Unitarity and Discrete Scale Invariance

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Abstract

While the complexity of some many-body systems may stem from a profusion of distinct scales, as we approach two-body unitarity (through experimental control or as a theoretical limit) rich structures exist even though there is no more than one essential scale. I comment, from the point of view of effective field theory, on some current problems in the transition from few to many bodies in bosonic and multi-state fermion systems, where order emerges from the discrete scale invariance associated with a single, contact three-body force.

1 Introduction

It is a trivial observation, of profound consequences, that nature is not scale invariant. Tracking how physics changes with resolution is the task of the renormalization group (RG), which underlies the paradigm of effective field theories (EFTs). In this paradigm, nature is described by a succession of EFTs, each one relevant in a certain energy regime. I focus here on the scales that appear in the EFTs for non-relativistic few-body systems with short-range interactions whose two-body subsystems are near unitarity. I will argue that only one scale is essential. At its 30th anniversary, *Few-Body Syst.* continues to provide a unique forum for discussions like this, which cut across the atomic, nuclear and other communities where one can hope to understand just *how* “more” becomes “different”.

Despite the obvious absence of scale invariance in nature as a whole, there have been frequent attempts to formulate one EFT or another using scale invariance as an approximate classical symmetry that is anomalous, that is, broken by quantum effects. The systems that I consider here lend themselves to such an attempt, because, by definition of “near”, the two-body dynamics near unitarity is approximately characterized by no scale other than the energy. Parameters that originate in the short-distance dynamics play only a minor role, which is amenable to perturbation theory. No finite-energy two-body bound state can exist. Yet, when the bodies are identical bosons and fermions with more than two spin and/or internal states, such as nucleons with spin 1/2 and isospin 1/2, more-body bound states exist with finite energy: continuous scale invariance is broken. In EFT, this breaking is anomalous: it is *required* by the RG at a quantum level in the three-body system [1, 2, 3]. But the breaking preserves *discrete* scale invariance, which implies a geometric tower of few-body bound states, as first discovered by Efimov in the case of three particles [4]. The lowest states of the Efimov tower have been seen near Feshbach resonances [5, 6] and in naturally occurring atomic ^4He [7, 8], while triton is the nuclear Efimov ground state.

I offer here a *Comment* on the growing effort to study the structure of systems with more particles near unitarity. A significant motivation for this enterprise is the promise of new, similarly striking discoveries. Another is the simplicity associated with universality. Existing evidence is that a single, three-body dimensional parameter is sufficient to describe the overall structure, with everything else, again, less important. This conclusion is not universally accepted but, if correct, it has far-reaching implications. For two-state fermions of mass m , it has long been realized that scale invariance means that energies are determined by the external constraints. For example, in the many-body system where one defines a Fermi momentum k_F , the energy can only be proportional to the free Fermi gas energy,

$$\lim_{N \rightarrow \infty} \frac{E_N}{N} \equiv \xi \frac{3k_F^2}{10m}. \quad (1)$$

ξ , known as the Bertsch parameter, is a pure, universal number that summarizes all the dynamics of this system. The sign of ξ , for example, determines the sign of the pressure. It makes for a meeting ground in the comparison of various approaches.

For bosons or multi-state fermions, I propose that similar focus be placed on the expression for N -body ground-state binding energies [9, 10, 11] that arises from the anomalous breaking of scale invariance,

$$B_N = \kappa_N B_3, \quad (2)$$

where κ_N are pure numbers ($\kappa_2 = 0$ and $\kappa_3 = 1$). As I discuss below, for small N it seems that

$$\kappa_N \approx (N - 2)^2. \quad (3)$$

I suggest that, if the growth slows down, the discrete scale invariance underlying Eq. (2) would imply, as an analog of Eq. (1), that

$$\lim_{N \rightarrow \infty} \frac{E_N}{N} = \frac{k_\rho^2}{m} \Upsilon \left[s_0 \ln \left(k_\rho / \sqrt{m B_3} \right) \right], \quad (4)$$

where k_ρ is the momentum associated with the density, s_0 is a constant, and Υ is a periodic function. What happens when N grows?

2 Unitarity and Scale Invariance

Non-relativistic particles interacting through a force of range R can be described at distances $r \gg R$ by a Hamiltonian containing contact interactions only. The interaction potential can be expanded in powers of R/r , starting with non-derivative contact interactions and continuing with successively more derivatives. It is important to realize that even if there is an underlying potential that is mostly two-body, its expansion at large distances will contain higher-body components, stemming from successive two-body encounters at short distances and times. In most situations these higher-body forces are relatively small, but not always. All contact interactions allowed by symmetries are non-vanishing. The problem for the theorist is to figure out which interactions are dominant.

One way to formulate the dynamics starts from the most general action consistent with the symmetries and built out of a field ψ that annihilates the particles of interest:

$$S = \int \frac{dt}{2m} \int d^3x \left\{ \psi^\dagger \left(2im \frac{\partial}{\partial t} + \nabla^2 \right) \psi - 4\pi C_0 (\psi^\dagger \psi)^2 - \frac{(4\pi)^2}{3} D_0 (\psi^\dagger \psi)^3 + \dots \right\}, \quad (5)$$

where C_0 , D_0 , *etc.* are interaction strengths or, in EFT jargon, low-energy constants (LECs). The “...” include terms with more derivatives. Note that I use units where $\hbar = c = 1$. Since, with the above definitions of the LECs¹, m enters Eq. (5) only in the combination t/m , it appears in observables together with the energy E as $mE = k^2$. As a consequence, binding energies $B \propto 1/m$, which is at the root of the alternative units frequently used in atomic physics, where one makes $m = 1$.

