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CALCULATIONS OF MIE BACK-SCATTERING FROM
MELTING ICE SPHERES

by

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ABSTRACT

Calculations of the normalized back-scattering cross-section, σ_b , of ice spheres surrounded by shells of liquid water have been made from an extension of the Mie theory to a two-layer model. Curves of σ_b as a function of the thickness of the liquid water shell are presented for various sized spheres for 3.21, 4.67 and 10.0 cm radiation. It is shown that, depending upon the size of the sphere and the wavelength of the incident radiation, the back-scattering may either increase or decrease as the ice acquires a liquid water shell. For certain sized spheres, interference phenomena, which in some instances may lower the value of σ_b by several orders of magnitude, are in evidence during the course of melting. Comparisons are made between the theoretical results presented here and experimental measurements of σ_b for melting ice spheres performed by Atlas et al., (1960).

I. INTRODUCTION.

In an earlier report (1959) the authors presented calculations of Mie back-scattering from ice spheres for 3.21 cm wavelength radiation. It was shown that for large values of α ($= \frac{2\pi b}{\lambda}$ where b = sphere radius and λ = the wavelength of the incident radiation) the back-scattering cross-sections of ice spheres are larger than those of water spheres of the same diameter, the ratio of the two values showing a general increase with increasing α , Figure 1. This result is in direct contrast to the back-scattering properties of the two media in the Rayleigh region (i.e., where $\alpha \ll 1$) where a water sphere has a higher back-scattering cross-section than an ice sphere of the same diameter.

It was further shown in the above-mentioned report (see Figure 1) that the curve of back-scattering of water spheres as a function of α shows rather regular, almost sinusoidal oscillations which become rapidly damped with increasing α . By the time an α of 10 is reached, the deviations of the back-scattering from the limiting value for $\alpha = \infty$ (i.e., a flat-water surface) of 0.63, are quite small. The same curve for ice spheres, on the other hand, shows highly irregular oscillations

with no tendency for any damping, up to an α of 60. At this point the normalized back-scattering cross-section of an ice sphere has a maximum value of 38.5. It then slowly decreases and approaches the limiting value (not shown in Figure 1) for a flat ice surface of 0.078. At an α of 500, it is approximately 0.30. It should be pointed out here that it is not at all obvious from the definition of σ_b that this parameter should approach the value of the normal incidence reflectivity from a flat plate of the material in question, as the radius of the sphere approaches ∞ . However, a formal proof of this point has been worked out by J. E. McDonald (to be published elsewhere) which confirms this rather peculiar property of σ_b .

During the course of the research described above and following correspondence with Dr. D. Atlas giving preliminary results of experimental work at Imperial College, it became apparent that it would be very desirable to know the back-scattering properties of frozen particles as they descended below the freezing level and acquired a layer of liquid water about them. Kerker et al., (1951) had examined the variation of back-scattering of ice spheres in the Rayleigh region. Back-scattering experiments carried on by Atlas et al., (1960) on large ice spheres suspended from a balloon, produced results which they could attribute to the melting of the ice. In view of the large differences in the back-scattering properties of water and ice, it is quite apparent that as an ice sphere acquires a layer of liquid water, its back-scatter should undergo quite marked changes. A knowledge of such changes is of considerable importance in radar studies of precipitation processes, particularly those associated with thunderstorms.

II. MATHEMATICAL MODEL.

The mathematical model used in the calculations presented here was originally described by Aden (1952). A brief review of the analysis will be given here.

The actual physical problem is replaced by a mathematical model in which it is desired to find the reflection of electromagnetic radiation from two concentric spheres with different complex indices of refraction. The complex index of refraction N is given by $N = m(1 - ki)$ where m is the ordinary index of refraction, k is the Mie absorption coefficient and is related to the ordinary absorption coefficient, a , by the expression $k = \frac{a}{m} \left(\frac{\lambda}{4\pi} \right)$ where λ is the wavelength of the incident radiation. Let the inner sphere be designated by the subscript 1, the outer sphere by 2, and the surrounding medium by 3, Figure 2. Let the incident electromagnetic wave vector, $\vec{S}_i$, be propagated along the Z-axis and the incident electric vector, $\vec{E}_i$, be linearly polarized parallel to the X-axis. Then the incident magnetic field vector, $\vec{H}_i$, will be parallel to the Y-axis. Here ϵ is the dielectric constant and μ the magnetic permeability.

