[\omega_3 - (\omega_1^2 + \omega_2^2)/2k]$, where δ is the Dirac delta function and $k = 2\pi/\lambda$. This relation is used to establish sampling theorems for u and for g (the image amplitude) and to express $g(x, y, z)$ as a double integral over U and O .

TABLE OF CONTENTS

	<u>Page</u>
I. INTRODUCTION	1
II. POSTULATES	4
A. Stationarity	4
B. Superposition	5
III. CONVOLUTION THEOREMS	7
IV. IMAGE SPECTRUM	8
V. TRANSFER THEOREM	10
VI. TRANSFER FUNCTION: $F(\vec{\Omega})$	12
VII. DEPENDENCE OF $F(\vec{\Omega})$ ON PUPIL FUNCTION	14
A. Derivation	14
B. Experimental Determination of $F(\vec{\Omega})$ by Use of a Shearing Interferometer	17
C. Numerical Calculation of $F(\vec{\Omega})$	17
D. Normalization of $F(\vec{\Omega})$	18
E. Diffraction-Limited Case	19
VIII. BANDPASS REGION FOR F AND $\bar{F}$	21
IX. SAMPLING THEOREMS.	23
A. Derivations	23
B. Experimental Determination of $F(\vec{\Omega})$ by Use of Sampling Theorems	24
X. SUMMARY AND DISCUSSION	25
XI. ACKNOWLEDGMENTS	29
XII. REFERENCES	30
* * *	
Appendix I. ISOTOMICITY IN THE DIFFRACTION-LIMITED CASE	31
Appendix II. CONVERGENCE OF THE INVERSE FOURIER TRANSFORMS	33
A. Transform (6.4)	33
B. Transform (6.2b)	34
C. Transform (4.5a)	34

I. INTRODUCTION

A real object, such as a ball, has three-dimensional extent in space. At present, the image of a real object cannot be described by a transfer theory unless the object approximates one of two idealized limits. These are the "transverse" object and the "longitudinal" object.

The "transverse" object¹ is defined as the radiance distribution within a given plane normal to the optical axis. The transfer theory that applies, due to Duffieux,² is familiar to workers in the field of image evaluation.³ For purposes of distinction, this theory will be called the "lateral" or "transverse" transfer theory. In an approximate sense, lateral transfer theory also applies to any three-dimensional object that is sufficiently distant from the optics. In this case the optics-detector arrangement has a depth of focus that is sufficiently large to mask any spatial departures, over the object, from the transverse plane. Hence, the measured image gives no information about depth variation across the object.

The "longitudinal" object¹ is defined as the radiance distribution that exists along a line which is directed in an arbitrary field direction (from the nodal point for that direction). The irradiance distribution that exists along the Gaussian image of the longitudinal object obeys a "longitudinal" transfer theory. Longitudinal transfer theory also applies if the object has a finite thickness, but then the radiance must be assumed to be uniform within any cross section of the object, even if it is not. Consequently, all information is lost about the radiance variation within any cross section.

The aim of this paper is to describe the image and image spectrum of a three-dimensional object by means of a suitably generalized transfer

theory. A surface object is probably the most common type of object observed by the unaided eye. When the object is itself an aerial image, which has three-dimensional extent, it must be represented as a volume distribution. The generalized transfer theory will apply to objects of both types.

The development is similar to that of lateral transfer theory,³ and corresponding results of the two developments are frequently compared. Since the generalized transfer theory does not require the object to approximate an idealized model, such as a plane or a line, there are no "information losses" of the types previously considered.

Fig. 1. Object and image spaces for three-dimensional image formation. The region of isotomicity is exaggerated for emphasis. Image irradiance $i(r)$ is shown to arise from overlap of the point spread functions from differential areas $d\sigma_i$ of the object.

Fig. 1 identifies the main parameters of the problem. The optics are of focal length f and radius r_0 ; H and H' are the principal points.

We introduce the vector notation

$$\begin{aligned}\vec{r}' &= (x', y', z') \\ \vec{r}_G &= (x_G, y_G, z_G) \\ \vec{r} &= (x, y, z) \\ d\vec{r}' &= dx'dy'dz' \\ d\vec{r} &= dxdydz.\end{aligned}\tag{1.1}$$

In the above, $\vec{r}'$ is a general object point, $\vec{r}_G$ is the corresponding Gauss point, and $\vec{r}$ is a general point in the image space. The origins for primed and unprimed coordinates are H' and H , respectively.

Any given object may generally be described as either a surface distribution of radiance $a_O(\vec{r}')$ (as illustrated in Fig. 1), or a volume distribution of radiance $a_V(\vec{r}')$. The differential radiance dN from object point $\vec{r}'$ is then given as either

$$dN(\vec{r}') = a_O(\vec{r}')d\sigma(\vec{r}')\tag{1.2a}$$

or

$$dN(\vec{r}') = a_V(\vec{r}')d\vec{r}',\tag{1.2b}$$

where $d\sigma$ is a differential area on the surface σ . Comparing Eqs. (1.2a) and (1.2b), it is noted that a_O and a_V are dimensionally different quantities.

II. POSTULATES

Lateral transfer theory is based on two assumptions: stationarity and superposition.³ As is well known, these conditions are met by an object which radiates incoherently from an "isoplanatic"⁴ area of the object plane. It will be seen that three-dimensional generalizations of these assumptions imply a three-dimensional transfer theory.

A. Stationarity

Let $s(\vec{r}'; \vec{r})$ represent the irradiance at $\vec{r}$ due to a point source at $\vec{r}'$. Let $\vec{r}_G$ locate the Gauss point corresponding to $\vec{r}'$. We shall require, as our condition of stationarity, that

$$s(\vec{r}'; \vec{r}) = s(\vec{r} - \vec{r}_G). \quad (2.1)$$

The portion V of the object space that obeys (2.1) will be called an "isotome" (from the Greek "iso" and "tomos," meaning "equal" and "volume section," respectively). A surface σ that lies within V , as in Fig. 1, will be called "isotomic." If the entire object is not "isotomous," it must be subdivided into volumes V which are individually isotomous. Each volume V can then be separately treated by the generalized transfer theory. This technique is especially useful if each V is a meaningful fraction of the object.

Physically, (2.1) requires the point spread function to remain invariant under changes in position of the Gauss point. This will be true if each point within V has essentially the same wave aberrations. The existence of stationarity in two dimensions, within an "isoplanatic" patch, has been established;⁵ hence, the existence of usefully large (as described above) volumes of stationarity seems likely, in particular cases. Appendix I verifies isotomicity in the diffraction-limited case.

For a material object with longitudinal extent, some radiation from distant object points might be scattered or reflected by nearer object points. Such "radiation blockage" will affect $s(\vec{r}';\vec{r})$ and consequently must be taken into account when determining the isotomous volume. Unless the object is an aerial image, "radiation blockage" will always be present for object types a_v and will sometimes be present (depending on the shape of surface σ) when the object is of type a_σ .

