The Maximum k-Differential Coloring Problem [☆]

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Abstract

Given an n -vertex graph G and two positive integers $d, k \in \mathbb{N}$, the (d, kn) -differential coloring problem asks for a coloring of the vertices of G (if one exists) with distinct numbers from 1 to kn (treated as *colors*), such that the minimum difference between the two colors of any adjacent vertices is at least d . While it was known that the problem of determining whether a general graph is $(2, n)$ -differential colorable is NP-complete, our main contribution is a complete characterization of bipartite, planar and outerplanar graphs that admit $(2, n)$ -differential colorings. For practical reasons, we also consider color ranges larger than n , i.e., $k > 1$. We show that it is NP-complete to determine whether a graph admits a $(3, 2n)$ -differential coloring. The same negative result holds for the $(\lfloor 2n/3 \rfloor, 2n)$ -differential coloring problem, even in the case where the input graph is planar.

1. Introduction

Several methods for visualizing relational datasets use a map metaphor where objects, relations between objects and clusters are represented as cities, roads and countries, respectively. Clusters are usually represented by colored regions, whose boundaries are explicitly defined. The 4-coloring theorem states that four colors always suffice to color any map such that neighboring countries have distinct colors. However, if not all countries of the map are contiguous and the countries are not colored with unique colors, it would be impossible to distinguish whether two regions with the same color belong to the same country or to different countries. In order to avoid such ambiguity, this necessitates the use of a unique color for each country; see Fig. 1.

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Figure 1: Illustration of a map colored using the same set of colors obtained by the linear interpolation of blue and yellow. There is one country in the middle containing the vertices 40-49 which is fragmented into three small regions.

However, it is not enough to just assign different colors to each country. Although human perception of color is good and thousands of different colors can be easily distinguished, reading a map can be difficult due to color constancy and color context effects [1]. Dillencourt et al. [2] define a good coloring as one in which the colors assigned to the countries are visually distinct while also ensuring that the colors assigned to adjacent countries are as dissimilar as possible. However, not all colors make suitable choices for coloring countries [3] and a “good” color palette is often a gradation of certain map-like colors [4]. In more restricted scenarios, e.g., when a map is printed in gray scale, or when the countries in a given continent must use different shades of a predetermined color, the color space becomes 1-dimensional.

This 1-dimensional fragmented map coloring problem is nicely captured by the *maximum differential coloring problem* [5, 6, 7, 8], which we slightly generalize in this paper. Given a map, define the *country graph* $G = (V, E)$ in which vertices represent countries and two vertices are connected by an edge if their corresponding countries share a non-trivial geographic boundary. Given two positive integers $d, k \in \mathbb{N}$, we say that G is (d, kn) -differential colorable if and only if there is a coloring of the n vertices of G with distinct numbers from 1 to kn (treated as *colors*), so that the *minimum color distance* between adjacent vertices of G is at least d . The *maximum k -differential coloring* problem asks for the largest value of d , called the *k -differential chromatic number* of G , so that G is (d, kn) -differential colorable. Note that the traditional *maximum differential coloring problem* corresponds to $k = 1$.

A natural reason to study the maximum k -differential coloring problem for $k > 1$ is that using more colors can help produce maps with larger differential chromatic number. Note, for example, that a star graph on n vertices has 1-differential chromatic number (or simply *differential chromatic number*) one, whereas its 2-differential chromatic number is $n + 1$. That is, by doubling the number of colors used, we can improve the quality of the resulting coloring by a factor of n . This is our main motivation for studying the maximum k -differential coloring problem for $k > 1$.

1.1. Related Work

The maximum differential coloring problem is a well-studied problem, which dates back in 1984, when Leung et al. [6] introduced it under the name “separation number” and showed its NP-completeness. It is worth mentioning though that the maximum differential coloring problem is also known as “dual bandwidth” [8] and “anti-bandwidth” [5], since it is the complement of the *bandwidth minimization problem* [9]. Due to the hardness of the problem, heuristics are often used for coloring general graphs, e.g., LP-formulations [10], memetic algorithms [11] and spectral based methods [12]. The differential chromatic number is known only for special graph classes, such as Hamming graphs [13], meshes [14], hypercubes [14, 15], complete binary trees [16], complete m -ary trees for odd values of m [5], other special types of trees [16], and complements of interval graphs, threshold graphs and arborescent comparability graphs [17]. Upper bounds on the differential chromatic number are given by Leung et al. [6] for connected graphs and by Miller and Pritikin [7] for bipartite graphs. Note that closely related is also the *radio frequency assignment* problem, where n transmitters have to be assigned n frequencies, so that interfering transmitters have frequencies as far apart as possible, see e.g., [18, 19]. For a more detailed bibliographic overview related to the maximum differential coloring problem refer to [20].

1.2. Our Contribution

In Section 2, we present preliminary properties and bounds on the k -differential chromatic number, for $k \geq 1$. One of them guarantees that any graph is $(1, n)$ -differential colorable; an arbitrary assignment of distinct colors to the vertices of the input graph guarantees a minimum color distance of one (see Lemma 1). So, the next reasonable question to ask is whether a given graph is $(2, n)$ -differential colorable. Unfortunately, this is already an NP-complete problem (for general graphs), since a graph is $(2, n)$ -differential colorable if and only if its complement has a Hamiltonian path [6]. This motivates the study of the $(2, n)$ -differential coloring problem for special classes of graphs. In Section 3, we present a complete characterization of bipartite, outer-planar and planar graphs that admit $(2, n)$ -differential colorings.

In Section 4, we double the number of available colors. As any graph is $(2, 2n)$ -differential colorable (due to Lemma 1; Section 2), we study the $(3, 2n)$ -differential coloring problem and we prove that it is NP-complete for general graphs (Theorem 4; Section 4). More general, we show that for any given $k \geq 2$ testing whether a given graph is $(k + 1, kn)$ -differential colorable is NP-complete (Theorem 5; Section 4). On the other hand, all planar graphs are $(\lfloor n/3 \rfloor + 1, 2n)$ -differential colorable (see Lemma 3; Section 2) and testing whether a given planar graph is $(\lfloor 2n/3 \rfloor, 2n)$ -differential colorable is shown to be NP-complete (Theorem 6; Section 4). In Section 5, we provide a simple ILP-formulation for the maximum k -differential coloring problem and experimentally compare the optimal results obtained by the ILP formulation for $k = 1$ and $k = 2$ with the GMap coloring method, which relies on a heuristic based on spectral methods [21]. We conclude in Section 6 with open problems and future work.

2. Preliminaries

We assume familiarity with basic concepts of graph theory and refer to the textbooks [22] for background. The maximum k -differential coloring problem can be easily reduced to the ordinary differential coloring problem as follows: Consider the maximum k -differential coloring of an n -vertex graph G ; create a (disconnected) graph G' that contains all vertices and edges of G plus $(k-1)n$ isolated vertices. It is easy to see that the k -differential chromatic number of G is equal to the 1-differential chromatic number of G' . A drawback of this approach, however, is that few results are known for the ordinary differential coloring problem, when the input is a disconnected graph. In the following, we present some immediate upper and lower bounds on the k -differential chromatic number for connected graphs.

Lemma 1. *For $k \geq 1$, the k -differential chromatic number of a connected graph is at least k .*

Proof. Let G be a connected graph on n vertices. It suffices to prove that G is (k, kn) -differential colorable. Indeed, an arbitrary assignment of distinct colors from the set $\{k, 2k, \dots, kn\}$ to the vertices of G guarantees a minimum color distance of k . $\square$

Lemma 2. *For $k \geq 1$, the k -differential chromatic number of a connected graph $G = (V, E)$ on n vertices is at most $\lfloor \frac{n}{2} \rfloor + (k-1)n$.*

Proof. The proof is a straightforward generalization of the proof of Yixun and Jinjiang [8] for the ordinary maximum differential coloring problem. One of the vertices of G has to be assigned with a color in the interval $[\lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil + (k-1)n]$, as the size of this interval is $(k-1)n + 1$ and there can be only $(k-1)n$ unassigned colors. Since G is connected, that vertex must have at least one neighbor which (regardless of its color) would make the difference along that edge at most $kn - \lceil \frac{n}{2} \rceil = \lfloor \frac{n}{2} \rfloor + (k-1)n$. $\square$

Lemma 3. *For $k \geq 1$, the k -differential chromatic number of a connected m -colorable graph $G = (V, E)$ on n vertices is at least $\lfloor \frac{(k-1)n}{m-1} \rfloor + 1$.*

Proof. Let $C_i \subseteq V$ be the set of vertices of G with color i and c_i be the number of vertices with color i , $i = 1, \dots, m$. We can show that G is $(\lfloor \frac{(k-1)n}{m-1} \rfloor + 1, kn)$ -differential colorable by coloring the vertices of C_i with colors from the following set: $[(\sum_{j=1}^{i-1} c_j) + 1 + (i-1)\lfloor \frac{(k-1)n}{m-1} \rfloor, (\sum_{j=1}^i c_j) + (i-1)\lfloor \frac{(k-1)n}{m-1} \rfloor]$ $\square$

3. The (2,n)-Differential Coloring Problem

In this section, we provide a complete characterization of (i) bipartite graphs, (ii) outerplanar graphs and (iii) planar graphs that admit $(2, n)$ -differential coloring. Central to our approach is a result of Leung et al. [6] who showed that a graph G has $(2, n)$ -differential coloring if and only if the complement G^c of G is Hamiltonian. As a consequence, if the complement of G is disconnected, then G has no $(2, n)$ -differential coloring.

In order to simplify our notation scheme, we introduce the notion of *ordered differential coloring* (or simply *ordered coloring*) of a graph, which is defined as follows.

Given a graph $G = (V, E)$ and a sequence $S_1 \rightarrow S_2 \rightarrow \dots \rightarrow S_m$ of m disjoint subsets of V , such that $\cup_{i=1}^m S_i = V$, an *ordered coloring* of G implied by the sequence $S_1 \rightarrow S_2 \rightarrow \dots \rightarrow S_k$ is one in which the vertices of S_i are assigned colors from $(\sum_{j=1}^{i-1} |S_j|) + 1$ to $\sum_{j=1}^i |S_j|$, $i = 1, 2, \dots, m$.

3.1. The $(2, n)$ -Differential Coloring Problem for Bipartite Graphs

In this subsection, we consider the class of bipartite graphs. It is not difficult to see that the complement of a complete bipartite graphs is disconnected. Therefore, such a graph is not $(2, n)$ -differential colorable. In the following, we prove that the reverse also holds.

Theorem 1. *A bipartite graph admits a $(2, n)$ -differential coloring if and only if it is not a complete bipartite graph.*

Proof. Let $G = (V, E)$ be an n -vertex bipartite graph, with $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$ and $E \subseteq V_1 \times V_2$. If G is a complete bipartite graph, then its complement is disconnected. Therefore, G does not admit a $(2, n)$ -differential coloring. Now, assume that G is not a complete bipartite graph. Then, there exist at least two vertices, say $u \in V_1$ and $v \in V_2$, that are not adjacent, i.e., $(u, v) \notin E$. Consider the ordered coloring of G implied by the sequence $V_1 \setminus \{u\} \rightarrow \{u\} \rightarrow \{v\} \rightarrow V_2 \setminus \{v\}$. As u and v are not adjacent, it follows that the color difference between any two vertices of G is at least two. Hence, G admits a $(2, n)$ -differential coloring. $\square$

3.2. The $(2, n)$ -Differential Coloring Problem for Outerplanar Graphs

In order to characterize which outerplanar graphs admit $(2, n)$ -differential colorings, we follow an approach that consists of two main steps: First, we prove a special property of any 3-coloring of an outerplanar graph that does not contain as subgraph $K_{1, n-1}$, where $n \geq 6$ is the number of the graph's vertices. In a second step, we use this property to prove that such a graph always admits a $(2, n)$ -differential coloring and vice versa.

Lemma 4. *An outerplanar graph with $n \geq 6$ vertices, that does not contain $K_{1, n-1}$ as a subgraph, admits a 3-coloring, in which each color set contains at least 2 vertices.*

Proof. Let $G = (V, E)$ be an outerplanar graph with $n \geq 6$ vertices, that does not contain $K_{1, n-1}$ as a subgraph. As G is outerplanar, it admits a 3-coloring [23]. Let $C_i \subseteq V$ be the set of vertices of G with color i and c_i be the number of vertices with color i , that is $c_i = |C_i|$, for $i = 1, 2, 3$. Without loss of generality let $c_1 \leq c_2 \leq c_3$. We further assume that each color set contains at least one vertex, that is $c_i \geq 1$, $i = 1, 2, 3$. If there is no set with less than 2 vertices, then the lemma clearly holds. Otherwise, we distinguish three cases:

C.1: $c_1 = c_2 = 1$ and $c_3 \geq 4$. Without loss of generality assume that $C_1 = \{a\}$ and $C_2 = \{b\}$. As G is outerplanar, vertices a and b can have at most 2 common neighbors. On the other hand, since G has at least 6 vertices, there exists at least one vertex, say $c \in C_3$, which is not a common neighbor of a and b . Without loss of generality assume that $(b, c) \notin E$. Then, vertex c can be colored with color 2. Therefore, we derive a new 3-coloring of G for which we have that $c_1 = 1$, $c_2 = 2$ and $c_3 \geq 3$.