The expressions for the incident plane wave here will be the same as for the case of a single solid sphere, and have been shown by Stratton (1941) to be given by

$$\vec{E}_i = E_0 \sum_{n=1}^{\infty} (-i)^n \left[\frac{2n+1}{n(n+1)} \right] (\vec{m}_{oin}^{(1)} + i \vec{n}_{ein}^{(1)}) \quad (1a)$$

$$\vec{H}_i = - \frac{k_3 E_0}{\omega_3 \mu_3} \sum_{n=1}^{\infty} (-i)^n \left[\frac{2n+1}{n(n+1)} \right] (\vec{m}_{ein}^{(1)} - i \vec{n}_{oin}^{(1)}) \quad (1b)$$

where $\omega_3 = \frac{2\pi V_3}{\lambda} = \frac{2\pi}{\lambda} (\mu_3 \epsilon_3)^{-\frac{1}{2}}$, V_3 is the velocity of the wave in medium 3, E_0 is the magnitude of the electric vector at its origin, and

the $\vec{m}$ and $\vec{n}$ vectors are the so-called spherical vector wave functions, having the property that they are solutions to the vector-wave equation describing all electromagnetic fields. These vector-wave functions are composed of various combinations of sines and cosines, associated Legendre Polynomials, and spherical Bessel functions. The subscripts e and o refer to even or odd dependence on the angle ϕ , where ϕ is the angle in the X-Y plane measured counterclockwise from the X-axis. The subscript l refers to the order of the associated Legendre Polynomials (we are thus restricted to Legendre Polynomials of order 1), while n refers to the degree of the associated Legendre Polynomials. The superscript (1) refers to the type of spherical Bessel function; in this case we use Bessel functions of the first kind in order that the incident wave expression remains finite at radius $r = 0$.

The induced secondary field is now expanded into three parts, one applying to the inner sphere called the transmitted field and denoted by the subscript t, one applying to the outer shell denoted by the subscript 2, and one applying to the surrounding medium, called the scattered field, and denoted by the subscript s. The scattered field, outside of the sphere, may be written as

$$\vec{E}_s = E_0 \sum_{n=1}^{\infty} (-i)^n \left[\frac{2n+1}{n(n+1)} \right] (a_n^s \vec{m}_{oin}^{(3)} + i b_n^s \vec{n}_{ein}^{(3)}) \quad (2a)$$

$$\vec{H}_s = -\frac{k_3 E_0}{\omega_3 \mu_3} \sum_{n=1}^{\infty} (-i)^n \left[\frac{2n+1}{n(n+1)} \right] (b_n^s \vec{m}_{ein}^{(3)} - i a_n^s \vec{n}_{oin}^{(3)}) \quad (2b)$$

where a_n^s and b_n^s are the scattering coefficients to be determined. The superscripts (3) here indicate that the particular Bessel functions to be used are Hankel functions of the second kind, in order to keep the scattered field finite at $r = \infty$.

Inside the inner sphere the transmitted field may be written as

$$\vec{E}_t = E_0 \sum_{n=1}^{\infty} (-i)^n \left[\frac{2n+1}{n(n+1)} \right] (a_n^t \vec{m}_{oin}^{(1)} + i b_n^t \vec{n}_{ein}^{(1)}) \quad (3a)$$

$$\vec{H}_t = -\frac{k_1 E_0}{\omega_1 \mu_1} \sum_{n=1}^{\infty} (-i)^n \left[\frac{2n+1}{n(n+1)} \right] (b_n^t \vec{m}_{ein}^{(1)} - i a_n^t \vec{n}_{oin}^{(1)}) \quad (3b)$$

where a_n^t and b_n^t are the transmission coefficients. Note here that Bessel functions of the first kind (indicated by the superscripts (1) after the $\vec{m}$ and $\vec{n}$ vectors) are used since the transmitted field must remain finite at the origin, as was the case for the incident field.

