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Bayesian economic cost model for a variable sampling plan for fraction defective and manufacturing process control

Jalbout, Fouad Noaman, Ph.D.
The University of Arizona, 1989

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BAYESIAN ECONOMIC COST MODEL FOR A VARIABLE SAMPLING PLAN FOR FRACTION DEFECTIVE AND MANUFACTURING PROCESS CONTROL

by

Fouad Jalbout

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For the Degree of

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WITH A MAJOR IN MECHANICAL ENGINEERING

In the Graduate College

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entitled BAYESIAN ECONOMIC COST MODEL FOR A VARIABLE SAMPLING PLAN
FOR FRACTION DEFECTIVE AND MANUFACTURING PROCESS CONTROL

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for the Degree of DOCTOR OF PHILOSOPHY

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Final approval and acceptance of this dissertation is contingent upon the
candidate's submission of the final copy of the dissertation to the Graduate
College.

I hereby certify that I have read this dissertation prepared under my
direction and recommend that it be accepted as fulfilling the dissertation
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DEDICATED TO

HELEN QUINN CORONATO

AND

IRMA AND ABRAHAM JALBOUT
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>LIST OF ILLUSTRATIONS</td>
<td>12</td>
</tr>
<tr>
<td>LIST OF TABLES</td>
<td>15</td>
</tr>
<tr>
<td>ABSTRACT</td>
<td>16</td>
</tr>
<tr>
<td>CHAPTER 1. INTRODUCTION AND LITERATURE REVIEW</td>
<td>18</td>
</tr>
<tr>
<td>1.1. Purpose</td>
<td>18</td>
</tr>
<tr>
<td>1.2. Background</td>
<td>19</td>
</tr>
<tr>
<td>1.2.1. Acceptance Sampling Plans</td>
<td>19</td>
</tr>
<tr>
<td>(Classical Approach)</td>
<td></td>
</tr>
<tr>
<td>1.2.2. Acceptance Sampling Plans</td>
<td>21</td>
</tr>
<tr>
<td>(Bayesian not Economic)</td>
<td></td>
</tr>
<tr>
<td>1.2.3. Acceptance Sampling Plans</td>
<td>22</td>
</tr>
<tr>
<td>(Bayesian Economic Models)</td>
<td></td>
</tr>
<tr>
<td>1.2.4. Acceptance Sampling Plans</td>
<td>25</td>
</tr>
<tr>
<td>(Incentive Reward)</td>
<td></td>
</tr>
<tr>
<td>1.2.5. Acceptance Sampling Plans (Others)</td>
<td>26</td>
</tr>
<tr>
<td>1.2.6. Inspection Errors</td>
<td>28</td>
</tr>
<tr>
<td>1.2.7. Process Control</td>
<td>33</td>
</tr>
<tr>
<td>1.3. Technical Approach, Assumptions and Parameters</td>
<td>37</td>
</tr>
<tr>
<td>1.4. Organization</td>
<td>42</td>
</tr>
<tr>
<td>CHAPTER 2. INSPECTION AND SELECTION TECHNIQUES</td>
<td>45</td>
</tr>
<tr>
<td>2.1. Background</td>
<td>45</td>
</tr>
<tr>
<td>2.2. Variable Sampling for Fraction Defective</td>
<td>45</td>
</tr>
<tr>
<td>2.3. Distribution of ( w(p</td>
<td>x) ) Given ( f(x</td>
</tr>
<tr>
<td>2.4. Distribution of the Fraction of Items Defective Given the Distribution of ( \mu )</td>
<td>48</td>
</tr>
<tr>
<td>2.5. Distribution of ( \mu ) Based Upon a Prior Distribution of ( p )</td>
<td>52</td>
</tr>
<tr>
<td>2.6. Double Specification for the Measureable Quality Characteristic</td>
<td>54</td>
</tr>
<tr>
<td>2.7. General Approach</td>
<td>56</td>
</tr>
</tbody>
</table>
**TABLE OF CONTENTS—Continued**

**CHAPTER 3. OPTICAL ECONOMIC COST MODELS**

**DESTRUCTIVE TESTING** ........................................ 60

3.1. Introduction ............................................. 60
3.2. Assumptions and Economic Parameters ....................... 61
3.3. Prior Cost Analysis ....................................... 62
   3.3.1. Accept Outright ................................... 62
   4.3.1. Reject Outright and Scrap .......................... 63
3.4. Posterior Cost Analysis .................................. 64
   3.4.1. Step by Step Procedure for Cost Estimation .......... 71
3.5. Cost Equation for Destructive
      Sampling Plan ........................................... 67
3.6. Expected Sampling Cost Per Unit .......................... 73
3.7. Operating Characteristic Curve (OC) ....................... 74

**CHAPTER 4. OPTIMUM ECONOMIC COST MODEL**

**NON DESTRUCTIVE TESTING** .................................. 84

4.1. Introduction ............................................. 84
4.2. Prior Cost Analysis ....................................... 84
4.3. Posterior Cost Analysis .................................. 90
   4.3.1. Disposition Chart ................................... 92
   4.3.2. Economic Parameters and
           Probabilities ....................................... 93
   4.3.3. Expected Cost of Acceptance ......................... 96
   4.3.4. Expected Cost of Screening ........................ 98
   4.3.5. Expected Cost of Scrapping ........................ 99
4.4. Model Optimization Algorithm ............................. 101
4.5. Calculation Procedures ................................... 108
4.6. Cost of Rejecting and Reworking
      Defective Items .......................................... 110
   4.6.1. Prior Cost for the Decision to
           Reject Outright and Reworking
           Defective Items ................................... 109
   4.6.2. The Posterior Cost of Rejecting
           and Reworking Defectives .......................... 112
4.7. A General Model for Rejection and
      Acceptance Costs ......................................... 120
TABLE OF CONTENTS—Continued

