NORMAL FORMS FOR THE HOPF BIFURCATIONS IN THE LORENZ SYSTEM

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Abstract. In this essay we compute the the twist number for the Hopf bifurcations in the Lorenz equation. This number is the ratio of the real and the imaginary part of the first coefficient of the normal form. Our purpose is to investigate the possibility of rank chaos for periodically kicked Lorenz system.

In this essay we study the Hopf bifurcation around one of the fixed points in Lorenz butterfly. Our objective is to compute the normal form to obtain the twist number to determine if rank one chaos is likely to occur when the equation is periodically kicked.

Let

$$\frac{dx}{dt} = A_\mu x + f_\mu(x) \tag{0.1}$$

be an ordinary differential equation where $x \in \mathbb{R}^n, n \geq 2$ is the phase variable and μ is a parameter. A_μ is a $n \times n$ matrix, and $f_\mu(x)$ are high order terms at $x = 0$. We assume that all eigenvalues of A_μ is with a negative real part except a conjugating pair, which we denote as $\lambda_{1,2}$. We assume that $\lambda_{1,2} = a(\mu) \pm \omega(\mu)i$ where $a(0) = 0, a'(0) > 0$ and $\omega(0) \neq 0$. Corresponding to $\lambda_{1,2}$, equation (0.1) has a center manifold at $x = 0$, which we denote as W^c. On W^c, we can write the induced flow of equation (0.1) in a complex variable and derived a normal form. The first coefficient of this normal form, which we denote as $k_1(\mu)$, determines what happens around $x = 0$ at $\mu = 0$. In general, there is a *generic Hopf bifurcation* at $\mu = 0$ around $x = 0$ if $Re(k_1(0)) \neq 0$.

In the case of a generic Hopf bifurcation, a periodic solutions emerges out of $x = 0$ as μ passes $\mu = 0$. This periodic solution is the *Hopf limit cycle.* If $Re(k_1(0)) < 0$, then $x = 0$ is a stable fixed point for $\mu < 0$. It becomes unstable for $\mu > 0$, and the Hopf limit cycle is a stable limit cycle. This is the case of a *sup-critical Hopf bifurcation.* If $Re(k_1(0)) > 0$. Then $x = 0$ changes from unstable to stable and the periodic solution emerged is unstable. This is the case of a *sub-critical Hopf bifurcation.*

As an important mathematical phenomenon, Hopf bifurcation has been been studied extensively in both theory and application [GH], [MM]. When a Hopf bifurcation occurs, it is critically important to determine if it is *sup-critical* or *sub-critical.* Therefore all attentions has been focused on the sign of $Re(k_1(0))$. In this essay we go one step further to compute the twist number, i.e., the ratio of $Im(k_1(0))$ and $Re(k_1(0))$. This number represents the strength of shearing around the Hopf limit cycle. The importance of the twist number is recently stressed in [WY], [WO] where the possibility of rank one chaos in periodically kicked Hopf bifurcations are investigated. Rank one chaos only occur around parameters with a large twist number.

In this essay we compute the the twist number for the Hopf bifurcations in the Lorenz equation. Our purpose is to investigate the possibility of the rank chaos for periodically kicked Lorenz system.

1. Lorenz Equation and the Fixed Point

The Lorenz equation was introduced by Lorenz in 1963 [L]. It is presented as an analysis of a coupled set of three quadratic ordinary differential equations representing three modes of the Oberbeck-Boussinesq equations for fluid convection in a two dimensional layer heated from below. The equations are

$$\begin{aligned} \frac{dx_1}{dt} &= -\sigma x_1 + \sigma y_1 \\ \frac{dy_1}{dt} &= r x_1 - y_1 - x_1 w_1 \\ \frac{dw_1}{dt} &= -b w_1 + x_1 y_1 \end{aligned} \tag{1.1}$$

This equation has a fixed point at $(x_1, y_1, w_1) = (\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1)$. Our study will be restricted to a small neighborhood of this fixed point.

First we move the the fixed point to the center of the coordinate system. Let

$$\begin{aligned} x_2 &= x_1 - \sqrt{(b(r-1)} \\ y_2 &= y_1 - \sqrt{(b(r-1)} \\ w_2 w_1 &- r + 1 \end{aligned} \tag{1.2}$$

The equations for the new variables (x_2, y_2, w_2) are

$$\begin{aligned} \frac{dx_2}{dt} &= -\sigma x_2 + \sigma y_2 \\ \frac{dy_2}{dt} &= x_2 - y_2 - w_2\sqrt{b(r-1)} - x_2 w_2 \\ \frac{dw_2}{dt} &= \sqrt{b(r-1)}x_2 + \sqrt{b(r-1)}y_2 - b w_2 + x_2 y_2 \end{aligned} \tag{1.3}$$

In matrix form, this gives the new system of ODEs:

$$\frac{d}{dt}\begin{bmatrix} x_2 \\ y_2 \\ w_2 \end{bmatrix} = M \begin{bmatrix} x_2 \\ y_2 \\ w_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -x_2 w_2 \\ x_2 y_2 \end{bmatrix}$$

where

$$M = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -\sqrt{b(r-1)} \\ \sqrt{b(r-1)} & \sqrt{b(r-1)} & -b \end{bmatrix}$$

