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CALCULATIONS OF MIE BACK-SCATTERING OF MICROWAVES
FROM ICE SPHERES

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Abstract

Thunderstorms containing hail have been found to have extremely large radar reflectivities. Experimental work by Atlas *et al.*, has shown that hailstones may have unexpectedly large back-scattering cross sections. In order to shed additional light on this problem, the normalized back-scattering cross sections (σ_b) of spherical ice particles with $N = 1.78 - 0.0024i$ were calculated from the complete Mie equations. Small

intervals of $\alpha (= \frac{2\pi a}{\lambda})$ were taken up to a value of 30. The back-scattering from large ice spheres is shown to be considerably higher than that from water spheres of the same size. In general, the differences between the reflectivities of ice and water spheres increase with increasing α up to an α of about 60.

Calculations of σ_b were made for various values of the complex index of refraction in order to better understand the properties of the back-scattering equations.

1. Introduction

It is well-known that when an electromagnetic wave encounters a particle in the free atmosphere, a certain amount of power is intercepted. The particle absorbs a fraction of the power in the form of heat and reradiates the remainder in various directions. That part of the power scattered back towards the source of the wave is referred to as "back-scattered" and represents a quantity of primary interest in all radar applications. The remainder of the power, namely that absorbed and that scattered in other directions, also is of importance in some microwave problems because it is related to attenuation of the wave.

It has been found that when the particle diameter is of the order of magnitude of the wave length involved, the scattering properties of the particles are complex functions of the particle diameters and the complex index of refraction. Studies of this problem were made in the late

eighteen hundreds by Lord Rayleigh who considered particles much smaller than the wave length. Mie (1908) published the first exact theoretical treatment of the scattering by spherical particles. Since that time, a substantial amount of research has been done on various aspects of the problem. The reader is referred to Stratton (1941) for a detailed discussion of a concise treatment of the theory involved.

2. Back-scattering cross section

The back-scattering cross section, σ , has been defined as the area which, when multiplied by the incident power intensity, gives the total power radiated by an isotropic source which radiates the same power in the backward direction as the scatterer. Thus, one can write

$$\sigma = 4\pi r^2 \frac{P_s}{P_i} \quad (1)$$

where P_i is the power density (power per unit area) in the incident plane wave at the position of the scatterer, and P_s is the power density of the scattered field at a distance r from the scatterer in the direction of the source.

Quite evidently σ has the units of length squared. It is common to express the back-scattering cross section of a particle of radius, a , as a nondimensional quantity by dividing σ by the area of the particle. This leads to

$$\sigma_b = \frac{4r^2}{a^2} \frac{P_s}{P_i} \quad (2)$$

a quantity referred to as the normalized back-scattering cross section.

As shown by Aden (1951) and others, when spherical particles and linearly polarized electromagnetic waves are involved

$$\sigma_b = \frac{1}{\alpha^2} \left| \sum_{n=1}^{\infty} (2n+1)(-1)^n (a_n^s - b_n^s) \right|^2 \quad (3)$$

where α is $2\pi a/\lambda$ and λ is wave length. The terms a_n^s and b_n^s are referred to as scattering coefficients and involve spherical Bessel and Hankel functions. The quantity n refers to the term of the expansion of the coefficients.

It can be shown that the scattering coefficients are given by

$$\begin{aligned} a_n^s &= - \left[\frac{J_n(\alpha)}{H_n(\alpha)} \right] \cdot \left[\frac{\sigma_n(\alpha) - N\sigma_n(N\alpha)}{P_n(\alpha) - N\sigma_n(N\alpha)} \right] \\ b_n^s &= - \left[\frac{J_n(\alpha)}{H_n(\alpha)} \right] \cdot \left[\frac{\sigma_n(N\alpha) - N\sigma_n(\alpha)}{P_n(N\alpha) - NP_n(\alpha)} \right] \end{aligned} \quad (4)$$

where $J_n(\alpha)$ is the spherical Bessel function and $H_n(\alpha)$ is a Hankel function of the second kind. The quantity N is the complex index of refraction of the scattering medium expressed in the form

$$N = m - mki$$

where the imaginary part is referred to as the absorption coefficient. The quantities σ_n and P_n depend on appropriate Bessel and Hankel functions respectively.