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.7.1. Assumptions</td>
<td>120</td>
</tr>
<tr>
<td>4.7.2. Expected Inspection Cost Per Lot</td>
<td>120</td>
</tr>
<tr>
<td>4.7.3. Expected Cost of Rejection When Sampling is Destructive and theRejected Lots Are Scrapped</td>
<td>121</td>
</tr>
<tr>
<td>4.7.4. Expected Cost of Rejection When Sampling is Nondestructive andthe Rejected Lots Are Scrapped</td>
<td>122</td>
</tr>
<tr>
<td>4.8. Expected Cost of Acceptance</td>
<td>123</td>
</tr>
</tbody>
</table>

CHAPTER 5. CONDITIONAL EXPECTATIONS AND MOMENTS OF FRACTION DEFECTIVE \( p \) RAISED TO THE \( k \)th POWER

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.1. Introduction</td>
<td>141</td>
</tr>
<tr>
<td>5.2. Model Development</td>
<td>142</td>
</tr>
<tr>
<td>5.2.1. Expected Value of Fraction Defective ( p )</td>
<td>142</td>
</tr>
<tr>
<td>5.2.2. Covariance of ( x_n ) and ( \mu )</td>
<td>146</td>
</tr>
<tr>
<td>5.3. Expected Value of ( p^k ) (( k &gt; 1 ))</td>
<td>150</td>
</tr>
<tr>
<td>5.3.1. Upper and Lower Limits for ( E(p^k) )</td>
<td>150</td>
</tr>
<tr>
<td>5.4. The Expected Value of ( p^k ), an Alternative Approach</td>
<td>153</td>
</tr>
<tr>
<td>5.5. Expected Value of ( p ) Given the Results of a Sample</td>
<td>166</td>
</tr>
</tbody>
</table>

CHAPTER 6. INSPECTION ERRORS IN ATTRIBUTE AND VARIABLE SAMPLING PLANS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.1. Attribute Sampling Plans</td>
<td>179</td>
</tr>
<tr>
<td>6.2. Probability of Acceptance</td>
<td>180</td>
</tr>
<tr>
<td>6.3. The Conditional and Marginal Probabilities of the Observed Numberof Defectives</td>
<td>181</td>
</tr>
<tr>
<td>6.3.1. Mixed Binomial Prior</td>
<td>183</td>
</tr>
<tr>
<td>6.4. The Effect of Inspection Error on AOQ and ATI</td>
<td>184</td>
</tr>
</tbody>
</table>
TABLE OF CONTENTS—Continued

6.4.1. AOQ Sampling With Replacement ........................................ 184
6.4.2. AOQ Sampling Without Replacement .................................... 186
6.4.3. ATI Sampling With Replacement ......................................... 186
6.4.4. ATI Sampling Without Replacement ..................................... 187
6.4.5. AOQ Under Both Error Free and Error Prone Sampling ............ 188
6.4.6. ATI Under Error Free and Error Prone Sampling ..................... 188

6.5. Variables Sampling Plans for Fraction Defective ...................... 189
6.5.1. Distribution of $\mu$ ..................................................... 195

6.6. Cost Equation and the Total Cost Per Unit .......................... 196
6.6.1. Expected Fraction Defective $p$ ..................................... 197

6.7. Optimum Economic Cost Model ........................................... 198

6.8. Conditional Distribution of Material Fraction Defective ............ 201
6.8.1. Marginal Distribution of $y_e$ ....................................... 201
6.8.2. The Conditional Probability of $w(p|y_e)$ .......................... 203

6.9. Conditional Expectation of $p$ given $\bar{X}$ ......................... 208
6.10. Conditional Expectation of $\mu_0$ Given $\bar{X}$ .................... 212
6.11. Conditional Expectation of $P_0(\mu_0)$ given $\bar{X}$ .............. 214

CHAPTER 7. EXPECTED COST OF KEEPING A PRODUCTION PROCESS UNDER CONTROL .................. 222

7.1. Background ................................................................. 222
7.2. Single Assignable Cause of Failure Production Cycle ............... 224
7.2.1. Model A ................................................................. 224
7.2.2. Model 2A ............................................................... 227