Next, we find the eigenvalues of the matrix M. Let

$$\begin{aligned}
det(M-\lambda I) &= det\begin{bmatrix} -\sigma-\lambda & \sigma & 0 \\ 1 & -1-\lambda & -\sqrt{b(r-1)} \\ \sqrt{b(r-1)} & \sqrt{b(r-1)} & -b-\lambda \end{bmatrix} \\
&= (-\sigma-\lambda)[(-1-\lambda)(-b-\lambda)+b(r-1)] - \sigma[(-b-\lambda)+b(r-1)] = \\
&= (-\sigma-\lambda)(b+b\lambda+\lambda+\lambda^2+br-b) - \sigma(-b-\lambda+br-b) \\
&= -(\lambda+\sigma)(\lambda^2+(b+1)\lambda+br) + \sigma(\lambda-br+2b) \\
&= -(\lambda^3+(b+1)\lambda^2+br\lambda+\sigma\lambda^2+\sigma(b+1)\lambda+\sigma br) + \sigma\lambda - \sigma br + 2b\sigma \\
&= -\lambda^3-(b+\sigma+1)\lambda^2-(br+\sigma b+\sigma)\lambda - \sigma br + \sigma\lambda - \sigma br + 2b\sigma \\
&= -\lambda^3-(b+\sigma+1)\lambda^2-(br+\sigma b+\sigma-\sigma)\lambda - 2\sigma br + 2b\sigma \\
&= -\lambda^3-(b+\sigma+1)\lambda^2-b(\sigma+r)\lambda - 2b\sigma(r-1)
\end{aligned}$$

This polynomial has three roots, one real negative root and two complex conjugate roots. Call them λ and $\alpha \pm \beta i$. Next we compute their eigenvectors.

Let

$$v_1 = (v_{11}, v_{12}, v_{13})$$

be the eigenvector of λ. Then, by the definition of eigenvector, we get the equations:

$$\begin{aligned}
-\sigma v_{11} + \sigma v_{12} &= \lambda v_{11} \\
v_{11} - v_{12} - \sqrt{b(r-1)}v_{13} &= \lambda v_{12} \\
\sqrt{b(r-1)}v_{11} + \sqrt{b(r-1)}v_{12} - bv_{13} &= \lambda v_{13}
\end{aligned} \tag{1.4}$$

From the first item of (1.4),

$$v_{11} = \frac{\sigma}{\lambda+\sigma}v_{12}.$$

Substituting this back into (1.4) yields new equations:

$$\begin{aligned}
\frac{\sigma}{\lambda+\sigma}v_{12} - v_{12} - \sqrt{b(r-1)}v_{13} &= \lambda v_{12} \\
\sqrt{b(r-1)}\frac{\sigma}{\lambda+\sigma}v_{12} + \sqrt{b(r-1)}v_{12} - bv_{13} &= \lambda v_{13}
\end{aligned}$$

which, when simplified, are:

$$\begin{aligned}
\frac{-\lambda}{\lambda+\sigma}v_{12} - \sqrt{b(r-1)}v_{13} &= \lambda v_{12} \\
\sqrt{b(r-1)}\frac{\lambda+2\sigma}{\lambda+\sigma}v_{12} - bv_{13} &= \lambda v_{13}
\end{aligned} \tag{1.5}$$

Solving (1.5) for v_{13} produces:

$$v_{13} = \frac{(\lambda+2\sigma)\sqrt{b(r-1)}}{(\lambda+b)(\lambda+\sigma)}v_{12} \tag{1.6}$$

This determines the one-dimensional space of eigenvectors. For simplicity, I will choose the eigenvector where

$$v_{12} = (\lambda+b)(\lambda+\sigma).$$

Then, we obtain

$$v_1 = (\sigma(\lambda + b), \quad (\lambda + b)(\lambda + \sigma), \quad (\lambda + 2\sigma)\sqrt{b(r-1)}).$$

Next, we use the definition to determine the eigenvectors for $\alpha+\beta i$. Call the real component of the eigenvector

$$p = (\ p_1, \quad p_2, \quad p_3\ )$$

and the imaginary component

$$q = (\ q_1, \quad q_2, \quad q_3\ ).$$

For simplicity, I will denote

$$\gamma = \sqrt{b(r-1)}$$

from now on. We now have equations:

$$\begin{aligned} (-\sigma - \alpha - \beta i)(p_1 + iq_1) + \sigma(p_2 + iq_2) &= 0 \\ p_1 + iq_1 - (1 + \alpha + \beta i)(p_2 + iq_2) - \gamma(p_3 + iq_3) &= 0 \\ \gamma(p_1 + iq_1 + p_2 + iq_2) - (b + \alpha + \beta i)(p_3 + iq_3) &= 0 \end{aligned} \tag{1.7}$$

which simplify to:

$$\begin{aligned} -\sigma p_1 - \alpha p_1 + \sigma p_2 + \beta q_1 + i(-\beta p_1 - \sigma q_1 - \alpha q_1 + \sigma q_2) &= 0 + 0i \\ p_1 - p_2 - \alpha p_2 + \beta q_2 - \gamma p_3 + i(q_1 - \beta p_2 - q_2 - \alpha q_2 - \gamma q_3) &= 0 + 0i \\ \gamma p_1 + \gamma p_2 - b p_3 - \alpha p_3 + \beta q_3 + i(\gamma q_1 + \gamma q_2 - \beta p_3 - b q_3 - \alpha q_3) &= 0 + 0i \end{aligned} \tag{1.8}$$

Equating the real and imaginary coefficients gives a system of six real equations in six unknowns. First, we will solve for the q_i in the equations from the real parts, which gives the equations:

$$\begin{aligned} q_1 &= \frac{(\sigma + \alpha)p_1 - \sigma p_2}{\beta} \\ q_2 &= \frac{-p_1 + (1 + \alpha)p_2 + \gamma p_3}{\beta} \\ q_3 &= \frac{-\gamma p_1 - \gamma p_2 + (b + \alpha)p_3}{\beta} \end{aligned} \tag{1.9}$$