In writing Eq. (5) I neglected spin projections that are important when particles have spin. For a two-state fermion, the C_0 interaction operates only between two particles in

¹My choice here differs from the usual definition found in the literature.

different states, and the D_0 interaction vanishes because of the exclusion principle. For more-state fermions, more than one C_0 - and/or D_0 -type interaction is possible, depending on the symmetries. For simplicity of notation, I will ignore such complications in the following. I assume there are sufficiently many approximate symmetries in the space of fermion states so that, as for bosons, only a single D_0 -type interaction is important. This is, for example, the case for nucleons which, in addition to (presumably exact) spatial-rotation invariance, have an approximate invariance under rotations in isospin space — that is, rotations that make a nucleon a superposition of proton and neutron. In this case the isospin-breaking interaction is small and there is a single relevant D_0 -type interaction, which is invariant [3] under Wigner’s $SU(4)_W$ symmetry in spin-isospin space. (The approximate $SU(4)_W$ symmetry of this EFT was elaborated upon in Refs. [12, 13].) The discussion can be generalized to more complicated cases in a straightforward way.

An essential aspect of this expansion around the zero-range limit is that the resulting interactions are singular. They demand an arbitrary procedure that — at intermediate steps of the calculation of observables, that is, of the S matrix — regularizes the short-range behavior of amplitudes by the introduction of a parameter Λ^{-1} with dimensions of distance. The delta functions become smeared over distances $r \lesssim \Lambda^{-1}$, which I denote by $\delta_\Lambda^{(3)}(r_{ij})$. This change must be compensated by a dependence of interaction strengths on Λ , in such a way that observables are not sensitive to the arbitrary regularization procedure. This second step — renormalization — is crucial to ensure that no detailed assumptions — other than symmetries — are being made about the short-range dynamics.

For $N = 2$, the action (5) gives rise to a potential

$$V_2(r_{ij}; \Lambda) = \frac{4\pi}{m} C_0(\Lambda) \delta_\Lambda^{(3)}(r_{ij}) \quad (6)$$

among the two particles labeled i and j , separated by the distance r_{ij} . It can be shown [14, 15, 16, 17] that after renormalization the scattering amplitude is equivalent to the effective range expansion (ERE) [18], Fermi’s pseudopotential [19], and boundary conditions at short distances [20, 21]. (The equivalence to the ERE in dimensional regularization with a particular subtraction scheme was shown in Refs. [22, 23].) In particular, we can write $C_0(\Lambda)$ as

$$C_0(\Lambda) = -\frac{1}{\theta_0 \Lambda} \left[1 + \frac{1}{\theta_0 \Lambda a_2} + \mathcal{O}\left(\frac{1}{(\Lambda a_2)^2}\right) \right], \quad (7)$$

where a_2 is the two-body scattering length (in the appropriate S -wave channel, if more than one exists), and $\theta_0 = \mathcal{O}(1)$ depends on the specific regularization employed. By dimensional analysis we expect the effective range and higher ERE parameters to be set by the potential range R , $|r_2| \sim R$, *etc.* In general the same is true for the scattering length, but $|a_2| \gg R$ when there exists a shallow real or virtual S -wave bound state. The scattering amplitude for on-shell momentum $k \ll 1/R$ is

$$T_2(k) = \frac{4\pi}{m} (a_2^{-1} + ik)^{-1} [1 + \mathcal{O}(kR, k\Lambda^{-1})]. \quad (8)$$

There is a pole at a binding energy $B_2 = (ma_2^2)^{-1}$. The unitarity limit $a_2^{-1} \rightarrow 0$ (and $r_2 \rightarrow 0$, *etc.*) is an idealization of the region $|a_2^{-1}| \ll k \ll 1/R$, which I will refer to as

the “unitarity window”, where there is no important dimensionful parameter other than k itself. The unitarity limit corresponds to a non-trivial fixed point of the RG [24].

The vanishing of the binding energy in the unitarity limit is a reflection of scale invariance. Under a change of scales [25] with parameter $\alpha > 0$,

$$r \rightarrow \alpha r, \quad t/m \rightarrow \alpha^2 t/m, \quad \Lambda \rightarrow \alpha^{-1} \Lambda, \quad \psi \rightarrow \alpha^{-3/2} \psi, \quad (9)$$

the first two terms in Eq. (5) are invariant, but only when $a_2^{-1} \rightarrow 0$ in Eq. (7). Under a scale change, $mE \rightarrow \alpha^{-2} mE$, but in the unitarity limit there is no scale, so $B_2 = 0$. In this limit the $N = 2$ system is also conformally invariant [26].

Away from the unitarity limit, scale symmetry is explicitly broken by a dimensionful parameter, a_2 . We can use the “spurion field” method [27], which is designed to exploit the consequences of an approximate symmetry, to determine the dependence of B_2 on a_2 . The idea is that *if* under scale invariance a_2 changed to αa_2 , then the first two terms in Eq. (5) would be invariant. In that case, the energy after the transformation should equal the transformed energy: $B_2(\alpha a_2) = \alpha^{-2} B_2(a_2)$. This implies $B_2(a_2) \propto (m a_2^2)^{-1}$. Now, since a_2 is fixed, $B_2(a_2)$ represents the specific way a_2 breaks scale invariance. In this particular case the spurion method is just dimensional analysis, since by allowing a_2 to vary we are changing all dimensionful quantities according to their (inverse mass) dimension. Of course this relation can be obtained directly from Eq. (8), but the spurion method illustrates how considerations of symmetries underlie dynamical results.

