OPTICAL SCIENCES

THE UNIVERSITY OF ARIZONA
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DYNAMIC RELAXATION: A GENERAL METHOD FOR DETERMINATION
OF ELASTIC DEFORMATION OF MIRRORS

Allan J. Malvick

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ABSTRACT

The tensor equations of elasticity in nonorthogonal curvilinear coordinates are presented in a form suitable for the method of dynamic relaxation. This method is described briefly and then is applied to the solution of the problem of elastic deformation of curved mirrors.

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INTRODUCTION

If a body is of sufficiently simple shape, a coordinate system may be found such that the coordinate surfaces conform to the surfaces of the body. This may be done for most solid mirrors, i.e. those without a rib structure. To take a simple example, a circular flat mirror with or without a concentric circular hole may be described by a cylindrical coordinate system. For a dished mirror, an orthogonal curvilinear coordinate system may be used if one does not mind the curving edge of the mirror. Such a coordinate system is not suitable for a mirror which is nearly flat because the coordinate transformation behaves badly in that limiting case. Non-orthogonal curvilinear coordinates are even more general than orthogonal curvilinear coordinates and therefore include a wider class of body shapes. Nonorthogonal coordinate transformations are generally easier to produce and offer better mathematical behavior than the orthogonal transformations. On the other hand, the equations of elasticity and motion are somewhat more complex in nonorthogonal coordinates, and this results in about 20% more terms in the equations for circular dished mirrors.

Dynamic relaxation as applied to orthogonal coordinates has been presented recently (Malvick and Pearson, 1968). The method was developed by Day (1965) and Otter et al. (1966). It is a numerical method that employs first-order central difference approximations to the equations of elasticity and motion involving stress and displacement. It thus avoids the precision problems associated with higher-order differential equations involving stress functions. The local detail of deflection and stress that the method provides is remarkable. There is complete freedom of choice of boundary conditions. The number of elements required for a reasonably correct solution is moderate, and so is the computer time. A mirror with a

diametric plane of symmetry divided into 500 elements--10 radial, 10 angular around half the mirror, and 5 thick--requires about 30 minutes of CDC 3200 computing time. In effect, some 1700 simultaneous displacement equations for the half mirror are solved.

This paper presents the dynamic relaxation method as applied in non-orthogonal curvilinear coordinates by Hobbs (1967) and illustrates the method with an analysis of the 158-inch telescope mirror of Kitt Peak National Observatory.

NONORTHOGONAL CURVILINEAR COORDINATES AND THE EQUATIONS OF ELASTICITY

Nonorthogonal curvilinear coordinates $x_1 x_2 x_3$ may be defined in terms of Cartesian coordinates $y_1 y_2 y_3$ by the relations

$$y_1 = y_1 (x_1, x_2, x_3)$$

$$y_2 = y_2 (x_1, x_2, x_3)$$

$$y_3 = y_3 (x_1, x_2, x_3)$$

or

$$x_1 = x_1 (y_1, y_2, y_3)$$

$$x_2 = x_2 (y_1, y_2, y_3)$$

$$x_3 = x_3 (y_1, y_2, y_3)$$

A differential element of arc ds has length

$$ds^2 = dy_1^2 + dy_2^2 + dy_3^2 = \sum_{i=1}^3 \sum_{j=1}^3 g_{ij} dx_i dx_j$$

in which $g_{ij} = \sum_{k=1}^3 \frac{\partial y_k}{\partial x_i} \cdot \frac{\partial y_k}{\partial x_j}$ is the covariant metric tensor.

If the summation and affix conventions of tensor analysis are introduced, we have $ds^2 = g_{ij} dx_i dx_j$. Whenever an index such as i is repeated in an expression, summation is assumed over all values of i . If $g_{ij} = 0$ for $i \neq j$, the coordinates $x_1 x_2 x_3$ are orthogonal. The inverse of the covariant metric tensor is the contravariant metric tensor g^{ij} , or $g_{ij}^{-1} = g^{ij}$.

The Jacobian of the coordinate transformation is $(g)^{\frac{1}{2}}$ where $g = |g_{ij}|$, the determinant of g_{ij} .

The tensor equations of elasticity are developed in a text by Green and Zerna (1954). Hooke's law relating stress and displacement is given by

$$\frac{\tau^{ij}}{G} = g^{j\alpha} U_{,\alpha}^i + g^{i\beta} U_{,\beta}^j + \frac{2\nu}{1-2\nu} g^{ij} U_{,\beta}^\beta \quad (1)$$

in which τ^{ij} is the stress tensor, G is the shear modulus of the material, ν is Poisson's ratio, U^i is the deflection vector, and $U_{,\alpha}^i$ denotes covariant differentiation of U^i with respect to the x_α coordinate. α and β are dummy indices.

