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Modeling and identification of nonlinear oscillations

Head, Kenneth Larry, Ph.D.
The University of Arizona, 1989
MODELING AND IDENTIFICATION OF NONLINEAR OSCILLATIONS

by

Kenneth Larry Head

A Dissertation Submitted to the Faculty of the DEPARTMENT OF SYSTEMS AND INDUSTRIAL ENGINEERING
In Partial Fulfillment of the Requirements For the Degree of DOCTOR OF PHILOSOPHY WITH A MAJOR IN SYSTEMS ENGINEERING In the Graduate College THE UNIVERSITY OF ARIZONA

1989
As members of the Final Examination Committee, we certify that we have read the dissertation prepared by Kenneth Larry Head entitled Modeling and Identification of Nonlinear Oscillations and recommend that it be accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy.

Donald G. Schultz
Date Jan 9, 1989

Ferenc Szidarovszky
Date 1/9/89

Suvrajeet Sen
Date

Final approval and acceptance of this dissertation is contingent upon the candidate's submission of the final copy of the dissertation to the Graduate College.

I hereby certify that I have read this dissertation prepared under my direction and recommend that it be accepted as fulfilling the dissertation requirement.

Dissertation Director
Donald G. Schultz
Date Jan 9, 1989
STATEMENT BY AUTHOR

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The topic of this dissertation, modeling and identification of nonlinear oscillation, represents an area of mathematical systems theory that has received little attention in the past. Primarily, the types of oscillation of interest are those found in biological systems where theoretical foundations for mathematical models are insufficient. These oscillations are also observed in other systems including electrical, mechanical, and chemical. The contributions of this dissertation are a generalized class of autonomous differential equations that are found to exhibit stable limit cycles, and an investigation of a method of system identification that can be used to estimate the model parameters.

Here the observed signal is modeled as the response of a nonlinear system that can be described by differential equations. Modeling the signal in the way shifts the emphasis from signal characteristics, such as spectral content, to system characteristics, such as parameter values and system structure. This shift in emphasis may provide a better method for monitoring complex systems that exhibit periodic behavior such as patients under anesthesia.

A class of autonomous differential equations, called the generalized oscillator models, are presented as one \( n^{th} \)-order differential equations with nonlinear coefficients. The coefficients are chosen to change sign depending on the magnitude of the phase variables. The coefficients are negative near the origin and positive away from the origin. Motivated by the generalized Routh-Hurwitz criterion, this coefficient sign changing produces the desired oscillation. Properties of the generalized oscillator model are investigated using the describing function method of analysis and numerical simulation. Several descriptive examples are presented.
Based on the generalized oscillator model as a set of candidate models, the system identification problem is formed as a mathematical programming problem. The method of quasilinearization is investigated as method of solving the identification problem. Two examples are presented that demonstrate the method. It is shown that in general, the method of quasilinearization as a solution to the system identification problem will not converge regardless of the initial starting point. This result indicates that although the quasilinearization method is useful for solving two-point boundary value problems, it is not useful (in its present form) for solving the system identification problem.
CHAPTER 1

INTRODUCTION

1.0 Introduction and Organization of the Chapter

The topic of this dissertation, modeling and identification of nonlinear oscillations, represents an area of mathematical systems theory that has received only little attention in the past. Primarily the types of oscillations of interest are observed in biological systems where theoretical foundations for mathematical models are insufficient. Oscillatory behavior is also observed in other systems including mechanical, electrical and chemical. The contributions of this dissertation are a class of mathematical models that can be used to model the oscillation phenomenon, and an investigation of a method for selecting the model parameters. The class of models that is proposed places the emphasis on modeling the oscillating signal as the response of a nonlinear system, hence shifting the emphasis from signal characteristics to system characteristics.

The purpose of this chapter is to establish the motivation for this research and to outline the organization of the thesis. In the following section the motivation and relationship to a real engineering problem are discussed. In the last section the contents are outlined.

1.1 Motivation

This research was motivated by the need for improved early warning and diagnostic monitoring capability in anesthesia delivery. Inadequate monitoring has been recognized as a potential contributor to the high rate of morbidity and mortality from anesthesia (Cooper 1978, Cooper 1984). Investigations suggest that
current monitoring techniques provide insufficient early warning and inadequate diagnostic information (Waterson 1986).

Much of the information used in monitoring is derived from "periodic", or oscillating, signals measured from the patient and equipment. Included are heart rate, derived from the electrocardiogram, systolic and diastolic blood pressure, derived from the maximum and minimum arterial blood pressure, breathing rate and volume, derived from the maximum and minimum airway pressures, to name a few. This information fails to utilize the additional information that is contained in the signal shape. The shape of the signal contains information that may improve detection and diagnostic methods. The simple fact that interpretation of these waveforms is included in the standard medical education indicates their importance (Smalhout 1983).

Methods intended for utilizing waveshape information have traditionally concentrated on pattern classification schemes, and have not attempted to utilize dynamic system models. One reason for this has been that dynamic models that exhibit periodic behavior are either harmonic oscillators or nonlinear systems, both of which require knowledge not traditional in the medical profession. Harmonic oscillators contain information similar to the Fourier series representation of a periodic signal. Although this representation contains additional signal information that can be used in monitoring, it only shifts the pattern recognition problem from the time domain to the frequency domain.

Nonlinear system models shift the representation from signal characteristics to system characteristics. System characteristics include system structure and parameter values. As an example consider an electronic waveshaping circuit. The circuit may include a clipper, differentiator, amplifier and a rectifier. These elements, together with the way that they are coupled, compose the structure of the system. The clipping level, amplifier gain, resistor and capacitor values compose
the parametric values of the system. The detection and diagnosis problem can then be discussed in terms of the system model, instead of the signal model.

The idea of using system models for detection and diagnosis is not new. Willsky (1976) presents a review of the methods ranging from failure detection filter theory to the theory of statistical tests of filter innovations. Failure detection filter theory (Beard 1970, Jones 1973, White 1987, Chow and Willsky 1984, Massoumnia 1986) prescribes the design of a special class of observers that are sensitive to structural changes and input disturbances in linear systems. Applications include failure detection and diagnosis of a lateral mode autopilot (Jones 1973), sensor failure in the NASA F-8 digital fly-by-wire aircraft (Deckert 1977), cardiac arrhythmia detection and classification (Gustafson 1975), a chemical reactor (Park 1983), for the detection of incidents on freeways (Willsky 1980), and for the detection and diagnosis of faults in an electronically controlled internal combustion engine (Min 1987).

Although the systems in each of these applications are inherently nonlinear, failure detection and diagnosis is performed using a linearized system model. The linearization is justified under the assumption that the system is operating in a relatively constant state. Changes in the system structure or parameters can be detected by shifts from this constant operating state. In the case of oscillating systems, the constant state assumption is no longer valid since the system states are not constant, but are oscillating. The distinction here is very important. In general, oscillating behavior is an undesirable system characteristic. Traditional design and analysis methods concentrate on prediction and prevention of this type of behavior. Hence, there is a need for a nonlinear system model that exhibits oscillator behavior, and will reflect significant changes in the underlying system structure or parameters.

Some nonlinear systems exhibit a stable periodic behavior, called a limit cycle, that can resemble the oscillations of interest. A stable limit cycle is a
periodic solution to a system of nonlinear differential equations that is stable in
the sense that any small disturbance will not change the steady state period or
magnitude of the solution. The fact that small disturbances do not affect the
solution is a desirable property for failure detection and diagnosis. If a significant
change does occur in the underlying system, then the change would be reflected in
the system model structure and/or parameters. Ideally, the changes in the system
model parameters will lead to associated changes in the physical characteristics of
the underlying system.

Nonlinear systems that exhibit stable limit cycles are a possible candidate
class of models for design of a failure detection and diagnosis methodology for
oscillating systems. Given this possibility, two areas of research are of importance.
First, an investigation of the possibility of modeling oscillations as the response
of a nonlinear system with a stable limit cycle must be determined. Next, a
methodology for the detection and diagnosis of failures in these systems is needed.

This dissertation focuses on the first topic. A class of nonlinear system
models is proposed and conditions for the existence of a stable limit cycle are in-
vestigated. Based on this class of system models, a system identification method-
ology for parameter identification is discussed. As yet there is no claim that this
class of models will prove valuable in the solution to the failure detection and
diagnosis problem, but they provide at least one design alternative.

1.2 Outline of the Dissertation

Following this introductory material, Chapter 2 is a brief review of some
definitions and techniques from nonlinear systems theory, a survey of some math-
ematical system models that are known to have limit cycles, and a review of
nonlinear system identification. The emphasis in this chapter is the development
of a basic understanding of the limit cycle and its properties, and of the problem
of nonlinear system identification.
Chapter 3 is devoted to the presentation of a class of differential equations that can be used to model oscillations. The properties of these models are investigated and several examples are presented. The emphasis is placed on the ability to shape the oscillation response by manipulating the equation parameters.

Chapter 4 is devoted to the investigation of a method of identifying the best model given a set of data. In this chapter, the system identification problem is presented as a mathematical programming problem. Here, the identification problem is stated as a minimization of the integral squared error between an observed data set and a parameterized model response, over an allowable range of system parameters. The emphasis is on problem formulation and the presentation of an algorithm that can be used to solve the identification problem.

Chapter 5 is a short summary of the dissertation and identifies several areas of future research.
CHAPTER 2

BACKGROUND

2.0 Introduction and Organization of the Chapter

This chapter provides background information relative to the topic of this dissertation. No attempt has been made to be complete, but only to introduce results that are used in later chapters, or that are necessary for understanding material presented in later chapters.

The chapter is divided into four sections. The first section is devoted to adopting notation and reviewing definitions. The second section concentrates on the properties of limit cycles. Several theoretical methods for determining the existence and stability of limit cycles are discussed. In addition to the theoretical methods discussed in this section, the describing function approximation technique and numerical simulation are used as analysis tools. The third section introduces several classes of systems that exhibit limit cycle behavior. The describing function method is introduced in this section since in addition to its usefulness as an analysis tool it describes a class of system models that exhibit limit cycles. The fourth section surveys the field of nonlinear system identification.

2.1 Notation and Definitions

The following notation is used throughout. Vectors are designated by underlined quantities, as \( \mathbf{z} \). A scalar valued function of a vector is denoted \( f(\mathbf{z}) \). A vector valued function of a vector is denoted \( \mathbf{f}({\mathbf{z}}) \). Matrices are denoted by capital letters, as \( A \) or \( X \), and their dimensions are stated unless it is obvious from the context.
Systems of differential equations can be represented in several forms, including one $n^{th}$ order equation, coupled sets of first and second-order equations, and $n$ first-order equations. Here only autonomous ordinary differential equations are considered. One $n^{th}$ order equation is of the form

$$\frac{d^n y}{dt^n} + a_n \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_2 \frac{dy}{dt} + a_1 y = 0, \quad (2.1)$$

or more conveniently as

$$y^{(n)} + a_n y^{(n-1)} + \cdots + a_2 \dot{y} + a_1 y = 0,$$

where $a_n, a_{n-1}, \ldots, a_1$ may be either constants or functions of $y, \dot{y}, \ldots, y^{(n-1)}$, and $y$ is a function of time.

Coupled sets of first and second order equations can take many different forms since these often arise from physical systems. For example,

$$\dot{y} + a_1 (\dot{y} - \dot{z}) + a_2 (y - z) = 0$$

$$a_1 (\ddot{z} - \dot{y}) + a_2 (z - y) = 0 \quad (2.2)$$

represents a coupled set of one second order and one first order equation. Again, $a_1$ and $a_2$ may be either constants or functions of $y, \dot{y}, z, \ddot{z}$, and $y$ and $z$ are functions of time.

A system of $n$ first-order equations are represented as:

$$\dot{x} = f(x)$$

or,

$$\dot{x}_1 = f_1(x)$$

$$\dot{x}_2 = f_2(x)$$

$$\vdots$$

$$\dot{x}_n = f_n(x) \quad (2.3)$$
where each function, \( f_1, f_2, \ldots, f_n \) of the vector valued function \( \mathbf{f} \) may be either linear or nonlinear, and each element \( x_1, x_2, \ldots, x_n \) of the vector \( \mathbf{x} \) are functions of time.

Knowledge of the vector \( \mathbf{x} \) for any time completely describes the state of the system, hence equations (2.3) are called the state equations. At any point in time the state of the system, \( \mathbf{x} \), can be thought of as a point in the \( n \)-dimensional Euclidean space, called state space. The most familiar state space is the two dimensional plane, called the phase-plane. The phase plane has the coordinate system \((x, \dot{x})\). A general two dimensional state plane has the coordinate system \((x_1, x_2)\), where \( x_1 \) and \( x_2 \) may be any state variables, not necessarily physical variables. A third order system would occupy a three dimensional space, a fourth order a four dimensional space, etc.

As the system evolves over time the state takes on values, denoted \( \mathbf{x}(t) \). The collection over time of these points traces out a trajectory in state space, called the state trajectory. Time is implicit in state space, hence time is not a dimension in state space but can be recovered from the state trajectory, at least in the case of two dimensional systems (Gibson 1963). In the case of linear, and some nonlinear, systems it is possible to obtain an analytic solution to the trajectory in state space, but in the general case of nonlinear systems an analytic solution is impossible to obtain.

In cases when explicit analytic solutions are not available, it is sometimes possible to obtain qualitative properties about the state trajectories. Some of these properties include information about the state trajectory shapes, existence of singular points and limit cycles, and stability. Consider a system represented by (2.3), then for each point \( \mathbf{x}^* \) in state space, there exists a vector \( \mathbf{f}(\mathbf{x}^*) \) that is an element of the vector field \( \mathbf{f}(\mathbf{x}) \). If \( C \) is a state trajectory passing through the point \( \mathbf{x}^* \), then \( \mathbf{f}(\mathbf{x}^*) \) is tangent to \( C \) at \( \mathbf{x}^* \). It is possible to graphically construct the state trajectories by plotting the elements of the vector field \( \mathbf{f}(\mathbf{x}) \) (Vidyasagar
1978). Of course this is only possible for systems of dimensions two or three, and not higher order systems.

Two important characteristics of state space are the existence of singular points and limit cycles. Any point \( x^* \) of (2.3) such that

\[ f(x^*) = 0 \]

is called a singular point or an equilibrium state. If a system is started in an equilibrium state then it will remain in that state for all time. Systems in the form (2.3) are usually assumed to have the origin as a singular point, that is \( f(0) = 0 \). If the origin is not a singular point a simple change of coordinates can be used to shift the singular point to the origin. A limit cycle is defined as any closed curve, \( C \), with any one of the following properties (Hsu and Meyer 1968):

i) All state trajectories in the vicinity of \( C \) tend toward \( C \) as \( t \to \infty \),

ii) All state trajectories in the vicinity of \( C \) tend away from \( C \) as \( t \to \infty \),

iii) All state trajectories in the vicinity of \( C \) fall into two mutually exclusive families, in the first family all trajectories tend towards \( C \) as \( t \to \infty \), and in the other family all trajectories tend away from \( C \) as \( t \to \infty \).

Limit cycles are a particularly interesting class of solutions that exist only in nonlinear systems. Because of their importance to this dissertation, the discussion of limit cycles is considered in greater depth in the next section.

Another important qualitative system property is stability. Stability can be thought of as the least amount of information that is needed to be known about a system. That is, if any one thing was to be known about a system, it would be whether the system is stable or unstable (Schultz and Melsa 1967). There are many definitions of stability, however many of these apply to nonautonomous systems. In linear systems, the definition of stability is basic to all control engineering.
A necessary and sufficient condition for a linear system to be stable is that the impulse response be absolutely integrable from zero to infinity, that is

\[ \int_{0}^{\infty} |h(t)| dt < \infty \]

where \( h(t) \) is the impulse response of the linear system. Equivalently, it is a necessary and sufficient condition for a linear system to be stable that all of the poles have negative real parts. A well known method for determining stability of linear systems is application of the Routh-Hurwitz criterion.

The Routh-Hurwitz criterion is reviewed here and is used in Chapter 3. The criterion is a two step procedure (Melsa and Schultz 1969). If the system is represented by one \( n^{th} \) order differential equation

\[ y^{(n)} + a_n y^{(n-1)} + \cdots + a_2 \dot{y} + a_1 y = 0, \]

or by its characteristic equation as

\[ p(\lambda) = \lambda^n + a_n \lambda^{n-1} + \cdots + a_2 \lambda + a_1, \]

then the first step in the procedure is application of the Hurwitz test. The Hurwitz test consists of a trivial examination of the coefficients of the characteristic polynomial to ensure:

i) All the \( a_i \) are present.

ii) All the \( a_i \) are positive.

If the Hurwitz test fails then the characteristic equation must have at least one pole that does not lie in the left half plane, hence the system is unstable. If the Hurwitz test passes, then no conclusion can be made about the location of the poles, hence the second part of the Routh-Hurwitz procedure is needed.
The second step in the procedure is application of the Routh test. The Routh test is based on the following formation of the Routh array:

\[
\begin{array}{cccccccc}
\text{ } & d_{01} = 1 & d_{02} = a_{n-1} & d_{03} = a_{n-3} & \ldots \\
\text{ } & d_{11} = a_n & d_{12} = a_{n-2} & d_{13} = a_{n-4} & \ldots \\
\text{ } & d_{21} & d_{22} & d_{23} & \ldots \\
\text{ } & d_{31} & d_{32} & d_{33} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ldots \\
d_{n-1,1} & 0 & 0 & \ldots \\
d_{n,1} & 0 & 0 & \ldots 
\end{array}
\]

where

\[d_{ij} = \frac{d_{i-1,1}d_{i-2,j+1} - d_{i-2,1}d_{i-1,j+1}}{d_{i-1,1}}\]

for \(i = 2, 3, \ldots, n\) and \(j = 1, 2, \ldots\). The Routh criterion is as follows:

If there are any sign changes or zeroes in the first column of the Routh array, then the characteristic equation has poles either on the \(j\omega\) axis or in the right half plane. Hence the system is unstable.

If a linear system passes the Routh-Hurwitz stability criterion it can be determined to be stable, and if the test fails the system is determined to be unstable.

The Routh-Hurwitz criterion provides a systematic test for determining stability of linear systems, but does not provide any detailed information as to the precise location of poles or shape of the response.