B. Superposition

The condition of superposition requires that the irradiance i at $\vec{r}$ due to point sources at $\vec{r}'_1$ and $\vec{r}'_2$ be given as the sum

$$i(\vec{r}) = s(\vec{r}'_1;\vec{r}) + s(\vec{r}'_2;\vec{r}). \quad (2.2)$$

As in the transverse case, this condition is satisfied by point sources that radiate in a mutually incoherent manner. From (2.2), if $\vec{r}'_1 = \vec{r}'_2$, that is, if a "strength" of two point sources is present at $\vec{r}'_1$, then

$$i(\vec{r}) = 2s(\vec{r}'_1;\vec{r}). \quad (2.3)$$

By continuing the process we establish that source strengths $o(\vec{r}'_i)$, where $i = 1, 2, \dots, N$, cause an

$$i(\vec{r}) = \sum_{i=1}^N o(\vec{r}'_i) s(\vec{r}'_i;\vec{r}). \quad (2.4)$$

The $o(\vec{r}'_i)$ are necessarily unitless, so that i has the (irradiance) unit of s . Eq. (2.4) is sometimes called the principal of linearity, and may be generalized to include a continuum of source, or object, strengths. If the object is characterized by a volume distribution $o_v(\vec{r}')$ over volume V ,

$$i(\vec{r}) = \int_V o_v(\vec{r}') s(\vec{r}';\vec{r}) d\vec{r}', \quad (2.5a)$$

whereas an object distribution $o_{\sigma}(\vec{r}')$ on surface σ causes an

$$i(\vec{r}) = \int_{\sigma} o_{\sigma}(\vec{r}') s(\vec{r}'; \vec{r}) d\sigma(\vec{r}'). \quad (2.5b)$$

Objects o_v and o_{σ} must have units of (volume) $^{-1}$ and (area) $^{-1}$, respectively, so that i and s have identical units.

On physical grounds,

$$\begin{aligned} o_v &\propto a_v \\ \text{and} \quad o_{\sigma} &\propto a_{\sigma} \end{aligned} \quad (2.6)$$

that is, object strength is proportional to object radiance. Hence o_v and o_{σ} may be experimentally determined by measuring the radiance distribution of the object.

Since i is due to the superposition of real images, any object must obey either

$$\begin{aligned} o_{\sigma}(\vec{r}') &= 0 \\ \text{or } o_v(\vec{r}') &= 0 \end{aligned} \quad \left. \vphantom{\begin{aligned} o_{\sigma}(\vec{r}') &= 0 \\ o_v(\vec{r}') &= 0 \end{aligned}} \right\} \text{ for } z' \leq f. \quad (2.7)$$

Constraint (2.7) will be assumed throughout this paper.

An object which is partially type o_v and partially type o_{σ} produces an $i(\vec{r})$ which is the sum of Eqs. (2.5a) and (2.5b).

III. CONVOLUTION THEOREMS

Assuming an isotomous, incoherently radiating object, we may use identity (2.1) in Eqs. (2.5). Then

$$i(\vec{r}) = \begin{cases} \int_V o_V(\vec{r}') s(\vec{r}-\vec{r}_G) d\vec{r}' & \text{or} \\ \int_\sigma o_\sigma(\vec{r}') s(\vec{r}-\vec{r}_G) d\sigma(\vec{r}'), & \end{cases} \quad (3.1a)$$

$$(3.1b)$$

depending on the object type.

In order to integrate over the primed coordinates it is necessary to express $\vec{r}_G$ in terms of $\vec{r}'$. From first-order optics,

$$\vec{r}_G = f\vec{r}'(z'-f)^{-1}. \quad (3.1c)$$

Eqs. (3.1) are the required convolution integrals. They may be used in the mathematical determination of $i(\vec{r})$ when the object and point spread function are given.

In the case of an object o_o located in a plane $z' = \text{constant}$, Eq. (3.1b) becomes

$$i(\vec{r}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} o_o(\vec{r}') s(\vec{r}-m\vec{r}') dx' dy' \quad (3.2a)$$

$$\text{where } m = f(z'-f)^{-1}. \quad (3.2b)$$

Eqs. (3.2) describe transverse image formation in any plane z .

IV. IMAGE SPECTRUM

Continuing a three-dimensional development, we define the image spectrum $I(\vec{\Omega})$ as

$$I(\vec{\Omega}) = (2\pi)^{-3/2} \int_{\text{Image}} i(\vec{r}) \exp(-j\vec{\Omega} \cdot \vec{r}) d\vec{r}, \quad (4.1a)$$

$$\text{where } \vec{\Omega} = (\omega_1, \omega_2, \omega_3) \quad (4.1b)$$

$$\text{and } j = (-1)^{1/2}. \quad (4.1c)$$

Parameter $\vec{\Omega}$ represents a triplicate of spatial frequencies. For later use, we define

$$\vec{\omega} = (\omega_1, \omega_2) \quad (4.2)$$

Since $i(\vec{r})$ is generally a function of three spatial coordinates, $I(\vec{\Omega})$ represents a spatially "complete" Fourier spectrum of $i(\vec{r})$. It will be shown in Section V that definition (4.1a) implies a three-dimensional transfer theorem.

In any plane z beyond the exit pupil, the total image power P_i must be conserved. Algebraically,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} i(x, y, z) dx dy = P_i \quad (4.3)$$

for all $z \geq 0$. Using (4.3) and (4.1a) we find that, in particular,

$$I(0, 0, \omega_3) = (2\pi)^{-1/2} P_i \delta(\omega_3) \quad (4.4)$$

where δ is the Dirac delta function.⁶ Eq. (4.4) shows that I is singular at $\vec{\Omega} = (0, 0, 0)$, and that $I(0, 0, \omega_3)$ is independent of the nature of the object and of the point spread function. This independence will not be true at any other triplicate $\vec{\Omega}$ of frequencies.

By the Fourier inversion theorem,⁷ Eq. (4.1a) yields

$$i(\vec{r}) = (2\pi)^{-3/2} \int_{\Sigma} I(\vec{\Omega}) \exp(j\vec{\Omega} \cdot \vec{r}) d\vec{\Omega}, \quad (4.5a)$$

$$\text{where } d\vec{\Omega} = d\omega_1 d\omega_2 d\omega_3, \quad (4.5b)$$

and Σ represents the three-dimensional "passband" for frequencies $\vec{\Omega}$, which is established in Section VIII. Eq. (4.5a) allows $i(\vec{r})$ to be computed from a known $I(\vec{\Omega})$. It is proven in Appendix II that integral (4.5a) generally converges, in the Fraunhofer approximation,⁸ even though $I(0,0,0)$ is singular.