For $N \geq 2$ two-state fermions, the potential is approximately

$$V = \sum_{\{ij\}} V_2(r_{ij}; \Lambda), \quad (10)$$

where $V_2(r_{ij}; \Lambda)$ is given by Eq. (6) and the sum extends over pairs of particles in different states. The N -body T matrix is properly renormalized without further interactions [14, 16]. At unitarity, scale invariance then ensures that there cannot be any finite, negative energy B_N . In order to bind the system, some sort of “external” interaction must be imposed, such as gravity in a neutron star. More simply, we can place the system in a cubic box of volume V . The corresponding breaking of scale invariance can be treated like the breaking from a_2 . The system now can have a ground state with energy [28]

$$E_N = \xi_N \left(\left(\frac{V}{3\pi^2 N a_2^3} \right)^{1/3} \right) \varepsilon_{FG} = \left[\xi_N(0) + \xi'_N(0) \left(\frac{V}{3\pi^2 N a_2^3} \right)^{1/3} + \dots \right] \varepsilon_{FG}, \quad (11)$$

where

$$\varepsilon_{FG} = \frac{3}{10m} \left(\frac{3\pi^2 N}{V} \right)^{2/3} \quad (12)$$

is the energy per particle of a free Fermi gas, and ξ_N is a dimensionless, universal function of its dimensionless argument, which I expanded within the unitarity window $R^3 \ll V/(3\pi^2 N) \ll |a_2|^3$. Monte Carlo calculations [29, 30, 31] with simple potentials give a nearly constant even-odd oscillation in N (a pairing gap) superimposed on a linear growth,

$$\xi_N(0) \approx N\xi, \quad \xi'_N(0) \approx -N\zeta, \quad \dots, \quad (13)$$

with $\xi \simeq 0.4$, $\zeta \simeq 1$, *etc.*. At unitarity, a large system described by the number density $\rho = N/V = k_F^3/(3\pi^2)$ does not collapse, and obeys Eq. (1). Away from unitarity, $\xi'_N(0) < 0$ means the energy increases as $-a_2^{-1}$ increases for $a_2 < 0$. Conversely, as $a_2^{-1} > 0$ increases the energy decreases and is consistent with that of a gas of spin-0 dimers interacting with a scattering length $a_{dd} \simeq 0.6a_2$ [32]. The $a_2 > 0$ Fermi gas is metastable due to dimer formation.

Obviously one can consider additional scales such as the effective range, temperature and polarization, leading to a dimensionless function of several dimensionless ratios and rich phase diagrams. For the effective range, for example, a generalization of the above argument leads to

$$\lim_{N \rightarrow \infty} \frac{E_N}{N} \equiv \frac{3k_F^2}{10m} \left(\xi - \frac{\zeta}{a_2 k_F} + \eta r_2 k_F + \dots \right), \quad (14)$$

where $\eta \simeq 0.1$ [33].

How relevant are these results for nuclear physics? Evidence was presented in Ref. [34] that the momenta relevant for the ground states of light nuclei are within the unitarity window of the 1S_0 channel, $|a_{1S_0}^{-1}| \ll k \ll r_{1S_0}^{-1}$, where $a_{1S_0} \simeq -23.7$ fm and $r_{1S_0} \simeq 2.73$ fm. Starting from unitarity in this channel, binding energies can be obtained in an expansion in both $(k a_{1S_0})^{-1}$ and $k r_{1S_0}$ — and for Z protons, also in $Z\alpha m_N/k$, where α is the fine-structure constant and $m_N \simeq 940$ MeV the nucleon mass, and other isospin-breaking corrections. Likewise, at finite density, Eq. (14) provides an approximation for neutron matter in the unitarity window (*cf.*, for example, Ref. [35].)

3 Three Bodies and Discrete Scale Invariance

Physics is quite different for multi-state fermions (such as protons and neutrons of spin up or down) or bosons (such as atomic ^4He), where the exclusion principle does not forbid three particles in the same state. The first surprise was probably the observation by Thomas [36] of a “collapse” in the $N = 3$ ground state: under a potential like (10) that gives rise to a shallow two-body (real or virtual) bound state, the ground-state binding energy $B_3 \propto \Lambda^2/m$ for $\Lambda \gg |a_2^{-1}|$. As Λ increases, excited bound states emerge and collapse as well. This is a consequence of a bizarre behavior of the half-off-shell amplitude for scattering of a particle on the two-particle bound state, which oscillates as a function of the off-shell momentum with a phase that depends on $\ln \Lambda$ [1, 2, 3]. Small cutoff variations result in large changes at low momentum. This regulator dependence indicates that the first two terms in Eq. (5) are not renormalizable beyond $N = 2$. Although they differ in detail, the situation is the same for multi-state fermions and bosons. For nucleons, for example, the coupled three-body integral equations for the half-off-shell amplitude reduce [3] at unitarity to an equation with a well-defined solution, plus an equation identical to that for bosons.

Since two-body interactions with more derivatives are smaller at low momentum, the appropriate counterterm must be a three-body force. Indeed, if we add in the potential

the non-derivative three-body force with parameter D_0 (the third term in Eq. (5)),

$$V = \sum_{\{ij\}} V_2(r_{ij}; \Lambda) + \sum_{\{ijk\}} V_3(r_{ijk}; \Lambda), \quad (15)$$

where $\{ijk\}$ stands for a sum over all triplets and

$$V_3(r_{ijk}; \Lambda) = \frac{(4\pi)^2}{m} D_0(\Lambda) \delta_\Lambda^{(3)}(r_{ij}) \delta_\Lambda^{(3)}(r_{jk}), \quad (16)$$

then $D_0(\Lambda)$ can exactly counterbalance the cutoff variation for $N = 3$ if [1, 2, 3]

$$D_0(\Lambda) \propto \frac{1}{\Lambda^4} \frac{\sin(s_0 \ln(\Lambda/\Lambda_\star) - \arctan s_0^{-1})}{\sin(s_0 \ln(\Lambda/\Lambda_\star) + \arctan s_0^{-1})} [1 + \mathcal{O}((a\Lambda)^{-1})], \quad (17)$$