Calculations of σ_b have been made by Ryde (1946). Tables of the scattering coefficients have been published by Lowan (1949), Penndorf (1956), and Haugen (1953). Aden (1951) has presented a detailed investigation of water particles of dielectric constant $\epsilon = 81 - 78i$ (where $\epsilon = N^2$) and $\lambda = 16.230$. He demonstrated that up to $\alpha = 6.0$, experimental values of σ_b correspond very closely to theoretical values calculated from (3). See solid curve on figure 3.

All of the published back-scattering functions have referred to water particles or to particles having diameters such that values of α have been smaller than about 6. Water particles in the atmosphere are usually of such sizes that α is less than 6. Ice particles, and particularly hail, reach sizes which lead to large values of α over the wave length band commonly used with weather radar. This study has been entirely concerned with the scattering of ice particles and has involved α spanning the range likely to be encountered in nature.

Calculations of a_n^s and b_n^s have been made with an IBM-650 and were combined to obtain the values of normalized back-scattering cross sections. The complex index of refraction of the ice particles was taken as $N = 1.78 - 0.0024i$, a typical value for ice at a temperature of 0°C . The results were quite different from those expected. See table 1 and figure 1.

On the basis of Aden's results with water particles it was expected that σ_b would have been quite small for values of α as large as 30. From the back-scattering equations it can be shown that with a hypothetical drop of infinite radius of curvature, i.e., with a flat surface, σ_b should have a value of 0.078.

In order to learn more about the relationship of σ_b to α , calculations were made at various values of α ranging up to 500. See table 2 and figure 2. It is evident that σ_b reaches a maximum in the vicinity of $\alpha = 60$ and then begins to decrease. At $\alpha = 500$, it has a value of about 0.3.

The calculations show that over a large range of particle sizes, the back-scattering from large hailstones exceeds that from water drops of the same size. Quite obviously, all-water particles having diameters of several centimeters will not be encountered in the atmosphere, but it is reasonable

Table 1

Calculated values of σ_b for ice spheres of $m = 1.78$, $k = 0.00135$

α	σ_b	α	σ_b	α	σ_b	α	σ_b	α	σ_b	α	σ_b	α	σ_b
0.1	7×10^{-5}	2.6	1.4646	7.2	6.7028	12.2	10.2694	17.2	22.9722	22.2	22.6530	27.2	20.0157
0.2	1.1×10^{-3}	2.7	1.7704	7.4	10.3707	12.4	22.0480	17.4	18.5164	22.4	14.5250	27.4	31.9916
0.3	5.6×10^{-3}	2.8	2.2135	7.6	6.5771	12.6	21.1806	17.6	15.8850	22.6	22.6408	27.6	26.9828
0.4	1.72×10^{-2}	2.9	3.1091	7.8	4.7213	12.8	12.3596	17.8	26.8321	22.8	24.5290	27.8	17.1326
0.5	4.08×10^{-2}	3.0	3.8304	8.0	10.9309	13.0	8.8786	18.0	26.2438	23.0	19.8164	28.0	26.6273
0.6	8.10×10^{-2}	3.2	2.4544	8.2	4.2887	13.2	17.0460	18.2	14.0078	23.2	26.3726	28.2	32.2838
0.7	0.1412	3.4	2.7091	8.4	6.2415	13.4	21.0595	18.4	16.6284	23.4	21.3766	28.4	23.5485
0.8	0.2205	3.6	6.4172	8.6	3.0834	13.6	9.7223	18.6	29.5433	23.6	19.1324	28.6	25.0345
0.9	0.3106	3.8	8.8841	8.8	5.1995	13.8	14.2656	18.8	22.5677	23.8	23.1163	28.8	24.5703
1.0	0.3928	4.0	4.6144	9.0	9.5341	14.0	14.9815	19.0	14.3629	24.0	33.3929	29.0	25.0680
1.1	0.4384	4.2	3.8994	9.2	6.2737	14.2	12.3239	19.2	25.3810	24.2	23.5812	29.2	24.9565
1.2	0.4168	4.4	13.9184	9.4	5.1482	14.4	17.5082	19.4	28.9221	24.4	17.9974	29.4	31.8320
1.3	0.3131	4.6	11.6072	9.6	7.4351	14.6	10.5048	19.6	18.8585	24.6	33.6263	29.6	25.5204
1.4	0.1580	4.8	6.0102	9.8	13.2813	14.8	11.7126	19.8	21.7305	24.8	34.7672	29.8	20.8244
1.5	0.0598	5.0	12.3088	10.0	10.0378	15.0	10.4418	20.0	22.0843	25.0	21.0351	30.0	35.8092
1.6	0.1530	5.2	11.3175	10.2	7.1086	15.2	16.0916	20.2	19.9911	25.2	23.0087		
1.7	0.3827	5.4	11.0038	10.4	13.9627	15.4	16.5382	20.4	16.2073	25.4	33.1696		
1.8	0.5564	5.6	7.2166	10.6	16.2087	15.6	7.9470	20.6	23.8476	25.6	27.1167		
1.9	0.6316	5.8	9.6191	10.8	11.4612	15.8	17.1191	20.8	21.8578	25.8	22.5253		
2.0	0.6655	6.0	12.0030	11.0	12.5049	16.0	14.2622	21.0	14.0746	26.0	30.5862		
2.1	0.6839	6.2	5.5433	11.2	18.0036	16.2	12.3911	21.2	24.5234	26.2	27.8026		
2.2	0.6579	6.4	11.9571	11.4	13.7645	16.4	14.0905	21.4	22.9111	26.4	21.9221		
2.3	0.6259	6.6	9.2789	11.6	12.3802	16.6	16.9274	21.6	15.5188	26.6	32.3319		
2.4	0.7869	6.8	5.1204	11.8	22.5442	16.8	17.3167	21.8	17.2873	26.8	32.0174		
2.5	1.1241	7.0	11.0214	12.0	16.3903	17.0	16.5161	22.0	21.7763	27.0	19.8481		

