OPTICAL SCIENCES

THE UNIVERSITY OF ARIZONA
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Technical Report 29

MAXIMUM ATTAINABLE MTF
FOR ROTATIONALLY SYMMETRIC LENS SYSTEMS

B. Roy Frieden

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ABSTRACT

Most optical systems have rotational symmetry. For such systems, we establish a method of finding (a) the maximum attainable modulation transfer function (MTF) at arbitrary frequency ω_0 ; and (b) the required pupil function $U(\omega_0; \rho)$. Physically, the latter comprises absorbing films in the pupil of diffraction-limited optics. The method of solution is numerical and iterative, based on the Newton-Raphson algorithm. Solutions (a) and (b) are established at frequencies $\omega_0 = 0.1, 0.2, \dots, 0.9$ ($\times$ optical cutoff). The computed (a) are correct to ± 0.0001 over all ω_0 indicated. Quantities (b) have an average error over each pupil of ± 0.002 for frequencies $\omega_0 \leq 0.5$. With $0.6 \leq \omega_0 \leq 0.9$, the error is 0.01. The curve of maximum $MTF(\omega_0)$ seems smooth enough to allow for accurate interpolation. Solutions (a) and (b) were also found over the finer subdivision $\omega_0 = 0.05, 0.1, 0.15, \dots, 0.8$ with slightly less accuracy than above, in order to allow for interpolation of pupils $U(\omega_0; \rho)$ over values ω_0 . This seems possible for $0.05 \leq \omega_0 \leq 0.40$. The maximum $MTF(\omega_0)$ shows appreciable gain (e.g., 8% at $\omega_0 = 0.2$) over the MTF for uncoated, diffraction-limited optics at all ω_0 except in the intermediate region $0.4 \leq \omega_0 \leq 0.6$. In the high-frequency band $0.5 \leq \omega_0 \leq 1.0$, however, the maximum $MTF(\omega_0)$ shows little gain over the MTF due to an uncoated, diffraction-limited pupil with the proper central obscuration. The light loss due to each $U(\omega_0; \rho)$ may be measured by the total energy transmission and the Strehl flux ratio. These are plotted against ω_0 , and indicate moderate light loss.

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BACKGROUND

The image quality of an optical system is often measured by the modulation transfer function (MTF). Most optical systems have rotational symmetry. It is important, then, to establish how large the MTF may be at a given spatial frequency ω_0 , for the class of rotationally symmetric optical systems.

Aside from its use as a criterion of image quality, the curve of maximum $\text{MTF}(\omega_0)$ may also be used for optimizing the detection of particles of a single spatial extent $2\pi/\omega_0$.¹ And, by analogy, the solution has immediate application to antenna theory.

It has already been proved² that $\text{MTF}(\omega_0)$, ω_0 arbitrary, is maximized by a purely real pupil function. Physically, this is an absorbing film in the exit pupil of a diffraction-limited lens system. Hence, the problem is immediately simplified to consideration of diffraction-limited lens systems.

MacDonald³ analyzed the more general problem where lens system and pupil coating are not constrained to have rotational symmetry, and where aberrations are generally present. Unfortunately, this problem was not analytically solvable. Instead, the required pupil function and the maximized MTF were shown to obey an integral equation whose solution could be approximated at small frequencies ω_0 . In this way, an asymptotic solution for $\omega_0 \rightarrow 0$ was found for our specific problem. However, no estimate was made of the maximum value of ω_0 for which the solution is valid, and in fact we have found this maximum value to be only about 0.1 times the optical cutoff frequency.

Other workers^{4,5} have found the maximum $\text{MTF}(\omega_0)$, ω_0 arbitrary, without the constraint of rotational symmetry. However, not only are the required pupil functions not rotationally symmetric, but they also cannot be found by the given method.