Inside the spherical shell, since r goes to neither zero nor infinity in this region, both Bessel functions of the first kind and Hankel functions of the second kind must be retained. Therefore, the expression for the secondary field in this region may be written as

$$\vec{E}_2 = E_0 \sum_{n=1}^{\infty} (-i)^n \left[\frac{2n+1}{n(n+1)} \right] [a_n^2 \vec{m}_{oin}^{(1)} + A_n^2 \vec{m}_{oin}^{(3)} + i(b_n^2 \vec{n}_{ein}^{(1)} + B_n^2 \vec{n}_{ein}^{(3)})] \quad (4a)$$

$$\vec{H}_2 = -\frac{k_2 E_0}{\omega_2 \mu_2} \sum_{n=1}^{\infty} (-i)^n \left[\frac{2n+1}{n(n+1)} \right] [b_n^2 \vec{m}_{ein}^{(1)} + B_n^2 \vec{m}_{ein}^{(3)} - i(a_n^2 \vec{n}_{oin}^{(1)} + A_n^2 \vec{n}_{oin}^{(3)})] \quad (4b)$$

At the boundaries, $r = a$ and $r = b$, we now apply the conditions that the tangential components of both the $\vec{E}$ and $\vec{H}$ fields must be continuous. Thus, at $r = a$

$$E_{2\theta} = E_{t\theta} \quad (5a)$$

$$H_{2\theta} = H_{t\theta} \quad (5c)$$

$$E_{2\phi} = E_{t\phi} \quad (5b)$$

$$H_{2\phi} = H_{t\phi} \quad (5d)$$

at $r = b$,

$$E_{i\theta} + E_{s\theta} = E_{\theta\theta} \quad (6a)$$

$$H_{i\theta} + H_{s\theta} = H_{\theta\theta} \quad (6c)$$

$$E_{i\phi} + E_{s\phi} = E_{\phi\phi} \quad (6b)$$

$$H_{i\phi} + H_{s\phi} = H_{\phi\phi} \quad (6d)$$

where $E_{\theta\theta}$ is the θ component of the $\vec{E}_\theta$ field, etc.

When the indicated components of the $\vec{E}$ and $\vec{H}$ fields are substituted into the above boundary conditions, it leads to two sets of simultaneous, nonhomogeneous equations, each involving four of the unknown coefficients, and thus, these coefficients may be obtained. The resulting expressions for the coefficients are rather long and cumbersome, involving the spherical Bessel and Hankel functions. The reader is referred to the original article by Aden (1952) for a more complete description of the theory.

The back-scattering cross-section, σ , as has been shown by Aden (1952) and others, is given by

$$\sigma = \frac{\lambda^2}{4\pi} \left| \sum_{n=1}^{\infty} (2n+1)(-1)^n (a_n^s - b_n^s) \right|^2 \quad (7)$$

The normalized back-scattering cross-section, σ_b , is obtained by dividing the expression for σ by the cross-sectional area of the scatterer, and is thus given by

$$\sigma_b = \frac{1}{\nu^2} \left| \sum_{n=1}^{\infty} (2n+1)(-1)^n (a_n^s - b_n^s) \right|^2 \quad (8)$$

where $\nu = \frac{2\pi b}{\lambda}$ *. The values of back-scattering presented in this paper are the normalized back-scattering cross-section as defined above. The

*The symbols ν and α are defined as $\frac{2\pi b}{\lambda}$. The former refers to melting ice spheres. The latter refers to homogeneous spheres.

calculations were performed on an IBM 650 computer by the Numerical Analysis Laboratory of The University of Arizona.

