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TURBULENT HYDRODYNAMIC LINE-STRETCHING:

THE RANDOM-WALK LIMIT

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Steward Observatory, University of Arizona, Tucson, Arizona 85721

Recent results on the central limit theorem for sums of dependent random variables are used to evaluate the asymptotic rates of turbulent stretching of material line and surface elements in the limit $t \rightarrow \infty$. The turbulence is assumed to be incompressible and statistically stationary and isotropic, and it is shown that the expectation value of the logarithmic rate of increase of the magnitude of the material elements vanishes as $t \rightarrow \infty$. We note possibilities for application to weak-field magnetohydrodynamic turbulence, and mention some associated difficulties.

I. INTRODUCTION.

This paper deals with the problem of the stretching of material line and surface elements in a turbulent fluid. In particular, we consider the asymptotic rates of stretching of these material elements in the limit $t \rightarrow \infty$, as an application of the central limit theorem for sums of mutually dependent random variables. The material presented here is an extension of previous work^{1,2}, which treated more general properties of turbulent hydrodynamic line stretching.

In Sec. II of this paper we derive the Lagrangian equation of motion of the parameters which determine the elongation (and contraction) of these material elements, and discuss the applicability of recent developments of central limit theory for sums of statistically dependent random variables. In an Appendix we discuss a particularly useful form of the central limit theorem for vector-valued variables; and we show that for incompressible and statistically isotropic and stationary turbulence, the distribution function for the stretching parameters satisfies the requirements of this theorem, provided that certain rather mild conditions are fulfilled.

In Sec. III we compute the asymptotic limit of the logarithmic rate of lengthening of a line element, and show that the expectation value of this quantity vanishes as $t \rightarrow \infty$, contrary to a conjecture by Batchelor³, provided that our application of the central limit theorem is valid.

We note that the equation of motion of a general passive vector quantity $\underline{F}$ (for example, a weak magnetic field in an infinitely conducting fluid) has the same form as that for a line element, but that many quantities of physical interest cannot be evaluated without further strengthening these results. In particular, an estimate of $\langle |\underline{F}(t)|^m \rangle$ ($m=1,2,\dots$) as $t \rightarrow \infty$ cannot be made without investigating the rate of convergence to the normal of the probability distributions involved.

Let us recall previous notation and results¹ which will be useful in the rest of the paper. The Lagrangian matrix equation of motion of an infinitesimal line element $\delta \underline{l}$ (a column vector) is, following its motion with the fluid,

$$\frac{d\delta \underline{l}}{dt} = T(t)\delta \underline{l} \quad , \quad (1)$$

where T is the matrix $T_{ab} \equiv \partial u_a / \partial x^b$, with u_a the fluid velocity.

Equation (1) has the integral solution¹ $\delta \underline{l}(t) = U(t,0) \delta \underline{l}_0$, where $U(t_2,0) = U(t_2,t_1) U(t_1,0)$, and

$$\frac{dU}{dt} = TU. \quad (2)$$

It was shown that if the fluid is incompressible, the determinant of U is unity, and that the square magnitude

of the infinitesimal line element is

$$\delta \mathcal{L}(t)^2 = \delta \mathcal{L}_0 \tilde{U} U \mathcal{L}_0.$$

The matrix $W \equiv \tilde{U}U$, being real and symmetric, has real eigenvalues w_a , with $w_a > 0$. In the rotating coordinate system in which $W(t)$ is always diagonal, we have the result

$$\delta \mathcal{L}(t)^2 = \delta \mathcal{L}_0^2 [\sin^2 \theta (\cos^2 \psi w_1 + \sin^2 \psi w_2) + \cos^2 \theta w_3], \quad (3)$$

where $\theta(t)$ and $\psi(t)$ are the position angles of $\delta \mathcal{L}_0$ in the rotating system. If the stationary coordinate system is chosen so that $\delta \mathcal{L}_0$ points along the z-axis, then θ and ψ are two of the three Euler angles (θ, ϕ, ψ) through which the coordinate system is rotated to diagonalize $W(t)$.