Next, we substitute these values into the equations for the imaginary parts to get:

$$\begin{aligned} -\beta p_1 - (\sigma + \alpha)\frac{(\sigma + \alpha)p_1 - \sigma p_2}{\beta} + \sigma\frac{-p_1 + (1 + \alpha)p_2 + \gamma p_3}{\beta} &= 0 \\ \frac{(\sigma + \alpha)p_1 - \sigma p_2}{\beta} - \beta p_2 - (1 + \alpha)\frac{-p_1 + (1 + \alpha)p_2 + \gamma p_3}{\beta} - \gamma\frac{-\gamma p_1 - \gamma p_2 + (b + \alpha)p_3}{\beta} &= 0 \\ \gamma\frac{(\sigma + \alpha)p_1 - \sigma p_2}{\beta} + \gamma\frac{-p_1 + (1 + \alpha)p_2 + \gamma p_3}{\beta} - \beta p_3 - (b + \alpha)\frac{-\gamma p_1 - \gamma p_2 + (b + \alpha)p_3}{\beta} &= 0 \end{aligned} \tag{1.10}$$

These equations are simplified (including multiplying through by β) to get:

$$\begin{aligned} &[-\beta^2 - (\sigma+\alpha)^2 - \sigma]p_1 + \sigma(1+2\alpha+\sigma)p_2 + \sigma\gamma p_3 = 0 \\ &(1+2\alpha+\sigma+\gamma^2)p_1 + [-\sigma-(1+\alpha)^2-\beta^2+\gamma^2]p_2 - \gamma(1+2\alpha+b)p_3 = 0 \\ &\gamma(\sigma+2\alpha+b-1)p_1 + \gamma(1+2\alpha+b-\sigma)p_2 + [\gamma^2-\beta^2-(b+\alpha)^2]p_3 = 0. \end{aligned} \tag{1.11}$$

We obtain for p_3,

$$p_3 = \frac{[\beta^2+(\sigma+\alpha)^2+\sigma]p_1 - \sigma(1+2\alpha+\sigma)p_2}{\sigma\gamma}$$

which is substituted back into (1.11) to produce:

$$\begin{aligned} &(1+2\alpha+\sigma+\gamma^2)p_1 + [-\sigma-(1+\alpha)^2-\beta^2+\gamma^2]p_2 \\ &-\gamma(1+2\alpha+b)\frac{[\beta^2+(\sigma+\alpha)^2+\sigma]p_1 - \sigma(1+2\alpha+\sigma)p_2}{\sigma\gamma} = 0; \\ &\gamma(\sigma+2\alpha+b-1)p_1 + \gamma(1+2\alpha+b-\sigma)p_2 + [\gamma^2-\beta^2 \\ &-(b+\alpha)^2]\frac{[\beta^2+(\sigma+\alpha)^2+\sigma]p_1 - \sigma(1+2\alpha+\sigma)p_2}{\sigma\gamma} = 0 \end{aligned} \tag{1.12}$$

Equations (1.12) are the simplified to give us:

$$\begin{aligned} &[(1+2\alpha+\sigma+\gamma^2) - \frac{(1+2\alpha+b)[\beta^2+(\sigma+\alpha)^2+\sigma]}{\sigma}]p_1 \\ &+[-\sigma-(1+\alpha)^2-\beta^2+\gamma^2+(1+2\alpha+b)(1+2\alpha+\sigma)]p_2 = 0; \\ &(\gamma(\sigma+2\alpha+b-1) + \frac{[\gamma^2-\beta^2-(b+\alpha)^2][\beta^2+(\sigma+\alpha)^2+\sigma]}{\sigma\gamma})p_1 \\ &+(\gamma(1+2\alpha+b-\sigma) - \frac{[\gamma^2-\beta^2-(b+\alpha)^2](1+2\alpha+\sigma)}{\gamma})p_2 = 0 \end{aligned} \tag{1.13}$$

Further simplification, including multiplying through by γ in equation (1.13), yields:

$$\begin{aligned} &[(\sigma+\gamma^2-b) - \frac{(1+2\alpha+b)[\beta^2+(\sigma+\alpha)^2]}{\sigma}]p_1 \\ &+(-\beta^2+\gamma^2+1+b+3\alpha^2+2\alpha b+2\alpha\sigma+b\sigma)p_2 = 0; \\ &(\gamma^2(\sigma+2\alpha+b) - \beta^2 - (b+\alpha)^2 + \frac{[\gamma^2-\beta^2-(b+\alpha)^2][\beta^2+(\sigma+\alpha)^2]}{\sigma})p_1 \\ &+(\gamma^2(b-2\sigma) + (1+2\alpha+\sigma)[\beta^2+(b+\alpha)^2])p_2 = 0 \end{aligned} \tag{1.14}$$

Next, we solve for p_2 in equation (1.14) to show

$$p_2 = \frac{\sigma\beta^2+\sigma(b+\alpha)^2-\sigma\gamma^2(\sigma+2\alpha+b) - [\gamma^2-\beta^2-(b+\alpha)^2][\beta^2+(\sigma+\alpha)^2]}{\sigma(\gamma^2(b-2\sigma)+(1+2\alpha+\sigma)[\beta^2+(b+\alpha)^2])}p_1.$$

Now, we can choose any $p_1$that satisfies this in order to form our eigenvectors. We choose

$$p_1 = \sigma(\gamma^2(b-2\sigma)+(1+2\alpha+\sigma)[\beta^2+(b+\alpha)^2])$$

which gives the eigenvectors:

