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Citation: **1786**, 070009 (2016); doi: 10.1063/1.4967585

View online: <http://dx.doi.org/10.1063/1.4967585>

View Table of Contents: <http://aip.scitation.org/toc/apc/1786/1>

Published by the [American Institute of Physics](#)

Poiseuille Channel Flow by Adding and Doubling

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Abstract. The flow of a rarefied gas between two plates of a micro channel, commonly called Poiseuille flow, has been extensively studied in the BGK approximation. In particular, the Analytical Discrete Ordinates (ADO) method determines the velocity profile and flow rate efficiently and accurately. Here, an equally efficient and precise solution is presented using the method of doubling, which however is arguably a numerically simpler and a more natural approach avoiding eigenvalues and eigenvectors. Highly accurate benchmark results for Poiseuille flow are reported for use as a benchmark.

INTRODUCTION

Poiseuille flow is one of, if not, the simplest of all RGD flows. It is molecular flow down a pipe or between plates as shown in Fig. 1. Here the flow is in the z -direction in a transversely infinite channel of height d . The flowing

FIGURE 1. Poiseuille micro channel flow.

molecules slip along the upper and lower channel plates to varying degrees. The accommodation coefficient α is a flow parameter indicating the strength of slip at the wall. The fundamental assumption for Poiseuille flow is a long wide channel with laminar flow down the channel as described by the BGK scattering model approximation. The challenge to Poiseuille flow is to determine the molecular distribution in the x -direction perpendicular to the flow and the corresponding flow rate.

The most straightforward treatment for Poiseuille flow is to linearize the density distribution function as

$$f(x, c) = f_0(x, c)[1 + h(x, c)] \quad (1)$$

where $f_0(x, c)$ is a Maxwellian and $h(x, c)$ is a perturbation from the Maxwellian to be determined from the linearized Boltzmann equation

$$kc_z + c_x \frac{\partial}{\partial x} h(x, c) = Lh(x, c), \quad (2a)$$

where L is the BGK operator¹ and

$$k \equiv \frac{1}{P} \frac{\partial P}{\partial x} \quad (2b)$$

is the pressure gradient driving the flow. The analysis continues with model reduction through integration and the definition of the moment

$$Z(x, c_x) = \pi^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(c_y^2 + c_z^2)} c_z h(x, c_x, c_y, c_z) dc_y dc_z \quad (3a)$$

leading to

$$\frac{1}{2} k \theta + \theta c_x \frac{\partial}{\partial x} Z(x, c_x) + Z(x, c_x) = \pi^{-1/2} \int_{-\infty}^{\infty} e^{-u^2} Z(x, u) du \quad (3b)$$

for $x \in (-d/2, d/2)$ where θ is the mean free path of molecular scattering. The boundary conditions are

$$Z(-d/2, c_x) = (1 - \alpha) Z(-d/2, -c_x) \quad (3c)$$

$$Z(d/2, -c_x) = (1 - \alpha) Z(d/2, c_x) \quad (3d)$$

where α is the accommodation coefficient specifying the degree of slip satisfying $0 \leq \alpha \leq 1$. On further change of variables $\tau = x/\theta$, $\mu = c_x$, we find

$$\frac{1}{2} k \theta + \mu \frac{\partial}{\partial \tau} Z(\tau, \mu) + Z(\tau, \mu) = \pi^{-1/2} \int_{-\infty}^{\infty} e^{-u^2} Z(\tau, u) du \quad (4a)$$

with $\tau \in (-\delta/2, \delta/2)$, $\mu \in (-\infty, \infty)$ and boundary conditions

$$Z(-\delta/2, \mu) = (1 - \alpha) Z(-\delta/2, -\mu) \quad (4b)$$

$$Z(\delta/2, -\mu) = (1 - \alpha) Z(\delta/2, \mu) \quad (4c)$$

where

$$\delta \equiv \frac{d}{\theta} \equiv 2a \quad (4d)$$

is the inverse of the Knudsen number.