It was shown rigorously that if the turbulence is statistically isotropic, the probability density for (θ, ϕ, ψ) at any time $t \geq 0$ is

$$dP(\theta, \phi, \psi) = \sin \theta d\theta d\phi d\psi / 8 \pi^2. \quad (4)$$

II. ASYMPTOTIC PROBABILITY DENSITY FUNCTION FOR w_a :
 AN APPLICATION OF THE CENTRAL LIMIT THEOREM
 FOR MUTUALLY DEPENDENT RANDOM VARIABLES

In this section we use recent results on the central limit theorem for sums of mutually dependent random variables to show that the triplet $(x_1, x_2, x_3) \equiv (\ln w_1, \ln w_2, \ln w_3)$ becomes normally distributed in the limit $t \rightarrow \infty$, provided that two mild conditions are satisfied. A proof that these conditions are indeed satisfied must await further developments either in central limit theory or in the theory of turbulence itself. We assume throughout, as in previous work³, that the turbulence is statistically stationary.

First, we examine the equation of motion of w_a . As in the previous paper¹, let $A(t)$ be the rotation matrix which diagonalizes W , so that

$$\bar{W} = A\tilde{W}A^T, \quad \bar{W}_{ab} = w_a \delta_{ab}, \quad A\tilde{A}^T = I.$$

By differentiating these equations, it is easily shown that $\dot{w}_a = (\dot{A}\tilde{W}A^T)_{aa}$, and then Eqn. (2) may be used to find

$$\dot{w}_a = 2 \sum_{m,n} \bar{U}_{ma} \bar{T}^{s}_{mn} \bar{U}_{na}, \quad \text{where } T^s \equiv \frac{1}{2}(T + \tilde{T}).$$

However, $w_a = (\tilde{U}\tilde{U})_{aa} = \sum_m (\bar{U}_{ma})^2$, and if we define the orthonormal column quantities $N_k^{(a)} \equiv \bar{U}_{ka} [\sum_m (\bar{U}_{ma})^2]^{-1/2}$,

it follows that

$$\frac{d}{dt} \ln w_a \equiv \dot{x}_a = 2 \tilde{N}^{(a)} \bar{T}^s N^{(a)}. \quad (5)$$

Hence, over the j th time interval of length Δt ,
the change in x_a is

$$\Delta_j x_a = \int_{t_j}^{t_j + \Delta t} \tilde{N}^{(a)} \tilde{T}^S N^{(a)} dt, \text{ where } t_j = (j-1)\Delta t,$$

and the total $\underline{x}$ is then $\underline{x}(n\Delta t) = \sum_{j=1}^n \Delta_j \underline{x}$.

The matrix $T^S(t)$ is presumed to be a random continuous function of t , as are the directional quantities $N^{(a)}$. However, the variables $\Delta_j \underline{x}$ are certainly not statistically independent, so that the classical forms of the central limit theorem for sums of independent random variables cannot be applied here.

Fortunately, various extensions to the dependent variable case have been made recently, and the central limit property has been shown to hold under a variety of intuitively appealing conditions. Of particular interest is the work of Serfling⁴, who showed the central limit property, given certain types of first- and second-moment decoupling from the "past": The influence of past occurrences of $\Delta_j \underline{x}$ on future values of $\langle \Delta_j \underline{x} \rangle$ and $\langle (\Delta_j \underline{x})^2 \rangle$ disappears as the time separating past and future becomes large. Various other conditions are also discussed, including "strong mixing" conditions⁵, and theorems involving them are proved also. These results would clearly be applicable to our problem except for the fact that certain additional assumptions involving moments higher than second are used and also that the theorems are proved only for one-dimensional

random variables. Additional work has been done by Phillip⁶, who uses somewhat different sets of assumptions and a more abstract formalism.

