Geometry's Fundamental Role in the Stability of Stochastic Differential Equations

by

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Signed: David P. Herzog

Dedication

To Charles, Phyllis, and Brenda.

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Abstract

We study dynamical systems in the complex plane under the effect of constant noise. We show for a wide class of polynomial equations that the ergodic property is valid in the associated stochastic perturbation if and only if the noise added is in the direction transversal to all unstable trajectories of the deterministic system. This has the interpretation that noise in the "right" direction prevents the process from being unstable: a fundamental, but not well-understood, geometric principle which seems to underlie many other similar equations. In view of [Has80, JK85, Jur97, MT93b, RB06, SV72], the result is proven by using Lyapunov functions and geometric control theory.

Chapter 1
Introduction and History

1.1 Introduction

The main purpose of this dissertation is to study dynamical systems under the effect of *noise*. More precisely, we investigate families of stochastic differential equations (SDEs) that are slight perturbations of deterministic differential equations. For fixed $n \geq 2$, the equation

$$\frac{dz(t)}{dt} = (z(t))^n; \; z(0) = z \in \mathbb{C} \tag{1.1}$$

is the primary focus. In particular, we find the maximal class of parameters $(\kappa_1, \kappa_2) \in \mathbb{C}^2$ such that the associated SDE:

$$dz(t) = (z(t))^n \, dt + \kappa_1 \, dW^{(1)}(t) + \kappa_2 \, dW^{(2)}(t) \tag{1.2}$$

has the *ergodic property*. In other words, we find all $(\kappa_1, \kappa_2) \in \mathbb{C}^2$ such that

1. For all initial conditions $z \in \mathbb{C}$, solutions of (1.2) exist for all finite times $t \geq 0$.
2. There exists a unique steady-state distribution μ to which the dynamics converges in the long-time regardless of the initial condition.

It is important to point out that $W^{(1)}(t)$ and $W^{(2)}(t)$ are indeed independent standard REAL Wiener processes defined on a probability space $(\Omega, \mathcal{F}, P)$. The infinitesimals $\kappa_1 \, dW^{(1)}(t)$ and $\kappa_2 \, dW^{(2)}(t)$ thus represent independent "kicks" in the directions of κ_1 and κ_2, respectively. The reason we allow noise in this form is that it will permit us to obtain and state the full results in terms of the geometry of the deterministic system (1.1). Specifically, one has the phase portrait (see Figure 1.1) of (1.1).

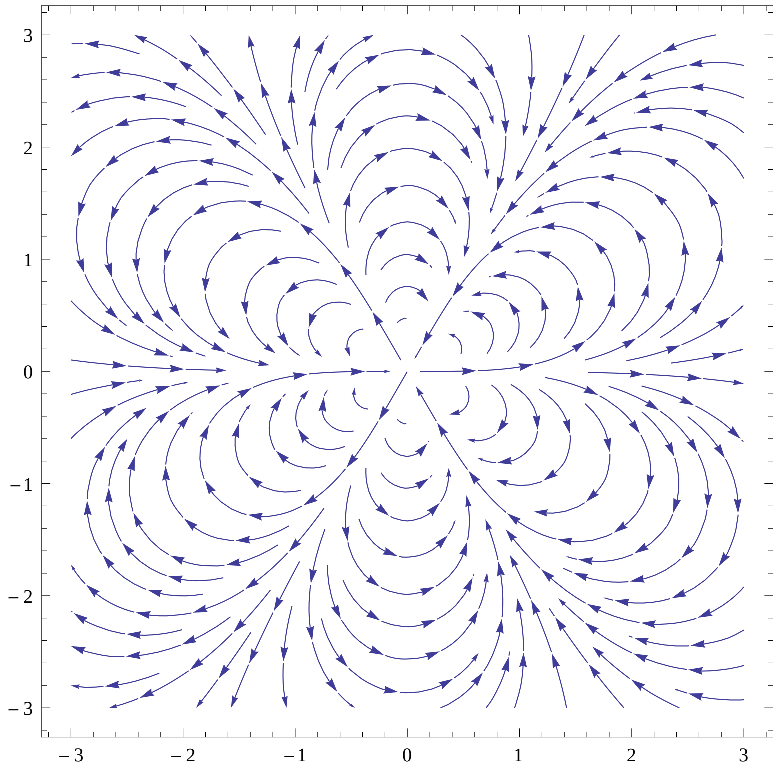


FIGURE 1.1. Phase portrait for the ODE (1.1) with $n = 4$. The only solutions that are unstable in time begin in $D_3 = \{z^3 > 0\}$. The rest approach the equilibrium point $z = 0$ as $t \to \infty$. For general $n \geq 2$, a similar picture is valid except that the unstable solutions begin in $D_{n-1} = \{z^{n-1} > 0\}$.

From this, it is intuitively clear that to obtain the ergodic property, we must at least require noise in the direction transversal to the rays

$$R_{n-1}(k) = \left\{\arg(z) = \frac{2\pi k}{n-1}\right\},$$

for all $k \in \mathbb{Z}$. If, for example, $\kappa_2 = 0$ and κ_1 is such that $\kappa_1^{n-1} \neq 0 \in \mathbb{R}$: for some $k \in \mathbb{Z}$, solutions that begin in $R_{n-1}(k)$ cannot leave $R_{n-1}(k)$. Thus if g is a primitive $(n-1)$st root of unity, for some $j \in \mathbb{Z}$ the process $x(t) := g^j z(t) > 0$ evolves according to the real-valued SDE:

$$dx(t) = (x(t))^n \, dt + g^j \kappa_1 \, dW^{(1)}(t), \tag{1.3}$$

which, by way of Feller's test [Dur96], is seen to have a positive probability of reaching infinity in finite time. Using this, we hence have a candidate for the permissible class of $(\kappa_1, \kappa_2) \in \mathbb{C}^2$:

Definition 1.4. We say the pair $(\kappa_1, \kappa_2) \in \mathbb{C}^2$ is **transversal** to D_{n-1} if either κ_1 and κ_2 are linearly independent over $\mathbb{R}$ or the set $\{\kappa_1^{n-1}, \kappa_2^{n-1}\}$ contains a non-real number.

It seems plausible that within the class of parameters transversal to D_{n-1}, equation (1.2) should have the ergodic property. After all, if a solution starts in the set D_{n-1} such noise guarantees the process must exit. In view of the trajectory plot (Figure 1.1), the stable dynamics should then take over. We are; however, reminded the effect noise can have on a well-behaved system. For example, it is shown in [Sch93] that there are asymptotically stable systems in $\mathbb{R}^2$ such that when any amount of constant noise is added, solutions of the stochastic perturbation starting anywhere reach infinity in finite time almost surely. Thus the noise that initially helps the process $z(t)$ out of D_{n-1} could in principle guide it back to D_{n-1}, or find an alternate route to infinity. A partial argument in this work is that the example given in [Sch93] is an exception, as the dynamics is tailored to specification. Outside the realm of such examples, we suggest there are no surprises. In particular, we prove:

Theorem 1.5. *For all $n \geq 2$, equation* (1.2) *has the ergodic property if and only if $(\kappa_1, \kappa_2) \in \mathbb{C}^2$ is transversal to D_{n-1}.*

This theorem serves also as an illustration of the difference between the stability of SDEs in one and higher dimensions. As mentioned earlier, the real-valued counterpart equation (1.3) has solutions which reach infinity in finite time. This is primarily because noise cannot moderate the instability by "pushing" the process off of the real axis and onto a stable region. It therefore seems that the dimension of the instability as a sub-manifold in the ambient space plays a fundamental role. This is exemplified in

[BHW11] where the stochastic dynamics has a critical pair of parameters $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ such that if $\alpha_1 < \alpha_2$, the ergodic property holds and if $\alpha_2 < \alpha_1$, there are solutions which reach infinity in finite time. If $\alpha_1 < \alpha_2$, the deterministic system has a single isolated unstable trajectory. When, however, $\alpha_2 < \alpha_1$ the unperturbed dynamics yields an entire open sub-manifold of unstable initial conditions of the state space.

Although there appears to be a fundamental geometric principle underlying the stability of stochastic differential equations, we are far from a general understanding of this. For instance, to show Theorem 1.5 for the innocuous family of equations (1.2), many careful so-called Lyapunov estimates like those performed in [GHW10, Sch93] are required. Additionally, for certain values of $(\kappa_1, \kappa_2) \in \mathbb{C}^2$, deep theorems, e.g. Hörmander's theorem [Hör67] and the support theorem [SV72], are employed. This is not to say that general results cannot be proven; rather, it is reminder that it is easy to go beyond the scope of existing theory, outside of which there is little guidance.

To effectively study SDEs with locally-Lipschitz coefficients like the system (1.2), the most difficult issue to resolve is usually that of global existence. Unfortunately, there is no known general theorem that can be immediately applied in this setting to conclude this. There are guiding principles, however. See, for example, the classical treatment in [Has80], or the more general prescription in the series of works [MT92, MT93a, MT93b, MT09]. All operate under the assumption that there exists a certain test function, called a Lyapunov function, which helps prove existence. Consequently, we must exhibit such a function, a task easier said than done. With the system (1.2) in mind, here we propose an algorithmic procedure to produce a Lyapunov function for an SDE. We do not claim this is a general result; however, these methods have been useful in many instances where existence is non-trivial [BHW11, GHW10, Sch93]. An additional benefit of this procedure is that we are easily able to infer the existence of a steady state distribution μ.

After moderating the above, we must settle the question of uniqueness of μ. If κ_1 and κ_2 span the entire complex plane over $\mathbb{R}$, uniqueness can be immediately concluded using classical methods from partial differential equations [Has80]. This follows intuitively since the process defined by equation (1.2) is Markovian and is, by the non-degeneracy of the pair (κ_1, κ_2), supported everywhere in $\mathbb{C}$. Thus, regardless of where the process begins, it "mixes" well-enough so that in the long-time the dynamics is unique. If, on the other hand, $\kappa_1 = c\,\kappa_2$ for some $c \in \mathbb{R}$, uniqueness of μ no longer follows by the same methods. Using similar ideas, one can establish uniqueness by proving smoothness of the transition measures $P(z, t, \,\cdot\,)$ of $z(t)$ via Hörmander's theorem [Hör67] and by showing that processes originating from distinct initial states still "mix" with perhaps less strength than before. The latter is done by using methods from control theory via the support theorem [SV72].

Before we proceed onto the main body of this work, we first give a brief history as to how this project originated.

1.2 History

The primary motivation of this work is to use experiences with systems such as (1.2) to not only generate new mathematical understanding, but also apply learned techniques to equations in order to gain insight into other scientific disciplines. With this motivation in place, it is thus natural to begin with a specific application in mind, as equations born here are not only interesting but also appear to exhibit a wide range of behaviors. It is not surprising then that the family (1.2) originated in a similar fashion, which we now describe.

To this day, fully understanding turbulence remains a challenge. One way to attack this problem is to study how the fluid transports small particles. For example, if the particle acts as a simple tracer of the flow, we have the following relation:

$$\dot{y}(t) = v(t, y(t)), \tag{1.6}$$

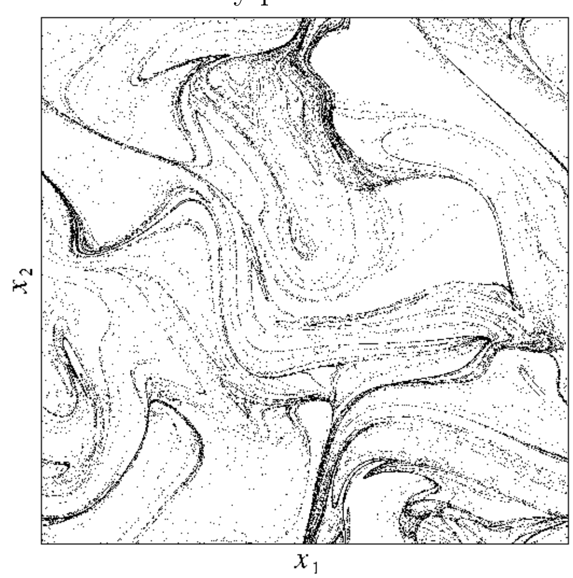
where $y(t) \in \mathbb{R}^d$ is the particle's position at time $t \geq 0$ and v is fluid velocity field. If, however, $y(t) \in \mathbb{R}^d$ has inertia it is subject to frictional forces. In particular, $y(t)$

now evolves according to the Newtonian equation:

$$\ddot{y}(t) = -\frac{1}{\tau}\left(\dot{y}(t) - v(t, y(t))\right), \tag{1.7}$$

where the constant $\tau > 0$ is called the **Stokes time**. One interest in equation (1.7) as opposed to (1.6) is the presence of spatial inhomogeneities in the distribution of particles when the mass of the particle is much larger than that of the carrier fluid. In particular, we have the image (Figure 1.2) courtesy of J. Bec [Bec05]. Thus the small but heavy particles, called **inertial particles**, separate or cluster in an irregular manner in the flow over time.

FIGURE 1.2. 10^5 small heavy particles for Stokes time $\tau = 10^{-2}$



We can capture this phenomenon by considering the dynamics of the particle separation $\rho(t) = \delta\, y(t)$ which obeys the linearized equation:

$$\ddot{\rho}(t) = -\frac{1}{\tau}\left[\dot{\rho}(t) - (\rho \cdot \nabla) v(t, y(t))\right], \tag{1.8}$$

and we may assume to good approximation [BCH07]:

$$\nabla_j v^i(t, y(t)) = S^i_j(t),$$

where $S(t)$ is a matrix-valued white noise with covariance structure:

$$E\left[S^i_j(s)S^k_l(t)\right] = D^{ik}_{jl}\,\delta(s-t),$$

where the constants D^{ik}_{jl} are chosen such that the covariance is isotropic and non-negative. Specifically, we set

$$D^{ik}_{jl} = A\delta^i_k\delta^j_l + B(\delta^i_j\delta^k_l + \delta^i_l\delta^k_j)$$

where $A, B \in \mathbb{R}$ are such that $A \geq |B|$ and $A+(d+1)B \geq 0$. Under these assumptions, equation (1.8) becomes the following linear SDE:

$$\ddot{\rho}(t) = -\frac{1}{\tau}\left[\dot{\rho}(t) - S(t)\rho(t)\right], \tag{1.9}$$

which can be interpreted using invariably the Itô or Stratonovich conventions. Writing this in the first-order form:

$$\begin{aligned} \dot{\rho}(t) &= \frac{1}{\tau}\chi(t) \\ \dot{\chi}(t) &= -\frac{1}{\tau}\chi(t) + S(t)\rho(t), \end{aligned}$$

we study the joint process $p(t) = (\rho(t), \chi(t))$.

To understand how particles cluster or separate over time, it is convenient to introduce the quantity (assuming it exists):

$$\lambda = \lim_{T\to\infty} \frac{1}{T} E_{p(0)}\left[\ln(|p(T)|)\right],$$

which is called the **(top) Lyapunov exponent** of the process $p(t)$. In dimensions $d \geq 2$, certain ergodic properties of $p(t)$ are assumed in [BCH07, WM03] to validate formulas for λ which are used to extract information on particle clustering. In [GHW10], we prove these formulas using similar techniques to those described in the previous section. Indispensable components of these arguments are the substitutions:

$$x(t) = \frac{\rho(t)\cdot\chi(t)}{|\rho(t)|^2}, \qquad y(t) = \frac{\rho^{(1)}(t)\chi^{(2)}(t) - \rho^{(2)}(t)\chi^{(1)}(t)}{|\rho(t)|^2}$$

in dimension $d = 2$ and

$$x(t) = \frac{\rho(t) \cdot \chi(t)}{|\rho(t)|^2}, \qquad y(t) = \frac{\sqrt{|\rho(t)|^2 |\chi(t)|^2 - (\rho(t) \cdot \chi(t))^2}}{|\rho(t)|^2}$$

in dimensions $d \geq 3$. Using the complex variable $z(t) = x(t) + iy(t)$, the process $z(t)$ evolves in $\mathbb{C}$ for $d = 2$ and in $\mathbb{H}_+ = \{\text{Im}(z) > 0\}$ for $d \geq 3$ according to the equation:

$$\begin{aligned} dz(t) \quad &= \quad -\frac{1}{\tau}\left((z(t))^2 + z(t) - i\frac{\tau A(d-2)}{2\,\text{Im}(z)}\right) dt \\ &\quad + \sqrt{A + 2B}\, dW^{(1)}(t) + i\sqrt{A}\, dW^{(2)}(t), \end{aligned} \tag{1.10}$$

where $W^{(1)}(t)$ and $W^{(2)}(t)$ are independent standard Wiener processes. When $d = 2$, the term $i\frac{\tau A(d-2)}{2\,\text{Im}(z)}$ is absent from the expression above.

Using these equations, as done in [GHW10] one can effectively prove the assumed ergodic properties in [Bec05, BCH07] by doing so for the slightly modified version of (1.10)

$$dz(t) = (z(t))^2\, dt + \epsilon_1\, dW^{(1)}(t) + i\epsilon_2\, dW^{(2)}(t), \tag{1.11}$$

where $\epsilon_1 \geq 0$ and $\epsilon_2 > 0$ are positive constants. Note that the above relation certainly falls within the class of equations (1.2) with noise (κ_1, κ_2) transversal to $D_1 = \mathbb{R}_{>0}$.

Assuming one has not seen the phase portrait of the associated ODE for $n = 2$, the fact that $z(t)$ satisfies the ergodic property is rather surprising as it has some comparable features of its real-valued, highly unstable, counterpart equation:

$$dx(t) = (x(t))^2\, dt + \epsilon\, dW(t).$$

Both have coefficients which are polynomial of degree two, hence grow at infinity relatively fast, and both are one-dimensional equations in some sense. As emphasized before, the difference is really in the geometry of the phase portrait of the non-random dynamical system. This is precisely why one conjectures the same stability to hold for the family (1.2) within the class of noise $(\kappa_1, \kappa_2) \in \mathbb{C}^2$ transversal to D_{n-1}. In this work, we provide a short testament to this.

The layout of the dissertation is as follows. In Chapter 2, we highlight methods that are used to infer or disprove the ergodic property for time-homogeneous stochastic differential equations in $\mathbb{R}^d$. It is possible to operate more generally under the assumption that the state space is a manifold, but we prefer to use $\mathbb{R}^d$ since such generality is not necessary to conclude the main results for the system (1.2). Sections 2.1-2.3 provide standard techniques, while Sections 2.4-2.5 illustrate some methods for proving uniqueness of invariant probability measures which are more esoteric. Section 2.6 provides sufficient conditions under which one can quantify a rate of convergence to the equilibrium μ. In Chapter 3, we prove Theorem 1.5 and note moreover that if (κ_1, κ_2) is transversal to D_{n-1}, by the results of Section 2.6, we may also prove that the transition measures approach the limiting distribution μ exponentially fast in the total variation norm.

Chapter 2
Stability of Stochastic Differential Equations

2.1 Introduction

In this dissertation, we determine if the ergodic property is valid for possibly degenerate stochastic differential equations. The goal of this chapter is to thus familiarize the reader with some techniques that can be used to prove or disprove such stability, both in this work and in general. We do not promise what follows to be a comprehensive study; rather, we hope to illustrate methods that were useful in our efforts.

Since the results of this chapter can be applied to a wider family of equations than (1.2), we consider the more general time-homogeneous stochastic system:

$$x(t) - x(0) = \int_0^t b(x(s))\,ds + \int_0^t \sigma(x(s))\,dW(s), \tag{2.1}$$

which will be written equivalently using differentials as:

$$dx(t) = b(x(t))\,dt + \sigma(x(t))\,dW(t).$$

Denoting the set of $d \times d$ matrices with real entries by $M_d(\mathbb{R})$, unless stated otherwise we make the following assumptions on equation (2.1):

(A1) $b : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \to M_d(\mathbb{R})$ are smooth functions on their respective spaces.

(A2) $W(t) = (W^{(1)}(t), W^{(2)}(t), \ldots, W^{(d)}(t))$ is a d-dimensional standard Wiener process defined on a probability space $(\Omega, \mathcal{F}, P)$ to which we attach the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ where for $t \geq 0$, $\mathcal{F}_t$ is the sigma algebra generated by $(W(t) : s \leq t)$.

(A3) The stochastic integral $\int \sigma\, dW$ is defined in the Itô sense.

(A4) The initial condition $x(0) = x \in \mathbb{R}^d$ is deterministic.

By the dimensions of b, σ, and $W(t)$, any solution $x(t)$ of (2.1) is a random process that evolves in $\mathbb{R}^d$. The first problem we will address is that of existence and uniqueness of solutions of (2.1). To this end, in the following section we show how to estimate the time in which the process $x(t)$ leaves every bounded domain in $\mathbb{R}^d$.

Before we proceed further, let us first fix some notation. We use $B_r(x) \subset \mathbb{R}^d$ to denote the open ball centered at $x \in \mathbb{R}^d$ of radius $r > 0$. For $x \in \mathbb{R}^d$, let P_x be the probability law of the process $x(t)$ determined by (2.1) with $x(0) = x$ and let E_x be its corresponding expectation. We use $\mathcal{B}([0,\infty))$ and $\mathcal{B}(\mathbb{R}^d)$ to denote the Borel sigma-algebra of subsets on $[0,\infty)$ and $\mathbb{R}^d$, respectively. For $A \in \mathcal{B}([0,\infty))$, $U \in \mathcal{B}(\mathbb{R}^d)$, and $k \in \mathbb{N}$, let $C_1^k(A \times U)$ be the set of functions $\Phi : A \times U \to \mathbb{R}$ which are once continuously differentiable on A and k times continuously differentiable on U, let $C_0^k(U)$ denote the set of functions $\Phi : U \to \mathbb{R}$ which are k times continuously differentiable on U and compactly supported in U, and let $C^k(U)$ be the set functions $\Phi : U \to \mathbb{R}$ which are k times continuously differentiable on U.

2.2 Absence or Presence of Explosions

2.2.1 Absence of Explosions

In order to prove the ergodic property holds, one must first show that solutions exist regardless of the initial point $x \in \mathbb{R}^d$, and are unique in some sense, for all finite times $t \geq 0$. To this effect, there is a general existence and uniqueness theorem for stochastic differential equations which we state without proof.

Theorem 2.2 (Existence and Uniqueness). *Let b and σ satisfy the following additional condition:*

(B1) $$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq C|x - y|; \; x, y \in \mathbb{R}^d$$

for some positive constant C. Then for every $x(0) = x \in \mathbb{R}^d$, there exists an almost surely continuous solution $x(t)$ of equation (2.1) *which is defined for all finite* *times $t \geq 0$. Morover, $x(t)$ is adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and unique up to* **equivalence**.[1]

Proof. See [Øks03]. □

Although interesting, as stated this theorem certainly cannot be applied in the general setting (2.1) since the bound **(B1)** is not necessarily satisfied. This is the case for our system (1.2) and many other systems where one expects global stability. For a simple example, consider the real-valued stochastic differential equation:

$$dx(t) = -(x(t))^3 + dW(t). \tag{2.3}$$

It is clear that for every initial condition $x(0) = x \neq 0 \in \mathbb{R}$ the dynamics, on average, is directed inward towards the origin, yet $b(x) = -x^3$ is not globally Lipschitz. In his influential work [Has80], Has′minskiĭ stresses that "It is therefore of paramount importance to find other, broader conditions for the existence and uniqueness of the solution of equation [(2.1)]" [Has80]. In the very same text, he provided such conditions that were phrased in terms of test functions and these methods are still widely used to gauge the stability of diffusion processes such as $x(t)$. See, for example, [EM01, GHW10, MSH02, MT93b, Sch93]. We now outline Has′minskiĭ's approach since it serves us well throughout this work.

Under assumptions **(A1)-(A4)**, one can always define continuous solutions $x(t)$ of equation (2.1) until the (random) time in which the process leaves every bounded domain in $\mathbb{R}^d$ [Has80]. To see this formally, for $n \in \mathbb{N}$ choose smooth functions $b_{(n)}$ and $\sigma_{(n)}$ on $\mathbb{R}^d$ such that on $B_n(0)$:

$$\begin{aligned} b_{(n)}(x) &= b(x) \\ \sigma_{(n)}(x) &= \sigma(x), \end{aligned}$$

[1] Two solutions $x_1(t)$ and $x_2(t)$ of equation (2.1) are *equivalent* if $P\{x_1(t) = x_2(t) \text{ for all } t \geq 0\} = 1$.

and $b_{(n)}$ and $\sigma_{(n)}$ satisfy **(B1)**. Thus by Theorem 2.2, for each fixed $n \in \mathbb{N}$, for all initial conditions $x \in \mathbb{R}^d$ there exists an almost surely continuous process $x_{(n)}(t)$ such that:

$$dx_{(n)}(t) = b_{(n)}(x_{(n)}(t))\, dt + \sigma_{(n)}(x_{(n)}(t))\, dW(t),$$

for all $t \geq 0$. Moreover, $x_{(n)}(t)$ is unique up to equivalence and adapted to the Wiener filtration $\mathcal{F}_t$. For $m, n \in \mathbb{N}$, we define stopping times

$$\xi_n^{(m)} = \inf_{t>0} \left\{ x_{(m)}(t) \in B_n(0)^c \right\},$$

and one can show (see [Dyn65]) that for $m, m' \geq n$, $x \in \mathbb{R}^d$:

$$\xi_n^{(m)} = \xi_n^{(m')} \qquad P_x \text{ - a.s.}$$

Thus for $n \in \mathbb{N}$ let $\xi_n = \xi_n^{(n)}$. Moreover it follows that for $m, m' \geq n$ and $x \in \mathbb{R}^d$:

$$P_x \left\{ \sup_{t \geq 0} |x_{(m)}(t \wedge \xi_n) - x_{(m')}(t \wedge \xi_n)| = 0 \right\} = 1.$$

Thus we may define a process $x(t)$ by:

$$x(\tau) = x_{(n)}(\tau), \ \text{ whenever } \tau < \xi_n$$

and we note that for all $n \in \mathbb{N}$ the equation:

$$dx(t \wedge \xi_n) = b(x(t \wedge \xi_n))\, dt + \sigma(x(t \wedge \xi_n))\, dW(t)$$

holds. Let ξ be the increasing limit of ξ_n as $n \to \infty$. If we can prove that

$$P_x \{\xi < \infty\} = 0$$

for all $x \in \mathbb{R}^d$, then for all initial conditions $x \in \mathbb{R}^d$ we have a unique solution $x(t)$ of equation (2.1) which is defined and continuous for all finite times $t \geq 0$. Moreover, $x(t)$ is adapted to $\{\mathcal{F}_t\}_{t \geq 0}$. To this end, our goal is to estimate the time ξ, called the **explosion time** of the process $x(t)$.

To prove ergodicity, we hope that for all $x \in \mathbb{R}^d$:

$$P_x\{\xi < \infty\} = 0;$$

in which case, we call the process $x(t)$ **non-explosive**. To prove $x(t)$ is non-explosive, we require the following lemma due to Dynkin.

Lemma 2.4 (Dynkin's Formula)**.** *Let $\Phi \in C_1^2([0,\infty) \times \mathbb{R}^d)$ and $\xi_n(t) = t \wedge \xi_n$. Then for $t \geq 0$ and $x \in \mathbb{R}^d$ we have:*

$$E_x\left[\Phi(\xi_n(t), x(\xi_n(t)))\right] - \Phi(0,x) = E_x\left[\int_0^{\xi_n(t)} L\Phi(u, x(u))\, du\right] \tag{2.5}$$

where

$$L\Phi(t,x) = \frac{\partial \Phi(t,x)}{\partial t} + \sum_{i=1}^d b^{(i)}(x)\frac{\partial \Phi(t,x)}{\partial x^{(i)}} + \frac{1}{2}\sum_{i,j=1}^d (\sigma\sigma^*)^{(ij)}(x)\frac{\partial^2 \Phi(t,x)}{\partial x^{(i)}\partial x^{(j)}}. \tag{2.6}$$

Proof. Using the discussion above, we may apply Itô's lemma to the process $\Phi(t, x(t))$ to obtain

$$\Phi(\xi_n(t), x(\xi_n(t))) - \Phi(0, x(0)) = \int_0^{\xi_n(t)} L\Phi(u, x(u))\, du + \text{ bounded martingale.} \tag{2.7}$$

Since the martingale starts at 0, after taking expectations E_x of both sides of equation (2.7) we have the result. □

The above relation (2.5) is perhaps the most beautiful in all of stochastic differential equations. On the left-hand side, we have a random process $x(t)$. On the right-hand side, we have a partial differential operator L. Such a relation provides just a hint of the intimate connection between the probabilistic theory of stochastic differential equations and the classical theory of partial differential equations. As we shall see, both points of view provide equally-valuable insights into the other.