V. TRANSFER THEOREM

We shall consider the three-dimensional frequency spectrum of Eq. (3.1b). Using definition (4.1a) on the left side, and rearranging integration orders on the right side, we obtain

$$I(\vec{\Omega}) = (2\pi)^{-3/2} \int_{\sigma} d\sigma(\vec{r}') o(\vec{r}') \int_{\text{All space}} d\vec{r} s(\vec{r} - \vec{r}_G) \exp(-j\vec{\Omega} \cdot \vec{r}). \quad (5.1)$$

The change of variable,

$$\vec{\rho} = \vec{r} - \vec{r}_G \equiv (\alpha, \beta, \gamma), \quad d\vec{\rho} \equiv d\alpha d\beta d\gamma, \quad (5.2)$$

is employed in (5.1), along with (3.1c). There results

$$I(\vec{\Omega}) = O(\vec{\Omega}) \cdot F(\vec{\Omega}), \quad (5.3)$$

which is the required transfer theorem. In (5.3)

$$O(\vec{\Omega}) = \int_{\sigma} d\sigma(\vec{r}') o_{\sigma}(\vec{r}') \exp[-jf(z'-f)^{-1}\vec{\Omega} \cdot \vec{r}'] \quad (5.4)$$

$$\text{and} \quad F(\vec{\Omega}) = (2\pi)^{-3/2} \int_{\text{All space}} d\vec{\rho} s(\vec{\rho}) \exp(-j\vec{\Omega} \cdot \vec{\rho}). \quad (5.5)$$

Eq. (5.3) therefore exhibits a separation in the effects of the object and the optics upon I . Were I defined as a one- or two-dimensional transform of i , this separation would not generally occur.

In the case of a volume distribution $o_V(\vec{r}')$, Eq. (5.4) is replaced by

$$O(\vec{\Omega}) = \int_V d\vec{r}' o_V(\vec{r}') \exp[-jf(z'-f)^{-1}\vec{\Omega} \cdot \vec{r}']. \quad (5.6)$$

We note from Eqs. (5.4) and (5.6) that $O(\vec{\Omega})$ is not the true Fourier spectrum of $o(\vec{r}')$ in general. Eqs. (5.4) and (5.6) define a new type of transform. Because of constraint (2.7) the quantity $(z' - f)^{-1}$ cannot be singular, so that O is well defined.

The transfer theorem in the transverse case is well known:

$$I_t(\vec{\omega}; z) = O_t(\vec{\omega}; z') \tau(\vec{\omega}; z - mz'), \quad (5.7a)$$

$$\text{where } I_t(\vec{\omega}; z) = (2\pi)^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} i(\vec{r}) \exp[-j(\omega_1 x + \omega_2 y)] dx dy, \quad (5.7b)$$

$$O_t(\vec{\omega}; z') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} o(\vec{r}') \exp[-jm(\omega_1 x' + \omega_2 y')] dx' dy', \quad (5.7c)$$

$$\tau(\vec{\omega}; z - mz') = (2\pi)^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s(\alpha, \beta, z - mz') \exp[-j(\omega_1 \alpha + \omega_2 \beta)] d\alpha d\beta, \quad (5.7d)$$

and z' locates the object plane. Eq. (5.7a) results from the use of definition (5.7b) in Eq. (3.2a). Eq. (5.7a) cannot be sequentially used in the optical relay of an image through a series of lens systems. This is because all successive images of a plane object have three-dimensional extent, whereas Eq. (5.7a) assumes the object to be confined in one plane z' . Hence, Eq. (5.3) is the only transfer theorem which can be sequentially used in the case of serial imagery as described.

VI. TRANSFER FUNCTION: $F(\vec{\Omega})$

The "complete" optical transfer function $F(\vec{\Omega})$ is observed, in Eq. (5.5), to be the three-dimensional Fourier transform of s . The origin of coordinates (α, β, γ) is the Gauss point.

As in derivation of Eq. (4.4), we have

$$F(0,0,\omega_3) = (2\pi)^{-1/2} P \delta(\omega_3), \quad (6.1)$$

where P is the total power in any receiving plane γ within the spread function. Eq. (6.1) is consistent with the implication of Eqs. (4.4), (5.3), and either (5.4) or (5.6).

By using Eqs. (5.7d) and (5.5), as supplemented by identities (3.1c), (3.2b) and (5.2), we have

$$F(\vec{\Omega}) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} d\gamma \tau(\vec{\omega}; \gamma) \exp(-j\omega_3 \gamma). \quad (6.2a)$$

Then by the Fourier inversion theorem,

$$\tau(\vec{\omega}; \gamma) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} d\omega_3 F(\vec{\Omega}) \exp(j\omega_3 \gamma), \quad (6.2b)$$

which establishes that F and τ are Fourier transform pairs. Although $F(0,0,0)$ is infinite (as discussed previously), the integral (6.2b) generally converges. This is proved in Appendix II.

Eq. (6.2a), or an equivalent sampling expression in Section IX, may be used to compute $F(\vec{\Omega})$ from knowledge of $\tau(\vec{\omega}; \gamma)$. Other methods for the determination of F are derived in Sections VII and IX.

From definition (5.5), $F(\vec{\Omega})$ is a generally complex function. Accordingly, $F(\vec{\Omega})$ has a modulus $|F(\vec{\Omega})|$ and a phase $\Lambda(\vec{\Omega})$ defined by

$$F(\vec{\Omega}) = |F(\vec{\Omega})| \exp[-j\Lambda(\vec{\Omega})], \quad (6.3)$$

where F and Λ are real.

By applying the Fourier inversion theorem to Eq. (5.5), we have

$$s(\vec{\rho}) = (2\pi)^{-3/2} \int_{\Sigma} F(\vec{\Omega}) \exp(j\vec{\Omega} \cdot \vec{\rho}) d\vec{\Omega}. \quad (6.4)$$

It is shown in Appendix II that integral (6.4) converges generally.

VII. DEPENDENCE OF $F(\vec{\Omega})$ ON PUPIL FUNCTION

A. Derivation

Fig. 2. Parameters of the pupil and of the point spread function space.