In the Appendix, we present a central limit theorem for sums of dependent random vector variables of founded dimensionality, assuming the existence of no moments higher than second. The theorem is stated in terms of three assumptions (A1), (A2), and (A3), and a lemma concerning (A1) is presented. Condition (A1) is the analog of the Lindeberg condition⁷ in the classical central limit theorem, and the Lemma shows that (A1) holds for a strictly stationary random process. The process of interest here is not strictly stationary, but it is certainly what we might call "asymptotically stationary," since the dependence of the $\Delta_j x$ on the given initial direction δl_0 becomes lost as $j \rightarrow \infty$, and since we assume that the turbulence itself is strictly stationary (in the statistical sense), as mentioned before. However, the Lemma is clearly applicable also to our asymptotically stationary process, since the distribution $F_n(y)$ may be chosen to be the distribution for the situation in which one has no knowledge at all of the initial direction of δl_0 . Then for some appropriately large n_0 , the lemma might be satisfied for $C=2$, say, unless something pathological occurs as $|y_{nj}| \rightarrow \infty$.

Condition (A2) refers to the device used in the proof, and may certainly be set up to hold here. Thus the only remaining condition is (A3), which codifies the assumption

that the blocks of variables used become independent from each other as the separations between them go to infinity.

We state the following sufficient condition for (A3) to hold: Let A_{nj} be any measurable set in the n, j variable block. Then if for all such sets $A_{n1}, A_{n2}, \dots, A_{nr}$, there exists a number $\phi(n)$, such that $\phi(n) \rightarrow 0$ as $n \rightarrow \infty$ and such that

$$|P(A_{n1} \cap A_{n2} \cap \dots \cap A_{nr}) - P(A_{n1})P(A_{n2}) \dots P(A_{nr})| \leq \phi(n)P(A_{n1})P(A_{n2}) \dots P(A_{nr}),$$

then (A3) obviously follows. The above decoupling condition is somewhat like a regularity condition discussed by Serfling⁴, and is a reasonable restriction for our application.

We proceed, then, under the assumption that the variable $\underline{x}$ does have the central limit property. But before writing down the limiting density function for $\underline{x}$, we exhibit some simple moment relations which follow from incompressibility and isotropy. Incompressibility implies¹ that for a change over any time interval Dt , $Dx_1 + Dx_2 + Dx_3 = 0$, and hence $\langle Dx_1 \rangle + \langle Dx_2 \rangle + \langle Dx_3 \rangle = 0$. But isotropy implies that these three moments are equal, and therefore $\langle Dx_a \rangle = 0$. Further, we likewise have $\langle (Dx_a)^2 \rangle \equiv \bar{D}^2$, and defining the correlation coefficient ρ by $\langle Dx_1 Dx_2 \rangle = \dots \equiv \rho \bar{D}^2$, we have

$$\langle (Dx_1 + Dx_2 + Dx_3)^2 \rangle = 3\bar{D}^2 + 6\rho\bar{D}^2 = 0,$$

yielding $\rho = -\frac{1}{2}$, independent of Dt .

At this point it is convenient to state the limiting

probability density for $\underline{x}$ in terms of only two of the variables, say x_1 and x_2 , since the third is simply a linear combination of the first two. The limiting density is then the bivariate normal density⁸, which because of the relation $\rho = -\frac{1}{2}$ may be written, as $n \rightarrow \infty$,

$$F(x_1, x_2; n) \sim (\pi D_n^2 \sqrt{3})^{-1} \exp[-2(x_1^2 + x_1 x_2 + x_2^2)/3D_n^2],$$

where $D_n^2 \equiv \langle (\sum_{j=1}^n \Delta_j x_a)^2 \rangle$, as required by the formalism in the Appendix.

In order to facilitate the discussion in Sec. III, we need to show that if $D(t) \equiv D_{(t/\Delta t)}$, then $dD/dt \rightarrow 0$ as $t \rightarrow \infty$. Normally, we might expect $D_n \sim 0$ ($n^{\frac{1}{2}}$), but it is not necessary to assume that this is so. Now, for $|j - k| \rightarrow \infty$, we must have $\langle \Delta_j x_a \Delta_k x_a \rangle \rightarrow 0$. Therefore it follows that $D_n/n \rightarrow 0$ as $n \rightarrow \infty$, as may be shown by considering D_n^2 as a double sum and using a simple argument by contradiction.