Because the operator L plays a fundamental role in this work, we adopt the common nomenclature and call it the **generator** of the process $x(t)$. The choice of this terminology will become more transparent in the next section when we discuss Markov processes.

We now use Lemma 2.4 as a means by which to verify existence and uniqueness in (2.1) when **(B1)** fails. The intuition behind what follows is to insert a suitable function Φ into equation (2.5). For simplicity, assume that $\Phi := \Phi(x) \in C^2(\mathbb{R}^d)$ is only a function of the spatial variables. To utilize relation (2.5), Φ should approach infinity as $|x| \to \infty$, so, without loss of generality, we can assume $\Phi \geq 0$. This often called "norm-like" property is to assure that Φ hits infinity when the process $x(t)$ does. If one can then control the right-hand side of (2.5) using properties of $L\Phi(x)$, non-explosivity of the process $x(t)$ should follow.

Theorem 2.8. *Let $\Phi \in C^2(\mathbb{R}^d)$ be a non-negative function and suppose*

$$\Phi(x) \to \infty \text{ as } |x| \to \infty,$$

and there exist positive constants C, D such that

$$L\Phi(x) \quad \leq \quad C\Phi(x) + D \quad \text{for all } x \in \mathbb{R}^d.$$

Then the process $x(t)$ is non-explosive, i.e., for all $x(0) = x \in \mathbb{R}^d$, $P_x\{\xi < \infty\} = 0$.

Proof. Let $x(0) = x \in \mathbb{R}^d$ and define a function $\Psi(t,x) = e^{-Ct}(\Phi(x) + D/C)$. Choose $N \in \mathbb{N}$ sufficiently large so that $\Phi(y) \geq 1$ for all $|y| \geq N$. By Lemma 2.4, we have for all $n \geq N$:

$$\begin{aligned} E_x\left[\Psi(\xi_n(t)), x(\xi_n(t)))\right] - \Psi(0,x) &= E_x\left[\int_0^{\xi_n(t)} L\Psi(u, x(u))\, du\right] \\ &= E_x\left[\int_0^{\xi_n(t)} -C\Psi(u,x(u)) + e^{-Ct}L\Phi(x(u))\, du\right] \\ &\leq E_x\left[\int_0^{\xi_n(t)} -C\Psi(u,x(u)) + C\Psi(u,x(u))\, du\right] \\ &= 0. \end{aligned}$$

Thus

$$E_x\left[\Psi(\xi_n(t)), x(\xi_n(t)))\right] \quad \leq \quad \Psi(0, x). \tag{2.9}$$

Estimating the left-hand side of (2.9), we see that

$$\begin{aligned} E_x\left[\Psi(\xi_n(t)), x(\xi_n(t)))\right] &\geq E_x\left[\mathbf{1}_{\{\xi_n \leq t\}}\Psi(\xi_n(t)), x(\xi_n(t))\right] \\ &\geq e^{-Ct}\inf_{|y|=n}\Phi(y)P_x\left\{\xi_n \leq t\right\}. \end{aligned}$$

Thus we have for $n \geq N$

$$P_x\left\{\xi_n \leq t\right\} \quad \leq \quad \frac{e^{Ct}\Psi(0, x)}{\inf_{|y|=n}\Phi(y)}.$$

Letting $n \to \infty$, we have that $P_x\{\xi \leq t\} = 0$ for all $t \geq 0$. Hence $P_x\{\xi < \infty\} = 0$. Since $x(0) = x \in \mathbb{R}^d$ was arbitrary, this finishes the proof. □

Let us return to equation (2.3). Recall that the coefficients do not satisfy the assumptions of Theorem 2.2, but it is intuitively clear that the resulting process is non-explosive. Let us verify this rigorously using the previous theorem. Define $\Phi(x) = x^2$ and note that $L = \partial_t - x^3\partial_x + \frac{1}{2}\partial_{xx}$. Thus $\Phi(x) \to \infty$ as $|x| \to \infty$ and

$$L\Phi(x) \quad = \quad -2x^4 + 1 \tag{2.10}$$

$$\leq \quad x^2 + 1 \tag{2.11}$$

$$= \quad 1 \cdot \Phi(x) + 1.$$

Hence, by Theorem 2.8, the process defined by (2.3) is non-explosive.

This theorem, although clearly useful, does not tell the whole story. First, no where does it instruct one on how to obtain the function Φ. It was very simple to find one for the system (2.3); but as the dynamics of the general equation (2.1) becomes increasingly complex, so does discovering Φ. Second: from equation (2.10) to (2.11) we have thrown away valuable information about the solution. In taking a careful look at relation (2.5) with $\Phi(x) = x^2$ and the bound (2.10), we see that, in essence, the process $x(t)$ should return quickly to a large ball about the origin. As we shall see, this will be of central importance in proving the existence of and convergence to a steady state since any solution $x(t)$ of equation (2.1) is a *strong Markov* process.

2.2.2 Presence of Explosions

In the previous subsection, we provided a means by which to verify the process $x(t)$ is non-explosive. Here, we do the contrary, i.e., we find sufficient conditions to prove:

$$P_x\{\xi < \infty\} > 0 \text{ for some } x \in \mathbb{R}^d. \tag{2.12}$$

If relation (2.12) is satisfied we say the process $x(t)$ is **explosive**. In a similar manner to before, one can prove explosivity of $x(t)$ using test functions.

Theorem 2.13. *Suppose that $\Phi \in C^2(\mathbb{R}^d)$ is a bounded non-negative function such that*

$$L\Phi(x) \geq C\Phi(x) \text{ for all } x \in \mathbb{R}^d,$$

for some $C > 0$. Then for all $x_0 \in \mathbb{R}^d$ such that $\Phi(x_0) > 0$, we have:

$$P_{x_0}\{\xi < \infty\} > 0. \tag{2.14}$$

Remark 2.15. We can actually be more specific than the estimate (2.14) and prove for all $\epsilon > 0$

$$P\left\{\xi^{(x_0)} < \frac{1}{C}\ln\left(\frac{\sup_{y\in\mathbb{R}^d}\Phi(y)}{\Phi(x_0)}\right) + \epsilon\right\} > 0,$$

whenever $\Phi(x_0) > 0$. Here, we recall $\xi^{(x_0)}$ is the explosion time for the process $x(t)$ with $x(0) = x_0$.

Proof. Let $x_0 \in \mathbb{R}^d$ be such that $\Phi(x_0) > 0$. Upon setting $\Psi(t, x) = e^{-Ct}\Phi(x)$, Dynkin's formula implies:

$$E_{x_0}\left[\Psi(\xi_n(t), x(\xi_n(t)))\right] \geq \Phi(x_0).$$

Estimating the left-hand side above, we have:

$$E_{x_0}\left[\Psi(\xi_n(t), x(\xi_n(t)))\right] \quad \leq \quad \sup_{y\in\mathbb{R}^d} \Phi(y) E_{x_0}\left[e^{-C\xi_n(t)}\right],$$

hence:

$$E\left[e^{-C\xi_n^{(x_0)}(t)}\right] \geq \frac{\Phi(x_0)}{\sup_{y\in\mathbb{R}^d}\Phi(y)}.$$

Since the bound above holds for all n, by the dominated convergence theorem we have

$$E\left[e^{-C\xi^{(x_0)}(t)}\right] \geq \frac{\Phi(x_0)}{\sup_{y\in\mathbb{R}^d}\Phi(y)}. \tag{2.16}$$

If for some $\epsilon > 0$,

$$P\left\{\xi^{(x_0)} < \frac{1}{C}\ln\left(\frac{\sup_{y\in\mathbb{R}^d}\Phi(y)}{\Phi(x_0)}\right) + \epsilon\right\} = 0,$$

by splitting the left-hand side of the bound (2.16) into:

$$E\left[e^{-C\xi^{(x_0)}(t)}\right] = E\left[\mathbf{1}_{\{t\leq\xi^{x_0}\}}e^{-Ct}\right] + E\left[\mathbf{1}_{\left\{t>\xi^{(x_0)}\right\}}e^{-C\xi^{x_0}}\right]$$

and setting

$$t = \frac{1}{C}\ln\left(\frac{\sup_{y\in\mathbb{R}^d}\Phi(y)}{\Phi(x_0)}\right) + \epsilon,$$

we violate the bound (2.16). Note that this finishes the proof. □

This is a very interesting theorem, but it displays a similar weakness to Theorem 2.8. Indeed, there is no instruction manual one can follow to produce a test function $\Phi \in C^2(\mathbb{R}^d)$ which is non-negative and bounded such that

$$L\Phi(x) \geq C\Phi(x) \text{ for all } x \in \mathbb{R}^d,$$

for some $C > 0$. In [BHW11], we were able to exhibit such a function, but it requires both the correct intuition and challenging estimates.

Let us now consider equation (1.2) in the case when $(\kappa_1, \kappa_2) \in \mathbb{C}^2$ is not transversal to D_{n-1}. We recall this implies for some $k \in \mathbb{Z}$, solutions of (1.2) starting in $R_{n-1}(k)$, when rotated by an appropriate $(n-1)$st root of unity, evolve on $\mathbb{R}_{>0}$. Determining explosion in such a case is far simpler than constructing a test function like Φ above. This is because in $\mathbb{R}$ we have Feller's test which we now discuss.

For the remainder of this subsection, we assume $x(t)$ defined by equation (2.1) evolves on $\mathbb{R}_{>0}$ such that there exists $\epsilon > 0$ such that $\sigma^2(x) \geq \epsilon$ for all $x \geq 0$. We will illustrate how one proves:

$$P_x\{\xi < \infty\} > 0 \text{ for some } x > 0. \tag{2.17}$$

Instead of working directly with $x(t)$, we define a strictly increasing function

$$\phi(x) = \int_0^x \exp\left(\int_0^y -\frac{2b(\tilde{y})}{\sigma^2(\tilde{y})}\,d\tilde{y}\right)dy$$

and use the induced process $y(t)$ given by:

$$y(t) = \phi(x(t)).$$

The benefit of working with $y(t)$ is seen by noting:

$$L\phi(x) = b(x)\phi'(x) + \frac{\sigma^2(x)}{2}\phi''(x) = 0.$$

In particular by Itô's formula, the process $y(t)$ is a local martingale. For $c > 0$, letting:

$$\begin{aligned}\xi_{c+} &= \inf_{t>0}\{x(t) > c\}\\ \tilde{\xi}_{c+} &= \inf_{t>0}\{y(t) > c\},\end{aligned}$$

$$\begin{aligned}\xi_0 &= \inf_{t>0}\{x(t) = 0\}\\ \tilde{\xi}_0 &= \inf_{t>0}\{y(t) = 0\},\end{aligned}$$

and

$$\begin{aligned}\xi_+ &= \lim_{c\uparrow\infty}\xi_{c+}\\ \tilde{\xi}_+ &= \lim_{c\uparrow\infty}\tilde{\xi}_{c+},\end{aligned}$$

we have:

Theorem 2.18 (Feller's Test)**.** *If* $\psi(x) = 1/(\phi'(x)\sigma^2(x))$*:*

(a) $P_x\{\xi_{c+} < \xi_0\} > 0$ *for* $x \in (0, c)$ *if*

$$\int_0^c \psi(x)\,(\phi(c) - \phi(x))\,dx < \infty.$$

(b) $P_x\{\xi_+ < \xi_0\} > 0$ *for* $x \in (0, \infty)$ *if*

$$\int_0^\infty \psi(x)\,(\phi(\infty) - \phi(x))\,dx < \infty.$$

Consequently,

Corollary 2.19. $P_x\{\xi < \infty\} > 0$ *for* $x \in (0, \infty)$ *if*

$$\int_0^\infty \psi(x)\,(\phi(\infty) - \phi(x))\,dx < \infty. \tag{2.20}$$

Proof of Corollary 2.20. Note that this follows immediately from (b) since for $x \in (0, \infty)$ we have:

$$P_x\{\xi < \infty\} \geq P_x\{\xi_+ < \xi_0\} > 0.$$

□

Proof of Theorem 2.18. We shall only prove (a) as (b) follows similarly. To prove (a), let $c > 0$ and fix $x \in (0, c)$. We claim that it suffices to show that

$$f(x) = \int_0^{\phi^{-1}(x)} \psi(y)\,(x - \phi(y))\,dy$$

is a strictly increasing function on $(0, \phi(c))$ such that

$$g(t, y(t)) = e^{-t} f(y(t))$$

is a local martingale. To see this, suppose that $f(\phi(c)) < \infty$. Let $y_n \geq y = \phi(x)$ be such that $y_n \uparrow \phi(c)$ as $n \to \infty$. Define $\tau_n = \tilde{\xi}_0 \wedge \tilde{\xi}_{y_n+}$. We then have:

$$\begin{aligned} 0 < f(y) &= E_y\left[g(\tau_n, y(\tau_n)))\right] \\ &\leq f(\phi(c))E_y\left[\mathbf{1}_{\{\tilde{\xi}_{y_n+} < \tilde{\xi}_0\}} e^{-\tilde{\xi}_{y_n+}}\right] \downarrow f(\phi(c))E_y\left[\mathbf{1}_{\{\tilde{\xi}_{\phi(c)+} < \tilde{\xi}_0\}} e^{-\tilde{\xi}_{\phi(c)+}}\right], \end{aligned}$$

as $n \to \infty$ which clearly implies that $0 < P_y \left\{ \tilde{\xi}_{\phi(c)+} < \tilde{\xi}_0 \right\} = P_x \left\{ \xi_{c+} < \xi_0 \right\}$, as required. Thus we have left to show that $f(x)$ is strictly increasing on $(0, \phi(c))$ such that $g(t, y(t))$ is a local martingale. For this we refer the reader to pages 216-218 in [Dur96]. □

Hence to prove that the ergodic property is not satisfied when $(\kappa_1, \kappa_2) \in \mathbb{C}$ is not transversal to D_{n-1}, we will use part (b) of Feller's test to conclude the process obtained by rotating by an appropriate $(n-1)$st root of unity is explosive. The benefit of this method is that the issue of explosion reduces to proving the integral in (b) is finite, which is far simpler than constructing a test function on $\mathbb{R}^2$ like the one in Theorem 2.13.

2.3 Markov Processes and Invariant Measures

Perhaps one of the most important properties of solutions of stochastic differential equations is that they are **strongly Markovian**. That is; imprecisely, the future only depends on the past through the present instant in time, even when one considers bounded stopping times. For this section, we assume that the solution $x(t)$ of (2.1) is non-explosive and note the following two lemmata:

Lemma 2.21 (**Markov Property**). *Let $\Phi : \mathbb{R}^d \to \mathbb{R}$ be a bounded, Borel measurable function. We have for $s, t \geq 0$:*

$$E_x \left[\Phi(x(s+t)) \,|\, \mathcal{F}_s \right] = E_y \left[\Phi(x(t)) \right] |_{y=x(s)}.$$

Proof. See [Øks03]. □

Lemma 2.22 (**Strong Markov Property**). *Let $\Phi : \mathbb{R}^d \to \mathbb{R}$ be a bounded, Borel measurable function and υ be an almost surely bounded stopping time with respect to $\{\mathcal{F}_t\}_{t \geq 0}$. Then for $t \geq 0$*

$$E_x \left[\Phi(x(\upsilon+t)) \,|\, \mathcal{F}_\upsilon \right] = E_y \left[\Phi(x(t)) \right] |_{y=x(\upsilon)}$$

where $\mathcal{F}_v$ is the sigma algebra generated by $(W(s \wedge v), s \geq 0)$.

Proof. See [Øks03]. □

Thus, using the words of McKean, the process $x(t)$ "begins afresh" at bounded stopping times v [McK05].

There is a whole theory of Markov processes outside of the realm of stochastic differential equations. Although extremely important for this work, we will not delve too deeply into this well-established area; rather, we will view the solution $x(t)$ of equation (2.1) as a process that has the (strong) Markov property and use this information to study $x(t)$. At times during this dissertation, however, the reader will notice that one can operate purely under the assumption that the process $x(t)$ is Markovian.

Pedagogically, introducing the Markov property as stated above is probably not the best way to initially think of such processes. Often, it is much more informative to define **transition kernels**

$$P(x, t, A) := P_x \{x(t) \in A\}, \tag{2.23}$$

for $x \in \mathbb{R}^d$, $t \geq 0$, and $A \in \mathcal{B}(\mathbb{R}^d)$ where $\mathcal{B}(\mathbb{R}^d)$ denotes the Borel sigma algebra of subsets on $\mathbb{R}^d$. We note that, by the Markov property, we have the so-called Chapman-Kolmogorov equations:

$$P(x, s+t, A) = \int_{\mathbb{R}^d} P(x, s, dy) P(y, t, A), \tag{2.24}$$

for all $s, t \geq 0$, $A \in \mathcal{B}(\mathbb{R}^d)$. We will often use relation (2.24) and not Lemma 2.22.

The Chapman-Kolmogorov equations coupled with the use of transition kernels help connect probabilistic notions to functional analysis. To see this, we define for $t \geq 0$ operators P_t which act on bounded measurable functions $\Phi : \mathbb{R}^d \to \mathbb{R}$ and on finite Borel measures μ on $\mathbb{R}^d$ in the following way:

$$\begin{aligned} P_t\Phi(x) &= \int P(x,t,dy)\Phi(y) \\ \mu P_t(A) &= \int \mu(dy)P(y,t,A), \;\; A \in \mathcal{B}(\mathbb{R}^d). \end{aligned} \tag{2.25}$$

It occurs regularly that the above can be defined for a more general class of functions and/or measures; in which case, we interpret the expressions $P_t\Phi$ and μP_t in the same way. By (2.24), the family $\{P_t\}_{t\geq 0}$ forms a semigroup on $B(\mathbb{R}^d)$, the set of bounded Borel measurable functions, and a (dual) semigroup on finite Borel measures. We now record some of its properties which we will use later without further comment.

Proposition 2.26. *Let $\Phi(x)$ be a bounded Borel measurable function. We have:*

1. $P_t\Phi(x) \geq 0$ *if* $\Phi(x) \geq 0$.

2. $P_tC = C$ *for constants* C.

Proof. This is an easy exercise. □

We now consider the quantity:

$$\mathcal{L}\Phi(x) = \lim_{t\downarrow 0} \frac{P_t\Phi(x) - \Phi(x)}{t}.$$

The set of functions $\Phi : \mathbb{R}^d \to \mathbb{R}$ for which the limit exists point-wise on $\mathbb{R}^d$ is called the **domain** of $\mathcal{L}$. We denote the domain of $\mathcal{L}$ by $\mathcal{D}_{\mathcal{L}}$.

Definition 2.27. $\mathcal{L}$ is called the **generator** of the Markov process $x(t)$.

In the previous section, we were sloppy in calling L the generator of $x(t)$. L is more like an *special* version of $\mathcal{L}$, as we see by the following.

Proposition 2.28. $\mathcal{D}_{\mathcal{L}} \supset C_0^2(\mathbb{R}^d)$. *Moreover, for* $\Phi \in C_0^2(\mathbb{R}^d)$ *we have*

$$\mathcal{L}\Phi(x) = L\Phi(x),$$

where L *is as before.*

Proof. This follows from Dynkin's formula and the fact that $x(t)$ is non-explosive. $\square$

As we shall see, even though Φ may not be in the domain of $\mathcal{L}$, we can still relate $L\Phi(x)$ with $P_t\Phi(x)$ is some fashion. This will be useful later when we extract a convergence rate to equilibrium. For now, however, we discuss what we have been referring to as the "long-time" behavior of the process $x(t)$.

We ideally hope there exists a Borel probability measure μ such that:

$$\lim_{t\to\infty} P(x,t,A) = \mu(A), \tag{2.29}$$

for all $x \in \mathbb{R}^d$, $A \in \mathcal{B}(\mathbb{R}^d)$. This would be perfect as we see that (1) there is a limiting distribution and (2) this distribution does not depend on the starting point $x(0) = x \in \mathbb{R}^d$. As noted above, this is ideal since (1) a limiting distribution may not exist and (2) even if one does exist, there may be many.

For practical purposes, proving the limit above exists is, in general, difficult. If one yields the strength of the limit (2.29) and proves the Cesàro mean

$$\frac{1}{t}\int_0^t P(x,s,\,\cdot\,)\,ds \tag{2.30}$$

has limit points in the weak topology as $t \to \infty$, we can still recover, on average, the long-time behavior of the process.

As one should expect, there is a general property that these limits have in common. For illustrative purposes, suppose that there exists a sequence of times $t_n \uparrow \infty$ as $n \to \infty$ such that for some $x \in \mathbb{R}^d$ the sequence of measures:

$$\frac{1}{t_n}\int_0^{t_n} P(x,s,\,\cdot\,)\,ds$$

converges weakly to some Borel probability measure μ_x. Suppose moreover that P_t is **weak Feller**, i.e., P_t maps bounded continuous functions to bounded continuous functions. We then have for all bounded continuous functions f on $\mathbb{R}^d$:

$$\begin{aligned}
\int f(y)\,\mu_x P_t(y) &= \int f(y) \int \mu_x(d\tilde{y}) P(\tilde{y}, t, dy) \\
&= \int P_t f(y)\,\mu_x(dy) \\
&= \lim_{n\to\infty} \frac{1}{t_n} \int P_t f(y) \int_0^{t_n} P(x, s, dy)\,ds \\
&= \lim_{n\to\infty} \frac{1}{t_n} \int_0^{t_n} P_{s+t} f(x)\,ds \\
&= \lim_{n\to\infty} \frac{1}{t_n} \left[\int_0^{t_n} P_u f(x)\,du + \int_{t_n}^{t_n+t} P_u f(x)\,du - \int_0^t P_u f(x)\,du \right] \\
&= \int f(y)\,\mu_x(dy).
\end{aligned}$$

From this, we infer $\mu_x P_t(A) = \mu_x(A)$ for all $A \in \mathcal{B}(\mathbb{R}^d)$, i.e., μ_x is invariant under the action of the semigroup $\{P_t\}_{t\geq 0}$. This leads us to the following definition.

Definition 2.31. Let μ be a Borel measure such that for all $t \geq 0$

$$\mu P_t = \mu.$$

We call μ an **invariant measure**. If $\mu(\mathbb{R}^d) < \infty$, then it can be normalized to a probability measure ν which also has $\nu P_t = \nu$. We call ν an **invariant probability measure**.

An invariant probability measure is precisely the notion of limiting behavior we desire. We now give a necessary and sufficient condition for the existence of such a measure.

Theorem 2.32. *Suppose that P_t is weak Feller. Then there exists an invariant probability measure if and only if for some $x \in \mathbb{R}^d$:*

$$\lim_{r\to\infty} \liminf_{t\to\infty} \frac{1}{t} \int_0^t P(x, s, B_r(0)^c)\,ds = 0. \tag{2.33}$$

Proof. Let μ be an invariant probability measure. Suppose that for all $x \in \mathbb{R}^d$ we have:

$$\lim_{r\to\infty} \liminf_{t\to\infty} \frac{1}{t} \int_0^t P(x,s,B_r(0)^c)\, ds \quad = \quad p(x) > 0.$$

Then by Tonelli,

$$\begin{aligned} 0 &= \lim_{r\to\infty} \liminf_{t\to\infty} \frac{1}{t} \int_0^t \mu(B_r(0)^c)\, ds \\ &= \lim_{r\to\infty} \liminf_{t\to\infty} \frac{1}{t} \int_0^t \int \mu(dy) P(y,s,B_r(0)^c)\, ds \\ &= \int \mu(dy) \lim_{r\to\infty} \liminf_{t\to\infty} \frac{1}{t} \int_0^t P(y,s,B_r(0)^c)\, ds \\ &= \int \mu(dy) p(y) > 0. \end{aligned}$$

This proves one direction. Suppose conversely that for some $x \in \mathbb{R}^d$ we have

$$\lim_{r\to\infty} \liminf_{t\to\infty} \frac{1}{t} \int_0^t P(x,s,B_r(0)^c)\, ds = 0.$$

Note that this implies there exists a sequence of times $t_n \uparrow \infty$ as $n \to \infty$ such that:

$$\lim_{r\to\infty} \lim_{n\to\infty} \frac{1}{t_n} \int_0^{t_n} P(x,s,B_r(0)^c)\, ds = 0,$$

where the limit as $r \to \infty$ is uniform in n. Thus the sequence of probability measures:

$$\nu_n(\,\cdot\,) = \frac{1}{t_n} \int_0^{t_n} P(x,s,\,\cdot\,)\, ds,$$

is tight. By Prokhorov's theorem, $\nu_n(\,\cdot\,)$ is weakly compact and hence has a subsequence ν_{n_k} that converges to some probability measure ν. By the same computation above, we see that $\nu P_t = \nu$. This finishes the proof. □

It follows that under the assumptions **(A1)-(A4)** the semi-group P_t is weak Feller [Has80]. We thus want to verify condition (2.33) for all $x \in \mathbb{R}^d$. We will use a test function to do this.

Theorem 2.34. *Let $\Phi \in C^2(\mathbb{R}^d)$ be a non-negative function such that*

$$L\Phi(x) \to -\infty \text{ as } |x| \to \infty.$$

Then condition (2.33) is satisfied for all $x \in \mathbb{R}^d$ and hence, there exists an invariant probability measure for $\{P_t\}_{t\geq 0}$.

Proof. Let $x \in \mathbb{R}^d$. Note that by Lemma 2.4 we have:

$$E_x\left[\Phi(x(\xi_n(t)))\right] - \Phi(x) = E_x\left[\int_0^{\xi_n(t)} L\Phi(x(s))\, ds\right] \tag{2.35}$$

Moreover, for $r > 0$ sufficiently large:

$$\begin{aligned} L\Phi(x(s)) &\leq \sup_{|x|>r} L\Phi(x) \cdot \mathbf{1}_{\{|x(s)|>r\}} + \sup_{x\in\mathbb{R}^d} L\Phi(x) \\ &\leq -c_r \cdot \mathbf{1}_{\{|x(s)|>r\}} + d, \end{aligned} \tag{2.36}$$

for some $c_r, d > 0$ such that $c_r \to \infty$ as $r \to \infty$. Combining (2.35) with (2.36) we obtain:

$$c_r E_x\left[\int_0^{\xi_n(t)} \mathbf{1}_{\{x(s)|>r\}}\, ds\right] \leq d \cdot t + \Phi(x).$$

By nonexplosivity of the process $x(t)$, $\xi_n(t) \to t$ as $n \to \infty$ almost surely. Thus

$$\frac{1}{t}\int_0^t P(x, s, B_r(0)^c)\, ds \leq \frac{d}{c_r} + \frac{\Phi(x)}{c_r t}.$$

This implies the result after taking the lim inf as $t \to \infty$ and then taking $r \to \infty$. □

Remark 2.37. We emphasize that Theorem 2.34 assumes the process $x(t)$ is non-explosive. One can verify the hypotheses of both Theorem 2.8 and Theorem 2.34 simultaneously by proving there exists a smooth function $\Phi : \mathbb{R}^d \to [0, \infty)$ such that

(C1) $\Phi(x) \to \infty$ as $|x| \to \infty$,

(C2) $L\Phi(x) \to -\infty$ as $|x| \to \infty$.

We will do this later for the system (1.2), except that we prove a more explicit form of **(C2)**. This form is only needed to prove exponential convergence to the invariant probability measure. Before diving into this, we first handle the question of uniqueness.