Referring to Fig. 2, let axes $\alpha'\beta'\gamma'$ have their origin at Gauss point G . Let axis γ' lie along $\vec{r}_G$, and let axis β' be in the plane that passes through $\vec{r}_G$ and the optical axis. A reference sphere of radius r_G intersects the optical axis at H' . Axes PQ are oriented such that P is perpendicular to the $\alpha'\gamma'$ plane. As is customary, we assume field angle $(x_G^2 + y_G^2)^{1/2}/z_G$ to be so small that axes $\alpha'\beta'\gamma'$ coincide with axes $\alpha\beta\gamma$, respectively.⁹ Then $\alpha' = \alpha$, $\beta' = \beta$, $\gamma' = \gamma$, and the image amplitude u is given by the Fraunhofer approximation⁸ as

$$u(\alpha, \beta, \gamma) = (\lambda r_G)^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dp dq U(p, q) \exp[jk(2r_G^2)^{-1}(p^2 + q^2)\gamma + jkr_G^{-1}(p\alpha + q\beta)], \quad (7.1)$$

where λ is the wavelength of light, and $k = 2\pi/\lambda$. Using (7.1) and the well-known relation

$$s = u \cdot u^* \quad (7.2)$$

in definition (5.7d), we establish the well-known autocorrelation representation

$$\tau(\vec{\omega}; \gamma) = (2\pi)^{-1} \int_{-\infty}^{\infty} dp dq U(p, q) U^*[p - r_G \omega_1/k, q - r_G \omega_2/k] \times \exp[j\gamma(p\omega_1/r_G + q\omega_2/r_G - \omega^2/2k)]. \quad (7.3)$$

Eq. (7.3) differs from the usual autocorrelation formula¹⁰ in its explicit presentation of the γ dependence. This dependence is traditionally absorbed into the pupil function dependence of Eq. (7.3) because the γ dependence is usually of secondary interest. We now substitute Eq. (7.3) into Eq. (6.1a). After an interchange of the orders of integration,

$$F(\vec{\Omega}) = (2\pi)^{-3/2} \int_{-\infty}^{\infty} dp dq U(p, q) U^*[p - r_G \omega_1/k, q - r_G \omega_2/k] \times \int_{-\infty}^{\infty} d\gamma \exp[j\gamma(p\omega_1/r_G + q\omega_2/r_G - \omega^2/2k - \omega_3)]. \quad (7.4)$$

By two identities¹¹ the γ integral in (7.4) is

$$2\pi r_G |\omega_1|^{-1} \delta[p + \omega_1^{-1} (\omega_2 q - r_G \omega^2/2k - r_G \omega_3)]. \quad (7.5)$$

Substitution of (7.5) into (7.4) yields

$$F(\vec{\Omega}) = (2\pi)^{-1/2} r_G |\omega_1|^{-1} \int_{-\infty}^{\infty} dq U[p(q), q] U^*[p(q) - r_G \omega_1 k^{-1}, q - r_G \omega_2 k^{-1}] \quad (7.6a)$$

$$\text{where } p(q) = \omega_1^{-1} (r_G \omega^2/2k + r_G \omega_3 - q\omega_2). \quad (7.6b)$$

In Eq. (7.6a), U and U^* are evaluated along two parallel straight lines in the pupil. This may be further simplified by evaluating U in the PQ coordinate system and U^* in a system $P'Q'$ such that

$$p' = p - r_G \omega_1 k^{-1} \quad (7.7)$$

and $q' = q - r_G \omega_2 k^{-1}.$

Fig. 3. Transfer function F represented as a line integral across the pupil. Integration path Γ is confined to overlap region R .

Referring to Fig. 3, $F(\vec{\Omega})$ is then evaluated as one line integral along the straight path Γ between points A and A' . Path Γ extends over overlap region R of the (generally vignettted) pupil PQ and its displaced equivalent $P'Q'$. From Eq. (7.6b), Γ is perpendicular to OO' and is located distance $b(\vec{\Omega})$ from origin O , where

$$b(\vec{\Omega}) = r_G \omega (2k)^{-1} + r_G (\omega_3 / \omega). \quad (7.8)$$

Integration path Γ is thereby determined for any $\vec{\Omega}$. It is interesting to note that

$$b(\omega_1, \omega_2, 0) = \frac{1}{2} OO', \quad (7.9)$$

in terms of the relative displacement OO' of the two pupils.

Result (7.6a) may be further simplified through the replacement of integration variable q by a coordinate ℓ along Γ . From Fig. 3 and Eq. (7.6b),

$$dq = (\omega_1/\omega)d\ell. \quad (7.10)$$

Hence

$$F(\vec{\Omega}) = (2\pi)^{-1/2} r_G \omega^{-1} \int_A^{A'} d\ell (U \cdot U^*)_{\Gamma} \quad (7.11a)$$

$$\text{where } \vec{\Omega} \text{ is in passband } \Sigma. \quad (7.11b)$$

The notation in (7.11a) is meant to imply that U and U^* are evaluated in their respective coordinate systems.

In summary, Eqs. (7.8) and (7.11) determine $F(\vec{\Omega})$ for a general pupil function. Examination of these equations shows that ω_3 contributes to F only in its capacity to locate integration path Γ . We see from (7.11a) that F contains a simple pole at $\omega = 0$; this is the essential reason for the behavior noted in Eq. (6.1).

B. Experimental Determination of $F(\vec{\Omega})$ by Use of a Shearing Interferometer

Eq. (7.11a) suggests that $F(\vec{\Omega})$ may be experimentally determined by the use of a shearing interferometer.¹² Pupil function U is physically sheared, or displaced, from itself by an amount $r_G \omega/k$, and a slit is placed upon path Γ , as located by Eq. (7.8). The light flux passing through the slit is, according to Eq. (7.11a), proportional to $F(\vec{\Omega})$. This method is already being used in experimental determination of $\tau(\vec{\omega}; \gamma)$,¹² where the light flux passing through overlap region R is instead measured.

C. Numerical Calculation of $F(\vec{\Omega})$

Eq. (7.11a) may be numerically processed by an electronic computer. The speed of computation is facilitated by the one-dimensional nature of

the integral. By contrast, the pupil function representation (7.3) for $\tau(\vec{\omega}; \gamma)$ involves a double integration and, hence, considerably more computation time.

D. Normalization of $F(\vec{\Omega})$

The normalized transfer function $\tau(\vec{\omega}; \gamma)$ has several properties which facilitate its practical use: τ is (1) solely a function of the optics; (2) unitless, and (3) bounded at all $\vec{\omega}$. We note from definition (5.5) and Eq. (6.1) that F does not obey any of these properties. For example, (1) is violated because F is proportional to the source strength, according to (5.5).

Using Eq. (7.11a), we can define a new function $F(\vec{\Omega})$ that obeys all three properties and is proportional to $F(\vec{\Omega})$:

$$F(\vec{\Omega}) = \omega K(\vec{\omega}) F(\vec{\Omega}), \quad (7.12a)$$

$$\text{where} \quad K(\vec{\omega}) = (2\pi)^{1/2} r_G^{-1} \int_{-\infty}^{\infty} |U(\eta, \theta)|^2 d\eta. \quad (7.12b)$$

In the above, (η, θ) are polar coordinates corresponding to (p, q) , and

$$\theta = \tan^{-1}(\omega_1 / \omega_2). \quad (7.12c)$$

By substitution of (7.11a) into (7.12a),

$$F(\vec{\Omega}) = \int_A^{A'} d\ell (U \cdot U^*)_{\Gamma} / \int_{-\infty}^{\infty} |U(\eta, \theta)|^2 d\eta. \quad (7.13)$$

It can be seen from this relation that $F(\vec{\Omega})$ obeys requirements (1)-(3).