Stability in nonlinear systems is not simply a question of stable or unstable, since different regions of state space may exhibit different modes of behavior. Stability must be discussed with respect to the characteristics of the associated state space. For example, equilibrium points can be either stable or unstable, and limit cycles can be stable, unstable, or semistable.

In the following discussion it is assumed that the systems are of the form (2.3) and the origin is a singular point. Let \(S(R)\) denote a spherical region in state space such that \(||x|| < R\). Then, the origin is stable if there exists a \(r\) such that \(x(t)\) remains inside \(S(R)\) for all \(t > 0\) and \(x(t)\) starting inside \(S(r)\). The origin is
asymptotically stable if for some \( r > 0 \) every trajectory, \( \mathbf{z}(t) \) starting inside \( S(r) \) tends to the origin as \( t \to \infty \). The origin is unstable if it is not stable. These concepts are illustrated in Figure 2.1 (La Salle and Lefschetz 1961).

One method for determining the stability of a singular point is to linearize the model about the singular point and determine the stability of the linearized system. Of course this method is only applicable to small neighborhoods around the singular point and cannot be used to make statements about global stability. Stability in the sense of Liapunov aims to deduce the properties of system stability from properties of other functions, called Liapunov functions, that summarize system behavior.

The following theorem, called La Salle's theorem, is a general form of Liapunov stability that is applicable to systems that have limit cycles.

**Theorem 2.1** Let \( \Omega \) be a compact set with the property that every solution of (2.3) that begins in \( \Omega \) will remain for all time. Suppose that there is a function \( V(\mathbf{z}) \) that has continuous first partials in \( \Omega \) and that \( dV/dt \leq 0 \) in \( \Omega \). Let \( E \) be the set of all points in \( \Omega \) such that \( dV/dt = 0 \). Let \( M \) be the largest invariant set in \( E \). Then every solution starting in \( \Omega \) approaches \( M \) as \( t \to \infty \).

Liapunov theory depends on knowing or finding a function \( V(\mathbf{z}) \) that will satisfy the properties of La Salles's theorem for a given system. If such a function can be found qualitative statements of stability can be made, otherwise no such conclusions are possible. Liapunov theory can be used to demonstrate the existence and stability of limit cycles if a proper \( V(\mathbf{z}) \) can be found. In the case of two dimensional systems a special form of La Salle's theorem, the Poincare'-Bendixon

\[\uparrow\] A set \( M \) is invariant if each solution starting in \( M \) remains in \( M \) for all time.
Figure 2.1. Concepts of Stability.
(Adapted from La Salle and Lefschetz 1961).
Theorem, establishes sufficient conditions for the existence of a stable limit cycle. The Poincare'-Bendixon theorem is stated as follows:

**Theorem 2.2** If there exists an annular region A in the state plane such that all trajectories of the system cross towards the interior of the region and A contains no singular points, then the region contains a stable limit cycle.

While the Poincare'-Bendixon theorem applies to two dimensional systems, there is no equivalent generalization to higher order systems. Li (1981) presents a review of the current state of attempts to determine an analogous criterion for higher order systems. Results have been limited to a few special cases, but no general theory is currently available.

### 2.2 Limit Cycles

In this section, properties of limit cycles and methods for determining stability are considered. In the literature the discussion of limit cycles is often restricted to two dimensional systems since graphical analysis is applicable and the Poincare'-Bendixon theorem can be used to characterize limit cycle behavior. In higher order systems limit cycle behavior is often referred to as a closed orbit or a periodic attractor (Hirsch and Smale 1974). In the definition presented above, a limit cycle was defined with no reference to the dimension of the system, hence the term limit cycle will be used to refer to this type of behavior for systems of any dimension.

A point on a limit cycle differs from a singular point since any point $x^*$ on the limit cycle $C$ has a direction of movement $f(x^*)$, where at a singular point there is no movement. An important property of limit cycles is periodicity. Since the trajectory of the system is closed, the system must traverse the exact same trajectory every $T$ time units. Hence, if $x^*$ is a point on $C$ at some time $t$, then $x^*$ will be revisited every $t + nT$, $n = 1, 2 \ldots$ time units.
Stability of limit cycles is a difficult property to determine for higher order systems. Application of Liapunov theory requires the generation of functions \( V(x) \) that satisfy the conditions of LaSalle's theorem. Wozny and Schultz (1966) have applied Liapunov theory to a generalized form of Zubov's equations.

Attempts to generalize the Poincare'-Bendixon theorem and to determine conditions for the stability of limit cycles have been discussed by Smith (1980, 1986, 1987). The results obtained are applicable to a class of systems that are dissipative. A system is said to be dissipative if there exists a bounded open subset of the state space, denoted \( D \), such that all trajectories enter and remain in this set. Smith (1987) presents the following theorem for dissipative systems:

Theorem 2.3 If a system of the form (2.3) is dissipative, there exists positive real constants \( \lambda \) and \( \epsilon \), a constant real symmetric non-singular matrix, \( P \), with exactly two negative eigenvalues such that

\[
(x - y)'P(f(x) - f(y) + \lambda(x - y)) \leq -\epsilon|x - y|^2 \tag{2.4}
\]

for every \( x \) and \( y \) in \( D \), \( D \) contains only one singular point at the origin, and the system linearized about the origin has exactly two positive eigenvalues and \( n - 2 \) negative eigenvalues then, \( D \) contains at least one stable limit cycle.

Smith's theorem provides sufficient conditions for the existence of a stable limit cycle, but like Liapunov theory it depends on finding a function (in this case a matrix \( P \)) that will satisfy specific conditions. Also, Smith's theorem allows the possibility of multiple limit cycles instead of only a single limit cycle.

Although Smith's theorem requires the satisfaction of several special conditions, such as the existence of a nonsingular matrix \( P \) and constants \( \lambda \) and \( \epsilon \), that satisfy (2.4), the theorem does provide useful information about the location of the eigenvalues of the system linearized about the origin. This information may be useful in choosing a model that exhibits a stable limit cycle. There is a strong
connection between this requirement and the describing function method that is
discussed in the next section. This relationship is discussed further in Chapter 3.

Another method of determining the existence and stability of a limit cycle
is by the construction of a Poincaré map, and determining if the map has a fixed
point (Hirsh and Smale 1974). The idea of a Poincaré map is illustrated in Figure
2.2 and is described in the following. Consider an $n - 1$ dimensional section of the
state space, denoted $S_0$, such that the system trajectory intersects this section at
some time $t_0$. Denote this point by $x_0$. If $x_0$ is a point on the limit cycle then $T$
time units later the system returns to this point. If $x_0$ is in some neighborhood of
the limit cycle then there is some time $T(x_0)$ where the trajectory intersects $S_0$.
Denote this new point as $x_1$. Continuing in this fashion a sequence of points, $x_k$, $k = 0, 1, \ldots$ can be defined. Then the Poincaré map can be defined as a discrete
time system that describes the sequence of points on $S_0$. Let $h$ denote the Poincaré
map. Then this defines a discrete time system of the form

$$x_{k+1} = h(x_k).$$

If this system has a stable singular point, $x^*$, such that

$$x^* = h(x^*)$$

then the limit cycle is stable, otherwise the limit cycle is unstable.

The difficulty of this approach is the construction of the Poincaré map $h$.
The method does provide a useful approach to determining the existence of limit
cycles when coupled with numerical simulation. If the intersection of the computed
trajectory can be shown to intersect an $n - 1$ dimensional section of the state space
such that the sequence of intersection points converge to a single fixed point, then
a stable limit cycle may be suspected.
Figure 2.2. Illustration of the Poincaré Map
(Adapted from Hirsh and Smale 1974).
To illustrate these concepts consider a simple two-dimensional system

\[
\begin{align*}
\dot{x}_1 &= x_2 + (1 - \dot{x}_1^2 - \dot{x}_2^2)x_1 \\
\dot{x}_2 &= -x_1 + (1 - \dot{x}_1^2 - \dot{x}_2^2)x_2
\end{align*}
\] (2.5)

A typical phase plot is shown in Figure 2.3.

This system is a contrived example that has received considerable treatment in introductory texts that discuss limit cycles (Luenberger 1979, Boyce and DiPrima 1977). The example is contrived since the limit cycle can be seen by inspection to be the unit circle, \(x_1^2 + x_2^2 = 1\). Despite this fact, the example is useful since it can be solved analytically, where most other systems that exhibit limit cycles cannot.

The methods discussed above can be used to determine the existence and stability of the limit cycle in this simple system. To demonstrate the use of Liapunov functions choose \(V(x)\) to be of the form

\[V(x) = \frac{1}{2}(x_1^2 + x_2^2)\]

Then

\[
\frac{d}{dt}V(x) = \dot{V}(x) = (1 - x_1^2 - x_2^2)(x_1^2 + x_2^2).
\]

Clearly, \(V(x)\) is positive definite and \(\dot{V}(x)\) is negative for all \(x\) such that \(|x| > 1\), is positive for all \(|x| < 1\), and is zero for all \(|x| = 1\). From this it can be concluded that all trajectories are approaching the unit circle where the stable limit cycle exists.

To demonstrate the use of Smith's theorem (Theorem 2.3) note that the Liapunov function used above can be used to determine that the system is dissipative. A region \(D\) can be defined as

\[D = \{x : |x| < K\}\]
Figure 2.3. Phase Plot of a Simple 2-Dimensional System.
for some \( K > 1 \), then all system trajectories enter and remain in \( D \). Since all trajectories remain in \( D \) there exists a \( \rho > 0 \) such that

\[
|f(x) - f(y)| \leq \rho|x - y|
\]

for every \( x \) and \( y \) in \( D \). \( P \) can be chosen (Li 1981) as

\[
P = \frac{1}{2} (\lambda - \rho)^{-1} I.
\]

If \( \lambda > \rho \), \( P \) has exactly two negative eigenvalues and satisfies condition (2.4). The origin is the only singular point of (2.5) and the eigenvalues of the system linearized about this point are at \( 1 \pm j \). By each of these facts the conditions of Smith’s theorem are satisfied and it can be concluded that \( D \) contains at least one limit cycle.

To demonstrate the use of the Poincaré map it is useful to change to polar coordinates \( r \) and \( \theta \) where

\[
x = r \cos \theta \\
y = r \sin \theta.
\]

The resulting system is

\[
\dot{r} = r(1 - r^2) \\
\dot{\theta} = -1.
\]

One solution is \( r = 1 \) and \( \theta = -t + t_0 \), thus the system will remain on the unit circle. The other solutions can be found to be

\[
r(t) = \frac{1}{\sqrt{1 + [(1/r_0^2) - 1]\exp^{-2t}}} \\
\theta(t) = -t + t_0
\]

for initial conditions \( r(t_0) = r_0 \) and \( \theta(t_0) = t_0 \). Choose a section, for example the positive half line, such that at some time \( t_s \) the system trajectories intersect
this section. Denote this point as \( z_0 = r(t_*) \). Since \( r \) and \( \theta \) are not coupled, the trajectory will intersect this section every \( 2\pi \) time units, hence

\[
z_{k+1} = \frac{1}{\sqrt{1 + [(1/z_k^2) - 1] \exp^{-2(t_++2\pi)}}}
\]

for \( k = 0, 1, \ldots \). If \( z_k > 1 \), then \( z_{k+1} < z_k \) so the sequence is decreasing. If \( z_k < 1 \), then \( z_{k+1} > z_k \) so the sequence is increasing. If \( z_k = 1 \), then \( z_{k+1} = z_k \), therefore \( z^* = 1 \) is a stable singular point of the discrete time system \( z_{k+1} = h(z_k) \), and the system has a stable limit cycle.

Each of these methods demonstrate a theoretical method for determining the existence and stability of limit cycles. While each method demonstrates an intuitive method, application is limited due to the construction difficulties. When complexity prevents the application of these methods, approximation methods must be used. One approach is to use numerical simulation. Often the results obtained are useful, but in general no behavioral properties can be concluded. Another approximation technique that is particularly useful in determining the existence and stability of limit cycles is describing function analysis. Because of the close relationship between the technique of describing function analysis and the class of system models for which it is applicable, the discussion of this technique is addressed in the next section.

2.3 Models of Nonlinear Oscillations

The purpose of this section is to introduce and classify some system models that have stable limit cycles. Although there are oscillators that are either not structurally stable, such as the simple harmonic oscillator, or that have unstable limit cycles, the discussion here will be limited to systems that have stable limit cycles. These systems are important since any small perturbation about the state trajectory will not destroy the existence of oscillations.
Many physical systems, including mechanical, electrical, thermal, and hydraulic, are inherently nonlinear and exhibit oscillatory behavior. Sometimes these systems can be modeled by systems of differential equations, that arise from standard modeling techniques such as application of Newton's second law, Kirchhoff's voltage and current laws, or in more general terms by the use of Lagrange's equations. The most famous of these models is Van der Pol's model of an electrical circuit with nonlinear resistance. The describing differential equation is

\[ \ddot{y} + \mu(y^2 - 1)\dot{y} + y = 0, \]

and one possible state space model is

\[ \dot{x}_1 = x_2 \]
\[ \dot{x}_2 = -x_1 - \mu(x_1^2 - 1)x_2 \]

The nonlinear resistance term, \( \mu(y^2 - 1) \), changes sign based on the magnitude of \( y \). That is, for \( y \) small the resistance is negative and by the Hurwitz test for stability it can be seen that the response is unstable. For \( y \) large, at least \( y > 1 \), the resistance is positive, hence the response is stable. Of course the Hurwitz criterion is not properly applicable to a nonlinear model, and is used only as a guide in predicting the system behavior.

Although no analytic solution to Van der Pol's equation exists, the phase-plane plot can be examined by either graphical or numerical techniques. A typical phase-plane plot when \( \mu = 4 \) is shown in Figure 2.4.

It is well known that Van der Pol's oscillator has a single stable limit cycle. The parameter, \( \mu \), affects the shape of the oscillation. For \( \mu \) very small, the effect of the nonlinear element is negligible and the system response closely resembles that of a simple harmonic oscillator. As \( \mu \) is increased, the effect of the nonlinear element becomes more evident. Figures 2.5 (a) and (b), Figure 2.6 (a) and (b), and Figure 2.7 demonstrate the effect of the parameter \( \mu \). Figure 2.5 (a) shows
Figure 2.4. Phase Plot of Van der Pol's Oscillator for $\mu = 4$. 
the time response for $\mu = 0.1$. The associated phase plot is shown in Figure 2.5 (b). Figure 2.6 (a) shows the time response for $\mu = 1$. The associated phase plot is shown in Figure 2.6 (b). Figure 2.7 shows the time response for $\mu = 4$. The associated phase plot is shown in Figure 2.4.

Following the rational of Van der Pol's oscillator, several attempts have been made to relate this type of behavior to that observed in other systems. One of these attempts was by Dewan (1964). Dewan drew analogies between the oscillations observed in Van der Pol's model and the oscillations observed in electroencephalography (EEG analysis). To enhance the analogy, Dewan elaborated the mathematical form of the nonlinear resistance, or nonlinear damping coefficient, of Van der Pol's model. One form that resulted was

$$\ddot{y} + (a + by^2 - cy^3 + dy^5)\dot{y} + y = 0.$$  

where the constants $a$, $b$, $c$, and $d$ are all positive. The resulting system exhibits three limit cycles, two stable and one unstable. The unstable limit cycle lies between the two stable limit cycles. To complete the analogy it is assumed that the system is under the influence of outside disturbances that will produce the irregular behavior that is observed in the electroencephalograph.

Another attempt to relate the behavior observed in Van der Pol's oscillator and another system was by Zeeman (1970). Zeeman used Van der Pol's equation to explain the relationship between the physical events observed in a heartbeat and those of the model. Analogies were drawn between "fast", or "action", events and "slow", or "resting" events. In Zeeman's investigation there was no attempt to modify the mathematical model. In neither of these investigations was any attempt made to fit experimental data to the mathematical model response by adjusting the model parameters.
Figure 2.5. (a) Typical Time Responses of Van der Pol's Oscillator for $\mu = 0.1$. 
Figure 2.5 (b). Phase Plane Plot of Van der Pol's Oscillator for $\mu = 0.1$. 
Figure 2.6. (a) Typical Time Responses of Van der Pol's Oscillator for $\mu = 1$. 
Figure 2.6. (b) Phase Plane Plot of Van der Pol's Oscillator for $\mu = 1$. 
Figure 2.7. Typical Time Responses of Van der Pol's Oscillator for
$\mu = 4$. 
Another class of systems that exhibit stable limit cycles are those models used to facilitate mathematical understanding and engineering design. One particular class of model are those that arise through the describing function method of analysis (Hsu and Meyer 1968, Gelb and Vander Velde 1968).

The describing function method is an approximation method for determining stability of nonlinear systems, and to predict the existence of limit cycles. The describing function method also predicts the magnitude and frequency of the fundamental harmonic of the limit cycle. The method is a frequency domain method rather than a time domain method. Consider a system of the form shown in Figure 2.8, where $G_p(s)$ is the transfer function of a linear system and $N(\cdot)$ is a nonlinear element. The input to the system, $r(t)$ is assumed to be zero for all time, and $y(t)$ is the output. The nonlinear element, $N(\cdot)$, is assumed to be any nonlinear time invariant element. Typical nonlinear elements include ideal relays, saturation, deadband and elements with hysteresis. Other nonlinearities might include nonlinear dynamics such as the nonlinear resistance found in Van der Pol’s equation.