Hence, we have $d(D_n/n)/dn = n^{-1} dD_n/dn - D_n/n^2 \rightarrow 0$, and thus

$$\frac{dD(t)}{dt} = \frac{1}{\Delta t} \frac{dD_n}{dn} \sim \frac{D_n}{n \Delta t} \rightarrow 0. \quad (6)$$

We may introduce the Dirac delta function to give $F(\underline{x}; t)$ finally as

$$F(\underline{x}; t) \sim [\pi D(t)^2]^{-1} \exp[-\underline{x}^2/3D(t)^2] \delta(x_1 + x_2 + x_3). \quad (7)$$

III. EVALUATION OF ASYMPTOTIC LOGARITHMIC RATES OF EXTENSION

In the previous paper¹, we showed rigorously that, if we define $\mu(t) \equiv \ln [|\delta \underline{\underline{L}}(t)| / |\delta \underline{\underline{L}}_0|]$ and $\sigma(t) \equiv \ln [|\delta \underline{\underline{S}}(t)| / |\delta \underline{\underline{S}}_0|]$, where $\delta \underline{\underline{S}}(t)$ is an infinitesimal material surface element moving with the fluid, then for incompressible, isotropic turbulence

$$\langle \mu(t) \rangle > 0 \quad \text{and} \quad \langle \sigma(t) \rangle > 0 \quad \text{for } t > 0.$$

Batchelor³ has discussed the quantities $\langle d\mu/dt \rangle$ and $\langle d\sigma/dt \rangle$ and conjectured that for statistically stationary turbulence there exists a constant $\xi > 0$ such that as $t \rightarrow \infty$, $\langle d\mu/dt \rangle \rightarrow \xi$ and $\langle d\sigma/dt \rangle \rightarrow \frac{1}{2}\xi$.

However, in this section we use the asymptotic form given by Eqn. (7) to show that actually $\xi = 0$.

In order to find the expectation value of a time derivative, we need to introduce the joint probability density $F(X', t+\delta t; X, t)$, which gives the probability of finding the sextuplet $X \equiv (\underline{x}, \theta, \phi, \psi)$ at t and then X' at $t+\delta t$. This joint density must satisfy

$$F(X; t) = \int d^6 X' F(X', t'; X, t) = \int d^6 X' F(X, t; X', t').$$

Then $\langle d\mu/dt \rangle$ may be evaluated by finding, with the help of the above equation,

$$\begin{aligned} \left\langle \frac{d\mu}{dt} \right\rangle &= \int d^6 X \int d^6 X' \lim_{dt \rightarrow 0} F(X', t+dt; X, t) \left[\frac{\mu(X') - \mu(X)}{dt} \right] \\ &= \lim_{dt \rightarrow 0} \frac{1}{dt} \int d^6 X' [F(X'; t+dt) \mu(X') - F(X; t) \mu(X)] \\ &= \frac{d}{dt} \langle \mu(t) \rangle. \end{aligned}$$

We may write, using Eqns. (3), (4), and (7),

$$\left\langle \frac{d\mu}{dt} \right\rangle \rightarrow \xi = \lim_{t \rightarrow \infty} \frac{d}{dt} \left\{ (8\pi^2 D^2 \sqrt{3})^{-1} \int d^3x e^{-x^2/3D^2} \delta(x_1+x_2+x_3) \right. \\ \left. \times \iint d\psi d\theta \sin\theta \ln [\sin^2\theta (\cos^2\psi e^{x_1} + \sin^2\psi e^{x_2}) + \cos^2\theta e^{x_3}] \right\}.$$