C.2: $c_1 = 1, c_2 = 2$ and $c_3 \geq 3$. Without loss of generality assume that $C_1 = \{a\}$ and $C_2 = \{b, b'\}$. First, consider the case where there exists at least one vertex, say $c \in C_3$, which is not a neighbor of vertex a . In this case, vertex c can be colored with color 1 and a new 3-coloring of G is derived with $c_1 = c_2 = 2$ and $c_3 \geq 3$, as desired. Now consider the more interesting case, where vertex a is a neighbor of all vertices of C_3 . As G does not contain $K_{1,n-1}$ as a subgraph, either vertex b or vertex b' is not a neighbor of vertex a . Without loss of generality let that vertex be b , that is $(a, b) \notin E$. As G is outerplanar, vertices a and b' can have at most 2 common neighbors. Since G has at least 6 vertices and vertex a is a neighbor of all vertices of C_3 , there exist at least one vertex, say $c \in C_3$, which is not adjacent to vertex b' , that is $(b', c) \notin E$. Therefore, we can color vertex c with color 2 and vertex b with color 1 and derive a new 3-coloring of G for which we have that $c_1 = c_2 = 2$ and $c_3 \geq 2$, as desired.

C.3: $c_1 = 1, c_2 \geq 3$ and $c_3 \geq 3$. Without loss of generality assume that $C_1 = \{a\}$. Then, there exists at least one vertex, say $c \in C_2 \cup C_3$, which is not a neighbor of vertex a . In this case, vertex c can be colored with color 1 and a new 3-coloring of G is derived with $c_1 = c_2 = 2$ and $c_3 \geq 3$, as desired.

From the above case analysis, it follows that indeed any outerplanar graph with $n \geq 6$ vertices, that does not contain $K_{1,n-1}$ as a subgraph, admits a 3-coloring, in which each color set contains at least 2 vertices. $\square$

Lemma 5. *Let $G = (V, E)$ be an outerplanar graph and let V' and V'' be two disjoint subsets of V , such that $|V'| \geq 2$ and $|V''| \geq 3$. Then, there exist two vertices $u \in V'$ and $v \in V''$, such that $(u, v) \notin E$.*

Proof. The lemma follows from the fact that an outerplanar graph does not contain $K_{2,3}$ as a subgraph [24]. $\square$

Theorem 2. *An outerplanar graph with $n \geq 8$ vertices has $(2, n)$ -differential coloring if and only if it does not contain $K_{1,n-1}$ as subgraph.*

Proof. Let $G = (V, E)$ be an outerplanar graph with $n \geq 8$ vertices. If G contains $K_{1,n-1}$ as subgraph, then the complement G^c of G is disconnected. Therefore, G does not admit a $(2, n)$ -differential coloring. Now, assume that G does not contain $K_{1,n-1}$ as subgraph. By Lemma 4, it follows that G admits a 3-coloring, in which each color set contains at least two vertices. Let $C_i \subseteq V$ be the set of vertices with color i and $c_i = |C_i|$, for $i = 1, 2, 3$, such that $2 \leq c_1 \leq c_2 \leq c_3$. We distinguish the following cases:

C.1: $c_1 = 2, c_2 = 2$ and $c_3 \geq 4$. Since $|C_1| = 2$ and $|C_3| \geq 4$, by Lemma 5 it follows that there exist two vertices $a \in C_1$ and $c \in C_3$, such that $(a, c) \notin E$. Similarly, since $|C_2| = 2$ and $|C_3 \setminus \{c\}| \geq 3$, by Lemma 5 it follows that there exist two vertices $b \in C_2$ and $c' \in C_3$, such that $c \neq c'$ and $(b, c') \notin E$; see Fig. 2a-2b.

C.2: $c_1 \geq 2, c_2 \geq 3$ and $c_3 \geq 3$. Since $|C_1| = 2$ and $|C_3| \geq 3$, by Lemma 5 it follows that there exist two vertices $a \in C_1$ and $c \in C_3$, such that $(a, c) \notin E$. Similarly,

Figure 2: (a) An outerplanar graph colored with 3 colors, white, black and gray (Case 1 of Thm. 2), and, (b) its $(2, n)$ -differential coloring. (c) Another outerplanar graph also colored with 3 colors, white, black and gray (Case 2 of Thm. 2), and, (d) its $(2, n)$ -differential coloring.

since $|C_2| \geq 3$ and $|C_3 \setminus \{c\}| \geq 2$, by Lemma 5 it follows that there exist two vertices $b \in C_2$ and $c' \in C_3$, such that $c \neq c'$ and $(b, c') \notin E$; see Fig. 2c-2d.

For both cases, consider the ordered coloring implied by the sequence $C_1 \setminus \{a\} \rightarrow \{a\} \rightarrow \{c\} \rightarrow C_3 \setminus \{c, c'\} \rightarrow \{c'\} \rightarrow \{b\} \rightarrow C_2 \setminus \{b\}$. As $(a, c) \notin E$ and $(b, c') \notin E$, it follows that the color difference between any two vertices of G is at least two. Hence, G admits a $(2, n)$ -differential coloring. $\square$

Remark. Using the ILP formulation of Section 5, it can be shown that Theorem 2 holds for all outerplanar graphs with $n = 7$ vertices. For $n \leq 6$ there exists in total five graphs for which Theorem 2 does not hold (see Figure 3).

Figure 3: Different graphs with $n \leq 6$ vertices which do not contain $K_{1,n-1}$ as subgraph but do not admit $(2, n)$ -differential colorings.

In the particular case that the outerplanar graph is biconnected, we present an interesting property (that we will use in Section 3.3) regarding the hamiltonicity of the complement of a biconnected outerplanar graph on $n \geq 6$ vertices. However, before we proceed with the detailed proof of this property, we need first to introduce some necessary terminology. The *weak-dual* of a plane graph is the subgraph of its dual induced by neglecting the face-vertex corresponding to its unbounded face. A maximal outerplanar graph is called *outer zig-zag*, if its weak-dual is a path.

Lemma 6. Let $G = (V, E)$ be biconnected outerplanar on $n \geq 6$ vertices that (i) does not contain $K_{1,n-1}$ as a subgraph, and (ii) is not isomorphic to the one of Figure 3e. Then, G^c contains a Hamiltonian path whose end-points are not consecutive on the outerface of G .

Proof. We first augment G by adding as many edges as possible without deviating outerplanarity and without introducing $K_{1,n-1}$ or the graph of Figure 3e, if $n = 6$. If $n \geq 7$, then the resulting graph is always maximal outerplanar. In the case where $n = 6$, the augmentation will result to a graph that is either isomorphic to an outer zig-zag on 6 vertices or to the graph of Figure 3e without one chord. So, in the following we will assume w.l.o.g. that G is the result of the aforementioned augmentation.

We will prove the lemma by induction on the number n of vertices of G . The base of the induction corresponds to the case where $n = 6$. In this particular case, it is not difficult to check that the complements of the two graphs described above both contain Hamiltonian paths whose end-points are not consecutive on their outerfaces.

For the inductive step, consider a maximal outerplanar graph G with $n > 6$ vertices and assume that all outerplanar graphs with $n - 1$ vertices that do not contain $K_{1,n-2}$ as subgraph, and are not isomorphic to the graph of Figure 3e contain Hamiltonian paths whose end-points are not consecutive on their outerfaces. Since G is outerplanar, it contains at least two vertices, say v and v' , of degree at most two. If both $G - v$ and $G - v'$ contain $K_{1,n-2}$, then it immediately follows that G contains $K_{1,n-1}$ as a subgraph, a contradiction. So, we may assume w.l.o.g. that $G - v$ does not contain $K_{1,n-2}$ as a subgraph. We distinguish two cases: (C.1) $G - v$ is isomorphic to the graph of Figure 3e, and (C.2) $G - v$ is not isomorphic to the graph of Figure 3e. Clearly in case C.1, G has exactly seven vertices. Let u be the vertex of degree two in G which is not incident to v and has a common neighbour with v in G . Then, $G - u$ is an outer zig-zag on 6 vertices. In case C.2, graph $G - v$ is biconnected and does not contain $K_{1,n-2}$ as a subgraph.

It follows that in both cases there is always a vertex w of degree at most two in G , such that $G - w$ is biconnected and satisfies the requirements of the Theorem (in case C.1 w is u , while in case C.2 w is v). By the induction hypothesis, $(G - w)^c$ has a Hamiltonian path, say P , whose endpoints are not consecutive on the outerface of $G - w$ and therefore in G . Since G has at least 7 vertices and the degree of v is at most two in G , there exist two consecutive internal vertices of P , say w_1 and w_2 , such that $(w, w_1) \notin E$ and $(w, w_2) \notin E$. This implies that G^c has also a Hamiltonian path whose end-points are not consecutive on the outerface of G and the proof is complete. $\square$

Lemma 7. *Let $G = (V, E)$ be biconnected outerplanar on $n \geq 6$ with $\text{diam}(G) \geq 3$. Then, G^c contains a Hamiltonian path whose end-points are not consecutive on the outerface of G .*

Proof. Since $\text{diam}(G) \geq 3$, graph G cannot contain $K_{1,n-1}$ as a subgraph. In addition, G cannot be isomorphic to the graph of Figure 3e, as the later graph has diameter two. Hence, by Theorem 6 graph G contains a Hamiltonian path whose end-points are not consecutive on the outerface of G . $\square$

3.3. The $(2, n)$ -Differential Coloring Problem for Planar Graphs

In this subsection, we provide a complete characterization for planar graphs admitting $(2, n)$ -differential colorings. First, we give a simple and quite intuitive proof of this characterization, which however holds for planar graphs with $n \geq 36$ vertices. Inspired from this proof and using more complicated arguments we prove that our

characterization can be extended to planar graphs with $n \geq 12$ vertices. As in the case of outerplanar graphs, our first proof consists of two main steps: First, we prove a special property of any 4-coloring of a planar graph that does not contain as subgraphs $K_{1,1,n-3}$, $K_{1,n-1}$ and $K_{2,n-2}$, where $n \geq 36$ is the number of the graph's vertices. Then, we employ this property to prove that such a graph always admits a $(2, n)$ -differential coloring.

Lemma 8. *A planar graph with $n \geq 36$ vertices, that does not contain as subgraphs $K_{1,1,n-3}$, $K_{1,n-1}$ and $K_{2,n-2}$, admits a 4-coloring, in which two color sets contain at least 2 vertices and the remaining two at least 5 vertices.*

Proof. Let $G = (V, E)$ be a planar graph with $n \geq 36$ vertices, that does not contain as subgraphs $K_{1,1,n-3}$, $K_{1,n-1}$ and $K_{2,n-2}$. As G is planar, it admits a 4-coloring [25]. Let $C_i \subseteq V$ be the set of vertices of G with color i and c_i be the number of vertices with color i , that is $c_i = |C_i|$, for $i = 1, 2, 3, 4$. Without loss of generality let $c_1 \leq c_2 \leq c_3 \leq c_4$. We further assume that each color set contains at least one vertex, that is $c_i \geq 1$, $i = 1, 2, 3, 4$. We distinguish the following cases:

C.1: $c_1 = 1$, $c_2 = 1$ and $c_3 \leq 7$. Since G has at least 36 vertices, it follows that $c_4 \geq 27$. Observe that a vertex of C_4 cannot be assigned a color other than 4 if and only if it is adjacent to at least one vertex in C_1 , one vertex in C_2 and one vertex in C_3 . Let $C_4^* \subseteq C_4$ be the set of such vertices and $c_4^* = |C_4^*|$. We claim that $c_4^* \leq 14$. To prove the claim, we construct an auxiliary bipartite graph $G_{\text{aux}} = (V_{\text{aux}}, E_{\text{aux}})$ where $V_{\text{aux}} = C_1 \cup C_2 \cup C_3 \cup C_4^*$ and $E_{\text{aux}} = E \cap (C_4^* \times (C_1 \cup C_2 \cup C_3))$. Clearly, G_{aux} is planar, since it is subgraph of G . Since all vertices of C_4^* have degree at least 3, it holds that $3c_4^* \leq |E_{\text{aux}}|$. On the other hand, it is known that a planar bipartite graph on $n \geq 3$ vertices cannot have more than $2n - 4$ edges [26]. Therefore, $|E_{\text{aux}}| \leq 2(c_1 + c_2 + c_3 + c_4^*) - 4$, which implies that $3c_4^* \leq 2(c_1 + c_2 + c_3 + c_4^*) - 4$. As $c_1 + c_2 + c_3 \leq 9$, it follows that our claim indeed holds. As a consequence, we can change the color of several vertices initially in C_4 and obtain a new 4-coloring of G in which C_4 has exactly 14 vertices. This implies that the number of vertices in C_1 , C_2 and C_3 is at least 22 or equivalently that one out of C_1 , C_2 and C_3 must contain strictly more than 7 vertices, say C_3 . So, in the new coloring it holds that $c_1 \geq 1$, $c_2 \geq 1$ and $c_3 \geq 8$; a case that is covered in the following.

C.2: $c_1 = 1$, $c_2 = 1$ and $c_3 \geq 8$. Assume without loss of generality that $C_1 = \{a\}$ and $C_2 = \{b\}$. Since G does not contain $K_{2,n-2}$ as subgraph, there exists at least one vertex, say $c \in C_3 \cup C_4$, which is not connected to both a and b ; say, $(b, c) \notin E$. Then, vertex c can be safely colored with color 2. In the new coloring, it holds that $c_1 = 1$, $c_2 = 2$ and $c_3 \geq 7$; refer to Case 4.

C.3: $c_1 = 1$, $c_2 = 2$ and $c_3 \leq 6$. Since G has at least 36 vertices, it follows that $c_4 \geq 27$. Now, observe that $c_1 + c_2 + c_3 \leq 9$. Hence, following similar arguments as in Case 1, we can prove that there is a 4-coloring of G in which C_4 has exactly 14 vertices. So, again the number of vertices in C_1 , C_2 and C_3 is at least 22 and consequently at least one out of C_1 , C_2 and C_3 must contain strictly more than 7

vertices, say C_3 . In the new coloring, it holds that $c_1 \geq 1$, $c_2 \geq 2$ and $c_3 \geq 8$; refer to Case 4.