The describing function method assumes a sinusoidal output of arbitrary magnitude and frequency, $y(t) = A \sin \omega t$. Based on the sinusoidal assumption, the magnitude and frequency necessary to sustain oscillations are established. The fundamental approximating assumption made in describing function analysis is that the linear transfer function has sufficient low-pass characteristics to allow approximation of the output of the nonlinear element by a pure sinusoid. This approximation allows the nonlinear element to be replaced by an “equivalent gain” that does not generate higher signal harmonics. The “equivalent gain” provides the same gain and phase shift of the pure sinusoid, fundamental harmonic, as the original nonlinear element, but can be considered as a linear element in the analysis of the system. The “equivalent gain” is denoted $\hat{N}(A, \omega)$, where $A$ and $\omega$ are the magnitude and frequency of the sinusoidal input to the nonlinear element.
Figure 2.8. Block Diagram of System Analyzed Using Describing Function Analysis.
Conditions for the existence of sustained oscillations can be determined from the closed loop transfer function, where the nonlinearity is replaced by the "equivalent gain". The condition is

$$\bar{N}(A, \omega)G_p(j\omega) = -1.$$  

Hence, the magnitude and frequency of the fundamental harmonic necessary for sustained oscillation can be determined such that this condition is satisfied. Several techniques can be used to determine these parameters including graphical treatments utilizing the familiar Nyquist diagram, the Bode plot, and if the "equivalent gain" depends only on magnitude, the root locus plot. If the Nyquist diagrams for the plant, \(G_p(j\omega)\), and the equivalent gain, \(-1/\bar{N}(A, \omega)\) are plotted on the same diagram, then any point where the two intersect will satisfy the condition for sustained oscillation.

Thus, one class of systems that can exhibit limit cycles are those that have a linear plant with nonlinear gain. Since the linear plant can be represented in the time domain by an infinite number of state variable systems, such as phase variables, canonic variables, or any state variables, this class of systems is very general. One important characteristic of this class of systems is that they allow the possibility of limit cycles in higher order systems. Since there is no restriction between the degree of the linear transfer function, any linear system that is sufficiently low-pass is possible. In fact, higher order systems can provide better low-pass quality than lower order systems, hence the low-pass assumption is more easily satisfied.

For example, consider the system shown in Figure 2.9. The system consists of a 3\(^{rd}\) order linear plant with transfer function

$$G_p(s) = \frac{10}{(s + 1)[(s + 3)^2 + 1^2]}.$$
Figure 2.9. Block Diagram of a 3rd Order System with a Limit Cycle.
and a nonlinear unit relay in the feedback loop. Since the plant is 3rd order, one possible state variable representation is

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= -10x_1 - 7x_2 - 16x_3 + 10u \\
u &= -\text{sgn}(x_1)
\end{align*}
\]

Of course the choice of state variables is arbitrary since there are an infinite number of state variable representations.

A typical time response for this system is shown in Figure 2.10. Using describing function analysis, the equivalent gain of the unit relay can be found to be

\[
\hat{N}(A, \omega) = \frac{4}{A\pi}.
\]

Figure 2.11 shows the Nyquist diagram of $-1/\hat{N}(A, \omega)$ and $G_p(j\omega)$. The two curves intersect at the point where $(A, \omega) = (0.125, 4)$, hence it is predicted that a limit cycle exists with fundamental frequency $\omega = 4$, and amplitude $A = 0.125$.

Since the equivalent gain is real, the root locus criterion can be used to predict the existence of a limit cycle. Figure 2.12 shows the root locus diagram for this system. Since two of the closed loop poles intersect the imaginary axis, a limit cycle is predicted to exist. The root locus crosses the imaginary axis at the point $\omega = 4$, which coincides with the prediction using the Nyquist criterion.

The describing function method can also be used to predict the stability of the limit cycle. If the magnitude of the fundamental harmonic, $A$, is increased, the equivalent gain, $\hat{N}(A, \omega)$, decreases. Using the root locus, it can be seen that when this happens the two closed loop poles move into the left half plane. Hence the

---

† Most nonlinear system books contain tables of common describing functions. For example see Hsu and Meyer (1968)
Figure 2.10. Time Response of 3rd Order System with Relay.
Figure 2.11. Polar Plot of $G_p(j\omega)$ and $-1/\hat{N}(\hat{z}_1, \omega)$ for the 3rd Order System.
Figure 2.12. Root Locus for the 3rd Order System.
system behaves in a stable manner and $A$ will decrease. If $A$ decreases, $\hat{N}(A, \omega)$ increases and the closed loop poles move into the right half plane. Hence the system behaves in an unstable manner and $A$ will increase. It can be concluded that the limit cycle is stable.

The describing function method provides both a class of system models that exhibit limit cycles and a powerful approximation technique for determining the existence of limit cycles. Although the method depends on the sinusoidal assumption many appropriate results have been obtained for systems that are not sinusoidal, for example Van der Pol's oscillator (Hsu and Meyer 1968).

2.4 Identification of Nonlinear Systems

System identification deals with building mathematical models of dynamical systems from observed data. Such models can be used for control and study of many kinds of systems including mechanical, electrical, and biological. Linear system models are the most common models used for system identification, hence there are many system identification methodologies that apply to linear systems. Sometimes the models must contain nonlinearities, such as systems with limit cycles. Methodologies for the identification of nonlinear systems are not as common as in the linear system case since the analysis of nonlinear systems is more difficult. The purpose of this section is to survey some previous work on the nonlinear system identification problem and to identify methods that may be useful in the identification of nonlinear oscillations.

The nonlinear system identification problem can be divided into three general classes: functional series methods, block oriented methods, and parameter estimation methods (Billings 1980). Common among all of these classes is the desire to define a mathematical relationship between inputs to the system and outputs from the system. It is usually assumed that the systems are nonautonomous, hence the input is independent from the state of the system. Although
this assumption represents many real situations it is a complicating assumption that is not needed in the identification of nonlinear oscillations.

The most general nonlinear system identification approach is the functional series approach. It is assumed that the relationship between the input and the output is of the form

\[ y(t) = \sum_{n=1}^{\infty} \int_{\Omega} \cdots \int h_{n}(\tau_1, \tau_2, \ldots, \tau_n) \prod_{i=1}^{n} u(t - \tau_i) d\tau_i. \]

This representation has become known as the Volterra series and the functions \( h_{n}(\tau_1, \tau_2, \ldots, \tau_n) \) as the Volterra kernels. This model can be considered as a generalization of the convolution integral familiar in linear systems.

Given the Volterra series representation, the system identification problem becomes one of estimating the kernel functions. Most of the methods based on Volterra series assume different forms of the kernel functions including statistical correlation functions and sets of orthogonal functions (Billings, Gray and Owens 1984). In general these approaches require large amounts of input-output data and require considerable computational effort. For the purpose of modeling nonlinear oscillations, the method is not applicable since it depends on the independent input \( u(t) \), and would provide no additional information beyond the standard signal modeling techniques such as the Fourier series.

The second general class of nonlinear system identification methods are the block oriented methods. Block oriented methods assume that the system can be represented by interconnected linear and static nonlinear elements. The emphasis in these methods is to reduce the computational burden required by the functional series methods by utilizing the underlying system structure. One of the earlier and simplest application of these ideas was by Narendra and Gallman (1966). In their
application, a system structure of the form shown in Figure 2.13 is assumed. The nonlinear element, $N(\cdot)$, is assumed to be an $m^{th}$ degree polynomial

$$p(u) = a_1 u + a_2 u^2 + \cdots + a_m u^m$$

where the coefficients $a_1, a_2, \ldots, a_m$ are unknown. The linear element $G_p(s)$ is replaced by its digital simulation

$$G_p(z) = \frac{c_0 + c_1 z^{-1} + \cdots + c_{n-1} z^{n-1}}{b_0 + b_1 z^{-1} + \cdots + b_n z^{-n}}.$$ 

Based on this discretized model the unknown coefficients, $a_1, a_2, \ldots, a_m, c_0, c_1, \ldots, c_{n-1}$, and $b_0, b_1, \ldots, b_n$ can be determined by linear least squares analysis such that the squared error between the observed data and the model response is minimized.

Block oriented methods provide system structural information as desired in this dissertation, but in general the system is nonautonomous, hence it is under the influence of the input that is independent of the system state. If the input was formed by using state variable feedback and no external input was assumed, the system would have a structure very similar to those previously discussed in association with describing function analysis. However this modification would produce a complex relationship between the unknown coefficients that could not be solved using linear least squares analysis.

The third class of nonlinear system identification approaches are parameter estimation methods. The model assumed in parameter estimation methods are more general than those assumed in block oriented methods, but the general idea is the same. It is assumed that the model is of the form

$$\dot{x} = f(x, u, \theta), \quad x(t_0) = x_0$$

$$y = g(x, u, \theta)$$

where $x(t)$ is the system input, and $\theta$ is the vector of unknown system parameters.
Figure 2.13. Block Oriented System of Narendra and Gallman (1966).
Approaches to solving the parameter estimation problem include the extended Kalman filter (Seinfeld 1970) and the method of quasilinearization (Bellman and Roth 1983). The extended Kalman filter approach assumes that the state vector $z(t)$ and the vector of unknown parameters, $\theta$, be augmented into a single state vector

$$x = \begin{pmatrix} z \\ \theta \end{pmatrix}$$

Given this augmented state vector, a new system of the form

$$\dot{x} = \begin{pmatrix} f(z, u, \theta) \\ \dot{\theta} \end{pmatrix}$$

is defined. The new system is then linearized about some operating point and a Kalman filter is used to estimate the state $\dot{z}$. From the estimates of $\dot{z}$ the estimates of the parameter values can be determined.

One drawback to this method is the requirement that the system be linearized about a static operating point. As previously mentioned in Chapter 1, systems that exhibit limit cycles are in a continuous state of change and hence linearization about a static operating point is not a valid assumption.

The method of quasilinearization (Bellman and Roth 1983) is another parameter estimation method that has received considerable attention. In this method the system is linearized about a trajectory rather than a single point. This method is discussed in greater depth in Chapter 4. It is the one method of system identification that has been applied to the identification of a system with a limit cycle. Bellman and Kalaba (1965) applied the method to estimating the parameter in Van der Pol's equation with some success.

The parameter estimation methods require that considerable a priori information be known about the structure of the underlying system. This requirement may be considered a drawback to these methods. Considering that relatively small
amounts of data are required and the methods are applicable to autonomous systems, these methods are a potential approach to the identification of nonlinear oscillations.

In this section a brief review of the methods of nonlinear system identification has been presented. From this review it can be determined that functional series methods are not applicable to the goals of this dissertation. The most promising approaches are the block oriented and the parameter estimation methods. Because of the difficulty associated with nonlinear system analysis, the scope of system identification methods is limited. Except for a single treatment by Bellman and Kalaba (1965), the application of nonlinear system identification to modeling systems with limit cycles, or nonlinear oscillations, has been ignored.
CHAPTER 3

MODELING NONLINEAR OSCILLATIONS

3.0 Introduction and Organization of the Chapter

This chapter is devoted to the presentation of a class of nonlinear differential equations that can be used to model nonlinear oscillations. A generalized class of models is needed if one model form is to be used to model many different oscillator responses. This point can best be made by considering the difference between linear and nonlinear systems. Every linear autonomous system can be represented as one $n^{th}$ order differential equation with constant coefficients, where a system of nonlinear differential equations cannot always be represented similarly. There are no standard model forms for nonlinear systems, and hence if one hopes to model many different system responses without proposing a new model form for each response, a generalized class of models is needed. The class of models presented here is general in the sense that many different response shapes can be generated by choosing different parameter values.

The chapter is divided into two parts. In the first part a general class of differential equations that exhibit stable limit cycles is defined. Properties of the generalized oscillator model are discussed and several examples are presented that demonstrate the model properties. The describing function method is used as the primary tool of analysis. Traditionally the describing function method is applied to systems with static nonlinearities or nonlinearities with hysteresis. In the class of differential equations presented, here the nonlinear elements contain significant dynamics that are described by differential equations. It is these dynamics that give the range of response shapes that can be used as models of nonlinear oscillations. Because the traditional application of the describing function method does
not, in general, consider these nonlinearities, the details of this application are presented. In the second part of the chapter, the class of generalized oscillator models are modified to exhibit oscillations that are unsymmetric. Again, several examples are presented to demonstrate the behavior of these models.

3.1 Generalized Oscillator Models

The purpose of this section is to present a generalized mathematical model that can be used to model many different oscillations. It is assumed that an oscillating signal is measured from a real system. The signal is the only measurement available from the system, hence neither the input nor any of the states are available for measurement. Furthermore, it is assumed that the system dynamics are not understood well enough to propose a mathematical model based on physical principles. Hence, the model proposed is intended to model the phenomenon of oscillation.

To simplify the presentation assume

\[ \mathbf{z} = \begin{pmatrix} y \\ \dot{y} \\ \vdots \\ y^{(n-1)} \end{pmatrix} \]

Define

\[ \mathbf{z}^{(i)} = \begin{pmatrix} y \\ \dot{y} \\ \vdots \\ y^{(i-1)} \end{pmatrix} \]

to be the \( i \) dimensional subvector of \( \mathbf{z} \) consisting of the \( i \) phase variables. Note that \( \mathbf{z}^{(1)} = y \) and \( \mathbf{z}^{(n)} = \mathbf{z} \).

The proposed generalized oscillator model assumes the form

\[ y^{(n)} + a_n(\mathbf{z})y^{(n-1)} + a_{n-1}(\mathbf{z})y^{(n-2)} + \cdots + a_2(\mathbf{z})\dot{y} + a_1(\mathbf{z})y = 0. \] (3.1)
where each of the coefficients $a_n(z)$, $a_{n-1}(z)$, ..., $a_1(z)$ are chosen such that the system exhibits a stable limit cycle.

Although the coefficients could reasonably be chosen to be any functions, they are chosen as polynomials of the form

$$a_i(z) = \cases{ A_i(z) - \alpha_i \\
\alpha_i }$$

or

(3.2)

where $A_i(z)$ is a positive semidefinite polynomial, and $\alpha_i \geq 0$, for $i = 2, \ldots, n$, and $\alpha_1 < 0$. In particular, $A_i(z)$ is chosen to be a positive semidefinite quadratic function of the form

$$A_i(z) = z^i A_i z$$

(3.3)

where $A_i$ is a positive semidefinite matrix of the form

$$A_i = \begin{pmatrix}
a_{11}^{(i)} & a_{12}^{(i)} & \cdots & a_{1n}^{(i)} \\
a_{21}^{(i)} & a_{22}^{(i)} & \cdots & a_{2n}^{(i)} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1}^{(i)} & a_{n2}^{(i)} & \cdots & a_{nn}^{(i)}
\end{pmatrix}$$

$A_i(z)$ is chosen to be quadratic because in general it is difficult to ensure a more general function to be positive semidefinite.

The form of the model follows from a generalization of Van der Pol's equation. In Van der Pol's equation, the oscillation occurs due to the changing sign of the nonlinear coefficient of $\dot{y}$. In order to generalize to higher order equations, the generalized Routh-Hurwitz analogy (Schultz 1963) was used as a guiding principle. It was hypothesized that if the Hurwitz criterion was violated near the origin, then the origin would be unstable, and if the Routh criterion held for states a reasonable distance from the origin, then the system would oscillate. Unfortunately, this hypothesis fails to hold in general, but does produce an interesting class of differential equations that exhibit limit cycles.
3.2 Properties of the Generalized Oscillator Model

In this section, properties of the generalized oscillator model are examined. In particular, a structural identifiable model form is found and conditions for the existence of limit cycles are discussed.

3.2.1 Structural Identifiability

The concept of structural identifiability is central to the system identification problem (Bellman and Åström 1970). This concept is concerned with determining what model structures are identifiable from input-output data. To demonstrate, consider a linear system represented by the following state variable equations

\[
\begin{align*}
\dot{x}_1 &= a_1 x_1 + a_2 x_2 \\
\dot{x}_2 &= a_3 x_1 + a_4 x_2 + u \\
y &= x_1 
\end{align*}
\]

The transfer function for this system is

\[
G(s) = \frac{a_2}{s^2 - (a_1 + a_4)s + (a_1 a_4 - a_2 a_3)}
\]  

(3.4)

which is equivalent to a transfer function of the form

\[
G(s) = \frac{K}{s^2 + as + b},
\]  

(3.5)

that has the associated phase variable representation

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -bx_1 - ax_2 + Ku \\
y &= x_1.
\end{align*}
\]
Any set of parameters, \((a_1, a_2, a_3, a_4)\) such that
\[
a_2 = K \\
-(a_1 + a_4) = a \\
a_1a_4 - a_2a_3 = b
\]
will describe a model that produces exactly the same input-output behavior as a model described by the parameters \((a, b, K)\). It is desirable to choose a model such that a unique set of parameters can be chosen. Since many different parameter sets can determine the same model, a model of the form (3.5) is said to be structurally identifiable and a model of the form (3.4) is said to be structurally unidentifiable.

The generalized oscillator model, (3.1), with polynomial coefficients, (3.3), is not presented in a structurally identifiable form. This can easily be seen through an example. Consider the second order differential equation
\[
\ddot{y} + (a_{11}y^2 + 2a_{12}y\dot{y} + a_{22}\dot{y}^2 - a_2)\dot{y} + (b_{11}y^2 + 2b_{12}y\dot{y} + b_{22}\dot{y}^2 - a_1)y = 0.
\]
By simple algebraic manipulations it can be seen that this model is equivalent to the following
\[
\ddot{y} + ((a_{11} + 2b_{12})y^2 + (2a_{12} + b_{22})y\dot{y} + a_{22}\dot{y}^2 - a_2)\dot{y} + (b_{11}y^2 - a_1)y = 0.
\]
Clearly, there can exist an infinite number of equivalent model representations. This fact is undesirable since any attempt to perform a system identification would result in optimizing in a parameter space that has many equivalent minimum.

The above example suggests that any equivalent models of the form (3.1) can be represented by a structurally identifiable model. This result is stated in the following theorem.

**Theorem 3.1** Any model of the form (3.1) with polynomial coefficients (3.3) can be represented by an irreducible model of the form
\[
y^{(n)} + b_n(z^{(n)})y^{(n-1)} + b_{n-1}(z^{(n-1)})y^{(n-2)} + \cdots \\
+ b_2(z^{(2)})\dot{y} + b_1(z^{(1)})y
\]
where
\[ b_i(x^{(i)}) = \begin{cases} \mathcal{E}^{(i)} B_i x^{(i)} - \alpha_i & \text{or} \\ \alpha_i & \end{cases} \tag{3.7} \]

and \( B_i \) is an \( i \times i \) positive semidefinite matrix.

Basically, the theorem states that equation (3.1) can be rewritten into a unique form by grouping the coefficients by differential order. The proof of the theorem is simple but will be presented because it is constructive. The method of the proof follows the reasoning presented in the example presented above.