2.4 Uniqueness of Invariant Probability Measures

In the previous section, we introduced invariant probability measures and discussed how they help describe the process $x(t)$ for large times $t > 0$. By the proofs of Theorem 2.32 and Theorem 2.34, if we exhibit a smooth test function $\Phi : \mathbb{R}^d \to [0, \infty)$ such that **(C1)** and **(C2)** are satisfied: for all $x \in \mathbb{R}^d$, the sequence of measures

$$\mu_T(\,\cdot\,) := \frac{1}{T}\int_0^T P(x, s, \,\cdot\,)\, ds, \tag{2.38}$$

has limit points in the weak topology as $T \to \infty$. These limit points are invariant probability measures and, since they possibly depend on the initial condition $x(0) = x \in \mathbb{R}^d$, there could be many such measures. Using the Markov property, we find sufficient conditions to prove there is only one.

In sole pursuit of uniqueness, assume throughout this section that the process $x(t)$ is non-explosive and has invariant probability measures. We let $\mathcal{M}$ denote the set of all such measures. It is easy to see that $\mathcal{M}$ is convex and consequently, to prove $\mathcal{M} = \{\mu\}$ it is enough to show $\mathcal{M}$ has only one **extremal point**[2]. This follows from Choquet's Theorem [Cho69] which asserts that if $\nu \in \mathcal{M}$, there exists a probability measure a_ν supported in the extremal points $\mathcal{E}(\mathcal{M})$ of $\mathcal{M}$ such that

$$\nu = \int_{\mathcal{E}(\mathcal{M})} \mu \, da_\nu(\mu).$$

Using this, we focus our discussion on extremal points of $\mathcal{M}$ which we call **extremal invariant probability measures**. To conclude that $\mathcal{E}(\mathcal{M}) = \{\mu\}$, we require the following:

[2]A point μ in a convex set $\mathcal{M}$ is *extremal* if whenever $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$ for some $\lambda \in (0,1)$ and $\mu_1,\, \mu_2 \in \mathcal{M}$, then $\mu_1 = \mu_2 = \mu$.

Theorem 2.39. *Distinct extremal invariant probability measures are mutually singular.*

Proof. This follows from Birkoff's Ergodic Theorem. For a discussion in our context, see Theorem 5.1 of M. Hairer's notes [Hai08]. □

From this, we easily obtain:

Corollary 2.40. *Suppose that for all $\mu_1, \mu_2 \in \mathcal{E}(\mathcal{M})$ we have*

$$\text{supp}(\mu_1) \cap \text{supp}(\mu_2) \neq \emptyset.$$

Then $\mathcal{M} = \{\mu\}$, i.e., there is one and only one invariant probability measure.

To prove uniqueness later on, we will use the corollary above. To gain traction on the supports of extremal invariant probability measures, we will use ideas in [AK87]. The key ingredient in this work is the use of the support theorem [SV72] which provides an intimate and accessible connection between the process $x(t)$ and control theory. More precisely, let us assume that the generator L of the process $x(t)$ can be written in the form:

$$L = \frac{\partial}{\partial t} + X_0 + \sum_{j=1}^{r} X_j^2,$$

where X_j is a smooth vector field on $\mathbb{R}^d$ for all $j = 0, 1, \ldots, r$. Consider the family of ordinary differential equations:

$$\frac{d\tilde{x}(t)}{dt} = X_0(\tilde{x}(t)) + \sum_{j=1}^{r} u_j(t) X_j(\tilde{x}(t)), \tag{2.41}$$

where for all $j = 1, 2, \ldots, r$, $u_j : [0, \infty) \to \mathbb{R}$ is a piecewise constant mapping with at most finitely many discontinuities. We call such $u(t) = (u_1(t), u_2(t), \ldots, u_r(t))$ **admissible controls** and we denote the class of all admissible controls by $\mathcal{U}$. For $u \in \mathcal{U}$ fixed, let $\varphi(x, u, t)$ be the maximal right solution of equation (2.41) passing through x at time $t = 0$. Define for $x \in \mathbb{R}^d$, $T > 0$ the sets:

$$\begin{aligned} A(x,T) &= \bigcup_{u \in \mathcal{U}} \{ y \in \mathbb{R}^d \, : \, \varphi(x,u,T) = y \} \\ A(x, \leq T) &= \bigcup_{0<t\leq T} A(x,t) \\ A^+(x) &= \bigcup_{t>0} A(x,t). \end{aligned}$$

Thus, in words, $A(x,T)$, $A(x, \leq T)$, and $A^+(x)$ are respectively the accessible points starting from $x \in \mathbb{R}^d$ through the trajectories (2.41) for all $u \in \mathcal{U}$ at exactly time $t = T$, some time $0 < t \leq T$, and some positive time $t > 0$. It is useful to note that for $T > 0$, the $A(x, \leq T)$ are nested, while the $A(x,T)$ need not be.

In view of [SV72], by non-explosivity of the process $x(t)$ with $x(0) = x \in \mathbb{R}^d$ defined by equation (2.1), we can determine its support by studying the accessibility sets $A(x,T)$.

Theorem 2.42 (Stroock-Varadhan 1972)**.** *For all $x \in \mathbb{R}^d$ and $T > 0$:*

$$\mathrm{supp}(P(x,T,\,\cdot\,)) = \overline{A(x,T)}.$$

Proof. See [SV72]. □

In the spirit of the previous result, Arnold and Kliemann [AK87] introduce the notion of an **invariant control set**, defined below, to find an expression for supp(μ) where $\mu \in \mathcal{E}(\mathcal{M})$ in terms of the positive orbits $A^+(x)$.

Definition 2.43. A set $C \neq \emptyset \subset \mathbb{R}^d$ is an **invariant control set** for the system (2.41) if:

$$\overline{A^+(x)} = \overline{C} \text{ for all } x \in C,$$

and C is maximal with respect to inclusion.

By Proposition 1.1 of [AK87], for all $\mu \in \mathcal{E}(\mathcal{M})$, there exists an invariant control set C such that

$$\mathrm{supp}(\mu) = \overline{C}.$$

Thus:

Corollary 2.44. *For all $\mu \in \mathcal{E}(\mathcal{M})$, there exists $x \in \mathbb{R}^d$ such that*

$$\text{supp}(\mu) = \overline{A^+(x)}.$$

Remark 2.45. To verify the hypotheses of Corollary 2.40 and hence prove uniqueness of the invariant probability measure, it is enough to show that for all $x, y \in \mathbb{R}^d$:

$$\overline{A^+(x)} \cap \overline{A^+(y)} \neq \emptyset.$$

The benefit of using the above is that it provides the means to use techniques from control theory which are extremely tractable. Moreover, we do not really need to know much about the sets $\overline{A^+(x)}$. It many cases, however, it is still difficult to uncover these sets. As we shall see in the next section, geometric ideas are fruitful in this regard.

2.5 Geometric Control Theory

In hopes of understanding the accessibility sets $A(x, T)$, $A(x, \leq T)$, and $A^+(x)$ for the control system (2.41), we find it convenient to use geometric control theory. Most of what follows is in the book by Jurdjevic [Jur97] as well as his joint works with Kupka [JK81, JK85]. By Remark 2.45, to conclude uniqueness of the invariant probability measure, it suffices to show that for all x, $y \in \mathbb{R}^d$, $\overline{A^+(x)} \cap \overline{A^+(y)} \neq \emptyset$. However, we aim at extracting a convergence rate to this equilibrium. Consequently, we will need a solid grasp on the sets $A(x, \leq T)$ and $A(x, T)$ for $x \in \mathbb{R}^d$ and $T > 0$ as well. To mesh with the geometrical setting in these works, we will slightly adjust some previously used notation. We begin with a definition.

Definition 2.46. Let F be a collection of smooth vector fields on $\mathbb{R}^d$. We call F a **polysystem**.

Let F be a polysystem. For $Y \in F$, let $\exp(tY)(x)$ denote the maximal right integral curve of Y passing through $x \in \mathbb{R}^d$ at time $t = 0$. For $T > 0$, we define $A_F(x, T)$ to be the set of all points $y \in \mathbb{R}^d$ such that there exist vector fields $Y_1, Y_2, \ldots, Y_k \in F$ and corresponding times $t_1, t_2, \ldots, t_k \geq 0$ such that $t_1 + t_2 + \cdots + t_k = T$ and

$$y = \exp(t_k Y_k) \circ \exp(t_{k-1} Y_{k-1}) \circ \cdots \circ \exp(t_1 Y_1)(x).$$

For $x \in \mathbb{R}^d$ and $T > 0$, let

$$\begin{aligned} A_F(x, \leq T) &= \bigcup_{0<t\leq T} A_F(x, t) \\ A_F^+(x) &= \bigcup_{t>0} A_F(x, t), \end{aligned}$$

Relating this setup with the control system (2.41), we note that, by definition, if we let

$$F = \left\{ X_0 + \sum_{j=1}^{r} u_j X_j \;:\; u = (u_1, u_2, \ldots, u_r) \in \mathbb{R}^r \right\},$$

then for all $x \in \mathbb{R}^d$ and $T > 0$:

$$\begin{aligned} A_F(x, T) &= A(x, T), \\ A_F(x, \leq T) &= A(x, \leq T), \\ A_F^+(x) &= A^+(x), \end{aligned}$$

where $A(x, T)$, $A(x, \leq T)$ and $A^+(x)$ were defined in the previous section. The benefit of using geometric ideas is it allows us to modify the polysystem F without changing the accessibility sets too much. With this in mind, we introduce an equivalence relation $\sim$ on polysystems.

Definition 2.47. Two polysystems F_1 and F_2 are **equivalent**, denoted by $F_1 \sim F_2$, if for all $x \in \mathbb{R}^d$, $T > 0$

$$\overline{A_{F_1}(x, \leq T)} = \overline{A_{F_2}(x, \leq T)}.$$

It is easy to see that $\sim$ is an equivalence relation. Starting from an initial polysystem F, the idea is to find operations on this family of vector fields such that, when performed, we stay within the class of equivalent polysystems. To this end, we have the following theorem.

Theorem 2.48. *Suppose that F, F_1, and F_2 are polysystems such that $F \sim F_1$ and $F \sim F_2$. Then $F \sim (F_1 \cup F_2)$.*

Before we prove the theorem, we need an important proposition which is a consequence of the existence, uniqueness, and smoothness theorem of ordinary differential equations. For further information, we refer the reader to John M. Lee's book [Lee03].

Proposition 2.49. *Let $U \subset \mathbb{R}^d$ and F a polysystem. For $T > 0$ define*

$$A_F(U, \leq T) = \bigcup_{x \in U} A_F(x, \leq T).$$

Then for all $x \in \mathbb{R}^d$ and S, $T > 0$

$$A_F(\overline{A_F(x, \leq S)}, \leq T) \subset \overline{A_F(x, \leq S+T)}.$$

Proof. Let $y \in A_F(\overline{A_F(x, \leq S)}, \leq T)$. Thus there exist $Y_1, Y_2, \ldots, Y_k \in F$ and times $t_1, t_2, \ldots, t_k \geq 0$ such that $t_1 + t_2 + \cdots + t_k = s \leq T$ and

$$y = \exp(t_k Y_k) \circ \exp(t_{k-1} Y_{k-1}) \circ \cdots \circ \exp(t_1 Y_1)(\tilde{y}),$$

for some $\tilde{y} \in \overline{A_F(x, \leq S)}$. By definition, there exists a sequence $x_j \in A_F(x, \leq S)$ such that $x_j \to \tilde{y}$ as $j \to \infty$. By the existence, uniqueness, and smoothness theorem from ordinary differential equations since $Y_1, \ldots, Y_k$ are smooth, for j sufficiently large, the sequence:

$$y_j := \exp(t_k Y_k) \circ \exp(t_{k-1} Y_{k-1}) \circ \cdots \circ \exp(t_1 Y_1)(x_j) \in A_F(x, \leq S+T)$$

is defined. Moreover, $y_j \to y$ as $j \to \infty$. This finishes the proof. $\square$

Proof of Theorem 2.48. Let $x \in \mathbb{R}^d$ and $T > 0$. The inclusion

$$\overline{A_F(x, \leq T)} = \overline{A_{F_1}(x, \leq T)} \subset \overline{A_{F_1 \cup F_2}(x, \leq T)}$$

is clear. To prove the reverse inclusion, we will show that $A_{F_1 \cup F_2}(x, \leq T) \subset \overline{A_F(x, \leq T)}$. Let $y \in A_{F_1 \cup F_2}(x, \leq T)$. Thus there exist $Y_1, Y_2, \ldots, Y_k \in F_1 \cup F_2$ and corresponding times $t_1, t_2, \ldots, t_k \geq 0$ such that $t_1 + t_2 + \cdots + t_k = s \leq T$ and

$$y = \exp(t_k Y_k) \circ \exp(t_{k-1} Y_{k-1}) \circ \cdots \circ \exp(t_1 Y_1)(x).$$

Let $y_0 = x$. For $j = 1, 2, \ldots, k$, define inductively

$$y_j = \exp(t_j Y_j)(y_{j-1}).$$

By the equivalence assumptions, $y_j \in \overline{A_F(y_{j-1}, \leq t_j)}$ for $j = 1, 2, \ldots, k$. Hence $y_1 \in \overline{A_F(x, \leq t_1)}$ and consequently by the previous proposition,

$$y_2 \in \overline{A_F(y_1, \leq t_2)} \subset \overline{A_F(\overline{A_F(x, \leq t_1)}, \leq t_2)} \subset \overline{A_F(x, \leq t_1 + t_2)}.$$

Iterating this procedure yields the result since $t_1 + t_2 + \cdots + t_k \leq T$. □

Let now

$$\operatorname{Sat}(F) = \bigcup_{\tilde{F} \sim F} \tilde{F}.$$

We call $\operatorname{Sat}(F)$ the **saturate** of F. As a consequence of the previous theorem, we have:

Corollary 2.50.

$$F \sim \operatorname{Sat}(F).$$

Proof. Fix $x \in \mathbb{R}^d$ and $T > 0$. Since $F \sim F$, the inclusion $\overline{A_F(x, \leq T)} \subset \overline{A_{\operatorname{Sat}(F)}(x, \leq T)}$ is clear. We prove $A_{\operatorname{Sat}(F)}(x, \leq T) \subset \overline{A_F(x, \leq T)}$. Let $y \in A_{\operatorname{Sat}(F)}(x, \leq T)$. Then there exist $Y_1, \ldots, Y_k \in \operatorname{Sat}(F)$ and times $t_1, \ldots, t_k \geq 0$ such that $t_1 + \cdots + t_k = s \leq T$ and

$$y = \exp(t_k Y_k) \circ \cdots \circ \exp(t_1 Y_1)(x).$$

Note that for $j = 1, 2, \ldots, k$, $Y_j \in F_j$ for some polysystem $F_j \sim F$. By the previous theorem inductively we have:

$$F \sim \bigcup_{j=1}^{k} F_j.$$

Thus, in particular, $y \in \overline{A_F(x, \leq T)}$. □

For a polysystem F, $\mathrm{Sat}(F)$ serves as an enlargement of F with which we can work and still deduce properties of $\overline{A_F(x, \leq T)}$ for $x \in \mathbb{R}^d$ and $T > 0$. To determine more polysystems than F that are equivalent to F, we have the following theorem:

Theorem 2.51. *F is equivalent to the closed convex hull $\boldsymbol{\mathcal{C}}_F$ of the family*

$$\tilde{F} = \{\lambda Y \, : \, 0 \leq \lambda \leq 1, \, Y \in F\}.$$

Here the closure is in the C^∞-topology on compact subsets of $\mathbb{R}^d$.

To prove this assertion, we first need two propositions.

Proposition 2.52. *Let $\overline{F}$ be the closure of a polysystem F in the C^∞-topology on compact subsets of $\mathbb{R}^d$. Then for all $x \in \mathbb{R}^d$ and $T > 0$:*

$$A_{\overline{F}}(x, T) \subset \overline{A_F(x, T)}.$$

Proof. Recall that $A_F(x, T)$ is the set of points that can be reached from x using trajectorties in F at exactly time T. Fix $x \in \mathbb{R}^d$, $T > 0$, and let $y \in A_{\overline{F}}(x, T)$. Then there exist $Y_1, Y_2, \ldots, Y_k \in \overline{F}$ and times $t_1, t_2, \ldots, t_k \geq 0$ such that $t_1 + t_2 + \cdots + t_k = T$ and

$$y = \exp(t_k Y_k) \circ \cdots \circ \exp(t_1 Y_1)(x).$$

All we must show is that $y \in \overline{A_F(x, T)}$. For $j = 1, 2, \ldots, k$, let $Y_j^n \in F$ be such that $\lim_{n \to \infty} Y_j^n = Y_j$. One can show, see Theorem 4 in Chapter 3 of [Jur97], that this implies for fixed $x_0 \in \mathbb{R}^d$ and $j = 1, 2, \ldots, k$

$$\exp(tY_j)(x_0) = \lim_{n \to \infty} \exp(tY_j^n)(x_0),$$

uniformly in $t \in [0,T]$. From this, we see that for $j = 1, 2, \ldots, k$

$$\exp(t_j Y_j) \circ \cdots \circ \exp(t_1 Y_1)(x) \in \overline{A_F\left(x, \sum_{l=1}^{j} t_l\right)}.$$

Taking $j = k$, we obtain the result. $\square$

Proposition 2.53. *Suppose that $Y_1, Y_2, \ldots, Y_k \in F$. Then for all $\lambda_1, \lambda_2, \ldots, \lambda_k \in [0,1]$ such that $\sum_j \lambda_j = 1$, and all $x \in \mathbb{R}^d$*

$$\exp\left(T\left(\sum_j \lambda_j Y_j\right)\right)(x) \in \overline{A_F(x,T)},$$

for all $T > 0$ for which the left-hand side is defined.

Proof. This proof of this is long but not hard. See Theorem 7 of Chapter 3 in [Jur97]. $\square$

Using the previous two propositions, we now prove Theorem 2.51.

Proof of Theorem 2.51. Fix $x \in \mathbb{R}^d$ and $T > 0$. The inclusion

$$\overline{A_F(x, \leq T)} \subset \overline{A_{\mathcal{C}_F}(x, \leq T)}$$

is clear. Hence we show that

$$A_{\mathcal{C}_F}(x, \leq T) \subset \overline{A_F(x, \leq T)}.$$

To see this, we first let $\mathrm{Co}(\tilde{F})$ be the convex hull of the set

$$\tilde{F} = \{\lambda Y \,:\, 0 \leq \lambda \leq 1,\, Y \in F\}.$$

Note that, by Proposition 2.52, we have the inclusion

$$A_{\mathcal{C}_F}(x,T) \subset \overline{A_{\mathrm{Co}(\tilde{F})}(x,T)},$$

since $\boldsymbol{\mathcal{C}}_F = \overline{\text{Co}(\tilde{F})}$. Hence $A_{\boldsymbol{\mathcal{C}}_F}(x, \leq T) \subset \overline{A_{\text{Co}(\tilde{F})}(x, \leq T)}$. Moreover, by Proposition 2.53, $A_{\text{Co}(\tilde{F})}(x, \leq T) \subset \overline{A_{\tilde{F}}(x, \leq T)}$. Hence it is enough to show that

$$A_{\tilde{F}}(x, \leq T) \subset \overline{A_F(x, \leq T)}.$$

Let $y \in A_{\tilde{F}}(x, \leq T)$. Thus there exist $Y_1, Y_2, \ldots, Y_k \in F$, $\lambda_1, \lambda_2, \ldots, \lambda_k \in [0, 1]$ and times $t_1, t_2, \ldots, t_k \geq 0$ such that $t_1 + t_2 + \ldots + t_k = s \leq T$ and

$$y = \exp(t_k \lambda_k Y_k) \circ \exp(t_{k-1} \lambda_{k-1} Y_{k-1}) \circ \cdots \circ \exp(t_1 \lambda_1 Y_1)(x).$$

But note that taking $s_j = \lambda_j t_j$ for $j = 1, 2, \ldots, k$ we have $s_1 + s_2 + \cdots + s_k \leq T$ and

$$y = \exp(s_k Y_k) \circ \cdots \circ \exp(s_1 Y_1)(x).$$

Thus $y \in A_F(x, \leq T)$ which finishes the proof. $\square$

In what follows, it will be very convenient to use $\boldsymbol{\mathcal{C}}_F$ to help determine $\overline{A_F(x, \leq T)}$ for $x \in \mathbb{R}^d$ and $T > 0$. There is yet another operation we may perform on F and remain in the saturate. To describe it, we first need a definition and a remark.

Definition 2.54. Let F be an arbitrary polysystem and $\psi : \mathbb{R}^d \to \mathbb{R}^d$ be a diffeomorphism. We call ψ a **normalizer** of F if for all $x \in \mathbb{R}^d$ and $T > 0$:

$$\psi(\overline{A_F(\psi^{-1}(x), \leq T)}) \subset \overline{A_F(x, \leq T)}.$$

We denote the set of all normalizers of F by $\text{Norm}(F)$.

Remark 2.55. For a diffeomorphism $\psi : \mathbb{R}^d \to \mathbb{R}^d$ and a smooth vector field Y on $\mathbb{R}^d$ we may create another smooth vector field on $\mathbb{R}^d$, which we denote by $\psi_{\#}(Y)$, defined by

$$\psi_{\#}(Y) = \psi_* \circ Y \circ \psi^{-1},$$

where ψ_* is the differential of ψ. The motivation for introducing this operation and the notion of a normalizer is easily seen by the following theorem.

Theorem 2.56. *F is equivalent to*

$$F_{\#} = \{\psi_{\#}(Y) \,:\, \psi \in \mathrm{Norm}(F),\, Y \in F\}.$$

Proof. Fix $x \in \mathbb{R}^d$ and $T > 0$. The inclusion $\overline{A_F(x, \leq T)} \subset \overline{A_{F_{\#}}(x, \leq T)}$ follows by taking ψ to be the identity map on $\mathbb{R}^d$ and realizing the identity is a normalizer. For the reverse inclusion, let $y \in A_{F_{\#}}(x, \leq T)$. Choose $Y_1, Y_2, \ldots, Y_k \in F$ and $t_1, t_2, \ldots, t_k \geq 0$ such that $t_1 + t_2 + \ldots t_k = s \leq T$ and

$$y = \exp(t_k(\psi_k)_{\#}(Y_k)) \circ \exp(t_{k-1}(\psi_{k-1})_{\#}(Y_{k-1})) \circ \cdots \circ \exp(t_1(\psi_1)_{\#}(Y_1))(x), \quad (2.57)$$

for some $\psi_1, \psi_2, \ldots, \psi_k \in \mathrm{Norm}(F)$. By relation (2.57) and Proposition 2.49, it is enough to show that for all $\psi \in \mathrm{Norm}(F)$, $Y \in F$, and $x_0 \in \mathbb{R}^d$

$$\exp(t\psi_{\#}(Y))(x_0) \in \overline{A_F(x_0, \leq t)},$$

for all $t > 0$ for which the left-hand side is defined. Note that we have

$$\exp(t\psi_{\#}(Y)) = \psi \circ \exp(tY) \circ \psi^{-1}.$$

Since ψ is a normalizer of F,

$$\exp(t\psi_{\#}(Y))(x_0) = \psi(\exp(tY)(\psi^{-1}(x_0))) \in \overline{A_F(x_0, \leq t)}$$

as $\exp(tY)(\psi^{-1}(x_0)) \in A_F(\psi^{-1}(x_0), \leq t)$. $\square$

To determine easily which diffeomorphisms ψ are normalizers, we have:

Lemma 2.58. *Let F be an arbitrary polysystem and $\psi : \mathbb{R}^d \to \mathbb{R}^d$ be a diffeomorphism. Suppose that for all $x \in \mathbb{R}^d$ and $T > 0$, $\psi(x)$, $\psi^{-1}(x) \in \overline{A_F(x, \leq T)}$. Then $\psi \in \mathrm{Norm}(F)$.*

Proof. Let $x \in \mathbb{R}^d$, $T > 0$, and $\epsilon > 0$. We show that

$$\psi(\overline{A_F(\psi^{-1}(x), \leq T)}) \subset \overline{A_F(x, \leq T + \epsilon)}.$$

By Proposition 2.49, since $\psi^{-1}(x) \in \overline{A_F(x, \epsilon/2)}$ we have

$$\begin{aligned} \overline{A_F(\psi^{-1}(x), \leq T)} &\subset \overline{A_F(\overline{A_F(x, \leq \epsilon/2)}, \leq T)} \\ &\subset \overline{A_F(x, \leq T + \epsilon/2)}. \end{aligned}$$

Since $\psi(y) \in \overline{A_F(y, \leq \epsilon/2)}$ for all $y \in \overline{A_F(x, \leq T + \epsilon/2)}$, applying Proposition 2.49 again we have:

$$\begin{aligned} \psi(\overline{A_F(\psi^{-1}(x), \leq T)}) &\subset \psi(\overline{A_F(x, \leq T + \epsilon/2)}) \\ &\subset \overline{A_F(\overline{A_F(x, \leq T + \epsilon/2)}, \epsilon/2)} \\ &\subset \overline{A_F(x, \leq T + \epsilon)}. \end{aligned}$$

This finishes the proof of the lemma. □

We now stand in perfect position to determine the sets $\overline{A_F(x, \leq T)}$, but as we recall from Theorem 2.42, we need some understanding of $\overline{A_F(x, T)}$ as well. The transfer mechanism between the two is the next theorem.

Theorem 2.59. *Suppose that F is a polysystem such that the span of the Lie algebra generated by elements of F is the entire tangent space at all points $x \in \mathbb{R}^d$. Suppose moreover that for some $x \in \mathbb{R}^d$ and some $U \neq \emptyset$ open:*

$$\overline{A_F(x, \leq T)} \supset U \text{ and } x \in A_F(x, \leq T) \text{ for all } T > 0.$$

Then $\overline{A_F(x, T)} \supset U$ for all $T > 0$.

Proof. Fix $x \in \mathbb{R}^d$ and $U \neq \emptyset$ so that the assumptions are satisfied. Let $y \in U$, $T > 0$, and pick a small open set $V \subset U$ that contains y. By Theorem 2 on p. 68 of [Jur97], the spanning assumption of the Lie algebra allows us to conclude that

$$A_F(x, \leq S) \supset U,$$

for all $S > 0$. Therefore for some $0 < T' \leq T$ there exists a continuous trajectory $\gamma : [0, T'] \to \mathbb{R}^d$ defined by vector fields of F such that $\gamma(0) = x$ and $\gamma(T') = y$.

Suppose that $T' < T$ and pick $\epsilon > 0$ small enough such that $\epsilon < T - T'$ and $\gamma(t) \in V$ for $t \in [T' - \epsilon, T']$. Since $x \in A_F(x, \leq S)$ for all $S > 0$, for any positive numbers $S_1 < S_2$, there exists $S' > 0$ and a curve $\delta : [0, S'] \to \mathbb{R}^d$ defined by vector fields of F such that $S_1 < S' < S_2$ and $\delta(0) = \delta(S') = x$. Let $S_1 = T - T'$ and $S_2 = T - T' + \epsilon$ and note that the composite curve

$$\sigma(t) = \begin{cases} \delta(t), & \text{for } t \in [0, S'] \\ \gamma(t - S') & \text{for } t \in [S', S' + T'] \end{cases}$$

has $\sigma(0) = x$ and $\sigma(T) \in V$. This finishes the proof that $\overline{A_F(x,T)} \supset U$ for all $T > 0$. $\square$

Let us now enjoy the fruits of our labor. We prove of a classical theorem which we do not need for the main results of this dissertation. Still, however, the proof illustrates the use of the methods developed in this section.