The physical component u_i of deflection U^i in direction of x_i is

$$u_i = U^i (\underline{g_{ii}})^{\frac{1}{2}} \quad (2)$$

where the bar suppresses summation.

The physical component σ_{ij} of stress τ^{ij} in direction of x_j on surface $x_i = \text{constant}$ is

$$\sigma_{ij} = \left(\underline{g_{jj}} / \underline{g_{ii}} \right)^{\frac{1}{2}} \tau^{ij} \quad (3)$$

The covariant derivative $U_{,\alpha}^i$ may be expanded to the form

$$U_{,\alpha}^i = \frac{\partial U^i}{\partial x_\alpha} + \left\{ \begin{matrix} i \\ \beta \alpha \end{matrix} \right\} U^\beta \quad (4)$$

with summation on β .

Similarly the covariant derivative of the stress tensor $\tau_{,\alpha}^{ij}$ is given by

$$\tau_{,\alpha}^{ij} = \frac{\partial \tau^{ij}}{\partial x_\alpha} + \left\{ \begin{matrix} i \\ \alpha \beta \end{matrix} \right\} \tau^{\beta j} + \left\{ \begin{matrix} j \\ \alpha \beta \end{matrix} \right\} \tau^{i\beta} \quad (5)$$

with summation on β and summation on α if $\alpha = j$ or $\alpha = i$.

In the above formulas the symbol $\left\{ \begin{matrix} i \\ j \quad \alpha \end{matrix} \right\}$ is the Christoffel symbol of the second kind and is given by

$$\left\{ \begin{matrix} i \\ j \quad \alpha \end{matrix} \right\} = \frac{1}{2} g^{i\beta} \left(\frac{\partial g_{j\beta}}{\partial x_\alpha} + \frac{\partial g_{\alpha\beta}}{\partial x_j} - \frac{\partial g_{i\alpha}}{\partial x_\beta} \right) \quad (6)$$

with summation on β . Using (1), (2), (3), (4), and (6), Hooke's law in terms of physical components becomes

$$\begin{aligned} \sigma_{ij} = & G \left(g_{jj}/g_{ii} \right)^{1/2} \left[g^{j\alpha} \left(\frac{\partial}{\partial x_\alpha} \left(u_i / g_{ii}^{1/2} \right) + \left\{ \begin{matrix} i \\ \beta \quad \alpha \end{matrix} \right\} u_\beta / g_{\beta\beta}^{1/2} \right) \right. \\ & + g^{i\beta} \left(\frac{\partial}{\partial x_\beta} \left(u_j / g_{jj}^{1/2} \right) + \left\{ \begin{matrix} j \\ \alpha \quad \beta \end{matrix} \right\} u_\alpha / g_{\alpha\alpha}^{1/2} \right) \\ & \left. + \frac{2\nu}{1-2\nu} g^{ij} \left(\frac{\partial}{\partial x_\beta} \left(u_\beta / g_{\beta\beta}^{1/2} \right) + \left\{ \begin{matrix} \beta \\ \alpha \quad \beta \end{matrix} \right\} u_\alpha / g_{\alpha\alpha}^{1/2} \right) \right] . \end{aligned} \quad (7)$$

The equation of motion of the body is

$$\tau_{,i}^{ij} + F^j = \rho A^j \quad (8)$$

in which F^j is the body force per unit volume, ρ is the mass density, and A^j is the acceleration.

The physical components of body force and acceleration are

$$f_j = F^j g_{jj}^{1/2} \quad (9)$$

and

$$a_j = A^j g_{jj}^{1/2} \quad (10)$$

Using (3), (5), (8), (9), and (10), the equation of motion in terms of physical components becomes

$$\begin{aligned} \rho a_j = f_j + g_{jj}^{\frac{1}{2}} \left[\frac{\partial}{\partial x_i} \left(\sigma_{ij} \left(g_{ii}/g_{jj} \right)^{\frac{1}{2}} \right) + \begin{Bmatrix} i \\ i \quad \alpha \end{Bmatrix} \sigma_{\alpha j} \left(g_{\alpha\alpha}/g_{jj} \right)^{\frac{1}{2}} \right. \\ \left. + \begin{Bmatrix} j \\ i \quad \alpha \end{Bmatrix} \sigma_{i\alpha} \left(g_{ii}/g_{\alpha\alpha} \right)^{\frac{1}{2}} \right] \end{aligned} \quad (11)$$