C.4: $c_1 = 1$, $c_2 = 2$ and $c_3 \geq 7$. Assume without loss of generality that $C_1 = \{a\}$ and $C_2 = \{b, b'\}$. First, consider the case where there exists at least one vertex, say $c \in C_3 \cup C_4$, which is not a neighbor of vertex a . In this particular case, vertex c can be colored with color 1 and a new 4-coloring of G is derived for which it holds that $c_1 = 2$, $c_2 = 2$ and $c_3 \geq 6$, as desired. Now, consider the more interesting case, where vertex a is a neighbor of all vertices in $C_3 \cup C_4$. As G does not contain $K_{1,n-1}$ as subgraph, vertex a is not a neighbor of vertex b or vertex b' or both. In the latter case, vertex a can be colored with color 2. Therefore, if we select two arbitrary vertices of C_3 and color them with color 1, we derive a new 4-coloring of G for which it holds that $c_1 = 2$, $c_2 = 2$ and $c_3 \geq 5$, as desired. On the other hand, if vertex a is a neighbor of exactly one of b and b' , say without loss of generality b , that is $(a, b) \in E$ and $(a, b') \notin E$, then there exists a vertex $c \in C_3 \cup C_4$ such that $(b, c) \notin E$; as otherwise the complete tripartite graph formed by vertex a , vertex b and $C_3 \cup C_4$ forms a $K_{1,1,n-3}$. So, if we color vertex b' with color 1 and vertex c with color 2, we obtain a new coloring of G for which it holds that $c_1 = 2$, $c_2 = 2$ and $c_3 \geq 6$, as desired.

C.5: $c_1 = 1$, $c_2 \geq 3$ and $c_3 \geq 3$. Assume without loss of generality that $C_1 = \{a\}$. As G does not contain $K_{1,n-1}$ as subgraph, there is a vertex $a' \in C_2 \cup C_3 \cup C_4$ which is not a neighbor of a . Hence, a' can be colored with color 1. In the new coloring, it holds that $c_1 \geq 2$, $c_2 \geq 2$, $c_3 \geq 2$; refer to Case 6.

C.6: $c_1 \geq 2$, $c_2 \geq 2$ and $c_3 \leq 4$. Since G has at least 36 vertices, it follows that $c_4 \geq 24$. Since $c_1 + c_2 + c_3 \leq 12$, following similar arguments as in Case 1 we can prove that there is a 4-coloring of G in which C_4 has exactly 20 vertices. So, the number of vertices in C_1 , C_2 and C_3 is at least 16 and consequently at least one out of C_1 , C_2 and C_3 must contain strictly more than 5 vertices, say C_3 . In the new coloring, it holds that $c_1 \geq 2$, $c_2 \geq 2$ and $c_3 \geq 6$, as desired.

From the above case analysis, it follows that G has a 4-coloring, in which two color sets contain at least 2 vertices and the remaining two at least 5 vertices. $\square$

Lemma 9. *Let $G = (V, E)$ be a planar graph and let V' and V'' be two disjoint subsets of V , such that $|V'| \geq 3$ and $|V''| \geq 3$. Then, there exist two vertices $u \in V'$ and $v \in V''$, such that $(u, v) \notin E$.*

Proof. The lemma follows directly from the fact that a planar graph does not contain $K_{3,3}$ as subgraph [27]. $\square$

Theorem 3. *A planar graph with $n \geq 36$ vertices has a $(2, n)$ -differential coloring if and only if it does not contain as subgraphs $K_{1,1,n-3}$, $K_{1,n-1}$ and $K_{2,n-2}$.*

Proof. Let $G = (V, E)$ be an n -vertex planar graph, with $n \geq 36$. If G contains $K_{l,n-l}$ as subgraph, then the complement G^c of G is disconnected, $l = 1, 2, \dots, \lceil n/2 \rceil$. Therefore, G has no $(2, n)$ -differential coloring. Note, however, that since G is planar,

Figure 4: (a) A planar graph colored with 4 colors, black (C_1), white (C_2), gray (C_3) and blue (C_4); Case 1 of Thm. 3, and, (b) its $(2, n)$ -differential coloring. (c) Another planar graph also colored with 4 colors, black, gray, blue and white; Case 2 of Thm. 3, and, (d) its $(2, n)$ -differential coloring.

it cannot contain $K_{l, n-l}$, $l = 3, 4, \dots, \lceil n/2 \rceil$, as subgraph. On the other hand, if G contains $K_{1,1,n-3}$ as subgraph, then G^c has two vertices of degree at most one, whose neighborhoods are subsets of the same vertex (as a singleton set). Hence, G^c is not Hamiltonian and as a consequence G has no $(2, n)$ -differential coloring, as well.

Now, assume that G does not contain as subgraphs $K_{1,1,n-3}$, $K_{1,n-1}$ and $K_{2,n-2}$. By Lemma 8, it follows that G admits a 4-coloring, in which two color sets contain at least 2 vertices and the remaining two at least 5 vertices. The main strategy of our approach is to compute ordered colorings implied by sequences of vertices such that adjacent vertices in the sequence are not connected by an edge. Let $C_i \subseteq V$ be the set of vertices with color i and $c_i = |C_i|$, for $i = 1, 2, 3, 4$, such that $c_1 \leq c_2 \leq c_3 \leq c_4$, $c_1, c_2 \geq 2$ and $c_3, c_4 \geq 5$. We distinguish the following cases:

C.1: $c_1 \geq 3$, $c_2 \geq 3$, $c_3 \geq 5$ and $c_4 \geq 5$. Since $|C_1| \geq 3$ and $|C_3| \geq 5$, by Lemma 9 it follows that there exist two vertices $a \in C_1$ and $c \in C_3$, such that $(a, c) \notin E$. Similarly, since $|C_2| \geq 3$ and $|C_4| \geq 5$, by Lemma 9 it follows that there exist two vertices $b \in C_2$ and $d \in C_4$, such that $(b, d) \notin E$. Finally, since $|C_3 \setminus \{c\}| \geq 4$ and $|C_4 \setminus \{d\}| \geq 4$, by Lemma 9 it follows that there exist two vertices $c' \in C_3$ and $d' \in C_4$, such that $c \neq c'$, $d \neq d'$ and $(c', d') \notin E$. Now consider the ordered coloring implied by the sequence $C_1 \setminus \{a\} \rightarrow \{a\} \rightarrow \{c\} \rightarrow C_3 \setminus \{c, c'\} \rightarrow \{c'\} \rightarrow \{d'\} \rightarrow C_4 \setminus \{d, d'\} \rightarrow \{d\} \rightarrow \{b\} \rightarrow C_2 \setminus \{b\}$. As $(a, c) \notin E$, $(b, d) \notin E$ and $(c', d') \notin E$, it follows that the color difference between any two vertices of G is at least two. Hence, G has a $(2, n)$ -differential coloring.

C.2: $c_1 = 2$, $c_2 \geq 4$, $c_3 \geq 5$ and $c_4 \geq 5$. Since G does not contain $K_{2,n-2}$ as subgraph, there exist two vertices, say $a \in C_1$ and $b \in C_2 \cup C_3 \cup C_4$ such that $(a, b) \notin E$. Assume without loss of generality that $b \in C_2$. Since $|C_2 \setminus \{b\}| \geq 3$ and $|C_3| \geq 5$, by Lemma 9 it follows that there exist two vertices $b' \in C_2$ and $c \in C_3$, such that $b' \neq b$ and $(b', c) \notin E$. Similarly, since $|C_3 \setminus \{c\}| \geq 4$ and $|C_4| \geq 5$, by Lemma 9 it follows that there exist two vertices $c' \in C_3$ and $d \in C_4$, such that $c' \neq c$ and $(c', d) \notin E$. As in the previous case, the ordered

coloring implied by the sequence $C_1 \setminus \{a\} \rightarrow \{b\} \rightarrow C_2 \setminus \{b\} \rightarrow \{b'\} \rightarrow \{c\} \rightarrow C_3 \setminus \{c, c'\} \rightarrow \{c'\} \rightarrow \{d\} \rightarrow C_4 \setminus \{d\}$ guarantees that G has a $(2, n)$ -differential coloring.

C.3: $c_1 = 2, c_2 = 3, c_3 \geq 5$ and $c_4 \geq 5$. Assume without loss of generality that $C_1 = \{a, a'\}$ and $C_2 = \{b, b', b''\}$. We distinguish two sub-cases:

C.3.1: *The subgraph of G induced by $C_1 \cup C_2$ is $K_{2,3}$, that is $C_1 \times C_2 \subseteq E$.* Since G does not contain $K_{2,n-2}$ as subgraph, there exists at least one vertex of C_1 , say vertex a , that is not a neighbor of a vertex of $C_3 \cup C_4$, say vertex $c \in C_3$, that is $(a, c) \notin E$. Since $|C_2| = 3$ and $|C_4| \geq 5$, by Lemma 9 it follows that there exist a vertex of C_2 , say vertex b , and a vertex of C_4 , say vertex d , that are not adjacent, that is $(b, d) \notin E$. Similarly, since $|C_3 \setminus \{c\}| \geq 4$ and $|C_4 \setminus \{d\}| \geq 4$, by Lemma 9 it follows that there exist two vertices $c' \in C_3$ and $d' \in C_4$, such that $c \neq c'$, $d \neq d'$ and $(c', d') \notin E$. Thus in the ordered coloring implied by the sequence $C_1 \setminus \{a\} \rightarrow \{a\} \rightarrow \{c\} \rightarrow C_3 \setminus \{c, c'\} \rightarrow \{c'\} \rightarrow \{d'\} \rightarrow C_4 \setminus \{d, d'\} \rightarrow \{d\} \rightarrow \{b\} \rightarrow C_2 \setminus \{b\}$, it holds that the color difference between any two vertices of G is greater or equal to two. Hence, G admits a $(2, n)$ -differential coloring.

C.3.2: *The subgraph of G induced by $C_1 \cup C_2$ is not $K_{2,3}$.* Assume without loss of generality that $(a, b) \notin E$. Since $|C_3| \geq 5$ and $|C_4| \geq 5$, by Lemma 9 it follows that there exist a vertex of C_3 , say vertex c , and a vertex of C_4 , say vertex d , that are not adjacent, that is $(c, d) \notin E$. Similarly, since $|C_3 \setminus \{c\}| \geq 4$ and $|\{a', b', b''\}| = 3$, by Lemma 9 it follows that there exist a vertex of $C_3 \setminus \{c\}$, say vertex c' , and a vertex of $\{a', b', b''\}$, say vertex x , that are not adjacent, that is $(x, c') \notin E$. First consider the case where $x = a'$. In this case, the ordered coloring implied by the sequence $\{b'\} \rightarrow \{b''\} \rightarrow \{b\} \rightarrow \{a\} \rightarrow \{a'\} \rightarrow \{c'\} \rightarrow C_3 \setminus \{c, c'\} \rightarrow \{c\} \rightarrow \{d\} \rightarrow C_4 \setminus \{d\}$ guarantees that G has a $(2, n)$ -differential coloring. Now consider the case, where $x \in \{b', b''\}$. As both cases are symmetric, we assume without loss of generality that $x = b'$. In this case, the ordered coloring implied by the sequence $\{a'\} \rightarrow \{a\} \rightarrow \{b\} \rightarrow \{b''\} \rightarrow \{b'\} \rightarrow \{c'\} \rightarrow C_3 \setminus \{c, c'\} \rightarrow \{c\} \rightarrow \{d\} \rightarrow C_4 \setminus \{d\}$ guarantees that G has a $(2, n)$ -differential coloring.

C.4: $c_1 = 2, c_2 = 2, c_3 \geq 5$ and $c_4 \geq 5$. Assume without loss of generality that $C_1 = \{a, a'\}$ and $C_2 = \{b, b'\}$. Again, we distinguish two sub-cases:

C.4.1: *The subgraph of G induced by $C_1 \cup C_2$ is $K_{2,2}$, that is $C_1 \times C_2 \subseteq E$.* Since G does not contain $K_{2,n-2}$ as subgraph, there exists at least one vertex of C_1 , say vertex a , that is not a neighbor of a vertex of $C_3 \cup C_4$, say vertex $c \in C_3$, that is $(a, c) \notin E$. Similarly, there exists a vertex of C_2 , say vertex b , that is not a neighbor of a vertex of $C_3 \cup C_4$, say vertex w , that is $(b, w) \notin E$.

First, consider the case where the subgraph of G induced by $C_2 \cup C_4$ is not K_{2,c_4} that is $w \in C_4$. Since $|C_3 \setminus \{c\}| \geq 4$ and $|C_4 \setminus \{w\}| \geq 4$, by

Lemma 9 it follows that there exists a vertex of C_3 , say c' , and a vertex of C_4 , say d , that are not adjacent, that is $c \neq c'$, $w \neq d$ and $(c', d) \notin E$. In this case, the ordered coloring implied by the sequence $C_1 \setminus \{a\} \rightarrow \{a\} \rightarrow \{c\} \rightarrow C_3 \setminus \{c, c'\} \rightarrow \{c'\} \rightarrow \{d\} \rightarrow C_4 \setminus \{d, w\} \rightarrow \{w\} \rightarrow \{b\} \rightarrow C_2 \setminus \{b\}$ guarantees that G has a $(2, n)$ -differential coloring.