where $s_0 \simeq 1.00624$ and $\Lambda_\star$ is a physical parameter, which is however only defined up to a factor $\exp(n_\star \pi/s_0)$, with $n_\star$ an integer. This is an RG limit cycle. Once one low-momentum datum is reproduced by a choice of $\Lambda_\star$, the phase of the half-off-shell scattering amplitude oscillation is fixed and all other low-momentum observables attain finite values as Λ increases. This is, in particular, true of bound states. Instead of the periodic emergence of bound states at zero energy before renormalization, after renormalization one observes the periodic emergence of *deeper* bound states, which achieve finite binding energies as Λ increases. At unitarity, in addition, the tower of states is geometric, extending down to threshold, with successive states having a ratio of binding energies $B_{3;n+1}/B_{3;n} = \exp(-2n\pi/s_0) \simeq 1/515$ [4]. At finite a_2 and R , only a few ($\sim \ln(|a_2|/R)/\pi$ [37]) of these states will be in the range of applicability of the theory. For atomic ^4He , for example, both the ground [7] and first-excited [8] states have been detected. For nucleons, only the ground state (triton/ ^3He) is observed, but one can show that there is a virtual state in neutron-deuteron (nd) scattering that becomes the triton excited state as the deuteron binding energy is decreased [38, 39]. Here and below, I label bound states with an integer $n \geq 0$, starting at the ground state, that is, the lowest state within the EFT.

Like in the two-body sector, derivative three-body interactions provide relatively small effects [2, 40, 41, 42, 43, 44]. Because the three-body force (17) contains a single parameter, one expects correlations among the observables that are sensitive to this force. Clearly that is not the case for every three-body observable; for example, the scattering length for nd scattering in the spin-3/2 channel, where the two neutrons are aligned, is determined to a very good accuracy by two-nucleon physics [14, 16]. But observables in channels not affected by the exclusion principle are sensitive to this one parameter. The classic example is the Phillips line [45]: a line in the plane spanned by the triton binding energy and the spin-1/2 nd scattering length. This correlation was first discovered empirically, as a line formed by points representing various phenomenological potentials, which describe two-nucleon data up to relatively high momenta. As $\Lambda_\star$ is varied, the EFT also produces a line [1, 2, 3], which lies close not only to the experimental point but also to the empirical line. This means that the various phenomenological potentials, each with its many parameters,

are equivalent to the same EFT with different values of $\Lambda_\star$. The conclusion that three-body systems at or near unitarity are determined by a single three-body force is confirmed.

The dimensionful parameter $\Lambda_\star$ arises from renormalization, even though at unitarity we start without any scale — an example of “dimensional transmutation”. Scale (as well as conformal) invariance is “anomalously” broken, but a discrete scale invariance [46] remains [1, 2, 3]. Because of the characteristic dependence on Λ in Eq. (17), the first *three* terms in Eq. (5) are invariant under a transformation (9), but only for discrete values

$$\alpha_l = e^{l\pi/s_0} \simeq (22.7)^l, \quad (18)$$

with l an integer. $\Lambda_\star$ offers a dimensionful scale upon which binding energies can be built in the unitarity limit. By dimensional analysis, we can write

$$B_{3;n} = \frac{\kappa_\star^2}{m} \beta_{3;n}((a_2\kappa_\star)^{-1}) = \frac{\kappa_\star^2}{m} \left[\beta_{3;n}(0) + \frac{\beta'_{3;n}(0)}{a_2\kappa_\star} + \dots \right], \quad \beta_{3;0}(0) = 1, \quad (19)$$

where $\beta_{3;n}((a_2\kappa_\star)^{-1})$ are universal, dimensionless functions and $\beta_{3;n}(0)$, $\beta'_{3;n}(0)$, *etc.* are pure numbers. Because $\Lambda_\star$ is only defined up to a factor, it was traded above by a fixed scale $\kappa_\star$ defined from the ground-state energy at unitarity:

$$s_0 \ln(\kappa_\star) = s_0 \ln(\beta \Lambda_\star) \pmod{\pi}, \quad (20)$$

with $\beta \simeq 0.383$ [47].

The energy of a bound state after a discrete scale transformation should equal the transformed energy but not necessarily of the same level, so that

$$\beta_{3;n+l}(0) = \alpha_l^{-2} \beta_{3;n}(0) \quad \Leftrightarrow \quad \beta_{3;n}(0) = e^{-2n\pi/s_0} \beta_{3;0}(0). \quad (21)$$

Thus discrete scale invariance leads to Efimov’s geometric tower. Using the spurion field method, the deviation from unitarity due to the two-body scattering length gives additionally

$$\beta_{3;n+l}((\alpha_l a_2 \kappa_\star)^{-1}) = \alpha_l^{-2} \beta_{3;n}((a_2 \kappa_\star)^{-1}). \quad (22)$$

This relation gives information about how Efimov’s tower evolves as $|a_2^{-1}|$ grows ². For example, taking a derivative and expanding in $(a_2 \kappa_\star)^{-1}$, we see the leading effect of tower deformation:

$$\beta'_{3;n+l}(0) = \alpha_l^{-1} \beta'_{3;n}(0) \quad \Leftrightarrow \quad \beta'_{3;n}(0) = e^{-n\pi/s_0} \beta'_{3;0}(0), \quad (23)$$

where $\beta'_{3;0}(0) \simeq 2.11$ [47]. Note that here the spurion method is not simply dimensional analysis because $\Lambda_\star$ is kept fixed. It instead tracks how the two-body scattering length explicitly breaks discrete scale invariance. Equation (22) is only valid to the extent that the three-body force retains discrete scale invariance except for $(a_2 \Lambda)^{-1}$ terms — that is, as long as Eq. (17) contains no $a_2 \Lambda_\star$ dependence, which would require in the spurion method that we scaled $\Lambda_\star$ as well.