The aim of this paper is to numerically generate the maximum MTF, and the required pupil function U , at a subdivision of values ω_0 throughout the optical passband. Since a different pupil U is required at each ω_0 , we use the double-argument notation $U(\omega_0; \rho)$, ρ being the "reduced" pupil radius.⁶ The modulation transfer at arbitrary frequency ω due to pupil $U(\omega_0; \rho)$ is denoted as $MTF(\omega_0; \omega)$. Solutions were checked by substitution into the formal solution, Eq. (7a).

METHOD OF SOLUTION

Unknown pupil $U(\omega_0; \rho)$ is bounded by both inside and outside radii, ρ_i and ρ_o , respectively. Radius ρ_o is simply the finite radius associated with the numerical aperture. Radius ρ_i defines a central obscuration which is found next.

It is well known⁶ that $MTF(\omega_0)$ is the normalized autocorrelation of $U(\omega_0; \rho)$. We consider the case $\omega_0 > \rho_o$. In the autocorrelation expression, points $\rho < (\omega_0 - \rho_o)$ contribute amounts $[U(\omega_0; \rho)]^2$ additively to the integrand in the denominator, but not at all to the numerator. Thus, for any non-zero $U(\omega_0; \rho)$, the effect is to decrease $MTF(\omega_0)$. Since, by hypothesis, $U(\omega_0; \rho)$ results in a maximum $MTF(\omega_0)$, it follows that

$$U(\omega_0; \rho) = 0 \quad \text{for } \rho < (\omega_0 - \rho_o) \quad (1a)$$

if

$$\omega_0 > \rho_o . \quad (1b)$$

Recalling that $2\rho_o$ is the optical cutoff frequency,⁶ constraint (1b) restricts attention to frequencies greater than half of the optical cutoff. From Eqs. (1), central obscuration

$$\rho_i = \begin{cases} \omega_0 - \rho_o & \text{if } \omega_0 > \rho_o \\ 0 & \text{if } \omega_0 \leq \rho_o \end{cases} . \quad (2a)$$

Since $U(\omega_0; \rho)$ is non-zero over a limited interval, it may be represented by a Fourier-Bessel series:

$$U(\omega_0; \rho) = \begin{cases} \sum_{m=0}^{M-1} b_m J_0(\nu_m \rho) & \text{for } \rho_i \leq \rho \leq \rho_o , \\ 0 & \text{for } \rho < \rho_i \text{ and } \rho > \rho_o . \end{cases} \quad (2b)$$

For each ω_0 we can expect a different function $U(\omega_0; \rho)$, so that in general $b_m = b_m(\omega_0)$.

In Eq. (2b), the μ_m are the zeros

$$J_1(\mu_m \rho_0) = 0, \quad m = 1, 2, \dots, M-1$$

and (2c)

$$\mu_0 = 0.$$

(Note that if instead of zeros (2c) we use the more common requirement $J_0(\mu_m \rho_0) = 0$, then no matter what b_m are used, $U(\omega_0; \rho_0) = 0$ identically. This would be an artificial constraint on solutions $U(\omega_0; \rho)$, whose behavior as $\rho \rightarrow \rho_0$ is at this point unknown.)

The b_m of Eq. (2b) which maximize $MTF(\omega_0)$ are to be found. For brevity, we denote any set

$$b_0, b_1, \dots, b_{M-1} \text{ by } \vec{b}, \quad (3)$$

and the solution set by $\vec{B}$. Since $MTF(\omega_0)$ is maximized by set $\vec{B}$, then

$$\left. \partial MTF(\omega_0) / \partial b_m \right|_{\vec{b}=\vec{B}} = 0, \quad m = 0, 1, \dots, M-1. \quad (4)$$

In order to find the $\vec{B}$, we use the "merit function" approach described previously.² This method is applied to the present problem as follows:

First, form a merit function:

$$W = W(\vec{b}) \equiv \sum_{m=0}^{M-1} [\partial MTF(\omega_0) / \partial b_m]^2. \quad (5)$$

Then the M conditions (4) are satisfied if and only if $W = 0$. In this manner, M requirements are replaced by a single requirement, greatly simplifying the problem.