Now consider the more interesting case where the subgraph of G induced by $C_2 \cup C_4$ is K_{2, c_4} , that is $w \in C_3$. We distinguish two sub cases:

C.4.1.1: $w \neq c$. Since $|C_3 \setminus \{c, w\}| \geq 3$ and $|C_4| \geq 5$, by Lemma 9 it follows that there exist vertices $p \in C_3$ and $q \in C_4$, that are not adjacent, that is $c \neq w \neq p$, and $(p, q) \notin E$. Similarly, since $|C_3 \setminus \{p, w\}| \geq 3$ and $|C_4 \setminus \{q\}| \geq 4$, by Lemma 9 it follows that there exist vertices $p' \in C_3$ and $q' \in C_4$, that are not adjacent, that is $p \neq w \neq p'$, $q \neq q'$ and $(p', q') \notin E$. If $p' \neq c$, the ordered coloring implied by the sequence $C_1 \setminus \{a\} \rightarrow \{a\} \rightarrow \{c\} \rightarrow \{p'\} \rightarrow \{q'\} \rightarrow C_4 \setminus \{q, q'\} \rightarrow \{q\} \rightarrow \{p\} \rightarrow C_3 \setminus \{p, p', c, w\} \rightarrow \{w\} \rightarrow \{b\} \rightarrow C_2 \setminus \{b\}$ guarantees that G has a $(2, n)$ -differential coloring. If $p' = c$, the ordered coloring implied by the sequence $C_1 \setminus \{a\} \rightarrow \{a\} \rightarrow \{p'\} \rightarrow \{q'\} \rightarrow C_4 \setminus \{q, q'\} \rightarrow \{q\} \rightarrow \{p\} \rightarrow C_3 \setminus \{p, p', w\} \rightarrow \{w\} \rightarrow \{b\} \rightarrow C_2 \setminus \{b\}$ guarantees that G has a $(2, n)$ -differential coloring.

C.4.1.2: $w = c$. Since the subgraph of G induced by $C_2 \cup C_4$ is K_{2, c_4} and $|C_2 \cup \{a\}| = 3$ and $|C_4| \geq 4$, by Lemma 9 it follows that there exist a vertex of C_4 , say q , not adjacent to vertex a , that is $(a, q) \notin E$. Similarly, since $|C_3 \setminus \{c\}| \geq 4$ and $|C_4 \setminus \{q\}| \geq 4$, by Lemma 9 it follows that there exist vertices $p' \in C_3$ and $q' \in C_4$, that are not adjacent, that is $q' \neq q$, $p' \neq c$ and $(p', q') \notin E$. Then the ordered coloring implied by the sequence $C_1 \setminus \{a\} \rightarrow \{a\} \rightarrow \{q\} \rightarrow C_4 \setminus \{q, q'\} \rightarrow \{q'\} \rightarrow \{p'\} \rightarrow C_3 \setminus \{p', c\} \rightarrow \{c\} \rightarrow \{b\} \rightarrow C_2 \setminus \{b\}$ guarantees that G is $(2, n)$ -differential colorable.

C.4.2: *The subgraph of G induced by $C_1 \cup C_2$ is not $K_{2, 2}$.* Assume without loss of generality that $(a, b) \notin E$.

First, consider the case where the subgraph of G induced by $\{a', b'\} \cup C_3 \cup C_4$ is K_{2, c_3+c_4} that is $\{a', b'\} \times C_3 \cup C_4 \subseteq E$. Since $|\{a, a', b'\}| = 3$ and $|C_3| \geq 4$, by Lemma 9 it follows that there exists a vertex of C_3 , say vertex c , not adjacent to vertex a , that is $(a, c) \notin E$. Similarly, since $|\{b, a', b'\}| = 3$ and $|C_4| \geq 4$, by Lemma 9 it follows that there exists a vertex of C_4 , say vertex d , not adjacent to vertex b , that is $(b, d) \notin E$. Since $|C_3 \setminus \{c\}| \geq 3$ and $|C_4 \setminus \{d\}| \geq 4$, by Lemma 9 it follows that there exist a vertex of C_3 , say vertex c' , and a vertex of C_4 , say vertex d' , that are not adjacent, that is $c \neq c'$, $d \neq d'$ and $(c', d') \notin E$. Then the ordered coloring implied by the sequence $C_1 \setminus \{a\} \rightarrow \{a\} \rightarrow \{c\} \rightarrow C_3 \setminus \{c, c'\} \rightarrow \{c'\} \rightarrow \{d'\} \rightarrow C_4 \setminus \{d, d'\} \rightarrow \{d\} \rightarrow \{b\} \rightarrow C_2 \setminus \{b\}$ guarantees that G has a $(2, n)$ -differential coloring.

Now we consider the case where the subgraph of G induced by $\{a', b'\} \cup C_3 \cup C_4$ is not K_{2, c_3+c_4} . There exist vertices $c \in C_3 \cup C_4$ and $p \in \{a', b'\}$,

that are not adjacent, that is $(p, c) \notin E$. Assume without loss of generality $c \in C_3$. Since $|C_3 \setminus \{c\}| \geq 3$ and $|C_4| \geq 4$, by Lemma 9 it follows that there exist vertices $c' \in C_3$ and $d \in C_4$, that are not adjacent, that is $c \neq c'$ and $(c', d') \notin E$. If $p = b'$ the ordered coloring implied by the sequence $\{a'\} \rightarrow \{a\} \rightarrow \{b\} \rightarrow \{b'\} \rightarrow \{c\} \rightarrow C_3 \setminus \{c, c'\} \rightarrow \{c'\} \rightarrow \{d'\} \rightarrow C_4 \setminus \{d'\}$ guarantees that G has a $(2, n)$ -differential coloring. If $p = a'$ the ordered coloring implied by the sequence $\{b'\} \rightarrow \{b\} \rightarrow \{a\} \rightarrow \{a'\} \rightarrow \{c\} \rightarrow C_3 \setminus \{c, c'\} \rightarrow \{c'\} \rightarrow \{d'\} \rightarrow C_4 \setminus \{d'\}$ guarantees that G has a $(2, n)$ -differential coloring.

From the above case analysis, it follows that G is $(2, n)$ -differential colorable. $\square$

As already stated, with more complicated arguments we are able to show that Theorem 3 holds for all planar graphs with $n \geq 12$ vertices. Recall that Theorem 3 guarantees the existence of a $(2, n)$ -differential coloring for graphs with $n \geq 36$ vertices. The proof consists of two main parts: In the first part of the proof, we show that all planar graphs with $n \geq 12$ vertices that do not contain as subgraphs $K_{1,1,n-3}$, $K_{1,n-1}$ and $K_{2,n-2}$ admit 6-colorings with the following two properties:

- P.1 Each color class contains at least two vertices.
- P.2 There exists a Hamiltonian path in the complement of the graph, which visits all vertices of the i -th colored class before visiting the vertices of the $(i+1)$ -th colored class, $i = 1, \dots, 5$, such that the edges that connect vertices of consecutive color classes are independent.

We refer to a 6-coloring satisfying properties P.1 and P.2 as a *good 6-coloring*. In the second part of the proof, we show how a good 6-coloring can be employed to show that any planar graph with $n \geq 12$ vertices that does not contain as subgraphs $K_{1,1,n-3}$, $K_{1,n-1}$ and $K_{2,n-2}$ admits $(2, n)$ -differential coloring.

We use induction to prove that all planar graphs with $n \geq 12$ vertices that do not contain as subgraphs $K_{1,1,n-3}$, $K_{1,n-1}$ and $K_{2,n-2}$ admit good 6-colorings. The base of the induction corresponds to the case where $n = 12$. To prove that all planar graphs with $n = 12$ vertices that do not contain as subgraphs $K_{1,1,9}$, $K_{1,11}$ and $K_{2,10}$ admit good 6-colorings is a tedious task, which we postpone for the time being (as many different cases need to be considered; see Corollary 1 of Lemma 13) and instead we present a more intuitive computer-aided approach: First, we generated all planar (non-isomorphic to each other) triangulations with $n = 12$ vertices (there are 7,595 of them) using plantri [28]. We used a simple integer linear program to test whether there is an assignment of the vertices of each of these graphs to one of the two available colors of each color class, such that both properties of a good 6-coloring hold (details follow). In total, we found 233 infeasible instances, all of which contained one of $K_{1,1,9}$, $K_{1,11}$, $K_{2,10}$ as a subgraph. From this we can conclude that the claim holds for all triangulations with $n = 12$ vertices. Note that this implies that it also holds for all planar graphs with 12 vertices, as one can always augment a planar graph to maximal planar without introducing all forbidden subgraphs mentioned above.

For the sake of completeness, in the following we give an outline of the integer linear program above. For a vertex $u \in V$ of the input graph G and a color c of color

set $C = \{1, \dots, 6\}$, the boolean variable $x_{u,c}^{(k)}$ indicates whether u is the k -th vertex with color c , where $k = 1, 2$. The first three constraints ensure that all vertices of G are properly colored (with colors from 1 to 6). The last two constraints guarantee property P.2 of a good 6-coloring.

$$\begin{array}{ll}
\text{maximize} & OPT \\
\text{subject to} & \sum_{u \in V} x_{u,c}^{(k)} = 1 \quad c \in C; k \in \{1, 2\} \\
& \sum_{c \in C} x_{u,c}^{(k)} = 1 \quad u \in V; k \in \{1, 2\} \\
& \sum_{k=1}^2 x_{u,c}^{(k)} + x_{v,c}^{(k)} \leq 1 \quad \forall (u, v) \in E; c \in C \\
& x_{u,c}^{(1)} + x_{u,c+1}^{(1)} + x_{v,c}^{(1)} + x_{v,c+1}^{(1)} \leq 1 \quad \forall (u, v) \in E; c \in \{1, 3, 5\} \\
& x_{u,c}^{(2)} + x_{u,c+1}^{(2)} + x_{v,c}^{(2)} + x_{v,c+1}^{(2)} \leq 1 \quad \forall (u, v) \in E; c \in \{2, 4\}
\end{array}$$

For the inductive step, assume that our claim holds for all planar graphs with a certain amount of vertices, say $n_0 \geq 12$, and consider a planar graph, say G , with $n_0 + 1$ vertices that does not contain as subgraphs $K_{1,1,n_0-2}$, K_{1,n_0} and K_{2,n_0-1} . By Euler's formula, it follows that G contains at least four vertices, say v_1, v_2, v_3 and v_4 , of degree less than 5. First, consider the case where at least one of the graphs $G - v_1$, $G - v_2$, $G - v_3$ and $G - v_4$ contains neither $K_{1,1,n_0-3}$ nor K_{1,n_0-1} nor K_{2,n_0-2} as subgraphs, say w.l.o.g. $G - v_1$. By the inductive hypothesis, it follows that $G - v_1$ admits a good 6-coloring. Since $\deg(v_1) \leq 5$, it follows that v_1 is incident to at most 5 vertices of potentially different colors in $G - v_1$. Hence, we can color v_1 with the sixth available color, say c . Observe that in the coloring of G each color class will still have at least two elements, satisfying property P.1 of a good 6-coloring. To obtain a Hamiltonian path in the complement graph G^c , which complies with property P.2 of a good 6-coloring, we will appropriately augment the corresponding Hamiltonian path of $G - v_1$ (which exists, by the induction hypothesis). To do so, it suffices to place v_1 anywhere between a pair of c -colored vertices along the Hamiltonian path of $G - v_1$. Hence, G admits a good 6-coloring.

To complete the proof in the inductive step, it remains to consider the case where each of $G - v_1$, $G - v_2$, $G - v_3$ and $G - v_4$ contain as subgraphs one of $K_{1,1,n_0-3}$, K_{1,n_0-1} or K_{2,n_0-2} . By the pigeonhole principle, it follows that at least two out of them will contain the same subgraph. Assume w.l.o.g. that $G - v_1$ and $G - v_2$ contain the same subgraph. Since $\deg(v_1) \leq 5$ and $\deg(v_2) \leq 5$, it follows that v_1 and v_2 will belong to the big parts of $K_{1,1,n_0-3}$, K_{1,n_0-1} or K_{2,n_0-2} . We consider the following subcases:

- C.1: $G - v_1$ and $G - v_2$ contain $K_{1,1,n_0-3}$ as subgraphs: In this case, by planarity it follows that both copies of $K_{1,1,n_0-3}$ have the same *poles*, that is, their small parts consist of the same pair of vertices. This implies that G contains $K_{1,1,n_0-2}$ as a subgraph, which is a contradiction.
- C.2: $G - v_1$ and $G - v_2$ contain K_{2,n_0-2} as subgraphs: In this case, planarity implies that both copies of K_{2,n_0-2} have the same *poles*, which implies that G contains K_{2,n_0-1} as a subgraph, which is also a contradiction.

C.3: $G - v_1$ and $G - v_2$ contain K_{1,n_0-1} as subgraphs: Let u_i be the vertex of $G - v_i$ that is adjacent to all the rest of its vertices, $i = 1, 2$. As already stated, $u_i \neq v_i$ holds, $i = 1, 2$. If $u_1 = u_2$, then it follows that G contains K_{1,n_0} as subgraph, which is a contradiction. On the other hand, if $u_1 \neq u_2$, then $\{u_1\}$, $\{u_2\}$ and $V - \{u_1, u_2\}$ form a $K_{1,1,n_0-3}$ in G . By planarity, it follows that not both $G - v_3$ and $G - v_4$ can contain as subgraph one of $K_{1,1,n_0-3}$, K_{1,n_0-1} or K_{2,n_0-2} ; a contradiction.

From the above case analysis it follows that $G - v_1$, $G - v_2$, $G - v_3$ and $G - v_4$ cannot simultaneously contain as subgraphs one of $K_{1,1,n_0-3}$, K_{1,n_0-1} or K_{2,n_0-2} and the claim follows.

Consider now a planar graph G with $n \geq 12$ vertices that admits a good 6-coloring. Next, we prove that G is $(2, n)$ -differential colorable. We color the vertices of G with increasing numbers from 1 to n according to the order in which they appear along the Hamiltonian path, say P , implied by the good 6-coloring. We claim that this coloring corresponds to a valid $(2, n)$ -differential coloring. Indeed, any two same-colored vertices that are consecutive along P are not adjacent in G (by the definition of 6-coloring). In addition, when two edges of different colors are consecutive along P , then they are not connected by an edge in G , due to property P.2 of a good 6-coloring. Hence, the coloring implied by the good 6-coloring of G is indeed a valid $(2, n)$ -differential coloring for G , as desired. We conclude with the following remark.