7.3. Multiplicity of Assignable Causes ....................................... 228
TABLE OF CONTENTS—Continued

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.3.1. Model 1B</td>
<td>228</td>
</tr>
<tr>
<td>7.3.1.1. Computational Procedures</td>
<td>231</td>
</tr>
<tr>
<td>7.3.2. Model 2B</td>
<td>233</td>
</tr>
<tr>
<td>7.3.2.1. Assumptions</td>
<td>233</td>
</tr>
<tr>
<td>7.3.2.2. Probabilities</td>
<td>233</td>
</tr>
<tr>
<td>7.3.2.3. Matrix Probabilities of Detection</td>
<td>234</td>
</tr>
<tr>
<td>7.4. Multivariate Statistical Process</td>
<td>236</td>
</tr>
<tr>
<td>7.5. Components of the Total Expected Cost Per Unit</td>
<td>238</td>
</tr>
<tr>
<td>7.6. Transition Matrix of Probabilities</td>
<td>239</td>
</tr>
<tr>
<td>7.6.1. Assumptions</td>
<td>241</td>
</tr>
<tr>
<td>7.6.2. Two States Statistical Process</td>
<td>242</td>
</tr>
<tr>
<td>7.7. Elements of Vector ( \gamma_i )</td>
<td>244</td>
</tr>
<tr>
<td>7.8. Probability Vector ( F )</td>
<td>246</td>
</tr>
<tr>
<td>7.9. Expected Length of Time of a Production Process</td>
<td>246</td>
</tr>
<tr>
<td>7.9.1. Expected Time Process as in Control</td>
<td>247</td>
</tr>
<tr>
<td>7.9.2. Expected Time for a Process Out of Control</td>
<td>249</td>
</tr>
<tr>
<td>7.9.3. Expected Time for a Production Cycle</td>
<td>249</td>
</tr>
<tr>
<td>7.9.4. Expected Loss-cost Per Hour</td>
<td>249</td>
</tr>
<tr>
<td>CHAPTER 8</td>
<td>260</td>
</tr>
<tr>
<td>8.1. Summary</td>
<td>260</td>
</tr>
<tr>
<td>8.2. Recommendations</td>
<td>263</td>
</tr>
<tr>
<td>8.2.1. Producer’s and Consumer’s Costs</td>
<td>263</td>
</tr>
<tr>
<td>8.2.2. Producer’s and Consumer’s Risks</td>
<td>264</td>
</tr>
<tr>
<td>8.2.2.1. Producer’s Risk</td>
<td>264</td>
</tr>
<tr>
<td>8.2.2.2. Consumer’s Risk</td>
<td>265</td>
</tr>
<tr>
<td>8.2.3. Workstations and a Series of Transfer Lines</td>
<td>266</td>
</tr>
<tr>
<td>8.2.4. Variances of ( X, OQ, TI )</td>
<td>268</td>
</tr>
<tr>
<td>8.2.5. Computer Programs</td>
<td>269</td>
</tr>
</tbody>
</table>
# TABLE OF CONTENTS—Continued

<table>
<thead>
<tr>
<th>Appendix</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>APPENDIX A.</td>
<td>Abbreviations and Notations</td>
<td>271</td>
</tr>
<tr>
<td>APPENDIX B.</td>
<td>Mean of a Lot Quality Characteristic</td>
<td>280</td>
</tr>
<tr>
<td>APPENDIX C.</td>
<td>Program FINAL1</td>
<td>287</td>
</tr>
<tr>
<td>APPENDIX D.</td>
<td>Program 2AA</td>
<td>297</td>
</tr>
<tr>
<td>APPENDIX E.</td>
<td>Program 3AA</td>
<td>328</td>
</tr>
<tr>
<td>APPENDIX F.</td>
<td>Program 4AA and 5AA</td>
<td>349</td>
</tr>
<tr>
<td>APPENDIX G.</td>
<td>Program TKEVAL</td>
<td>369</td>
</tr>
<tr>
<td>APPENDIX H.</td>
<td>Program MULTSTOP</td>
<td>406</td>
</tr>
<tr>
<td>APPENDIX I.</td>
<td>Program MULTRUN</td>
<td>414</td>
</tr>
<tr>
<td>REFERENCES</td>
<td></td>
<td>421</td>
</tr>
</tbody>
</table>
# LIST OF ILLUSTRATIONS