**Proof.** Since the coefficients of (3.1) are polynomials, each term can be uniquely factored into the following form
\[
a_i(x)y^{(i-1)} = \nu_{i,n}(x^{(n)})y^{(n-1)} + \nu_{i,n-1}(x^{(n-1)})y^{(n-2)} + \cdots + \nu_{i,1}(x^{(1)})y\]

for \( i = 1, 2, \ldots, n - 1 \). Define
\[
b_i(x^{(i-1)}) = \begin{cases} a_n(x^{(n-1)}) + \sum_{k=1}^{n} \nu_{k,i}(x^{(k-1)}), & \text{if } i = n \\ \sum_{k=1}^{n} \nu_{k,i}(x^{(k-1)}), & \text{otherwise} \end{cases} \tag{3.8} \]

then,
\[
y^{(n)} + a_n(x)y^{(n-1)} + a_{n-1}(x)y^{(n-2)} + \cdots + a_2(x)y + a_1(x)y = y^{(n)} + b_n(x^{(n)})y^{(n-1)} + b_{n-1}(x^{(n-1)})y^{(n-2)} + \cdots + b_2(x^{(2)})y + b_1(x^{(1)})y. \]

The irreducibility is a result of the unique factorization of the coefficients (3.8).

The constructability of the proof can be demonstrated by considering the earlier example. The generalized model is
\[
\ddot{y} + (a_{11}y^2 + 2a_{12}y\dot{y} + a_{22}\dot{y}^2 - \alpha_2)\dot{y} + (b_{11}y^2 + 2b_{12}y\dot{y} + b_{22}\dot{y}^2 - \alpha_1)y = 0. \]

Let
\[
a_2(y, \dot{y}) = (a_{11}y^2 + 2a_{12}y\dot{y} + a_{22}\dot{y}^2 - \alpha_2) \]
and

\[ a_1(y, \dot{y}) = (b_{11}y^2 + 2b_{12}y\dot{y} + b_{22}\dot{y}^2 - \alpha_1) \]

then the factored form of \(a_1(y, \dot{y})\) is

\[ a_1(y, \dot{y})y = (b_{11}y^2 - \alpha_1)y + (2b_{12}y^2 + b_{22}y\dot{y})\dot{y} \]

so,

\[ b_2(y, \dot{y}) = a_2(y, \dot{y}) + (2b_{12}y^2 + b_{22}y\dot{y}) \]

\[ = (a_{11} + 2b_{12})y^2 + (2a_{12} + b_{22})y\dot{y} + b_{22}\dot{y}^2 - \alpha_2. \]

and,

\[ b_1(y) = b_{11}y^2 - \alpha_1. \]

Therefore,

\[ \ddot{y} + b_2(y, \dot{y})\dot{y} + b_1(y)y = \]

\[ \ddot{y} + ((a_{11} + 2b_{12})y^2 + (2a_{12} + b_{22})y\dot{y} + a_{22}\dot{y}^2 - \alpha_2)\dot{y} + (b_{11}y^2 - \alpha_1)y = 0, \]

which demonstrates the desired results.

From this point on only generalized oscillator models that are structurally identifiable are considered. This does not limit the scope of the models considered since it is only desired to model the oscillation phenomenon and not the physics of the real systems.

### 3.2.2 System Analysis

The purpose of this section is to investigate the behavioral properties of the class of generalized oscillator models presented in the previous section. Particular properties of interest include existence and stability of singular points and the existence of limit cycles. While it would be desirable to prove theorems regarding the existence and uniqueness of limit cycles in this class of models, it is recognized that this task is beyond the scope of this investigation. The discussion presented
here relies on the describing function method as its guiding principle, but it is recognized that these results should be carefully interpreted.

To discuss the properties of these systems it is necessary to consider them represented in state variable form. This is easily accomplished using the familiar change to phase variables

\[
x_1 = y
\]
\[
x_2 = y
\]
\[
\vdots
\]
\[
x_n = y^{(n-1)}.
\]

Then an equivalent state variable representation of (3.6) is

\[
\dot{x}_1 = x_2 \\
\dot{x}_2 = x_3 \\
\vdots \\
\dot{x}_n = -a_1(x^{(1)})x_1 - a_2(x^{(2)})x_2 - \cdots - a_n(x^{(n)})x_n.
\]
\[
y = x_1
\]

Since no equilibrium points may exist in the neighborhood of a limit cycle this can be assured by assuming that the origin is the only equilibrium point. Under this assumption it is necessary that

\[
0 = x_2
\]
\[
0 = x_3
\]
\[
\vdots
\]
\[
0 = -a_1(x^{(1)})x_1 - a_2(x^{(2)})x_2 - \cdots - a_n(x^{(n)})x_n.
\]

which reduces to the condition

\[-a_1(x_1)x_1 = 0.\]
In this case, either \( x_1 = 0 \) or \( a_1(x_1) = 0 \). Since it is desired that \( x_1 = 0 \) then it must be assured that \( a_1(x_1) = 0 \) only if \( x_1 = 0 \), that is \( a_1(x_1) \) has at most one real root, and it is located at \( x_1 = 0 \). From (3.10) this condition requires that \( A_1 \geq 0 \), which also is required by the positive semidefinite condition, and the additional condition that \( \alpha_1 < 0 \).

It is also essential that the origin be an unstable equilibrium point. Conditions necessary to ensure this can be established by examining the system linearized about the origin,

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\vdots \\
\dot{x}_n
\end{pmatrix} =
\begin{pmatrix}
0 & 1 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-a_1(0) & -a_2(0,0) & \cdots & -a_n(0,0,\ldots,0)
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix}.
\]

Stability of the origin can be determined by applying the Routh-Hurwitz criterion. The linearized system has the characteristic equation

\[
\lambda^n + a_n(0,0,\ldots,0)\lambda^{n-1} + \cdots + a_1(0) = 0.
\]

By the Hurwitz criterion, the linearized system is unstable if \( a_i(0,0,\ldots,0) < 0 \) for any \( i = 1,\ldots,n \). Thus a necessary condition for the origin to be unstable is that at least one of the coefficient functions must be less than zero at the origin.

The conditions discussed above and Smith’s theorem (Theorem 2.3) have a strong connection. First, both Smith’s theorem and the discussion presented above require that the origin be the only singular point of the system. In addition, Smith’s theorem requires that the system linearized about the origin have exactly two positive eigenvalues. While this was not required in the previous discussion, it could easily be an additional requirement on the generalized oscillator model. In fact, the examples presented in Section 3.3 each satisfy this requirement, as does Van der Pol’s equation.

Describing function analysis provides a method of determining the existence and stability of limit cycles. The method, as discussed in section 2.4, is based on
the assumption that the output of the system can be approximated by a sinusoid with unknown frequency and magnitude. The discussion presented here follows closely that of Gelb and Vander Velde (1968) but extends their results to apply to the particular form of the generalized oscillator model.

While the describing function method provides useful insight into the behavior of some nonlinear system problems, it is recognized that the method is an approximation technique. In the following section several examples are presented, and each example is discussed utilizing the describing function method. To complement and validate this analysis, numerical simulation is used as an analysis tool. Together these methods provide useful insight into the behavior of this class of models.

To perform describing function analysis, the system is first divided into a linear and a nonlinear part. For the systems considered here the state equations can be written as

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\vdots \\
\dot{x}_n
\end{pmatrix} =
\begin{pmatrix}
0 & 1 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_1 & \alpha_2 & \cdots & \alpha_n
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix} +
\begin{pmatrix}
0 \\
0 \\
\vdots \\
-N(x)
\end{pmatrix}
\]

(3.11)

where

\[
N(x) = \sum_{k=1}^{n} (x^{(k)})' A_k x^{(k)} x_k
\]

(3.12)

is the nonlinear portion of the system. Since the state variables are phase variables, the nonlinear element, \(N(\cdot)\), can be written in terms of the differential operator as

\[
N(x) = N(x_1, \frac{dx_1}{dt}, \ldots, \frac{d^n x_1}{dt^n}).
\]
This notation allows the system to be represented by the block diagram shown in Figure 3.1, where the linear part of the system is represented by its associated transfer function denoted,

$$ G_p(s) = \frac{1}{s^n - \alpha_n s^{n-1} - \cdots - \alpha_1} \tag{3.13} $$

The describing function can be derived assuming

$$ x_1 = \hat{x}_1 \sin \omega t, $$

where the magnitude, $\hat{x}_1$, and the frequency, $\omega$, are determined as part of the analysis. Under this assumption, the input to the plant is

$$ u = -N(\hat{x}_1 \sin \omega t, \hat{x}_1 \omega \cos \omega t, \ldots ). $$

It is desired to replace the nonlinear element, $N(\cdot)$, with a transfer element, $\tilde{N}(\hat{x}, \omega)$, that produces the equivalent attenuation and phase shift of the fundamental harmonic of the output as the nonlinear element. Then

$$ u \approx -\tilde{N}(\hat{x}_1, \omega) x_1. $$

To accomplish this, express $N(\hat{x}_1 \sin \omega t, \hat{x}_2 \omega \cos \omega t, \ldots )$ as a Fourier series of the form

$$ N(\hat{x}_1 \sin \omega t, \hat{x}_2 \omega \cos \omega t, \ldots ) = \sum_{k=0}^{\infty} A_k(\hat{x}_1, \omega) \sin(k\omega t + \theta_k(\hat{x}, \omega)). \tag{3.14} $$

The dc-component is

$$ A_0(\hat{x}_1, \omega) = \frac{1}{2\pi} \int_{0}^{2\pi} N(\hat{x}_1 \sin \omega t, \hat{x}_2 \omega \cos \omega t, \ldots )d\omega $$

and is zero for the nonlinearities discussed in this section.
Figure 3.1. Block Diagram of the Generalized Oscillator Model.
The magnitude and the phase of the fundamental harmonic can be found by multiplying (3.14) by \( \cos \omega t \) and \( \sin \omega t \) and integrating to yield, respectively

\[
A_1(\tilde{x}_1, \omega) \sin \theta_1(\tilde{x}_1, \omega) = \frac{1}{\pi} \int_{0}^{2\pi} N(\tilde{x}_1 \sin \omega t, \tilde{x}_1 \omega \cos \omega t, \ldots) \cos \omega t \, dwt \tag{3.15}
\]

and

\[
A_1(\tilde{x}_1, \omega) \cos \theta_1(\tilde{x}_1, \omega) = \frac{1}{\pi} \int_{0}^{2\pi} N(\tilde{x}_1 \sin \omega t, \tilde{x}_1 \omega \cos \omega t, \ldots) \sin \omega t \, dwt. \tag{3.16}
\]

Multiply (3.15) by \( j \) and add it to (3.16) and divide by \( \tilde{x}_1 \), the magnitude of the input, then the describing function is

\[
\hat{N}(\tilde{x}_1, \omega) = \frac{A_1(\tilde{x}_1, \omega)}{\tilde{x}_1} e^{j \theta(\tilde{x}_1, \omega)} = \frac{j}{\pi \tilde{x}_1} \int_{0}^{2\pi} N(\tilde{x}_1 \sin \omega t, \tilde{x}_1 \omega \cos \omega t, \ldots) e^{-j \omega t} \, dwt. \tag{3.17}
\]

To demonstrate the derivation of the describing function, consider the general second order oscillator, in state variable form

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2 
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
-\alpha_1 x_1 - \alpha_2 x_2 & -\alpha_1 x_1 - \alpha_2 x_2 
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2 
\end{pmatrix}.
\]

The block diagram of this system is shown in Figure 3.2. The nonlinear element is

\[
N(x_1, x_2) = a_{11}^{(2)} x_1^2 x_2 + 2a_{12}^{(2)} x_1 x_2^2 + a_{22}^{(2)} x_2^3 + a_{11}^{(1)} x_1^3.
\]

Applying (3.17), under the sinusoidal assumption, \( \hat{N}(\tilde{x}, \omega) \) is found to be

\[
\hat{N}(\tilde{x}_1, \omega) = \frac{j}{\tilde{x}_1 \pi} \int_{0}^{2\pi} \tilde{x}_1^3 \left[ a_{11}^{(2)} \omega \sin^2 \omega t \cos \omega t + 2a_{12}^{(2)} \omega^2 \sin \omega t \cos^2 \omega t + a_{22}^{(2)} \omega^3 \cos^3 \omega t + a_{11}^{(1)} \sin^3 \omega t \right] e^{i \omega t} \, dwt
\]

\[
= \frac{\tilde{x}_1^2}{4} \left( [2a_{12}^{(2)} \omega^2 + 3a_{11}^{(1)}] + j[a_{11}^{(1)} \omega + 3a_{22}^{(2)} \omega^3] \right).
\]
Figure 3.2. Block Diagram of the Generalized 2\textsuperscript{nd} Order Oscillator.
The condition for existence of limit cycles, assuming that the transfer function of the linear part of the system is denoted \( G_p(s) \), is

\[
1 + \hat{N}(\hat{x}_1, \omega)G_p(j\omega) = 0. \tag{3.18}
\]

It is desired to determine pairs \((\hat{x}_1, \omega)\) such that (3.18) is satisfied. It is possible that there are no pairs, a single pair, or many pairs that satisfy this condition. If there are no solution pairs, then the describing function analysis predicts that the system does not exhibit a limit cycle. If there is a single pair, then it is predicted that there is a single limit cycle. Many pairs indicate the possible existence of multiple limit cycles.

Solution of (3.18) can be accomplished by either graphical or analytical techniques. To find the solution by the graphical method, rewrite (3.18), as

\[
G(j\omega) = \frac{-1}{\hat{N}(\hat{x}_1, \omega)}. \tag{3.19}
\]

Then either a magnitude-phase or a polar plot of each side of this equation is constructed. If the polar plot is chosen, the plot of \( G(j\omega) \) is the familiar Nyquist diagram. The right hand side of (3.19) forms a two parameter family of curves. Any point where the two curves intersect is a candidate for a limit cycle. The point of intersection is characterized by a frequency, \( \omega \), for \( G(j\omega) \) and a magnitude-frequency pair \((\hat{x}_1, \omega)\) for \( \hat{N}(\hat{x}_1, \omega) \). If the frequency of both curves is equal at the intersection, then the describing function analysis predicts a limit cycle at the frequency and magnitude at the intersection. In general, the graphical method provides useful intuition into the solution, but is cumbersome as a solution technique.

If \( \hat{N}(\hat{x}_1, \omega) \) is purely real (or imaginary), the root locus criterion can be used to determine the existence of limit cycles. This approach, while limited to this special case, provides a very intuitive relationship between the existence of a limit cycle and the structure of the linear portion of the system. If \( \hat{N}(\hat{x}_1, \omega) \) is
purely real, then the criteria for existence of a limit cycle becomes associated with
the closed loop system poles crossing the \( j\omega \) axis. If \( \tilde{N}(\hat{x}_1, \omega) \) is purely imaginary,
then a zero can be placed at the origin in the linear portion and the root locus
criteria described above can be used.

The special structure of the nonlinear element can be used to simplify the
analytic solution of (3.18). The describing function can be represented as

\[
\tilde{N}(\hat{x}_1, \omega) = K_N(\hat{x}_1)[R_N(\omega) + jI_N(\omega)]
\]

and

\[
G_p(j\omega) = \frac{1}{R_G(\omega) + jI_G(\omega)}.
\]

Then (3.18) becomes

\[
1 + \frac{K_N(\hat{x}_1)[R_N(\omega) + jI_N(\omega)]}{R_G(\omega) + jI_G(\omega)} = 0.
\]

After simplification this can be written as

\[
(R_G(\omega) + K_N(\hat{x}_1)R_N(\omega)) + j(I_G(\omega) + K_N(\hat{x}_1)I_N(\omega)) = 0
\]

The condition for existence of a limit cycle can be satisfied by setting the real
and imaginary parts equal to zero. Hence, the following set of equations must be
solved for the pair \((\hat{x}_1, \omega)\)

\[
R_G(\omega) + K_N(\hat{x}_1)R_N(\omega) = 0
\]

and

\[
I_G(\omega) + K_N(\hat{x}_1)I_N(\omega) = 0.
\]

Either of these equations can be solved for \( K_N(\hat{x}_1) \) and the result substituted
into the other to reduce the problem to one of finding the roots of a polynomial
equation in \( \omega \).
Solving the first equation of (3.23) for $K_N(\omega)$ gives

$$K_N(\omega) = \frac{R_{G_p}(\omega)}{R_N(\omega)}. \quad (3.24)$$

Then the polynomial in $\omega$ is

$$R_N(\omega)I_{G_p}(\omega) - R_{G_p}(\omega)I_N(\omega) = 0. \quad (3.25)$$

The roots of this polynomial form the set of candidate frequencies for existence of a limit cycle. If the roots are complex or negative they can be eliminated from the set of possible frequencies. All positive real roots correspond to limit cycles. The magnitude $\dot{x}_1$ can be determined by the relation (3.24). For all of the nonlinear elements found in the generalized oscillator models, the magnitude dependent gain, $K_N(\dot{x}_1)$ is

$$K_N(\dot{x}_1) = \frac{\dot{x}_1^2}{4},$$

hence $\dot{x}_1$ can easily be determined once $\omega$ is known.