Remark. Using a simple ILP-formulation of the problem of finding a good 6-coloring, it can be shown that Theorem 3 holds for all planar graphs with $n \geq 12$ vertices.

It remains now to prove in detail the base of our induction. To this end, we first need to introduce some necessary notions. Let W be a subset of the set of vertices of a graph $G = (V, E)$. A vertex v is called *simplicial* to W , if every vertex in W is a neighbour of v . When $W = V \setminus \{v\}$, we call v a simplicial vertex of G . We denote the subgraph that W induces in G by $G[W]$. For $v, u \in V$ and S a subgraph of G , by writing vSu we mean a path in G with vertex set $\{v, u\} \cup V(S)$ and endpoints v, u . We define vS and Su similarly.

Lemma 10. Let $T = (V_T, E_T)$ be a forest on n vertices. Then, exactly one of the following is true:

- i. T is a tree with a vertex of degree $n - 1$ (i.e., T is a star), which implies that T^c has no Hamiltonian path.
- ii. T is a tree with a vertex of degree $n - 2$. In this case, T^c contains a Hamiltonian path with v as an endpoint. If in addition $|V_T| \geq 5$, then T^c contains a path with v as one endpoint and any other neighbour of v in T as second endpoint.
- iii. T has maximum degree at most $n - 3$ and T^c contains a Hamiltonian cycle.

Proof. The implication of (i) is straight-forward. So, let us first prove that the implications of (ii) are also correct. To this end, assume that T has a vertex v of degree $n - 2$. Since T is tree and the degree of v is $n - 2$, it follows that T has at least 4 vertices.

More precisely, if T has exactly 4 vertices, then T is a path and therefore its complement is also a path, which starts or ends at v . If $|V_T| \geq 5$, let u be the non-neighbour of v . Observe that $T^c \setminus \{v, u\}$ is a clique with at least three vertices. Let $w \in V_T \setminus \{v, u\}$. Since $|V_T \setminus \{v, u\}| \geq 3$, there is a $w' \in V_T \setminus \{v, u\}$ with $w' \neq w$, such that w' is a non-neighbour of u . Then $vw'w(T^c \setminus \{v, u\})w$ is a Hamiltonian path of T^c , which proves that the implications of (ii) is also correct.

To complete the proof it remains to prove that the complement of any forest T with n vertices, that contains no vertex of degree $n - 1$ or $n - 2$, has a Hamiltonian cycle. Our proof is by induction. The base of the induction corresponds to the case where $|V_T| \leq 4$. In this case, the maximum degree of T is at most one and it is straightforward to see that T^c is Hamiltonian.

For the inductive step, consider a forest T with $|V_T| \geq 5$ and assume that the complements of all forests with $|V_T| - 1$ vertices that do not satisfy (i) and (ii) are Hamiltonian. In this case, the radius of T is at least 2. Let r be a vertex of T that achieves the radius of T . Since T has no vertex of degree $n - 2$, it follows that there are at least two leaves, say u and v , that have distance to r at least two. So, $T - v$ is not a star. If $T - v$ has a vertex of degree $n - 3$, then $T \setminus \{u, v\}$ is a star with center r . Let u' and v' be the neighbours of u and v in T . Notice that $T \setminus \{r, u, v\}$ induces a clique in T^c . Hence, $rvu'(T^c \setminus \{r, u, v\})v'ur$ is a Hamiltonian cycle in T^c . Consider now the case, where $T - v$ has no vertex of degree $n - 3$. By the induction hypothesis, $T^c - v$ has a Hamiltonian cycle, say C . Since $|V(C)| \geq 4$ and the degree of v in T is one, there are two consecutive vertices u and u' of C that are non-adjacent to v in T . Extending C with v by deleting edge (u, u') and connecting u and u' to v implies a Hamiltonian cycle for T^c , which completes our inductive proof. $\square$

The *contraction* of an edge $e = (x, y)$ of G is the identification of x and y into a new vertex v_e , adjacent to all former neighbours of x and y . We say that H is a *minor* of G if H can be obtained from G by deleting edges and vertices, and by contracting edges. It is well-known that a graph G is planar if and only if it does not contain K_5 or $K_{3,3}$ as a minor.

Lemma 11. *Let $G = (V, E)$ be a planar graph and assume that the vertices of G can be partitioned into two sets C and H such that: (i) $G[C]$ is biconnected outerplanar with at least 6 vertices and diameter at least 3, and (ii) $G[H]$ has at least three vertices and $G[H]^c$ contains a Hamiltonian path. Then, G^c contains a Hamiltonian path with at least one end-point in C .*

Proof. By Lemma 6, there exists a Hamiltonian path P_c of $G[C]^c$ with end-points c and c' , which are not consecutive on the outerface of $G[C]$. Let P_h be a Hamiltonian path of $G[H]^c$ and denote by h and h' its end-points. For a proof by contradiction assume that G^c contains no Hamiltonian path with at least one end-point in C .

First, consider the case where $(h, h') \in E$. We claim that in graph G vertices h and h' cannot have more than two common neighbours from set C . To prove the claim, assume to the contrary that $w, w', w'' \in C$ are three common neighbours of h and h' . Contract C into a triangle with vertices w, w', w'' . Then, h, h', w, w' , and w'' form a K_5 -minor in G , contradicting the planarity of G . Hence, h and h' have at most two common neighbours from set C . Since P_c has at least five edges, by pigeonhole

principle it follows that there are two consecutive vertices, say z and z' of P_c (where c, z, z' and c' appear in this order in P_c) such that $(h, z) \notin E$ and $(h', z') \notin E$. Hence, $cP_czhP_hh'z'P_c c'$ is a Hamiltonian path of G^c ; a contradiction.

Next, consider the case where $(h, h') \notin E$. In this case, $P'_h = P_hhh'$ is a Hamiltonian cycle of $G[H]^c$. If c and c' are not simplicial to H in G , then we can just concatenate P_c and P'_h through a non-edge in G between an endpoint of P_c and a vertex of P'_h , and obtain a Hamiltonian path in G^c . Similarly, if $(c, c') \notin E(G)$, the $P'_c = P_cc'$ is a Hamiltonian cycle of $G[C]^c$ and since G excludes $K_{3,3}$ by planarity, we can always concatenate cycles P'_c and P'_h with a non-edge of G between them and obtain a Hamiltonian path of G^c .

Therefore, we may assume that $(c, c') \in E$ and that for every $u \in H$, we have $(u, c) \in E$ and $(u, c') \in E$. Then, (c, c') is a chord of $G[C]$. Let A and B be the two connected components of $G[C] \setminus \{c, c'\}$. Notice that since the vertices of $A \cup B$ are the internal vertices of P_c , there exist $\alpha \in A$ and $\beta \in B$ such that α and β are consecutive in P_c , where c, α, β, c' appear with this order in P_c . Next, we make two claims. Our first claim is that both α and β have at most two neighbours each from set H in G . To prove the first claim assume to the contrary that one of α or β , say w.l.o.g α , has three neighbours w, w' and w'' from set H in G . Then, $\{\alpha, c, c'\} \times \{w, w', w''\}$ is a $K_{3,3}$ in G , which contradicts the planarity of G . So, the first claim holds. Our second claim is that α and β cannot have two common neighbours in H . Again, the proof is by contradiction. Let w and w' be two such common neighbours of α and β . Then, vertices α, c, c', w and w' form a K_5 -minor in G , contradicting the planarity of G . Since P'_h has at least three edges, it follows that there are two consecutive vertices γ and δ of P'_h such that $(\alpha, \gamma) \notin E$ and $(\beta, \delta) \notin E$. This implies that $cP_c\alpha\gamma P'_h\delta\beta P_cc'$ is a Hamiltonian path of G^c , a contradiction. Since both cases, where $(h, h') \in E$ and $(h, h') \notin E$ were lead to a contradiction, the proof of the theorem follows. $\square$

The following lemma by Yang and Yuan [29] describes a structural property of planar graphs of maximum diameter two.

Lemma 12 (Yang and Yuan [29]). *Let G be a planar graph of diameter at most two 2. Then, the vertices of G can be partitioned into two sets such that each one induces a forest in G .*

Finally, the following lemma will allow us to obtain a good 6-coloring of a planar graph with $n = 12$ vertices, which does not contain as subgraphs $K_{1,11}$, $K_{2,10}$ and $K_{1,1,9}$.

Lemma 13. *Let $G = (V, E)$ be planar on $n = 12$ vertices, which does not contain as subgraphs $K_{1,11}$, $K_{2,10}$ and $K_{1,1,9}$. Then, G^c contains a Hamiltonian path.*

Proof. It is enough to prove the lemma for maximal planar graphs with $n = 12$ vertices, since any planar graph, which does not contain as subgraphs $K_{1,11}$, $K_{2,10}$ and $K_{1,1,9}$, can be augmented to maximal by adding edges and simultaneously avoiding creating $K_{1,11}$, $K_{2,10}$ and $K_{1,1,9}$. So, we will assume w.l.o.g. that G is maximal planar with a fixed planar embedding. Moreover, to keep notational consistency with the rest of the section so far, from now on we will prefer to refer to the obstructions in the

statement of the Lemma as $K_{1,n-1}$, $K_{2,n-2}$ and $K_{1,1,n-3}$ instead of $K_{1,11}$, $K_{2,10}$ and $K_{1,1,9}$. We distinguish two main cases based on the diameter of G :

C.1: *The diameter of G is at least 3:* Let u and v be two vertices of G such that $d(u, v) = \text{diam}(G)$. Since G is maximal planar, it is 3-connected. Hence, there are at least three internally vertex-disjoint paths from u to v . Let P_1 , P_2 and P_3 be three such paths chosen such that $|V(P_1)| + |V(P_2)| + |V(P_3)|$ is minimum, where $V(P_i)$ denotes the set of vertices of path P_i , $i = 1, 2, 3$. Since $d(u, v) \geq 3$, it follows that each of P_1 , P_2 and P_3 consists of at least 4 vertices and by their choice, they must be induced in G .

Let $i, j \in \{1, 2, 3\}$ be such that $i \neq j$. Denote by C_{ij} , to be called the *interior* of $P_i \cup P_j$, the vertices of G contained in the open region of $G \setminus (P_i \cup P_j)$ that does not contain P_{6-i-j} . Similarly, denote by D_{ij} , which we call the *exterior* of $P_i \cup P_j$, the set of all vertices of G contained neither in the interior of $P_i \cup P_j$ nor in $P_i \cup P_j$. In other words, $D_{ij} := V \setminus (V(P_i) \cup V(P_j) \cup C_{ij})$. By definition and the planarity of G , $P_i \cup P_j$ separates C_{ij} from D_{ij} . Since D_{ij} contains the internal vertices of P_{6-i-j} , it follows that $|D_{ij}| \geq 2$. Additionally, $C_{12} \cup C_{23} \cup C_{31} = V \setminus (V(P_1) \cup V(P_2) \cup V(P_3))$. We also denote by G_{ij} the graph induced by $P_i \cup P_j$, that is, $G_{ij} = G[P_i \cup P_j]$. Recall that the paths P_1, P_2, P_3 are induced and that P_{6-i-j} separates the region that contains the vertices of D_{ij} , hence all the edges of G_{ij} not on P_i or P_j must be embedded in the region containing the vertices of C_{ij} . It follows that G_{ij} is outerplanar. In addition, G_{ij} does not contain $K_{1,n-1}$ as a subgraph (because G_{ij} is a subgraph of G) and is of diameter at least three (because of the choice of u and v). By Lemma 6, it follows that G_{ij}^c contains a Hamiltonian path (observe that the graph of Figure 3e has diameter two, so it cannot be isomorphic to G_{ij}).

We define the *ij-discrepancy* to be the quantity $\delta_{ij} := ||C_{ij}| - |D_{ij}||$. The existence of an *ij-discrepancy* at most 1 immediately allows us to find a Hamiltonian path in G^c as follows: alternate between the vertices of C_{ij} and D_{ij} to obtain a Hamiltonian path of $G^c[C_{ij} \cup D_{ij}]$ and apply Lemma 11 to G_{ij} and $G[C_{ij} \cup D_{ij}]$. Whenever one of the three discrepancies is at most 1, we will call a path constructed as above a *standard path* in G^c . It remains to consider the cases where all three *ij-discrepancies* are at least 2. We distinguish three main subcases based on the maximum length of the paths P_1, P_2 and P_3 .

C.1.1: $\max\{|V(P_i)| : i = 1, 2, 3\} = 4$: Since $d(u, v) \geq 3$, it follows that $|V(P_1)| = |V(P_2)| = |V(P_3)| = 4$. Since all three exteriors contain at least two vertices, the only possible way that all *ij-discrepancies* are at least 2 is that an interior, say C_{12} , contains all 4 remaining vertices v_1, v_2, v_3, v_4 of G not on a P_i path. If the four vertices induce a K_4 , then by the planarity of G , one of them, say v_1 , will be the only vertex in one of the two open regions in the plane defined by removing the triangle of the remaining three vertices v_2, v_3, v_4 . Therefore, v_1 is only adjacent to v_2, v_3, v_4 . In this case, since the discrepancy between $C_{12} \setminus \{v_1\}$ and D_{12} is 1, we can consider a standard path in $G^c \setminus \{v_1\}$ with one end-point in $V(G_{12})$ and complete it to a Hamiltonian path for G^c with v_1 .

Otherwise, there are two of the four vertices, say v_1, v_2 , that are non-adjacent in G . Let $D_{12} = \{u_1, u_2\}$. Then $G^c[C_{ij} \cup D_{ij}]$ has the Hamiltonian path $v_3u_1v_1v_2u_2v_4$ and the Hamiltonian path of G^c follows by Lemma 11.