We iteratively reduce W to zero, and hence determine $\vec{B}$, by the following Newton-Raphson algorithm: Arbitrarily choose M and a set $\vec{b}$. Compute $MTF(\omega_0)$ and the $\partial MTF(\omega_0)/\partial b_m$, with $m = 0, 1, \dots, M-1$. Compute W and the $\partial W/\partial b_m$, $m = 0, 1, \dots, M-1$. Compute $\Delta \vec{b}$, where

$$\Delta b_m \equiv - \frac{W \cdot \partial W / \partial b_m}{\sum_{n=0}^M (\partial W / \partial b_n)^2}, \quad m = 0, 1, \dots, M-1. \quad (6)$$

Form a new set of $\vec{b} = \text{old } \vec{b} + \Delta \vec{b}$. Recycle. Stop the calculation when two successive values of $MTF(\omega_0)$ differ by less than 10^{-7} (typically).

The above quantities are computed as follows. The Fraunhofer approximation⁶ is used to find the point amplitude response at radial sampling points.⁷ Next, $MTF(\omega_0)$ is computed from the point amplitude response by use of the radial sampling theorem.⁷ Each $\partial MTF(\omega_0)/\partial b_m$ is evaluated by analytic differentiation of the sampling expression for $MTF(\omega_0)$. This allows W to be computed from Eq. (5). Finally, the $\partial W/\partial b_n$ are computed by analytic differentiation of Eq. (5); the mixed partials $\partial^2 MTF(\omega_0)/\partial b_n \partial b_m$ resulting from this differentiation are evaluated from the sampling expression for $MTF(\omega_0)$.

All of the preceding operations were evaluated on a CDC 6400 electronic computer. In a typical run, $MTF(\omega_0)$ was maximized at a fine subdivision of values $\omega_0 = 0.05, 0.1, 0.15, \dots, 0.9$ ($\times$ cutoff $2\rho_0$). This allowed the set $\vec{B}$ for $\omega_0 = 0.05$ to be used as a good initial set $\vec{b}$ for $\omega_0 = 0.1$, etc. Such interlacing of solutions tends to minimize the required number of iterations for convergence at each ω_0 .

ACCURACY

The formal solution to this problem obeys an integral equation:

$$\int_0^\theta d\phi U(\omega_0; \rho') = \pi \text{MTF}(\omega_0) U(\omega_0; \rho) \quad (7a)$$

where

$$\rho' = (\rho^2 + \omega_0^2 - 2\rho\omega_0 \cos \phi)^{1/2} \quad (7b)$$

and

$$\theta = \begin{cases} \cos^{-1} [(2\rho\omega_0)^{-1}(\rho^2 + \omega_0^2 - \rho_0^2)] & \text{for } \rho \geq (\rho_0 - \omega_0) \\ \pi & \text{for } \rho \leq (\rho_0 - \omega_0) \end{cases} \quad (7c)$$

This may be established by conventional use of the calculus of variations (and is similar to Eq. (3.8) of MacDonald³).

Each approximate solution $U(\omega_0; \rho)$, $\text{MTF}(\omega_0)$ was checked by substitution into Eqs. (7). The integral in Eq. (7a) is evaluated by the Gauss-Lobatto scheme.⁸

Only in the limit $M \rightarrow \infty$ can series (2b) represent an arbitrary function. Thus, for M finite, Eq. (2b) can only approximate the true $U(\omega_0; \rho)$. Hence, as larger values of M are used, equality (7a) should be more accurately satisfied by our solutions. This was the case.

The largest value of M permitted by the storage limitations of our computer is $M = 50$. For this case, and with $\omega_0 \leq 0.5$, equality (7a) is satisfied by our solutions to 0.2%, on the average, over the pupil. However, for each ω_0 between 0.6 and 0.9 the agreement is not so good, averaging over the pupil to about 1.0%. This error appears to be mainly in the approximation to $U(\omega_0; \rho)$, since the latter suffers about 1.0% fluctuations from an average curve in each case. (See Fig. 1 on page 8.)