It is convenient in applying (7) and (11) to reduce the nine physical stress components to six. This is done by taking (3) to write

$$\tau^{ij} = \left(g_{ii}/g_{jj} \right)^{\frac{1}{2}} \sigma_{ij} = \left(g_{jj}/g_{ii} \right)^{\frac{1}{2}} \sigma_{ji}$$

or

$$\sigma_{ji} = \left(g_{ii}g_{jj}/g_{jj}g_{jj} \right)^{\frac{1}{2}} \sigma_{ij}$$

THE DYNAMIC RELAXATION METHOD

For simplicity we write (11) in the form

$$\rho a = f + L(\sigma)$$

where L denotes the linear differential operation given by (11). A damping term is introduced into the equation:

$$L(\sigma) + f - cv = \rho a$$

where v is velocity and c is a damping coefficient. The damping coefficient should produce slightly-under-critical damped motion in the dynamic relaxation procedure; otherwise an excessive number of iterations is required for convergence to the static equilibrium solution. The value of critical damping must be found by trial for each body geometry considered.

Consider the motion over a time interval Δt . At the beginning of the time interval t_0 , let $v = v_0$. At the end, let $v = v_1$. The v represents the array of velocity components of all grid points. Then the equation of motion may be approximated by

$$L(\sigma) + f - c \frac{v_0 + v_1}{2} = \rho \frac{v_1 - v_0}{\Delta t}$$

If we let $c/\rho = k/\Delta t$ and solve for v_1 , we obtain

$$v_1 = \frac{1 - (k/2)}{1 + (k/2)} v_0 + \frac{\Delta t}{\rho[1 + (k/2)]} (L(\sigma) + f)$$

Thus, given v_0 everywhere at a given instant t_0 , and the state of stress everywhere at $t_0 + \frac{1}{2}\Delta t$, the velocity v_1 is found everywhere at $t_0 + \Delta t$. Given the displacement u_0 everywhere at $t_0 + \frac{1}{2}\Delta t$, the displacement u_1 at $t_0 + \frac{3}{2}\Delta t$ may be found by

$$u_1 = u_0 + v_1 \Delta t.$$

The stress may be found everywhere at $t_0 + \frac{3}{2}\Delta t$ by the stress-displacement relations. The procedure is repeated until all velocities damp out to negligible values, leaving the static displacements corresponding to equilibrium.

The time increment Δt must satisfy the Forsyth-Wasow criterion (see Otter et al., 1966):

$$\Delta t \leq \left\{ \frac{\rho(1+\nu)(1-2\nu)}{E(1-\nu)} \cdot \frac{1}{1/dx^2 + 1/dy^2 + 1/dz^2} \right\}^{\frac{1}{2}}$$

in which dx , dy , dz are the physical dimensions of that element (usually the smallest one of the body) which produces the largest value of $(1/dx^2 + 1/dy^2 + 1/dz^2)$. The criterion as expressed here is true for orthogonal coordinates, but only approximate for nonorthogonal coordinates. The ultimate test is that if Δt is at all too large, the procedure diverges very rapidly. The stress-displacement relations are expressed in central difference form (see Fig. 1) by such relations as

$$\left. \frac{\partial u}{\partial x_2} \right|_A = \frac{u_2 - u_1}{\Delta c_2} \quad \left. \frac{\partial \sigma}{\partial x_1} \right|_B = \frac{\sigma_2 - \sigma_1}{\Delta c_1} \quad \left. \frac{\partial \tau}{\partial x_2} \right|_B = \frac{\tau_2 - \tau_1}{\Delta c_2}$$

Thus the stresses, displacements, and velocities are interlocked with their neighbors in space and time by means of central difference equations.

Fig. 1. Nonorthogonal curvilinear coordinates. Location of normal and shear stresses, and displacements.

BOUNDARY CONDITIONS

The boundary conditions are handled in the following manner, noting that normal stresses appear at the middle of the elements and shear stresses at the edge in the dynamic relaxation technique.

A typical boundary force P is shown in Fig. 2a. In Fig. 2b it is divided into two forces, Q_1 and Q_2 , acting at the centers of the neighboring elements so that the resultant of Q_1 and Q_2 is P . In Fig. 2c each force Q_1 and Q_2 is resolved into "normal" N and "shear" S components in the coordinate directions. Each is divided by its element area to obtain the physical components of stress. In Fig. 2d the shear components are shifted to the element corner nearest P . The procedure does introduce small errors manifested by a small drift in the dynamic relaxation solution.

Fig. 2. Handling boundary forces.