C.1.2: $\max\{|V(P_i)| : i = 1, 2, 3\} = 5$: In this case, there are only three cases for all three ij -discrepancies to be at least 2:

- (i) $|V(P_1)| = 5, |V(P_2)| = |V(P_3)| = 4, |C_{12}| = |C_{23}| = |C_{31}| = 1,$
- (ii) $|V(P_1)| = 4, |V(P_2)| = |V(P_3)| = 5, |C_{12}| = |C_{31}| = 1, \text{ and}$
- (iii) $|V(P_1)| = |V(P_2)| = |V(P_3)| = 5, |C_{12}| = 1, |C_{23}| = |C_{31}| = 0.$

We study each of these three cases separately in the following.

- (i) Since each exterior consists of a single vertex, we may assume that $C_{ij} = \{v_{ij}\}$. Note that v_{12}, v_{23}, v_{31} form an independent set in G . Now, let u' and v' be the unique neighbours of u and v in P_1 , respectively. Let also w be the middle vertex of P_1 . Recall that P_1 is induced in G . Hence, u' and v' are not adjacent. If v_{12}, u', v_{31} and v' induce a cycle C in G , then w is the unique vertex contained in one of the two open regions of the plane after removing C . Hence, w is adjacent only to the vertices of C . Notice that vertices u', v', v_{23}, v_{12} and v_{31} form a path in G^c . By Lemma 11 for G_{23} and the rest of the vertices of $G - w$, we obtain a Hamiltonian path P for $G - w$ with one endpoint in $V(G_{23})$. We complete P by connecting the end-point of P in $V(G_{23})$ to w to obtain a Hamiltonian path of G^c . On the other hand, if v_{12}, u', v_{31} and v' do not induce a cycle C in G , then there is an edge of C missing. Assume w.l.o.g. that $(v_{23}, u') \notin E$. In this case, vertices $v_{31}, v_{12}, u', v', v_{23}$ and w form a path in G^c and we apply Lemma 11 to obtain a Hamiltonian path for G^c .
- (ii) As in the previous subcase, let $C_{12} = \{v_{12}\}$ and $C_{31} = \{v_{23}\}$. Let u', w and v' be the internal vertices of P_3 , where u' and v' are the neighbours of u and v in P_3 , respectively. Recall that P_3 is induced. Hence, $(u', v') \notin E$. It follows that vertices w, v_{12}, u' and v' form a Hamiltonian path of $G^c[C_{12} \cup D_{12}]$. To obtain a Hamiltonian path for G^c , we apply Lemma 11 to G_{12} and $G[C_{12} \cup D_{12}]$.
- (iii) Exactly as in the previous subcase, vertices w, v_{12}, u' and v' form a Hamiltonian path of $G^c[C_{12} \cup D_{12}]$. So, in order to obtain a Hamiltonian path for G^c , we apply Lemma 11 to G_{12} and $G[C_{12} \cup D_{12}]$.

C.1.3: $\max\{|V(P_i)| : i = 1, 2, 3\} \geq 6$: Suppose w.l.o.g. that $|V(P_1)| \geq 6$. Let P'_1 be the path consisting of the internal vertices of P_1 . Since P_1 is induced, P'_1 contains a Hamiltonian path by Lemma 10. Now, observe that $|C_{12} \cup C_{23} \cup C_{31}| = |V \setminus V(P_1 \cup P_2 \cup P_3)| \leq 2$. We now claim that every vertex in $C_{12} \cup C_{23} \cup C_{31}$ has at most three neighbours in $V(P'_1)$. To prove this claim, assume to the contrary that $w \in C_{12} \cup C_{31}$ has at least four neighbours in $V(P'_1)$. If this is the case, we can replace P_1 with a shorter path between u and v through w , which contradicts the choice of

P_1 , P_2 and P_3 . Hence, every vertex in $C_{12} \cup C_{23} \cup C_{31}$ has at most three neighbours in $V(P_1')$. We study each of these three cases implied by our earlier observation that $|C_{12} \cup C_{23} \cup C_{31}| \leq 2$ in the following.

(i) $|C_{12} \cup C_{23} \cup C_{31}| = 0$: In this case, $G^c[C_{23} \cup D_{23}]$ is the graph $P_1'^c$. Hence, $G^c[C_{23} \cup D_{23}]$ contains a Hamiltonian path. To obtain a Hamiltonian path for G^c , we apply Lemma 11 to G_{23} and $G[C_{23} \cup D_{23}]$.

(ii) $|C_{12} \cup C_{23} \cup C_{31}| = 1$: Let x be the unique vertex of $C_{12} \cup C_{23} \cup C_{31}$. We consider the two cases where P_1' has either at least 5 vertices or exactly 4 vertices.

In the former case, $P_1'^c$ contains a Hamiltonian cycle and we connect x to a vertex of $P_1'^c$ in G^c . Since $G^c[C_{23} \cup D_{23}]$ is the graph $G^c[V(P_1') \cup \{x\}]$, we apply Lemma 11 to G_{23} and $G[C_{23} \cup D_{23}]$ to obtain a Hamiltonian path for G^c .

Consider now the latter case, in which P_1' has exactly 4 vertices. In this case, P_2 has exactly five vertices, while P_3 has four vertices. We distinguish three subcases based on whether x belongs to C_{12} , to C_{23} or to C_{31} . Assume first that x belongs to C_{23} , that is, $C_{23} = \{x\}$. It follows that $G^c[C_{23} \cup D_{23}]$ is the graph $G^c[V(P_1') \cup \{x\}]$ and clearly $G^c[C_{23} \cup D_{23}]$ contains a Hamiltonian path. A Hamiltonian path for G^c follows by Lemma 11 for G_{23} and $G[C_{23} \cup D_{23}]$. So, next assume that x belongs to C_{12} , that is, $C_{12} = \{x\}$. In this case, the 12-discrepancy is 1 and as stated earlier G^c has a Hamiltonian path. Finally, we consider the case where x belongs to C_{31} , that is, $C_{31} = \{x\}$. Let P_2' be the path consisting of the internal vertices, say u' , w and v' , of P_2 , where u' and v' are the neighbors of u and v , respectively. Then, vertices w , x , u' and v' form a Hamiltonian path of $G^c[C_{31} \cup D_{31}]$. To obtain a Hamiltonian path for G^c , we apply Lemma 11 to G_{31} and $G[C_{31} \cup D_{31}]$.

(iii) $|C_{12} \cup C_{23} \cup C_{31}| = 2$: Assume that $C_{12} \cup C_{23} \cup C_{31} = \{w, w'\}$. The only possible way to have that all three discrepancies are at least 2, is when $C_{23} = \{w, w'\}$. Hence there are no edges in G between $\{w, w'\}$ and $V(P_1)$. Since $P_1'^c$ has a Hamiltonian path, $G^c[C_{23} \cup D_{23}]$, which is the graph $G^c[V(P_1') \cup \{w, w'\}]$, contains a Hamiltonian path. Hence, to obtain a Hamiltonian path for G^c , we apply Lemma 11 to G_{23} and $G[C_{23} \cup D_{23}]$.

C.2: *The diameter of G is 2:* By Lemma 12, there is a partition of V into two sets V_1 and V_2 , such that both $T_1 = G[V_1]$ and $T_2 = G[V_2]$ are forests. Let $n_1 = |V(T_1)|$ and $n_2 = |V(T_2)|$. The degenerate cases where $\min\{n_1, n_2\} \leq 2$ are straightforward and left as an exercise to the reader. So, we may assume in what follows that $\min\{n_1, n_2\} \geq 3$. We consider the following subcases:

C.2.1: *T_1 and T_2 are stars:* Let r_1 and r_2 be the centers of T_1 and T_2 , respectively. Then, $T_1^c - r_1$ and $T_2^c - r_2$ are complete subgraphs of G^c . Hence, they have Hamiltonian cycles where the vertices appear in any possible order.

A *linking path* of G^c is a path of the form $t_1 r_2 r_1 t_2$ in G^c , where t_1 and t_2 are two vertices of T_1 and T_2 , respectively. If a linking path exists, then $(T_1^c - r_1) t_1 r_2 r_1 t_2 (T_2^c - r_2)$ gives us a Hamiltonian path in G^c . So, we may assume that such a path does not exist in G^c . On the other hand, since G does not contain simplicial vertices, none of r_1 and r_2 are simplicial to $V(T_2)$ and $V(T_1)$, respectively. So, there are two vertices t_1 and t_2 of T_1 and T_2 , respectively, such that $(r_1, t_2) \notin E$ and $(r_2, t_1) \notin E$. Since G does not contain $K_{2,n-2}$ as a subgraph, it follows that $\{t_1, t_2\} \neq \{r_1, r_2\}$. We assume w.l.o.g. that $t_1 \neq r_1$. A *linking edge* of G^c is an edge (s_1, s_2) of G^c , where s_1 is a vertex of T_1 different from r_1 and t_1 and s_2 is a vertex of T_2 different from r_2 and t_2 . We distinguish two cases based on whether t_2 is the vertex r_2 or not.

- (i) $t_2 \neq r_2$: In this case, (r_1, r_2) is an edge of G , as otherwise $t_1 r_2 r_1 t_2$ is a linking path in G^c . Suppose that there are no linking edges in G^c , that is, if τ_1 and τ_2 are two vertices of T_1 and T_2 not in $\{r_1, t_1\}$ and $\{r_2, t_2\}$, respectively, then (τ_1, τ_2) is an edge of G . If $\min\{n_1, n_2\} = 3$ (w.l.o.g. say $T_1 = \{r_1, t_1, s_1\}$), then G contains a $K_{1,1,n-3}$ with parts $\{s_1\}, \{r_2\}$ and $V \setminus \{s_1, r_2, t_2\}$, a contradiction. If $\min\{n_1, n_2\} \geq 4$, then G clearly contains a $K_{3,3}$, a contradiction to its planarity. Therefore, there exists a linking edge (s_1, s_2) in G^c . Then G^c contains a Hamiltonian path of the form $r_2 t_1 (T_1^c - r_1) s_1 s_2 (T_2^c - r_2) t_2 r_1$.
- (ii) $t_2 = r_2$: In this case, we may assume that r_1 is adjacent to all leaves of T_2 , as otherwise we can transform this case into the previous one. Again, the presence of a linking edge (s_1, s_2) in G^c allows for a Hamiltonian path in G^c that is of the form $r_1 r_2 t_1 (T_1^c - r_1) s_1 s_2 (T_2^c - r_2)$. Hence, we may assume that no linking edge exists. By symmetry, r_2 must also be adjacent to all vertices of T_1 not in $\{r_1, t_1\}$. Then, if $n_2 \geq 4$, then G contains $K_{3,3}$ (with parts $\{r_1, t_1, s_1\}$ and r_2 together with two any leaves of T_2) as a subgraph, a contradiction to its planarity. Otherwise, $n_2 = 3$ implies that T_2 consists of r_2, t_2 and s_2 . In this case, G contains again $K_{3,3}$ (with parts $\{r_2, t_2, s_2\}$ and any three leaves of T_1) as a subgraph, again a contradiction. So, G must have at least one linking edge and therefore G^c contains a Hamiltonian path.

C.2.2: *Neither T_1 nor T_2 is a star*: In this case, each of T_1 and T_2 has to satisfy either condition (ii) or condition (iii) of Lemma 10. We consider the corresponding cases in the following.

- (i) Both T_1 and T_2 satisfy condition (iii) of Lemma 10: Let $u_1 \dots u_{n_1} u_1$ and $v_1 \dots v_{n_2} v_1$ be the Hamiltonian cycles implied by Lemma 10 of T_1^c and T_2^c , respectively. Since $n_1, n_2 \geq 3$, it follows by the planarity of G that not all edges between the vertices of T_1 and the vertices of T_2 are present in G . Assume w.l.o.g. that $(u_{n_1}, v_1) \notin E$. In this case, $u_1 \dots u_{n_1} v_1 \dots v_{n_2}$ is a Hamiltonian path of G^c .

(ii) T_1 satisfies condition (ii) and T_2 satisfies condition (iii): As in the previous case, let $C = v_1 \dots v_{n_2} v_1$ be the Hamiltonian cycle of T_2^c implied by Lemma 10. Since T_1 is not a star, it follows that $n_1 \geq 4$. Let r_1 be the vertex of T_1 with degree $n_1 - 2$ and s_1 the vertex of T_1 , which is not neighboring with r_1 . Then, T_1^c has a Hamiltonian path P which starts from vertex r_1 , then visits vertex s_1 , then visits any leaf of T_1 other than s_1 , then the common neighbor of r_1 and s_1 and then the remaining leaves of T_1 in any order (if any). Denote by u_1 the end-point P , that is different from r_1 . We can assume that all edges between $\{r_1, u_1\}$ and the vertices of T_2 exist in G , as otherwise we could appropriately concatenate P and C to obtain a Hamiltonian path of G^c . Since G does not contain $K_{3,3}$ as a subgraph, all vertices of T_1 other than r_1 and u_1 must have at most two neighbours in T_2 . Observe that if there are two consecutive internal vertices x_1 and x_2 of P and two consecutive vertices y_1 and y_2 of C such that $(x_1, y_1) \notin E$ and $(x_2, y_2) \notin E$, then $r_1 P x_1 y_1 C y_2 x_2 P u_1$ is a Hamiltonian path of G^c . If $n_2 \geq 4$, this is always true by the pigeonhole principle for any two internal vertices of P . Consider now the case where $n_2 = 3$ and assume that the vertices of T_2 are t_1, t_2 and t_3 . In this case, the approach mentioned above is not applicable only when every internal vertex of P is adjacent to exactly the same two vertices of T_2 , say w.l.o.g. to t_1 and t_2 . In this particular case, however, G contains $K_{3,3}$ (with parts $\{r_1, t_1, t_2\}$ and tree internal vertices of P) as a subgraph, a contradiction to its planarity. Note that the case where T_2 satisfies condition (ii) and T_1 satisfies condition (iii) is symmetric.