Table 2

Calculated values of σ_b for ice spheres
 $m = 1.78$, $k = 0.00135$ at some very large
values of α .

α	σ_b	α	σ_b
40	24.9543	90	15.5643
50	31.0070	100	13.5398
60	38.5239	150	11.0306
70	24.3862	200	7.4799
80	18.8127	500	0.2957

to expect that a large hailstone may develop a layer of water on its surface when it starts melting. Also, Ludlam (1950) has pointed out that a hailstone may develop a layer of supercooled water if it sweeps up excessive amounts of supercooled water by collision with smaller cloud and raindrops.

It is important at this point to ask whether or not the equations can be accepted as applicable over the assumed conditions. Fortunately, some experimental data are available with which to compare the theoretical results. Atlas et al.,* have been carrying out measurements of the back-scattering cross section of ice spheres. As a matter of fact, a communication from him dealing with his work encouraged the authors to carry out the research described in this report. The Mie equations had already been programmed on the IBM-650 for another investigation and it was a straightforward job to make the ice sphere calculations.

*Atlas, D., W. G. Harper, F. H. Ludlam, and W. Macklin, 1959: Radar Scatter by Large Hail. Technical Report under Air Force Contract No. 61(052)-354, Imperial College, London.

Figure 1 Calculated values of σ_b for ice spheres of $m = 1.78$ and $k = 0.00135$.

Figure 2 Smoothed curve of σ_b as function of α for spheres of $m = 1.78$ and $k = 0.00135$.

The measurements by Atlas et al.,* extended over a range of α from about 0.4 to 10. Although precise values of their measurements are not available, they fit the theoretical curves quite closely. In the paper cited they shall report their findings and show the close relation between their experimental results and the theoretical findings discussed here.

The conclusion that an ice sphere has a higher radar reflectivity than a water sphere is surprising because with particles much smaller than the wave length, the reverse is true. When $\alpha \ll 1$, that is, when Rayleigh scattering prevails, the normalized back-scattering of a single spherical particle is given by

$$\sigma_b = 4\alpha^4 \left| \frac{N^2 - 1}{N^2 + 2} \right|^2 \quad (5)$$

The term inside the brackets has values of about 0.93 and about 0.19 for water and ice respectively. The ratio of about 5:1 in favor of water has long been known. The calculations show that outside the Rayleigh region, the ratio of back-scattering of ice and water spheres is a highly variable quantity and cannot be generalized in a simple equation. This is clearly demonstrated in table 3 which shows the back-scattering cross sections of water and ice spheres of various sizes. The results are applicable to 3-cm radiation. This table also shows that a single large hailstone may have a reflectivity some 8 orders of magnitude higher than a typical raindrop.