(1.15)

$$
\begin{aligned}
p_1 &= \sigma(\gamma^2(b-2\sigma) + (1+2\alpha+\sigma)[\beta^2 + (b+\alpha)^2]) \\
p_2 &= (\sigma\beta^2 + \sigma(b+\alpha)^2 - \sigma\gamma^2(\sigma+2\alpha+b) - [\gamma^2 - \beta^2 - (b+\alpha)^2][\beta^2 + (\sigma+\alpha)^2]) \\
p_3 &= \frac{1}{\gamma}\{[\beta^2 + (\sigma+\alpha)^2 + \sigma](\gamma^2(b-2\sigma) + (1+2\alpha+\sigma)[\beta^2 + (b+\alpha)^2]) \\
&\quad - (1+2\alpha+\sigma)(\sigma\beta^2 + \sigma(b+\alpha)^2 - \sigma\gamma^2(\sigma+2\alpha+b) - [\gamma^2 - \beta^2 - (b+\alpha)^2][\beta^2 + (\sigma+\alpha)^2])\} \\
q_1 &= \frac{1}{\beta}\{\sigma(\sigma+\alpha)(\gamma^2[b-2\sigma] + (1+2\alpha+\sigma)[\beta^2 + (b+\alpha)^2]) \\
&\quad - \sigma(\sigma\beta^2 + \sigma(b+\alpha)^2 - \sigma\gamma^2(\sigma+2\alpha+b) - [\gamma^2 - \beta^2 - (b+\alpha)^2][\beta^2 + (\sigma+\alpha)^2])\} \\
q_2 &= \frac{1}{\beta}\{-p_1 + (1+\alpha)p_2 + \gamma p_3\} \\
q_3 &= \frac{}{\beta}\{-\gamma p_1 - \gamma p_2 + (b+\alpha)p_3\}
\end{aligned}
$$

Once the eigenvectors have been found, we define a new set of variables by multiplying by matrix C:

$$
\begin{bmatrix} x \\ y \\ w \end{bmatrix} = C \begin{bmatrix} x_2 \\ y_2 \\ w_2 \end{bmatrix}
$$

where

$$
C = \begin{bmatrix} p_1 & q_1 & \sigma(\lambda+b) \\ p_2 & q_2 & (\lambda+b)(\lambda+\sigma) \\ p_3 & q_3 & \gamma(\lambda+2\sigma) \end{bmatrix}
$$

In order to find the ODEs in this form, we must find C^{-1}. First, define c_{ij} with i, j from 1 to 3 so that:

$$
\begin{aligned}
C^{-1} &:= \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \\
&= \frac{1}{det(C)} \begin{bmatrix} q_2\gamma(\lambda+2\sigma) - q_3(\lambda+b)(\lambda+\sigma) & p_2\gamma(\lambda+2\sigma) - p_3(\lambda+b)(\lambda+\sigma) & p_2q_3 - p_3q_2 \\ q_1\gamma(\lambda+2\sigma) - q_3\sigma(\lambda+b) & p_1\gamma(\lambda+2\sigma) - p_3\sigma(\lambda+b) & p_1q_3 - p_3q_1 \\ q_1(\lambda+b)(\lambda+\sigma) - q_2\sigma(\lambda+b) & p_1(\lambda+b)(\lambda+\sigma) - p_2\sigma(\lambda+b) & p_1q_2 - p_2q_1 \end{bmatrix}
\end{aligned}
$$

Using matrix C and its inverse, and knowing $C^{-1}AC$, we can convert the system of differential equations:

$$
\frac{d}{dt}\begin{bmatrix} x_2 \\ y_2 \\ w_2 \end{bmatrix} = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -\sqrt{b(r-1)} \\ \sqrt{b(r-1)} & \sqrt{b(r-1)} & -b \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ w_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -x_2w_2 \\ x_2y_2 \end{bmatrix}
$$

to

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ w \end{bmatrix} = C^{-1}AC\begin{bmatrix} x \\ y \\ w \end{bmatrix} + C^{-1}\begin{bmatrix} 0 \\ -x_2w_2 \\ x_2y_2 \end{bmatrix}$$

Note also that this defines x_2, y_2, and w_2 in terms of x, y and w:

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}\begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} x_2 \\ y_2 \\ w_2 \end{bmatrix} => \begin{bmatrix} x_2 \\ y_2 \\ w_2 \end{bmatrix} = \begin{bmatrix} c_{11}x + c_{12}y + c_{13}w \\ c_{21}x + c_{22}y + c_{23}w \\ c_{31}x + c_{32}y + c_{33}w \end{bmatrix}$$

Then, we get the new system:

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} \alpha & \beta & 0 \\ -\beta & \alpha & 0 \\ 0 & 0 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ w \end{bmatrix} + C^{-1}\begin{bmatrix} 0 \\ -x_2w_2 \\ x_2y_2 \end{bmatrix}$$