III. RESULTS.

The values of the normalized back-scattering cross-section as a function of water thickness were calculated for various size spheres and for 3.21, 4.67 and 10 cm radiation. The results are shown in Figures 3, 4 and 5. It should be pointed out that for these calculations, the outer radius, b , was held constant for each curve, while the water thickness, $b-a$, was varied. The last point on each curve is, in each of the cases presented, the value of σ_b for an all-water drop of radius b .

An inspection of the curves reveals that in most instances the presence of the water layer first begins to affect the value of σ_b when it has attained a thickness of approximately 10^{-4} cm. The thickness of water necessary to cause the sphere to have a back-scattering cross-section essentially that of an all-water drop of the same diameter, however, is different for each of the wavelengths considered. The thicknesses are approximately 5×10^{-1} , 7×10^{-1} and between 1 and 2 cm for wavelengths of 3.21, 4.67 and 10.0 cm, respectively. For a sphere whose initial radius is less than the above values, the all-water value of σ_b is not reached until the melting is complete, although for the smallest size spheres the departure of σ_b from that of an all-water drop is quite small once the major portion of the sphere has melted.

The variation with λ of the thickness of liquid water necessary to cause the sphere to act essentially as an all-water drop is quite reasonable when one considers the Mie absorption coefficients, k , for the three wavelengths: 0.405, 0.277 and 0.164 at wavelengths of 3.21, 4.67 and 10.0 cm, respectively. Thus, as the wavelength increases, k decreases, and therefore

at the longer wavelengths a thicker layer of water is necessary to absorb essentially all of the incident radiation which penetrates the front surface of the sphere, so that there may be no contribution to the back-scatter from internal reflections at the water-ice interfaces. When this thickness is reached, the back-scatter is essentially a surface phenomena, the only contribution coming from reflections at the initial air-water interface, and thus the sphere optically is no different from an all-water sphere of the same diameter.

The curves for the two shorter wavelengths of incident radiation (3.21 and 4.67 cm) may clearly be divided into two distinct groups, one for which the back-scatter increases as the sphere melts, and one for which the back-scatter decreases as the sphere melts. These are labeled groups A and B respectively in Figures 3 and 4. For $\lambda = 3.21$ cm, the division of these classes occurs somewhere in the region between a sphere radius b of 1.0 and 1.5 cm, which corresponds to $\alpha \approx 3$, while for $\lambda = 4.67$ cm, the division occurs for a sphere radius between 1.5 and 2.0 cm. This is to be expected since a value of $\alpha = 3$ ($\alpha = \frac{2\pi b}{\lambda}$) represents the approximate point which separates the zone in which water has a higher normalized back-scattering cross-section than does ice from the zone for which the reverse is true for 3.21 cm radiation (see Figure 1). For $\lambda = 4.67$ cm, a value of $\alpha = 3$ corresponds to a radius of approximately 2.2 cm. The small discrepancy at 4.67 cm arises from the fact that at this wavelength the complex index of refraction of water is different from that at 3.21 cm, and thus the curve of σ_b of water as a function of α will differ from that at 3.21 cm, causing a slightly different cross-over value.

Inspection of the 10 cm radiation curves shows that only for radii larger than approximately 4.0 cm does the back-scattering decrease as the

drops melt. Thus, at this wavelength, for ranges of particle sizes normally encountered in the atmosphere, one would always expect a higher return signal from liquid spheres than from their frozen counterparts, assuming the shape factor to be unimportant. It should be pointed out here that the 10 cm curve for $b = 0.20$ cm is essentially the same as that presented by Kerker et al., (1951). There are minor differences in the actual magnitudes of σ_b , but these are attributed to the slightly different indices of refraction used.

The curves of σ_b for those size ranges for which the back-scattering increases as the sphere melts show, as has been known for many years, that melting plays an important part in the formation of the radar bright-band. An inspection of the curves reveals that the back-scattering may increase as much as one order of magnitude as the sphere goes from the solid to the liquid state.