The stability of a limit cycle can also be investigated using the describing function method. Again, since the method is one of approximations, the results must be interpreted carefully. The condition for a stable limit cycle is developed (Gelb and Van der Velde 1968) by considering small perturbations in the limit cycle amplitude, rate of change of the amplitude, and the frequency. Let

$$\dot{x}_1 + \Delta \dot{x}_1$$

and

$$\omega + \Delta \omega + j \Delta \sigma$$

denote the change in amplitude ($\Delta \dot{x}_1$), frequency ($\Delta \omega$), and rate of change amplitude ($\Delta \sigma$), respectively. If the condition for existence of a limit cycle, (3.18), is considered in the following form

$$R_{NG_p}(\dot{x}_1, \omega) + j I_{NG_p}(\dot{x}_1, \omega) = 0,$$
under the perturbations this condition becomes

\[ R_{NG_x}(\dot{z}_1 + \Delta \dot{z}_1, \omega + \Delta \omega + j \Delta \sigma) + j I_{NG_x}(\dot{z}_1 + \Delta \dot{z}_1, \omega + \Delta \omega + j \Delta \sigma) = 0. \]

Since the perturbations are assumed to be small, the perturbed existence condition can be written as a Taylor series, ignoring the higher order terms, as

\[ \frac{\partial R_{NG_x}}{\partial \dot{z}_1} \Delta \dot{z}_1 + \frac{\partial R_{NG_x}}{\partial \omega} (\Delta \omega + j \Delta \sigma) + \frac{\partial I_{NG_x}}{\partial \dot{z}_1} \Delta \dot{z}_1 + \frac{\partial I_{NG_x}}{\partial \omega} (\Delta \omega + j \Delta \sigma) = 0. \]

Setting the real and imaginary parts equal to zero, eliminating \( \Delta \omega \) yeilds the following relationship between \( \Delta \dot{z}_1 \) and \( \Delta \sigma \),

\[ \left( \frac{\partial R_{NG_x}}{\partial \dot{z}_1} \frac{\partial I_{NG_x}}{\partial \omega} - \frac{\partial R_{NG_x}}{\partial \omega} \frac{\partial I_{NG_x}}{\partial \dot{z}_1} \right) \Delta \dot{z}_1 = \left[ \left( \frac{\partial R_{NG_x}}{\partial \omega} \right)^2 + \left( \frac{\partial I_{NG_x}}{\partial \omega} \right)^2 \right] \Delta \sigma. \]

For the limit cycle to be stable a positive increment in \( \Delta \dot{z}_1 \) must lead to a positive increment in \( \Delta \sigma \), and a negative increment in \( \Delta \dot{z}_1 \) must lead to a negative increment in \( \Delta \sigma \). Hence, a necessary condition for a stable limit cycle is

\[ \frac{\partial R_{NG_x}}{\partial \dot{z}_1} \frac{\partial I_{NG_x}}{\partial \omega} - \frac{\partial R_{NG_x}}{\partial \omega} \frac{\partial I_{NG_x}}{\partial \dot{z}_1} > 0. \quad (3.26) \]

The results stated above present conditions for the existence and stability of limit cycles based on the describing function method of approximation. These results must be carefully interpreted since the fundamental assumption may not hold in many cases. The describing function method has proved a useful analysis tool for over two decades and for most cases does provide useful information about system behavior.

To demonstrate the describing function technique, consider Van der Pol's equation

\[ \ddot{y} + (4y^2 - 4)\dot{y} + y = 0. \]
In terms of the general second order oscillator model the system parameters are

\[ a^{(2)}_{11} = 4 \]
\[ a^{(2)}_{12} = 0 \]
\[ a^{(2)}_{22} = 0 \]
\[ a^{(1)}_{11} = 0 \]
\[ a_{12} = 0 \]
\[ \alpha_2 = 4 \]
\[ \alpha_1 = -1. \]

Since \( \alpha_1 < 0 \) and

\[ a_2(0, 0) = -4 \]

the origin is the only singular point and it is unstable. The describing function is

\[ \hat{N}(\dot{x}_1, \omega) = j \dot{x}_1^2 \omega. \]

The transfer function of the linear portion is

\[ G_p(s) = \frac{1}{s^2 - 4s + 1}, \]

so

\[ G_p(j\omega) = \frac{1}{(1 - \omega^2) - j4\omega}. \]

Thus, the conditions for existence of a limit cycle from (3.23) are

\[ 1 - \omega^2 = 0 \]
\[ (\dot{x}_1^2 - 4)\omega = 0. \]
A solution to this can be seen to be $\omega = 1$ and $x_1 = 2$. Figure 3.3 shows the polar plot of $G(j\omega)$ and $-1/\tilde{N}(\tilde{x}, \omega)$. The point where these two curves intersect can be seen to be the solution.

Since $\tilde{N}(\tilde{x}_1, \omega)$ is purely complex,

$$\tilde{N}(\tilde{x}_1, \omega) = j\tilde{x}_1^2\omega,$$

the root locus can be used to determine the existence of a limit cycle. This can be accomplished by assuming

$$\tilde{N}(\tilde{x}_1, \omega) = \tilde{x}_1^2s,$$

where $s = j\omega$. The differentiator, $s$, can be included in the linear plant. The modified gain is

$$\tilde{N}(\tilde{x}_1) = \tilde{x}_1^2$$

and the modified plant is

$$\tilde{G}_p(s) = \frac{s}{s^2 - 4s + 1}.$$

The feedback gain is proportional to $\tilde{x}_1^2$, so as the magnitude of the first harmonic increases, so does the gain. The root locus is shown in Figure 3.4. The root locus crosses the imaginary axis at $\omega = \pm 1$ which corresponds to the result obtained above.

Applying (3.26) the limit cycle is determined to be stable since

$$4\omega^2\tilde{x}_1 > 0$$

at the solution point $(\tilde{x}_1, \omega) = (2, 1)$. The root locus also indicates that the limit cycle is stable since increasing the gain moves the closed loop poles deeper into the left half plane, hence the system behaves in a stable manner when $\tilde{x}_1$ increases. The closed loop poles move into the right half plane when $\tilde{x}_1$ decreases, hence
Figure 3.3. Polar Plot of $G_p(j\omega)$ and $-1/\dot{N}(\dot{x}_1, \omega)$ for Van der Pol's Oscillator.
Figure 3.4. Root Locus of Van der Pol's Equation with the Describing Function Replacing the Nonlinear Element.
the system behaves in an unstable manner. Together these facts indicate that the
limit cycle is stable.

In the next section several examples are studied. These examples demon­
strate the effect of the nonlinear coefficients in shaping the oscillation. The anal­
ysis presented in this section is used to discuss the existence and stability of limit
cycles.

3.3 Examples

This section is devoted to the presentation of several interesting examples. The purpose of these examples is to demonstrate the types of responses that the
generalized oscillator models can produce. The first example is a simple general­
ization of Van der Pol’s equation and is intended to demonstrate the basic idea
of the generalized oscillator model. The second example is a third order form of
the generalized oscillator model and is primarily useful in that it demonstrates a
limit cycle in three dimensional state space. The third and fourth examples are
particular forms of third order systems that are used to investigate the effect of
the nonlinear coefficients. The final example is a fourth order system that is used
to demonstrate some of the complexities introduced by higher order models.

3.3.1 Second Order Example

Assume that the system is represented as one second order equation

\[ \ddot{y} + \left( \begin{array}{c} \dot{y} \\ y \end{array} \right) \left( \begin{array}{cc} 4 & 1 \\ 1 & 2 \end{array} \right) \left( \begin{array}{c} y \\ \dot{y} \end{array} \right) - 4 \dot{y} + y = 0. \]

The state variable representation is

\[ \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1 - (4x_1^2 + 2x_1x_2 + 2x_2^2 - 4)x_2 \end{pmatrix}. \]
The origin can be seen to be unstable since

\[
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - 4 = -4 < 0
\]

when \((x_1, x_2) = (0, 0)\). For \(||x||\) large, the damping term is positive, hence the system exhibits stable behavior. Since the system exhibits unstable behavior near the origin and stable behavior a sufficient distance away from the origin, it can be concluded that a stable limit cycle must exist.

The phase plane is shown in Figure 3.5 and the time response for both states is shown in Figure 3.6. The describing function is

\[
\tilde{N}(\tilde{x}_1, \omega) = \frac{\tilde{x}_1^2}{4} \left[(2\omega^2) + j(4\omega + 6\omega^3)\right],
\]

and \(G_p(j\omega)\) is

\[
G_p(j\omega) = \frac{1}{(1 - \omega^2) - j4\omega}.
\]

Since the describing function is neither purely real or complex, either the graphical or analytical methods must be used to determine conditions for existence and stability. Analytically, the equations that must be satisfied to determine existence are

\[
(1 - \omega^2) + \frac{\tilde{x}_1^2}{4}(2\omega^2) = 0
\]

and

\[
-4\omega + \frac{\tilde{x}_1^2}{4}(4\omega + 6\omega^3) = 0.
\]

A solution to these equations is \(\omega = \sqrt{2}\) and \(\tilde{x}_1 = 1\). Figure 3.7 shows the polar plot of \(G(j\omega)\) and \(-1/\tilde{N}(\tilde{x}_1, \omega)\) for \(\tilde{x}_1 = 1\). The point where the two curves intersect is the point \((\tilde{x}_1, \omega) = (1, \sqrt{2})\), which agrees with the analytic results.

From (3.26) the condition for stability is

\[
(\tilde{x}_1 \omega^2)(-4 + \tilde{x}_1^2(1 + 3\omega^2)) - (-2\omega + \tilde{x}_1^2 \omega)(\tilde{x}_1^2(2\omega + 3\omega^2)) = 4(4\sqrt{2} - 1).
\]
Figure 3.5. Phase Plane of System in Example 3.3.1.
Figure 3.6. Time Response of $y$, (solid), and $\dot{y}$, (dashed), of Example 3.3.1.
Figure 3.7. Polar Plot of $G_p(j\omega)$ and $-1/\hat{N}(\hat{x}_1,\omega)$ for System of Example 3.3.1.
Since $4\sqrt{2} - 1 > 0$ the limit cycle is stable, as was predicted by the sign of the damping term discussed at the beginning of this example.

This example presents a simple modification of Van der Pol's equation. Recall that Van der Pol's equation behaved in an unstable manner when $\|y\| < 1$, hence $\|\dot{y}\|$ could increase without bound while $y$ was in this region. In this example, if either $y$ or $\dot{y}$ increases such that $\|4y^2 + 2y\dot{y} + 2\dot{y}^2\| > 4$ the system behaves in a stable manner. This dependence on both variables produces a response that is much smoother and is not subject to "fast" transitions.

### 3.3.2 Third Order Example

Assume that the system is represented as one third order equation

$$y^{(3)} + ((y \quad \dot{y} \quad \ddot{y}) \begin{pmatrix} 10 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}) \begin{pmatrix} y \\ \dot{y} \\ \ddot{y} \end{pmatrix} - 10\ddot{y} + ((y \quad \dot{y}) \begin{pmatrix} 4 & 0 \\ 0 & 16 \end{pmatrix} \begin{pmatrix} y \\ \dot{y} \end{pmatrix} - 4\dot{y} + 2y = 0,$$

or equivalently by the state variable model

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ -2x_1 - (4x_1^2 + 16x_2^2 - 4)x_2 - (10x_1^2 + 2x_1x_2 + 2x_2^2 + 5x_3^2 - 10)x_3 \end{pmatrix}.$$  

The block diagram is shown in Figure 3.8. This example is the first third order example considered, hence it will be used to discuss the three dimensional state space.

The origin is the only equilibrium point and it is unstable since both the coefficients of $\ddot{y}$ and $\dot{y}$ are negative at the origin. The time response of $y$ and $\dot{y}$ are shown in Figure 3.9. The three dimensional phase space plot is shown in Figure 3.10. Interpretation of the phase plot in three dimensions must be done with care. It is important to choose a viewpoint, the point in three dimensional space from which the viewer would look toward the origin, such that the curve
Figure 3.8. Block Diagram of the System in Example 3.3.2.
Figure 3.9. Time Response of System in Example 3.3.2.
Figure 3.10. Phase Space Plot of Limit Cycle in Example 3.3.2.
observed correctly represents the true state trajectory. In some cases it may be necessary to present several views of the same trajectory.

The describing function is

\[ \hat{N}(\hat{x}_1, \omega) = \frac{\hat{x}_1}{4} \left( [-30\omega^2 - 2\omega^4 - 15\omega^6] + j[4\omega + 46\omega^3] \right). \]

A polar plot of \( G_p(j\omega) \) and \(-1/\hat{N}(\hat{x}_1, \omega)\) is shown in Figure 3.11. Analytically the point of intersection is found to be \((\hat{x}_1, \omega) = (0.422, 1.91)\). Using (3.26), the limit cycle is found to be stable.

This example demonstrates a stable limit cycle in a third order system. This is an important example in that it introduces the graphical presentation of a limit cycle in three dimensional space, and it demonstrates the existence of a limit cycle for a third order generalized oscillator model. The relationship between the shape of the oscillation and the parameter values, in this case the numerical values, can not be readily determined due to the complexity of the nonlinear element. In the next two examples two particular forms of this equation are investigated. In each example the model consists of a single parameter. The effect of this parameter on the existence of the limit cycle and the shape of the oscillation are investigated.

### 3.3.3 Third Order Example

In this and the following example, two similar models forms are presented. These examples are intended to investigate the effect of the nonlinear coefficients on the shape of the oscillation. Consider a generalized oscillator model of the form

\[ y^{(3)} + \ddot{y} + \mu(y^2 - 1)\dot{y} + y = 0, \]

where the parameter \( \mu \) can be chosen to shape the oscillation. It interesting to note the resemblance between this model and Van der Pol's equation.
Figure 3.11. Polar Plot of $G_p(j\omega)$ and $-1/\hat{N}(\bar{z}_1, \omega)$ for Example 3.3.2.
The equivalent state variable form is

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{pmatrix} =
\begin{pmatrix}
x_2 \\
x_3 \\
-x_1 - \mu (x_1^2 - 1)x_2 - x_3
\end{pmatrix}.
\]

The origin can be seen to be unstable since the coefficient of \( j \) is negative at the origin. The block diagram of the system is shown in Figure 3.12. The describing function is

\[
\hat{N}(\dot{x}_1, \omega) = \frac{x_1^2}{4} j \omega.
\]

Since the describing function is purely imaginary, a zero can be placed at the origin in the plant, \( G_p(s) \), and the describing function can be considered to be real. The plant transfer function is

\[
G_p(s) = \frac{\mu s}{s^3 + s^2 - \mu s + 1}.
\]

The block diagram of the system, with the nonlinear element replaced with the appropriate describing function is shown in Figure 3.13.

The effect of the parameter \( \mu \) can be investigated by considering the location of the closed-loop poles of the system shown in Figure 3.13. The closed-loop transfer function is

\[
\frac{Y(s)}{R(s)} = \frac{\mu s}{s^3 + s^2 + \mu (\hat{N} - 1)s + 1}
\]

where \( \hat{N} = \frac{\dot{x}_1^2}{4} \).

The Routh-Hurwitz criteria can be used to determine the stability of the closed-loop system. If \( \hat{N} < 1 \), the coefficient of \( s \) is negative, hence the closed-loop system has at least one pole in the right half plane. If \( \hat{N} > 1 \), all of the coefficients are positive and the Routh criterion must be used. The Routh table is

\[
\begin{array}{cccc}
1 & \mu(\hat{N} - 1) \\
1 & 1 \\
\mu(\hat{N} - 1) - 1 & 0 \\
1 & \\
\end{array}
\]
\( r(t) = 0 \)

\[
\begin{align*}
y(t) &= \frac{\mu}{s^3 + s^2 - \mu s + 1} \\
\end{align*}
\]

Figure 3.12. Block Diagram of the Nonlinear System in Example 3.3.3.
Figure 3.13. Block Diagram of the Equivalent Linear System in Example 3.4.3.
If \( \mu(\tilde{N} - 1) - 1 > 0 \) there are no sign changes in the first column, hence all of the closed-loop poles must lie in the left half plane. If \( \mu(\tilde{N} - 1) - 1 < 0 \) there are two sign changes, hence there must be two poles in the right half plane. It can be concluded that for large \( \tilde{N} \) the system will behave in a stable manner, and for small \( \tilde{N} \) the system will behave in an unstable manner. Hence, there is a value of \( \tilde{N} \) such that the system will oscillate.

To establish a relationship between the shape of the oscillation and the parameter \( \mu \), recall that \( \tilde{N} \) is proportional to the square of the magnitude of the fundamental harmonic \( (\hat{x}_1) \) of the system output. Using this fact and the critical case

\[
\mu(\tilde{N} - 1) - 1 = 0
\]

from the Routh table, it can be seen that

\[
\frac{\hat{x}_1^2}{4} = 1 + \frac{1}{\mu}
\]

As \( \mu \) increases \( \hat{x}_1^2 \) decreases.

To determine the relationship between \( \mu \) and the frequency, \( \omega \), of the oscillation, the criterion at (3.18) can be used. The condition for the existence of a limit cycle is

\[
1 + \tilde{N}(\hat{x}_1, \omega)G_p(j\omega) = 0.
\]

Setting the real and imaginary parts to zero it can be seen that \( \omega = 1 \) is the approximation of the fundamental frequency. Since \( \omega \) does not depend on \( \mu \), it can be concluded that the describing function analysis predicts no relationship between \( \omega \) and \( \mu \).

Figure 3.14 shows the time response of this system for two values of \( \mu \), \( \mu = 0.3 \) and \( \mu = 3.0 \). The effect of \( \mu \) on the magnitude of the oscillation is evident. The time response also shows a relationship between \( \mu \) and \( \omega \). This relationship was not predicted by the describing function analysis. Since the response is not
Figure 3.14. Time Response of the System in Example 3.3.3 for $\mu = 0.3$ (solid) and $\mu = 3.0$ (dashed).
sinusoidal, the describing function technique is not expected to yield good results. The key point here is not the performance of the describing function method, but the shape of the oscillation generated by the complex dynamics in the nonlinear element and the relationship between the shape and the parameter \( \mu \).

Figure 3.15 shows two different views of the limit cycle in the three dimensional state space for \( \mu = 3.0 \). Both view points are necessary to determine that the trajectories are not interlocked. It is interesting to note the complex behavior that can develop in a non-planar system such as the one discussed in this example.

3.3.4 Third Order Example

Consider a system of the form

\[
y^{(3)} + \mu(y^2 - 1)y \dot{y} + \dot{y} + y = 0.
\]

This system is similar to the system presented in the previous example, except that the nonlinear coefficient is associated with \( \ddot{y} \) instead of \( \dot{y} \). Again, the effect of the parameter \( \mu \) on the shape of the oscillation is investigated.