The equation we finally solve comes with the substitution

$$Z(\tau, \mu) = \frac{1}{2} k \theta [\tau^2 - 2\tau\mu + 2\mu^2 - a^2 - 2Y(\tau, \mu)] \quad (5)$$

to give

$$\left[\mu \frac{\partial}{\partial \tau} + 1 \right] Y(\tau, \mu) = \int_{-\infty}^{\infty} d\mu' \Psi(\mu') Y(\tau, \mu') \quad (6a)$$

where $\mu \in (-\infty, \infty)$, $\tau \in (0, a)$, and the boundary conditions are

$$\begin{aligned}
Y(0, \mu) &= Y(0, -\mu) \\
Y(a, -\mu) &= (1 - \alpha)Y(a, \mu) + g(\mu) \\
g(\mu) &\equiv \alpha\mu^2 + a(2 - \alpha)\mu
\end{aligned} \tag{6b,c,d}$$

with $\mu \in [0, \infty)$ and $\Psi(\mu) \equiv \pi^{-1/2} e^{-\mu^2}$.

THEORY OF ADDING AND DOUBLING

There have been several different approaches to solve Eqs(6) including the variational method of Cercignani and Loranani¹, Elementary solutions by Loyalka², et. al., Fourier transform by Boffi³, et. al. and the FN method of Siewert, Garcia and Barichello⁴. All these approaches involve relatively intricate mathematical constructions as well as elaborate numerical methods. What is proposed here is more fundamental and natural, yet provides as accurate, or more accurate results than the precise methods existing today.

Discrete Ordinates Balance Equation

In this section, we derive the discrete ordinates balance equation by specifying a more familiar form of Eq(6a) through the variable u , $0 \leq u \leq 1$, and introducing the result into Eq(6a)

$$\left[\mu(u) \frac{\partial}{\partial \tau} + 1 \right] Y(\tau, \mu(u)) = \int_{-\infty}^{\infty} d\mu' \Psi(\mu(u')) Y(\tau, \mu(u')), \tag{7}$$

with dependence on u and u' made explicit. One can then partition the integral into its half range integrations

$$\int_{-\infty}^{\infty} d\mu' \Psi(\mu') Y(\tau, \mu') = \int_0^1 du' \frac{d\mu'(u')}{du'} \Psi(\mu'(u')) \left[\begin{array}{l} Y(\tau, \mu'(u')) + \\ + Y(\tau, -\mu'(u')) \end{array} \right]. \tag{8}$$

Therefore, for $\mu \in (0, \infty)^4$, and

$$u \equiv \frac{1}{1 + \mu}, \tag{9a}$$

we have

$$\int_{-\infty}^{\infty} d\mu' \Psi(\mu') Y(\tau, \mu') = \int_0^1 du' u'^{-2} \Psi(\mu'(u')) \left[\begin{array}{l} Y(\tau, \mu'(u')) + \\ + Y(\tau, -\mu'(u')) \end{array} \right]. \tag{9b}$$

When we approximate the half-range integration by a Gauss quadrature of order N , there results

$$\int_{-\infty}^{\infty} d\mu' \Psi(\mu') Y(\tau, \mu') = \sum_{m'=0}^N \omega_{m'} \Psi_{m'} \left[Y_{m'}^+(\tau) + Y_{m'}^-(\tau) \right] + E_N(\tau), \tag{10a}$$

where

$$Y(\tau, \pm \mu_m) = Y_m^{\pm}(\tau) + \varepsilon_m(\tau). \tag{10b}$$

In the above expressions, $E_N(\tau)$ is the quadrature error and $\varepsilon_m(\tau)$ is the local error in Y .

The abscissae come from zeros of the Legendre polynomial of degree N

$$\begin{aligned} P_N(x_m) &= 0, \quad x_m \equiv 2u_m - 1, \quad m = 1, \dots, N \\ u_m &= \frac{1}{2}(1 + x_m) \end{aligned} \quad (11a,b)$$

giving the half-range ordinates for $\mu \in (-\infty, \infty)$

$$\mu_m = \frac{u_m}{1 - u_m}, \quad \mu_{N+m} = -u_m, \quad (11c)$$

and weights

$$\omega_m = \omega_{N+m} \equiv \frac{2}{(N+1)^2 [P_{N+1}(u_m)]^2} \frac{1}{u_m^2}. \quad (11d)$$

Introducing Eq(10a) into Eq(7), evaluating at $\pm\mu_m$ and dropping the error term gives the following the discrete ordinates approximation in the two half-ranges:

$$\begin{aligned} \left[\mu_m \frac{\partial}{\partial \tau} + 1 \right] Y_m^+(\tau) &= \sum_{m'=0}^N \omega_{m'} \Psi_{m'} [Y_{m'}^+(\tau) + Y_{m'}^-(\tau)] \\ \left[-\mu_m \frac{\partial}{\partial \tau} + 1 \right] Y_m^-(\tau) &= \sum_{m'=0}^N \omega_{m'} \Psi_{m'} [Y_{m'}^+(\tau) + Y_{m'}^-(\tau)]. \end{aligned} \quad (12a,b)$$