<table>
<thead>
<tr>
<th>Figure</th>
<th>Title</th>
<th>page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Disposition of a Rejected Lot</td>
<td>40</td>
</tr>
<tr>
<td>2.</td>
<td>Diagramatic Representation of an In-Control Out-of-Control Production Process</td>
<td>41</td>
</tr>
<tr>
<td>3.</td>
<td>The Sampling Cost Per Unit Vs. the Sample Size for a Destructive Variable Sampling Plan for a Lot of Size 1000</td>
<td>79</td>
</tr>
<tr>
<td>4.</td>
<td>The Sampling Cost Per Unit Vs. the Sample Size for a Destructive Variable Sampling Plan for Fraction Defective for a Lot of Size 35</td>
<td>80</td>
</tr>
<tr>
<td>5.</td>
<td>The Acceptance Cost Per Unit Vs. the Sample Size for a Destructive Variable Sampling Plan for Fraction Defective</td>
<td>81</td>
</tr>
<tr>
<td>6.</td>
<td>The Rejection Cost Per Unit Vs. the Sample for a Destructive Variable Sampling Plan for Fraction Defective</td>
<td>82</td>
</tr>
<tr>
<td>7.</td>
<td>The Posterior Cost Per Unit Vs. the Sample Size for a Destructive Variable Sampling Plan for Fraction Defective</td>
<td>83</td>
</tr>
<tr>
<td>8.</td>
<td>The Lower Limit of the Sample Mean for Lot Acceptance Vs. the Sample Size for a Non-Destructive Variable Sampling Plan for Fraction Defective</td>
<td>131</td>
</tr>
<tr>
<td>9.</td>
<td>The Upper Limit of the Sample Mean for Lot Acceptance Vs. the Sample Size for Non-Destructive Variable Sampling Plan for Fraction Defective</td>
<td>132</td>
</tr>
<tr>
<td>10.</td>
<td>The Cost of Acceptance Per Lot Vs. the Sample Size for Nondestructive Variables Sampling Plan For Fraction Defective</td>
<td>133</td>
</tr>
<tr>
<td>Figure</td>
<td>Title</td>
<td>page</td>
</tr>
<tr>
<td>--------</td>
<td>-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>11.</td>
<td>The Cost of Screening Per Lot Vs. the Sample Size for Nondestructive Variables Sampling Plan for Fraction Defective for a Lot of Size 1000</td>
<td>136</td>
</tr>
<tr>
<td>12.</td>
<td>Total Scrapping Cost Per Lot for a Nondestructive Variables Sampling Plan for Fraction Defective for a Lot of Size 1000</td>
<td>139</td>
</tr>
<tr>
<td>13.</td>
<td>The Expected Total Cost Vs. the Sample Size for a Nondestructive Variable Sampling Plan for Fraction Defective for a Lot of Size 1000</td>
<td>140</td>
</tr>
<tr>
<td>14.</td>
<td>The V Values Vs. the First and Second Factors of the Kernel of the Expected Moment of the Fraction Defective Raised to the $K - \theta$ Power</td>
<td>177</td>
</tr>
<tr>
<td>15.</td>
<td>The Combined Values of the First and Second Factors of the Kernel of the Expected Value of Fraction Defective $p$</td>
<td>178</td>
</tr>
<tr>
<td>16.</td>
<td>The First Root of the Cost Equation for a Destructive Variable Sampling Plan for Fraction Defective for Both Error Free and Error Prone Inspection for a Lot of Size 1000</td>
<td>217</td>
</tr>
<tr>
<td>17.</td>
<td>The Second Root of the Cost Equation for a Destructive Variable Sampling Plan for Fraction Defective for Both Error Free and Error Prone Inspection for a Lot of Size 1000</td>
<td>218</td>
</tr>
<tr>
<td>18.</td>
<td>The Acceptance Cost Per Unit for a Destructive Variable Sampling Plan for Fraction Defective for Both Error Free and Error Prone Inspection for a Lot of Size 1000</td>
<td>219</td>
</tr>
<tr>
<td>Figure</td>
<td>Title</td>
<td>page</td>
</tr>
<tr>
<td>--------</td>
<td>-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>19.</td>
<td>The Rejection Cost Per Unit for a Destructive Variable Sampling Plan for Fraction Defective for Both Error Free and Error Prone Inspection for a Lot of Size 1000</td>
<td>220</td>
</tr>
<tr>
<td>20.</td>
<td>The Total Expected Cost Per Unit for a Destructive Variable Sampling Plan for Fraction Defective for Both Error Free and Error Prone Inspection for a Lot of Size 1000</td>
<td>221</td>
</tr>
<tr>
<td>21.</td>
<td>Transfer Line for a Series of Workstations</td>
<td>267</td>
</tr>
</tbody>
</table>
## LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Title</th>
<th>page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Special Cases for $B_\alpha(v)$</td>
<td>145</td>
</tr>
<tr>
<td>2.</td>
<td>Special Cases for $\rho_{T_n}, \alpha$</td>
<td>150</td>
</tr>
<tr>
<td>3.</td>
<td>Single Parameter AOG and ATI equations</td>
<td>190</td>
</tr>
<tr>
<td></td>
<td>for Nine Rectifying Inspection Policies</td>
<td></td>
</tr>
<tr>
<td>4.</td>
<td>Some Probability Density Functions</td>
<td>200</td>
</tr>
<tr>
<td></td>
<td>(All Normally Distributed) Employed</td>
<td></td>
</tr>
<tr>
<td></td>
<td>in Chapters 1-4 When Sampling is Error</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Free or Error Prone</td>
<td></td>
</tr>
<tr>
<td>B.1.</td>
<td>Values of $z^{(k)}(0)$ and $h^{(k)}(0)$ for</td>
<td>284</td>
</tr>
<tr>
<td></td>
<td>$k = 1, 2, \ldots, 12$</td>
<td></td>
</tr>
</tbody>
</table>
ABSTRACT

Acceptance sampling plans by variables are a basic quality control technique. These plans provide economical procedures to determine the acceptability of batches of product. Most of these plans are based on a single quality characteristic and are of the classical type.

This work concentrates on Bayesian variable acceptance sampling plans for fraction defective. Both destructive and non-destructive sampling procedures are considered. A set of decision points are estimated and employed to make decisions about the inspected lots. Techniques to dispose of the rejected lots are provided. Components of the expected total cost relative to various decisions are estimated. The sample size required to obtain the expected optimum cost is found.

An untrue assumption implicit in the measurement of the quality characteristic of items sampled is that the observed dimensions are error free. The distributions, means, and variances of a set of parameters for error free and error prone sampling is listed.

Computer programs written in FORTRAN 77 are developed to compute the decision points and the costs for both destructive and nondestructive testing.

Precise Bays estimate of the costs and other economic parameters involve the moments of the fraction defective $p$ raised to the $k^{th}$ power. Mathematical expressions for the conditional expectations of $p|x$ and $p|x$ are derived and a computer program to estimate these moments is provided.

Producing quality items with minimum cost requires keeping a production process under control. The quality characteristic $X$ of each item
produced is determined and the sample means are plotted on an $\bar{X}$-control chart. A production process is assumed to start in control at time $t = 0$ with specific values of the mean and standard deviation ($\sigma$ is assumed to be constant). The occurrence of a single or multiple cause-failures shift the process mean outside the control limits. During the search for the causes of failure, the process is either allowed to continue in operation or shut down until the assignable cause or causes are discovered. The expected duration of time during which the process is shut down and the additional time to be taken to repair the process are considered. Computer programs are provided to estimate the optimal sample size, the interval between successive samples, the control limits, the probability of type I error, the power of the chart, and the average time the process operates in the presence of an assignable cause. The parameters estimated are employed to estimate the optimal loss-cost.