In general, the denominator $D(\theta)$ of (7.13) must be evaluated at all azimuths θ . However, because $|U|^2 = |T|^2$ (the energy transmittance), $D(\theta)$ does not depend on the aberrations and should be simple to compute in general.

In most instances $T = 1$; that is, the optics are uncoated. Then D is independent of θ , unless there is vignetting. If the pupil is round and of radius r_o , then $D(\theta) = 2r_o$.

From Eq. (7.13), $F(\vec{\Omega}) < 1$ for a general function U , and $F(0,0,0) = 1$.

Also, by Schwarz' inequality,

$$|F(\vec{\Omega})| < |F_o(\vec{\Omega})|, \quad (7.14)$$

where subscript o indicates diffraction-limited optics. The optics are assumed to be uncoated.

E. Diffraction-Limited Case

Fig. 4. Determination of F for a diffraction-limited, circular pupil. $F = AA'/BB'$.

In the diffraction limit

$$U(\eta, \theta) = \begin{cases} 1 & \text{for } |\eta| \leq r_o \\ 0 & \text{for } |\eta| > r_o, \end{cases} \quad (7.15)$$

independent of θ . The pupil is assumed to be nonvignetted, with radius

r_o . For a general $\vec{\omega}$ we may shift pupils along the P axis because, by (7.15), U is radially symmetric. Then, by substituting (7.15) into Eq. (7.13),

$$F_o(\vec{\Omega}) = AA'/BB', \quad (7.16)$$

as defined in Fig. 4. By sight,

$$BB' = 2r_o. \quad (7.17)$$

By elementary geometry

$$AA' = 2[r_o^2 - (r_G\omega/2k + r_G|\omega_3|/\omega)^2]^{1/2}, \quad (7.18)$$

where $|\omega_3|$ is taken because, by inspection of Fig. 4, AA' is even in ω_3 .

By substitution of (7.17) and (7.18) into Eq. (7.16),

$$F_o(\vec{\Omega}) = \begin{cases} [1 - (r_G/r_o)^2(\omega/2k + |\omega_3|/\omega)^2]^{1/2} & \text{for } \vec{\Omega} \text{ in } \Sigma \\ 0 & \text{for } \vec{\Omega} \text{ not in } \Sigma. \end{cases} \quad (7.19)$$

Bandpass region Σ is derived in Section VIII.

VIII. BANDPASS REGION FOR F AND F

We seek a region Σ in $\vec{\Omega}$ -space within which F (or $\bar{F}$) is generally zero. From Eqs. (7.8) and (7.11), F is zero whenever (1) region R is zero, or (2) Γ falls outside of R.

Condition (1) is first considered. From Fig. 3,

$$F(\vec{\Omega}) = 0 \text{ when } \omega \geq 2\alpha_0, \text{ where } \alpha_0 = kr_0/r_G. \quad (8.1)$$

In the above, r_0 is either the pupil radius or a radius that contains a generally vignetted pupil. We note that (8.1) is independent of ω_3 .

Condition (2) is accomplished, according to Eq. (7.8), when, for a given $\omega \leq 2\alpha_0$,

$$|\omega_3| \geq (\omega/2k)(2\alpha_0 - \omega); \quad (8.2a)$$

or, for a given ω_3 that obeys $|\omega_3| \leq \alpha_0^2/2k$, when either

$$\begin{aligned} \omega &\leq \alpha_0 - (\alpha_0^2 - 2k\omega_3)^{1/2} \\ \text{or} \quad \omega &\geq \alpha_0 + (\alpha_0^2 - 2k\omega_3)^{1/2}. \end{aligned} \quad (8.2b)$$

Eqs. (8.2b) indicate that, except at $\omega_3 = 0$, F is bandwidth-limited at both large and small values of ω . The latter property has no parallel in the dependence of $\tau(\omega; \gamma)$ upon $\vec{\omega}$.

Conditions (8.2) may be combined to define a volume Σ in $\vec{\Omega}$ -space within which F is generally nonzero. As shown in Fig. 5, region Σ is enclosed within the surfaces

$$\omega_3 = \pm(\omega/2k)(2\alpha_0 - \omega) \quad (8.3)$$

Fig. 5. Bandpass volume Σ in $\vec{\omega}$ -space for $F(\vec{\omega})$ and $I(\vec{\omega})$. Region Σ is generated by rotation of the curve $\omega_3 = \omega(2k)^{-1}(2\alpha_0 - \omega)$ about the ω_3 axis.

From Eqs. (8.1) and (8.2), F is zero when independently

$$\begin{aligned} |\omega_1| &> 2\pi R \\ \text{or } |\omega_2| &> 2\pi R \\ \text{or } |\omega_3| &> 2\pi R_3, \end{aligned} \tag{8.4}$$

$$\text{where } R = \alpha_0/\pi \quad \text{and} \quad R_3 = \alpha_0^2/(4\pi k). \tag{8.5}$$

Eqs. (8.4) will be used later in the derivation of sampling expressions.

It is interesting to note from (8.5) that ratio

$$R_3/R = \alpha_0/4k = (8f\#)^{-1} \tag{8.6}$$

$$\text{where } f\# = r_G/2r_0. \tag{8.7}$$

Hence in all practical cases

$$R_3 \ll R. \tag{8.8}$$

Consequences of (8.8) are discussed in Section X.

IX. SAMPLING THEOREMS

A. Derivations

By use of Eqs. (8.4) and (5.5), we have the following sampling theorems:¹³

$$F(\vec{\Omega}) = \begin{cases} (2\pi)^{-3/2} \sum_{\ell, m, n} s(\ell\pi/2\alpha_o, m\pi/2\alpha_o, 2n\pi k/\alpha_o^2) \\ \quad \times \exp[-j\pi(\ell\omega_1/2\alpha_o + m\omega_2/2\alpha_o + 2nk\omega_3/\alpha_o^2)], \\ \quad \text{for } |\omega_1| \leq 2\alpha_o, |\omega_2| \leq 2\alpha_o, |\omega_3| \leq \alpha_o^2/2k; \\ \text{or } 0, \text{ for } \vec{\Omega} \text{ not in } \Sigma; \end{cases} \quad (9.1)$$

and

$$s(\alpha, \beta, \gamma) = \sum_{\ell, m, n} s(\ell\pi/2\alpha_o, m\pi/2\alpha_o, 2n\pi k/\alpha_o^2) \\ \times \text{sinc}(2\alpha_o\alpha - \ell\pi) \text{sinc}(2\alpha_o\beta - m\pi) \text{sinc}(\alpha_o^2\gamma/2k - n\pi). \quad (9.2)$$

All summation indices are integers that range from $-\infty$ to $+\infty$. Spatial points $(\ell\pi/2\alpha_o, m\pi/2\alpha_o, 2n\pi k/\alpha_o^2)$ will be called "sampling points," as is customary.