An examination of the curves of σ_b reveals that for certain sizes -- i.e., for $\lambda = 3.21$ cm, $b = 0.5$ and 1.0 cm, and for $\lambda = 4.67$, $b = 1.0$ and 1.5 cm -- the values of σ_b do not increase steadily from the all-ice value to the all-water value as the thickness of the water shell is increased. Instead, the back-scattering is first observed to decrease to a value less than that of the all-ice value, and then, as the liquid layer thickens, it rises toward the all-water value. In some instances, after the above-mentioned minimum, a maximum is reached, for which σ_b is greater than that for an all-water sphere, e.g., for $\lambda = 3.21$ cm, $b = 1.0$ cm, after which σ_b decreases toward the all-water value. While a complete physical interpretation of these results cannot be offered, these maxima and minima are undoubtedly due to interference phenomena between the externally reflected ray at the front surface of the sphere and the internally reflected rays

from the two water-ice interfaces and from the back surface of the sphere. The possibility that similar interference effects are responsible for the anomalous behavior of σ_b for all ice spheres has already been suggested by Atlas et al., (1960). A further factor which may contribute to the formation of the maxima and minima as the spheres melt, is the changing radius of the ice part of the sphere as the melting process is continued. If it is recalled that, for these calculations, the radius of the sphere, b , is held constant while the thickness of the water shell is increased, it is evident that there is a decrease of the radius of the ice sphere within the water shell. If one now inspects the curve of back-scattering for all-ice spheres, Figure 1, it is seen that for small changes of α (i.e., small changes of sphere radius), the value of σ_b may undergo very marked changes. It should be pointed out here that the values of σ_b for ice were calculated for increments of α of 0.2. There is reason to suspect that were smaller increments employed, oscillations even more abrupt and marked than those shown in Figure 1 would have been found. Thus it is reasonable to suspect that for certain sizes, as the melting process proceeds, thereby causing a decrease in the radius of the frozen part of the sphere, the contribution to the back-scattering by the ice alone may undergo considerable change, thus contributing to the maxima and minima of σ_b observed for some sphere sizes.

If we now focus our attention on the curves of σ_b for the larger size spheres for incident radiation of wavelengths of 3.21 and 4.67 cm, it is seen that as the spheres melt and acquire a shell of liquid water, the back-scattering decreases. In several of these curves the interference phenomena mentioned above is evident. The most extreme case of interference is observed on the curve for $\lambda = 3.21$ cm and $b = 5.0$ cm. Here, a sharp drop of over 3 orders of magnitude is observed, after which a rapid recovery

is made to a value of σ_b close to that of all water. Undoubtedly, if calculations were made for smaller increments of water thickness, other such remarkable examples of interference would have been obtained.

Atlas et al., (1960) have carried on experimental measurements of the back-scattering from melting ice spheres. Figure 6 is reproduced from their work. It should be noted that the authors assumed that as the ice spheres melted, the overall sphere radius, (b), remained constant until a thickness of water of 0.01 cm had been attained. Thereafter, the thickness of the water shell was assumed to remain constant at 0.01 cm, while the radius (b) decreased as the ice melted. This assumption was made on the basis of laboratory measurements which indicated that 0.01 cm was the maximum thickness of liquid water which would remain on an ice sphere. Thus, the vertical portions of the curves in Figure 6 (indicating constant diameter) represent that portion of the melting process during which the water layer has not yet reached a thickness of 0.01 cm. After 0.01 cm of water is reached, the curves slope toward decreasing diameter. The final sphere diameter was measured, and intermediate diameters were determined from the relationship

$$a_0^2 - a^2 = kt \quad (9)$$

where a_0 is the initial radius, a the radius at time t , and k is a constant determined from the final measured radius.