We must now use equations for C^{-1}, x_2, y_2, w_2 to find the system in terms of x, y, and w. Let

$$\begin{bmatrix} \eta_1(x,y,w) \\ \eta_2(x,y,w) \\ \eta_3(x,y,w) \end{bmatrix} = C^{-1}\begin{bmatrix} 0 \\ -x_2w_2 \\ x_2y_2 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}\begin{bmatrix} 0 \\ -x_2w_2 \\ x_2y_2 \end{bmatrix}$$
$$= \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}\begin{bmatrix} 0 \\ -(c_{11}x + c_{12}y + c_{13}w)(c_{31}x + c_{32}y + c_{33}w) \\ (c_{11}x + c_{12}y + c_{13}w)(c_{21}x + c_{22}y + c_{23}w) \end{bmatrix}$$
$$= \begin{bmatrix} -c_{12}(c_{11}x + c_{12}y + c_{13}w)(c_{31}x + c_{32}y + c_{33}w) + c_{13}(c_{11}x + c_{12}y + c_{13}w)(c_{21}x + c_{22}y + c_{23}w) \\ -c_{22}(c_{11}x + c_{12}y + c_{13}w)(c_{31}x + c_{32}y + c_{33}w) + c_{23}(c_{11}x + c_{12}y + c_{13}w)(c_{21}x + c_{22}y + c_{23}w) \\ -c_{32}(c_{11}x + c_{12}y + c_{13}w)(c_{31}x + c_{32}y + c_{33}w) + c_{33}(c_{11}x + c_{12}y + c_{13}w)(c_{21}x + c_{22}y + c_{23}w) \end{bmatrix}$$
$$= \begin{bmatrix} -c_{12}(c_{11}x + c_{12}y + c_{13}w)(c_{31}x + c_{32}y + c_{33}w) + c_{13}(c_{11}x + c_{12}y + c_{13}w)(c_{21}x + c_{22}y + c_{23}w) \\ -c_{22}(c_{11}x + c_{12}y + c_{13}w)(c_{31}x + c_{32}y + c_{33}w) + c_{23}(c_{11}x + c_{12}y + c_{13}w)(c_{21}x + c_{22}y + c_{23}w) \\ -c_{32}(c_{11}x + c_{12}y + c_{13}w)(c_{31}x + c_{32}y + c_{33}w) + c_{33}(c_{11}x + c_{12}y + c_{13}w)(c_{21}x + c_{22}y + c_{23}w) \end{bmatrix}$$

I will now separate each component, and evaluate individually:

(1.16)
$$\begin{aligned} \eta_1(x,y,w) = &-c_{12}[c_{11}c_{31}x^2 + c_{11}c_{32}xy + c_{11}c_{33}xw + c_{12}c_{31}xy + c_{12}c_{32}y^2 \\ &+ c_{12}c_{33}yw + c_{13}c_{31}xw + c_{13}c_{32}yw + c_{13}c_{33}w^2]+ \\ &+ c_{13}[c_{11}c_{21}x^2 + c_{11}c_{22}xy + c_{11}c_{23}xw + c_{12}c_{21}xy+ \\ &c_{12}c_{22}y^2 + c_{12}c_{23}yw + c_{13}c_{21}xw + c_{13}c_{22}yw + c_{13}c_{23}w^2] \\ = &(-c_{11}c_{12}c_{31} + c_{11}c_{13}c_{21})x^2 + (-c_{11}c_{12}c_{32} - c_{12}^2c_{31} + c_{11}c_{13}c_{22} + c_{12}c_{13}c_{21})xy \\ &+ (-c_{11}c_{12}c_{33} - c_{12}c_{13}c_{31} + c_{11}c_{13}c_{23} + c_{13}^2c_{21})xw + (-c_{12}^2c_{32} + c_{12}c_{13}c_{22})y^2 \\ &+ (-c_{12}^2c_{33} - c_{12}c_{13}c_{32} + c_{12}c_{13}c_{23} + c_{13}^2c_{22})yw + (-c_{12}c_{13}c_{33} + c_{13}^2c_{23})w^2 \end{aligned}$$

For $\eta_2(x,y,w)$ and $\eta_3(x,y,w)$, we have

(1.17)
$$\begin{aligned}\eta_2(x,y,w) &= -c_{22}[c_{11}c_{31}x^2 + c_{11}c_{32}xy + c_{11}c_{33}xw + c_{12}c_{31}xy + c_{12}c_{32}y^2 + c_{12}c_{33}yw \\ &\quad + c_{13}c_{31}xw + c_{13}c_{32}yw + c_{13}c_{33}w^2]+ \\ &\quad + c_{23}[c_{11}c_{21}x^2 + c_{11}c_{22}xy + c_{11}c_{23}xw + c_{12}c_{21}xy \\ &\quad + c_{12}c_{22}y^2 + c_{12}c_{23}yw + c_{13}c_{21}xw + c_{13}c_{22}yw + c_{13}c_{23}w^2] \\ &= (-c_{11}c_{22}c_{31} + c_{11}c_{23}c_{21})x^2 + (-c_{11}c_{22}c_{32} - c_{12}c_{22}c_{31} + c_{11}c_{22}c_{23} + c_{12}c_{21}c_{23})xy \\ &\quad + (-c_{11}c_{22}c_{33} - c_{13}c_{22}c_{31} + c_{11}c_{23}^2 + c_{13}c_{21}c_{23})xw+ \\ &\quad + (-c_{12}c_{22}c_{32} + c_{12}c_{22}c_{23})y^2 + (-c_{12}c_{22}c_{33} - c_{13}c_{22}c_{32} + c_{12}c_{23}^2 + c_{13}c_{22}c_{23})yw \\ &\quad + (-c_{13}c_{22}c_{33} + c_{13}c_{23}^2)w^2\end{aligned}$$