(iii) Both T_1 and T_2 satisfy condition (ii) of Lemma 10: Since T_1 and T_2 are not stars, it follows that $n_1, n_2 \geq 4$. Let r_1 and r_2 be the vertices of T_1 and T_2 with degrees $n_1 - 2$ and $n_2 - 2$, respectively. Let also s_1 and s_2 be the vertices of T_1 and T_2 that are not neighbors of r_1 and r_2 in T_1 and T_2 , respectively. If both n_1 and n_2 are at least 5, then T_1 and T_2 contain at least three vertices each that are all different from r_1, s_1, r_2, s_2 . Since G does not contain $K_{3,3}$ as a subgraph, it follows that there exist two vertices w_1 and w_2 in T_1 and T_2 , respectively, such that $w_1 \notin \{r_1, s_1\}$, $w_2 \notin \{r_2, s_2\}$ and $(w_1, w_2) \notin E$. By Lemma 10.(ii), there are Hamiltonian paths of T_1^c and T_2^c with end-points r_1 and w_1 on one hand, and r_2 and w_2 on the second hand. Concatenating them through w_1 and w_2 , we obtain a Hamiltonian path of G^c .

If n_1 or n_2 is less than 5, then either T_1 or T_2 , say w.l.o.g. T_1 , is a path on four vertices (recall that $n_1, n_2 \geq 4$). Let v_1, v_2, v_3 and v_4 be the vertices of T_1 as they appear along the path and define r_2 and s_2 for T_2 as above. Notice that T_1^c is uniquely defined as the path connecting vertices v_2, v_4, v_1 and v_3 in this order. We now claim that v_2 and v_3 cannot be both simplicial to the vertices of T_2 other than r_2 and s_2 . Assume that this is not the case. Then, G would contain

$K_{3,3}$ (with parts $\{v_2, v_3, r_2\}$ and three vertices of T_2 other than r_2 and s_2) as a subgraph, a contradiction to its planarity. So, the claim holds and we can assume w.l.o.g. that there exists a vertex w of T_2 other than r_2 and s_2 such that $(w, v_1) \notin E$. Then we concatenate T_1^c with a Hamiltonian path of T_2^c through v_1 and w and obtain a Hamiltonian path of G^c .

C.2.3: T_1 is a star and T_2 is not a star: Let r_1 be the center of T_1 and observe again that $T_1^c - r_1$ is a complete graph. Since G does not contain $K_{1,n-1}$ as a subgraph, there is a vertex t_2 of T_2 that is not adjacent to r_1 in G , that is, $(r_1, t_2) \notin E$. Moreover, if T_2 has three vertices it is always a subgraph of a star, hence this case would be an easier version of C.2.1. Therefore, we may assume that T_2 has at least four vertices. Note that T_2 satisfies either condition (ii) or condition (iii) of Lemma 10. We investigate each of these two cases in what follows.

- (i) T_2 satisfies condition (ii) of Lemma 10: Let r_2 be the vertex of T_2 with degree $n_2 - 2$ and denote by $\{\tau_1, \dots, \tau_{n_2-2}\}$ the neighbourhood of r_2 in T_2 . Let also s_2 be the vertex of T_2 that is not neighboring r_2 in T_2 . Hence, s_2 is either a leaf or an isolated vertex of T_2 . So, we can assume w.l.o.g. that s_2 is not adjacent to $\{\tau_1, \dots, \tau_{n_2-3}\}$. Moreover, notice that the neighbourhood of r_2 in T_2 consists of at least two vertices and that it induces a clique in T^c . We now claim that not all edges between $\{\tau_1, \dots, \tau_{n_2-2}\}$ and the leaves of T_1 are present in G . For a proof by contradiction, assume that this is not the case. We consider two subcases. If T_1 has more than two leaves, then G would contain $K_{3,3}$ as a subgraph (indeed, any subset of $\{r_1, \tau_1, \dots, \tau_{n_2-2}\}$ with three elements together with any three leaves of T_1 would yield a $K_{3,3}$ in G), a contradiction to its planarity. If T_1 has exactly two leaves, then $n_2 = 9$ and G contains a $K_{3,3}$ with parts the two leaves of T_1 together with r_2 and any three vertices of $\{\tau_1, \dots, \tau_{n_2-2}\}$. So, our claim holds. Hence, there exist a leaf w of T_1 and a vertex $z \in \{\tau_1, \dots, \tau_{n_2-2}\}$ that are not adjacent in G , that is, $(w, z) \notin E$.

Observe that r_2 cannot be adjacent to all the vertices of T_1 , because G would contain a $K_{1,1,n-3}$. If $t_2 = r_2$, then there is a Hamiltonian path in G^c that is of the form $r_1 r_2 T_2^c z w (T_1^c - r_1)$, as desired. Otherwise, r_2 is adjacent to r_1 in G . In this case, by planarity it follows that there exists a leaf w' of T_1 that is different from w , such that r_2 is not adjacent to w' , that is $(r_2, w') \notin E$. If $t_2 \neq s_2$, then G^c contains a Hamiltonian path that is of the form $r_1 t_2 T_2^c r_1 w (T_1^c - r_1)$. Otherwise if $t_2 = s_2$, G^c contains a Hamiltonian path that is of the form $r_2 w' (T_1^c - r_1) w z (T_2^c - r_2) t_2 r_1$. From the above, it follows that if T_2 satisfies condition (ii) of Lemma 10, then there is always a Hamiltonian path in G^c .

- (ii) T_2 satisfies condition (iii) of Lemma 10: Let $C = v_1 \dots v_{n_2} v_1$ be a Hamiltonian cycle of T_2^c . We will assume w.l.o.g. that $v_1 = t_2$. Sup-

pose first that $n_1 \geq 4$. Then, if both v_2 and v_{n_2} are simplicial to all vertices of T_1 other than r_1 , then G has $K_{3,3}$ (with parts $\{r_1, v_2, v_{n_2}\}$ and three vertices of T_1 other than r_1) as subgraph, a contradiction to its planarity. Therefore, there is a vertex w of T_1 other than r_1 such that w.l.o.g. $(w, v_{n_2}) \notin E$. We find a Hamiltonian cycle in G^c by starting with $r_1 v_1 v_2 \dots v_{n_2} w$ and traversing the rest of $T_1^c - r_1$. Otherwise, $n_1 = 3$ and we can assume that T_1 consists of vertices r_1, t_1 and s_1 . Notice that if both t_1 and s_1 are simplicial to the vertices of T_2 , then G contains $K_{2, n-2}$ as a subgraph (with poles t_1 and s_1). Hence, there is a vertex v_i of C , such that w.l.o.g. $(s_1, v_i) \notin E$. Now, observe that if there is a vertex x of T_2 that has at most one neighbour in T_1 , then $T_1 \cup \{x\}$ and $T_2 - x$ are two forests that define a partition of V , which belongs to a case that we have already considered. So, we may assume that every vertex of T_2 has at least two neighbours in T_1 .

It follows that for $v_1 = t_2$, we have that $(t_1, v_1) \in E$ and $(s_1, v_1) \in E$. Moreover, if there are two consecutive vertices v_j and v_{j+1} of C that are not adjacent to r_1 , then $t_1 s_1 v_i C v_j r_1 v_{j+1} C v_{i-1}$ is a Hamiltonian path of G^c . Since $(r_1, v_1) \notin E$, we may assume that $(r_1, v_2) \in E$ and $(r_1, v_{n_2}) \in E$. In the following, we may assume that all edges between $\{t_1, s_1\}$ and $\{v_2, v_{n_2}\}$ must be present in G . To see this assume that $(s_1, v_2) \notin E$. The remaining cases are symmetric. Then $t_1 s_1 v_2 C v_1 r_1$ is a hamiltonian path of G^c . Now, suppose that $(v_2, v_{n_2}) \in E$. Then by contracting v_1 on s_2 we obtain a K_5 -minor in G with vertices $r_1, s_1, t_1, v_2, v_{n_2}$, a contradiction. It follows that $(v_2, v_{n_2}) \notin E$. If $(v_3, s_1) \notin E$, then $t_1 s_1 v_3 C v_{n_2} v_2 v_1 r_1$ is a hamiltonian path of G^c . Symmetrically, if $(v_3, t_1) \notin E$, then $s_1 t_1 v_3 C v_{n_2} v_2 v_1 r_1$ is a hamiltonian path of G^c . So, we may further assume that $(v_3, t_1) \in E$ and $(v_3, s_1) \in E$. So, since the graph on vertices r_1, t_1, s_1 and v_3, v_2, v_{n_2} cannot contain $K_{3,3}$ as subgraph, it must be $(r_1, v_3) \notin E$. Recall that there are no two consecutive vertices of C that are non-adjacent to r_1 , therefore $(r_1, v_4) \in E$. Again, since the graph on vertices r_1, t_1, s_1 and v_4, v_2, v_{n_2} cannot contain $K_{3,3}$ as subgraph, w.l.o.g it must be $(s_1, v_4) \notin E$. But then $t_1 s_1 v_4 C v_3 r_1$ is a hamiltonian path of G^c and we are done.

From the above case analysis, it follows that G^c has always a Hamiltonian path, as desired. $\square$

Clearly, by taking every two consecutive vertices of the Hamiltonian path of G^c as a color class, a good 6-coloring of G is implied. So, we may conclude with the following corollary.

Corollary 1. *A planar graph with $n = 12$ vertices that does not contain as subgraphs $K_{1,1,9}$, $K_{1,11}$ and $K_{2,10}$ admits a good 6-coloring.*

Figure 5: (a) An instance of the $(3, n)$ -differential coloring problem for $n = 6$; (b) An instance of the $(3, 2n')$ -differential coloring problem constructed based on graph G .

4. NP-completeness Results

As already stated, all graphs are $(2, 2n)$ -differential colorable (recall Lemma 1). So, the next reasonable question to ask is whether a given graph is $(3, 2n)$ -differential colorable. In the following theorem, we prove that this problem is NP-complete.

Theorem 4. *Given a graph $G = (V, E)$ on n vertices, it is NP-complete to determine whether G has a $(3, 2n)$ -differential coloring.*

Proof. The problem is clearly in NP, since a non-deterministic algorithm needs only to guess an assignment of distinct colors (out of $2n$ available colors) to the vertices of the graph and then it is easy to verify in polynomial time whether this assignment corresponds to a differential coloring of color difference at least 3.

In order to prove that the problem is NP-hard, we employ a reduction from the $(3, n)$ -differential coloring problem, which is known to be NP-complete [6]. More precisely, let $G = (V, E)$ be an instance of the $(3, n)$ -differential coloring problem, i.e., graph G is an n -vertex graph with vertex set $V = \{v_1, v_2, \dots, v_n\}$. We will construct a new graph G' with $n' = 2n$ vertices, so that G' is $(3, 2n')$ -differential colorable if and only if G is $(3, n)$ -differential colorable; see Fig. 5.

Graph $G' = (V', E')$ is constructed by attaching n new vertices to G that form a clique; see the gray colored vertices of Fig. 5b. That is, $V' = V \cup U$, where $U = \{u_1, u_2, \dots, u_n\}$ and $(u, u') \in E'$ for any pair of vertices u and $u' \in U$. In addition, for each pair of vertices $v \in V$ and $u \in U$ there is an edge connecting them in G' , that is $(v, u) \in E'$. In other words:

1. the subgraph, say G_U , of G' induced by U is complete, and,
2. the bipartite graph, say $G_{U \times V}$, with bipartition V and U is also complete.

Note that construction of G' is known as the *join* of the two graphs G and K_n .

First, suppose that G has a $(3, n)$ -differential coloring and let $l : V \rightarrow \{1, \dots, n\}$ be the respective coloring. We compute a coloring $l' : V' \rightarrow \{1, \dots, 4n\}$ of G' as follows: (i) $l'(v) = l(v)$, for all $v \in V' \cap V$ and (ii) $l'(u_i) = n + 3i$, $i = 1, 2, \dots, n$. We claim that l' is a $(3, 2n')$ -differential coloring of G' . To realize this first observe that the label difference between any two vertices of $V' \cap V$ is at least three, since G

is $(3, n)$ -differential colorable and l is the respective coloring. By the definition of l' , the label difference between any two vertices of U is exactly three. Finally, the label difference between a vertex in $V' \cap V$ and a vertex in U is also at least three, since the vertices of $V' \cap V$ are labeled with colors from the set $\{1, 2, \dots, n\}$ and the vertices of U with colors from the set $\{n+3, \dots, 4n\}$.

Now, suppose that G' is $(3, 2n')$ -differential colorable and let $l' : V' \rightarrow \{1, \dots, 2n'\}$ be the respective coloring (recall that $n' = 2n$). We next show how to compute the $(3, n)$ -differential coloring for G . Without loss of generality, let $V = \{v_1, \dots, v_n\}$ contain the vertices of G , such that $l'(v_1) < \dots < l'(v_n)$, and $U = \{u_1, \dots, u_n\}$ contain the newly added vertices of G' , such that $l'(u_1) < \dots < l'(u_n)$. Since G_U is complete, it follows that the color difference between any two vertices of U is at least three. Similarly, since $G_{U \times V}$ is complete bipartite, the color difference between any two vertices of U and V is also at least three. We claim that l' can be converted to an equivalent $(3, 2n')$ -differential coloring for G' , in which all vertices of V are colored with numbers from 1 to n , and all vertices of U with numbers from $n+3$ to $4n$.