While exact comparisons between their experimental results and the theoretical results presented here are not possible, inasmuch as the exact water thicknesses present on the ice spheres used by Atlas et al., are not known, certain of their results may, qualitatively at least, be verified. On Figure 6 it can be seen, that in several instances, the final experimental values of back-scattering fall below the value of an all-water sphere of the

same diameter. Presumably, the particular water thicknesses and sphere diameters involved were such as to cause destructive interference. An inspection of the theoretical curves for those spheres for which the back-scattering decreases as the sphere melts, shows that some of the destructive interference phenomena occur for water thicknesses near 0.01 cm. However, Figures 3 and 4 show that values of back-scattering less than those of all-water spheres occur over a finite range of water thicknesses around the actual minima. Thus, for example, for $\lambda = 3.21$ cm, $b = 3.0$ cm, values of σ_b less than that of an all-water sphere of the same radius occur for water thicknesses between approximately 8×10^{-3} and 2.5×10^{-2} cm. Thus, a sphere of radius 3.0 cm composed of a shell of water whose thickness lies between the above values, surrounding a sphere of ice, will back-scatter less radiation than will an all-water sphere of the same radius.

Figures 7a and 7b show σ_b for all-water spheres and for ice spheres surrounded by 0.01 cm of water for wavelengths of 3.21 and 4.67 cm. It is quite apparent from these curves that, except for radii less than approximately 1.0 cm, and between approximately 2.8 and 3.5 cm, for 3.21 cm radiation, and for radii less than approximately 1.5 cm and certain values between 4.0 and 5.0 cm for 4.67 cm radiation, an ice sphere with 0.01 cm of liquid water surrounding it still is a better back-scatterer than is an all-water sphere of equal radius. In general, a shell of liquid water of 0.01 cm is not thick enough to cause the sphere to act like an all-water drop. This result is rather obvious from an inspection of Figures 3, 4 and 5 also. Thus, if as stated by Atlas et al., (1960), 0.01 cm liquid water is the maximum amount which will remain on a melting ice sphere, then, at least over most size ranges, the sphere will continue to scatter more than an equivalent water sphere as the melting process proceeds.

As a final bit of comparison between theory and experiment, calculations of σ_b as a function of water thickness, for 3.21 and 4.67 cm radiation, were made for spheres of 2.95, 3.33, 4.63 and 3.90 cm radii. These are shown in Figure 8. These radii are the initial radii of the spheres for the curves labeled D and D', C and C', E and E', B and B' respectively in Figure 6. For these four targets, Atlas et al., (1960) have presented curves of simultaneous echo intensity measurements for 3.3 cm and 4.7 cm radiation, as a function of time. These curves are shown in Figure 9. It may be seen that for targets B, C and E, as the melting progresses and the water shell thickens (to some maximum value after which it begins to drip off), the echo intensity drops off much more rapidly at 3.3 cm than at 4.7 cm. An inspection of Figure 8 reveals that these results are in agreement with the theoretical curves. If we assume, as stated by Atlas et al., (1960) that 0.01 cm of liquid water is the maximum thickness attainable, then from the theoretical curves we see that for this thickness of liquid water, the value of σ_b for 3.21 cm has undergone a larger drop from the all-ice value than has σ_b for 4.67 cm radiation for all four radii. For two of the spheres, 2.95 and 3.33 cm, a water thickness of 0.01 cm is at a sharp minimum in the σ_b curve, apparently due to interference phenomena. That such sharp minima are not detected in the experimental curves is not surprising, since it is unlikely that exactly 0.01 cm of liquid water is the maximum shell obtainable, and slight variations in the water thickness are enough to remove the value of σ_b from the sharp minimum on the theoretical curves.