Theorem 2.60 (Rank Theorem)**.** *Let $A \in M_d(\mathbb{R})$, $b \in \mathbb{R}^d$ and consider the control system*

$$\frac{dy(t)}{dt} = Ay(t) + u(t)b,$$

where $u : [0, \infty) \to \mathbb{R}$ is a piecewise constant mapping with at most finitely many discontinuities. Let $\boldsymbol{A}$ be the vector field determined by $\boldsymbol{A}(x) = Ax$ and $\boldsymbol{b}$ be the constant vector field $\boldsymbol{b}(x) = b$. Consider the polysystem $F = \{\boldsymbol{A} + u\boldsymbol{b} : u \in \mathbb{R}\}$. If $\operatorname{span}\{b, Ab, \ldots, A^{d-1}b\} = \mathbb{R}^d$*, then $\overline{A_F(x,T)} = \mathbb{R}^d$ for all $x \in \mathbb{R}^d$ and $T > 0$.*

Proof. By Theorem 2.51, for all $u \in \mathbb{R}$:

$$\lim_{\lambda \to \infty} \frac{1}{\lambda}(\boldsymbol{A} + u\lambda\boldsymbol{b}) = u\boldsymbol{b} \in \operatorname{Sat}(F).$$

Since $\psi_u(x)(t) = x + tub$ is the integral curve of $u\boldsymbol{b}$ with inverse $\psi_u(x)^{-1}(t) = \psi_{-u}(x)(t)$, we note that for all $u \in \mathbb{R}$, the map $\psi_u(x) := \psi_u(x)(1) \in \operatorname{Norm}(\operatorname{Sat}(F))$

for all $u \in \mathbb{R}$. Hence

$$\begin{aligned}(\psi_u)_\#(\boldsymbol{A})(x) &= (\psi_u)_*(\boldsymbol{A}(\psi_u^{-1}(x)) \\ &= A(x - ub) \\ &= Ax - uAb.\end{aligned}$$

Thus $(\psi_u)_\#(\boldsymbol{A}) = \boldsymbol{A} - u\boldsymbol{Ab} \in \mathrm{Sat}(F)$ for all $u \in \mathbb{R}$ where $u\boldsymbol{Ab}$ is the constant vector field $u\boldsymbol{Ab}(x) = uAb$. Moreover,

$$\lim_{\lambda\to\infty} \frac{1}{\lambda}(\boldsymbol{A} + u\lambda\boldsymbol{Ab}) = u\boldsymbol{Ab} \in \mathrm{Sat}(F),$$

for all $u \in \mathbb{R}$. Note by Theorem 2.51, we have

$$\overline{A_F(x, \leq T)} \supset \{x\} + \mathrm{span}\{b, Ab\}.$$

We can continue this procedure to see that the constant vector fields $u_2\boldsymbol{A^2b}$, ..., $u_{d-1}\boldsymbol{A^{d-1}b} \in \mathrm{Sat}(F)$ for all $u_2, \ldots, u_{d-1} \in \mathbb{R}$. Note that by the observation above and the spanning assumption, this is sufficient to conclude

$$\overline{A_F(x, \leq T)} = \mathbb{R}^d,$$

for all $x \in \mathbb{R}^d$ and all $T > 0$. By Theorem 2 on p. 68 of [Jur97], we obtain

$$A_F(x, \leq T) = \mathbb{R}^d,$$

for all $x \in \mathbb{R}^d$ and all $T > 0$. By the previous theorem since $x \in A_F(x, \leq T)$ for all $T > 0$ and all $x \in \mathbb{R}^d$ we may conclude

$$\overline{A_F(x, T)} = \mathbb{R}^d,$$

for all $x \in \mathbb{R}^d$ and all $T > 0$, as required. □

It is important to point out that one can prove this theorem without using the methods of this section. In cases where one cannot infer accessibility properties easily, geometric control theory allows us to do so with more ease.

2.6 Geometric Ergodicity

Let $\|\cdot\|_{TV}$ be the total variation norm on $\mathcal{B}(\mathbb{R}^d)$-measures and suppose that the Markov process $x(t)$ defined by equation (2.1) with transition kernel $P(x,t,A)$ is non-explosive and has a unique invariant probability measure μ. In this section, we provide sufficient conditions under which we can quantify a rate of convergence to the equilibrium μ. Specifically, we are concerned with the case when this rate is geometric. In this section, we will thus show that under two minimal assumptions that the process $x(t)$ has the following property:

Property 2.61. There exists a constant $\rho \in (0,1)$ and a function $\Psi : \mathbb{R}^d \to [0,\infty)$ such that

$$\|P(x,t,\cdot\,) - \mu(\,\cdot\,)\|_{TV} \leq \Psi(x)\rho^t,$$

for all $t \geq 0$ and all $x \in \mathbb{R}^d$.

Thus for all fixed $x \in \mathbb{R}^d$, the transition measures $P(x,t,\cdot\,)$ approach the invariant probability measure exponentially fast in time in the total variation norm. This is an extremely strong form of convergence which surprisingly can, in our case, be improved. If the process $x(t)$ satisfies Property 2.61, we call $x(t)$ **exponentially ergodic**.

There are many different presentations on exponential ergodicity of Markov processes. See, for example, the works of Meyn and Tweedie [MT92, MT93a, MT93b, MT09] or the treatments in the context of stochastic differential equations of Rey-Bellet [RB06] and Higham, Mattingly, and Stuart [MSH02]. Appealing partially to these, we will primarily use the elegant and concise notes of Hairer and Mattingly [HM08]. The benefit of their work over others is their proof of exponential convergence is short. Moreover, one can apply their methods as they do to extremely degenerate (and even infinite dimensional) stochastic differential equations.

Throughout this section, we make the two assumptions below. One should not think of them as being entirely separate as they are intertwined through the test function Φ.

Assumption 2.62. There exists a non-negative function $\Phi \in C^2(\mathbb{R}^d)$ and positive constants C, D such that

$$L\Phi(x) \leq -C\Phi(x) + D \text{ for all } x \in \mathbb{R}^d.$$

Assumption 2.63. There exists a distinguished time $T_0 > 0$ such that for all $R > 0$ sufficiently large, there exists $\alpha_R \in (0,1)$ and a probability measure ν such that

$$\inf_{x \in C_R} P(x, T_0, \cdot\,) \geq \alpha_R \nu(\,\cdot\,),$$

where $C_R = \{x \in \mathbb{R}^d : \Phi(x) \leq R\}$ and Φ is as in Assumption 2.62.

If the sets $C_R = \{x : \Phi(x) \leq R\}$ for $R > 0$ are pre-compact, Assumption 2.62 essentially implies that the dynamics is focused in a possibly large, but bounded region. One can of course guarantee pre-compactness if $\Phi(x) \to \infty$ as $|x| \to \infty$, which is sometimes also a standard assumption. With this idea in place, Assumption 2.63 is only non-standard in the sense in which it is expressed, even though we should expect an exponentially ergodic Markov process to satisfy it as "mixing" is crucial in this regard.

To state the main theorem of this section which implies Property 2.61, we introduce a weighted norm $\|\cdot\|$ on $\mathcal{B}(\mathbb{R}^d)$-measureable real-valued functions φ:

$$\|\varphi\| = \sup_{x \in \mathbb{R}^d} \frac{|\varphi(x)|}{1 + \Phi(x)},$$

where φ is such that $\|\varphi\| < \infty$. The norm $\|\cdot\|$ comes equipped with a dual norm on signed $\mathcal{B}(\mathbb{R}^d)$-measures ν:

$$\|\nu\| = \sup_{\|\varphi\| \leq 1} \int \varphi(x)\nu(dx).$$

We will prove:

Theorem 2.64. *Under Assumption 2.62 and Assumption 2.63, there exist constants $\rho \in (0,1)$ and $E > 0$ such that*

$$\|P(x,t,\,\cdot\,) - \mu\| \le E\rho^t(1+\Phi(x)),$$

for all $t \ge 0$ and all $x \in \mathbb{R}^d$.

In many works, the proof of the above is quite non-trivial. To circumambulate these difficulties, Hairer and Mattingly introduce a family of norms that are slightly tweaked versions of the norm $\|\cdot\|$ above. To this end, let $\gamma > 0$ and define

$$\|\varphi\|_\gamma = \sup_{x\in\mathbb{R}^d} \frac{|\varphi(x)|}{1+\gamma\Phi(x)},$$

where φ is such that $\|\varphi\|_\gamma < \infty$. Of course, like $\|\cdot\|$ above, the norm $\|\cdot\|_\gamma$ has a dual norm on signed $\mathcal{B}(\mathbb{R}^d)$-measures ν:

$$\|\nu\|_\gamma = \sup_{\|\varphi\|_\gamma \le 1} \int \varphi(x)\nu(dx).$$

The goal is to choose the parameter $\gamma > 0$ so that a version of Theorem 2.64 holds in this norm in the discrete setting, for (1) the norms $\|\cdot\|$ and $\|\cdot\|_\gamma$ on measures are equivalent and (2) we have a natural time $T_0 > 0$ to define an embedded Markov chain. In view of this, let $\{x_n\}_{n\in\mathbb{N}}$ be the Markov chain with n-step transitions $P^n(x,A)$ given by:

$$P^n(x,A) := P(x, nT_0, A),$$

for $x \in \mathbb{R}^d$ and $A \in \mathcal{B}(\mathbb{R}^d)$. To conclude Theorem 2.64, we will show:

Theorem 2.65. *Under Assumption 2.62 and Assumption 2.63, there exists $\rho \in (0,1)$ and $\gamma > 0$ such that*

$$\|\nu_1 P - \nu_2 P\|_\gamma \le \rho\|\nu_1 - \nu_2\|_\gamma, \tag{2.66}$$

for all $\mathcal{B}(\mathbb{R}^d)$-probability measures ν_1, ν_2 where $P = P^1$. Moreover,

$$\int_{\mathbb{R}^d} \Phi(y)\mu(dy) < \infty,$$

where μ is the invariant probability measure.

Before we prove that Theorem 2.65 implies Theorem 2.64, we first relate Assumption 2.62 to a bound on the semigroup P_t.

Lemma 2.67. *Suppose that Assumption 2.62 holds. Then for all $t \geq 0$*

$$P_t\Phi \leq e^{-Ct}\Phi + D/C.$$

Proof. Let $\Psi(t,x) = e^{Ct}(\Phi(x) - D/C)$. Applying Dynkin's formula, we obtain:

$$E_x\left[\Psi(\xi_n(t), x(\xi_n(t)))\right] \;\; \leq \;\; \Phi(x) - D/C.$$

Estimating the left-hand side of the above, we note that:

$$\begin{aligned} E_x\left[\Psi(\xi_n(t), x(\xi_n(t)))\right] \;\; &\geq \;\; E_x\left[\mathbf{1}_{\{t<\xi_n\}}\Psi(\xi_n(t), x(\xi_n(t)))\right] \\ &= \;\; e^{Ct}E_x\left[\mathbf{1}_{\{t<\xi_n\}}(\Phi(x(t)) - D/C)\right] \end{aligned}$$

Combining this with the first estimate, we obtain:

$$E_x\left[\mathbf{1}_{\{t<\xi_n\}}(\Phi(x(t)) - D/C)\right] \;\; \leq \;\; e^{-Ct}\Phi(x) - e^{-Ct}D/C.$$

Since the process is non-explosive and the bound above holds for all n, we have:

$$\begin{aligned} P_t\Phi(x) \;\; &\leq \;\; e^{-Ct}\Phi(x) + (1 - e^{-Ct})D/C \\ &\leq \;\; e^{-Ct}\Phi(x) + D/C, \end{aligned}$$

for all $t \geq 0$ and $x \in \mathbb{R}^d$. □

Proof that Theorem 2.65 $\implies$ Theorem 2.64. We note that since μ is invariant, by (2.66) we have

$$\|P^n(x,\,\cdot\,) - \mu\|_\gamma \leq \rho^n \|P(x,\,\cdot\,) - \mu\|_\gamma,$$

for all $n \in \mathbb{N}$ and all $x \in \mathbb{R}^d$. We first show that there exists a constant $E' > 0$ independent of x such that

$$\|P(x,\,\cdot\,) - \mu\|_\gamma \leq E'(1 + \Phi(x)).$$

Since $\int \Phi(y)\mu(dy) = C' < \infty$, by the previous lemma we obtain

$$\begin{aligned}
\|P(x,\,\cdot\,) - \mu\|_\gamma &\leq \sup_{\|\varphi_1\|_\gamma,\, \|\varphi_2\|_\gamma \leq 1} \int [|\varphi_1(y)| P(x,dy) + |\varphi_2(y)|\mu(dy)] \\
&\leq \int (1 + \gamma\Phi(y))[P(x,dy) + \mu(dy)] \\
&\leq 2 + \gamma e^{-CT_0}\Phi(x) + \gamma D/C + \gamma C' \\
&\leq E'[1 + \Phi(x)],
\end{aligned}$$

for some $E' > 0$ independent of x. We now show that

$$\|P^n(x,\,\cdot\,) - \mu\|_\gamma \geq \min(1,\gamma)\|P^n(x,\,\cdot\,) - \mu\|.$$

To see this, note that

$$\begin{aligned}
\|P^n(x,\,\cdot\,) - \mu\|_\gamma &= \sup_{\|\varphi\|_\gamma \leq 1} \int \varphi(y)(P^n(x,dy) - \mu(dy)) \\
&\geq \begin{cases} \sup_{\|\varphi\| \leq 1} \int \varphi(y)(P^n(x,dy) - \mu(dy)) & \text{if } \gamma \geq 1, \\ \gamma \sup_{\|\varphi\| \leq 1} \int \varphi(y)(P^n(x,dy) - \mu(dy)) & \text{if } \gamma \in (0,1) \end{cases} \\
&\geq \min(1,\gamma)\|P^n(x,\,\cdot\,) - \mu\|.
\end{aligned}$$

Thus, in particular, we have shown that

$$\|P^n(x,\,\cdot\,) - \mu\| \leq E' \min(1,\gamma)^{-1}\rho^n[1 + \Phi(x)],$$

for all $x \in \mathbb{R}^d$, $n \in \mathbb{N}$. To introduce the continuous paramter $t \geq 0$ in the above, for $t \geq 0$ write $t = nT_0 + \delta$ where $\delta \in [0, T_0)$. By the Chapman-Kolmogorov equations, we obtain

$$\begin{aligned}
\|P(x,t,\,\cdot\,)-\mu\| &= \sup_{\|\varphi\|\le 1}\int \varphi(y)(P(x,t,dy)-\mu(dy))\\
&= \sup_{\|\varphi\|\le 1}\int \varphi(y)\left(\int P(x,\delta,dx')P(x',nT_0,dy)-\mu(dy)\right)\\
&= \sup_{\|\varphi\|\le 1}\int P(x,\delta,dx')\int \varphi(y)\left(P^n(x',dy)-\mu(dy)\right)\\
&= \int P(x,\delta,dx')\|P^n(x',\,\cdot\,)-\mu\|\\
&\le \int P(x,\delta,dx')E'\min(1,\gamma)^{-1}\rho^n[1+\Phi(x')]\\
&= E'\min(1,\gamma)^{-1}\rho^n\left[1+P_\delta\Phi(x)\right]\\
&\le E'\min(1,\gamma)^{-1}\rho^n\left[1+e^{-C\delta}\Phi(x)+D/C\right]
\end{aligned}$$

Using this, we can choose $\tilde{\rho}\in(\rho,1)$ and $E>0$ independent of x such that

$$\|P(x,t,\cdot)-\mu\|\le E\tilde{\rho}^t[1+\Phi(x)],$$

for all $t\ge 0$ and all $x\in\mathbb{R}^d$. This finishes the proof. □

Sometimes it is useful to know a precise rate of convergence and estimate. For this, we note:

Corollary 2.68. *Under the assumptions of the previous theorem, we have the bound:*

$$\|P(x,t,\,\cdot\,)-\mu\|\le \frac{1}{\min(1,\gamma)\rho}(\rho^{1/T_0})^t\left[2+\gamma e^{-CT_0}\Phi(x)+\gamma D/C(1+e^{-CT_0})+\gamma C'\right],$$

where $C'=\int\Phi(y)\,d\mu(y)$.

Proof. This follows easily from the proof above. □

With this implication in place, we now are in position to setup the main components of the proof of Theorem 2.66. With the norm $\|\cdot\|_\gamma$ on functions in mind, we define a metric on $\mathbb{R}^d$:

$$d_\gamma(x,y)=\begin{cases}0 & \text{if } x=y;\\ 2+\gamma\Phi(x)+\gamma\Phi(y) & \text{if } x\ne y.\end{cases}$$

Since $\Phi \geq 0$, it is easy to show that this is indeed a metric. Moreover, we define a Lipschitz semi-norm:

$$|||\varphi|||_\gamma = \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{d_\gamma(x,y)}.$$

Note that we have the following relationship between $|||\varphi|||_\gamma$ and $\|\varphi\|_\gamma$.

Lemma 2.69. *Suppose that $\|\varphi\|_\gamma < \infty$. We have*

$$|||\varphi|||_\gamma = \inf_{c \in \mathbb{R}} \|\varphi + c\|_\gamma.$$

Proof. It is easy to see that $|||\varphi|||_\gamma \leq \|\varphi\|_\gamma$ for all functions φ. Thus by the shift invariance of $|||\cdot|||$, we have $|||\varphi|||_\gamma \leq \inf_{c\in\mathbb{R}} \|\varphi + c\|_\gamma$. To see the other inequality, let $|||\varphi|||_\gamma = 1$ and define

$$c = \inf_{y \in \mathbb{R}^d} (1 + \gamma\Phi(y) - \varphi(y)).$$

Since $\|\varphi\|_\gamma < \infty$, it is easy to see that $c \in \mathbb{R}$. Note that

$$\varphi(x) + c \leq \varphi(x) + 1 + \gamma\Phi(x) - \varphi(x) = 1 + \gamma\Phi(x),$$

and

$$\begin{aligned}
\varphi(x) + c &= \inf_{y \in \mathbb{R}^d} (\varphi(x) + 1 + \gamma\Phi(y) - \varphi(y)) \\
&\geq \inf_{y \in \mathbb{R}^d} (-|\varphi(x) - \varphi(y)| + 1 + \gamma\Phi(y)) \\
&\geq \inf_{y \in \mathbb{R}^d} (-|||\varphi|||_\gamma d_\gamma(x,y) + 1 + \gamma\Phi(y)) \\
&= -1 - \gamma\Phi(x).
\end{aligned}$$

Thus $\|\varphi+c\|_\gamma \leq 1 = |||\varphi|||_c$ and hence we have $\inf_{c\in\mathbb{R}} \|\varphi+c\| \leq |||\varphi|||_\gamma$ as claimed. $\square$

We will now use $|||\cdot|||_\gamma$ to prove Theorem 2.66.

Proof of Theorem 2.66. We will first prove that

$$\int \Phi(y)\mu(dy) < \infty,$$

where μ is the unique invariant measure. By the proof of Theorem 2.32, since μ is unique it can be defined as a limit point in the weak topology of the sequence:

$$\mu_T(\,\cdot\,) = \frac{1}{T}\int_0^T P(x', s, \,\cdot\,)\, ds,$$

for some $x' \in \mathbb{R}^d$. Thus take a subsequence μ_{T_k} that converges weakly to μ as $k \to \infty$ and note that for all $k = 1, 2, \ldots$

$$\begin{aligned}
\int \Phi(y)\mu_{T_k}(dy) &= \frac{1}{T_k}\int_0^{T_k}\int P(x', s, dy)\Phi(y)\, ds \\
&= \frac{1}{T_k}\int_0^{T_k} P_s\Phi(x')\, ds \\
&\leq \frac{1}{T_k}D' + D/C,
\end{aligned}$$

for some $D' > 0$ independent of k. Note that the last inequality follows from Lemma 2.67. Applying Fatou's lemma, finishes the proof that $\int \Phi(y)\mu(dy) < \infty$. Moreover we can obtain the precise estimate:

$$C' = \int \Phi(y)\, \mu(dy) \leq D/C,$$

To show the contraction property, we first claim that it is enough to prove that there exist constants $\rho \in (0, 1)$ and $\gamma > 0$ such that

$$|||P\varphi|||_\gamma \leq \rho|||\varphi|||_\gamma.$$

Note that if ν_1, ν_2 are Borel probability measures we have for $c_\varphi = \inf_{y\in\mathbb{R}^d}(1+\gamma\Phi(y)-$

$\varphi(y)$):

$$\begin{aligned}
\|\nu_1 P - \nu_2 P\|_\gamma &= \sup_{\|\varphi\|_\gamma \le 1} \int \varphi(y)\,(\nu_1 P(dy) - \nu_2 P(dy)) \\
&= \sup_{\|\varphi\|_\gamma \le 1} \int P\varphi(x)(\nu_1(dx) - \nu_2(dx)) \\
&\le \sup_{|||\varphi|||_\gamma \le 1} \int P\varphi(x)(\nu_1(dx) - \nu_2(dx)) \\
&\le \sup_{|||\varphi|||_\gamma \le 1} \int \rho\varphi(x)(\nu_1(dx) - \nu_2(dx)) \\
&\le \sup_{||\varphi + c_\varphi||_\gamma \le 1} \int \rho(\varphi(x) + c_\varphi - c_\varphi)(\nu_1(dx) - \nu_2(dx)) \\
&= \sup_{\|\varphi + c_\varphi\|_\gamma \le 1} \int \rho(\varphi(x) + c_\varphi)(\nu_1(dx) - \nu_2(dx)) \\
&\le \rho\|\nu_1 - \nu_2\|_\gamma.
\end{aligned}$$

Thus all we have left to show is that there exist constants $\rho \in (0,1)$ and $\gamma > 0$ such that $|||P\varphi|||_\gamma \le \rho|||\varphi|||_\gamma$. By Lemma 2.67, we have:

$$P_{T_0}\Phi(x) \le e^{-CT_0}\Phi(x) + D/C.$$

Set $\alpha = e^{-CT_0} \in (0,1)$ and $\beta = D/C$. First suppose that $x \neq y$ are such that $\Phi(x) + \Phi(y) \ge R$. Then we have:

$$\begin{aligned}
|P\varphi(x) - P\varphi(y)| &\le P(|\varphi(x) - \varphi(y)|) \\
&\le |||\varphi|||_\gamma P(d_\gamma(x,y)) \\
&= (P(2 + \gamma\Phi(x) + \gamma\Phi(y)))|||\varphi|||_\gamma \\
&\le (2 + \gamma\alpha\Phi(x) + \gamma\alpha\Phi(y) + 2\gamma\beta)|||\varphi|||_\gamma.
\end{aligned}$$

Let $\epsilon > 0$ be small enough such that $\alpha(\epsilon) = \alpha + \epsilon < 1$. We then have the estimate:

$$|P\varphi(x) - P\varphi(y)| \le (2 + 2\gamma\beta - 2\gamma\epsilon R + \gamma\alpha(\epsilon)\Phi(x) + \gamma\alpha(\epsilon)\Phi(y))|||\varphi|||_\gamma.$$

Note that, if we choose $R = R(\beta, \epsilon) > 0$ large enough such that:

$$R > \beta/\epsilon,$$

we have the bound

$$\sup_{\{x\neq y, \Phi(x)+\Phi(y)\geq R\}} \frac{|P\Phi(x)-P\Phi(y)|}{d_\gamma(x,y)} \leq \rho_1 |||\varphi|||_\gamma,$$

where

$$\rho_1(\beta,\epsilon,\gamma) = \max\left\{1-\gamma(R\epsilon-\beta), \alpha(\epsilon)\right\}.$$

We now suppose that $x \neq y$ are such that $\Phi(x)+\Phi(y) \leq R$. Hence $x, y \in C_R$. By Lemma 2.69, we may assume that $\|\varphi\|_\gamma = 1$. Note then we can decompose φ into two functions $\varphi = \varphi_1 + \varphi_2$ such that $|\varphi_1(x)| \leq 1$ and $|\varphi_2(x)| \leq \gamma\Phi(x)$ for all $x \in \mathbb{R}^d$. Moreover, by Assumption 2.63, for $x \in C_R$ we may define another transition kernel:

$$Q(x,\,\cdot\,) = \frac{1}{1-\alpha_R}P(x,\,\cdot\,) - \frac{\alpha_R}{1-\alpha_R}\nu(\,\cdot\,).$$

Hence we have

$$\begin{aligned} |P\varphi(x)-P\varphi(y)| &\leq (1-\alpha_R)|Q\varphi_1(x)-Q\varphi_1(y)| + (1-\alpha_R)|Q\varphi_2(x)-Q\varphi_2(y)| \\ &\leq 2(1-\alpha_R) + \gamma\alpha\Phi(x) + \gamma\alpha\Phi(y) + 2\gamma\alpha\beta. \end{aligned}$$

Hence choose $\gamma > 0$ small enough such that

$$\gamma < \frac{\alpha_R}{\alpha\beta}.$$

Hence we obtain:

$$\sup_{\{x\neq y,\, \Phi(x)+\Phi(y)\leq R\}} \frac{|P\varphi(x)-P\varphi(y)|}{d\gamma(x,y)} \leq \rho_2,$$

where

$$\rho_2 = \max\left\{1-\alpha_R+\gamma\alpha\beta, \alpha\right\}.$$

Hence we have shown the result for

$$\rho = \max\left\{1-\gamma(R\epsilon - D/C),\, e^{-CT_0}+\epsilon,\, 1-\alpha_R+\gamma e^{-CT_0}D/C\right\},$$

where ϵ, R, $\gamma > 0$ are such that

$$e^{-CT_0} + \epsilon < 1$$

$$R > D/(C\epsilon)$$

$$\gamma < \frac{\alpha_R}{e^{-CT_0} D/C}.$$

□

Remark 2.70. We note that if Assumption 2.62 and Assumption 2.63 are satisfied with Φ such that $\Phi(x) \to \infty$ as $|x| \to \infty$, the conclusion of Theorem 2.64 is much stronger than Property 2.61. This is because the norm $\|\cdot\|$ which quantifies the convergence rate is taken over the supremum of a wider class of functions than just bounded functions. These functions are moreover permitted to approach infinity (of course no faster than $1 + \Phi(x)$) as $|x| \to \infty$. In the case of equation 1.2, we will be able to prove the existence of such a Φ and hence conclude not only Property 2.61, but also the bound in Theorem 2.64.

With the ideas of Chapter 2 in place, we now proceed onto proving properties about the system 1.2.

Chapter 3
Proof of Main Theorem

3.1 Introduction

Let $n \geq 2$. We now prove Theorem 1.5. To do so, we will show the following two lemmata:

Lemma 3.1. *Suppose that $(\kappa_1, \kappa_2) \in \mathbb{C}^2$ is transversal to D_{n-1}, then*

1. *The process $z(t)$ is non-explosive.*
2. *There is one and only one invariant probability measure μ.*
3. *$z(t)$ satisfies Property 2.61.*

Lemma 3.2. *If $(\kappa_1, \kappa_2) \in \mathbb{C}^2$ is not transversal to D_{n-1}, there exists a set of initial conditions $A \neq \emptyset \subset \mathbb{C}$ such that*

$$P_z\{\xi < \infty\} > 0,$$

for all $z \in A$.

We first attend to Lemma 3.1; the proof of which comprises five sections. Identifying $\mathbb{C}$ with $\mathbb{R}^2$ throughout, the first four sections focus entirely on exhibiting a smooth function $\Phi : \mathbb{C} \to [0, \infty)$ with positive constants C and D such that:

(C1) $$\Phi(z, \bar{z}) \to \infty \text{ as } |z| \to \infty,$$

(C3) $$L\Phi(z, \bar{z}) \leq -C\Phi(z, \bar{z}) + D \text{ for all } z \in \mathbb{C}.$$

By Theorem 2.8 and Theorem 2.34, the existence of Φ proves the process $z(t)$ is non-explosive and has invariant probability measures. If, in addition, κ_1 and κ_2 span the entire complex plane over $\mathbb{R}$, it is not hard to see that the existence of Φ also guarantees exponential ergodicity of $z(t)$ [RB06]. However, we are also interested in cases where κ_1 and κ_2 are linearly dependent over $\mathbb{R}$. Hence different methods are employed in Section 3.6 to validate parts 2 and 3 of Lemma 3.1.

To prove Lemma 3.2, we spend a short sixth section using Feller's test as in Section 2.2.2.