Eq. (9.2) shows that s is known throughout all space once it is determined at the sampling points. From (9.2),

$$s(\alpha, \beta, 0) = \sum_{\ell, m} s(\ell\pi/2\alpha_o, m\pi/2\alpha_o, 0) \text{sinc}(2\alpha_o\alpha - \ell\pi) \text{sinc}(2\alpha_o\beta - m\pi) \quad (9.3)$$

and

$$s(0, 0, \gamma) = \sum_n s(0, 0, 2n\pi k/\alpha_o^2) \text{sinc}(\alpha_o^2\gamma/2k - n\pi). \quad (9.4)$$

Eq. (9.3) is known from transverse theory, and Eq. (9.4), which relates values of the Strehl definition, has recently been derived.¹

By use of Eqs. (8.4) and (6.1), we have the sampling expressions:¹³

$$F(\vec{\Omega}) = (2\pi)^{-1/2} \sum_n \tau(\vec{\omega}; 2k\pi n/\alpha_o^2) \exp(-2jk\pi n\omega_3/\alpha_o^2) \quad (9.5)$$

and

$$\tau(\vec{\omega}; \gamma) = \sum_n \tau(\vec{\omega}; 2k\pi n / \alpha_o^2) \text{sinc}(\alpha_o^2 \gamma / 2k - n\pi). \quad (9.6)$$

Eq. (9.6) indicates that the curves $\tau(\vec{\omega})$, as determined in different receiving planes γ , are interrelated by a sampling theorem.

Owing to relations (4.1a) and (5.3), I is limited to the same pass-band as is F , and we may make the replacements

$$\begin{aligned} (\alpha, \beta, \gamma) &\rightarrow (x, y, z) \\ s &\rightarrow i \\ \text{and } F &\rightarrow I \end{aligned} \quad (9.7)$$

in Eqs. (9.1) through (9.6). Eq. (9.2) then shows that any image space is degenerate, and is, in fact, known everywhere if it is known at the sampling points. This is an important limitation on, for example, the information content within an emulsion.

B. Experimental Determination of $F(\vec{\Omega})$ by Use of Sampling Theorems

Eq. (9.1) may be used to determine $F(\vec{\Omega})$ for a fabricated optical system if the sampled values of s are experimentally determined. Because (9.1) is a summation, which may be exactly evaluated on the electronic computer, the error in any resulting value of F is due solely to experimental error in values of s ; there is no computational error.

Alternatively, Eq. (9.5) shows that $F(\vec{\Omega})$ may be experimentally determined by determining $\tau(\vec{\omega}; \gamma)$ in the "sampling planes" $\gamma_n = 2k\pi n / \alpha_o^2$.

X. SUMMARY AND DISCUSSION

A transfer theory has been established to describe the image of a three-dimensional object. This theory is based on the existence of "isotomicity," as defined in Eq. (2.1). According to Eqs. (3.1) and (5.3), the image-forming ability of an optical system is determined by either the point spread function $s(\rho)$ or the "complete" optical transfer function $F(\omega_1, \omega_2, \omega_3)$. Function F is shown, in Eq. (6.2a), to be the Fourier transform with respect to the γ (longitudinal) direction of the "lateral" transfer function $\tau(\vec{\omega}; \gamma)$.

The generalized transfer theory becomes identical with the lateral transfer theory in the limit

$$o_V(\vec{r}') \rightarrow o(x', y') \delta(z' - z'_0) \quad (10.1)$$

of a transverse object $o(x', y')$, where $z' = z'_0$ is the object plane. This can be shown by substituting (10.1) into Eq. (5.6), with subsequent use of Eqs. (5.3), (4.5a), and (6.2b). In this case, $i(\vec{r})$ is determined by the Fourier transform of F , as in Eq. (6.2b), and therefore by $\tau(\vec{\omega}; \gamma)$.

Practical use can be made of the convolution theorem (3.1) and the transfer property (5.3) if the object can be subdivided into isotomes which are not too numerous. (The order of 10 would be desirable.) Appendix I establishes a method for estimating the size of isotomes in the diffraction-limited case.

Eq. (9.2) and its analogy in image space (based on substitutions [9.7]) establish a limit on the information density of three-dimensional image space. Regarding each independent image value as a degree of freedom for the image, there can be no higher information density than

$$1/\lambda^3 f_{\#}^4 \text{ degrees of freedom/volume} \quad (10.2)$$

in image space. The value (10.2) corresponds to a diffraction-limited system, and is based on cutoff frequencies R and R_3 of Eq. (8.5). Because $R_3 \ll R$ in general, the z -distance $1/2R_3$ between sampling points is much greater than the corresponding x and y distances of $1/2R$. Hence, a much higher density of image information is contained in a direction normal to the optical axis than parallel to it.

A lens system designed such that F is constant at all $\vec{\Omega}$ will, according to Eqs. (5.3) and (4.5a) (with infinite limits), produce an image that is the Gaussian distortion of the object. Hence, any transverse object is perfectly imaged, and with a magnification determined by position z of the object plane. This property would be most advantageous for a lens that is to operate at a variety of conjugates (e.g., the relay lens). Use of $F(\vec{\Omega})$ as a design criterion seems, therefore, to be indicated. In this regard, it is interesting to note relations between moments of the $F(\vec{\Omega})$ distribution and derivatives of $s(\vec{\rho})$ at $\vec{\rho} = 0$, as implied by Eq. (6.4). Also, to compute $F(\vec{\Omega})$ from ray-trace data, the geometrical approximation to $F(\vec{\Omega})$ must be expressed in terms of ray intercepts in the spot diagram. This may be accomplished by taking the limit $\lambda \rightarrow 0$ in Eq. (7.11a) and using l'Hôpital's rule.

Three-dimensional image processing is suggested by the following. We may invert Eq. (5.6), expressing object o_V in terms of its spectrum O :

$$o_V(\vec{r}') = (2\pi)^{-3} f^4 (z'-f)^{-4} \int_{-\infty}^{\infty} d\vec{\Omega} O(\vec{\Omega}) \exp[jf(z'-f)^{-1} \vec{\Omega} \cdot \vec{r}']. \quad (10.3)$$

If $O(\vec{\Omega})$ of Eq. (5.3) is substituted into Eq. (10.3), and the infinite limits in (10.3) are replaced by bandpass region Σ , a method is indicated for the approximate reconstruction of a three-dimensional object by observation of its image.