The maximized values for $MTF(\omega_0)$ using $M = 50$ are believed accurate to 0.0001 at all $0.1 \leq \omega_0 \leq 0.9$. This estimate is based upon two observations. First, at any ω_0 the solutions $MTF(\omega_0)$ for values of $M = 5, 10, 20$, and 50 are observed to rapidly saturate with increasing M . The change in MTF from $M = 10$ to $M = 50$ is, for any ω_0 , at most in the 4th decimal place. Hence, significantly higher values of MTF should not occur for values $M \geq 50$. The second observation is that the 1% fluctuations in U described previously do not seem to affect the MTF significantly. For fixed M and ω_0 , with $\omega_0 > 0.5$, successive runs of the computer yield solutions $U(\omega_0; \rho)$ which differ at any ρ by up to 1%, but which yield values $MTF(\omega_0)$ that differ only in the 5th decimal place! Such insensitivity in MTF to pupil U is probably because the MTF is the autocorrelation of U ,⁶ and this operation is insensitive to local fluctuations in U .

NUMERICAL SOLUTIONS

We show results for the case $M = 50$, i.e., a 50-term Fourier-Bessel expansion for $U(\omega_0; \rho)$. Fig. 1 shows optimal pupil $U(\omega_0; \rho)$ (solid curve) and the resulting MTF($\omega_0; \omega$) (dashed curve) for subdivision of frequencies $\omega_0 = 0.1, 0.2, \dots, 0.9 \times 2\rho_0$ (optical cutoff).

Fig. 1. Optimal pupil $U(\omega_0; \rho)$ (solid) and resulting MTF($\omega_0; \omega$) (dashed) for subdivision of frequencies ω_0 . Dotted curves are the MTF due to uncoated, circular, diffraction-limited optics.

In Fig. 1, note the smooth behavior of curves U for $\omega_0 \leq 0.5$, as contrasted with the somewhat erratic behavior of U for $\omega_0 > 0.5$. The latter effect is believed to be about 1% erroneous, as discussed above; the true curve U for such ω_0 is a smoothed average of the fluctuations.

Curves U are observed to behave as classic "apodizers"⁹ for values $\omega_0 \leq 0.5$ although they do not fall to zero at the pupil rim. This corroborates the discussion following Eq. (2c). For values $\omega_0 > 0.5$, curves U behave as "superresolvers,"⁹ most of the enhancement of $\text{MTF}(\omega_0)$ being due to the central obscuration shown.

The dotted curve in each box of the figure represents the MTF due to uncoated, circular, diffraction-limited optics.⁶ Comparison of the dashed and dotted curve for each ω_0 shows that pupil U enhances a region of the MTF in the vicinity of ω_0 but attenuates the MTF almost everywhere else. However, the enhanced region can be appreciable, for example region $0 \leq \omega \leq 0.35$ for the case $\omega_0 = 0.1$.

Fig. 2 (next page) shows five MTF curves: (1) our solutions for the maximum $\text{MTF}(\omega_0)$, based on values for frequencies $\omega_0 = 0.1, 0.2, \dots, 0.9 \times \text{cutoff}$, (2) maximum attainable $\text{MTF}(\omega_0)$ without constraint of rotational symmetry, as found by Lukosz,⁴ when there is no requirement of rotational symmetry, (3) maximum $\text{MTF}(\omega_0)$ for small ω_0 , due to MacDonald,³ (4) MTF due to uncoated, circular, diffraction-limited optics,⁶ and (5) MTF, for frequencies $\omega_0 > \rho_0$, due to circular, diffraction-limited optics, with a central obscuration ρ_i given by Eq. (2a).