(1.18)
$$\begin{aligned}\eta_3(x,y,w) &= -c_{32}[c_{11}c_{31}x^2 + c_{11}c_{32}xy + c_{11}c_{33}xw + c_{12}c_{31}xy + c_{12}c_{32}y^2 \\ &\quad + c_{12}c_{33}yw + c_{13}c_{31}xw + c_{13}c_{32}yw + c_{13}c_{33}w^2]+ \\ &\quad + c_{33}[c_{11}c_{21}x^2 + c_{11}c_{22}xy + c_{11}c_{23}xw + c_{12}c_{21}xy \\ &\quad + c_{12}c_{22}y^2 + c_{12}c_{23}yw + c_{13}c_{21}xw + c_{13}c_{22}yw + c_{13}c_{23}w^2] \\ &= (-c_{11}c_{31}c_{32} + c_{11}c_{21}c_{33})x^2 + (-c_{11}c_{32}^2 - c_{12}c_{31}c_{32} + c_{11}c_{22}c_{33} + c_{12}c_{21}c_{33})xy \\ &\quad + (-c_{11}c_{32}c_{33} - c_{13}c_{31}c_{32} + c_{11}c_{23}c_{33} + c_{13}c_{21}c_{33})xw+ \\ &\quad + (-c_{12}c_{32}^2 + c_{12}c_{22}c_{33})y^2 + (-c_{12}c_{32}c_{33} - c_{13}c_{31}c_{32} + c_{12}c_{23}c_{33} + c_{13}c_{22} + c_{33})yw \\ &+ (-c_{13}c_{32}c_{33} + c_{13}c_{23}c_{33})w^2\end{aligned}$$

So, the system is now:

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} \alpha & \beta & 0 \\ -\beta & \alpha & 0 \\ 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} + \begin{bmatrix} \eta_1(x,y,w) \\ \eta_2(x,y,w) \\ \eta_3(x,y,w) \end{bmatrix}$$

where$\eta_1(x,y,w), \eta_2(x,y,w)$ and $\eta_3(x,y,w)$ are as in (1.16)-(1.18).

2. Center Manifold and Normal Form

Next, we let $z = x + iy$ and derive the equation in z. Together with $\bar{z} = x - iy$ we have

$$x = \frac{1}{2}(z + \bar{z}), \quad y = \frac{1}{2i}(z - \bar{z}).$$

We now must find $\frac{dz}{dt}$ (note $\frac{d\bar{z}}{dt}$ is the conjugate of $\frac{dz}{dt}$):

$$
\begin{aligned}
\frac{dz}{dt} &= \frac{dx}{dt} + i\frac{dy}{dt} = (\alpha x + \beta y + \eta_1(x,y,w)) + i(-\beta x + \alpha y + \eta_2(x,y,w)) \\
&= (\alpha - \beta i)x + (\alpha i + \beta)y + \eta_1(x,y,w) + i\eta_2(x,y,w) \\
&= (\alpha - \beta i) * \frac{1}{2}(z + \bar{z}) + (\alpha i + \beta) * \frac{1}{2i}(z - \bar{z}) + \eta_1(x,y,w) + i\eta_2(x,y,w) \\
&= \frac{1}{2}[\alpha - \beta i + \alpha + \frac{\beta}{i}]z + \frac{1}{2}[\alpha - \beta i - \alpha - \frac{\beta}{i}]\bar{z} + \eta_1(x,y,w) + i\eta_2(x,y,w) \\
&= (\alpha - \beta i)z + \eta_1(x,y,w) + i\eta_2(x,y,w)
\end{aligned} \tag{2.1}
$$

Thus, the system of (now complex) differential equations is:

$$
\begin{aligned}
\frac{dz}{dt} &= (\alpha - \beta i)z + \eta_1(z,w) + i\bar{\eta_1}(z,w) \\
\frac{d\bar{z}}{dt} &= (\alpha + \beta i)\bar{z} + \bar{\eta_1}(z,w) - i\eta_1(z,w) \\
\frac{dw}{dt} &= \lambda w + \eta_3(z,w)
\end{aligned} \tag{2.2}
$$

where

$$
\begin{aligned}
\eta_1(z,w) &= g_{11}z^2 + g_{12}z\bar{z} + g_{22}\bar{z}^2 + g_{13}zw + g_{23}\bar{z}w + g_{33}w^2 \\
\eta_3(z,w) &= h_{11}z^2 + h_{12}z\bar{z} + h_{22}\bar{z}^2 + h_{13}zw + h_{23}\bar{z}w + h_{33}w^2
\end{aligned}
$$

and we have

$$
\begin{aligned}
g_{11} &= [\frac{1}{4}(-c_{11}c_{12}c_{31} + c_{11}c_{13}c_{21}) + \frac{1}{4i}(-c_{11}c_{12}c_{32} - c_{12}^2c_{31} + c_{11}c_{13}c_{22} + c_{12}c_{13}c_{21}) + \frac{1}{4}(c_{12}^2c_{32} - c_{12}c_{13}c_{22})] \\
g_{12} &= +[\frac{1}{2}(-c_{11}c_{12}c_{31} + c_{11}c_{13}c_{21}) - \frac{1}{2}(c_{12}^2c_{32} - c_{12}c_{13}c_{22})]z\bar{z} \\
g_{22} &= [\frac{1}{4}(-c_{11}c_{12}c_{31} + c_{11}c_{13}c_{21}) - \frac{1}{4i}(-c_{11}c_{12}c_{32} - c_{12}^2c_{31} + c_{11}c_{13}c_{22} + c_{12}c_{13}c_{21}) + \frac{1}{4}(c_{12}^2c_{32} - c_{12}c_{13}c_{22})] \\
g_{13} &= [\frac{1}{2}(-c_{11}c_{12}c_{33} - c_{12}c_{13}c_{31} + c_{11}c_{13}c_{23} + c_{13}^2c_{21}) + \frac{1}{2i}(-c_{12}^2c_{33} - c_{12}c_{13}c_{32} + c_{12}c_{13}c_{23} + c_{13}^2c_{22})] \\
g_{23} &= [\frac{1}{2}(-c_{11}c_{12}c_{33} - c_{12}c_{13}c_{31} + c_{11}c_{13}c_{23} + c_{13}^2c_{21}) - \frac{1}{2i}(-c_{12}^2c_{33} - c_{12}c_{13}c_{32} + c_{12}c_{13}c_{23} + c_{13}^2c_{22})] \\
g_{33} &= [(-c_{12}c_{13}c_{33} + c_{13}^2c_{23})];
\end{aligned}
$$