But a change of variables to $y = 2x/3D^2$, with $C^2 \equiv 3D^2/2$, yields

$$\xi = \lim_{t \rightarrow \infty} \iint dy_1 dy_2 e^{-(y_1^2 + y_1 y_2 + y_2^2)} I(y_1, y_2; C) \frac{dC}{dt}$$

where

$$I(y_1, y_2; C) \equiv \iint \frac{d\psi d\theta \sin\theta [\sin^2\theta (\cos^2\psi e^{y_1 C} y_2 + \sin^2\psi e^{y_2 C} y_1) - \cos^2\theta e^{-(y_1+y_2)C} (y_1+y_2)]}{\sin^2\theta (\cos^2\psi e^{y_1 C} + \sin^2\psi e^{y_2 C}) + \cos^2\theta e^{-(y_1+y_2)C}}$$

But since $C \rightarrow \infty$, we can write

$$\xi = \frac{\sqrt{3}}{4\pi} \iint dy_1 dy_2 e^{-(y_1^2 + y_1 y_2 + y_2^2)} \max(y_1, y_2, -y_1 - y_2) \lim_{t \rightarrow \infty} \frac{dC}{dt} \\ \sim 0.0133 \lim_{t \rightarrow \infty} \frac{dC}{dt}.$$

From Eqn. (6), we see that $dC/dt \rightarrow 0$, and therefore

$\xi = 0$. The behavior of these integrals for large C has been checked by direct numerical evaluation of $\langle \mu \rangle$ as a function

of C on the University of Arizona CDC 6400 computer. As expected from the above expression for ξ , it was found that

$$\langle \mu(t) \rangle \sim 0.0133 C(t) \text{ as } C \rightarrow \infty.$$

It is a simple matter to show further that $\langle \sigma(t) \rangle$ is identical in all respects to $\langle \mu(t) \rangle$, in so far as the asymptotic expression Eqn. (7) is valid. Therefore $\langle d\sigma(t)/dt \rangle \rightarrow 0$ likewise as $t \rightarrow \infty$.

It would be useful to be able to use Eqn. (7) to evaluate such quantities as $\langle \delta \mathcal{L}(t)^2 \rangle$ for large t. However, $\delta \mathcal{L}(t)^2$ is a linear combination of exponentials of x_a , and the convergence of the exact density $F(x;t)$ to that given by Eqn. (7) may, for large $|x|$, not be fast enough ever to permit an accurate limiting value for $\langle \exp(x_a) \rangle$. Most of the contribution to $\langle \exp(x_1) \rangle$, for example, comes from around the point $(x_1, x_2) = (1, -\frac{1}{2}) D$, where the argument of the exponential in the probability density is $-D^2/2$. Thus our conclusion that the real density of the variable $(x_1, x_2)/D$ converges uniformly to the bivariate normal density of unit dispersion does not permit the asymptotic evaluation of $\langle \delta \mathcal{L}(t)^2 \rangle$ by this method.

IV. REMARKS AND CONCLUSIONS

With an eye to possible physical applications of these results, we note that Eq. (1) is the equation of motion of a general passive vector quantity $\vec{F}$, without diffusion. A weak magnetic field $\vec{B}$ frozen into an infinitely conducting incompressible fluid is an example of such a quantity, but unfortunately we cannot apply the central limit theorem to evaluate such averages as

$\langle |\vec{B}(t)| \rangle$ and $\langle |\vec{B}(t)|^2 \rangle$, because of the arguments stated at the end of Sec. III. Further, in the magnetic field case, an increase in field strength would cause a reaction back on the fluid flow and ruin the asymptotically stationary character of the quantities $\Delta_j \vec{x}$.

We see that further work remains to be done if this line of attack is to be made rigorous, even with the relatively straight-forward line-stretching problem. Rigorous proofs are needed about the validity of conditions (A1) and (A3), and it would be good to have information about the rate of convergence of the distribution of $\vec{x}$ to the normal distribution.

It is hoped that this paper will stimulate further applications of the central limit theorem for the dependent variable case in turbulence theory.