Corrections in $r_2 \kappa_\star$ can be introduced as in Eq. (14). As long as Eq. (17) contains no $r_2 \Lambda_\star$ dependence, the coefficients of linear corrections should scale with α_l^{-3} [42], as can be easily verified with the spurion method. (However, an explicit calculation [42] says that these coefficients vanish.) Other corrections can be handled similarly.

²I thank H.-W. Hammer for stressing to me the importance of this relation.

4 More Bodies But No More Scales?

The crucial issue now is whether a new essential scale appears for $N \geq 4$. By “essential” I mean, not accountable for perturbatively: it is essential for the very formulation of the theory. A four-body force carrying a new dimensionful parameter, for example, is not forbidden by symmetries, thus it appears at some level of precision. The question is thus whether $N \geq 4$ systems are well defined with the potential (15), in the absence of higher-body forces. If so, higher-body forces and their scales can be accounted for in perturbation theory at higher orders. In this section I review the evidence which suggests that no new essential scales are needed for systems with short-range forces near unitarity. If that is the case, the spectrum of bound states is entirely fixed by $\Lambda_\star$, except for small corrections.

A many-body force is generated by the elimination of short-distance physics, and if it is an essential ingredient it will show up as did the three-body force (17): many-body observables will not be renormalized properly without such a force; instead, they will fail to converge as the cutoff Λ increases. If there is one regulator for which lack of convergence is seen, renormalizability is disproved. By now, there exist a few calculations with different regulators, which display no lack of convergence.

Let me start with $N = 4$. The pioneering calculations of Refs. [48, 49, 50] have found apparent convergence in the binding energies of the ground states for bosons and nucleons (away from unitarity), as well as of the first excited state for bosons. A separable Gaussian regulator was used with the cutoff varied over values that correspond to a full cycle of the three-body force. Such a result was confirmed *i)* with the same regulator in Ref. [51] (at unitarity), and *ii)* with a local Gaussian regulator (under a similar cutoff variation) for bosons (very close to unitarity) in Ref. [11] and for nucleons (away from unitarity) in Refs. [52, 53, 54]. The apparent convergence in these calculations implies a correlation between four- and three-body observables. The classic example is the Tjon line [55] in the plane of the four- and three-body ground-state energies, $B_{4;0}$ and $B_{3;0}$. As with the Phillips line, this correlation was discovered empirically by plotting results of phenomenological nuclear potentials. It also exists for ^4He atoms [56]. It materializes in EFT as a variation of $\Lambda_\star$ [48, 49, 50].

In contrast, in Refs. [57, 58, 59] a contact model suggested the emergence of a four-body scale. However, a subtraction procedure was employed which has no transparent connection to field theory. It is unclear at this point whether the new scale is essential in the sense defined above, or a consequence of an unessential subtraction. If the former possibility is true, one has no reason to expect a Tjon line. Nor would one expect the EFT Tjon line, which would presumably be cutoff dependent, to be close to the empirical Tjon line. The fact that it is means that the diverse physics of all the phenomenological potentials that form the Tjon line is captured by the single scale $\Lambda_\star$. There is at least a very large class of potentials that do not seem to have an extra, essential parameter. This result is confirmed by other potential-model calculations [60, 61, 62, 63] and an independent RG analysis [64]. However, one cannot, of course, exclude *a priori* the existence of an exotic system where a many-body force is very large.

This analysis has been extended to $N = 5, 6, 16$. A calculation [65] of the ground-state energy for $N = 6$ nucleons, with a cutoff provided by the number of shells in a harmonic-oscillator basis, converged without five- or six-body forces. Similarly, there was no sign of extraneous cutoff dependence for $N = 16$ nucleons with a local Gaussian regulator [54]. The four spin-isospin states of nucleons require five- and more-body forces to include derivatives, which should suppress them. This suppression is absent for bosons, but calculations [11] with a local Gaussian regulator showed no evidence of the need for those forces, either. The absence of higher-body forces at leading order implies the existence of correlations of five- and six-body observables with three-body observables. For example, variation of Λ_* creates “generalized Tjon lines” in the planes spanned by the ground-state energies $B_{5;0}$ or $B_{6;0}$ and $B_{3;0}$ [11]. Again, such correlations had already been discovered in the context of potential models [66, 67, 60].

A limitation in the renormalization argument for the absence of new essential scales has been the range of cutoff values used. This limitation ³ has been mitigated recently. As indicated in Eq. (8) for $N = 2$, one expects the fall off of observables with cutoff to be governed by the characteristic momentum of the process. One possible way to estimate the binding momentum per particle is

$$Q_N \equiv \sqrt{2mB_N/N}. \quad (24)$$

The rationale for this estimate is that Q_2 defined this way agrees with the location of the two-body bound state in the complex plane for the relative on-shell momentum; $Q_{N \gg 1}$ amounts to each particle contributing equally to the binding energy in the center of mass; and Q_∞ gives a finite momentum for a saturating system, such as nuclear matter. Thus we would expect for the n th state of the N -body system

$$B_{N;n}(\Lambda) = B_{N;n} \left[1 + \alpha_N \frac{Q_{N;n}}{\Lambda} + \beta_N \left(\frac{Q_{N;n}}{\Lambda} \right)^2 + \dots \right], \quad (25)$$

where the dimensionless parameters α_N , β_N , *etc.* are expected to be of $\mathcal{O}(1)$ if there is convergence to the asymptotic value $B_{N;n}$ (from which $Q_{N;n}$ is obtained via Eq. (24)). Results for the ground states of $N \leq 6$ ^4He atoms display exactly such a behavior [11]. Thus, within the range of cutoffs studied, these observables are entirely consistent with convergence at higher cutoff values. Although one cannot completely exclude the opposite, there is quite a bit of evidence against the emergence of new, essential scales in these systems.