3.2 Lyapunov Coverings

The existence of a smooth function $\Phi : \mathbb{C} \to [0, \infty)$ that satisfies **(C1)** and **(C3)** is shown by an explicit construction; the first part of which is to find locally-defined test functions which satisfy local versions of **(C1)** and **(C3)** on their respective domains. If the union of these domains essentially covers $\mathbb{C}$, we have almost, but not quite, assured stability of the process $z(t)$. To proceed, we need the following terminology:

Definition 3.3. Let $U \subset \mathbb{C}$ be an unbounded domain with continuous boundary ∂U. Suppose that $\varphi : U \to \mathbb{R}$ satisfies:

(I) $\varphi \in C^\infty(U)$.

(II) $\varphi(z, \bar{z}) \to \infty$ as $|z| \to \infty$, $z \in U$.

(III) There exist positive constants c and d such that

$$L\varphi(z, \bar{z}) \leq -c\varphi(z, \bar{z}) + d \text{ for all } z \in U.$$

We call φ a **Lyapunov function** on U. If there exists a sequence of Lyapunov functions $\varphi_1, \varphi_2, \ldots, \varphi_k$ on $U_1, U_2, \ldots, U_k$ respectively such that

$$\mathbb{C} = \bigcup_{j=1}^{k} U_j \cup B_R(0),$$

for some $R > 0$, we call $\{(\varphi_1, U_1), (\varphi_2, U_2), \ldots, (\varphi_k, U_k)\}$ a **Lyapunov covering**.

From this definition, it is intuitively clear that the existence of a Lyapunov covering provides some handle on the behavior of the process $z(t)$. To illustrate this, we prove the next lemma. Let us use ξ_n to denote the first exit time of $z(t)$ from $B_n(0)$ and ξ to denote the finite or infinite limit of ξ_n as $n \to \infty$.

Lemma 3.4. *Suppose φ is a Lyapunov function on U. Let $V \subset U$ be another unbounded region such that ∂V is continuous and $\partial U \cap \partial V = \emptyset$. If $\xi_V = \inf_{t>0}\{z(t) \in V^c\}$, then for $z \in V$*

$$P_z\{\xi_V < \xi\} = 1.$$

Proof. Let $\xi_U = \inf_{t>0}\{z(t) \in U^c\}$ and $\xi_{n,U}(t) = \xi_n \wedge \xi_U \wedge t$. By Definition 3.3, it is clear that the much weaker bound $L\varphi(z,\bar{z}) \le c\varphi(z,\bar{z}) + d$ for some $c, d > 0$ is satisfied on U. Letting $\psi(t,z,\bar{z}) = e^{-ct}(\varphi(z,\bar{z}) + \frac{d}{c})$, for $z \in V$ we obtain by Dynkin's formula

$$\begin{aligned} E_z\left[\psi(\xi_{n,U}(t), z(\xi_{n,U}(t)), \overline{z(\xi_{n,U}(t))})\right] - \psi(0,z,\bar{z}) &= E_z\left[\int_0^{\xi_{n,U}(t)} L\psi(s, z(s), \overline{z(s)})\, ds\right] \\ &\le 0. \end{aligned}$$

We see that for n sufficiently large:

$$E_z\left[\psi(\xi_{n,U}(t), z(\xi_{n,U}(t)), \overline{z(\xi_{n,U}(t))})\right] \;\ge\; e^{-ct}\varphi_n P_z\{\xi_n \le \xi_U \wedge t\},$$

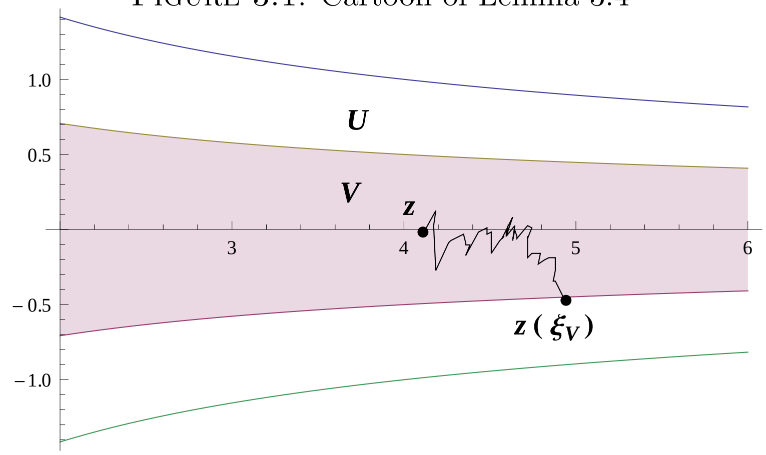
where $\varphi_n = \inf_{z \in U \cap \partial B_n} \varphi(z,\bar{z}) \to \infty$ as $n \to \infty$. Combining the previous two estimates yields

$$P_z\{\xi_n \le \xi_U \wedge t\} \le \varphi_n^{-1} e^{ct}\psi(0,z,\bar{z}).$$

Letting $n \to \infty$ we see that $P_z\{\xi \le \xi_U \wedge t\} = 0$ for all $t \ge 0$. Whence $P_z\{\xi \ge \xi_U\} = 1$. By path continuity of $z(t)$, $P_z\{\xi_U > \xi_V\} = 1$. Hence $P_z\{\xi > \xi_V\} = 1$ as claimed. □

Remark 3.5. Note that the same behavior is true if **(III)** of Definition 3.3 is replaced by a much weaker bound. Indeed, suppose U is as in Definition 3.3 and $\varphi : U \to \mathbb{R}$

Figure 3.1. Cartoon of Lemma 3.4



satisfies **(I)** and **(II)** of Definition 3.3 on U. If φ satisfies the bound

$$L\varphi(z,\bar{z}) \le c\varphi(z,\bar{z}) + d,$$

for all $z \in U$ for some $c, d > 0$, then the conclusion of Lemma 3.4 remains valid.

The existence of a Lyapunov covering $\{(\varphi_j, U_j\}_{j=1}^{k}$ guarantees $z(t)$ cannot leave $\mathbb{C}$ directly through any unbounded region $V \subset U_j$ with continuous boundary such that $\partial V \cap \partial U_j = \emptyset$. As we shall see in our case, it is possible to extract subsets $V_j \subset U_j$ with continuous boundary such that $\partial U_j \cap \partial V_j = \emptyset$ for all $j = 1, 2, \dots, k$ and $\{(\varphi_j, V_j)\}_{j=1}^{k}$ is a Lyapnov covering.

Definition 3.6. Suppose that $\{(\varphi_j, V_j)\}_{j=1}^{k}$ is as above. We call $\{(\varphi_j, V_j)\}_{j=1}^{k}$ a **strong Lyapunov covering subordinate to** $\{(\varphi_j, U_j)\}_{j=1}^{k}$.

Hence, given the existence of a strong Lyapunov covering $\{(\varphi_j, V_j)\}_{j=1}^{N}$, $z(t)$ cannot exit $\mathbb{C}$ directly through any V_j. Moreover

$$\mathbb{C} = \bigcup_{j=1}^{N} V_j \cup B_R(0),$$

for some $R > 0$. Thus it seems likely that $z(t)$ cannot exit $\mathbb{C}$ in finite time. However, it is possible $z(t)$ could oscillate between two or more regions on its way to infinity in finite time. As we shall see, to eliminate this rare event we must glue our Lyapunov covering together so that we have a Lyapunov function Φ on all of $\mathbb{C}$. If necessary, by adding a sufficiently large constant to Φ, we have a nonnegative smooth function that satisfies **(C1)** and **(C3)**.

The construction of a Lyapunov function Φ on $\mathbb{C}$ naturally splits into two stages:

Stage 1: Existence of a Lyapunov covering from which we can extract a subordinate strong Lyapunov covering.

Stage 2: Existence of a Lyapunov function Φ on all of $\mathbb{C}$ as a glued version of the Lyapunov covering.

Because the Lyapunov covering presented here involves some non-standard regions in the complex plane, we will first define, graph, and prove some properties of these sets separate from their associated Lyapunov functions. This is done to amend the procedure as a whole. Also, we prove a lemma which allows us to vary the magnitude of the diffusion present in (1.2). We then proceed onto Stage 1 and Stage 2 in that order.

3.3 Lyapunov Regions and Scaling

3.3.1 Lyapunov Regions

We now provide the regions of definition for the Lyapunov functions that will follow in the next section. To this end, for $z \neq 0 \in \mathbb{C}$ let $\theta_z = \arg(z)$. Of course, θ_z is multi-valued; however, its multi-valuedness plays no significant role in the arguments. Since there is some symmetry in (1.2), it is convenient to use g to denote a primitive $(n-1)$st root of unity. Moreover, we choose $R > 0$ sufficiently large which will be made precise later.

For the first region, we define a constant:

$$\eta_n = \frac{2(n-1)}{\pi(e-1)} > 0,$$

and the reason for its chosen value will be more apparent later in this section.

Region 1.

$$U_1 = B_R(0)^c \cap \left\{ z \neq 0 \in \mathbb{C} \,:\, \cos((n-1)\theta_z) \leq -\eta_n |z|^{-1} \right\}$$

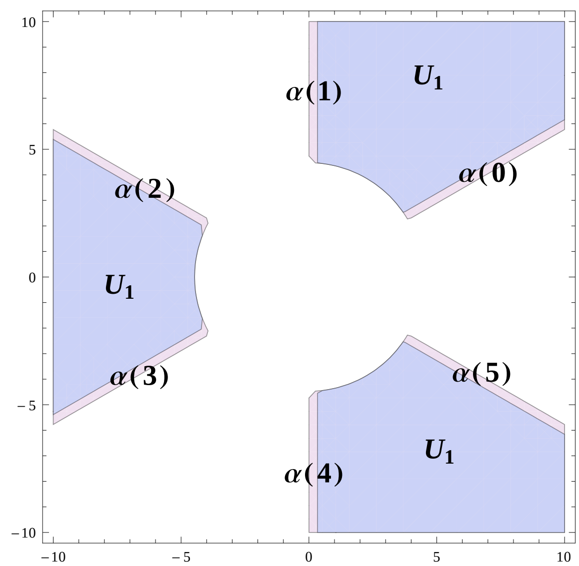


FIGURE 3.2. The region U_1 (in blue) for $n = 4$. The pink represents the distance to the rays $\alpha(k)$ defined below.

Remark 3.7. For $|z|$ large, one should think of U_1 as essentially all points where

$$\cos((n-1)\theta_z) < 0.$$

The power -1 in $|z|^{-1}$ allows us to get reasonably close to where $\cos((n-1)\theta_z) = 0$. The constant η_n allows us to be precise about how close.

To define the second region, for $k \in \mathbb{Z}$ we introduce angles

$$\alpha(k) = \frac{\pi}{2(n-1)} + \frac{\pi k}{n-1},$$

and let

$$U_2(\alpha(k)) = B_2(0) + \{z \neq 0 \in \mathbb{C} : \theta_z = \alpha(k)\}.$$

We now have:

Region 2.

$$U_2 = \bigcup_{k\in\mathbb{Z}} U_2(\alpha(k)) \cap B_R(0)^c.$$

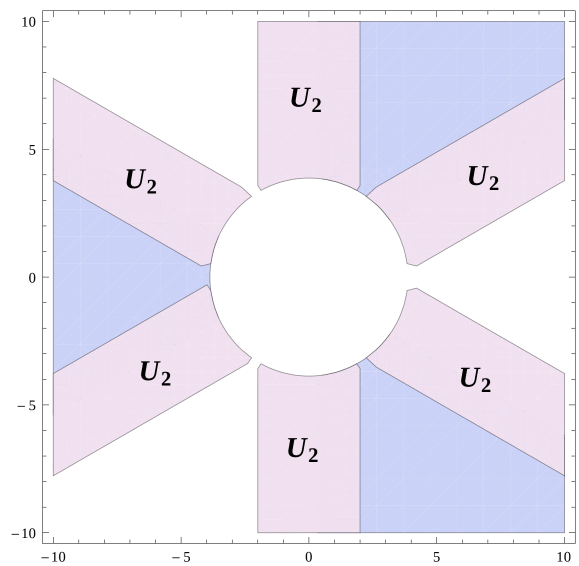


FIGURE 3.3. The region U_2 for $n = 4$. It covers the rays $\alpha(k)$ for all $k \in \mathbb{Z}$ and also overlaps U_1 by the choice of η_n.

Region 3.

$$U_3 = \left\{z \neq 0 \in \mathbb{C} : \min\{|\sin((n-1)\theta_z)|, \cos((n-1)\theta_z)\} \geq \eta_n |z|^{-1}\right\}.$$

Remark 3.8. Hence U_3 is similar in some sense to U_1. We see that we are within the realm where $\cos((n-1)\theta_z) > 0$ and can get close to the rays where $\cos((n-1)\theta_z) = 0$ or $\sin((n-1)\theta_z) = 0$, but only as close as $\eta_n|z|^{-1}$ permits.

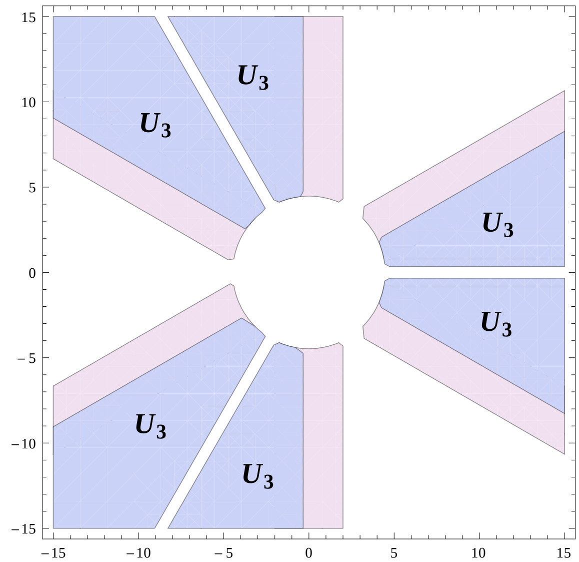


FIGURE 3.4. The region U_3 with $n = 4$. With the choice of η_n, U_3 overlaps U_2.

For the fourth region, we use the variables $(x, y) \in \mathbb{R}^2$ instead of $z = x + iy$. We first define a ground region:

$$U_4(0) = \{x \geq 1\} \cap \left\{|x|^{-(n-1)/2} \leq |y| \leq 2\right\}$$

and rotate it to produce the entire U_4.

Region 4.

$$U_4 = \bigcup_{k \in \mathbb{Z}} (g^k U_4(0)) \cap B_R(0)^c.$$

Remark 3.9. Thus we take $U_4(0)$ and rotate it by integer multiples of the angle $2\pi/(n-1)$. We do the same for the next region.

Let

$$U_5(0) = \{x \geq 1\} \cap \left\{|y| \leq 2|x|^{-(n-1)/2}\right\}$$

Region 5.

$$U_5 = \bigcup_{k \in \mathbb{Z}} (g^k U_5(0)) \cap B_R(0)^c.$$

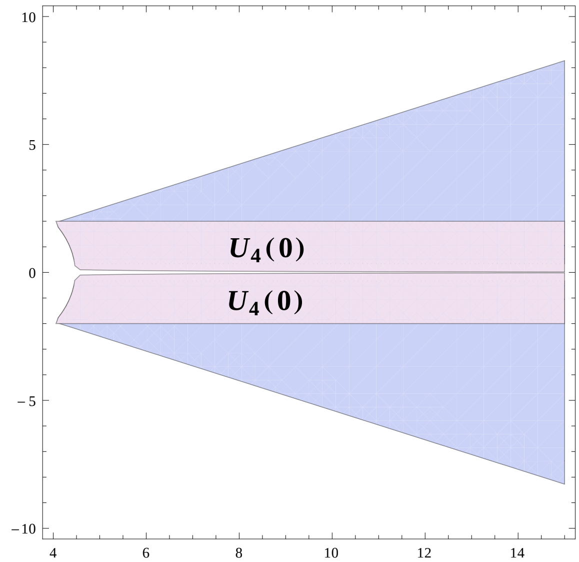


FIGURE 3.5. The region $U_4(0)$ with $n = 4$. Although it is hard to see, there is a tiny space that still needs to be covered.

With these definitions in place, we prove there exists $S > 0$ such that

$$\mathbb{C} = \bigcup_{j=1}^{5} U_j \cup B_S(0).$$

Before we do this, we quantify the word "close" used above in the remarks. For $k \in \mathbb{Z}$, define angles

$$\beta(k) = \frac{\pi k}{n-1}.$$

Proposition 3.10. *For $K, R > 0$ and $l \in \mathbb{Z}$, consider the sets*

$$\begin{aligned}
V_1(K,R,l) &= \left\{z \neq 0 \in \mathbb{C} : |\cos((n-1)\theta_z)| = K|z|^{-1}\right\} \cap \left\{|\theta_z - \alpha(l)| \leq R^{-1}\right\} \\
V_2(K,R,l) &= \left\{z \neq 0 \in \mathbb{C} : |\sin((n-1)\theta_z)| = K|z|^{-1}\right\} \cap \left\{|\theta_z - \beta(l)| \leq R^{-1}\right\} \\
W_1(l) &= \{z \neq 0 \in \mathbb{C} : \theta_z = \alpha(l)\} \\
W_2(l) &= \{z \neq 0 \in \mathbb{C} : \theta_z = \beta(l)\}.
\end{aligned}$$

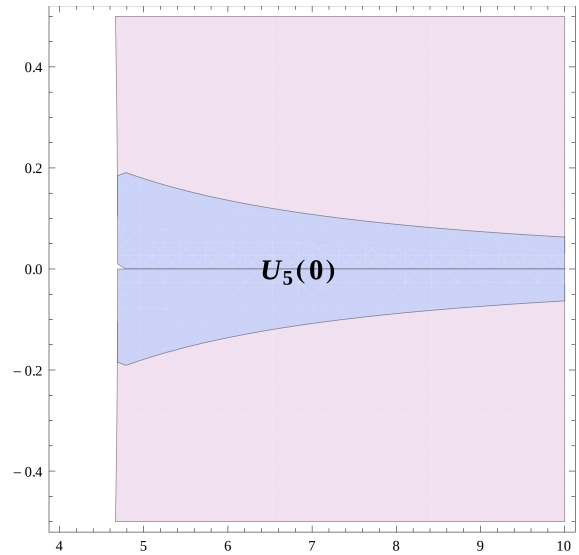


FIGURE 3.6. The region $U_5(0)$ for $n = 4$

Then there exists $R' > 0$ such that for all $R \geq R'$ and all $l \in \mathbb{Z}$

$$\begin{aligned} \operatorname{dist}\left(V_1(K,R,l), W_1(l)\right) &\leq K\frac{\pi(e-1)}{2(n-1)} \\ \operatorname{dist}\left(V_2(K,R,l), W_2(l)\right) &\leq K\frac{\pi(e-1)}{2(n-1)}. \end{aligned}$$

Proof. We will only prove there exists $R' > 0$ such that for all $R \geq R'$:

$$\operatorname{dist}\left(V_1(K,R,0), W_1(0)\right) \leq K\frac{\pi(e-1)}{2(n-1)},$$

as the other proofs are nearly identical. By definition, for $R' > 1$ sufficiently large any $z \in V_1(K,R,0)$ for $R \geq R'$ has $|z| \geq 2K$. Fix $R' > 1$ such that the prior occurs and note that for $R \geq R'$ and $z \in V_1(K,R,0)$ we have the estimates:

$$\begin{aligned}
||z|e^{i\theta_z} - |z|e^{i\alpha(0)}| &= |z||e^{i(\theta_z-\alpha(0))} - 1| \\
&= |z||\theta_z - \alpha(0)| \left| \sum_{j=1}^{\infty} i^n \frac{(\theta_z - \alpha(0))^{n-1}}{n!} \right| \\
&\leq (e-1)|z||\theta_z - \alpha(0)| \\
&= \frac{e-1}{n-1}|z| \left| (n-1)\theta_z - \frac{\pi}{2} \right| \\
&= \frac{e-1}{n-1}|z| \left| \arccos(\pm K|z|^{-1}) - \frac{\pi}{2} \right| \\
&= \frac{e-1}{n-1}|z| \left| \arcsin(\pm K|z|^{-1}) \right| \\
&= \frac{e-1}{n-1}|z| \left| \sum_{j=0}^{\infty} \left(\frac{(2j)!}{2^{2j}(j!)^2} \frac{(\pm K|z|^{-1})^{2j+1}}{(2j+1)} \right) \right| \\
&\leq K\frac{e-1}{n-1}|\arcsin(1)| \\
&= K\frac{\pi(e-1)}{2(n-1)}.
\end{aligned}$$

Thus since $z' = |z|e^{i\alpha(0)} \in W_1(0)$, we have finished the proof. □

Thus the choice of $K = \eta_n$ now becomes clear. We now are in position to prove:

Proposition 3.11. *There exists $S > 0$ such that*

$$\mathbb{C} = \bigcup_{j=1}^{5} U_j \cup B_S(0).$$

Proof. Let $z \neq 0 \in \mathbb{C}$. There are three cases:

Case 1. $\cos((n-1)\theta_z) = 0$.

Case 2. $\cos((n-1)\theta_z) < 0$.

Case 3. $\cos((n-1)\theta_z) > 0$.

Case 1. Hence $\theta_z = \alpha(k)$ for some $k \in \mathbb{Z}$. Thus for $|z| \geq R$, $z \in U_2$.

Case 2. For $|z| \geq R$, either $\cos((n-1)\theta_z) \leq -\eta_n|z|^{-1}$ or $\cos((n-1)\theta_z) \in (-\eta_n|z|^{-1}, 0)$. In the first case, $z \in U_1$. In the second case, for $|z|$ sufficiently large, by the proposition above, z is within distance 1 unit to one of the rays $\alpha(k)$ for some $k \in \mathbb{Z}$. Hence, $z \in U_2$.

Case 3. For $|z| \geq R$, either $\cos((n-1)\theta_z) \in (0, \eta_n|z|^{-1})$ or $\cos((n-1)\theta_z) \geq \eta_n|z|^{-1}$. In the first case, by the proposition, for $|z|$ sufficiently large, z is within distance 1 unit of one of the rays $\alpha(k)$ for some $k \in \mathbb{Z}$. Hence $z \in U_2$. In the second case, either $\cos((n-1)\theta_z) \geq \eta_n|z|^{-1}$ and $|\sin((n-1)\theta_z)| \geq \eta_n|z|^{-1}$, or $\cos((n-1)\theta_z) \geq \eta_n|z|^{-1}$ and $|\sin((n-1)\theta_z)| < \eta_n|z|^{-1}$. In the first case, $z \in U_3$. In the second case, by the proposition for $|z|$ sufficiently large, z is within distance 1 unit of one of the rays $\beta(k)$ for some $k \in \mathbb{Z}$. Since $\cos((n-1)\theta_z) > 0$, z must be within distance 1 unit of one of the rays $2\pi k/(n-1)$ for some $k \in \mathbb{Z}$. Hence, $z \in U_4 \cup U_5$ for $|z|$ sufficiently large. □

3.3.2 Scaling

It is important to realize that the diffusion process $z(t)$ defined by equation (1.2) depends on $(\kappa_1, \kappa_2) \in \mathbb{C}^2$. It is; however, extremely easy to forget about this when constructing Φ. To emphasize, Φ can (and should) depend on κ_1 and κ_2. It will be convenient in the next section to work with diffusion parameters that are sufficiently small which is why we provide the next lemma. To note that the operator L depends on $\boldsymbol{\kappa} = (\kappa_1, \kappa_2)$ we will write $L^{\boldsymbol{\kappa}}$. We have the following:

Lemma 3.12. *Suppose that Φ is a Lyapunov function on $\mathbb{C}$ corresponding to the operator $L^{\boldsymbol{\kappa}}$. For all parameters $\boldsymbol{\iota} = (\iota_1, \iota_2)$ such that $\boldsymbol{\iota} = \lambda\boldsymbol{\kappa}$ for some $\lambda > 0$, there exists a Lyapunov function Ψ corresponding to the operator $L^{\boldsymbol{\iota}}$.*

Proof. Let $\tilde{\lambda} > 0$ be such that $\lambda = \tilde{\lambda}^{-(n+1)/2}$. Define $\Psi(z, \bar{z}) = \Phi(\tilde{\lambda}z, \tilde{\lambda}\bar{z})$. Note easily that Ψ satisfies properties **(I)** and **(II)**. To verify **(III)**, let $w = \tilde{\lambda} z$ and note:

$$\begin{aligned} L^{\boldsymbol{\iota}}\Psi(z,\bar{z}) &= \frac{1}{\tilde{\lambda}^{n-1}} L^{\boldsymbol{\kappa}}\Phi(w,\bar{w}) \\ &\leq -C\Psi(z,\bar{z}) + D, \end{aligned}$$

for some $C, D > 0$, as required. □

3.4 Stage 1: A (Strong) Lyapunov Covering

In this section we prove the existence of a Lyapunov covering. From what follows, it is very easy to extract a strong Lyapunov covering, so we will eliminate this minor detail to focus solely on constructing Lyapunov functions on U_1, U_2, U_3, U_4, and U_5.

We first define a multitude of constants which depend on the diffusion parameters κ_1 and κ_2. In view of Lemma 3.12, we will also choose $|\kappa_1|^2 + |\kappa_2|^2$ sufficiently small so that certain estimates that follow are valid. In what follows, it is perhaps easiest to skip this part and proceed onto the proofs that φ_1, φ_2, φ_3, φ_4, and φ_5 are Lyapunov functions on their respective domains. One can then refer back to the subsection of constants when needed.

3.4.1 Constants

Let

$$m(\kappa_1,\kappa_2) = \min_{j\in\mathbb{Z}} \left(\operatorname{Im}(g^j\kappa_1)^2 + \operatorname{Im}(g^j\kappa_2)^2\right).$$

Note that since the pair (κ_1,κ_2) is transversal to D_{n-1}, we have $m(\kappa_1,\kappa_2) > 0$. We can now define positive constants δ, d, and f from this:

$$\begin{aligned} \delta &= \frac{m(\kappa_1,\kappa_2)}{12\left(2+\frac{4}{n-1}\right)\left(1+\frac{3}{8}(n-1)\right)} \\ d &= \frac{m(\kappa_1,\kappa_2)}{12\left(1+\frac{3}{8}(n-1)\right)} = 2\delta + \frac{4\delta}{n-1} \\ f &= \frac{m(\kappa_1,\kappa_2)}{12} = d + \frac{3}{8}d(n-1). \end{aligned}$$

Choose $|\kappa_1|^2 + |\kappa_2|^2 > 0$ small enough such that

1. $\delta \in \left(0, \frac{1}{2}\right)$

2. $-\left(\frac{3}{4}n - 1\right) + (|\kappa_1|^2 + |\kappa_2|^2)\frac{3}{8}\left(\frac{3}{4}d + 1\right) < -\frac{1}{4}$

3. $2^{\frac{3}{4}d} \leq 2$

4. $\eta_n^{4\delta/(n-1)} \geq 1/2$.

We can then choose positive constants C_1, C_2, …, C_5 independent of κ_1 and κ_2:

1. $C_2 = 2 > 1 = C_1$

2. $C_3 = 3 > 2 = C_2$

3. $C_4 = 25 > 24 = 8C_3$.

4. $2C_5 = 26 < 25 = C_4$.

We choose $R > 0$ sufficiently large as in the previous section and moreover to assure that the regions $U_2(\alpha(k)) \cap B_R^c(0)$ are disjoint for all $k \in \mathbb{Z}$ and such that $|\sin((n-1)\theta_z)| \geq 1/2$ in U_2.

3.4.2 Lyapunov Functions

Lyapunov Function 1. Let $\varphi_1(z, \bar{z}) = C_1|z|^{2\delta}$. Then φ_1 is a Lyapunov function on U_1.