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APPENDIX: A CENTRAL LIMIT THEOREM FOR DEPENDENT VECTOR VARIABLES

We consider the set of dependent random variables $x_1, x_2, \dots$, each of which is a vector-valued quantity of fixed dimensionality N . The marginal distribution function for the first n of these variables is denoted by $P \{ x_1 \leq u_1, \dots, x_n \leq u_n \} \equiv F_n(u_1, \dots, u_n)$, where by $x_k \leq u_k$, we mean $x_{k1} \leq u_{k1}, \dots, x_{kN} \leq u_{kN}$. The first and second moments $\langle x_{kj} \rangle$ and $\langle x_{kj} x_{im} \rangle$ are assumed finite, but the existence of higher-order moments is not assumed.

Below, we state sufficient conditions (A1), (A2), and (A3), which imply that the vector sum $S_n \equiv \sum_{j=1}^n x_j$, suitably normalized, is distributed according to the multivariate normal distribution⁹ in the limit $n \rightarrow \infty$. The proof of this form of the central limit theorem is given elsewhere¹⁰, and relies on the fact that one can leave out some of the variables in the above sum S_n in such a way that as $n \rightarrow \infty$, the sum of all the variables becomes asymptotically equal to the sum of "almost all" of them. Further, this can be done in such a way that the omitted variables break up the remaining ones into groups which become independent from each other as $n \rightarrow \infty$. One of the methods of proof of the classical central limit theorem⁷ may then be used to complete the proof of the theorem in our case.

For each n , we denote the omitted set of integers as Q_n , and the included set is denoted by P_n . Further, we break up P_n into $r(n)$ subsets P_{nj} , with $P_n = \bigcup_{j=1}^{r(n)} P_{nj}$. Of course, for $n < m$, we will probably not have $P_n \subset P_m$, etc. Indeed, it will happen in most applications that the membership of the individual P_{nj} 's goes to infinity as $n \rightarrow \infty$. Also, we conceive that the membership of Q_n goes to infinity, but that the ratio of omitted to included variables goes to zero. Only then will the entire sum be asymptotically equal to the deficient sum.

We let F_{nj} ($1 \leq j \leq r(n)$) be the marginal distribution function for the set of variables P_{nj} , and $F_n^{(p)}$ be that for the whole set P_n . We denote

$$\delta F_n \equiv F_n^{(p)} - \prod_{j=1}^{r(n)} F_{nj},$$

and with $a_k \equiv \langle x_k \rangle = \int x_k dF_{nj}$ for $k \in P_{nj}$, define

$$y_{nj} \equiv \sum_{k \in P_{nj}} (x_k - a_k), \quad B_{nj}^2 \equiv \langle y_{nj}^2 \rangle,$$

and

$$B_n^2 \equiv \sum_{j=1}^{r(n)} B_{nj}^2.$$

Note that since y_{nj} is an N - dimensional vector, y_{nj}^2 denotes the scalar product $y_{nj} \cdot y_{nj}$. Likewise, in what follows, we use $|y_{nj}|$ to mean the square root of

this quantity. Further, we remark that B_{nj}^2 is the trace of the $N \times N$ covariance matrix $\bar{C}^{(nj)}$, with $\bar{C}^{(nj)}_{ab} = \langle y_{nja} y_{njb} \rangle$, for $a, b = 1, \dots, N$. Thus the N -dimensional variable y_{nj}/B_n is normalized to have a covariance matrix $C^{(nj)}$ such that $\text{tr } C^{(n)} = 1$, where $C^{(n)} = \sum_{j=1}^{r(n)} C^{(nj)}$

We now discuss the conditions under which our central limit theorem may be shown to hold.