Comparison of curves (1) and (2) shows how the constraint of circular symmetry affects the maximized MTF, a strong effect for nearly all frequencies.

Comparison of curves (1) and (3) shows the breakdown of MacDonald's asymptotic solution³ as ω_0 exceeds 0.1 of cutoff.

Comparison of curves (1) and (4) measures the enhancement in $MTF(\omega_0)$ due to $U(\omega_0; \rho)$, over $MTF(\omega_0)$ for an uncoated pupil. We see there is significant gain in the MTF at all ω_0 , except for $\omega_0 \approx 0.5 \times \text{cutoff}$. For example, at $\omega_0 = 0.2$ there is an 8% gain of contrast over the uncoated case.

Finally, comparison of curves (1) and (5) shows that for frequencies $\omega_0 > 0.5$ a simple obscuration of radius $\omega_0 - \rho_0$ is nearly equivalent to use of the optimal pupil $U(\omega_0; \rho)$. Hence, for $\omega_0 > 0.5$ it is the central obscuration of $U(\omega_0; \rho)$, and not the coating in the ring beyond the obscuration, that is the major factor in maximizing $MTF(\omega_0)$.

Fig. 2. Maximum $MTF(\omega_0)$, curve (1), compared with other relevant MTF curves:
 (2) Maximum attainable MTF, due to Lukosz;
 (3) Asymptotic solution, for small ω_0 , due to MacDonald;
 (4) MTF due to uncoated, circular, diffraction-limited optics;
 (5) For $\omega_0 > 0.5$, MTF due to preceding case with added central obscuration of radius $\omega_0 - \rho_0$.

Table I shows, to four places, the maximized $\text{MTF}(\omega_0)$ used to establish curve (1) in Fig. 2. Maximum $\text{MTF}(\omega_0)$ is shown for $M = 50$; estimated error (departure from case $M \rightarrow \infty$) is no more than ± 0.0001 at any ω_0 shown. For numerical comparison, values are shown for uncoated, diffraction-limited optics and for a simple obscuration in the uncoated pupil in the high frequency band $0.5 \leq \omega_0 \leq 0.9$. Figures in the first and third MTF columns are nearly identical, indicating that coatings $U(\omega_0; \rho)$ in the annulus $(\omega_0 - \rho_0) \leq \rho \leq 2\rho_0$ have little effect at these high frequencies.

Table I. Numerical comparison of MTF's for values of ω_0 .

ω_0	Modulation transfer function		
	Maximum	Uncoated	Obscuration
.1	.9502	.8729	--
.2	.8310	.7471	--
.3	.6819	.6238	--
.4	.5324	.5046	--
.5	.3937	.3910	.3910
.6	.3008	.2848	.2966
.7	.2317	.1881	.2240
.8	.1719	.1041	.1626
.9	.1118	.0374	.1039

It is remarkable that the maximum $\text{MTF}(\omega_0)$ for the case $M = 5$ are nearly equivalent to use of $M = 50$. For $0 \leq \omega_0 \leq 0.8$, corresponding values of $\text{MTF}(\omega_0)$ differ by a maximum of 0.0003 (with the value for $M = 5$ always the smaller, as is expected). The same is not true, however, in comparison of pupils $U(\omega_0; \rho)$, indicating once more than $\text{MTF}(\omega_0)$, at any ω_0 , is very nearly maximized by a definite variety of pupils.

Table II shows coefficients $\vec{B}$ and the maximum MTF, for $M = 5$, at fine subdivisions of frequencies $\omega_0 = .05, .1, .15, \dots, .8 \times \text{cutoff}$. Coefficients $\vec{B}$

may be used to generate new pupils $U(\omega_0; \rho)$, through Eqs. (2). It appears possible to use these coefficients $\vec{B}$ as the basis for interpolation to functions $U(\omega_0; \rho)$ for intermediate values of ω_0 in the range $.05 \leq \omega_0 \leq 0.4$. Unfortunately, we see that the $\vec{B}$, and hence the form of $U(\omega_0; \rho)$, change rapidly once ω_0 exceeds 0.4. Values of maximum MTF differ from those for $M = 50$ by, at most, 0.0003 (compare Table I).