THE MIRROR COORDINATE SYSTEM

A convenient coordinate system for a parabolic mirror with flat back is the following:

$$y_1 = x_1 \cos x_3$$

$$y_2 = x_1 \sin x_3$$

$$y_3 = x_2 [(k-1)x_1^2/r_0^2 + 1]$$

or

$$x_1 = (y_1^2 + y_2^2)^{1/2}$$

$$x_2 = y_3 / [(k-1)(y_1^2 + y_2^2)/r_0^2 + 1]$$

$$x_3 = \tan^{-1} \frac{y_2}{y_1}$$

in which y_1, y_2, y_3 are Cartesian coordinates with y_3 coinciding with the optical axis of the mirror; x_1, x_2, x_3 are nonorthogonal paraboloid-cylindrical coordinates with x_1 being the radial coordinate, x_2 the depth coordinate, and x_3 the angular coordinate. Also, r_0 is the outside radius of the mirror and k is a parameter describing the curvature of the upper surface. If $k=1$, the mirror is flat and the coordinates are the familiar cylindrical ones.

To evaluate the metric tensor g_{ij} we have:

$$\frac{\partial y_1}{\partial x_1} = \cos x_3$$

$$\frac{\partial y_2}{\partial x_1} = \sin x_3$$

$$\frac{\partial y_3}{\partial x_1} = 2x_1x_2(k-1)/r_0^2 = E$$

$$\frac{\partial y_1}{\partial x_2} = 0$$

$$\frac{\partial y_2}{\partial x_2} = 0$$

$$\frac{\partial y_3}{\partial x_2} = (k - 1) x_1^2 / r_0^2 + 1 = D$$

$$\frac{\partial y_1}{\partial x_3} = -x_1 \sin x_3$$

$$\frac{\partial y_2}{\partial x_3} = x_1 \cos x_3$$

$$\frac{\partial y_3}{\partial x_3} = 0$$

$$g_{ij} = \begin{bmatrix} (1 + E^2) & DE & 0 \\ DE & D^2 & 0 \\ 0 & 0 & x_1^2 \end{bmatrix}$$

The inverse of g_{ij} is:

$$g^{ij} = \begin{bmatrix} 1 & -E/D & 0 \\ -E/D & (1+E^2)/D^2 & 0 \\ 0 & 0 & 1/x_1^2 \end{bmatrix}$$

The Jacobian is $g^{\frac{1}{2}} = x_1 D$.

There are 27 Christoffel symbols of the second kind. The only non-zero ones are as follows:

$$\begin{aligned} \begin{Bmatrix} 2 \\ 2 \ 1 \end{Bmatrix} &= \begin{Bmatrix} 2 \\ 1 \ 2 \end{Bmatrix} = E/x_2 D \\ \begin{Bmatrix} 1 \\ 3 \ 3 \end{Bmatrix} &= -x_1 \\ \begin{Bmatrix} 2 \\ 1 \ 1 \end{Bmatrix} &= E/x_1 D \\ \begin{Bmatrix} 3 \\ 3 \ 1 \end{Bmatrix} &= \begin{Bmatrix} 3 \\ 1 \ 3 \end{Bmatrix} = 1/x_1 \\ \begin{Bmatrix} 2 \\ 3 \ 3 \end{Bmatrix} &= Ex_1/D \end{aligned}$$

A typical stress-displacement relation turns out to be:

$$\begin{aligned} \sigma_{11} = 2G (g_{11}/g^{11})^{1/2} &\left\{ \left[\frac{1-\nu}{1-2\nu} g^{11} \frac{\partial g_{11}^{-1/2}}{\partial x_1} + g^{12} \frac{\partial g_{11}^{-1/2}}{\partial x_2} + g^{11} g_{11}^{-1/2} \left(\frac{E}{x_2 D} + \frac{1}{x_1} \right) \frac{\nu}{1-2\nu} \right] u_1 \right. \\ &+ \left. \frac{1-\nu}{1-2\nu} g^{11} g_{11}^{-1/2} \frac{\partial u_1}{\partial x_1} + g^{12} g_{11}^{-1/2} \frac{\partial u_1}{\partial x_2} + \frac{\nu}{1-2\nu} g^{11} g_{22}^{-1/2} \frac{\partial u_2}{\partial x_2} + \frac{\nu}{1-2\nu} g^{11} g_{33}^{-1/2} \frac{\partial u_3}{\partial x_3} \right\} \end{aligned}$$

A typical equation of motion is:

$$\begin{aligned} \rho a_1 + c v_1 = f_1 + &\left\{ \left[g_{11}^{1/2} \frac{\partial g_{11}^{-1/2}}{\partial x_1} + g_{11}^{-1/2}/x_1 + 2(k-1)x_1/r_0^2 D \right] \sigma_{11} - (g_{11}^{1/2}/x_1) \sigma_{33} \right. \\ &+ \left. \frac{\partial \sigma_{11}}{\partial x_1} + (g^{22})^{1/2} \frac{\partial \sigma_{12}}{\partial x_2} + (g_{11}^{1/2}/x_1) \frac{\partial \sigma_{13}}{\partial x_3} \right\} \end{aligned}$$

The six stress-displacement equations and the three equations of motion are written in central difference form as described above, and they are solved by the dynamic relaxation method.