5 How Many Is Too Many for Universality?

This is a remarkably broad universality, which extends far beyond the three-body system. It means that all states within the validity of the EFT, *i.e.*, those states that are insensitive to the details of physics at distances $r \lesssim R$, are described essentially by a single

³Frequently emphasized by T. Frederico.

parameter Λ_* . This should include not only cold atoms around Feshbach resonances, but also naturally occurring systems near unitarity, such as atomic ^4He systems where the ratio $B_2/B_{3,0} \simeq 10^{-3}$ is close enough to vanishing for the trimer to accommodate an excited state [8]. A comparison between results for $N \leq 6$ in EFT and potential models [11] suggests that these systems are within this universality class. Recently we have argued [51], for example showing that the Tjon line can be obtained in perturbation theory from the Tjon line at unitarity, that even nuclei are part of this class of systems — even though $B_2/B_{3,0} \simeq 0.26$ is not as small in this case.

The absence at leading order of more-body forces with additional dimensionful parameters means that $N \geq 4$ systems should also display discrete scale invariance. Reference [50] discovered that an $N = 3$ state spawns two $N = 4$ states, one very close to the $N = 3$ threshold and one about four times deeper. According to the accurate calculation of Ref. [63], for the two lowest $N = 4$ states at unitarity, $B_{4;0;0}/B_{3,0} \simeq 4.611$ and $B_{4;0;1}/B_{3,0} \simeq 1.002$. These states have been spotted in atomic systems [68]. Note that the alpha particle has two states, $B_{4;0;0}/B_{3,0} \sim 3$ and $B_{4;0;1}/B_{3,0} \sim 1$, which are qualitatively similar to unitarity states: another evidence that light nuclei are perturbatively close to unitarity [51].

Remarkably, potential-model calculations show that this doubling process continues with increasing number of bosons [9, 69, 70, 71], so that for a given N there are $2^{(N-3)}$ “interlocking” towers of states. Generalizing Eq. (19),

$$B_{N;n;\{i\}} = \frac{\kappa_\star^2}{m} \beta_{N;n;\{i\}} ((a_2 \kappa_\star)^{-1}), \quad (26)$$

by labeling each state with the $N = 3$ ancestor state (n) and a set $\{i\} = \{i_1, i_2, \dots, i_{N-3}\}$ of $i_j = 0, 1$ labels tracking the doubling, with $i_j = 0$ (1) denoting the lower (higher) state. Just as before, the dimensionless functions $\beta_{N;n;\{i\}}$ of $(a_2 \kappa_\star)^{-1}$ reduce at unitarity to pure numbers $\beta_{N;n;\{i\}}(0)$, which obey

$$\beta_{N;n+l;\{i\}}(0) = \alpha_l^{-2} \beta_{N;n;\{i\}}(0) \quad \Leftrightarrow \quad \beta_{N;n;\{i\}}(0) = e^{-2n\pi/s_0} \beta_{N;0;\{i\}}(0). \quad (27)$$

Again the spurion method gives

$$\beta_{N;n+l;\{i\}}((\alpha_l a_2 \kappa_\star)^{-1}) = \alpha_l^{-2} \beta_{N;n;\{i\}}((a_2 \kappa_\star)^{-1}), \quad (28)$$

with similar implications as for $N = 3$.

I am not aware of an explanation for the doubling, which has a topological interpretation [72], but the replicating towers are a reflection of the surviving discrete scale invariance. For $N \geq 4$, all but the two lower states appear as resonances in the scattering of a particle on the $(N - 1)$ -particle ground state. Because of the tower structure, we can focus on these two lower states. The higher one is near the ground state of the system with one less particle,

$$\beta_{N;0;0,\dots,0,1}(0) \simeq \beta_{N-1;0;0,\dots,0}(0), \quad (29)$$

and thus can be thought of as a two-body system of a particle and an $(N - 1)$ cluster. More generally, this state and its cousins up the tower are the analog of halo nuclei: they

have a clusterized structure with a certain number of “valence” particles orbiting a tight cluster of the remaining particles. Their low-energy properties can be described by Halo EFT [73, 74]. In contrast, the ground states close to unitarity get deeper and deeper as summarized by

$$B_N \equiv B_{N;0;\{0\}} \simeq \beta_{N;0;\{0\}}(0) B_{3;0} \equiv \kappa_N B_3, \quad (30)$$

which are the generalized Tjon lines calculated for atomic ^4He in Ref. [11].

For $N > 4$ multi-state fermions the pattern of states is not clear, but towers must also exist. We can retain the above notation, leaving the structure of the set $\{i\}$ unspecified. For the ground states, Eq. (30) still holds, but with a different set of numbers κ_N . The four-state fermion system, for example, reduces to a bosonic system [75], similarly to the three-nucleon system [3]. κ_4 is then the same in both cases, but $\kappa_{N \geq 5}$ must differ on account of the exclusion principle. For example, in nuclei the $N = 5$ ground state is a shallow resonance, that is, the two poles that are bound states for bosons are instead in the lower half-plane of the complex nucleon- ^4He relative-momentum plane. Halo EFT [73, 74] provides a description of this type of state that is completely parallel to the Contact (or Pionless) EFT that describes identical particles near unitarity.