All of the foregoing has assumed the object to be radiating incoherently. If the opposite assumption is made, a transfer theory for image amplitude $g(\vec{r})$ can be constructed, where

$$i(\vec{r}) = |g(\vec{r})|^2. \quad (10.4)$$

A condition of stationarity and convolution, transfer, and sampling theorems can be constructed which are completely analogous to those of the incoherent case. Parameter F of the latter case corresponds to an "amplitude transfer function" $W(\vec{\Omega})$ of the coherent case,¹⁴ defined as

$$W(\vec{\Omega}) = (2\pi)^{-3/2} \int_{\text{All space}} d\vec{\rho} u(\vec{\rho}) \exp(-j\vec{\Omega} \cdot \vec{\rho}). \quad (10.5)$$

It is interesting to obtain a pupil-representation for $W(\vec{\Omega})$. By substituting Eq. (7.1) into Eq. (10.5), we find that $W(\vec{\Omega})$ is proportional to

$$U(r_G \omega_1/k, r_G \omega_2/k) \delta(\omega^2/2k - \omega_3). \quad (10.6)$$

This shows that W is bandwidth-limited to a region

$$\omega \leq \alpha_0, \omega_3 \leq \alpha_0^2/2k. \quad (10.7)$$

Therefore, both $u(\vec{\rho})$ and $g(\vec{r})$ obey sampling theorems that correspond to (9.2), once the sampling intervals are adjusted.

We may also use relation (10.6) in the coherent analog to Eqs. (4.5a) and (5.3) in combination. Amplitude $g(\vec{r})$ is found to be proportional to

$$\iint_{-\infty}^{\infty} d\omega_1 d\omega_2 U(r_G \omega_1/k, r_G \omega_2/k) \delta(\omega_1, \omega_2, \omega^2/2k) \exp[j(\omega_1 x + \omega_2 y) + jz\omega^2/2k]. \quad (10.8)$$

This equation is especially useful because it involves only a double integration and depends directly upon the pupil function.

XI. ACKNOWLEDGMENTS

This work was supported by Tropel, Inc., and by grants 5263-945-910 and 5263-945-130.

The author gratefully acknowledges valuable discussions with K. Sayanagi and N. A. Hochgraf. He also wishes to thank H. Gamo for critically reading the manuscript.

XII. REFERENCES

1. B. Roy Frieden, "Longitudinal image formation": J. Opt. Soc. Am. 56, 1495-1501 (1966).
2. P. M. Duffieux, L'Integrale de Fourier et Ses Applications à l'Optique (privately printed, Besancon, France, 1947).
3. See, for example, H. H. Hopkins, "The application of frequency response techniques in optics": In K. J. Habel (ed.), Proceedings of the Conference on Optical Instruments and Techniques, London, 1961 (John Wiley and Sons, Inc., New York, 1963), pp. 483, 484.
4. Max Born and Emil Wolf, Principles of Optics (Pergamon Press, New York, 1959), p. 481.
5. P. Dumontet, "Sur la correspondance objet-image en optique": Optica Acta 2, 53-63 (1955).
6. Ref. 4, p. 752.
7. Ron Bracewell, The Fourier Transform and Its Applications (McGraw Hill Book Co., San Francisco, 1965), p. 7.
8. Ref. 4, p. 382.
9. The orientation of axes $\alpha'\beta'\gamma'$ is as described by B. R. A. Nijboer, The Diffraction Theory of Aberrations: Thesis, Groningen, Netherlands, 1948. The approximate coincidence of axes α, β, γ and α', β', γ' is implicitly assumed in, for example, ref. 4, p. 480.
10. Ref. 4, p. 484.
11. Ref. 4, p. 754, Eq. (10); and p. 756, Eq. (23).
12. H. H. Hopkins, "Interferometric methods for the study of diffraction images": Optica Acta 2, 23-29 (1955).
13. The corresponding one-dimensional formulae are derived by Stanford Goldman, Information Theory (Prentice-Hall, New York, 1955), p. 82.
14. A similar concept was previously used by C. W. McCutchen, "Generalized aperture and the three-dimensional diffraction image": J. Opt. Soc. Am. 54, 240-244 (1964).
15. Ref. 7, pp. 9, 11, 91.

Appendix I. ISOTOMICITY IN THE DIFFRACTION-LIMITED CASE

We are to establish the extent to which condition (2.1) is obeyed as the position $\vec{r}'$ of a point source is changed. Eqs. (7.1) and (7.2) will be used in establishing $s(\vec{r}'; \vec{r})$.

Assume a point source, of total power output E , to be located at general field position $\vec{r}'$. Then the irradiance at the pupil is approximately $E/4\pi r'^2$, so that

$$U(p, q) = (E/4\pi)^{1/2} r'^{-1}. \quad (11.1)$$

Substituting (11.1) into (7.1) leads to

$$u = u(v, \gamma) = (\pi E)^{1/2} (\lambda r_G r')^{-1} \int_0^{r_0} d\mu \cdot \mu J_0(k\mu v/r_G) \exp(jk\mu^2 \gamma/2r_G^2) \quad (11.2a)$$

$$\text{where } v = (\alpha^2 + \beta^2)^{1/2} \text{ and } \mu = (p^2 + q^2)^{1/2} \quad (11.2b)$$

Since (11.2a) is dependent upon r' and r_G in modulus only, we suspect that any isotome will be in the form of a spherical shell of thickness $\Delta r'$.

This is shown, below, to be true at small field angle positions.

We now observe the effect upon $|u|^2$ of a radial change $\Delta r'$ in position of the point source. In particular, we examine resulting changes in the $|u(v, 0)|^2$ and $|u(0, \gamma)|^2$ distributions, since they are easy to estimate and, at the same time, correlate with changes in the overall $|u(v, \gamma)|^2$ distribution.

From Eq. (11.2a), we have

$$|u(v, 0)|^2 = s(v, 0) = (\pi E/4\lambda^2) (r_0/r')^4 (r' - f_\phi)^2 f_\phi^{-2} [2J_1(\alpha_0 v)/\alpha_0 v]^2, \quad (11.3a)$$

$$\text{where } f_\phi = f \sec(\phi), \quad (11.3b)$$

ϕ is the field angle between r' and the optical axis, and J_1 is the first-order Bessel function. Relation (3.1c) was used. Also, from Eq. (11.2a),

$$|u(0,\gamma)|^2 = s(0,\gamma) = (\pi E/4\lambda^2)(r_o/r')^4(r'-f_\phi)^2 f_\phi^{-2} [\sin(\alpha_o^2 \gamma/4k)/(\alpha_o^2 \gamma/4k)]^2. \quad (11.4)$$

The invariance of $s(v,0)$ and $s(0,\gamma)$ under changes $\Delta r'$ can be observed through behavior of the central maximum $s(0,0)$, the radial distance R_v to the first zero of $s(v,0)$, and the axial distance Z_γ to the first zero of $s(0,\gamma)$. From (11.3) and (11.4),

$$s(0,0) = (\pi E/4\lambda^2)(r_o/r')^4(r'-f_\phi)^2 f_\phi^{-2}, \quad (11.5a)$$

$$R_v = (1.64\lambda/r_o) f_\phi r' (r'-f_\phi)^{-1}, \quad (11.5b)$$

$$\text{and } Z_\gamma = (2\lambda/r_o^2)(f_\phi r')^2 (r'-f_\phi)^{-2}. \quad (11.5c)$$

Using these equations we can compute the relative changes in $s(0,0)$, R_v , and Z_γ due to changes $\Delta r'$. We note that these relative changes will depend upon field angle ϕ to a certain extent, because of relation (11.3b).