In applying boundary conditions at the outer or inner edge of the mirror, a stress Q acting radially has components in the nonorthogonal coordinate directions x_1, x_2 of $N = (1+E^2)^{1/2}Q$ and $S = -EQ$, respectively.

RESULTS

Deflections of the upper surface for various back and edge support systems are about the same as for the orthogonal coordinate program described previously (Malvick and Pearson, 1968). Differences of optical results of the two programs are on the order of .004 arc sec.

A. B. Meinel of the Optical Sciences Center has suggested a procedure that speeds up the solution for upper surface deflections perpendicular to the mirror surface due to edge supports. It requires the solutions for upper surface deflection $\delta_1(\rho, \phi)$ and $\delta_2(\rho, \phi)$ for the mirror resting on a knife-edge support on the lowest point of the outer edge and the highest point of the inner edge, respectively, when the mirror is in a vertical plane, as shown in Figs. 3a and 3b.

These solutions δ_1 and δ_2 may be superimposed to give the deflection due to an arbitrary edge support system $f_1(\theta_1)$ and $f_2(\theta_2)$, as shown in Fig. 3c. Of course the manner of distribution of the edge force over the width of the edge should be the same as for the edge forces of Figs. 3a and 3b.

Figs. 3a-3c. Mirror in vertical plane with (a) point support on bottom, (b) point support at top of hole, and (c) arbitrary knife-edge support.

The superposition works like this: For equilibrium of the mirror, the edge forces must obey the relations

$$\begin{aligned} \int_0^{2\pi} f_1(\theta_1) \cos \theta_1 d\theta_1 + \int_0^{2\pi} f_2(\theta_2) \cos \theta_2 d\theta_2 &= W \\ \int_0^{2\pi} f_1(\theta_1) \sin \theta_1 d\theta_1 + \int_0^{2\pi} f_2(\theta_2) \sin \theta_2 d\theta_2 &= 0 \end{aligned}$$

in which W is the weight of the mirror. The equation below gives the deflection $\Delta\delta(\rho, \phi)$ due to an edge force $f_1(\theta_1)\Delta\theta_1$ resisting a gravity field $\Delta g_1 = gf_1(\theta_1)\Delta\theta_1/W$ at angle θ_1 with the vertical (see Fig. 3d):

$$\Delta\delta(\rho, \phi) = f_1(\theta_1)\Delta\theta_1\delta_1(\rho, \phi - \theta_1)/W$$

The next equation gives the deflection $\Delta\delta(\rho, \phi)$ due to an edge force $f_2(\theta_2)\Delta\theta_2$ resisting a gravity field $\Delta g_2 = gf_2(\theta_2)\Delta\theta_2/W$ at angle θ_2 with the vertical (see Fig. 3e):

$$\Delta\delta(\rho, \phi) = f_2(\theta_2)\Delta\theta_2\delta_2(\rho, \phi - \theta_2)/W$$

Figs. 3d and 3e. Mirror in vertical plane with infinitesimal portions of edge load and gravity.

The validity of superposition is established by considering $\int_0^{2\pi} (dg_1 + dg_2) = g$, acting vertically downward. Thus the deflection due to the arbitrary edge load is as follows:

$$\delta(\rho, \theta) = \frac{1}{W} \left[\int_0^{2\pi} f_1(\theta_1) \delta_1(\rho, \phi - \theta_1) d\theta_1 + \int_0^{2\pi} f_2(\theta_2) \delta_2(\rho, \phi - \theta_2) d\theta_2 \right]$$

It is interesting to note that any unsymmetrical support system may be analyzed this way even though δ_1 and δ_2 are symmetrical functions.

If the mirror surface is divided into 200 area elements, 10 radial and 20 angular, and the integral expression is stated in finite difference form, the Kitt Peak National Observatory CDC 3200 computer evaluates $\delta(\rho, \phi)$ at the 200 points in about 4 seconds. This is compared to about 30 minutes by dynamic relaxation, yet the results are about the same.

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