we also have

$$\begin{aligned}
h_{11} &= \frac{1}{4}(-c_{11}c_{31}c_{32} + c_{11}c_{21}c_{33} + c_{12}c_{32}^2 - c_{12}c_{22}c_{33}) \\
&\quad + \frac{1}{4i}(-c_{11}c_{32}^2 - c_{12}c_{31}c_{32} + c_{11}c_{22}c_{33} + c_{12}c_{21}c_{33}) \\
h_{12} &= \frac{1}{2}(-c_{11}c_{31}c_{32} + c_{11}c_{21}c_{33} - c_{12}c_{32}^2 + c_{12}c_{22}c_{33}) \\
h_{13} &= \frac{1}{2}(-c_{11}c_{32}c_{33} - c_{13}c_{31}c_{32} + c_{11}c_{23}c_{33} + c_{13}c_{21}c_{33}) \\
&\quad + \frac{1}{2i}(-c_{12}c_{32}c_{33} - c_{13}c_{31}c_{32} + c_{12}c_{23}c_{33} + c_{13}c_{22} + c_{33}) \\
h_{22} &= \frac{1}{4}(-c_{11}c_{31}c_{32} + c_{11}c_{21}c_{33} + c_{12}c_{32}^2 - c_{12}c_{22}c_{33}) \\
&\quad - \frac{1}{4i}(-c_{11}c_{32}^2 - c_{12}c_{31}c_{32} + c_{11}c_{22}c_{33} + c_{12}c_{21}c_{33}) \\
h_{23} &= \frac{1}{2}(-c_{11}c_{32}c_{33} - c_{13}c_{31}c_{32} + c_{11}c_{23}c_{33} + c_{13}c_{21}c_{33}) \\
&\quad - \frac{1}{2i}(-c_{12}c_{32}c_{33} - c_{13}c_{31}c_{32} + c_{12}c_{23}c_{33} + c_{13}c_{22} + c_{33}) \\
h_{33} &= -c_{13}c_{32}c_{33} + c_{13}c_{23}c_{33}
\end{aligned}$$

We now move forward to find the center manifold, which we write as

$$w = w_{11}z^2 + w_{12}z\bar{z} + w_{22}\bar{z}^2. \tag{2.3}$$

We find w_{11}, w_{12}, and w_{22} by using equation (2.2). First we have from (2.3),

$$\frac{dw}{dt} = 2w_{11}z\frac{dz}{dt} + w_{12}(z\frac{d\bar{z}}{dt} + \bar{z}\frac{dz}{dt}) + 2w_{22}\bar{z}\frac{d\bar{z}}{dt}.$$

By using equation (2.2), we obtain,

$$\begin{aligned}
&(\lambda w_{11} + h_{11})z^2 + (\lambda w_{12} + h_{12})z\bar{z} + (\lambda w_{22} + h_{22})\bar{z}^2 + h_{13}zw + h_{23}\bar{z}w + h_{33}w^2 \\
&= (2w_{11}\alpha - 2w_{11}\beta i)z^2 + (2w_{12}\alpha)z\bar{z} + (2w_{22}\alpha + 2w_{22}\beta i)\bar{z}^2 + 2w_{11}z\eta_1 + 2iw_{11}z\bar{\eta_1} \\
&+ w_{12}\bar{z}\bar{\eta_1} - w_{12}i\bar{z}\eta_1 + w_{12}\bar{z}\eta_1 + w_{12}i\bar{z}\bar{\eta_1} + 2w_{22}\bar{z}\bar{\eta_1} - 2w_{22}i\bar{z}\eta_1
\end{aligned} \tag{2.4}$$

Now, we will equate the coefficients of the second-degree terms. The remaining terms in the above equation with w and those with η_1 or η_3 can thus be disregarded because they will all be of degree 3 or above.

This gives us the system of equations:

$$\begin{aligned}
\lambda w_{11} + h_{11} &= 2w_{11}\alpha - 2w_{11}\beta i \\
\lambda w_{12} + h_{12} &= 2w_{12}\alpha \\
\lambda w_{22} + h_{22} &= 2w_{22}\alpha + 2w_{22}\beta i
\end{aligned} \tag{2.5}$$

When solved for the w_{11}, w_{12}, and w_{22} values, we see that:

(2.6)
$$\begin{aligned} w_{11} &= \frac{h_{11}}{2\alpha - 2\beta i - \lambda} \\ w_{12} &= \frac{h_{12}}{2\alpha - \lambda} \\ w_{22} &= \frac{h_{22}}{2\alpha + 2\beta i - \lambda} \end{aligned}$$

We will also need their conjugates, which are:

(2.7)
$$\begin{aligned} \bar{w}_{11} &= \frac{\bar{h}_{11}}{2\alpha + 2\beta i - \lambda} \\ \bar{w}_{12} &= \frac{\bar{h}_{12}}{2\alpha - \lambda} \\ \bar{w}_{22} &= \frac{\bar{h}_{22}}{2\alpha - 2\beta i - \lambda} \end{aligned}$$