The conclusion that large hailstones have much larger radar reflectivities than previously assumed has important meteorological significance. Extreme echo intensities of the kind reported by Donaldson (1959) now seem to be more reasonable. Also, rapid changes of echo intensity

Table 3

Back-scattering cross sections of water and ice spheres in cm^2 . Results are applicable to 3-cm radiation.

Diameter cm	α	Water (Approximate)	Ice	Ratio
0.1	0.105	(8.32×10^{-6})	(1.70×10^{-6})	0.20
3.0	3.14	4.9	20	4
6.0	6.28	17.0	229	13
10.0	10.48	51.2	1170	23

of the type reported by Battan (1953) need reexamination in the light of this new information.

3. Calculations of σ_b for various values of N and k

The properties of the Mie equations over the region of interest certainly are poorly known. As a consequence, unusual results should probably have been expected. In order to learn more about them various additional calculations have been made.

Because of the large differences in the magnitudes of σ_b for water and ice, it was decided to examine the effects of changing the real and imaginary parts of the complex index of refraction. The results of the calculations are shown in figures 3 and 4. In the first diagram the absorption coefficient, k, was fixed at a value of 0.048, the value used by Aden, while the value of the real part was reduced from 8.99 to 2.0. The values of σ_b , for the most part, are of the same order of magnitude and the oscillations are damped as larger values of α are approached. As the largest α values are reached it is seen that the curves are approaching a value of σ_b which decrease as m decreases. This result is quite reasonable, as can easily be shown in the limit, by considering a flat surface.

Figure 3 Calculations of $\hat{\sigma}_b$ for spheres of $k = 0.00135$ and various values of m .

Figure 4 Calculations of σ_b for spheres of $m = 8.99$ and various values of k .

The curves shown in figure 4 are of greater interest because they throw some light on the question of the differences in the back-scattering of water and ice. For these calculations the real part of the refractive index was fixed at 8.99 while the absorption coefficient was changed. It can be seen that over the range from 0.05 to 0.01, the shape of the curve and the magnitudes of σ_b are quite close to one another. However, when the coefficient is reduced from 0.01 to 0.009, the curve suddenly undergoes a substantial change. The phase of the oscillations is very much different than those of the other curves, the amplitudes are larger, and there is little sign that the amplitudes are decreasing with increasing α . On the basis of figure 1, it might be expected that further increases of α would result in increases of σ_b .

With an absorption coefficient of 0.005, the frequency of the oscillations as well as the magnitudes is even greater than those shown for $k = 0.009$. For example, at α values of 3.75 and 5.5, σ_b has magnitudes of 3.3 and 2.8 respectively.

On the basis of these calculations, it is evident that the main factor in causing ice spheres to have such large back-scattering coefficients is the small absorption. It also is shown that there is a sudden transition from a scattering regime where the back-scattering decreases fairly rapidly with increasing sphere diameter to one where it oscillates between very large and small values, and where it may increase over a very large range of sphere sizes.

Atlas et al.,* have suggested that with a water sphere, the back-scattering is essentially a surface phenomena as with a metal sphere. However, when the absorption of the medium composing the sphere is

*Op. Cit.

extremely low, as is the case with ice, the sphere will act like a lens which focuses the incoming wave on the back side. This is recognized as a crude description, but for the time being, it at least permits visualization of a physical process which may have some reality. If, in fact, this is close to the actual mechanism, the calculations shown in figure 4 would lead one to conclude that back-scattering from the front side of the sphere is predominant when the absorption coefficients are greater than some critical value. It might be further speculated that until this value is reached, the quantity of energy penetrating the drop and being focused is virtually zero. When the absorption drops below the critical value, there is a marked increase of the fraction of the incoming energy which is focused and back-scattered.

The most obvious conclusion regarding the physical processes involved is that at this point we know very little with certainty. Further research is in progress and will be reported in subsequent papers.

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