The experimental measurements on target D indicate that the signal dropped off approximately at the same rate for both wavelengths. This result, however, is not borne out by the theoretical curves, the theoretical value of σ_b for 3.21 cm radiation being, in fact, at a sharp minimum for a

0.01 cm water thickness. It does not, however, follow that for all sphere sizes the signal drops off more rapidly at 3.21 than at 4.67 cm. For example, consider the curves at the two wavelengths for a sphere of radius 4.0 cm, Figures 3 and 4. Here the value of σ_b at 4.67 falls sharply to an interference minimum at a water thickness of 4×10^{-2} cm, while in this region of water thickness at 3.21 cm, the value of σ_b falls near a secondary maximum, the main minimum not being reached until a water thickness of 0.01 cm has been attained. However, in most instances, with the absence of pronounced interference effects, it is probably true that the signal drop-off is more rapid at 3.21 cm than at 4.67 cm, as the melting progresses. This effect was explained by Atlas et al., (1960) as being a result of the greater absorption of liquid water at the shorter wavelength. Thus, as the melting proceeds a given thickness of liquid water will have a greater effect at 3.21 cm than at 4.67 cm, until such time as the liquid water attains a thickness such that the spheres are acting essentially as all water, after which changes in the water thickness are of no further consequence.

It is unfortunate that a more critical comparison between the experimental work of Atlas et al., (1960) and the theoretical work presented here cannot be made, but as has already been pointed out, this is not possible because of an uncertainty as to the exact thickness of the liquid water shells in the experimental work, and also because of the decrease in sphere radius due to shedding of liquid water as the melting proceeded in the latter work. If one could say for sure that, say, 0.01 cm of liquid is the maximum thickness present on any melting ice sphere, and thereafter, as the melting proceeds, this thickness remains constant, the additional melt being shed off, then it would be a simple matter to combine the curves of

Figure 3 or 4 with those of Figure 7a or 7b to obtain σ_b for a melting ice sphere for 3.21 or 4.67 cm radiation. Thus for a given initial radius, one would pick the corresponding curve from Figure 3 or 4 and utilize that portion of the curve up to a water thickness of 0.01 cm. Thereafter, the back-scattering would follow a portion of Figure 7a or 7b, depending upon the wavelength of the incident radiation. Such a back-scattering curve for 3.21 and 4.67 cm radiation is shown in Figure 10 for a sphere of initial radius of 3.90 cm corresponding to curves B and B' of Figure 6. It is obvious from this figure that once the maximum thickness of liquid water has been attained, the value of σ_b should not continue a steady decrease, but rather should undergo marked oscillations. That the experimental results shown in Figure 9 indicate a continued steady decrease of σ_b (with the exceptions of curves A and A') has already been pointed out by Atlas et al., (1960) and is attributed to the fact that cracks were present in the artificial hailstones in which liquid water accumulated. Thus, not only was a shell of liquid water present during their experiments, but after this shell attained a maximum thickness, additional water continued to accumulate in the cracks, thereby causing the continuing decrease in the back-scattering.

It appears from the preceding discussion that it would be an extremely difficult task to experimentally reproduce the conditions assumed in the theoretical results presented here. However, many of the difficulties encountered by Atlas et al., (1960) in artificially manufacturing ice spheres are adequately handled in nature. The value of the maximum water thickness present on a hailstone is obviously still unknown, as is the effect of nonsphericity. Further, since natural hailstones are composed of layers, this factor may have an important effect in determining their back-scattering cross-section. Nevertheless, the theoretical results presented here should

represent a close enough approximation to actual conditions as to be of considerable value for future studies of radar reflectivity, particularly from hailstorms.

IV. SUMMARY.

Theoretical calculations of the normalized back-scattering cross-sections, σ_b , of melting ice spheres have been performed based on an extension of the Mie theory to a two-layer model. The resulting curves of σ_b as a function of the thickness of the liquid water shell show that the back-scattering may either increase or decrease as the melting proceeds, depending upon the wavelength of the incident radiation and the original size of the ice sphere. For 3.21 cm radiation, as the melting proceeds the value of σ_b decreases for spheres of radius 1.0 cm or smaller, while it increases for spheres of radius 1.5 cm or greater. For 4.67 cm radiation, the corresponding values of the radii are 1.5 and 2.0 cm, respectively. For 10 cm incident radiation, the back-scattering always increases as melting proceeds for particle sizes normally encountered in the atmosphere.