The equivalent state variable model is

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{pmatrix} = \begin{pmatrix}
x_2 \\
x_3 \\
-x_1 - x_2 - \mu(x_1^2 - 1)x_3
\end{pmatrix}.
\]

The origin is the only singular point and it is unstable. The block diagram of the system is shown in Figure 3.16. The describing function is

\[
\hat{N}(\dot{x}_1, \omega) = -\frac{3\ddot{x}_1^2 \omega^2}{4}.
\]

The describing function is purely real and is negative. Replacing the nonlinear element with its describing function the system becomes a positive feedback system.
Figure 3.15. Two Views of the Limit Cycle in State Space of the System in Example 3.4.3 for $\mu = 3.0$. 
Figure 3.16. Block Diagram of the System in Example 3.3.4.

\[ r(t) = 0 \]

\[ y(t) = \frac{\mu}{s^3 - \mu s^2 + s + 1} \]
The block diagram is shown in Figure 3.17. The poles of the closed loop system can be found by considering the closed-loop transfer function,

\[
\frac{Y(s)}{R(s)} = \frac{\mu \hat{N}}{s^3 - \mu s^2 + s + (1 - \mu \hat{N})}
\]

where \( \hat{N} = 3 \hat{z}_1^2 \omega^2 / 4 \).

The Routh criteria can be used to determine the stability of the system. The Routh table is

<table>
<thead>
<tr>
<th>( \mu )</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 - ( \hat{N} + \frac{1}{\mu} )</td>
<td>0</td>
</tr>
<tr>
<td>1 - ( \hat{N} )</td>
<td>( \mu )</td>
</tr>
</tbody>
</table>

Since \( \mu > 0 \), there is always at least one pole in the right half plane. This fact would generally result in the conclusion that the system was unstable, hence no limit cycle could exist. Further investigation demonstrates that despite this prediction, a stable limit cycle does exist.

Using numerical integration, the system was solved for several small values of \( \mu \). The time response for \( \mu = 0.1 \) is shown in Figure 3.18. Two important observations can be made based on the simulation results. First, for small values of \( \mu \) the system does exhibit limit cycle type behavior. Also, the system response is not symmetric with respect to the time axis. Further investigation of the symmetry shows that two limit cycle modes of behavior are possible. Each of these modes depends on the sign of the initial condition \( x_1(t_0) \). Figure 3.19 shows the three dimensional state space that contains two limit cycles. Further discussion of the unsteady behavior is reserved for the following section. The remainder of this example will be dedicated to investigating the existence of limit cycles for small values of \( \mu \).

From (3.23) the condition for the existence of a limit cycle becomes

\[
1 + \mu \omega^2 - \frac{3}{4} \mu \hat{z}_1^2 \omega^2 = 0
\]
\[ r(t) = 0 \]

\[ y(t) = \frac{\mu}{s^3 - \mu s^2 + s + 1} \]

\[ \frac{3\omega_1^2\omega^2}{4} \]

Figure 3.17. Block Diagram of the Equivalent Linear System in Example 3.3.4.
Figure 3.18. Time Response of the System in Example 3.3.4.
Figure 3.19. Two Limit Cycles in State Space of the System in Example 3.3.4.
and

\[ \omega - \omega^3 = 0. \]

From the second condition either \( \omega = 0 \) or \( \omega = 1 \). Since \( \omega = 0 \) will not satisfy the first condition, \( \omega = 1 \) is a possible solution. Then

\[ \hat{x}_1^2 = \frac{4}{3} \left( 1 + \frac{1}{\mu} \right). \]

This result predicts that for any \( \mu > 0 \), there exists a solution to the existence conditions stated above, hence it is predicted that a limit cycle exists. But, this conclusion contradicts the result obtained by examining the closed loop pole locations.

To help resolve this conflict, consider the effect of \( \mu \) on the location of the open loop poles. As \( \mu \) is varied \( (\mu > 0) \), a locus of open loop pole locations can be determined. This locus is shown in Figure 3.20. For each \( \mu \), a root locus of close loop pole locations can be determined by varying the magnitude of the describing function \( \hat{N} \). This produces a family of root loci, that depend on \( \mu \) and \( \hat{N} \). From this family of root loci it can be seen that for large values of \( \mu \) the system contains three real poles, two in the right half plane. As \( \hat{N} \) is increased, one pole continues to move deeper into the right half plane, hence the system is unstable for all \( \hat{N} > 0 \).

For small values of \( \mu \), the root locus shows a complex pair of poles moving from the left half plane and one real pole moving from the left half plane to the right half plane as the describing function is increased. For small values of \( \mu \), either the complex pair or the real pole, but not both, are in the right half plane, and all three are very near the imaginary axis. Since the describing function method of analysis is an approximation technique, it is possible that location of the three poles near the imaginary axis is a case when absolute application of the existence criteria fails.

This example has demonstrated a limit cycle in a generalized oscillator model that is different than was expected. The effects of the parameter \( \mu \) has been
Figure 3.20. Locus of Open Loop Pole Locations as $\mu$ is Increased.
seen to effect both the shape and the existence of a limit cycle. It has been found that the describing function method of analysis fails to provide consistent information regarding the existence of a limit cycle. It has also been found that the system demonstrates two limit cycles that are not symmetric with respect to the time axis, a characteristic that was unsuspected and are investigated further is the next section.

3.3.5 Fourth Order Example

This example presents a fourth order system that exhibits a stable limit cycle. The example is important because it also demonstrates the possibility of multiple limit cycles in higher order systems. Consider an oscillator model of the form

\[ y^{(iv)} + 4y^{(3)} + 3\dot{y} + 4(y^2 - 1)\dot{y} + y = 0. \]

Since the coefficient of \( \dot{y} \) is negative near the origin, it can be seen that the origin is the only singular point and it is unstable.

The nonlinear term in this model is the same as in Van der Pol's and the system of Example 3.3.3. The describing function is

\[ \hat{N}(\dot{x}_1, \omega) = \frac{\omega^2 \text{sgn}(\omega)}{4} \]

and the block diagram of the system, with the nonlinearity replaced with its describing function as shown in Figure 3.21. Since the describing function is purely imaginary, a zero can be placed at the origin in the plant and the root locus can be utilized to determine the possibility of a limit cycle. The root locus is shown in Figure 3.22. From the root locus it appears that there are two possible limit cycles, one at each value of \( \omega \) where the root locus crosses the imaginary axis. The first crossing occurs at \((\dot{x}_1, \omega) = (2.3511, 0.6180)\) and the second crossing occurs at \((\dot{x}_1, \omega) = (3.8047, 1.6181)\).
Figure 3.21. Block Diagram of the Nonlinear System in Example 3.3.5.
Figure 3.22. Root Locus for the System in Example 3.3.5.
The root locus also provides useful information regarding the stability of each of the limit cycles. Consider the limit cycle at \( \omega = 0.6180 \). If the magnitude of the fundamental harmonic, \( \hat{x}_1 \) is slightly increased the closed loop poles moves into the left half plane and the response will decrease, hence \( \hat{x}_1 \) decreases and the response returns to the limit cycle. If \( \hat{x}_1 \) is slightly decreased, the closed loop poles moves back into the right half plane and the response increases, hence \( \hat{x}_1 \) increases and the system returns to the limit cycle. This behavior predicts that the limit cycle with fundamental frequency \( \omega = 0.6180 \) is stable. Similarly, the limit cycle with fundamental frequency \( \omega = 1.6181 \) can be determined to be unstable. Thus, the describing function analysis predicts that the system exhibits two limit cycles, one unstable and one stable.

The time response for this system is shown in Figure 3.23. The system was started with the initial condition \( \mathbf{z}(t_0) = (1, 1, 1, 1)' \) for the response shown. Two observations are immediately evident. First, the oscillation appears to be unsymmetric with respect to the time axis. This behavior is very similar to that observed in the previous example. As in the previous example, there appears that there are two limit cycle that can be reached by changing the sign of the initial conditions. Hence, instead of a single limit cycle there appears two be two stable limit cycles. Next, the response appears to reach a stable oscillation at a frequency near that predicted for a stable limit cycle. When the system was started at an initial condition that had greater magnitude, for example at \( \mathbf{z}(t_0) = (5, 5, 5, 5)' \), the response appeared to be unstable, thus confirming the describing function prediction.

This example presents a system that exhibits two stable and at least one unstable limit cycle. The shape of the response closely resembles that observed in Example 3.3.4 in that both responses are unsymmetric with respect to the time axis. Since the system is fourth order it is not possible to present the response in state space, but from the behavior observed this space would be complex. This
Figure 3.23. Time Response of the System in Example 3.3.5.
example does serve to demonstrate that the generalized oscillator model can exhibit limit cycles in higher order systems, although as the order of the system is increased the dynamics become more complex.

3.4 Unsymmetric Generalized Oscillators

The purpose of this section is to develop a generalized mathematical model that can be used to model oscillations that are unsymmetric. By unsymmetric oscillations it is meant periodic functions, i.e. system time responses, that are unsymmetric with respect to the time axis. Many observed oscillations including the electrocardiogram, the piston position in the cylinder and the repetitive motion of a robot arm, exhibit this type of behavior. In this section, a generalized oscillator model of the form (3.6) is presented that can be used to model unsymmetric oscillations. The model is developed by exploring possible sources of the unsymmetric behavior and choosing a model structure that can be used for system identification. Several examples are presented at the end of the section.

3.4.1 Mathematical Model

Unsymmetric oscillations may exist in systems through two possible sources: natural dynamics and system structure. To demonstrate the occurrence of unsymmetric oscillations of through natural dynamics, consider the linear system

\[ y^{(3)} + \omega^2 y = 0. \]

The characteristic equation of this system is

\[ \lambda^3 + \omega^2 \lambda = 0, \]

so there is an integrator and a complex pair of poles. The solution is of the form

\[ y(t) = A + B \sin(\omega t + \theta). \]
Given the initial conditions \( y(0), \dot{y}(0), \) and \( \ddot{y}(0), \) the constants \( A, B \) and \( \theta \) can be determined. Clearly, if \( A \neq 0, \) the system exhibits an oscillation that is biased by an amount \( A, \) hence it is not symmetric with respect to the time axis. It is important to note that this system is linear and cannot have a limit cycle, but is useful for investigating the unsymmetric behavior.

The oscillations observed in example 3.3.4 and 3.3.5 exhibited behavior very similar to that observed in the linear system considered above. Recall, in Example 3.3.4, that for small values of \( \mu, \) the closed loop poles were all near the imaginary axis, one pole was near the origin and a complex pair near \( \pm j. \) In example 3.3.5, the closed loop system contained a zero at the origin, with a pole very near, and a complex pair near \( \pm j0.618. \) The similarity between these two examples and the linear system discussed above presents a possible explanation for the unsymmetric behavior observed in these systems.

Unsymmetric oscillations may also exists due to the underlying system structure. To investigate this possibility, consider Van der Pol's equation

\[
\dddot{y} + \mu(y^2 - 1)\dot{y} + y = 0.
\]

The symmetry of the time response can be seen in Figures 2.4 to 2.7. Van der Pol's equation is known to oscillate due to the nonlinear damping function \( \mu(y^2 - 1). \) Figure 3.24 is a graph of the nonlinear damping versus \( y. \) Clearly, the damping is symmetrical with respect to \( y = 0. \) By simply shifting the point of symmetry, to \( y = a, \) as shown in Figure 3.25, the damping is no longer symmetrical about \( y = 0, \) thus the time response is longer be symmetrical about the time axis. The unsymmetrical response is shown in Figure 3.26 for \( a = 0.95. \)

The principle of shifting the point of symmetry of the nonlinear damping can be extended to the generalized oscillator model (3.6). In this case if one or
Figure 3.24. Symmetric Nonlinear Damping Element of Van der Pol's Oscillator.
Figure 3.25. Unsymmetric Nonlinear Damping Element of Van der Pol's Oscillator.
Figure 3.26. Time Response of Van der Pol Type Oscillator with Unsymmetric Nonlinear Damping Element.
more of the nonlinear coefficients, $a(x^{(i)})$, is unsymmetric about the origin, then
the oscillation is unsymmetric. The general unsymmetric oscillator is of the form

$$y^{(n)} + a_n(x^{(n)} - \beta_n)y^{(n-1)} + \cdots + a_2(x^{(2)} - \beta_2)\dot{y} + a_1(x^{(1)} - \beta_1)y = 0,$$

(3.27)

where $\beta_j, j = 1, \ldots, n$ is the point of symmetry of the nonlinear coefficient $a_n(\cdot)$.

As defined for the generalized oscillator model, the coefficients are chosen to be quadratic and of the form

$$a_j(x^{(j)} - \beta_j) = \begin{cases} (x^{(j)} - \beta_j)'A_j(x^{(j)} - \beta_j) - \alpha_j & \text{or} \\ \alpha_j & \end{cases}$$

(3.28)

for $j = 1, 2, \ldots, n$. Again, $A_j$ is chosen to be a positive semidefinite $j \times j$ matrix and $\alpha_j \geq 0$ for $j = 2, 3, \ldots, n$ ($\alpha_1 < 0$).

For the purpose of system identification, the structural approach to modeling the unsymmetric oscillating behavior is more desirable. Control over the structural properties is obtained and the natural model dynamics are not required for modeling unsymmetric behavior. In the next section some properties of the unsymmetric generalized oscillator model are discussed.

### 3.4.2 System Analysis

In this section several properties of the unsymmetric oscillator model are investigated. The focus of this discussion is on establishing the origin as the only equilibrium point of the system and ensuring that it is unstable. The existence of stable limit cycles can be investigated by the describing function method but is not presented in this discussion since it closely follows that of Section 3.2.2. The generalized Routh-Hurwitz criterion (Schultz 1963) is used as the motivating principle for the existence of limit cycles.
The unsymmetric generalized oscillator model has the equivalent phase variable representation

\[ \begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
&\quad \vdots \\
\dot{x}_n &= -a_1(x^{(1)} - \beta_1)x_1 - a_2(x^{(2)} - \beta_2)x_2 - \cdots - a_n(x^{(n)} - \beta_n)x_n.
\end{align*} \tag{3.29} \]

To ensure that the origin as the only singular point, it is necessary that

\[ \begin{align*}
0 &= x_2 \\
0 &= x_3 \\
&\quad \vdots \\
0 &= -a_1(x^{(1)} - \beta_1)x_1 - a_2(x^{(2)} - \beta_2)x_2 - \cdots - a_n(x^{(n)} - \beta_n)x_n. 
\end{align*} \tag{3.30} \]

From these conditions, it follows that

\[ a_1(x^{(1)} - \beta_1) = 0 \]

only if \( x^{(1)} = 0 \) or equivalently only if \( x_1 = 0 \). Since \( a_1(\cdot) \) is a quadratic function of the form (3.28) this can be ensured if \( \alpha_1 < 0 \).

To determine if the origin is a stable or an unstable singular point, the Hurwitz criteria can be applied to the system (3.29) linearized about the origin. The characteristic equation of the linearized system is

\[ \lambda^n + (\beta_n' A_n \beta_n - \alpha_n)\lambda^{n-1} + \cdots + (\beta_2' A_2 \beta_2 - \alpha_2)\lambda + (\beta_1' A_1 \beta_1 - \alpha_1)\lambda = 0. \]

The origin is unstable if any of the coefficients of the characteristic equation are negative, i.e.

\[ \beta_j' A_j \beta_j - \alpha_j < 0 \tag{3.31} \]

for any \( j = 1, 2, \ldots, n \). Thus, (3.31) is a necessary condition for the origin to be an unstable singular point.
Following the generalized Routh-Hurwitz analogy it is desirable that any quadratic coefficient of the form

\[(x - \beta)'A(x - \beta) - \alpha\]  

(3.32)

change sign from negative to positive as \(|x| \to \infty\). Since \(A\) was required to be positive semidefinite the quadratic \((x - \beta)'A(x - \beta)\) is positive for all \(x \neq \beta\) and zero if \(x = \beta\). If \(\alpha > 0\), then (3.32) is negative for \(x\) in a neighborhood of \(\beta\) and positive when \(|x| \gg |\beta|\). This condition requires that \(\beta < \infty\) since it is desired that the coefficient change sign as \(x \to \infty\). Together with (3.31), the requirement that the origin be an unstable singular point, a necessary condition for the coefficient to change signs is

\[\beta'A\beta < \alpha.\]

In the following section two simple examples are presented to demonstrate the response shapes that can be generated using the unsymmetric generalized oscillator model. These examples are generated based on the simple principle developed in this section. Application of the describing function technique could be used to establish the existence and stability of the limit cycles in these examples but is omitted for the sake of simplicity.

3.5 Examples

3.5.1 Second Order Example

Consider a system of the form

\[\ddot{y} + \begin{pmatrix} y & 0.1 \\ \dot{y} & 0.95 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} y & 0.5 \\ \dot{y} - 0.95 \end{pmatrix} - 4\dot{y} + y = 0.\]

The state variable representation is

\[
\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -x_1 - (4(x_1 - 0.1)^2 + 2(x_1 - 0.1)(x_2 - 0.95) + 4(x_2 - 0.95)^2 - 4)x_2 \\ x_2 \end{pmatrix}
\]
The origin is the only singular point and it is unstable since

\[
\begin{pmatrix}
0.1 & 0.95 \\
1 & 4
\end{pmatrix}
\begin{pmatrix}
0.1 \\
0.95
\end{pmatrix} - 4 < 0.
\]

The time response for both states, \( y \) (solid) and \( \dot{y} \) (dashed) are shown in Figure 3.27. The phase plane is shown in Figure 3.28.

The effect of shifting the symmetry of the nonlinear damping term can be seen by comparing the response of this system with that of the system in Example 3.3.1. In Example 3.3.1 the structure of the nonlinear term is similar except that it is symmetric about the origin. With the symmetry about the origin the response is nearly circular, as shown in Figure 3.5. The responses shown in Figure 3.27 is far from circular. This example has shown that by a simple shift in the symmetry of the nonlinear element, the shape of the oscillation has been considerably effected.