The economic design of $\bar{X}$-charts assumes one quality characteristic of interest. However a product quality in most industrial products and processes is characterized by more than one quality characteristic where the application of a $\bar{X}$-control chart for each variable is inappropriate. In this work a Hotellings $T^2$ control chart is employed to handle cases of where products are tested relative to several quality characteristics.
CHAPTER 1

INTRODUCTION AND LITERATURE REVIEW

1.1. Purpose

Recent research on industrial quality control systems devotes considerable attention to process control and acceptance sampling. Acceptance sampling plans are developed to overcome inherent shortcomings of 100% inspection technique. The plans are designed to provide economical methods for inspecting products and to develop mathematical procedures for arriving at a decision to accept or reject an inspection lot. These plans deal with single attribute or several attributes or variables. Attempts to minimize the costs concentrate on reducing the expected sample size while meeting statistical criteria. Most of the statistical models developed are based on a single quality characteristic and are of the classical type. Bayesian acceptance sampling plans by variables for fraction defective have received little attention in the literature.

Control charts are widely used to maintain statistical control of a production process and are also effective devices for estimating process parameters and analyzing process capability. Past research formulated expressions of the loss-cost per hour based on statistical considerations of the classical type and dealt with a single quality characteristic. The objectives of this dissertation are to:

(1) Develop Bayesian variables sampling plans for fraction defective based upon economic considerations and inspection errors. Cost models are developed and optimized to obtain a set of decision points based on upper and lower limits of the sample means for lot acceptance screening or scrapping.

(2) Develop models for the loss-cost function for an in/out of control production process, based on single or multiple quality characteristics. Costs
are optimized to obtain the optimal sample sizes, the interval between successive samples and the control limits. The possibilities of shutting down the production facility or keeping it operating while searching for the single or multiple causes of failure are considered.

(3) Demonstrate the advantage of the Bayesian models over classical ones.

1.2. Background

1.2.1. Acceptance Sampling Plans (Classical Approach)

The main purpose of acceptance sampling is to determine a course of action. It is not an attempt to control the product quality. Classical acceptance sampling prescribes a procedure to specify the risk of accepting lots of a certain quality, in other words, it yields quality assurance. Two sampling plans are identical if they possess identical OC curves. For two OC curves to be identical, they should coincide at two chosen points. Variables and attribute sampling plans for fraction defective can be made to possess identical OC curves. This is done by insuring that the two curves possess the same slope at the indifference quality level (IQL), where the probability of acceptance $P_a = 0.5$. At the IQL both the producer and the consumer share the same risk. The principle of equivalence of two acceptance sampling plans is applied by Hamaker (1979) to study the relative efficiencies of variables and attributes single sampling plans. The investigation reveals that these relative efficiencies depend exclusively on the indifference quality level ($P_a = 0.5$).

Fertig, and Mann (1983) assumed a normal distributed process with an unknown mean and known and unknown variances. Their acceptance sampling plan examines nondestructive testing with an upper specification limit only. However, due to symmetry the lower specification limit could be defined in
terms of the upper one. They suggested that the economic cost of making a particular decision is measured by a loss function similar to the work of Hald (1960). They assumed:

1. Nonconforming items found in the sample are replaced by conforming items.
2. Rejected lots are screened and nonconforming items are replaced by conforming items.
3. The cost of sampling per item is equal to the cost of screening.
4. The decision rule used in these plans is to accept the lot if the expected loss of accepting the lot is less than that of the cost of rejecting.

Based on these assumptions Fertig and Mann developed tables of variables acceptance sampling plans for small lots. The theoretic approach used was the same as they used to minimize the sample size in their 1974 paper. In addition to the acceptable quality level (AQL); the rejectable quality level (RQL); and the two parameters $\alpha$ and $\beta$, they defined the average amount that one is willing to spend if the process is operating at the RQL. The sample size is chosen such that:

1. the average cost is no more than $R_{RQL}$ when the process is operating at or above RQL,
2. the average cost is minimized when the process is operating at the AQL.

The sample sizes required by variable sampling were compared with two types of an attribute plans. One type employed an infinite lot size and a binomial model while the other used a finite lot size and a hypergeometric model. It was assumed that the lot came from a process with normally distributed quality characteristics. Under these assumptions the sample sizes for variable plans were considerably smaller than for attribute plans.
1.2.2. Acceptance Sampling Plans (Bayesian not Economic)

The Bayesian approach to acceptance sampling combines the information provided by the random sample and the available pertinent prior information. The researcher usually uses his experience and knowledge as a basis to define the probabilities associated with prior information. Shafer (1967) assumed the following:

(1) The lot sampling process is hypergeometric.
(2) A sample is selected at random and after inspection the defective items are replaced in the lot to maintain the constant lot size.
(3) The lots are either accepted or rejected.
(4) Two posterior risks are required to select the pair \((n, c)\)
   a. Posterior producer’s risk \((\alpha)\); the probability that a given lot was rejected and the lot quality was acceptable.
   b. Posterior consumer’s risk \((\beta)\); the probability that a given lot was accepted and the lot quality was unacceptable.

Shafer presented his technique by using different prior distributions.