Proof. Since $|z| \neq 0$ on U_1, it follows that $\varphi_1 \in C^\infty(U_1)$. Moreover, it is clear that $\varphi_1 \to \infty$ as $|z| \to \infty$, $z \in U_1$. Thus φ_1 satisfies properties **(I)** and **(II)** on U_1. To see that property **(III)** is valid on U_1, note that

$$\begin{aligned}
\frac{1}{C_1} L\varphi_1(z,\bar{z}) &= 2\delta|z|^{2\delta-2}\operatorname{Re}\left(\bar{z}z^n\right) + \frac{1}{2}\delta(\delta-1)(\kappa_1^2+\kappa_2^2)|z|^{2\delta-4}\bar{z}^2 \\
&\quad + \delta^2(|\kappa_1|^2+|\kappa_2|^2)|z|^{2\delta-2} + \frac{1}{2}\delta(\delta-1)(\overline{\kappa_1}^2+\overline{\kappa_2}^2)z^2|z|^{2\delta-4} \\
&= 2\delta|z|^{2\delta}\operatorname{Re}(z^{n-1}) + \frac{1}{2}\delta(\delta-1)(\kappa_1^2+\kappa_2^2)|z|^{2\delta-4}\bar{z}^2 \\
&\quad + \delta^2(|\kappa_1|^2+|\kappa_2|^2)|z|^{2\delta-2} + \frac{1}{2}\delta(\delta-1)(\overline{\kappa_1}^2+\overline{\kappa_2}^2)z^2|z|^{2\delta-4} \\
&\leq 2\delta|z|^{2\delta}\operatorname{Re}\left(z^{n-1}\right) + \left(\delta|\delta-1|+\delta^2\right)\left(|\kappa_1|^2+|\kappa_2|^2\right)|z|^{2\delta-2} \\
&= 2\delta|z|^{n+2\delta-1}\cos((n-1)\theta_z) + \left(\delta|\delta-1|+\delta^2\right)\left(|\kappa_1|^2+|\kappa_2|^2\right)|z|^{2\delta-2} \\
&\leq -\frac{\delta\eta_n}{C_1}|z|^{n-2}\varphi_1 + \frac{d_1}{C_1}
\end{aligned}$$

for some constant $d_1 > 0$. Thus we have the bound

$$\begin{aligned}
L\varphi_1(z,\bar{z}) &\leq -\delta\eta_n|z|^{n-2}\varphi_1(z,\bar{z}) + d_1 \\
&\leq -\delta\eta_n\varphi(z,\bar{z}) + d_1
\end{aligned} \tag{3.13}$$

on U_1 since $n \geq 2$ and $|z| \geq 1$ on U_1. □

Lyapunov Function 2. Let $\varphi_2(z,\bar{z})$ be a function defined on U_2 by:

$$\varphi_2(z,\bar{z}) = C_2\left(\left|\operatorname{Im}(z^{n-1})\right|^{\frac{2\delta}{n-1}} + 2\operatorname{Re}(e^{i\gamma(k)}z)\right),\ z \in U_2(\alpha(k)) \cap B_R(0)^c$$

where

$$\gamma(k) = \begin{cases} \frac{\pi}{2} - \frac{\pi}{2(n-1)} - \frac{\pi k}{n-1} & \text{for } k \text{ even} \\ \frac{3\pi}{2} - \frac{\pi}{2(n-1)} - \frac{\pi k}{n-1} & \text{for } k \text{ odd.} \end{cases}$$

Then φ_2 is a Lyapunov function on U_2.

Remark 3.14. This is where we use the choice of $\delta \in (0, 1/2)$.

Proof. By the definition of U_2, we have $\operatorname{Im}(z^{n-1}) \neq 0$ and for $k \in \mathbb{Z}$ the regions $U_2(\alpha_1(k)) \cap B_R(0)^c$ are disjoint, hence $\varphi_2 \in C^\infty(U_2)$. For $z \in U_2$, write $z = e^{i\alpha(k)}r + w$

with $r > 0$ and $|w| \leq 2$. Since $|\sin((n-1)\theta_z)| \geq 1/2$ in U_2 and $\delta < 1/2$, we have:

$$\begin{aligned}\frac{1}{C_2}\varphi_2(z,\bar{z}) &\geq \frac{1}{2}|z|^{2\delta} + 2\,\mathrm{Re}\left(e^{i(\gamma(k)+\alpha_1(k))}r + e^{i\gamma(k)}w\right)\\ &\geq \frac{1}{2}|z|^{2\delta} + 0 - 4.\end{aligned}$$

Thus φ_2 satisfies **(I)** and **(II)** in U_2. Since $\delta \leq 1$, it is easy to check the second order terms in $C_2^{-1}L\varphi_2(z,\bar{z})$ are bounded above by a constant $E_2 > 0$. After setting $a = 2\delta/(n-1)$, we then have

$$\begin{aligned}\frac{1}{C_2}L\varphi_2(z,\bar{z}) &\leq E_2 + a(n-1)\,\mathrm{Sign}(\mathrm{Im}(z^{n-1}))\left|\mathrm{Im}(z^{n-1})\right|^{a-1}\mathrm{Im}(z^{2n-2})\\ &\quad + 2\,\mathrm{Re}(e^{i\gamma(k)}z^n)\\ &\leq E_2 + a(n-1)\left|\mathrm{Im}(z^{n-1})\right|^{a-1}|z|^{2n-2}\\ &\quad + 2|z|^n\cos(n\theta_z + \gamma(k))\\ &\leq E_2 + 2^{1/a}a(n-1)|z|^{n-1+2\delta}\\ &\quad + 2|z|^n\cos(n\theta_z + \gamma(k)).\end{aligned}$$

Note that as $|z| \to \infty$, $z \in U_2(\alpha(k))$, $n\theta_z + \gamma(k) \to \pi + 2\pi j$ for some $j \in \mathbb{Z}$. Thus for some $d_2' > 0$ we have the bound for $z \in U_2$:

$$\frac{1}{C_2}L\varphi_2(z,\bar{z}) \leq -|z|^n + \frac{d_2'}{C_2}.$$

Therefore there exists $d_2 > 0$ such that:

$$\begin{aligned}L\varphi_2(z,\bar{z}) &\leq -C_2|z|^n + d_2'\\ &\leq -|z|^{n-2\delta}\varphi_2(z,\bar{z}) + d_2 \qquad (3.15)\\ &\leq -\varphi_2(z,\bar{z}) + d_2,\end{aligned}$$

on U_2 since $|z| \geq 1$ on U_2. □

Lyapunov Function 3. Let

$$\varphi_3(z,\bar{z}) = C_3\frac{|z|^{6\delta}}{|\mathrm{Im}(z^{n-1})|^{\frac{4\delta}{n-1}}}.$$

Then φ_3 is a Lyapunov function on U_3.

Proof. Note that φ_3 is smooth in U_3 since $\operatorname{Im}(z^{n-1}) \neq 0$ in this region. Moreover, we have the bound

$$\varphi_3(z,\bar{z}) \geq C_3|z|^{2\delta}.$$

Thus φ_3 satisfies properties **(I)** and **(II)** in U_3. Let, for simplicity, $2b = 6\delta$ and $c = 4\delta/(n-1)$. To verify **(III)**, note that the second order terms in $C_3^{-1}L\varphi_3(z,\bar{z})$ are bounded above by

$$E_3(|\kappa_1|^2 + |\kappa_2|^2)\frac{|z|^{2n+2b-4}}{|\operatorname{Im}(z^{n-1})|^{c+2}},$$

for some constant $E_3 > 0$ independent of κ_1, κ_2. Thus we have

$$\begin{aligned} \frac{1}{C_3}L\varphi_3(z,\bar{z}) &\leq E_3(|\kappa_1|^2 + |\kappa_2|^2)\frac{|z|^{2n+2b-4}}{|\operatorname{Im}(z^{n-1})|^{c+2}} + 2b\frac{|z|^{2b}}{|\operatorname{Im}(z^{n-1})|^{c}}\operatorname{Re}\left(z^{n-1}\right) \\ &\quad - c(n-1)\operatorname{Sign}(\operatorname{Im}(z^{n-1}))\frac{|z|^{2b}}{|\operatorname{Im}(z^{n-1})|^{c+1}}\operatorname{Im}\left(z^{2n-2}\right) \\ &= \frac{1}{C_3}\varphi_3(z,\bar{z})\Big[E_3(|\kappa_1|^2 + |\kappa_2|^2)\frac{|z|^{2n-4}}{|\operatorname{Im}(z^{n-1})|^2} + 2b\operatorname{Re}(z^{n-1}) \\ &\quad - c(n-1)\operatorname{Sign}(\operatorname{Im}(z^{n-1}))\frac{\operatorname{Im}(z^{2n-2})}{|\operatorname{Im}(z^{n-1})|}\Big]. \end{aligned} \tag{3.16}$$

Note that, by the definition of U_3,

$$\begin{aligned} \operatorname{Sign}(\operatorname{Im}(z^{n-1}))\sin((2n-2)\theta_z) &= 2|\sin((n-1)\theta_z)|\cos((n-1)\theta_z) \\ &= 2|\sin((n-1)\theta_z)||\cos((n-1)\theta_z)|. \end{aligned}$$

We thus have the following estimate after combining the last two terms in (3.16):

$$\begin{aligned} \frac{1}{C_3}L\varphi_3(z,\bar{z}) &\leq \frac{1}{C_3}\varphi_3(z,\bar{z})\big[E_3(|\kappa_1|^2 + |\kappa_2|^2)\frac{|z|^{2n-4}}{|\operatorname{Im}(z^{n-1})|^2} \\ &\quad - 2\delta|\cos((n-1)\theta_z)||z|^{n-1}\big] \\ &= \frac{1}{C_3}\varphi_3(z,\bar{z})\big[E_3(|\kappa_1|^2 + |\kappa_2|^2)\frac{|z|^{-2}}{|\sin((n-1)\theta_z)|^2} \\ &\quad - 2\delta|\cos((n-1)\theta_z)||z|^{n-1}\big] \\ &= \frac{\varphi_3(z,\bar{z})}{C_3|\sin((n-1)\theta_z)|}\big[E_3(|\kappa_1|^2 + |\kappa_2|^2)\frac{|z|^{-2}}{|\sin((n-1)\theta_z)|} \\ &\quad - 2\delta|\sin((n-1)\theta_z)||\cos((n-1)\theta_z)||z|^{n-1}\big]. \end{aligned} \tag{3.17}$$

Let $x = |\cos((n-1)\theta_z)|$. Note that for $|z| > 2\eta_n$

$$\begin{aligned} |\sin((n-1)\theta_z)||\cos((n-1)\theta_z)| &\geq \min_{x\in\left[\eta_n|z|^{-1},\sqrt{1-\eta_n^2|z|^{-2}}\right]} x\sqrt{1-x^2} \\ &\geq \eta_n|z|^{-1}\sqrt{1-\eta_n^2|z|^{-2}} \\ &\geq \frac{\sqrt{3}}{2}\eta_n|z|^{-1}. \end{aligned}$$

Applying this to (3.17), we obtain:

$$\frac{1}{C_3}L\varphi_3(z,\bar{z}) \leq \frac{\varphi_3(z,\bar{z})}{C_3|\sin((n-1)\theta_z)|}\left[E_3(|\kappa_1|^2+|\kappa_2|^2)\eta_n^{-1}|z|^{-1} - \sqrt{3}\eta_n\delta|z|^{n-2}\right]$$

From this we infer that there exists a constant $d_3 > 0$ such that

$$\begin{aligned} L\varphi_3(z,\bar{z}) &\leq -\frac{\sqrt{3}\eta_n\delta}{2}|z|^{n-2}\varphi_3(z,\bar{z}) + d_3 \qquad (3.18) \\ &\leq -\frac{\sqrt{3}\eta_n\delta}{2}\varphi_3(z,\bar{z}) + d_3, \end{aligned}$$

as $|z| \geq 1$ on U_3. □

Before we proceed onto the fourth and fifth Lyapunov functions, we first need a lemma:

Lemma 3.19. *Let g be a primitive $(n-1)$st root of unity and for $k \in \mathbb{Z}$ let $g^k \cdot \boldsymbol{\kappa} = (g^k\kappa_1, g^k\kappa_2)$. Suppose that φ is a Lyapunov function on $U \subset \mathbb{C}$ corresponding to the operator $L^{g^k\cdot\boldsymbol{\kappa}}$ for all $k \in \mathbb{Z}$. Then $\psi_k(z,\bar{z}) = \varphi(g^{-k}z, \overline{g^{-k}}\bar{z})$ is a Lyapunov function on g^kU corresponding to the operator $L^{g^j\cdot\boldsymbol{\kappa}}$ for all $j \in \mathbb{Z}$.*

Proof. Fix $k \in \mathbb{Z}$ and note that ψ_k satisfies both **(I)** and **(II)** in g^kU. To see **(III)**, we have for $w = g^{-k}z$ and $z \in g^kU$:

$$\begin{aligned} L^{g^j\cdot\boldsymbol{\kappa}}\psi_k(z,\bar{z}) &= L^{g^{j-k}\cdot\boldsymbol{\kappa}}\varphi(w,\bar{w}) \\ &\leq -C\psi_k(z,\bar{z}) + D, \end{aligned}$$

on g^kU for some positive constants C, D. □

Note that the positive constants C, D in the previous lemma can depend on $j \in \mathbb{Z}$ as in the diffusion parameter $g^j \cdot \boldsymbol{\kappa}$. Since g is an $(n-1)$st root of unity, there are only finitely many distinct diffusion parameters. Hence we may make a uniform choice in these constants.

In what follows, it is more convenient to prove property **(III)** in the (x, y) variables. This is the case because, in $U_4(0)$ and $U_5(0)$, the variable x plays the dominant role since y is bounded. We shall thus use the expression:

$$\begin{aligned} L &= \operatorname{Re}(z^n)\frac{\partial}{\partial x} + \operatorname{Im}(z^n)\frac{\partial}{\partial y} + \frac{1}{2}(\operatorname{Re}(\kappa_1)^2 + \operatorname{Re}(\kappa_2)^2)\frac{\partial^2}{\partial x^2} \\ &\quad +(\operatorname{Re}(\kappa_1)\operatorname{Im}(\kappa_1) + \operatorname{Re}(\kappa_2)\operatorname{Im}(\kappa_2))\frac{\partial^2}{\partial x \partial y} + \frac{1}{2}(\operatorname{Im}(\kappa_1)^2 + \operatorname{Im}(\kappa_2)^2)\frac{\partial^2}{\partial y^2}. \end{aligned}$$

for the generator L of the process $z(t) = x(t) + iy(t)$.

Lyapunov Function 4. Let $d > 0$ and define

$$\varphi_{4,0}(z, \bar{z}) = \varphi_{4,0}(x, y) = C_4 \frac{x^d + |y|^d}{|y|^{\frac{3}{4}d}}.$$

For g, a primitive $(n-1)$st root of unity, define

$$\varphi_4(z, \bar{z}) = \varphi_{4,0}(g^{-k}z, \overline{g^{-k}}\bar{z}) \text{ if } z \in g^k U_4(0).$$

Then φ_4 is a Lyapunov function on U_4.

Remark 3.20. This is where we use the choice

$$-\left(\frac{3}{4}n - 1\right) + (|\kappa_1|^2 + |\kappa_2|^2)\frac{3}{8}\left(\frac{3}{4}d + 1\right) < -\frac{1}{4}$$

Proof. Note by the previous lemma, it suffices to prove that $\varphi_{4,0}$ is a Lyapunov function in $U_4(0)$ corresponding to the operator $L^{g^k \cdot \boldsymbol{\kappa}}$ for all $k \in \mathbb{Z}$. Note that $\varphi_{4,0}$

clearly satisfies **(I)** and **(II)** in $U_4(0)$. Note that we have:

$$
\begin{aligned}
& \frac{1}{C_4} L^{g^k\cdot\boldsymbol{\kappa}}\varphi_4(x,y) \\
= \; & d\frac{x^{d-1}}{|y|^{\frac{3}{4}d}}\operatorname{Re}(z^n) \\
& + \left(-\frac{3d}{4}\operatorname{Sign}(y)\frac{x^d}{|y|^{\frac{3d}{4}+1}} + \frac{d}{4}\operatorname{Sign}(y)|y|^{\frac{d}{4}-1}\right)\operatorname{Im}(z^n) \\
& + \frac{d(d-1)}{2}\left(\operatorname{Re}(g^k\kappa_1)^2 + \operatorname{Re}(g^k\kappa_2)^2\right)\frac{x^{d-2}}{|y|^{\frac{3}{4}d}} \\
& - \frac{3}{4}d^2\operatorname{Sign}(y)\left(\operatorname{Re}(g^k\kappa_1)\operatorname{Im}(g^k\kappa_1) + \operatorname{Re}(g^k\kappa_2)\operatorname{Im}(g^k\kappa_2)\right)\frac{x^{d-1}}{|y|^{\frac{3}{4}d+1}} \\
& + \left(\operatorname{Im}(g^k\kappa_1)^2 + \operatorname{Im}(g^k\kappa_2)^2\right)\left(\frac{3}{8}d\left(\frac{3}{4}d+1\right)\frac{x^d}{|y|^{\frac{3}{4}d+2}} + \frac{d}{8}\left(\frac{d}{4}-1\right)|y|^{\frac{d}{4}-2}\right) \\
= \; & \frac{x^d}{|y|^{\frac{3}{4}d}}\Bigg[\frac{d}{x}\operatorname{Re}(z^n) \\
& + \left(-\frac{3d}{4}\operatorname{Sign}(y)|y|^{-1} + \frac{d}{4}\operatorname{Sign}(y)|y|^{d-1}x^{-d}\right)\operatorname{Im}(z^n) \\
& + \frac{d(d-1)}{2}(\operatorname{Re}(g^k\kappa_1)^2 + \operatorname{Re}(g^k\kappa_2)^2)x^{-2} \\
& - \frac{3}{4}d^2\operatorname{Sign}(y)(\operatorname{Re}(g^k\kappa_1)\operatorname{Im}(g^k\kappa_1) + \operatorname{Re}(g^k\kappa_2)\operatorname{Im}(g^k\kappa_2))x^{-1}|y|^{-1} \\
& + (\operatorname{Im}(g^k\kappa_1)^2 + \operatorname{Im}(g^k\kappa_2)^2)\left(\frac{3}{8}d\left(\frac{3}{4}d+1\right)|y|^{-2} + \frac{d}{8}\left(\frac{d}{4}-1\right)x^{-d}|y|^{d-2}\right)\Bigg] \\
= \; & \frac{x^d}{|y|^{\frac{3}{4}d}}\Bigg(-d\left(\frac{3}{4}n-1\right)x^{n-1} + (\operatorname{Im}(g^k\kappa_1)^2 + \operatorname{Im}(g^k\kappa_2)^2)\frac{3}{8}d\left(\frac{3}{4}d+1\right)|y|^{-2} \\
& + \mathcal{O}(x^{n-1-\min(1,d)})\Bigg) \\
\leq \; & \frac{x^d}{|y|^{\frac{3}{4}d}}\left(-\frac{d}{4}x^{n-1} + \mathcal{O}(x^{n-1-\min(1,d)})\right).
\end{aligned}
$$

Thus there exists a constant $d_4 > 0$ such that

$$
\begin{aligned}
L\varphi_4(x,y) &\leq -\frac{d}{8}|z|^{n-1}\varphi_4(x,y) + d_4 \qquad (3.21)\\
&\leq -\frac{d}{8}\varphi_4(x,y) + d_4,
\end{aligned}
$$

as $|z| \geq 1$ in U_4. □

Lyapunov Function 5.

$$\varphi_{5,0}(z,\bar{z}) = \varphi_{5,0}(x,y) = C_5(6x^f - y^2x^{f+n-1}).$$

For g, a primitive $(n-1)$st root of unity, define

$$\varphi_5(z,\bar{z}) = \varphi_{5,0}(g^{-k}z, \overline{g^{-k}}\bar{z}), \text{ for } z \in g^k U_5(0).$$

Then φ_5 is a Lyapunov fuction on U_5.

Remark 3.22. So far, this is the only point in the argument where we need employ our assumption that $m(\kappa_1,\kappa_2) > 0$. Intuitively, since U_5 encloses the unstable trajectories of the ODE system (1.1), this should be expected. The assumption $m(\kappa_1,\kappa_2) > 0$ says that if we start on an unstable trajectory, noise kicks us off.

Here, we also use the choice

$$f = \frac{1}{12}m(\kappa_1,\kappa_2).$$

Proof. Again we apply the previous lemma and focus our attention on $\varphi_{5,0}$. Note clearly $\varphi_{5,0}$ satisfies **(I)** in $U_{5,0}$. Moreover,

$$\frac{1}{C_5}\varphi_5 \quad \geq \quad 2x^f.$$

Thus $\varphi_{5,0}$ satisfies **(II)** in U_5. Note that

$$\begin{aligned}
\frac{1}{C_5}L^{g^j\cdot\boldsymbol{\kappa}}\varphi_5(x,y) \quad &= \quad (6fx^{f-1} - (n+f-1)y^2x^{n+f-2})\,\mathrm{Re}(z^n) \\
&\qquad - 2yx^{n+f-1}\,\mathrm{Im}(z^n) \\
&\qquad - (\mathrm{Im}(g^j\kappa_1)^2 + \mathrm{Im}(g^j\kappa_2)^2)x^{n+f-1} + \mathcal{O}(x^{n+f-2}) \\
&\leq \quad (6f - (\mathrm{Im}\, g^j(\kappa_1)^2 + \mathrm{Im}(g^j\kappa_2)^2))x^{n+f-1} + \mathcal{O}(x^{n+f-2}) \\
&\leq \quad -\frac{1}{2}m(\kappa_1,\kappa_2)x^{n+f-1} + \mathcal{O}(x^{n+f-2}).
\end{aligned}$$

Thus there exists a constant $d_5 > 0$ such that

$$\begin{aligned} L\varphi_5(x,y) &\leq -\frac{1}{4}m(\kappa_1,\kappa_2)|z|^{n-1}\varphi_5(x,y) + d_5 \\ &\leq -\frac{1}{4}m(\kappa_1,\kappa_2)\varphi_5(x,y) + d_5, \end{aligned} \tag{3.23}$$

as $|z| \geq 1$ in U_5. □

3.5 Stage 2: Gluing

In this section, we finish constructing Φ by piecing together the Lyapunov covering $\{(\varphi_1, U_1), (\varphi_2, U_2), \ldots, (\varphi_5, U_5)\}$ exhibited in the previous section. For $j = 1, 2, 3, 4$, the idea is to define smooth auxiliary functions

$$\rho_{j,j+1} : U_j \cup U_{j+1} \to [0, 1]$$

such that $\rho_{j,j+1}|_{U_j \setminus U_j \cap U_{j+1}} = 0$ and $\rho_{j,j+1}|_{U_{j+1} \setminus U_j \cap U_{j+1}} = 1$ and argue that

$$\varphi_{j,j+1}(z,\bar{z}) := \rho_{j,j+1}\varphi_{j+1} + (1 - \rho_{j,j+1})\varphi_j \tag{3.24}$$

is now a Lyapunov function on the larger domain $U_j \cup U_{j+1}$. If this holds, for $R > 0$ sufficiently large we may then choose a smooth function Φ on $\mathbb{C}$ such that

$$\Phi(z,\bar{z}) = \begin{cases} \varphi_{j,j+1}(z,\bar{z}) & \text{for } z \in U_j \cup U_{j+1} \\ \text{arbitrary smooth} & \text{for } |z| \leq R - \epsilon, \end{cases}$$

for fixed $0 < \epsilon < R$. By construction, Φ is a Lyapunov function on all of $\mathbb{C}$.

By expression (3.24), we note that properties **(I)** and **(II)** are easily satisfied on $U_j \cap U_{j+1}$ for $j = 1, 2, 3, 4$. Thus to finish, all we must verify is property **(III)** on $U_j \cap U_{j+1}$ for $j = 1, 2, 3, 4$ and we shall do so in that order.

Patch 1. We first define $\rho_{1,2}$.

Choose $\mathfrak{f}(z,\bar{z}) \geq 0$ to be a smooth function on $R_{\geq 1} := \{\text{Re}(z) \geq 1\}$ such that

$$\mathfrak{f}(z,\bar{z}) = \begin{cases} 1 & \text{if } \text{Im}(z) \leq 1 \\ 0 & \text{if } \text{Im}(z) \geq 2, \end{cases}$$

and such that:

$$\begin{aligned}\frac{\partial}{\partial y}\mathfrak{f}(z,\bar{z}) &< 0 \text{ for } y \in (1,2)\\ \frac{\partial}{\partial x}\mathfrak{f}(z,\bar{z}) &= 0 \text{ for } x \geq 1,\end{aligned}$$

where $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$. Choose $\mathfrak{g}(z,\bar{z}) \geq 0$ to be a smooth function on $R_{\geq 1}$ such that

$$\mathfrak{g}(z,\bar{z}) = \begin{cases} 0 & \text{if } \mathrm{Im}(z) \leq -2 \\ 1 & \text{if } \mathrm{Im}(z) \geq -1, \end{cases}$$

and such that

$$\begin{aligned}\frac{\partial}{\partial y}\mathfrak{g}(z,\bar{z}) &> 0 \text{ for } y \in (-1,-2)\\ \frac{\partial}{\partial x}\mathfrak{g}(z,\bar{z}) &= 0 \text{ for } x \geq 1,\end{aligned}$$

where $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$. Hence we define

$$\rho_{1,2}(z,\bar{z}) = \begin{cases} \mathfrak{f}(e^{-i\alpha(k)}z, e^{i\alpha(k)}\bar{z}) & \text{for } z \in e^{i\alpha(k)}R_{\geq 1} \text{ if } k \text{ is even} \\ \mathfrak{g}(e^{-i\alpha(k)}z, e^{i\alpha(k)}\bar{z}) & \text{for } z \in e^{i\alpha(k)}R_{\geq 1} \text{ if } k \text{ is odd.} \end{cases}$$

Claim 1. *$\varphi_{1,2}$ given by equation (3.24) is a Lyapunov function on $U_1 \cup U_2$.*

Remark 3.25. Here we use the choice

$$C_2 = 2 > C_1 = 1.$$

Proof. Recall that it is sufficient to verify property **(III)** on the intersection $U_1 \cap U_2$. Note that we have:

$$\begin{aligned}L\varphi_{1,2}(z,\bar{z}) = \;& \rho_{1,2}L\varphi_2 + (1-\rho_{1,2})L\varphi_1 + (\varphi_2 - \varphi_1)(z^n\partial_z(\rho_{1,2}) + \bar{z}^n\partial_{\bar{z}}(\rho_{1,2}))\\ & + \frac{1}{2}(\kappa_1^2 + \kappa_2^2)\left((\varphi_2-\varphi_1)\partial_{zz}(\rho_{1,2}) + 2\partial_z(\varphi_2-\varphi_1)\partial_z(\rho_{1,2})\right)\\ & + (|\kappa_1|^2 + |\kappa_2|^2)\Big((\varphi_2-\varphi_1)\partial_{z\bar{z}}(\rho_{1,2}) + \partial_z(\varphi_2-\varphi_1)\partial_{\bar{z}}(\rho_{1,2})\\ & + \partial_{\bar{z}}(\varphi_2-\varphi_1)\partial_z(\rho_{1,2})\Big)\\ & + \frac{1}{2}(\overline{\kappa_1}^2 + \overline{\kappa_2}^2)\left((\varphi_2-\varphi_1)\partial_{\bar{z}\bar{z}}(\rho_{1,2}) + 2\partial_{\bar{z}}(\varphi_2-\varphi_1)\partial_{\bar{z}}(\rho_{1,2})\right).\end{aligned}$$

It is easy to check that the last four lines of the expression above are $\mathcal{O}(\rho_1(z,\bar{z})|z|^{2\delta})$ where ρ_1 is a bounded smooth function such that $\rho_1|_{U_1\setminus U_1\cap U_2} = \rho_1|_{U_2\setminus U_1\cap U_2} = 0$. Moreover, by construction of $\mathfrak{f}$ and $\mathfrak{g}$, $\rho_1(z,\bar{z}) \to 0$ uniformly in $|z|$ as $z \to (U_1 \setminus U_1 \cap U_2) \cup (U_2 \setminus U_1 \cap U_2)$. Combining this with estimates (3.13) and (3.15), we obtain:

$$\begin{aligned} L\varphi_{1,2}(z,\bar{z}) &\leq -\min(1,\delta\eta_n)|z|^{n-2}\varphi_{1,2}(z,\bar{z}) + (\varphi_2-\varphi_1)(z^n\partial_z(\rho_{1,2}) + \bar{z}^n\partial_{\bar{z}}(\rho_{1,2})) \\ &\quad + \mathcal{O}(\rho_1(z,\bar{z})|z|^{2\delta}) + \max(d_1,d_2). \end{aligned}$$