The thickness of the water shell necessary to make the sphere act as though it were all-water is likewise a function of the wavelength of the incident radiation. For 3.21 cm radiation, this thickness was shown to be approximately 5×10^{-1} cm, for 4.67 cm radiation it is 7×10^{-1} cm, while for 10.0 cm radiation it is approximately 1.0 to 2.0 cm. For thicknesses of water greater than those indicated, the spheres act the same, insofar as their back-scattering properties are concerned, as all-water spheres. For intermediate water thicknesses, back-scattering cross-sections are different than those of all-ice or all-water spheres, and vary in a highly erratic fashion. Cases have been presented for which σ_b drops abruptly by more than 3 orders of magnitude. Presumably this behavior is

due to interference effects between the rays reflected from the front surface of the sphere and those reflected internally from the two water-ice interfaces and the back surface of the sphere.

Finally, some qualitative comparisons are made between the theoretical results and the experimental results of Atlas et al., (1960). Values of σ_b determined experimentally for ice spheres surrounded by a shell of liquid water are lower than σ_b for an all-water sphere in some instances. These results are explained on the basis of the interference phenomena mentioned above. It was also found experimentally by Atlas et al., (1960) that the back-scattering cross-sections fall off more slowly at 4.67 cm than at 3.3 cm. This result is in good agreement with the theoretical results for 3.21 and 4.67 cm radiation for the particular size spheres involved.

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Figure 1: Normalized back-scattering cross-sections of water and ice spheres as a function of α for $\lambda = 3.21$ cm.

Figure 2: Model of sphere of radius b composed of an ice sphere of radius a surrounded by a concentric shell of water of thickness $b-a$.

Figure 3: Normalized back-scattering cross-sections for $\lambda = 3.21$ cm of spheres of various radii, b , as a function of the thickness of the water shell surrounding the inner ice sphere. (Curve for $b = 2.5$ cm is dashed for better clarity.)

Figure 4: Normalized back-scattering cross-sections for $\lambda = 4.67$ cm of spheres of various radii, b , as a function of the thickness of the water shell surrounding the inner ice sphere.

Figure 5: Same as Figure 4 but for $\lambda = 10.0 \text{ cm}$.

Figure 6: Experimental measurements of the normalized back-scattering cross-sections for $\lambda = 3.3$ and 4.67 cm as a function of D/λ for melting ice spheres. Note that the all-water curve shown in this figure is for $\lambda = 16.23$ cm. The applicable curves for water for $\lambda = 3.21$ and 4.67 cm differ primarily from that shown above in the locations of the maxima and minima. The range of magnitudes of σ_b however, do not differ significantly. (After Atlas et al., 1960.)

Figure 7a: Normalized back-scattering cross-sections for $\lambda = 3.21 \text{ cm}$ of all-water spheres and ice spheres surrounded by 0.01 cm of liquid water, as a function of sphere radius.

Figure 7b: Same as Figure 7a but for $\lambda = 4.67 \text{ cm}$.

Figure 8: Normalized back-scattering cross-sections of spheres of radii 3.90 cm, 3.33 cm, 2.95 cm, and 3.63 cm, composed of ice surrounded by a shell of water, as a function of water thickness.

Figure 9: Experimentally determined echo power for $\lambda = 3.21$ and 4.67 cm as a function of time for melting ice spheres of initial radii the same as in Figure 8. (After Atlas et al., 1960.)

Figure 10: Normalized back-scattering cross-sections for $\lambda = 3.21$ and 4.67 cm of a melting ice sphere of initial radius of 3.90 cm, as a function of the thickness of the melted layer. The sphere radius is assumed to remain constant until the melted layer reaches a shell thickness of 0.01 cm. Thereafter, the thickness of the melted layer is assumed to remain constant at 0.01 cm, the additional melt being shed, thereby decreasing the sphere radius.