Esterling (1970) considered an acceptance sampling plan where the parameter of interest \((\theta)\) is a random variable with a known distribution. He suggested the use of the posterior distribution arising from the outcome of an experiment as the prior for the succeeding experiment. Let \(\Omega\) be the parameter space that corresponds to two subspaces where \(\Omega_+\) represents the subspace corresponding to good quality and \(\Omega_-\) represents the subspace corresponding to bad quality. Easterling updated the work of Balaban (1969) who set up the following criteria for determining an acceptance rule of a product

\[
P(\text{accept}|\theta \in \Omega_+) \geq 1 - \alpha
\]

\[
P(\theta \in \Omega_-| \text{accept}) \leq \beta
\]
where $\alpha$ refers to the producer's risk and $\beta$ is the consumer's risk from the viewpoint of the quality distribution of the outgoing. However Esterling's acceptance rule based on the average of the producer's and consumer's risk seems more appropriate than that of Balaban. Let $g(\theta)$ be the prior density on the parameter $\theta$, then

$$
\int_{\Omega_+} p(\text{accept}|\theta)g(\theta|\Omega_+)d\theta \geq 1 - \alpha
$$

$$
\int_{\Omega_-} p(\text{accept}|\theta)g(\theta|\Omega_-)d\theta \leq \beta.
$$

The above relations specify average values rather than points on the OC curve.

Rectifying sampling plans for attributes are sometimes designed to satisfy predetermined constraints on the outgoing quality. Hall and Hassan (1981) developed a procedure for rectifying sampling plans by attributes based on an upper bound on the outgoing quality of the lots with a specified minimum probability $(1 - \alpha)$. No consideration was given to the process average. The AOQL was calculated according to the upper bound concept. A table of sampling plans were developed for $1 - \alpha = 0.95$.

1.2.3. Acceptance Sampling Plans (Bayesian Economic Models)

When single-sample lot by lot rectifying inspection plans for fraction defective are used, a sample is selected at random and the number of defectives ($x$) in the sample are compared to an acceptance number $c$. Conditionally, if $x \leq c$ accept the lot with no further testing and if $x > c$ reject. If $x > c$ all the items in the lot are inspected to determine the possible costs associated with rejection.

In variables sampling plans the quality characteristic ($x$) is compared to an upper limit, lower limit or both, to classify the items as defective or effective. Similar acceptance or rejection criteria as those discussed above are then used to formulate the mathematical model.
Consider a lot of size $N$ characterized by non-negative random variables, assumed to be independent, which represent the quality of the items. All the variables are assumed to follow the same probability distributions each depending on the single parameter ($\Theta$). Gauthrie and Johns (1959) developed a cost model to find an explicit asymptotic characterization for large $N$. Their investigations showed that decision procedures and sample sizes are optimal in the Bayes sense for various priori probability distributions with different parameter values. The main advantages of their model are:

1. the range of the quality random variable will be the same as the range for $\Theta$.

2. prior information can be expressed in terms of the expected value of the quality level defined before.

Hald (1960) considered a linear model based on prior distributions and costs. This work yielded some general theorems based on the compound hypergeometric and the mixed binomial distribution. The inspection plans are developed in terms of the prior distributions, and rejection and acceptance after sampling. Rejected lots are sorted for future disposition. Defective components are replaced by effective ones. The cost of defective items is the comparable effective items production cost plus an additional amount for rework or minus the cost of selling it as junk. Hald also considered the possibility of accepting or rejecting lots without sampling inspection.

Case, Schmidt and Bennet (1975) developed a model for multiple attributes each of which is assumed to have its own sample size and acceptance number $C_i$. This entails grouping the available items into batches of finite size prior to acceptance sampling. Here the lot fraction defective for each attribute is discrete. Therefore, as the sample size increase so does the number of defectives in the sample. In their model, the system can be approximated by a
continuous density function. This allows direct comparison between the discrete and continuous approximations.

Up to this point, the economic acceptance sampling plans considered have been developed for a single attribute or several attributes. Ailor, Schmidt and Bennett (1975) presented an economic plan for several attributes where several variables are subjected simultaneously to an acceptance plan.

In most of the previously discussed articles quality characteristics are assumed to be independently distributed random variables. The decision criteria are in general the sample size or the lower and upper acceptance limits for the sample mean. It is assumed that the inspection tests are either destructive or nondestructive. Consider the situation where the sample mean for each characteristic is used to accept or reject the lot. Chapman, Schmidt and Bennett (1978) investigated plans where the inspection tests are either destructive or nondestructive. In destructive testing, rejection because some characteristic is out of specification leads to the scrapping of the lot. In non destructive testing rejection because some characteristic leads to the screening of the lot. For the non destructive cases the cost of repairing or replacing defective items is a key factor. However, if inspection is destructive two possibilities arise:

1. The product is rendered unfit for its intended use and for further inspection on other characteristics.

2. Rejection based on some characteristic renders the product unfit for its intended use, once inspection on those characteristics takes place. However inspection on some other characteristic does not render the product unfit for further inspection.

Moriquti and Breakwell (1985) developed a cost model that includes a variety of sampling inspection and acceptance sampling procedures. Each provides different choices for cost parameters and distribution families. The
authors tried to identify an optimal solution for various sample sizes. This was done in terms of a minimax criterion for large lot sizes. The procedure seems to work in the case some prior distributions when either rejection or acceptance of a lot leads to the same expected cost.