The next example extends the idea of shifting the symmetry of the nonlinear coefficient in higher-order systems by examining a third order system

### 3.5.2 Third Order Example

Consider a system of the form

\[
y^{(3)} + \ddot{y} + \left( y - 0.5 \right) \begin{pmatrix}
0.5 & 0 \\
0 & 4
\end{pmatrix} \begin{pmatrix}
y - 0.5 \\
\dot{y} - 1
\end{pmatrix} - 8 \ddot{y} + y = 0.
\]

The state variable representation is

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} = \begin{pmatrix}
x_2 \\
-x_1 - (0.5(y - 0.5)^2 + 4(\dot{y} - 1)^2 - 8)x_2 - x_3
\end{pmatrix}.
\]

The nonlinear coefficient in this example closely resembles that of Example 3.5.1. The origin is the only singular point and is unstable since

\[
\begin{pmatrix}
0.5 & 1 \\
0 & 4
\end{pmatrix}
\begin{pmatrix}
0.5 \\
1
\end{pmatrix} - 8 < 0.
\]
Figure 3.27. Time Response for Both States $y$ (solid) and $\dot{y}$ (dashed) of the System in Example 3.5.1.
Figure 3.28. Phase Plane of the System in Example 3.5.1.
The time responses for the three states are shown in Figures 3.29 (a)-(c). The most striking feature of this example is the complex response that is generated by simply increasing the system from second to third order. This effect was also observed in comparing Example 3.3.1 to both Examples 3.3.3 and 3.3.4.

The effect of increasing the order of the system contradicts the traditional view of systems that can be analyzed by the describing function method. The traditional view is that higher order systems provide better filter characteristics, hence the response should be closer to a pure sinusoid. In the class of systems presented in this chapter the opposite behavior is observed. As the order of the generalized oscillator model increases, more complexity is added to the nonlinear element, consequently the response becomes less sinusoidal. The addition of the ability to shift the symmetry of the nonlinear element adds an additional level of complexity. This also adds an additional means of controlling the shape of the oscillation. The desire to control the shape of the oscillation is central to the problem of system identification. From the results presented in this chapter, the class of generalized nonlinear oscillators may provide a powerful vehicle for solving the system identification of nonlinear oscillations problem.
Figure 3.29. (a) Time Response of State $y = x_1$ of Example 3.5.2.
Figure 3.29. (b) Time Response of State $\dot{y} = x_2$ of Example 3.5.2.
Figure 3.29. (c) Time Response of State $\dot{y} = x_3$ of Example 3.5.2.
CHAPTER 4

SYSTEM IDENTIFICATION

4.0 Introduction and Organization of the Chapter

This chapter discusses the identification of nonlinear oscillations using the class of generalized oscillator models presented in Chapter 3. The focus of this chapter is on problem formulation and investigation of methodologies for solving the identification problem. As previously mentioned, the system identification problem is primarily concerned with building mathematical models of dynamical systems from observed data.

This chapter is divided into three parts. In the first part the system identification problem is formulated as a mathematical programming problem. The components of the identification problem are identified, and the quasilinearization method of parameter estimation is described in detail. The second part develops specific characteristics of the identification problem based on the generalized oscillator model as the set of candidate model structures. Particular requirements that must be satisfied for the generalized oscillator model to exhibit a limit cycles are stated in the format of the identification problem. Two examples are presented that demonstrate the application of the method of quasilinearization to the identification of nonlinear oscillations. The third part discusses the convergence properties of the method of quasilinearization.

4.1 The System Identification Problem

The system identification problem involves three basic elements: the set of data, the class of candidate models, and the performance criterion (Ljung 1987). The set of data is typically an input-output record of some experiment on the
system or process being modeled. If possible it is desirable to perform controlled experiments on the real system in order to maximize the amount of information available. Sometimes controlled experiments are not possible, hence the data set must represent the typical system behavior. Increased difficulty is added when the input to the system is unavailable for measurement.

The class of candidate models is probably the most difficult and important choice of the identification procedure. A priori knowledge and intuition must be used in selecting a class of models that can successfully represent the true system. General classes of models include linear time-invariant lumped parameter, linear time-invariant distributed parameter, linear time-variable lumped parameter, linear time-variant distributed parameter, nonlinear time-invariant, and nonlinear time-variable, plus numerous others. Within each of these classes are subclasses that depend on properties such as dimension and structure. Models can sometimes be generated using physical properties of the system or sometimes a black-box approach must be taken.

Another important distinction in choosing the class of models is between deterministic and stochastic models. Stochastic system models are used to model systems where there is uncertainty in the inputs or disturbances in the inputs or outputs. Models of economic systems are often modeled as stochastic systems since all of the inputs, or factors that affect the state and output, are not known and are not measurable. Systems where the inputs and the outputs are known, or at least can be assumed known, and are measurable can be modeled as deterministic systems.

The performance criterion is the decision rule by which a specific member of the chosen class of system models is selected, given the set of data. The performance criterion is used to guide the identification procedure in searching for the desired model. The most popular performance criterion is the minimum mean square error between the model response and the set of data.
Given these three elements, the system identification problem can be stated as a mathematical programming problem. Assume that a set of data is the observed system response \( \{y_d(t) : 0 \leq t \leq T\} \). For the purpose of identification of nonlinear oscillators it can be assumed that the input is a constant equal to zero, the system is deterministic, and the class of system models is of the form

\[
\begin{align*}
\dot{x}_\theta(t) &= f(x_\theta(t), \theta) \\
y(t, \theta) &= g(x_\theta(t), \theta)
\end{align*}
\]

where \( \theta \) is an \( m \)-dimensional vector of unknown parameters and \( x_\theta \) indicates that \( x \) depends on \( \theta \) through \( f \). The performance index is

\[
\int_0^T (y(t, \theta) - y_d(t))^2 dt.
\]

The system identification problem can then be stated as

\[
\begin{align*}
\text{Minimize} & \quad \int_0^T (y(t, \theta) - y_d(t))^2 dt \\
\text{s.t.} & \quad \dot{x}_\theta(t) = f(x_\theta(t), \theta) \\
& \quad y(t, \theta) = g(x_\theta(t), \theta) \\
& \quad \theta \in \Theta.
\end{align*}
\]

where the minimization is performed over \( \theta \) and \( \Theta \) is the set of feasible parameter values.

The mathematical programming problem (4.1) contains all of the elements of the system identification problem, but is not presented in a form that is solvable by traditional optimization techniques. The difficulty arises from the differential equations in the constraint set. It is these differential equations that relate the parameters \( \theta \) to the model output \( y(t, \theta) \), hence they must somehow be incorporated into the objective function.
In the case of a linear system with known input this incorporation is simple. The differential equations can be explicitly solved and the solution substituted into the objective. The resulting optimization problem is then the familiar nonlinear least-squares problem. In the case of a nonlinear system the equations cannot in general be explicitly solved, hence other methods of incorporation must be used. One technique for accomplishing this is the method of quasilinearization (Bellman, Kagiwada, and Kalaba 1965, Roth 1966, Roth 1981, Bellman and Roth 1984).

The method of quasilinearization was proposed as a method for solving nonlinear boundary-value problems, but it was also recognized that the method could also be applied to the system identification problem. The method is a generalization of Newton-Raphson method of finding roots of nonlinear equations to function space. In this section, the method of quasilinearization is presented. Numerous examples can be found in the references.

For the purpose of this presentation assume that

\[ y(t, \theta) = g(x_{d}(t), \theta) = x_{d,1}(t) \] (4.2)

It is also assumed that \( f \) is continuous in \( t, x, \) and \( \theta \). The function \( f(x_{d}(t), \theta) \) can be expanded in a Taylor series about a nominal state trajectory \( x_{d}^{0}(t) \) and the initial parameter estimates \( \theta^{0} \) as

\[
f(x_{d}, \theta) = f(x_{d}^{0}(t), \theta^{0}) + F_{x}(x_{d}^{0}(t), \theta^{0})(x_{d}(t) - x_{d}^{0}(t)) + \ldots
\] (4.3)

where the expressions \( F_{x} \) and \( F_{\theta} \) represent the Jacobian matrices of \( f \) with respect to \( x_{d} \) and \( \theta \), respectively.

The system of quasilinear differential equations can be formed by neglecting the higher order term in the expansion (4.3) and noting that the parameters \( \theta \) are
assumed constant and unknown, hence can be treated as states with unknown initial conditions. Then, the quasilinear system is

\[
\dot{x}_2 = f(x_2^0(t), \theta^0) + F_h(x_2^0(t), \theta^0)(x_2(t) - x_2^0(t)) + F_g(x_2^0(t), \theta^0)(\theta - \theta^0)
\]

\[\dot{\theta} = 0\]

\[x_2(0) = x_0\]

\[\theta(0) = \theta^0.\]

For a given set of initial conditions, \((x_0, \theta^0)\), the quasilinear system (4.4) can be solved\(^\dagger\). If the solution is used as the nominal trajectory, \(x_2^0\), and the system solved again, then a recursive relationship can be established of the form

\[
\dot{x}_{2k+1} = f(x_{2k}^k(t), \theta^k) + F_h(x_{2k}^k(t), \theta^k)(x_{2k+1}^k(t) - x_{2k}^k(t)) + F_g(x_{2k}^k(t), \theta^k)(\theta^{k+1} - \theta^k)
\]

\[\dot{\theta}^{k+1} = 0\]

\[x_{2k+1}(0) = x_0\]

\[\theta^{k+1}(0) = \theta.\]

The fundamental idea of the quasilinear technique is contained in the recursive application of the system (4.5).

Application of the quasilinear technique to the system identification problem is accomplished by using the linearity of the quasilinear system (4.5). The system output \(y(t, \theta)\) can be written as the sum of the particular solution, \(y_p(t, \theta)\), and the homogeneous solution, \(y_h(t, \theta)\),

\[y(t, \theta) = y_p(t, \theta) + y_h(t, \theta).\]

\(^\dagger\) Numerical solution is often the only available method of solution.
In addition the homogeneous solution can be written as the weighted sum of the response to the set of initial conditions

\[ \{e_i : e_{ij} = 0 \text{ if } i \neq j \text{ and } e_{ij} = 1 \text{ if } i = j \} . \]

The response is

\[ y(t, \theta) = y_p(t, \theta) + \sum_{j=1}^{n+m} c_j y_{h,j}(t, \theta), \]  

(4.6)

where the weights \( c_j, j = 1, \ldots, n + m \) are unknown and \( y_{h,i}(t, \theta) \) is the homogeneous response due to the initial condition \( e_i \).

To determine the weights, \( c_j \), the integral squared error between the model response, \( y(t, \theta^k) \), and the observed data \( y_d(t) \) is minimized over the set of allowable \( c_j \)'s. Then at the \( k^{th} \) iteration, the system identification problem becomes:

\[ \text{Minimize } \int_0^T (y_p(t, \theta^k) + \sum_{j=1}^{n+m} c_j y_{h,j}(t, \theta^k) - y_d(t))^2 dt \]

s.t.

\[ \theta \in \mathcal{C} \]

where \( \mathcal{C} = \mathbb{R}^n \times \Theta \). At each iteration of the quasilinear technique, this minimization is repeated, and \( (\theta_{k+1}^0, \theta_{k+1}^{k+1}) = \theta \). This process is repeated until some predetermined stopping condition, such as integral squared error less than some epsilon, is achieved.

4.2 Nonlinear Oscillator Identification

The purpose of this section is to develop the system identification problem using the generalized oscillator models as the set of candidate models. Based on this set of candidate models the method of quasilinearization is used to solve the identification problem. Since these models have structural requirements that determine the existence of limit cycles, specific feasible parameter sets must be
determined. In the following section two examples are presented that demonstrate the method.

The first step in utilizing the generalized oscillator model is to identify the set of system parameters. The generalized oscillator model can be represented by one $n^{th}$ order structurally identifiable equation of the form

$$y^{(n)} + a_n(x^{(n)})y^{(n-1)} + a_{n-1}(x^{(n-1)})y^{(n-2)} + \cdots + a_2(x^{(2)})y + a_1(x^{(1)})y = 0$$

where

$$a_i(x^{(i)}) = \begin{cases} x^{(i)}A_i x^{(i)} - \alpha_i & \text{or} \\ \alpha_i & \text{if } i \geq 0 \end{cases}$$

$\alpha_i \geq 0$ for $i = 2, \ldots, n$, $\alpha_1 < 0$, and $A_i$ is an $i \times i$ positive semidefinite matrix.

If the matrix $A_i$ is of the form

$$A_i = \begin{pmatrix} a_{11}^{(i)} & a_{12}^{(i)} & \cdots & a_{1i}^{(i)} \\ a_{12}^{(i)} & a_{22}^{(i)} & \cdots & a_{2i}^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1i}^{(i)} & a_{2i}^{(i)} & \cdots & a_{ii}^{(i)} \end{pmatrix}$$

then the set of system parameters can be written as

$$\theta = (a_{11}^{(n)} \ a_{12}^{(n)} \ \cdots \ \alpha_n \ a_{11}^{(n-1)} \ \cdots \ a_{n-1,n-1}^{(n-1)} \ \alpha_{n-1} \ \cdots \ a_{11}^{(1)} \ \alpha_1)'$$

Given this parameterized form of the generalized oscillator model, the model can be written in the state variable form as

$$\dot{x} = f(x, \theta)$$

where

$$f(x, \theta) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -b_1(x^{(1)}, \theta) & -b_2(x^{(2)}, \theta) & -b_3(x^{(3)}, \theta) & \cdots & -b_n(x^{(n)}, \theta) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix}.$$
The next step in utilizing the generalized oscillator model is the determination of the set of feasible system parameters. In the development of the generalized oscillator model, the nonlinear coefficients were used to switch between stable and unstable modes of behavior. In order to achieve this switching behavior, the coefficients were required to change sign as $\|z\|$ became large. To accomplish this it was required that the matrix $A_i(\theta)$ be positive semidefinite (denoted $A_i(\theta) \geq 0$). To ensure this, it is required that all of the principal minors of $A_i(\theta)$ have nonnegative determinants. It was also required that $\alpha_i \geq 0$, $i = 2, 3, \ldots, n$ and $\alpha_1 < 0$.

Together these two requirements constitute a set of feasible system parameters for the generalized oscillator model to be used as a candidate model for system identification. It should be noted that these conditions are only necessary and not sufficient for the generalized oscillator model to exhibit a limit cycle.

In addition to the structural parameters of the generalized oscillator model, the system identification problem must also be able to determine the proper initial conditions on the state variables. For identification problems where the true system structure is known it may be possible to estimate the initial conditions numerically from the data. But in cases where the true model structure is not known, and a class of models, such as the generalized oscillator models, are to be used to model the system, then appropriate initial conditions must be considered as system parameters. Considering the initial conditions of the state variables as system parameters, the formal set of parameters to be determined by the system identification problem is

$$\mathbb{R}^n \times \Theta = \{(y(0), \dot{y}(0), \ldots, y^{(n-1)}(0), b^{(n)}_{11}, \ldots, b^{(1)}_{11}, \alpha_1) :$$

$$B_i(\theta) \geq 0, i = 1, 2, \ldots, n;$$

$$a_i \geq 0, i = 2, 3, \ldots, n;$$

$$\alpha_1 < 0\}$$
Given the generalized oscillator models as a set of candidate models, a set of feasible parameter values \((\mathbb{R}^n \times \Theta)\), and a data set \(\{y_d(t) : 0 \leq t \leq T\}\) the nonlinear oscillator identification problem can be stated as:

\[
\text{Minimize } \int_0^T (y(t, \theta) - y_d(t))^2 dt \\
\text{s.t.} \\
\dot{x}_0(t) = f(x_0(t), \theta) \\
y(t, \theta) = g(x_0(t), \theta) \\
x_0(0) \in \mathbb{R}^n \\
B_i(\theta) \geq 0, \quad i = 1, 2, \ldots, n \\
\alpha_i \geq 0, \quad i = 2, 3, \ldots, n \\
\alpha_1 < 0.
\] (4.7)

The method of quasilinearization can be used to incorporate the differential equations into the objective function and solve the system identification problem. The system identification problem can be solved by solving a sequence of problems of the form:

\[
\text{Minimize } \int_0^T (y_p(t, \theta^k) + \sum_{j=1}^{n+m} c_{j}y_{h,j}(t, \theta^k) - y_d(t))^2 dt \\
\text{s.t.} \\
\theta \in \mathbb{R}^n \times \Theta
\] (4.8)
or equivalently by solving

\[
\begin{align*}
\text{Minimize} & \quad \frac{1}{2} c' Q_k c + d_k c + b_k \\
\text{s.t.} & \quad c = (x_{g^k,1}(0) \quad x_{g^k,2}(0) \quad \cdots \quad b^{(1)}_{11} \quad \alpha_1)' \\
& \quad B_i(g^k) \geq 0, \quad i = 1, 2, \ldots, n \\
& \quad \alpha_i \geq 0, \quad i = 2, 3, \ldots, n \\
& \quad \alpha_1 < 0, \\
& \quad x_{g^k}(0) \in \mathbb{R}^n,
\end{align*}
\]

where \(Q_k, d_k,\) and \(b_k\) are defined by the quadratic objective function in (4.8). This problem is a nonlinearly constrained quadratic programming problem that is in a form that can be solved using traditional optimization techniques.

In the following section, two examples are presented that demonstrate the formulation and solution of the system identification problem. Convergence properties of the method of quasilinearization are discussed in the section 4.4.

4.3 Examples

In this section, two examples are presented that demonstrate the solution of the system identification problem using the method of quasilinearization and the generalized oscillator models as the set of candidate models. In both of these examples, the data set is produced from a known model form. This known model form is a particular member of the class of generalized oscillator models. The set of candidate models is chosen to be a generalized oscillator model of the same dimension as the known model. System identification under these conditions provides for the greatest possibility of success. If the methods do not work under these
conditions, then there would be no hope for success under less controlled conditions. If the methods are successful under these conditions then the properties of the method can be studied.

4.3.1 Second Order Example

This example presents the application of the quasilinearization method to the identification of Van der Pol's oscillator using a 2nd order generalized oscillator model. This problem has been previously discussed by Bellman (1965). In his treatment of this problem, Bellman used a one parameter model of the form

\[ \ddot{y} + \mu \left( y^2 - 1 \right) \dot{y} + y = 0 \]

as the candidate model. The unknown parameter was \( \mu \). Initial estimates of the initial conditions were made using numerical differentiation. The method was successful if the initial estimate of \( \mu \) was chosen close to the true value. The method was not successful when the value of \( \mu \) was taken to be large.