Suppose first that k is even. Since we are only interested in estimating on the intersection $U_1 \cap U_2$, we may suppose that $n\theta_z - \alpha(k) \to (n-1)\alpha(k)$ as $|z| \to \infty$ for some $k \in \mathbb{Z}$ even. Hence for $w = e^{-i\alpha(k)}z$ we have:

$$z^n\partial_z(\rho_{1,2}) + \bar{z}^n\partial_{\bar{z}}(\rho_{1,2}) = |z|^n\left(e^{in\theta_z - i\alpha(k)}\mathfrak{f}_w(w,\bar{w}) + e^{-in\theta_z + i\alpha(k)}\mathfrak{f}_{\bar{w}}(w,\bar{w})\right).$$

Since $n\theta_z - \alpha(k) \to (n-1)\alpha(k) = \pi/2 + \pi k$ (and k is EVEN) as $|z| \to \infty$ and

$$i\mathfrak{f}_w(w,\bar{w}) - i\mathfrak{f}_{\bar{w}}(w,\bar{w}) = \frac{\partial}{\partial y}\mathfrak{f}(w,\bar{w}) < 0$$

where $y = \mathrm{Im}(w)$, we have that

$$\begin{aligned} L\varphi_{1,2}(z,\bar{z}) &\leq -\min(1,\delta\eta_n)|z|^{n-2}\varphi_{1,2}(z,\bar{z}) - \frac{1}{2}|z|^{n+2\delta}\left|\partial_y(\mathfrak{f}(w,\bar{w}))\right| \\ &\quad + \mathcal{O}(\rho_1(z,\bar{z})|z|^{2\delta}) + \max(d_1,d_2). \end{aligned}$$

Since $\partial_x(\mathfrak{f}(w,\bar{w})) = 0$ where $x = \mathrm{Re}(w)$, the term $-\frac{1}{2}|z|^{n+2\delta}\left|\partial_y(\mathfrak{f}(w,\bar{w}))\right|$ dominates the term $\mathcal{O}(\rho_1(z,\bar{z})|z|^{2\delta})$ until we are arbitrarily close to $\partial(U_1 \cap U_2)$; in which the first term $-\min(1,\delta\eta_n)|z|^{n-2}\varphi_{1,2}$ dominates $\mathcal{O}(\rho_1(z,\bar{z})|z|^{2\delta})$ since ρ_1 is smooth an vanishes on $U_1 \setminus U_1 \cap U_2$ and $U_2 \setminus U_1 \cap U_2$. From this it is clear that there exists a positive constant $d_{1,2}$ such that

$$L\varphi_{1,2}(z,\bar{z}) \leq -\frac{\min(1,\delta\eta_n)}{2}\varphi_{1,2} + d_{1,2}.$$

If k is odd, we may suppose that $n\theta_z - \alpha(k) \to (n-1)\alpha(k) = \pi/2 + \pi k$ as $|z| \to \infty$ for some $k \in \mathbb{Z}$ odd. Using the same argument, we realize that for $|z|$ sufficiently large and $w = e^{-i\alpha(k)}z$:

$$\begin{aligned} z^n \partial_z(\rho_{1,2}) + \bar{z}\partial_{\bar{z}}(\rho_{1,2}) &= |z|^n \left(e^{in\theta_z - i\alpha(k)}\mathfrak{g}_w(w,\bar{w}) + e^{-in\theta_z + i\alpha(k)}\mathfrak{g}_{\bar{w}}(w,\bar{w})\right) \\ &\leq -|z|^n \frac{\partial}{\partial y}\mathfrak{g}(w,\bar{w}) \\ &= -|z|^n \left|\frac{\partial}{\partial y}\mathfrak{g}(w,\bar{w})\right|. \end{aligned}$$

The rest of the argument follows through similarly. □

Patch 2. We now define $\rho_{2,3}$.

We will again use the functions $\mathfrak{f}$ and $\mathfrak{g}$ defined in Patch 1. Shifting $\mathfrak{f}$ down three units and $\mathfrak{g}$ up three units, let $\mathfrak{f}_{-3}(z,\bar{z}) = \mathfrak{f}(z+3i, \bar{z}-3i)$ and $\mathfrak{g}_{+3}(z,\bar{z}) = \mathfrak{g}(z-3i, \bar{z}+3i)$. Define

$$\rho_{2,3}(z,\bar{z}) = \begin{cases} \mathfrak{f}_{-3}(e^{-i\alpha(k)}z, e^{i\alpha(k)}\bar{z}) & \text{for } z \in e^{i\alpha(k)}R_{\geq 1} \text{ if } k \text{ is even} \\ \mathfrak{g}_{+3}(e^{-i\alpha(k)z}, e^{i\alpha(k)}\bar{z}) & \text{for } z \in e^{i\alpha(k)}R_{\geq 1} \text{ if } k \text{ is odd.} \end{cases}$$

Claim 2. *$\varphi_{2,3}$ defined by equation (3.24) is a Lyapunov function on $U_2 \cup U_3$.*

Remark 3.26. Here we use the choice

$$C_3 = 3 > C_2 = 2.$$

Proof. Proceeding in a similar fashion as in the proof of Claim 1, we see that by the estimates (3.15) and (3.18), since $0 < \delta < 1/2$ we have on $U_2 \cap U_3$:

$$\begin{aligned} L\varphi_{2,3}(z,\bar{z}) &\leq -\min\left(1, \frac{\sqrt{3}\eta_n\delta}{2}\right)|z|^{n-2}\varphi_{2,3}(z,\bar{z}) + (\varphi_3 - \varphi_2)(z^n\partial_z(\rho_{2,3}) + \bar{z}^n\partial_{\bar{z}}(\rho_{2,3})) \\ &\quad + \mathcal{O}(\rho_2(z,\bar{z})|z|^{2\delta}) + \max(d_2, d_3), \end{aligned}$$

where ρ_2 is a bounded smooth function such that $\rho_2|_{U_2 \setminus U_2 \cap U_3} = \rho_2|_{U_3 \setminus U_2 \cap U_3} = 0$. Moreover, by construction of $\mathfrak{f}$ and $\mathfrak{g}$, $\rho_2(z,\bar{z}) \to 0$ uniformly in $|z|$ as $z \to (U_2 \setminus U_2 \cap U_3) \cup (U_3 \setminus U_2 \cap U_3)$. Using exactly the same argument as before, we can find a constant $d_{2,3} > 0$ such that

$$L\varphi_{2,3}(z,\bar{z}) \leq -\min\left(\frac{1}{2}, \frac{\sqrt{3}\eta_n\delta}{4}\right)|z|^{n-2}\varphi_{2,3}(z,\bar{z}) + d_{2,3}$$

on $U_2 \cap U_3$, as claimed.

□

Patch 3. We now define $\rho_{3,4}$.

Let

$$\rho_{3,4}(z,\bar{z}) = \begin{cases} \mathfrak{f}(e^{-i\beta(k)}z, e^{i\beta(k)}\bar{z}) & \text{for } z \in e^{i\beta(k)}R_{\geq 1} \text{ if } k \text{ is even} \\ \mathfrak{g}(e^{-i\beta(k)}z, e^{i\beta(k)}\bar{z}) & \text{for } z \in e^{i\beta(k)}R_{\geq 1} \text{ if } k \text{ is even.} \end{cases}$$

Note that k is only even since $\cos((n-1)\theta_z) > 0$ on $U_3 \cup U_4$.

Claim 3. *$\varphi_{3,4}$ defined by equation (3.24) is a Lyapunov function on $U_3 \cup U_4$.*

Remark 3.27. Here we use the choice:

$$\frac{C_4}{2^{\frac{3}{4}d}} \geq \frac{C_4}{2} = 25/2 > 12 = 4C_3 > \frac{2C_3}{\eta_n^{4\delta/(n-1)}}$$

Proof. Applying Lemma 3.19, it suffices to show that $\varphi_{3,4}$ is a Lyapunov function on $U_3 \cap U_4(0)$ for all diffusion parameters $g^j \cdot \boldsymbol{\kappa}$ for $j \in \mathbb{Z}$ and $\boldsymbol{\kappa}$ fixed. Proceeding as in Claim 1 and Claim 2, by the estimates (3.18) and (3.21) we have:

$$\begin{aligned} L\varphi_{3,4}(z,\bar{z}) \;&\leq\; -2c_{3,4}|z|^{n-2}\varphi_{3,4} + (\varphi_4 - \varphi_3)(z^n \partial_z(\rho_{3,4}) + \bar{z}^n \partial_{\bar{z}}(\rho_{3,4})) \\ &\quad + \mathcal{O}(\rho_3(z,\bar{z})|z|^{2\delta + \frac{4\delta}{n-1}}) + \max(d_3, d_4) \end{aligned}$$

on $U_3 \cap U_4(0)$ where $c_{3,4} > 0$ is a positive constant and ρ_3 is a bounded smooth function on $U_3 \cup U_4(0)$ such that $\rho_3|_{U_3 \setminus U_3 \cap U_4(0))} = \rho_3|_{U_4(0) \setminus U_3 \cap U_4(0)} = 0$. Moreover, by construction of $\mathfrak{f}$ and $\mathfrak{g}$, $\rho_3(z,\bar{z}) \to 0$ uniformly in $|z|$ as $z \to (U_3 \setminus U_3 \cap U_4(0)) \cup (U_4(0) \setminus U_3 \cap U_4(0))$. Comparing with the proofs of Claim 1 and Claim 2, we see that the only difference thus far is the asymptotic behavior change in both $\varphi_{3,4}$ and the $\mathcal{O}$ terms which are, conveniently, both of order $2\delta + 4\delta/(n-1)$. To estimate the term $(\varphi_4 - \varphi_3)(z^n \partial_z(\rho_{3,4}) + \bar{z}^n \partial_{\bar{z}}(\rho_{3,4}))$, note that if $z \in U_3 \cap U_4(0)$ with $y = \mathrm{Im}(z) > 0$ and

$|z|$ sufficiently large we have

$$\begin{aligned} z^n\partial_z(\rho_{3,4}) + \bar{z}^n\partial_{\bar{z}}(\rho_{3,4}) &= |z|^n \sin(n\theta_z)\frac{\partial}{\partial y}\mathfrak{f}(z,\bar{z}) \\ &= |z|^n(\sin((n-1)\theta_z)\cos(\theta_z) + \sin(\theta_z)\cos((n-1)\theta_z))\frac{\partial}{\partial y}\mathfrak{f}(z,\bar{z}) \\ &\leq -\frac{\eta_n}{2}|z|^{n-1}\left|\frac{\partial}{\partial y}\mathfrak{f}(z,\bar{z})\right|, \end{aligned}$$

since $\theta_z \downarrow 0$ as $|z| \to \infty$ in this region. If on the other hand $y = \mathrm{Im}(z) < 0$, we have the same estimate upon replacing $\mathfrak{f}$ with $\mathfrak{g}$. From this point, using the same reasoning as in Claim 1 and Claim 2, we finish the proof of Claim 3. □

Patch 4. We now define $\rho_{4,5}$. Because $U_4 \cap U_5$ is a region that consists of decreasing strips instead of boxes, this will be the most involved patch.

Define an interpolating function $p : [1,\infty) \times \mathbb{R} \to [0,1]$ by

$$p_n(x,y) = \begin{cases} 1 & \text{if } |x|^{\frac{n-1}{2}}|y| \leq 1 \\ 2 - |x|^{\frac{n-1}{2}}|y| & \text{if } 1 < |x|^{\frac{n-1}{2}}|y| < 2 \\ 0 & \text{if } 2 \leq |x|^{\frac{n-1}{2}}|y|. \end{cases}$$

To smooth the sharp corners of this function, introduce

$$r(t) = \begin{cases} e^{-\frac{1}{t(1-t)}} & \text{if } 0 < t < 1 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$s(x) = \frac{1}{N}\int_{-\infty}^{x} r(t)dt,$$

where $N = \int_{\mathbb{R}} r(t)dt$. We then let

$$\rho_{4,5,0}(x,y) = s(p_n(x,y)) \text{ on } x \geq 1,$$

which is now a smooth function on $R_{\geq 1}$. Rotating $\rho_{4,5,0}(x,y) = \rho_{4,5,0}(z,\bar{z})$ we define

$$\rho_{4,5}(z,\bar{z}) = \rho_{4,5,0}(e^{-i\beta(k)}z, e^{i\beta(k)}\bar{z}) \text{ for } z \in e^{\beta(k)}R_{\geq 1} \text{ for } k \text{ even.}$$

Claim 4. *$\varphi_{4,5}$ defined by equation (3.24) is a Lyapunov function on $U_4 \cup U_5$.*

Remark 3.28. Here we use the choice

$$2C_5 = 26 < 25 = C_4.$$

Proof. We will again choose $|\kappa_1|^2 + |\kappa_2|^2$ smaller than before which is made precise below. Applying Lemma 3.12 and Lemma 3.19, it suffices to prove that $\varphi_{4,5,0}$ is a Lyapunov function on $U_4(0) \cap U_5(0)$ for all diffusion parameters $g^j \cdot \boldsymbol{\kappa}$ where j varies through $\mathbb{Z}$. Thus, using the variables (x, y), we obtain the expression on $U_4(0) \cap U_5(0)$:

$$\begin{aligned} L\varphi_{4,5,0}(x,y) \;=\; & \rho_{4,5,0} L\varphi_{5,0} + (1-\rho_{4,5,0}) L\varphi_{4,0} \\ & + \operatorname{Re}(z^n)(\varphi_{5,0} - \varphi_{4,0})\partial_x(\rho_{4,5,0}) + \operatorname{Im}(z^n)(\varphi_{5,0} - \varphi_{4,0})\partial_y(\rho_{4,5,0}) \\ & + \frac{1}{2}(\operatorname{Re}(\kappa_1)^2 + \operatorname{Re}(\kappa_2)^2)\Big[(\varphi_{5,0} - \varphi_{4,0})\partial_{xx}(\rho_{4,5,0}) \\ & + 2\partial_x(\varphi_{5,0} - \varphi_{4,0})\partial_x(\rho_{4,5,0})\Big] \\ & + (\operatorname{Re}(\kappa_1)\operatorname{Im}(\kappa_1) + \operatorname{Re}(\kappa_2)\operatorname{Im}(\kappa_2))\Big[(\varphi_{5,0} - \varphi_{4,0})\partial_{xy}(\rho_{4,5,0}) \\ & + \partial_y(\varphi_{5,0} - \varphi_{4,0})\partial_x(\rho_{4,5,0}) + \partial_x(\varphi_{5,0} - \varphi_{4,0})\partial_y(\rho_{4,5,0})\Big] \\ & + \frac{1}{2}(\operatorname{Im}(\kappa_1)^2 + \operatorname{Re}(\kappa_2)^2)\Big[(\varphi_{5,0} - \varphi_{4,0})\partial_{yy}(\rho_{4,5,0}) \\ & + 2\partial_y(\varphi_{5,0} - \varphi_{4,0})\partial_y(\rho_{4,5,0})\Big]. \end{aligned}$$

By definition of $\rho_{4,5,0}$, the term on the third and fourth lines above is $\mathcal{O}(|x|^{f-2})$ on $U_4(0) \cap U_5(0)$ and term on the fifth and sixth lines above is $\mathcal{O}(|x|^{f+\frac{n-3}{2}})$ on $U_4(0) \cap U_5(0)$. The last two terms are bounded above by:

$$E(|\kappa_1|^2 + |\kappa_2|^2)(n(p)q(p) + n(p))|x|^{f+n-1},$$

where $E > 0$ is a constant independent of κ_1 and κ_2, $n(x) = N^{-1}r(x)$, $q(x) = |1-2x|/(x-x^2)^2$ and $p = p_n(x,y)$. Hence using the strong bounds (3.21) and (3.23), there exists a constant $c_{4,5} > 0$ independent of κ_1 and κ_2 such that on $U_4(0) \cap U_5(0)$:

$$\begin{aligned} L\varphi_{4,5,0}(x,y) &\leq -c_{4,5}m(\kappa_1,\kappa_2)|x|^{f+n-1} + \mathrm{Re}(z^n)(\varphi_{5,0} - \varphi_{4,0})\partial_x(\rho_{4,5,0}) \\ &\quad + \mathrm{Im}(z^n)(\varphi_{5,0} - \varphi_{4,0})\partial_y(\rho_{4,5,0}) \\ &\quad + E(|\kappa_1|^2 + |\kappa_2|^2)(n(p)q(p) + n(p))|x|^{f+n-1} \\ &\quad + \mathcal{O}(|x|^{f+\frac{n-3}{2}}) \\ &\leq -c_{4,5}m_n(\kappa_1,\kappa_2)|x|^{f+n-1} - n(p)(\varphi_{5,0} - \varphi_{4,0})|x|^{n-1} \\ &\quad + E(|\kappa_1|^2 + |\kappa_2|^2)(n(p)q(p) + n(p))|x|^{f+n-1} \\ &\quad + \mathcal{O}(|x|^{f+\frac{n-3}{2}}). \end{aligned}$$

Note that for all $\lambda > 0$:

$$\frac{|\kappa_1|^2 + |\kappa_2|^2}{m(\kappa_1,\kappa_2)} = \frac{|\lambda\kappa_1|^2 + |\lambda\kappa_2|^2}{m(\lambda\kappa_1,\lambda\kappa_2)}.$$

Thus regardless of how much we decrease the magnitude $|\kappa_1|^2 + |\kappa_2|^2$, the ratio above remains unchanged. Since both $n(p)$ and $n(p)q(p)$ approach 0 as $|x|^{\frac{n-1}{2}}|y|$ approaches 1 or 2, the above relation implies there exists $\epsilon > 0$ independent of both κ_1 and κ_2 such that

$$-c_{4,5}m(\kappa_1,\kappa_2) + E(|\kappa_1|^2 + |\kappa_2|^2)(n(p)q(p) + n(p)) \leq -c'_{4,5} < 0,$$

for all (x,y) such that $|x|^{\frac{n-1}{2}}|y| \in [1, 1+\epsilon) \cup (2-\epsilon, 2]$. Moreover it is clear that choosing $|\kappa_1|^2 + |\kappa_2|^2$ sufficiently small we can assure that

$$-n(p)(\varphi_{5,0} - \varphi_{4,0})|x|^{n-1} + E(|\kappa_1|^2 + |\kappa_2|^2)(n(p)q(p) + n(p))|x|^{f+n-1} < 0,$$

for all (x,y) such that $|x|^{\frac{n-1}{2}}|y| \in [1+\epsilon, 2-\epsilon]$. Note that this finishes the result for now there exists $d_{4,5} > 0$ such that

$$L\varphi_{4,5,0}(x,y) \leq -\frac{c'_{4,5}}{2}\varphi_{4,5,0}(x,y) + d_{4,5},$$

as required. □

3.6 Uniqueness of μ and Geometric Ergodicity

Now that we have finished constructing a smooth function $\Phi : \mathbb{C} \to [0, \infty)$ that satisfies **(C1)** and **(C3)**, our goal is to prove that the invariant probability measure μ is unique and the process $z(t)$ is exponentially ergodic. From what follows, uniqueness is easily established by Remark 2.45 and the results of Section 2.5. We will thus focus on showing part 3 of Lemma 3.1. We note that by the existence of $\Phi : \mathbb{C} \to [0, \infty)$ that satisfies **(C1)** and **(C3)** and the results of Section 2.6, it sufficies to show Assumption 2.63 is valid for $z(t)$, i.e., we prove:

Theorem 3.29. *There exists a distinguished time $T_0 > 0$ such that for all $R > 0$ sufficiently large, there exists $\alpha_R \in (0, 1)$ and a probability measure ν_R such that*

$$\inf_{z \in C_R} P(z, T_0, \,\cdot\,) \geq \alpha_R \nu_R(\,\cdot\,),$$

where $C_R = \{z \in \mathbb{C} \,:\, \Phi(z, \bar{z}) \leq R\}$.

We split the proof of the theorem above into two lemmata:

Lemma 3.30. *There exists $T_0 > 0$ and non-empty open $U \subset \mathbb{C}$ such that*

$$\text{supp}(P(z, T_0, \,\cdot\,)) \supset U \text{ for all } z \in \mathbb{C}.$$

Lemma 3.31. *For all $t > 0$ and $z \in \mathbb{C}$*

$$P(z, t, dw) = p(z, t, w)\, dw,$$

where dw is Lebesgue measure on $\mathbb{R}^2$ and p is a smooth function on $\mathbb{R}^2 \times (0, \infty) \times \mathbb{R}^2$.

Proof that Lemma 3.30 and Lemma 3.31 $\implies$ Theorem 3.29. Here we follow the appendix in [MSH02]. Pick $R > 0$ large enough so that $U \cap \text{int}(C_R) \neq \emptyset$. Fix $z^* \in U \cap \text{int}(C_R)$ and $\delta > 0$ such that $B_\delta(z^*) \subset \text{int}(C_R)$. By Lemma 3.30, we have:

$$P(z^*, T_0, B_\delta(z^*)) > 0.$$

By Lemma 3.31, we have:

$$p(z^*, T_0, w^*) \geq 2\epsilon,$$

for some $\epsilon > 0$ and $w^* \in B_\delta(z^*)$. By Lemma 3.31 again, we obtain:

$$p(z, T_0, w) \geq \epsilon \text{ for all } (z, w) \in B_{\epsilon_1}(z^*) \times B_{\epsilon_2}(w^*),$$

for some $\epsilon_1, \epsilon_2 > 0$ where $\epsilon_2 > 0$ is also chosen such that $B_{\epsilon_2}(w^*) \subset C_R$. Thus for all $z \in B_{\epsilon_1}(z^*)$ and $A \in \mathcal{B}(\mathbb{C})$, we have:

$$\begin{aligned} P(z, T_0, A) &= \int_A p(z, T_0, w)\, dw \\ &\geq \int_{A \cap B_{\epsilon_2}(w^*)} p(z, T_0, w)\, dw \\ &\geq \epsilon\, \lambda\, (A \cap B_{\epsilon_2}(w^*)) \end{aligned}$$

where λ is Lebegue measure on $\mathbb{C}$. Since $\overline{C_R}$ is compact, by Lemma 3.30 and Lemma 3.31 we have:

$$\inf_{z \in \overline{C_R}} P(z, T_0, B_{\epsilon_1}(z^*)) \geq \zeta,$$

for some $\zeta > 0$. Define $\tilde{T}_0 = 2T_0$ and note that for all $z \in \overline{C_R}$ and A Borel, we have:

$$\begin{aligned} P(z, \tilde{T}_0, A) &= \int_{\mathbb{C}} P(z, T_0, dw) P(w, T_0, A) \\ &\geq \int_{B_{\epsilon_1}(z^*)} p(z, T_0, w) P(w, T_0, A)\, dw \\ &\geq \epsilon\, \lambda(A \cap B_{\epsilon_2}(w^*)) \int_{B_{\epsilon_1}(z^*)} p(z, T_0, w)\, dw \\ &= \epsilon\, \lambda(A \cap B_{\epsilon_2}(w^*)) P(z, T_0, B_{\epsilon_1}(z^*)) \\ &\geq \epsilon\, \zeta\, \lambda(A \cap B_{\epsilon_2}(w^*)) \\ &= \epsilon\, \zeta\, \lambda(B_{\epsilon_2}(w^*))\, \nu(A) \end{aligned}$$

where $\nu(A) = \lambda(B_{\epsilon_2}(w^*))^{-1} \lambda(A \cap B_{\epsilon_2}(w^*))$. Note that this finishes the proof since ν is indeed a probability measure. □

As we will see, Lemma 3.30 follows from the geometric techniques of Section 2.5 and Lemma 3.31 is a simple consequence of a deep result of Hörmander [Hör67]. Both arguments, however, employ the fact that the the for all $z \in \mathbb{C}$, the span of the Lie algebra generated by the polysystem:

$$F = \{Z_n + u_1 \boldsymbol{\kappa}_1 + u_2 \boldsymbol{\kappa}_2 \, : \, u_1, u_2 \in \mathbb{R}\},$$

at $z \in \mathbb{C}$, where Z_n, $\boldsymbol{\kappa}_1$, $\boldsymbol{\kappa}_2$ are vector fields on $\mathbb{R}^2$ determined by $Z_n(z) = z^n$, $\boldsymbol{\kappa}_1(z) = \kappa_1$, $\boldsymbol{\kappa}_2(z) = \kappa_2$, is the whole tangent space. In the first lemma, this is used to validate one hypothesis of Theorem 2.59. In the latter, it is used to show Lemma 3.31 as stated. With both results in mind, we first show Lemma 3.31 by verifying the Lie algebra generated by F spans the whole tangent space, which in this case is $\mathbb{C}$, at all points.

Proof of Lemma 3.31. Note that if κ_1 and κ_2 are linearly independent over $\mathbb{R}$, there is nothing to prove since $\boldsymbol{\kappa}_1$ and $\boldsymbol{\kappa}_2$ span the tangent space at all points. Suppose that $\kappa_1 = c\,\kappa_2$ for some $c \in \mathbb{R}$. Since (κ_1, κ_2) is transversal to D_{n-1} we may assume $\kappa_1^{n-1} \notin \mathbb{R}$. For vector fields X and Y, we let $\operatorname{ad} X(Y) = [X, Y]$ and for $k \in \mathbb{N}$ $k \geq 2$, let $\operatorname{ad}^k X(Y) = \operatorname{ad}^{k-1} X(\operatorname{ad} X(Y))$. Computing Lie brackets (in $\mathbb{R}^2$) we obtain $\operatorname{ad}^n \boldsymbol{\kappa}_1(Z_n) = n!\boldsymbol{\kappa}_1^n$, where $\boldsymbol{\kappa}_1^n$ is the vector field on $\mathbb{R}^2$ determined by $n!\boldsymbol{\kappa}_1^n(z) = n!\kappa_1^n$. Since κ_1 and κ_1^n are linearly independent over $\mathbb{R}$, this finishes the proof. □

To prove Lemma 3.30, let us distinguish between two cases; the first of which is more straightforward than the second.

Case 1. $n \geq 2$ is odd or κ_1 and κ_2 are linearly independent over $\mathbb{R}$.

Case 2. $n \geq 2$ is even and $\kappa_1 = c\,\kappa_2$ for some $c \in \mathbb{R}$.

We will first prove Lemma 3.30 in Case 1, as we now have the techniques to do so. Moreover, the argument will illustrate the difference between Case 1 and Case 2.