From this, we can determine that on the center manifold:

$$\begin{aligned} \frac{dz}{dt} &= (\alpha - \beta i)z + \eta_1(z,w) + i\bar{\eta}_1(z,w) \\ &= (\alpha - \beta i)z + g_{11}z^2 + g_{12}z\bar{z} + g_{22}\bar{z}^2 + g_{13}zw + g_{23}\bar{z}w + g_{33}w^2 \\ &\quad + i(\bar{g}_{11}\bar{z}^2 + \bar{g}_{12}z\bar{z} + \bar{g}_{22}z^2 + \bar{g}_{13}\bar{z}\bar{w} + \bar{g}_{23}z\bar{w} + \bar{g}_{33}\bar{w}^2) \\ &= (\alpha - \beta i)z + (g_{11} + i\bar{g}_{22})z^2 + (g_{12} + i\bar{g}_{12})z\bar{z} + (g_{22} + i\bar{g}_{11})\bar{z}^2 \\ &\quad + g_{13}zw + g_{23}\bar{z}w + i\bar{g}_{13}\bar{z}\bar{w} + i\bar{g}_{23}z\bar{w} + g_{33}w^2 + i\bar{g}_{33}\bar{w}^2 \\ &= (\alpha - \beta i)z + (g_{11} + i\bar{g}_{22})z^2 + (g_{12} + i\bar{g}_{12})z\bar{z} + (g_{22} + i\bar{g}_{11})\bar{z}^2 \\ &\quad + g_{13}z(w_{11}z^2 + w_{12}z\bar{z} + w_{22}\bar{z}^2) + g_{23}\bar{z}(w_{11}z^2 + w_{12}z\bar{z} + w_{22}\bar{z}^2) + \\ &\quad + i\bar{g}_{13}\bar{z}(\bar{w}_{11}\bar{z}^2 + \bar{w}_{12}z\bar{z} + \bar{w}_{22}z^2) + i\bar{g}_{23}z(\bar{w}_{11}\bar{z}^2 \\ &\quad + \bar{w}_{12}z\bar{z} + \bar{w}_{22}z^2) + g_{33}(w_{11}z^2 + w_{12}z\bar{z} + w_{22}\bar{z}^2)^2 + i\bar{g}_{33}(\bar{w}_{11}\bar{z}^2 + \bar{w}_{12}z\bar{z} + \bar{w}_{22}z^2)^2 \\ &= (\alpha - \beta i)z + (g_{11} + i\bar{g}_{22})z^2 + (g_{12} + i\bar{g}_{12})z\bar{z} + (g_{22} + i\bar{g}_{11})\bar{z}^2 \\ &\quad + g_{13}w_{11}z^3 + g_{13}w_{12}z^2\bar{z} + g_{13}w_{22}z\bar{z}^2 + g_{23}w_{11}z^2\bar{z} + g_{23}w_{12}z\bar{z}^2 + g_{23}w_{22}\bar{z}^3 + \\ &\quad + i\bar{g}_{13}\bar{w}_{11}\bar{z}^3 + i\bar{g}_{13}\bar{w}_{12}z\bar{z}^2 + i\bar{g}_{13}\bar{w}_{22}z^2\bar{z} + i\bar{g}_{23}\bar{w}_{11}z\bar{z}^2 + i\bar{g}_{23}\bar{w}_{12}z^2\bar{z} \\ &\quad + i\bar{g}_{23}\bar{w}_{22}z^3 + g_{33}(w_{11}z^2 + w_{12}z\bar{z} + w_{22}\bar{z}^2)^2 + i\bar{g}_{33}(\bar{w}_{11}\bar{z}^2 + \bar{w}_{12}z\bar{z} + \bar{w}_{22}z^2)^2 \end{aligned}$$

Therefore the equation on the induced flow on the center manifold is

(2.8)
$$\frac{dz}{dt} = (\alpha - \beta i)z + P_2(z,\bar{z}) + P_3(z,\bar{z})$$

where

$$P_2(z,\bar{z}) = (g_{11} + i\bar{g}_{22})z^2 + (g_{12} + i\bar{g}_{12})z\bar{z} + (g_{22} + i\bar{g}_{11})\bar{z}^2$$

and

$$P_3(z,\bar{z}) = p_{30}z^3 + p_{21}z^2\bar{z} + p_{12}z\bar{z}^2 + p_{03}\bar{z}^3,$$

in which

$$p_{30} = \frac{g_{13}h_{11} + i\bar{g}_{23}\bar{h}_{22}}{2\alpha - 2\beta i - \lambda}$$

$$p_{21} = \frac{g_{13}h_{12} + i\bar{g}_{23}\bar{h}_{12}}{2\alpha - \lambda} + \frac{i\bar{g}_{13}\bar{h}_{22} + g_{23}h_{11}}{2\alpha - 2\beta i - \lambda}$$

$$p_{12} = \frac{g_{13}h_{22} + i\bar{g}_{23}\bar{h}_{11}}{2\alpha + 2\beta i - \lambda} + \frac{g_{23}h_{12} + i\bar{g}_{13}\bar{h}_{12}}{2\alpha - \lambda}$$

$$p_{03} = \frac{g_{23}h_{22} + i\bar{g}_{13}\bar{h}_{11}}{2\alpha + 2\beta i - \lambda}$$

3. Normal Form and The Twist Number

To be completed:
(1) compute the normal form
(2) Formula for the twist number
(3) Putting a few set of parameter values into the entire process to find the corresponding twist number to see if they are large or not. Determine if rank one chaos is likely for periodically kicked equation.

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