Equation (30) is the counterpart for bosons and multi-state fermions of a relation like (11) for two-state fermions. The set of numbers κ_N encapsulates the dynamical information about the ground states at unitarity, and similarly to Eq. (19) deviations from unitarity can be considered in a $(a_2 \kappa_*)^{-1}$ expansion. There have been some attempts to determine the N dependence of B_N , mostly for bosons. Reference [10] extracted an N^2 growth from the statistical distribution of the two-particle propagator in Monte Carlo simulations. With a particular potential, Refs. [9, 70] obtained values that for $N \leq 6$ are well described by the number of triplets, $N(N-1)(N-2)/6$, slowing down to an N^2 growth for $7 \leq N \leq 10$ and even slower for $N \geq 11$. Ground-state ratios calculated from other potentials [76, 77] show different trends as N increases, but tend to agree with the form proposed in Ref. [78],

$$B_{N \geq 3} \approx \left[(N-3) \sqrt{\frac{B_4}{B_3}} + 4 - N \right]^2 B_3. \quad (31)$$

This relation is more general than (30), since it accommodates a separate four-body scale through B_4 , but of course is consistent with Eq. (30) if $B_4/B_3 = \kappa_4$ is a pure number. In EFT, Ref. [11] found Eq. (3) for $N \leq 6$, which is consistent with Eq. (31) for $B_4/B_3 \approx 4$.

The question immediately arises⁴, what is the large- N behavior of κ_N ? As κ_N quickly increases at small N , the Q_N in Eq. (24) increases, and the density increases. In EFT, finite energies for few-body systems arise from a balance between two- and three-body forces, and the latter cannot be separated from the high-momentum components of the former in a cutoff-independent way. But effectively the three-body force stops the Thomas collapse. We might expect the effectively repulsive effect of three-body forces to cause

⁴A discussion on this topic with J. Carlson and M.J. Savage is gratefully acknowledged.

saturation, that is, that there is a finite limit

$$\lim_{N \rightarrow \infty} \frac{B_N}{N} = \kappa \frac{B_3}{3} \quad (32)$$

with a universal, pure number κ . At this point there is no strong support for this possibility, other than the tapering off in growth observed in Ref. [9]. It could be, for example, that at some N the binding momentum per particle becomes large enough that we can no longer neglect short-range interaction details, and new essential scales emerge. (For bosons in two spatial dimensions, for example, it has been argued [79] that B_{N+1}/B_N approaches a constant for large N , quickly leading to the loss of universality.) In fact, the above estimate for Q_N would indicate that one is already probing distances comparable to the Van der Waals length in ^4He atomic systems for $N \sim 6$. Still, in Ref. [11] it was found that EFT and potential-model results are in reasonably good agreement, suggesting that the above estimate for Q_N is an overestimate. The limit of validity of the description in terms of the single parameter $\Lambda_\star$ and its associated discrete scale invariance remains an open question.

The arguments in the previous paragraph need indeed to be taken with a truck of salt. For one thing, we have little experience with treating the many-body problem from an EFT perspective where renormalization ties two- and three-body forces. But such a “superfluid Efimov liquid” phase of a Bose fluid has been argued for in Ref. [80] on the basis of a Monte Carlo simulation for a short-range two-body interaction supplemented by a hard-core three-body potential. (In contrast, much of the literature on the Bose gas, for example Ref. [81], focuses on the atomic phase. While for $a_2 < 0$ such a phase is mechanically unstable, for $a_2 > 0$ it is metastable, and the dynamics before collapse is universal in the sense of the two-component Fermi gas.)

To see the potential differences stemming from an EFT approach, let me assume that not only the many-body ground state is within the EFT, but also that the energy per particle, $\lim_{N \rightarrow \infty} E_N/N$, of the infinite problem at zero temperature, when excited states do not contribute, can in a first approximation be described solely by the number density ρ . This is the case in density functional theory [82] and in a class of Skyrme forces where nuclear-matter saturation arises from (unrenormalized) two- and three-body forces [83]. Such is also the case for Eq. (1), but there it is a consequence of scale invariance. Here we can ask instead for the constraints from discrete scale invariance under the extra assumption of dependence solely on

$$\rho = \frac{g}{6\pi^2} k_\rho^3, \quad (33)$$

where g is the degeneracy factor ($g = 1$ for one boson species and $g = 4$ for a nucleon) and k_ρ^{-1} measures the average particle separation ($k_\rho \equiv (9\pi/2)^{1/3} r_0^{-1}$ for bosons and $k_\rho = k_F$ for fermions). I am assuming here no polarization in the space of fermion states, *e.g.* no spin or isospin polarization for nucleons. Polarization and/or temperature can be accounted for along the lines of what is done for two-state fermions.

From dimensional analysis [84],

$$\lim_{N \rightarrow \infty} \frac{E_N[k_\rho]}{N} = \frac{\kappa_\star^2}{m} \varepsilon[k_\rho/\kappa_\star], \quad (34)$$

with ε a dimensionless function of the dimensionless combination $k_\rho/\kappa_\star$. Since under a scale transformation $\rho \rightarrow \alpha^{-3}\rho$, discrete scale invariance implies additionally:

$$\varepsilon[\alpha_l^{-1}k_\rho/\kappa_\star] = \alpha_l^{-2} \varepsilon[k_\rho/\kappa_\star] \quad \Leftrightarrow \quad \varepsilon[k_\rho/\kappa_\star] = \frac{k_\rho^2}{\kappa_\star^2} \mathfrak{T}[s_0 \ln(k_\rho/\kappa_\star)], \quad (35)$$

or

$$\lim_{N \rightarrow \infty} \frac{E_N[k_\rho]}{N} = \frac{k_\rho^2}{m} \mathfrak{T}[s_0 \ln(k_\rho/\kappa_\star)], \quad (36)$$

where $\mathfrak{T}$ is (real) periodic with period π ,

$$\mathfrak{T}[x + l\pi] = \mathfrak{T}[x] = \frac{3\gamma_0}{10} - \sum_{n=1}^{\infty} \gamma_n \sin(2nx + \phi_n). \quad (37)$$

In the last step I expanded $\mathfrak{T}$ in a Fourier series in terms of real coefficients $\gamma_{n \geq 0}$ and phases $\phi_{n \geq 1}$. Corrections in $(a_2\kappa_\star)^{-1}$ and $r_2\kappa_\star/2$ can be incorporated by the spurion formalism as before.