A more convenient form is found if one defines the vectors in the positive (+) and negative (−) half-ranges

$$Y^\pm(\tau) \equiv [Y_1^\pm(\tau) \quad Y_2^\pm(\tau) \quad \dots \quad Y_N^\pm(\tau)]^T \quad (13)$$

for which Eqs(12) becomes

$$\left[\frac{d}{d\tau} + \mathbf{M}^{-1}(\mathbf{I} - \mathbf{C}) \right] \mathbf{Y}^+(\tau) = \mathbf{M}^{-1} \mathbf{C} \mathbf{Y}^-(\tau) \quad (14a,b)$$

$$\begin{aligned} \left[-\frac{d}{d\tau} + \mathbf{M}^{-1}(\mathbf{I} - \mathbf{C}) \right] \mathbf{Y}^-(\tau) &= \mathbf{M}^{-1} \mathbf{C} \mathbf{Y}^+(\tau) \\ \mathbf{\Psi} &\equiv \text{diag} \{ \Psi_m; j = 1, \dots, J \} \\ \mathbf{W} &\equiv \text{diag} \{ \omega_m; j = 1, \dots, J \} \\ \mathbf{M} &\equiv \text{diag} \{ \mu_m; j = 1, \dots, J \} \\ \mathbf{C} &\equiv \mathbf{W} \mathbf{\Psi}. \end{aligned} \quad (14c,d)$$

From Eq(6b,c,d), the boundary conditions are

$$\begin{aligned}
Y_{N+m}(0) &= Y_m(0) \\
Y_m(a) &= (1-\alpha)Y_{N+m}(a) + g_{N+m} \\
g_{N+m} &\equiv \alpha\mu_{N+m}^2 + a(2-\alpha)\mu_{N+m}^2.
\end{aligned} \tag{14e,f,g}$$

Solution by Adding and Doubling

In this section, we will find the solution to Eqs(14) by adding and doubling.

If

$$\phi(\tau) \equiv \begin{bmatrix} Y^{T+}(\tau) & Y^{T-}(\tau) \end{bmatrix}^T, \tag{15a}$$

then

$$\frac{d\phi(\tau)}{d\tau} + A\phi(\tau) = 0, \tag{15b}$$

with

$$A = \begin{bmatrix} M^{-1}(I - C) & -M^{-1}C \\ M^{-1}C & -M^{-1}(I - C) \end{bmatrix}, \tag{15c}$$

where I is the (size N) identity matrix.

Assuming a uniform interval discretization h , the exact solution to Eq(15b) over the interval $[\tau_j, \tau_{j+1}]$ is

$$\phi_{j+1} = e^{-Ah} \phi_j. \tag{16a}$$

Certainly, one can evaluate the matrix exponential by finding the eigenvalues and eigenvectors of A , as is effectively done by the ADO method⁴, but this is unnecessary for the most convenient numerical result. All that is required is the Padé approximant

$$e^{-Ah} \simeq P^{*-1} P \tag{17a}$$

with

$$P^* \equiv [I + hA/2]^{-1} \tag{17b}$$

$$P \equiv [I - hA/2] \tag{17c}$$

to give the solution over the interval

$$\begin{bmatrix} Y_j^- \\ Y_{j+1}^+ \end{bmatrix} = R \begin{bmatrix} Y_{j+1}^- \\ Y_j^+ \end{bmatrix} \tag{18a}$$

where, after some algebra, the interval response matrix R is

$$\mathbf{R} = \begin{bmatrix} \alpha_+ - \beta & \alpha_+ - \beta \\ \alpha_+ + \beta & -(\alpha_+ + \beta) \end{bmatrix}^{-1} \begin{bmatrix} \alpha_- + \beta & \alpha_- + \beta \\ \alpha_- - \beta & -(\alpha_- - \beta) \end{bmatrix} \quad (18b)$$

with

$$\begin{aligned} \alpha_+ &\equiv I + \frac{h}{2} M^{-1} (I - C), \quad \beta \equiv \frac{h}{2} M^{-1} C \\ \alpha_- &\equiv I - \frac{h}{2} M^{-1} (I - C). \end{aligned} \quad (18c,d)$$

It is important to note that the RHS vector in Eq(18a) is what comes into the interval from either side, and the LHS vector is what leaves, or the response.