Proof of Lemma 3.30 in Case 1. Suppose first that κ_1 and κ_2 are linearly independent over $\mathbb{R}$. We note that by Theorem 2.51 for all $u_1, u_2 \in \mathbb{R}$

$$\begin{aligned} u_1\boldsymbol{\kappa}_1 &= \lim_{\lambda\to\infty} \frac{1}{\lambda}(Z_n + \lambda u_1\boldsymbol{\kappa}_1) \in \mathrm{Sat}(F) \\ u_2\boldsymbol{\kappa}_2 &= \lim_{\lambda\to\infty} \frac{1}{\lambda}(Z_n + \lambda u_2\boldsymbol{\kappa}_2) \in \mathrm{Sat}(F). \end{aligned}$$

Hence for all $z \in \mathbb{C}$ and all $T > 0$, by the linearly independence assumption

$$\overline{A_F(z, \leq T)} = \mathbb{C}.$$

By Theorem 2 on p. 68 of [Jur97],

$$A_F(z, \leq T) = \mathbb{C}.$$

Hence by Theorem 2.42 and Theorem 2.59,

$$\mathrm{supp}(P(z, T, \cdot)) = \overline{A_F(z, T)} = \mathbb{C},$$

for all $z \in \mathbb{C}$, $T > 0$. Suppose now that n is odd and $\kappa_1 = c\,\kappa_2$ for some $c \in \mathbb{R}$. We may suppose without loss of generality that $\kappa_1^{n-1} \notin \mathbb{R}$. Using Theorem 2.51 for all $u_1 \in \mathbb{R}$ we have:

$$u_1\boldsymbol{\kappa}_1 = \lim_{\lambda\to\infty} \frac{1}{\lambda}(Z_n + \lambda u_1\boldsymbol{\kappa}_1) \in \mathrm{Sat}(F).$$

From this, one can check that for all $u_1 \in \mathbb{R}$, $\exp(u_1\boldsymbol{\kappa}_1)(z) = z + u_1\kappa_1 \in \mathrm{Norm}(\mathrm{Sat}(F))$. Therefore, by Theorem 2.56 $\exp(u_1\boldsymbol{\kappa}_1)_\#(Z_n) \in \mathrm{Sat}(F)$. Computing the vector field $\exp(u_1\boldsymbol{\kappa}_1)_\#(Z_n)$, we obtain:

$$\begin{aligned} \exp(u_1\boldsymbol{\kappa}_1)_\#(Z_n)(z) &= Z_n(z - u_1\kappa_1) \\ &= \sum_{j=0}^{n} \binom{n}{j} z^{n-j}(-1)^j u_1^j \kappa_1^j. \end{aligned}$$

Thus we determine

$$\lim_{\lambda\to\infty} \frac{1}{\lambda^n} \exp(u_1\lambda\kappa_1)_\#(Z_n) = (-1)^n u_1^n \boldsymbol{\kappa}_1^n \in \mathrm{Sat}(F),$$

where $\boldsymbol{\kappa}_1^n(z) = \kappa_1^n$. Since n is odd, we infer that $u_1\boldsymbol{\kappa}_1^n \in \mathrm{Sat}(F)$ for all $u_1 \in \mathbb{R}$. Since κ_1 and κ_1^n are linearly independent over $\mathbb{R}$, we see that for all $z \in \mathbb{C}$ and $T > 0$:

$$\overline{A_F(z, \leq T)} = \mathbb{C}.$$

Using the same reasoning as before, we obtain:

$$\mathrm{supp}(P(z,t,\,\cdot\,)) = \overline{A_F(z,T)} = \mathbb{C}.$$

This finishes the proof of Case 1.

□

We note that in Case 1 of Lemma 3.30, $T_0 > 0$ can be chosen to be any positive time and U can be chosen to be the whole space. To see what changes in Case 2, suppose now that n is even and $\kappa_1 = c\,\kappa_2$ for some $c \in \mathbb{R}$. We may again suppose without loss of generality that $\kappa_1^{n-1} \notin \mathbb{R}$. Proceeding in exactly the same way as in the proof of the second part of Case 1, we see that $u_1\boldsymbol{\kappa}_1 \in \mathrm{Sat}(F)$ and $(-1)^n u_1^n \boldsymbol{\kappa}_1^n \in \mathrm{Sat}(F)$ for all $u_1 \in \mathbb{R}$. Since n is even, we may only deduce that $\lambda\,\boldsymbol{\kappa}_1^n \in \mathrm{Sat}(F)$ for all $\lambda \geq 0$. Hence for all $z \in \mathbb{C}$ and $T > 0$ we may only determine that $\overline{A_F(z, \leq T)}$ contains

$$H(z,\kappa_1) = \{z\} + \{u\kappa_1 + \lambda\kappa_1^n \,:\, u \in \mathbb{R},\, \lambda \geq 0\},$$

which, since κ_1 and κ_1^n are linearly independent over $\mathbb{R}$, is a half-plane that depends on z. Note that by Theorem 2 on p. 68 of [Jur97], we deduce

$$A_F(z, \leq T) \supset \mathrm{int}(H(z,\kappa_1))$$

for all $z \in \mathbb{C}$ and all $T > 0$. Thus by Remark 2.45, this is sufficient to conclude uniqueness of μ in Case 2, but we must work a little harder to obtain exponential ergodicity.

In pursuit of the conclusion of Lemma 3.30 in Case 2, there are two problems with the above. First, $H(z,\kappa_1)$ depends on the initial point $z \in \mathbb{C}$. Second, supposing that we are able to remove this dependence, we still must transfer between the sets $\overline{A_F(z, \leq T)}$ and $\overline{A_F(z, T)}$.

We will be able to get rid the dependence on z in $H(z, \kappa_1)$ in the following way. We will show that for all $\epsilon > 0$, there exists a time $T_0' > 0$ such that for all initial conditions $z \in \mathbb{C}$, we can use trajectories of F to control a solution starting from z into $B_\epsilon(0)$ for all $T \geq T_0'$. It is important to note that the choice of $T_0' > 0$ does not depend upon z, but only on the size ϵ of the ball. Hence if $z(\epsilon, \kappa_1) \in \partial B_\epsilon(0) \cap H(0, \kappa_1)$ is such that $z(\epsilon, \kappa_1) \perp \kappa_1$, we obtain for all $t > 0$ and for all $z \in \mathbb{C}$:

$$\overline{A_F(z, T_0' \leq t)} \supset H(z(\epsilon, \kappa_1), \kappa_1),$$

where $A_F(z, T_0' \leq t)$ is the set of points that can be reached from z using trajectories in F at some time in the interval $[T_0', T_0' + t]$.

Although it seems from this that we should be able to deduce that

$$\overline{A_F(z, T_0')} \supset H(z(\epsilon, \kappa_1), \kappa_1),$$

it is not immediate. By the proof of Theorem 2.59, what we have left to show is that

$$z_0 \in A_F(z_0, \leq T) \text{ for all } T > 0,$$

for all points $z_0 \in B_\epsilon(0)$ which are images of the trajectories of F that initially guided us into $B_\epsilon(0)$.

Let us now proceed using the ideas above. First note that by separation of variables for $z \neq 0 \in \mathbb{C}$:

$$(\exp(tZ_n)(z))^{n-1} = -\frac{1}{\left((n-1)t - \frac{\overline{z}^{n-1}}{|z|^{2n-2}}\right)}. \tag{3.32}$$

Thus if $z \in D_{n-1}$, $\exp(tZ_n)(z)$ is only defined locally in time. For $z \neq 0$ otherwise, however, $\exp(tZ_n)(z)$ is strongly dissipative. More precisely:

Proposition 3.33. *For all $\epsilon > 0$ there exists $T_1(\epsilon) > 0$ such that*

$$|\exp(tZ_n)(z)| \leq \epsilon \text{ for all } t \geq T_1(\epsilon) \tag{3.34}$$

for all $|z| \geq \epsilon$ *such that* $z \notin D_{n-1}$.

Proof. For $|z| \geq \epsilon$ such that $z \notin D_{n-1}$, let $w = \overline{z}^{n-1}/|z|^{2n-2}$. Note that $|\operatorname{Re}(w)| \leq 1/\epsilon^{n-1}$. Pick then $T_1(\epsilon) = \frac{2}{(n-1)\epsilon^{n-1}}$ and note that for $t \geq T_0(\epsilon)$ we have:

$$\begin{aligned} |\exp(tZ_n)(z)|^{2n-2} &= \frac{1}{((n-1)t - \operatorname{Re}(w))^2 + \operatorname{Im}(w)^2} \\ &\leq \epsilon^{2n-2}. \end{aligned}$$

□

In the previous proposition, we only used the vector field Z_n. For initial points z elsewhere besides $|z| \geq \epsilon$ and $z \notin D_{n-1}$, we will use more of the polysystem F to control z into the set where $|z| \geq \epsilon$ and $z \notin D_{n-1}$. This is illustrated in the next two propositions. Again we recall that $\kappa_1 = c\,\kappa_2$ for some $c \in \mathbb{R}$ and we assume without loss of generality that $\kappa_1^{n-1} \notin \mathbb{R}$.

Proposition 3.35. *For all* $\epsilon > 0$, $|z| \leq \epsilon$, *and* $T' > 0$ *there exists* $u > 0$ *large enough such that*

$$|\exp(t(Z_n + u\boldsymbol{\kappa}_1))(z)| > \epsilon \textit{ for some } t \in (0, T').$$

Proof. Let $\epsilon > 0$, $|z| \leq \epsilon$, and $T' > 0$. Suppose to the contrary that for all $u > 0$,

$$|\exp(t(Z_n + u\boldsymbol{\kappa}_1))(z)| \leq \epsilon \text{ for all } t \in (0, T').$$

We then have the estimate:

$$|\exp(t(Z_n + u\boldsymbol{\kappa}_1))(z) - z - u\kappa_1 t| \geq u|\kappa_1|t - 2\epsilon,$$

for all $u > 0$, $t \in (0, T')$. Since $\exp(t(Z_n + u\boldsymbol{\kappa}_1))$ is an integral curve, we obtain:

$$\begin{aligned} u|\kappa_1|t &\leq 2\epsilon + |\exp(t(Z_n + u\boldsymbol{\kappa}_1))(z) - z - u\kappa_1 t| \\ &= 2\epsilon + \left| \int_0^t Z_n\left(\exp(s(Z_n + u\boldsymbol{\kappa}_1))(z)\right)\, ds \right| \\ &= 2\epsilon + \left| \int_0^t \left(\exp(s(Z_n + u\boldsymbol{\kappa}_1))(z)\right)^n\, ds \right| \\ &\leq 2\epsilon + \epsilon^n t, \end{aligned}$$

for all $u > 0$, $t \in (0, T')$, a contradiction. $\square$

Proposition 3.36. *For all $\epsilon > 0$, $u > 0$ and $|z| \geq \epsilon$ such that $z \in D_{n-1}$, we have:*

$$\exp(t(Z_n + u\boldsymbol{\kappa}_1))(z) \notin D_{n-1} \text{ for some } t \in (0, T'),$$

for all $T' \leq T_{\max}$ where $T_{\max} > 0$ is the maximal time of definition for $\exp(t(Z_n + u\boldsymbol{\kappa}_1))(z)$.

Proof. Let $\epsilon > 0$, $u > 0$, $T' \leq T_{\max}$, and $|z| \geq \epsilon$ with $z \in D_{n-1}$. Suppose to the contrary that $(\exp(t(Z_n + u\boldsymbol{\kappa}_1))(z))^{n-1} > 0$ for all $t \in (0, T')$ and let g be a primitive $(n-1)$st root of unity. By continuity, there exists $j \in \mathbb{Z}$ such that $g^j \exp(t(Z_n + u\boldsymbol{\kappa}_1))(z) \in \mathbb{R}$ for all $t \in [0, T')$. But note this implies:

$$\begin{aligned} g^j \exp(t(Z_n + u\boldsymbol{\kappa}_1))(z) - g^j z &= g^j \int_0^t (\exp(s(Z_n + u\boldsymbol{\kappa}_1))(z))^n \, ds + g^j u\kappa t \in \mathbb{R} \\ &= \int_0^t \left(g^j \exp(s(Z_n + u\boldsymbol{\kappa}_1))(z)\right)^n \, ds + g^j u\kappa t \in \mathbb{R}, \end{aligned}$$

for all $t \in (0, T')$. In particular, $g^j \kappa_1 \in \mathbb{R}$, a contradiction. $\square$

Let us collect the previous three propositions into a Lemma.

Lemma 3.37. *For all $\epsilon > 0$, there exist a time $T_1' = T_1'(\epsilon) > 1$ such that for all $z \in \mathbb{C}$, there exist vector fields $Y_1, Y_2, Y_3 \in F$ and times $t_1, t_2 \geq 0$ such that $t_1 + t_2 \leq 1$ and*

$$|\exp(tY_3) \circ \exp(t_2 Y_2) \circ \exp(t_1 Y_1)(z)| \in (0, \epsilon] \text{ for all } t \geq T_1' - 1$$

where the last vector field Y_3 can always be chosen to be $Z_n \in F$. Moreover, for all $t \geq T_1' - 1$ the path

$$\exp(tY_3) \circ \exp(t_2 Y_2) \circ \exp(t_1 Y_1)(z) \in D_{n-1}^c.$$

Proof. Let $\epsilon > 0$ and pick $T_1(\epsilon)$ such that the first proposition holds. Take $T_1'(\epsilon) = T_1(\epsilon) + 1$. Thus for all $|z| \geq \epsilon$ with $z \notin D_{n-1}$, the conclusions of the lemma hold by taking $Y_1 = Y_2 = Y_3 = Z_n$ and $t_1 = t_2 = 0$. If $|z| \leq \epsilon + 1$, by the second proposition there exists a $u_1 > 0$ such that

$$|\exp(t_1(Z_n + u_1\boldsymbol{\kappa}_1))(z)| \in (\epsilon + 1, R),$$

for some $t_1 \in (0, 1/3)$ and $R > \epsilon + 1$. If $\exp(t_1(Z_n + u_1\boldsymbol{\kappa}_1))(z) \notin D_{n-1}$, let $Y_1 = Z_n + u_1\boldsymbol{\kappa}_1$ and $Y_2 = Y_3 = Z_n$ and $t_2 = 0$ and note that the conclusions hold. If $\exp(t_1(Z_n + u_1\boldsymbol{\kappa}_1))(z) \in D_{n-1}$, let $z_1 = \exp(t_1(Z_n + u_1\boldsymbol{\kappa}_1))(z)$ and $u_2 > 0$. By the third lemma, we have

$$\exp(t_2(Z_n + u_2\boldsymbol{\kappa}_1))(z_1) \notin D_{n-1} \text{ for some } t_2 \in (0, T').$$

for all $T' > 0$ for which $T' \leq T_{\max}$. By choosing $T' < 1/3$ smaller if necessary, we may assure that

$$|\exp(t_2(Z_n + u_2\boldsymbol{\kappa}_1))(z_1)| > \epsilon.$$

Thus we let $Y_1 = Z_n + u_1\boldsymbol{\kappa}_1$, $Y_2 = Z_n + u_2\boldsymbol{\kappa}_1$, and $Y_3 = Z_n$ and note that the conclusions hold. The only other case we must handle is when $|z| \geq \epsilon + 1$ and $z \in D_{n-1}$, but this follows easily from the above by replacing z_1 with z. □

We have now guided any initial point $z \in \mathbb{C}$ into the closed ball $\overline{B_\epsilon(0)}$ in a very specific manner. This will be extremely important to show

$$z_0 \in A_F(z_0, \leq T) \text{ for all } T > 0,$$

where $z_0 \in \overline{B_\epsilon(0)} \cap D_{n-1}^c$ belongs to the image of one of the trajectories defined in the lemma. Before we proceed further, we first note that we have shown:

Corollary 3.38. *For all $z \in \mathbb{C}$ and $t > 0$:*

$$\overline{A_F(z, T_1' \leq t)} \supset H(z(\epsilon, \kappa_1), \kappa_1).$$

Proof. This follows immediately by the previous lemma and by noting for all $z_0 \in \overline{B_\epsilon(0)}$ and all $T > 0$, $\overline{A_F(z_0, \leq T)} \supset H(z(\epsilon, \kappa_1), \kappa_1)$. □

We now hope to show

$$z_0 \in A_F(z_0, \leq T) \text{ for all } T > 0,$$

where $z_0 \in \overline{B_\epsilon(0)} \cap D_{n-1}^c$ belongs to the image of one of the trajectories defined in the lemma. To do this, we require a few more propositions.

Proposition 3.39. *For all $z \neq 0$ such that $z \notin D_{n-1}$:*

$$\lim_{t \to \infty} \arg\left(\exp(tZ_n)(z)^{n-1}\right) = [\pi],$$

where $[\theta]$ is the equivalence class of the angle θ under $\theta \sim \theta'$ iff $\theta = \theta + 2\pi k$ for some $k \in \mathbb{Z}$. If we further assume $|z| \geq \epsilon$, then the limit is uniform in the initial condition.

Proof. Using the expression 3.32, we obtain:

$$\arg\left(\exp(tZ_n)(z)^{n-1}\right) = [\pi] + \arg\left(\frac{1}{(n-1) - \frac{z^{n-1}}{t|z|^{2n-2}}}.\right)$$

Take $t \to \infty$ to obtain the result and note that if $|z| \geq \epsilon$, we can take the limit independent of $|z| \geq \epsilon$. □

Proposition 3.40. *For all $\epsilon > 0$, there exists a time $T_2(\epsilon) > 0$ such that*

$$\{\exp(tZ_n)(z) + s\,\kappa_1 \,:\, s \in \mathbb{R}\} \cap \{0\} = \emptyset$$

for all $t \geq T_2$, and all $|z| \geq \epsilon$ such that $z \notin D_{n-1}$. In particular, $T_2 > 0$ can be chosen so that the lines

$$l(z,t) = \{\exp(tZ_n)(z) + s\,\kappa_1 \,:\, s \in \mathbb{R}\}$$

intersect the lines $\{g^j s \,:\, s \in \mathbb{R}\}$ for all $j \in \mathbb{Z}$ away from the origin for all $t \geq T_2$ and all $|z| \geq \epsilon$, $z \notin D_{n-1}$. Here again g is a primitive $(n-1)$st root of unity.

Proof. By the previous proposition, for all $\delta > 0$, we may choose $t_2 = t_2(\epsilon) > 0$ such that for all $|z| \geq \epsilon$ and $z \notin D_{n-1}$

$$\arg(\exp(tZ_n)(z)) \in \left(\frac{\pi + 2\pi k}{n-1} - \delta, \frac{\pi + 2\pi k}{n-1} + \delta\right) \text{ for all } t \geq t_2$$

for some $k \in \mathbb{Z}$. Consider the set

$$S(k,\delta) = \left\{\arg(z) \in \left(\frac{\pi + 2\pi k}{n-1} - \delta, \frac{\pi + 2\pi k}{n-1} + \delta\right)\right\}$$

and suppose there exists a sequence $\delta_j \downarrow 0$ as $j \to \infty$ such that

$$0 \in S(k,\delta_j) + \{s\,\kappa_1 \,:\, s \in \mathbb{R}\}$$

for all j. Write $0 = z(\delta_j) + s(\delta_j)\kappa_1$ where $z(\delta_j) \in S(k,\delta)$ and note that $s(\delta_j) \neq 0$ and cannot change sign since $z(\delta_j) \neq 0 \in S(k,\delta_j)$ for all $\delta_j > 0$. But note that this implies $\arg(\kappa_1) = -\arg(z(\delta_j)/s(\delta_j)) \to \pm(\pi + 2\pi k)/(n-1)$ as $j \to \infty$. In particular, $\kappa_1^{n-1} \in \mathbb{R}$, a contradiction. Therefore there exists $\delta' > 0$ sufficiently small such that

$$0 \notin S(k,\delta') + \{s\,\kappa_1 \,:\, s \in \mathbb{R}\}$$

for all $k \in \mathbb{Z}$. Note this implies the first result after taking $T_2 = t_2(\epsilon) > 0$ where t_2 is chosen so that we are within δ'. The second result follows easily with the same choice of T_2 since κ_1 is transversal to D_{n-1}. □

Proposition 3.41. *Using the notation in the previous proposition, let $n \geq 4$ be even. For all $|z| \geq \epsilon$, $z \notin D_{n-1}$, and all $t \geq T_2(\epsilon)$ there exist $g^{j_1} \neq g^{j_2}$*

$$l(z,t) \cap \{g^{j_1}\,s \,:\, s > 0\} \neq \emptyset \text{ and } l(z,t) \cap \{g^{j_2}\,s \,:\, s < 0\} \neq \emptyset.$$

Proof. Fix $|z| \geq \epsilon$ such that $z \notin D_{n-1}$ and $t \geq T_2(\epsilon)$. By the previous proposition, $l(z,t)$ must intersect the lines $\{g^j\,s \,:\, s \in \mathbb{R}\}$ for all $j \in \mathbb{Z}$ away from the origin. In particular, this means $l(z,t)$ must intersect the sets $\{g^j\,s \,:\, s \neq 0 \in \mathbb{R}\}$ for all $j \in \mathbb{Z}$. Since $n \geq 4$, this implies the result since $l(z,t)$ is a line. □

Proposition 3.42. *Suppose that $n = 2$. For all $|z| \geq \epsilon$ such that $z \notin D_{n-1}$ and all $t \geq T_2(\epsilon)$, the line $l(z,t)$ intersects two trajectories of Z_n of opposing direction relative to $l(z,t)$.*

Proof. We know that by Proposition 3.40, $l(z,t)$ must intersect the set $\{s : s \neq 0 \in \mathbb{R}\}$. Suppose first that $l(z,t) \cap \{s : s < 0\} \neq \emptyset$ and that the slope of the line is positive. Since $l(z,t)$ is transversal to the line $\{s : s \in \mathbb{R}\}$, the associated trajectory of Z_n starting on $\{s < 0\}$ points to the right of the line $l(z,t)$. We thus need to find a trajectory of Z_n that points to the left of this line. If the line $l(z,t)$ passes through the first quadrant where $|y| > |x|$, all trajectories of Z_n in this quadrant are strictly increasing in the imaginary direction and strictly decreasing in the real direction, hence point to the left of the line. If the line passes through the first quadrant only where $|y| < |x|$, the trajectories of Z_n in the imaginary direction are strictly increasing, the rate of which increases as $x \to \infty$, hence the trajectories eventually point to the left of the line. If $l(z,t) \cap \{s : s < 0\} \neq \emptyset$ and the slope of the line is negative, the associated trajectory starting on $\{s < 0\}$ points to the right of the line. We thus need to find a trajectory of Z_n that points to the left of this line. If the line $l(z,t)$ passes through the fourth quadrant where $|y| > |x|$, all trajectories of Z_n in this quadrant are strictly decreasing in the imaginary direction and strictly decreasing in the real direction, hence point to the left of the line $l(z,t)$. If the line passes through the fourth quadrant only where $|y| < |x|$, the trajectories of Z_n in the imaginary direction are strictly decreasing, the rate of which strictly decreases as $x \to \infty$, hence the trajectories eventually point to the left of the line. If $l(z,t)$ is vertical, the trajectory of Z_n on $\{s < 0\}$ points to the right of the line. Moreover, there exists a trajectory in the second quadrant that points to the left of $l(z,t)$ since the trajectories are strictly decreasing in the real direction. This handles all cases when $l(z,t)$ intersects $\{s < 0\}$. The cases when $l(z,t)$ intersects $\{s > 0\}$ are done similarly. □

Lemma 3.43. *For all $n \geq 2$ even, $|z| \geq \epsilon$ such that $z \notin D_{n-1}$, $t \geq T_2(\epsilon)$, $z_1 \in l(z,t)$, and all $T > 0$:*

$$z_1 \in A_F(z_1, \leq T).$$

Proof. Fix $n \geq 2$ even, $|z| \geq \epsilon$ such that $z \notin D_{n-1}$, $t \geq T_2$, $z_1 \in l(z,t)$, and $T > 0$. We know that for all $t > 0$

$$\overline{A_F(z_1, \leq t)} \supset H(z_1, \kappa_1).$$

By Theorem 2 on p. 68 of [Jur97], we have for all $t > 0$:

$$A_F(z_1, \leq t) \supset \text{int}(H(z_1, \kappa_1)), \tag{3.44}$$

where int is the interior. By the previous two propositions, the line $l(z,t)$ intersects (at least) two trajectories of Z_n of opposing direction relative to $l(z,t)$. Hence pick the direction that opposes the direction of κ_1^n. We may get arbitrarily close to this trajectory via 3.44 within any amount of positive time $t > 0$. Hence we may choose $t > 0$ small enough such that we flow opposite of κ_1^n across the line $l(z,t)$ by or before time $S < T$. We realize that from this point, the accessibility set in time $\epsilon > 0$ or less must contain z_1 for all $\epsilon > 0$. Note that this finishes the proof. □

We note that this finishes the proof of Lemma 3.30 in Case 2.

3.7 Explosive Case

We now handle Theorem 3.2. Under these assumptions, our stochastic differential equation takes the form

$$dz(t) = z(t)^n \, dt + \kappa \, dW(t), \tag{3.45}$$

where $\kappa^{n-1} \in \mathbb{R} \setminus \{0\}$ and $W(t)$ is a one-dimensional standard Wiener process. For a primitive $(n-1)$st root of unity g, there exists $j \in \mathbb{Z}$ such that $g^j \kappa \in \mathbb{R}$. Hence, if we let $w(t) = g^j z(t)$, we obtain:

$$dw(t) = w(t)^n \, dt + g^j \kappa dW(t). \tag{3.46}$$

Rephrasing this, the solution $z(t)$ of equation 3.45 starting from $z_0 \in \mathbb{C}$ explodes if and only if the solution $w(t)$ of equation 3.46 starting from $g^j z_0$ explodes. We will thus argue from the second equation and prove:

Lemma 3.47. *For all $x > 0$,*

$$P_x \left\{ \xi_{w(t)} < \infty \right\} > 0,$$

where $\xi_{w(t)}$ is the explosion time of the process $w(t)$.

Proof. Note that since $x > 0$ and $g^j \kappa \in \mathbb{R}$, there exists a real-valued solution $x(t)$ with $x(0) = x$ of the equation:

$$dx(t) = x(t)^n \, dt + g^j \kappa \, dW(t)$$

which has the same distribution as $w(t)$ with $w(0) = x$. Thus it suffices to prove

$$P_x \left\{ \xi_{x(t)} < \infty \right\} > 0.$$

We thus apply Feller's test as in Section 2.2.2. Let $\alpha = g^j \kappa$ and note that

$$\begin{aligned} \phi(x) &= \int_0^x \exp\left(\int_0^y -2 \frac{\tilde{y}^n}{\alpha^2} d\tilde{y} \right) dy \\ &= \int_0^x \exp\left(-2 \frac{y^{n+1}}{(n+1)\alpha^2} \right) dy, \end{aligned}$$

and

$$\begin{aligned} m(x) &= \frac{1}{\phi'(x)\alpha^2} \\ &= \alpha^{-2} \exp\left(2 \frac{x^{n+1}}{(n+1)\alpha^2} \right). \end{aligned}$$

It is clear that $\phi(c) \uparrow c_\infty \in (0, \infty)$ as $c \to \infty$. Thus we must prove

$$\int_0^\infty dx \, m(x)(c_\infty - \phi(x)) dx < \infty.$$

It is clear that the integral $\int_0^1$ is finite. We shall prove then that there exists $\delta, C > 0$ such that

$$m(x)(c_\infty - \phi(x)) \leq \frac{C}{x^{1+\delta}} \text{ for all } x \geq 1.$$

Note by L'Hospital's rule for $\delta < n+1$ we have

$$\begin{aligned}
\lim_{x\to\infty} x^{1+\delta} m(x)(c_\infty - \phi(x)) &= \lim_{x\to\infty} \frac{(c_\infty - \phi(x))}{x^{-1-\delta}\alpha^2 \exp\left(-2\frac{x^{n+1}}{(n+1)\alpha^2}\right)} \\
&= \lim_{x\to\infty} \frac{1}{2x^{n-1-\delta} + \alpha^2(1+\delta)x^{-\delta-2}} \\
&= 0,
\end{aligned}$$

which finishes the proof. □

Chapter 4
Summary

In this work, we found that the maximal class of $(\kappa_1, \kappa_2) \in \mathbb{C}$ such that the SDE (1.2) has the ergodic property consists solely of pairs (κ_1, κ_2) that are transversal to D_{n-1}. Outside the realm of such noise, there are solutions of (1.2) which reach infinity in finite time with positive probability. In the case when $n = 2$, the problem was originally motived by applications to turbulent transport of inertial particles [GHW10]. For $n \geq 3$, the problem was driven by a simple geometric intuition that noise transversal to all isolated unstable trajectories stabilizes the system as a whole. This intuition is validated here; however, in [Sch93] this is not the case. The difference between the two is that the dynamics in [Sch93] "cooked up" to disagree with this intuition. We feel, therefore, there should be a general class of functions b and σ as in (2.1) for our intuition to hold. Using the results of this dissertation, one has a natural place to start to determine such a class.

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