1.2.4. Acceptance Sampling Plans (Incentive Reward)

This type of plan does not use the Bayesian approach. In this type of plan the consumer and the producer usually enter into an incentive type contract. The price paid for the product depends upon the outcome of an acceptance test. This test is a measure of the quality of the product. Without the test the consumer might incur a considerable penalty. However, if the consumer agrees to pay a fee depending on the test's results the producer can invest the money for improving his product.

Felhinger and Miller (1964) considered an approach to improve the product where each conventional test has a number of results. The probability of each result can be calculated as a function of the vector that represents the quality of the \( n \) product parameters. The objective is to identify an optimal strategy which specifies both a test procedure and a schedule of premium payments. The overall class of strategies is defined by the space of all results under consideration. Each member of this class is considered admissible if it yielded a maximum profit for the consumer and producer.

Foster and Perry (1972) introduced the concept of price adjusted single sampling plans (PASS) with linear indifference. In this concept, the producer and consumer agree upon an acceptable sampling plan. The single variable that affects cost and profit is the incoming fraction defective \( p' \). This variable is controlled by the producer most of the time. The consumer designates the level of quality he considers unacceptable and selects a plan that rejects lots at or
above this level. The sales are balanced by forcing the producer to submit lots at an acceptable value of $p'$. Adjusting the selling price of a lot according to sample inspection results helps to avoid the costs of rejecting lots and to retain the power to influence the producer's choice of $p'$. This theory is extended to include sampling risks incurred by the producer and consumer. The producer's risk is the probability that the producer receives a price lower than a specified lower limit ($L$) at an acceptable quality level $P_1$. Under this plan the consumer's risk is the probability that the consumer pays a price greater than a specified upper limit $U$ at a predetermined quality level of $P_2$ where $U > L$ and $P_2 > P_1$.

The PASS technique is somewhat similar to a technique developed by Rolcoffs (1967) called “price differential acceptance plan”. Rolcoffs (1967) plan is different from PASS in two ways:

(1) It applies the price adjustment concept to existing conventional single sampling plans;

(2) It does not eliminate the rejection process. The price adjustment occurs only when the lot is accepted.

Foster and Perry (1977) expanded the application of the PASS to include quadratic indifference. In this case the indifference function is defined in terms of the acceptable quality level $P$. This level is a function of the lot size and the average price per unit. Foster's and Perry's (1977) quadratic indifference function provides flexibility in PASS plans by allowing the price paid per acceptable unit to vary in different ways over the range of the quality level $P$.

1.2.5. Acceptance Sampling Plans (Others)

Shafer (1960) presented a method for evaluating a sampling plan in situations where the prior knowledge of quality can be estimated from past experience. The Bayes operating characteristic curve (BOC) is the analytical
tool used to assess this prior knowledge. In general the BOC curves are the posterior quality distributions depicting the conditional acceptance of lots.

Brush, Cautin and Lewin (1981) derived acceptance probabilities for sampling when the process average for the incoming lots follows a beta distribution. They also examined the problem of determining the distribution for the proportion defective in the accepted lots.

In attribute sampling with switching rules the long run costs will depend on the proportion of lots inspected under normal, reduced and tightened sampling rules. Few researchers considered the effect of switching rules. Koyoma (1969) studied switching by using signal graph theory.

Rutemiller (1974) considered three switching rules when several consecutive lots are submitted for inspection. A shift to less severe ("reduced") inspection or to more severe ("tightened") inspection depends on whether the quality of the product is high or low. Returning to normal sampling from either of these states is permitted in accordance to the number of consecutive lots accepted. Rutemiller examines the costs of the sampling schemes for the three inspection types by considering them as three markov chains with periodic transitions from chain to chain. He suggested ways to minimize the cost by proper choice of the quality control parameters. The expected sample size and the expected proportion of rejected products are determined as functions of the parameters controlled by the manufacturer, namely the lot size and the product quality. In general, lots which contain many defectives, may be returned to the manufacturer, purchased at a lower price or scrapped.

Brown and Rutemiller (1973, 1974) discussed several single sample plans and various quality levels. Their investigation focused on expected fraction of rejected lots, the expected sample size per lot and the expected number of lots processed before discontinuing sampling inspection. They derived equations
with appropriate cost parameters to compute the long term cost of sampling and inspection.

Case and Bennett (1977) used a variable sampling plan to determine the acceptability of an individual lot. This plan is defined by the sample size \( n \) and an acceptance point \( k \). A product dimension is compared to an upper and lower specifications. Procedurally, the sample mean \( \bar{x} \) is computed from the observed measurements then, a decision is made to accept the lot based on the sample mean. In this model the population variance is assumed to be constant and known. The decision variables \((n, k)\) and the sample mean's lower and upper limits are evaluated from \( \alpha, \beta \) or a mathematical model such as that developed by Schmidt (1974).

Schmidt, Bennett and Case (1980) considered a sampling plan for lot acceptance or rejection which depends on the steps for disposal of the lot. The cost for each step was factored into the overall cost. Depending on the minimum cost it was possible to estimate the sample size and the optimal limits of \( \bar{x} \) for accepting, screening or scrapping.

1.2.6. Inspection Errors

The development of most acceptance sampling plans is based on the implicit assumption that inspection is perfect. Many studies have established the existence of inspector error. In one of his experiments, Harris (1966) found that there is almost a perfect inverse relationship between equipment complexity and the percentage of defects. Here, the equipment complexity has a significant effect on inspection performance. This factor cannot be eliminated by extending inspection time.