(A1). This condition is the analog of the Lindeberg condition as used in proofs of the classical central limit theorem. It implies that the individual summands y_{nj}/B_n make uniformly small contributions to the total sum⁷, as $n \rightarrow \infty$: For any fixed $\tau > 0$, as $n \rightarrow \infty$, we must have

$$\frac{1}{B_n^2} \sum_{j=1}^{r(n)} \int_{|y_{nj}| > \tau B_n} y_{nj}^2 dF_{nj} \rightarrow 0.$$

(A2). The second condition is readily seen to be more of a restriction on the choice of the sets P_{nj} and Q_n , and is certainly only a weak assumption about the behavior of the random variables themselves: We must have, as $n \rightarrow \infty$,

$$\frac{1}{B_n} \left\langle \left| \sum_{i \in Q_n} (x_i - a_i) \right| \right\rangle \rightarrow 0.$$

(A3). The final assumption must deal with the fact the groups P_{nj} must become mutually independent as the class

of omitted variables Q_n becomes infinitely large: As

$n \rightarrow \infty$,

$$\int |d\delta F_n| \rightarrow 0.$$

It is now simple to state the central limit theorem for the entire sum $S_n \equiv \sum_{j=1}^n x_j$, the proof of which is to be published elsewhere.

Theorem: If there exists a sequence of sets P_{nj} ($1 \leq j \leq r(n)$) such that conditions (A1), (A2), and (A3) are satisfied, then it follows that the distribution function for the entire vector sum S_n is asymptotically multivariate normal, in the sense that, uniformly with respect to U , as $n \rightarrow \infty$,

$$P\{S_n/B_n < U\} \rightarrow (2\pi)^{-N/2} |C^{(n)}|^{-1} \int_{-\infty}^{u_1} \dots \int_{-\infty}^{u_N} dz_1 \dots dz_N \exp [-\tilde{z} C^{(n)-1} z/2] .$$

In applications of the classical central limit theorem, it is often quite easy to show the satisfaction of the Lindeberg condition. For example, for sums of identically distributed independent variables it is trivially easy. However, in our case the situation is not so simple, and it is useful to state the following Lemma concerning condition (A1). We then demonstrate that the Lemma is true for a strictly stationary process, in which case the F_{nj} may, for fixed n , all be functionally identical. This is obviously the analog of the identically distributed independent variable case.

Lemma: If there exists a sequence of distributions $F_n(y)$,
a positive constant C , and an integer n_0 , such that for
all $n > n_0$ and all $Y > 0$,

$$C \int_{|y| > Y} y^2 dF_n(y) \geq \int_{|y_{nj}| > Y} y_{nj}^2 dF_{nj} \text{ for } 1 \leq j \leq r(n) , \quad (A4)$$

$$\text{and } \frac{C}{r(n)} \sum_{j=1}^{r(n)} B_{nj}^2 = \frac{C}{r(n)} B_n^2 \geq \int y^2 dF_n(y). \quad (A5)$$

then the Lindeberg condition (A1) is satisfied.

We remark that for a strictly stationary process it is
certainly reasonable to choose each of the P_{nj} to be a continuous
block of integers, and, for fixed n , to choose them all to be of
the same size, and to be separated by equal intervals of Q -
integers. Then the F_{nj} will, for fixed n , all be functionally
identical, and we can take $F_n(y) = F_{n1}^*(y)$, where $F_{n1}^*(y)$
is the distribution function for the sum y_{n1} created in the
usual way from F_{n1} . In this case, equality signs hold in
(A4) and (A5) with $C = 1$, and the conditions of the Lemma
are satisfied.

FOOTNOTES

1. W. J. Cocke, Phys. Fluids 12, 2488 (1969).
2. S. A. Orszag, Phys. Fluids 13, 2203 (1970).
3. G. K. Batchelor, Proc. Roy. Soc. (London) A213, 349 (1952).
4. R. J. Serfling, Ann. Math. Statist. 39, 1158 (1968).
5. M. Rosenblatt, Random Processes (Oxford University Press, New York, 1962), p.110.
6. W. Phillip, Z. Wahrsch. verw. Geb. 12, 155 (1969).
7. B. W. Gnedenko, Wahrscheinlichkeitsrechnung (Akademie - Verlag, Berlin, 1958), p.228.
8. W. Feller, An Introduction to Probability Theory, Vol.II (John Wiley and Sons, Inc., New York, 1960), p. 69
9. Feller, ref. 8, p. 82.
10. W. J. Cocke, to be published.