Table II. Solution $\vec{B}$ and maximum MTF, for case $M = 5$, at frequencies ω_0 .

ω_0	B_0	B_1	B_2	B_3	B_4	Max MTF
.05	.490344	.696905	-.231833	.110003	-.068225	.986153
.1	.501649	.641526	-.181176	.089106	-.051105	.949915
.15	.527091	.595254	-.160472	.070897	-.032770	.896416
.2	.553007	.548009	-.135314	.044111	-.009813	.830711
.25	.602298	.515993	-.105424	.009002	-.022490	.757641
.3	.628942	.454993	-.056707	-.015103	-.093359	.681622
.35	.627035	.370155	.008824	.064798	-.070811	.605045
.4	.539544	.238343	.082410	.147859	-.008157	.532070
.45	.568086	.171132	.101969	.120266	.038547	.460658
.5	.875260	.147570	-.054467	.032063	-.000426	.393719
.55	.906828	-.043883	-.287984	-.027528	-.053450	.342231
.6	.863444	-.288644	-.472309	-.075748	-.108735	.300710
.65	.803039	-.523541	-.461345	.094217	-.015185	.264265
.7	.726909	-.762350	-.341965	.273084	-.045362	.231507
.75	.636197	-.948244	-.073948	.294900	-.261136	.200988
.8	.518551	-1.095873	.370139	.137939	-.569494	.171737

ILLUMINATION LOSSES

Since each pupil $U(\omega_0; \rho)$ is purely real, it absorbs a fraction of the incident radiation. In addition, the central obscuration of radius $\omega_0 - \rho_0$, for cases $\omega_0 > 0.5$, must cause further light losses.

Two well-known parameters that measure light loss through a pupil are the total radiant power transmittance T , and the Strehl⁶ flux ratio S . It is fortunate that representation (2b) for $U(\omega_0; \rho)$ allows both T and S to be readily computed for any coefficients $\vec{b}$.

A formula for S was derived by substituting representation (2b) into the Fraunhofer formula⁶ for point amplitude, evaluating this expression at zero radius, and squaring the result. Division by the same quantity for the pupil $U(\rho) = 1$ then produces S .

Parameter T was evaluated by squaring expression (2b) and integrating

$$2\pi \int_{\rho_i}^{\rho_0} d\rho \rho .$$

Division by pupil area $\pi\rho_0^2$ then produces T .

These formulas were programmed on the CDC 6400 computer and were evaluated for each solution $\vec{B}(\omega_0)$. Fig. 3 (next page) shows results for the case $M = 5$ (i.e., based on coefficients $\vec{B}$ given in Table II). Results for the solutions with $M = 50$ were similar, with a maximum difference between corresponding T values, or S values, of 0.03. The illustrated T and S curves indicate rapid changes in solution $U(\omega_0; \rho)$ over the band of frequencies 0.3 to 0.4 and especially over the band 0.4 to 0.5. This correlates with the behavior of coefficients $\vec{B}$ in Table II.

Most important, we note that the light losses manifest by T and S are not severe; e.g., T is greater than 0.3 over the entire range of ω_0 plotted.

Finally, we observe that at ω_0 for which T and S are maximal, the advantage of the maximum $\text{MTF}(\omega_0)$ over the uncoated $\text{MTF}(\omega_0)$ (curve (1) over curve (4) in Fig. 2) is minimal, and vice versa. In short, there is a tradeoff between illumination and contrast.

Fig. 3. Light loss for each pupil $U(\omega_0; \rho)$ with $M=5$, as measured by the total pupil power transmittance T , and the Strehl flux ratio S . Curves T and S for the case $M=50$ differ from the above by a maximum of 0.03 at any ω_0 .

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