Let U' be a maximal set of vertices $\{u_1, \dots, u_j\} \subseteq U$ so that there is no vertex $v \in V$ with $l'(u_1) < l'(v) < l'(u_j)$. If $U' = U$ and $l'(v) < l'(u_1), \forall v \in V$, then our claim trivially holds. If $U' = U$ and $l'(v) > l'(u_j), \forall v \in V$, then we can safely recolor all the vertices in V' in reverse order, resulting in a coloring that complies with our claim. Now consider the case where $U' \subsetneq U$. Next, we will show how to extend U' . There is a vertex $v_k \in V$ such that $l'(v_k) - l'(u_j) \geq 3$. Similarly, we define $V' = \{v_k, \dots, v_l \in V\}$ to be a maximal set of vertices of V , so that $l'(v_k) < \dots < l'(v_l)$ and there is no vertex $u \in U$ with $l'(v_k) < l'(u) < l'(v_l)$. Then, we can safely recolor all vertices of $U' \cup V'$, such that:

1. the relative order of the colors of U' and V' remains unchanged,
2. the color distance between v_l and u_1 is at least three, and
3. the colors of U' are strictly greater than the ones of V' .

Note that the color difference between u_j and u_{j+1} and between v_{k-1} and v_k is at least three after recoloring, i.e., $l'(u_{j+1}) - l'(u_j) \geq 3$ and $l'(v_k) - l'(v_{k-1}) \geq 3$. If we repeat this procedure until $U' = U$, then the resulting coloring complies with our claim. Thus, we obtain a $(3, n)$ -differential coloring l for G by assigning $l(v) = l'(v), \forall v \in V$. $\square$

Theorem 5. *Given a graph $G = (V, E)$ on n vertices, it is NP-complete to determine whether G has a $(k+1, kn)$ -differential coloring, where $k \geq 2$.*

Proof. Based on an instance $G = (V, E)$ of the $(k+1, n)$ -differential coloring problem, which is known to be NP-complete [6], construct a new graph $G' = (V', E')$ with $n' = kn$ vertices, by attaching $n(k-1)$ new vertices to G that form a clique of size $n(k-1)$, as in the proof of Theorem 4. Then, using a similar argument as above, we can show that G has a $(k+1, n)$ -differential coloring if and only if G' has a $(k+1, kn')$ -differential coloring. $\square$

The NP-completeness result of Theorem 4 was for general graphs. In the following, we consider the complexity of the problem for planar graphs. Note that from Lemma 2

Figure 6: (a) An instance of the 3-coloring problem; (b) An instance of the $(\lfloor 2n'/3 \rfloor, 2n')$ -differential coloring problem constructed based on graph G ; (c) The $(\lfloor 2n'/3 \rfloor, 2n')$ -differential coloring of G' , in the case where G is 3-colorable.

and Lemma 3, it follows that the 2-differential chromatic number of a planar graph on n vertices is between $\lfloor \frac{n}{3} \rfloor + 1$ and $\lfloor \frac{3n}{2} \rfloor$ (a planar graph is 4-colorable). The next theorem shows that testing whether a planar graph is $(\lfloor 2n/3 \rfloor, 2n)$ -differential colorable is NP-complete. Since this problem can be reduced to the general 2-differential chromatic number problem, it is NP-complete to determine the 2-differential chromatic number even for planar graphs.

Theorem 6. *Given an n -vertex planar graph $G = (V, E)$, it is NP-complete to determine if G has a $(\lfloor 2n/3 \rfloor, 2n)$ -differential coloring.*

Proof. The problem is clearly in NP; a non-deterministic algorithm needs only to guess an assignment of distinct colors (out of $2n$ available colors) to the vertices of the graph and then it is easy to verify in polynomial time whether this assignment corresponds to a differential coloring of minimum color difference at least $\lfloor 2n/3 \rfloor$.

To prove that the problem is NP-hard, we employ a reduction from the well-known 3-coloring problem, which is NP-complete for planar graphs [30]. Let $G = (V, E)$ be an instance of the 3-coloring problem, i.e., G is an n -vertex planar graph. We will construct a new planar graph G' with $n' = 3n$ vertices, so that G' is $(\lfloor 2n'/3 \rfloor, 2n')$ -differential colorable if and only if G is 3-colorable.

Graph $G' = (V', E')$ is constructed by attaching a path of length two to each vertex of G ; see Fig. 6a-6b. Hence, we can assume that $V' = V \cup V_1 \cup V_2$, where V is the vertex set of G , V_1 contains the middle vertex of each 2-vertex path and V_2 the last vertices. Clearly, G' is a planar graph on $n' = 3n$ vertices. Since G is a subgraph of G' , G is 3-colorable if G' is 3-colorable. On the other hand, if G is 3-colorable, then G' is also 3-colorable: for each vertex $v \in V$, simply color its neighbors $v_1 \in V_1$ and $v_2 \in V_2$ with two distinct colors different from the color of v . Next, we show that G' is 3-colorable if and only if G' has a $(\lfloor 2n'/3 \rfloor, 2n')$ -differential coloring.

First assume that G' has a $(\lfloor 2n'/3 \rfloor, 2n')$ -differential coloring and let $l : V' \rightarrow \{1, \dots, 2n'\}$ be the respective coloring. Let $u \in V'$ be a vertex of G' . We assign a color $c(u)$ to u as follows: $c(u) = i$, if $2(i-1)n + 1 \leq l(u) \leq 2in$, $i = 1, 2, 3$. Since l is a $(\lfloor 2n'/3 \rfloor, 2n')$ -differential coloring, no two vertices with the same color are adjacent. Hence, coloring c is a 3-coloring for G' .

Now, consider the case where G' is 3-colorable. Let $C_i \subseteq V$ be the set of vertices of the input graph G with color i , $i = 1, 2, 3$. Clearly, $C_1 \cup C_2 \cup C_3 = V$. We compute a coloring l of the vertices of graph G' as follows (see Fig. 6c):

- Vertices in C_1 are assigned colors from 1 to $|C_1|$.
- Vertices in C_2 are assigned colors from $3n + |C_1| + 1$ to $3n + |C_1| + |C_2|$.
- Vertices in C_3 are assigned colors from $5n + |C_1| + |C_2| + 1$ to $5n + |C_1| + |C_2| + |C_3|$.
- For a vertex $v_1 \in V_1$ that is a neighbor of a vertex $v \in C_1$, $l(v_1) = l(v) + 2n$.
- For a vertex $v_1 \in V_1$ that is a neighbor of a vertex $v \in C_2$, $l(v_1) = l(v) - 2n$.
- For a vertex $v_1 \in V_1$ that is a neighbor of a vertex $v \in C_3$, $l(v_1) = l(v) - 4n$.
- For a vertex $v_2 \in V_2$ that is a neighbor of a vertex $v_1 \in V_1$, $l(v_2) = l(v_1) + 3n + |C_2|$.

From this construction it follows that the color difference between (i) any two vertices in G , (ii) a vertex $v_1 \in V_1$ and its neighbor $v \in V$, and (iii) a vertex $v_1 \in V_1$ and its neighbor $v_2 \in V_2$, is at least $2n = \lfloor \frac{2n'}{3} \rfloor$. Thus, G' is $(\lfloor 2n'/3 \rfloor, 2n')$ -differential colorable. $\square$

5. An ILP for the Maximum k-Differential Coloring Problem

In this section, we describe an integer linear program (ILP) formulation for the maximum k-differential coloring problem. Recall that an input graph G to the maximum k-differential coloring problem can be easily converted to an input to the maximum 1-differential coloring by creating a disconnected graph G' that contains all vertices and edges of G plus $(k-1)n$ isolated vertices. In order to formulate the maximum 1-differential coloring problem as an integer linear program, we introduce for every vertex $v_i \in V$ of the input graph G a variable x_i , which represents the color assigned to vertex v_i . The 1-differential chromatic number of G is represented by a variable OPT , which is maximized in the objective function. The exact formulation is given below. The first two constraints ensure that all vertices are assigned colors from 1 to n . The third constraint guarantees that no two vertices are assigned the same color, and the fourth constraint maximizes the 1-differential chromatic number of the graph. The first three constraints also guarantee that the variables are assigned integer values.

$$\begin{array}{llll}
\text{maximize} & OPT \\
\text{subject to} & x_i & \leq & n \quad \forall v_i \in V \\
& x_i & \geq & 1 \quad \forall v_i \in V \\
& |x_i - x_j| & \geq & 1 \quad \forall (v_i, v_j) \in V^2 \\
& |x_i - x_j| & \geq & OPT \quad \forall (v_i, v_j) \in E
\end{array}$$

Note that a constraint that uses the absolute value is of the form $|X| \geq Z$ and therefore can be replaced by two new constraints: (i) $X + M \cdot b \geq Z$ and (ii) $-X + M \cdot (1 - b) \geq Z$, where b is a binary variable and M is the maximum value that can be assigned to the sum of the variables, $Z + X$. That is, $M = 2n$. If b is equal to zero, then the two constraints are $X \geq Z$ and $-X + M \geq Z$, with the second constraint always true. On the other hand, if b is equal to one, then the two constraints are $X + M \geq Z$ and $-X \geq Z$, with the first constraint always true.

Figure 7: A map with 16 countries colored by: (a) GMap [21], (b) ILP-n, (c) ILP-2n.

Next, we study two variants of the ILP formulation described above: ILP-n and ILP-2n, which correspond to $k = 1$ and $k = 2$, and experimentally compare them with the GMap coloring method, which relies on a heuristic based on spectral methods [21]. In order to evaluate the quality of the produced maps, in addition to measuring the minimum color difference between adjacent vertices, we also use the *average color difference between adjacent vertices*, which for a given coloring $c : V \rightarrow \{1, 2, \dots, kn\}$ of the vertices of a graph $G = (V, E)$ is defined as $\sum_{(i,j) \in E} |c(i) - c(j)| / |E|$. Note that this metric measures the average quality of a coloring, rather than the worst case pairwise value.

5.1. Experiment's Setup

We generate a collection of 1,200 synthetic maps and analyze the performance of ILP-n, ILP-2n and GMap, on an Intel Core i5 1.7GHz processor with 8GB RAM, using the CPLEX solver [31]. For each map a country graph $G_c = (V_c, E_c)$ with n countries is generated using the following procedure, consisting of four steps.

- S.1:** We generate $10n$ vertices and place an edge between pairs of vertices (i, j) such that $\lfloor \frac{i}{10} \rfloor = \lfloor \frac{j}{10} \rfloor$, with probability 0.5, thus resulting in a graph G with approximately n clusters.
- S.2:** More edges are added between all pairs of vertices with probability p , where p takes the values $1/2, 1/4, \dots, 2^{-10}$.
- S.3:** Ten random graphs are generated for different values of p .
- S.4:** G is used as an input to a map generating algorithm (available as the Graphviz [32] function `gvmap`), to obtain a map M with country graph G_c .

A sample map generated by the aforementioned procedure is shown in Fig. 7. Note that the value of p determines the “fragmentation” of the map M , i.e., the number of regions in each country, and hence, also affects the number of edges in the country graph. When p is equal to $1/2$, the country graph is a nearly complete graph, whereas for p equal to 2^{-10} , the country graph is nearly a tree. To determine a suitable range for

(a) The running time plotted against the number of clusters of the map.

(b) The differential coloring plotted against the number of clusters of the map.

(c) The average color difference plotted against the number of clusters of the map.

Figure 8: Illustration of: (a) running time results, (b) differential coloring performance, and, (c) average color difference.

the number of vertices in the country graph, we evaluated real world datasets, such as those available at `gmap.cs.arizona.edu`. Even for large graphs with over 1,000 vertices, the country graphs tend to be small, with less than 16 countries.

5.2. Evaluation Results

Fig. 8 summarizes the results of our experimental analysis. Since n is ranging from 5 to 16, the running times of both ILP-n and ILP-2n are reasonable, although still much higher than GMap; see Fig. 8a. The color assignments produced by ILP-n and GMap are comparable, while the ones produced by ILP-2n are of better minimum and average color difference; see Fig. 8b and 8c. It is worth mentioning, though, that in the presence of twice as many colors as the graph's vertices, it is easier to obtain higher color difference between adjacent vertices. However, this high difference comes at the cost of assigning pairs of colors that are more similar to each other for non-adjacent vertices.

6. Conclusion and Future Work

Even though the $(2, n)$ -differential coloring is NP-complete for general graphs, in this paper, we gave a complete characterization of bipartite, outerplanar and planar graphs that admit $(2, n)$ -differential colorings. Note that these characterizations directly lead to polynomial-time recognition algorithms. We also generalized the differential coloring problem for more colors than the number of vertices in the graph and showed that it is NP-complete to determine whether a general graph admits a $(k + 1, kn)$ -differential coloring, for $k \geq 2$. Even for planar graphs, the problem of determining whether a graph is $(\lfloor 2n/3 \rfloor, 2n)$ -differential colorable remains NP-hard. Several related problems are still open:

- Outerplanar and planar graphs form two special cases of the class of so-called *degenerate graphs* (recall that a k -degenerate graph is one in which every subgraph has a vertex of degree at most k). In particular, outerplanar graphs are 2-degenerate, while the planar ones are 5-degenerate. So, a reasonable question to ask is whether it is possible to characterize which degenerate graphs are $(2, n)$ -differential colorable.
- Is it possible to characterize which bipartite, outerplanar or planar graphs are $(3, n)$ -differential colorable?
- Extend the characterizations for those planar graphs that admit $(2, n)$ -differential colorings to *1-planar graphs*, which can be drawn on the plane such that no edge is crossed more than once.
- Extend the results above to (d, kn) -differential coloring problems with larger $k > 2$.
- As all planar graphs are $(\lfloor \frac{n}{3} \rfloor + 1, 2n)$ -differential colorable, is it possible to characterize which planar graphs are $(\lfloor \frac{n}{3} \rfloor + 2, 2n)$ -differential colorable?
- Since it is NP-complete to determine the 1-differential chromatic number of a planar graph [20], a natural question to ask is whether it is possible to compute in polynomial time the 1-differential chromatic number of an outerplanar graph.

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