I do not see a general argument to determine the form of $\mathfrak{T}$, which could be different for bosons and fermions. γ_0 is the direct analog of the Bertsch parameter ξ in Eq. (1), and m/γ_0 can be thought of as an in-medium or effective mass. If there were no three-body effects, $\gamma_{n \geq 1} = 0$ and the pressure (related to the first derivative of Eq. (36)) would not vanish anywhere — there would be no saturation. Thus, for stability at least one of the $\gamma_{n \geq 1}$ is non-vanishing and the dependence on k_ρ is non-analytic — qualitatively very different from the power-law dependence on ρ of simple mean-field models with two- and three-body forces [83]. The energy (36) is a curve in the $\lim_{N \rightarrow \infty} (mE_N/N) \times k_\rho$ plane, which is a minimum at the saturation point $k_\rho = k_0$. Since $\kappa_\star$ provides the scale for both axis on this plane, as $\Lambda_\star$ changes the minima lie on a curve $\lim_{N \rightarrow \infty} (mE_N/N)_{k_0} \propto -k_0^2$. Again, a correlation between the minimum energy per nucleon and the saturation density — the so-called Coester line — has been found [85] for phase-shift-equivalent two-nucleon potentials. The line is, indeed, approximately quadratic in k_0 [85, 84]. The EFT at unitarity, with its one essential parameter $\Lambda_\star$, provides a possible mechanism for this line, which would be displaced in nuclear systems by $(Qa_2)^{-1}$ and $r_2Q/2$ effects — just like the Phillips and Tjon lines.

As a minimal model to explore the saturation effects of the three-body force, let me consider the first two terms in Eq. (37),

$$\lim_{N \rightarrow \infty} \frac{E_N[k_\rho]}{N} = \frac{k_\rho^2}{m} \left[\frac{3\gamma_0}{10} - \gamma_1 \sin(2s_0 \ln(k_\rho/\kappa_\star) + \phi_1) \right]. \quad (38)$$

The position of the energy minimum and the energy at the saturation point are overdetermined by $\gamma_{0,1}$ and ϕ_1 . These parameters can be used to fix the compressibility,

$$K_\infty = \left(k_\rho^2 \frac{d^2}{dk_\rho^2} \lim_{N \rightarrow \infty} \frac{E_N[k_\rho]}{N} \right)_{k_\rho=k_0} = 4 \left[s_0^2 \gamma_0 \frac{3k_0^2}{10m} - (1 + s_0^2) \left(\lim_{N \rightarrow \infty} \frac{E_N[k_0]}{N} \right) \right], \quad (39)$$

as well. If I additionally take $\phi_1 = 0$, saturation in (unpolarized, isospin-symmetric) nuclear matter, where $\kappa_\star \simeq 90$ MeV, $k_0 \simeq 260$ MeV, and $\lim_{N \rightarrow \infty} (E_N[k_0]/N) \simeq -16$ MeV, results from $\gamma_0 = 0.50$ and $\gamma_1 = 0.43$. Both values are natural and, interestingly, γ_0 turns out to be numerically close to the Bertsch parameter. For the compressibility one finds $K_\infty \simeq 170$ MeV, or about 30% off the empirical value $K_\infty \simeq 250$ MeV. This is reasonable agreement considering that corrections are $\mathcal{O}((\kappa_\star a_2)^{-1}, \kappa_\star r_2/2) \sim 0.5$. At this moment, however, there is nothing but optimism to justify Eq. (38).

Given the large period $\simeq 22.7\kappa_\star$ in momentum, the periodic behavior of $\mathbb{T}$ would be difficult to observe, as are excited Efimov states⁵. I do not believe anybody has seen a hint of oscillation in the nuclear-matter equation of state, but it would have been surprising if they did: half of the period corresponds to a momentum above the maximum relative momentum used in the usual fits of data with phenomenological two-nucleon potentials. At such momenta it is unlikely that an expansion around unitarity converges. Minima that are deeper than the physical minimum must be outside the EFT, just as the states in the few-body towers that are below the physical ground states. To see the log-periodic behavior we would need more than one minimum within the EFT domain. In any case, we would be lucky if the expansion converges even at saturation. It would be interesting if for nuclear matter, for example, one could exclude Eq. (36), or else find a specific form that produces a realistic equation of state around the saturation density.

6 No Scales Left to Shed

There has been a lot of interest in systems that are universal in the sense of having dynamics depending in a first approximation on no parameter, and thus being ruled by scale invariance. Nowhere near as extensive is our understanding of the remarkable features of those systems that are characterized essentially by a single, three-body parameter, and ruled by discrete scale invariance.

Much of what I said above — with the possible exception of the speculation at the end — has been known to some people for a while. Still, my goal was to show that the latter class of systems is perhaps even more interesting than the former, because they present richer structures from a still pretty simple action. The discrete version of scale invariance allows for geometric towers of bound states in few-body systems and is conjectured to yield saturation in larger systems, which retains the log-periodic behavior observed in smaller systems. The simple model of Eq. (38), for example, should be contrasted with Eq. (1): while both contain a repulsive term that can be interpreted as a simple reduction

⁵This paragraph benefited from an interesting exchange with H.W. Griesshammer.

of the kinetic term via a quadruplication of the effective mass, discrete scale invariance allows for an additional log-periodic attraction which stems from the three-body force (17). Beyond the intrinsic theoretical beauty of the structures that emerge from discrete scale invariance, there is some tantalizing evidence that these structures survive in bosonic atom clusters and in nuclei. Hopefully this *Comment* encourages further study of the many-body consequences of discrete scale invariance.

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