In addition, one can write

$$\begin{Bmatrix} T_n \\ R_f \end{Bmatrix} = \frac{1}{2} \left[(\alpha_+ - \beta_+)^{-1} (\alpha_- + \beta_-) \pm (\alpha_+ + \beta_+)^{-1} (\alpha_- - \beta_-) \right] \quad (19a)$$

and

$$\mathbf{R} = \begin{bmatrix} T_n & R_f \\ R_f & T_n \end{bmatrix}. \quad (19b)$$

where $\mathbf{R}_n$ and $\mathbf{T}_n$ are the interval reflectance and transmittance, which are naturally symmetric.

It is now possible to add two homogeneous media together to find a composite response. Figure 2

FIGURE 2. Concatenation of two homogeneous media.

shows the combination of two slabs of different responses. By eliminating the densities at the interface, one finds, with some algebra,

$$\begin{bmatrix} Y_0^- \\ Y_l^+ \end{bmatrix} = \mathbf{Q}_l \begin{bmatrix} Y_l^- \\ Y_0^+ \end{bmatrix}, \quad (20a)$$

where

$$\mathbf{Q}_l = \begin{bmatrix} \mathbf{0} & \mathbf{Q}_{l-1,1} \\ \mathbf{T}_n & \mathbf{0} \end{bmatrix} \mathbf{U}_{l-1} + \begin{bmatrix} \mathbf{0} & \mathbf{Q}_{l-1,2} \\ \mathbf{R}_f & \mathbf{0} \end{bmatrix} \quad (20b)$$

$$\mathbf{U}_{l-1} = \begin{bmatrix} \mathbf{Q}_{l-1,2} [\mathbf{I} - \mathbf{R}_f \mathbf{Q}_{l-1,2}]^{-1} \mathbf{T}_n & \mathbf{Q}_{l-1,1} + \mathbf{Q}_{l-1,2} [\mathbf{I} - \mathbf{R}_f \mathbf{Q}_{l-1,2}]^{-1} \mathbf{R}_f \mathbf{Q}_{l-1,1} \\ [\mathbf{I} - \mathbf{R}_f \mathbf{Q}_{l-1,2}]^{-1} \mathbf{T}_n & [\mathbf{I} - \mathbf{R}_f \mathbf{Q}_{l-1,2}]^{-1} \mathbf{R}_f \mathbf{Q}_{l-1,1} \end{bmatrix} \quad (20c)$$

The solution procedure is as follows:

- + Divide the channel into 2^n uniform intervals and call the intervals $l = 1, 2, \dots, n$.
- + Find the response $\mathbf{Q}_1$ for interval $l = 1$.
- + Combine two such intervals using Eq(20b) and (20c) to form $\mathbf{Q}_2$.
- + Now combine the last 2 intervals with the next 2 intervals and continue doubling to the l th interval to form the response $\mathbf{Q}_{2^l}$.
- + Eventually cover the entire channel to give the channel response $\mathbf{Q}(n) = \mathbf{Q}_{2^n}$.

Now Eq(20a) represents the solution over the channel

$$\begin{bmatrix} \mathbf{Y}_0^- \\ \mathbf{Y}_n^+ \end{bmatrix} = \mathbf{Q}_n \begin{bmatrix} \mathbf{Y}_n^- \\ \mathbf{Y}_0^+ \end{bmatrix}, \quad (21a)$$

where, for convenience

$$\mathbf{Q}_n \equiv \mathbf{Q}(n) \quad (21b)$$

and from Eqs(14e,f,g)

$$\begin{bmatrix} \mathbf{Y}_n^- \\ \mathbf{Y}_0^+ \end{bmatrix} = \begin{bmatrix} (1-\alpha)\mathbf{Y}_n^+ + \mathbf{g}^+ \\ \mathbf{Y}_0^- \end{bmatrix} = \begin{bmatrix} \mathbf{0} & (1-\alpha)\mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{Y}_0^- \\ \mathbf{Y}_n^+ \end{bmatrix} + \begin{bmatrix} \mathbf{g}^+ \\ \mathbf{0} \end{bmatrix}. \quad (21c)$$

When we introduce Eq(21c) into Eq(21a) and solve for the exiting vector, we find the solution

$$\begin{bmatrix} \mathbf{Y}_0^- \\ \mathbf{Y}_n^+ \end{bmatrix} = \left[\mathbf{I} - \mathbf{Q}_n \begin{bmatrix} \mathbf{0} & (1-\alpha)\mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \right]^{-1} \begin{bmatrix} \mathbf{g}^+ \\ \mathbf{0} \end{bmatrix} \quad (22)$$

where $\mathbf{g}^+ \equiv \mathbf{M} [\alpha \mathbf{M} + a(2-\alpha)\mathbf{I}] \mathbf{1}$.