In this example, the set of observed data is generated according to the model

\[ \ddot{y} + 4(y^2 - 1)\dot{y} + y = 0, \]

\( y(0) = 0.5 \) and \( \dot{y}(0) = -0.5 \). The candidate model is chosen to be of the form

\[ \ddot{y} + \left( \begin{pmatrix} y & \dot{y} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{pmatrix} \begin{pmatrix} y \\ \dot{y} \end{pmatrix} - \alpha_2 \right) \dot{y} + \alpha_1 y = 0. \] (4.10)

The system parameters are

\[ \theta = \begin{pmatrix} b_{11} & b_{12} & b_{22} & \alpha_2 & \alpha_1 \end{pmatrix} \]
The set of feasible system parameters is

\[ \mathbb{R}^2 \times \Theta = \{ (y(0), \dot{y}(0), b_{11}, b_{12}, b_{22}, \alpha_2, \alpha_1) : \\
\begin{align*}
& b_{11} \geq 0 \\
& b_{22} \geq 0 \\
& b_{11}b_{22} - b_{12}^2 \geq 0 \\
& \alpha_2 \geq 0 \\
& \alpha_1 \geq 0 \} \]

For the purpose of this demonstration, the initial parameter estimates are chosen to be:

\[ y^0 = \begin{pmatrix} b_{11} \\ b_{12} \\ b_{22} \\ \alpha_2 \\ \alpha_1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 2 \\ 1 \end{pmatrix} \]

and

\[ z^0 = \begin{pmatrix} y(0) \\ \dot{y}(0) \end{pmatrix} = \begin{pmatrix} 0.5 \\ -0.5 \end{pmatrix} \]

Using these initial estimates, the nominal state trajectory, \( z^0(t) \), can be determined by integrating the candidate system. The quasilinear system (4.5) can be formed based on this estimate. For the purpose of this example, a time interval of \( T = 10 \) time units is considered. The quasilinear system is solved using a 4\(^{th}\) order Runge-Kutta algorithm and produces an observation at each time epoch corresponding to the observed data.

At each iteration of the quasilinearization identification procedure, the nonlinearly constrained quadratic programming problem (4.9) is solved by the following heuristic:

1.0 Solve the unconstrained quadratic programming problem:

\[ \text{Minimize} \quad \frac{1}{2} z' Q z' + d' P z + b_k \]
where $Q_k$, $d_k$ and $b_k$ are appropriately defined.

2.0 If $b_{11} > 0$ and $b_{11}b_{22} - b_{12}^2 \geq 0$, stop.

3.0 If $b_{11} > 0$, $b_{22} > 0$ and $b_{11}b_{22} - b_{12}^2 < 0$, then $b_{12} = b_{12}/2$, repeat 2.0.

4.0 If $b_{11} > 0$, and $b_{22} < 0$, set $b_{22} = 0$, $b_{12} = 0$, solve the linearly equality constrained quadratic programming problem:

$$\text{Minimize} \quad \frac{1}{2} \xi'Q_k\xi + d_k'\xi + b_k$$

s.t.

$$\xi = (y(0) \ \dot{y}(0) \ b_{11} \ b_{12} \ b_{22} \ \alpha_2 \ \alpha_1)'$$

$$b_{22} = 0$$

$$b_{12} = 0$$

Then repeat 2.0.

5.0 If $b_{11} < 0$. Error condition. Stop.

This heuristic does not consider the constraints $\alpha_2 \geq 0$, $\alpha_1 \geq 0$, and is not considered a rigorous approach to the identification problem, but it can been seen in the following discussion, it does successfully solve the identification problem.

Table 4.1 shows the values of the parameter estimates and the integral squared error at each iteration of the quasilinearization method. The true parameter values are also shown for comparison. Figures 4.1 (a)-(m) show the observed data (solid line) and the response generated by the quasilinear system (dotted line) based on the parameter values shown in Table 4.1. The responses shown in Figures (a)-(m) are shown to demonstrate the unusual shape of the quasilinear system response compared to the data.

This example presents a generalized approach to the identification of Van der Pol's equation. Although the problem is similar to Bellman's example, it is different in several ways. First, the candidate system was assumed to be a 2$^{nd}$ order generalized oscillator model as defined in Chapter 3. By choosing a more general model the identification problem becomes more complex. This added complexity
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Table 4.1. Summary of Parameter Estimates and Computed Error at Each Iteration of the Quasilinearization Identification Procedure of Example 4.4.1.
Figure 4.2. (a) Observed Data and Nominal System Response of at the $0^{th}$ Iteration.
Figure 4.1. (b) 1st Iteration.
Figure 4.1. (c) 2nd Iteration.
Figure 4.1. (d) 3rd Iteration.
Figure 4.1. (e) 4th Iteration.
Figure 4.1. (f) 5\textsuperscript{th} Iteration.
Figure 4.1. (g) 6th Iteration.
Figure 4.1. (h) $7^{th}$ Iteration.
Figure 4.1. (i) 8th Iteration.
Figure 4.1. (j) 9th Iteration.
Figure 4.1. (k) 10th Iteration.
Figure 4.1. (l) 11th Iteration.
requires that the structural requirements of the generalized oscillator be satisfied at each iteration of the method. These added requirements are incorporated as constraints in the system identification problem. The identification problem is solved using the method of quasilinearization and a simple heuristic to determine the parameter values at each iteration of the method.

By examining the errors in Table 4.1 it can be seen that the method provides good reduction in the integral squared error at each iteration, except the 9th. At the 9th iteration a slight increase occurs. Examining the parameter estimates at the previous iterations it can be seen that this increase occurs when the nonlinear constraint

\[ b_{11} b_{22} - b_{12}^2 \geq 0 \]

becomes tight. After the slight increase the reduction continues until the method stops. In this example the stopping criterion was integral squared error less than 0.01.

Comparison of the observed data and the response of the quasilinear system observed data even for relatively good initial parameter estimates. After several iterations, it can be seen that the quasilinear response becomes less abrupt and more accurately approximated the observed data until the two are indistinguishable. In the following example, a 3rd order problem in investigated.

4.3.2 Third Order Example

This example presents the application of the method to the identification of the following system:

\[ y^{(3)} + 6\dot{y} + 32(y^2 - 1)\ddot{y} + 80y = 0, \]

\[ y(0) = 1.0792, \dot{y}(0) = -3.8983, \text{ and } \ddot{y}(0) = 5.8633. \]
The candidate model is of the form
\[
y^{(3)} + \alpha_3 \ddot{y} + ((y \ y) \begin{pmatrix} b_{11} & b_{12} \\ b_{11} & b_{22} \end{pmatrix} \begin{pmatrix} y \\ \dot{y} \end{pmatrix} - \alpha_2 \dot{y} + \alpha_1 y = 0.
\]

The system parameters are
\[
\theta = (\alpha_3 \ b_{11} \ b_{12} \ b_{22} \ \alpha_2 \ \alpha_1)'.
\]

The set of feasible system parameters is
\[
\mathbb{R}^3 \times \Theta = \left\{ \begin{pmatrix} y(0) \\ \dot{y}(0) \end{pmatrix}, \begin{pmatrix} \ddot{y}(0) \\ b_{11} \ b_{12} \ b_{22} \ \alpha_2 \ \alpha_1 \end{pmatrix} : \begin{array}{l}
\alpha_3 \geq 0 \\
b_{11} \geq 0 \\
b_{22} \geq 0 \\
b_{11}b_{22} - b_{12}^2 \geq 0 \\
\alpha_2 \geq 0 \\
\alpha_1 \geq 0 \end{array} \right\}
\]

For the purpose of this demonstration the initial parameter estimates are chosen to be:
\[
\theta^0 = \begin{pmatrix} \alpha_3 \\ b_{11} \\ b_{12} \\ b_{22} \\ \alpha_2 \\ \alpha_1 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 0 \\ 0 \\ 4 \\ 50 \end{pmatrix}
\]

and
\[
x_0^0 = \begin{pmatrix} y(0) \\ \dot{y}(0) \\ \ddot{y}(0) \end{pmatrix} = \begin{pmatrix} 1.7092 \\ -3.8983 \\ 5.8633 \end{pmatrix}.
\]

A time interval of \( T = 2.5 \) time units is considered. To increase the ability of the method to generate a sequence of estimates that will converge to the data, the initial state values are chosen to be exactly those of the set of observed data.
The nominal state trajectory is generated by integrating the candidate system and the initial parameter estimates. Based on this trajectory the quasilinear system (4.5) is formed. At each iteration of the quasilinearization method the heuristic presented in Example 4.3.1 is used to solve the nonlinearly constrained quadratic programming problem. Again, the parameters $\alpha_3$, $\alpha_2$ and $\alpha_1$ are not considered in the solution heuristic.

Table 4.2 shows the parameter estimates and the integral squared error at each iteration of the quasilinearization method. The true parameter values are shown for comparison. Figures 4.2 (a)-(f) show the observed data (solid line) and the response generated by the quasilinear system (dotted line) based on the parameter values in Table 4.2.

This example presents the application of the quasilinearization method to the identification of a nonlinear 3rd order oscillator based on a simplified form of the generalized oscillator model. A simplified form of the generalized oscillator model is chosen since the solution of the nonlinearly constrained quadratic programming problem can be accomplished by the simple heuristic of Example 4.3.1. Solution of the more general problem is considerably more difficult and remains an open research problem.

From Table 4.2 it can be seen that the integral squared error is rapidly reduced until it becomes small. A stopping criteria of integral squared error less than 0.01 was used in this example, but the error failed to decrease to this level. From figures 4.2 (a)-(f) the approximation can be seen to converge to the data.

The two examples presented in this section demonstrate the solution to the nonlinear oscillator system identification problem. The generalized oscillator models proposed in Chapter 3 are used as the candidate system models and the method of quasilinearization is used to incorporate the system of differential equations into the performance function. It should be noted that these two examples are the success that were achieved in this investigation. In general, even
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Table 4.2. Summary of Parameter Estimates and Computed Error at Each Iteration of the Quasilinearization Identification Procedure of Example 4.4.2.
Figure 4.2. (a) Observed Data and Nominal System Response of at the $0^{th}$ Iteration.
Figure 4.2. (b) 1st Iteration.
Figure 4.2. (c) 2nd Iteration.
Figure 4.2. (d) 3rd Iteration.
Figure 4.2. (e) 4\textsuperscript{th} Iteration.
Figure 4.2. (f) 5th Iteration.
for different parameter values in these example, the approach presented here fails to produce acceptable results. In the following section the convergence properties of the method of quasilinearization as a method of system identification are discussed.

4.4 Convergence Discussion

In the previous section it was noted that in general the method of quasilinearization applied to the identification of nonlinear oscillations was not successful. The two example presented in the previous section demonstrate two cases when the method was successful. Many attempts were made when the method failed to produce successful results. In the example that were successful, the parameter estimates were chosen relatively close to the true values and the initial conditions on the state variable were chosen to be exact. This can be seen in Tables 4.1 and 4.2.

Several authors have noted the difficulty of the method (Lee 1968, Seinfeld 1970), and have attributed it to the requirement that the initial parameter estimates be sufficiently close to the true values. This requirement is assumed in the proof of convergence (Bellman and Roth 1983, Bellman and Kalaba 1965), and are analogous to the requirement of convexity in the proof of convergence of Newton's method of finding roots of nonlinear equations.

In this investigation the experience was that even if this closeness requirement were satisfied, in most cases the method would not converge. It can be shown (Szidarovszky 1988) that the method will not necessarily converge. The difficulty lies in the fact that the initial conditions on the state variable are assumed unknown. Since the solution of the differential equations are a function of the initial condition, and since the solution of the quasilinear system is only an approximation to true solution, both the solution and the initial conditions are
being estimated at the same time. In a sense the initial conditions and the solution are “competing”.

To better illustrate this behavior consider the case of finding roots of nonlinear equations by Newton’s method. Assume that the nonlinear function of interest is

\[ f(x/\theta) \]

where \( \theta \) is a parameter that effects the structure of the function \( f \). It is desired to estimate \( x^* \) such that \( f(x^*/\theta) \), for a fixed \( \theta \).

Assume that \( f(x/\theta) \) is convex and that \( f'(x/\theta) > 0 \). Newton’s method is

\[ x_{k+1} = x_k - \frac{f(x_k/\theta)}{f'(x_k/\theta)} \]

which under the assumed conditions will converge to \( x^* \) as \( k \to \infty \). If at each iteration \( \theta \) is also changed, perhaps to minimize some other performance index, then Newton’s method becomes

\[ x_{k+1} = x_k - \frac{f(x_k/\theta_k)}{f'(x_k/\theta_k)} \]

hence the function \( f \) changes at each iteration and there is no guarantee that the method will converge.

This behavior is shown in Figure 4.3. Assume \( x_0 \) is the initial estimate. Newton’s method produces \( x_1 \) as the next estimate. If \( \theta_0 \) then becomes \( \theta_1 \), then the next iteration of Newton’s method will produce \( x_2 \), and a new \( \theta_2 \) is generate. The process will continue in this fashion with no assurance that the method will converge.

In the proof of convergence of the quasilinearization method, Bellman and Roth (1983) do not consider this structural change associated with the initial condition. In fact the proof given for the application of the method to the system identification problem is identical to the proof given for the two point boundary value problems (Bellman and Kalaba 1965).
Figure 4.3. Illustration of the Convergence Difficulty of the Method of Quasilinearization.
CHAPTER 5

CONCLUSIONS

5.0 Introduction and Organization of the Chapter

In chapter 1 a real engineering problem, monitoring patients under anesthesia, is sighted as the motivating factor in this research. Central to this problem is the need for modeling techniques that could be used to facilitate failure detection and diagnosis in systems where the measured signals are "periodic" or oscillating. In this dissertation, the problem of modeling these oscillating signals is considered.

The problem is approached by modeling the signals as the response of a nonlinear system that exhibits a limit cycle. A new class of models that exhibit limit cycles is developed and methods of system identification based on these models are investigated.

In this chapter, the contributions of dissertation thesis are summarized. Several areas of future research are also identified.

5.1 Contributions of this Dissertation

The primary goal of this dissertation is to develop a methodology for modeling oscillations that are observed in many systems including biological, electrical, mechanical, and chemical. The need for this methodology arises from the desire to perform failure detection and diagnosis on systems where the primary information source is measured oscillating signals. Since these systems are inherently dynamic, it is desired to model the oscillations as the response of a dynamic system. Modeling the signal in this fashion shifts the emphasis from signal properties to system properties that could possibly be associated with physical properties of the underlying system.
As a first step in achieving this goal a new class of nonlinear oscillator models is proposed. This class of system models is represented by one $n^{th}$ order autonomous differential equation with nonlinear coefficients of the form

$$y^{(n)} + a_n(x)y^{(n-1)} + \cdots + a_2(x)y + a_1(x)y = 0$$

where $x$ denotes the vector of system states. This class of oscillator model is motivated by the generalized Routh-Hurwitz criteria for the stability of nonlinear systems of the form described above. In the generalized oscillator model the nonlinear coefficients are chosen such that at least one coefficient is negative for $x$ near the origin, and all of the coefficients are positive as $x$ tends towards infinity. This is accomplished by choosing the coefficients to be quadratics of the form

$$a_i(x) = x'A_i x - \alpha_i$$

where $A_i$ is a positive semidefinite matrix. The class of system models defined in the fashion are found, for a broad range of equation forms, to exhibit stable limit cycles.

Because of the complexity and limited applicability of theoretical methods of analysis, such as Liapunov theory, the Poincaré-Bendixon Theorem, or the Poincaré map, the describing function approximation method is used as an analysis tool. Although this class of systems fit the general describing function form, they differ considerably. Traditionally, the describing function method is applied to systems with static nonlinearities, such as saturation and dead bands, or systems with hysteresis. The method has been applied to some simple dynamic nonlinearities, such as the differentiator found in Van der Pol's oscillator, but in general these systems are not treated.

The nonlinearities presented here are highly dynamic and of complex structure. It is through the structure of these nonlinearities that the wide range of
oscillations can be generated. While the describing function method is not generally applied to this type of system, it proves to be a useful tool that aids the understanding of these systems. Several numerical examples are presented that demonstrate the system behavior and the effect of the choice of structure and parameters.

The second step in achieving this goal is the formulation of the system identification problem based on the class of generalized oscillator models. The system identification problem is formulated as a mathematical programming problem of the form

\[
\text{Minimize } J(\theta) \\
\text{s.t. } \theta \in \Theta
\]

where \( J(\theta) \) is the performance, chosen to be the integral squared error, and \( \Theta \) defines the set of feasible system parameters. The set of feasible system parameters is defined by the conditions established on the parameters of the generalized oscillator model.

The solution of the system identification problem is investigated by considering the popular parameter estimation method of quasilinearization. Although the quasilinearization method has intuitively appealing properties and has proved successful in solving boundary value problems, it proves unsuccessful at solving the nonlinear oscillator identification problem. In fact it can be shown that the quasilinearization method of system identification will not, in general, converge to a solution. Section 4.4 presents an intuitive argument of the difficulty encountered in this approach.

Overall the contributions of this dissertation have provided a positive step toward the solution of the problem of modeling nonlinear oscillations. The class of models developed here have promising possibilities as a candidate set of models to be used in the system identification problem. The identification still remains
a complex obstacle. In the following section this problem, and others that have been identified as areas of potential research, are discussed.

5.2 Future Research

The single most important problem left unsolved at this point is the system identification problem. The complexity of this problem lies in the need to incorporate the system of governing differential equations into the objective function. One approach to this problem would be to solve the optimization problem by numerically solving the differential equations for different values of the parameters and estimating the parameter sensitivity, i.e. the gradient. One problem with this approach might be the relative insensitivity of the integral squared error to small changes in the system parameters. The sensitivity of these parameters needs to be studied. A sensitivity study may determine parameters that are not useful to the identification problem, hence the dimensionality of the problem could be reduced.

Another important area of future research is the need for further analysis of the generalized oscillator model. Basically two facts warrant this investigation. First, the describing function method is an approximation method that assumes the system response can be represented by a sinusoid. Because of the types of oscillations of interest here, this assumption is not valid. Also, the work of Smith (1987, 1986, 1980) and the properties of the generalized oscillator model indicate a strong possibility of establishing theoretical conditions for the existence of limit cycles in this class of models. If these conditions can be established, perhaps they can be used to aid the solution of the identification problem.

A longer range area of future research, one that is depends on methods to solve the identification problem, is the application of these ideas to real systems and to study the application of these models to the problem of failure detection and diagnosis.
REFERENCES