However, in many cases ϕ is sufficiently small (less than about 20°) that $f_\phi \approx f$ over the object. In these cases the isotomes are then spherical shells. When ϕ is appreciable, relative changes in parameters (11.5) must be evaluated due to changes $\Delta\phi$ as well as changes $\Delta r'$.

As an example, we use relations (11.5) for the case $r' = 2f$, $\Delta r'/r' = 5\%$, and ϕ small. Using differentials we find that $\Delta s(0,0)/s(0,0) = 0$ identically; $\Delta R_v/R_v = -5\%$, and $\Delta Z_\gamma/Z_\gamma = -10\%$. This indicates that isotomicity holds fairly well within the shell $r' \pm 0.05r'$, ϕ small, in this case.

Appendix II. CONVERGENCE OF THE INVERSE FOURIER TRANSFORMS

Functions I and F are "generalized"¹⁵ in the limit of unit "modifying factors"¹⁵; this is indicated by Eqs. (4.4) and (6.1). For a generalized function f(x) the repeated Fourier integral

$$(2\pi)^{-1} \int_{-\infty}^{\infty} d\omega \exp(j\omega x') \int_{-\infty}^{\infty} dx' f(x') \exp(-j\omega x')$$

equals f(x) at each value of x for which it converges.¹⁵ Hence, in order to prove that inverse transforms (6.4), (6.2b) and (4.5a) converge to $s(\vec{\rho})$, $\tau(\vec{\omega};\gamma)$, and $i(\vec{r})$, respectively, we need only establish that the inverse transforms converge. For example, in the diffraction-limited case it can be shown that the known functions $s(\vec{\rho})$ and $\tau(\vec{\omega};0)$ are converged upon by integrals (6.4) and (6.2b), respectively. The general convergences of transforms (6.4), (6.2b), and (4.5a) are shown below.

A. Transform (6.4)

By defining

$$\epsilon(\theta) = \int_{-\infty}^{\infty} |U(\eta, \theta)|^2 d\eta \quad (11.6)$$

and substituting Eqs. (7.12) into Eq. (6.4), we have

$$s(\vec{\rho}) = (2\pi)^{-2} r_G \int_{\Sigma} F(\vec{\Omega})_{\omega}^{-1} \epsilon(\theta) \exp(j\vec{\Omega} \cdot \vec{\rho}) d\vec{\Omega}. \quad (11.7)$$

Owing to factor ω^{-1} , the integrand is singular at $\omega = 0$. We show below that integral (11.7) converges at all $\vec{\rho}$, despite this singularity.

Replace Cartesian coordinates $\vec{\rho}$ and $\vec{\Omega}$ in (11.7) by cylindrical coordinates (ν, ϕ, γ) and $(\omega, \theta, \omega_3)$, respectively, where

$$\left. \begin{aligned} \alpha &= \nu \sin \phi \\ \beta &= \nu \cos \phi \end{aligned} \right\} \quad (11.8a)$$

and

$$\left. \begin{aligned} \omega_1 &= \omega \sin \theta \\ \omega_2 &= \omega \cos \theta \end{aligned} \right\} \quad (11.8b)$$

Bandpass Σ is determined by Eqs. (8.1) and (8.3). Eq. (11.7) then becomes

$$s(\nu, \Phi, \gamma) = (2\pi)^{-2} r_G \int_0^{2\pi} \int_0^{2\alpha_0} \int_{-g(\omega)}^{g(\omega)} d\omega_3 d\omega d\theta F(\omega, \theta, \omega_3) \epsilon(\theta) \quad (11.9a)$$

$$\times \exp[j\omega\nu\cos(\Phi-\theta) + j\omega_3\gamma],$$

$$\text{where } g(\omega) = (\omega/2k)(2\alpha_0 - \omega) \text{ and } \omega \leq 2\alpha_0. \quad (11.9b)$$

By definitions (7.13) and (11.6), F and ϵ are bounded at all $\vec{\Omega}$. In addition, the integration limits in (11.9a) are finite. Hence, integral (11.9a) must be finite at all $\vec{\rho}$.

B. Transform (6.2b)

By the use of Eq. (8.3), integral transform (6.2b) becomes

$$\tau(\vec{\omega}; \gamma) = (2\pi)^{-1/2} \int_{-g(\omega)}^{g(\omega)} d\omega_3 F(\vec{\Omega}) \exp(j\omega_3\gamma). \quad (11.10)$$

By Eq. (7.11a), F is nonsingular, except at $\omega = 0$. Now

$$g(\omega) \leq \alpha_0^2/2k \quad (11.11)$$

because $\tau(\vec{\omega}; \gamma)$ is nonzero only when $\omega \leq 2\alpha_0$. Since integrand and limits in (11.10) are then finite, except at $\omega = 0$, (11.10) must be finite for $\omega \neq 0$.

The case $\omega = 0$ is now considered. Substitution of Eq. (6.1) into (11.10) yields

$$\tau(0, 0; \gamma) = (2\pi)^{-1} p, \quad (11.12)$$

a finite number.

C. Transform (4.5a)

By using polar coordinates $(r'; \psi)$, defined by

$$\left. \begin{aligned} x &= r' \sin \psi \\ y &= r' \cos \psi \end{aligned} \right\} \quad (11.13)$$

and Eqs. (5.3), (7.12), (11.6), and (11.8a), Eq. (4.5a) becomes

$$i(\vec{r}) = (2\pi)^{-2} r_G \int_0^{2\pi} \int_0^{2\alpha_0} \int_{-g(\omega)}^{g(\omega)} d\omega_3 d\omega d\theta \epsilon(\theta) Q(\omega, \theta, \omega_3) F(\omega, \theta, \omega_3) \quad (11.14)$$

$$X \exp[j\omega r' \cos(\theta - \psi) + j\omega_3 z].$$

By either definition (5.4) or (5.6), any Q due to a real object must be nonsingular at all $\vec{\Omega}$. Also, Eq. (7.13) shows that F is nonsingular at all $\vec{\Omega}$. The integral (11.14) must then be convergent, because its integrand and limits are finite.