Numerical Implementation and Demonstration

For this presentation, because of space limitation, it will suffice to determine only the channel flow rate to high precision.

After some manipulations, one can show the flow rate for a channel of width a is

$$\mathbf{Q}(a, \alpha; N, n) = B(a, \alpha) + \frac{2}{a^2 \sqrt{\pi}} [\Psi \mathbf{W}]^T [\alpha \mathbf{M}^3 + a(2-\alpha)\mathbf{M}^2] \mathbf{Y}_n^+ \quad (23a)$$

$$B(a, \alpha) \equiv \frac{a}{3} - \frac{1}{2a} + \frac{\alpha}{a} \left[\frac{3}{4} - \frac{2}{a\sqrt{\pi}} \right] - (2-\alpha) \left[\frac{3}{4a} - \frac{1}{\sqrt{\pi}} \right]. \quad (23b)$$

Note the dependence on the quadrature order N and channel discretization are explicitly included, since at this point, we have the solution as a function of the discretizations. For this reason, there is an opportunity for convergence acceleration⁶, which enables a high order evaluation of the following double limit:

$$Q(a, \alpha) = \lim_{N \rightarrow \infty} \lim_{n \rightarrow \infty} Q(a, \alpha; N, n) \quad (24a)$$

The idea is to use Wynn-epsilon in N and Richardsons convergence acceleration in n to form a surrogate sequence $\hat{Q}(a, \alpha; N, n)$, which, generally, converges faster to the limit of Eq(24) than the original sequence.

The surrogate sequence will usually satisfy

$$\frac{Q(a, \alpha) - \hat{Q}(a, \alpha; N, n)}{Q(a, \alpha) - Q(a, \alpha; N, n)} \rightarrow 0 \quad (24b)$$

indicating the numerator converges faster than the original sequence. Figure 3 shows the characteristic behavior of Q with decreasing Knudsen number¹ to fully developed flow.

Finally, Table 1 gives highly precise benchmark values to be used as a standard. The values are in complete agreement with those of Ref. 4 with two additional digits.

FIGURE 3. Flow variation with inverse Knudsen number.

Table 1. Benchmark values for flow rate.

α	0.5	0.8	0.88	0.96	1.00
5.000E-02	5.2232964E+00	3.0897113E+00	2.7383403E+00	2.4373544E+00	2.3022564E+00
1.000E-01	4.5564062E+00	2.7077408E+00	2.4060457E+00	2.1482414E+00	2.0327143E+00
3.000E-01	3.7784723E+00	2.2447708E+00	2.0010675E+00	1.7945088E+00	1.7024740E+00
5.000E-01	3.5443709E+00	2.1022657E+00	1.8766202E+00	1.6863424E+00	1.6018742E+00
7.000E-01	3.4376693E+00	2.0387670E+00	1.8220109E+00	1.6398495E+00	1.5591860E+00
9.000E-01	3.3838869E+00	2.0092408E+00	1.7976360E+00	1.6202230E+00	1.5417996E+00
1.000E+00	3.3682182E+00	2.0018669E+00	1.7920590E+00	1.6163124E+00	1.5386785E+00
2.000E+00	3.3765738E+00	2.0413852E+00	1.8385632E+00	1.6693655E+00	1.5948569E+00
5.000E+00	3.7744018E+00	2.4382339E+00	2.2350591E+00	2.0654781E+00	1.9907674E+00
7.000E+00	4.0881078E+00	2.7461124E+00	2.5414362E+00	2.3703751E+00	2.2949322E+00
9.000E+00	4.4101902E+00	3.0634644E+00	2.8575645E+00	2.6852950E+00	2.6092536E+00

CONCLUSION

The adding and doubling method is applied for the first time in RGD. The novelty of the approach is that no eigenvalues or eigenvectors are required. We only require quadrature and a Padé approximant. In spite of the method's simplicity, one finds highly precise flow rates.

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