In another study, Harris (1968) selected eighty inspectors to examine materials produced are at four different defect rates. Results showed that the
inspection accuracy decreased with reduction in defect rates.

Fox and Haslegrave (1969) and Wallack and Adams (1969) indicate that the conclusion of constant error is not valid in all cases. Collins, Case and Bennett (1973) present results for constant inspection error. Biegel (1974) questioned the idea of linearity between the errors and fraction defective in submitted lots. The linear model is useful when it results in a smaller standard error. To compensate for inspection errors, the quality control analyst must be able to model their occurrence. In addition, the errors must be included in the design of the procedure. Doris and Bobbie (1977) focused on modeling inspection errors. Their investigation led to methods for estimating error rates. In addition, they studied the effects of these errors on standard methods of evaluation. In their work it was concluded that it is necessary to model the process generating these errors.

Dorris and Fotte (1977) examined the effects of inspection errors on statistical quality control procedures. They assumed a statistical model for the process generating the errors. Two random variables $X$ and $Y$ are considered in their analysis. The random variable $X$ represents the number of actual defectives in a lot of $N$ items and $Y$ the number of items classified as defective by the inspector. Two error rates $\theta$ and $\phi$ are defined as follows: A defective item is accepted with probability $\theta$ and a non defective item is rejected with probability $\phi$. The error probability $\pi$ is defined as:

$$\pi = p(1 - \theta) + (1 - p)\phi.$$  

The variables $X$ and $Y$ are assumed to be binomially distributed with parameter $N$, $p$ and $N$, $\pi$ respectively. The constant error rate is limited to analysis of a single inspector or several inspectors who have the same error probabilities. Assuming that the inspectors have error rates $(\theta_1, \phi_1)$, $(\theta_2, \phi_2)$, $\cdots$, $(\theta_k, \phi_k)$ respectively then the error probabilities will be $\pi_1$, $\pi_2$, $\cdots$, $\pi_n$ respectively. In this
case the model of Dorris and Foote is not valid. However if $100\alpha_1\%$, $100\alpha_2\%$, $100\alpha_3\%$, $\cdots$, $100\alpha_k\%$ of the lots are inspected by inspectors $1$, $2$, $\cdots$, $k$, where $100\alpha_1 + 100\alpha_2 + \cdots + 100\alpha_k = 100$, then the probability distribution of $Y$ is a mixed binomial distribution with weights $\alpha_1, \alpha_2, \cdots, \alpha_k$.

Minton (1972) considered the case of misclassification of defective items as nondefectives for single sampling inspection plans for attributes. Formulas based on average outgoing quality and costs are provided to analyzing the effect of both inefficient inspection and to correct the power of the sampling plan to distinguish between acceptable and unacceptable quality levels. The term (applicable techniques) is used to emphasize the discussion concerning the effective power of the test to distinguish between good and poor quality when errors are made by sample inspectors and correctors.

Hoag, Foote and Campbell (1975) considered a new distribution of the number of observed defects in a sample of size $n$. This distribution is based on the following probabilities

a. The probability of the inspector observing a defect in the inspected item.

b. The probability the inspected item is defective.

The new distribution is a binomial distribution given in terms of these two probabilities. Each condition described by these probabilities and other factors cause changes in the AOQ value. Hoag's et al. technique was applied to single and sequential sampling models. Their study attempted to identify the effects of inspector errors on Type I and Type II errors.

Collins and Case (1976) used a mixed binomial prior constructed from the apparent fraction defectives for each of two processes and a special weighting factor. This weighting factor describes the frequencies where each apparent fraction defective exists. The mixed binomial consisting of $m$ weighted components
is generalized in the same manner.

The AOQ, ATI, and OC curve are the three most important measures to describe a sampling plan. In the literature, OC curves have been studied extensively for single and double sampling plans. However, not much is known about the AOQ or ATI. Bemoy and Case (1981) provided expressions and illustrations of the AOQ and ATI for both single and double sampling with and without errors. Their work led researchers to propose different sampling techniques concerned primarily with the statistical and economical approaches to the problem. For example, Davis (1966), McKnight (1967) and Minton (1970) considered the AOQ plan in which apparent defective items are replaced one-for-one. They assumed replacement components were good. In addition, these researchers considered the ATI plan for defective items which are removed from the sampled lot but not replaced. Collins, Case and Bennett (1972) allowed defective replacements in the formulation of ATI.

Bennett, Case and Schmidt (1974) considered an economic scheme based on three samples and two rejected lot dispositions. For the sample:

1. discard the entire sample
2. replace, repair or rework all defectives
3. only enumerate defectives for the rejected lots.

For the rejected lots

1. discard the entire lot
2. replace, repair or rework all defectives.

The model was a single sampling plan involving several cost components. They considered both types of errors and the apparent number of defectives in the sample. A certain kind of stochastic process governs the quality of the incoming lot. The optimal plan requires further inspection under both the free and error
prone procedures. This was done to provide a valid comparison between the two dispositions.

Mei, Case and Schmidt (1975) presented a one sided variables sampling plan. This plan compensated for measurement error to provide the desired OC curve. The lot distribution was characterized by a continuous density function. The true standard deviation of the lot was assumed to be known. Finally, the sample size is assumed to be large. This allowed the sample average to be described by a normal distribution in accordance with the central limit theorem.

Case and Bennett (1977) combined economics with the error effects of imperfect measurement. Their techniques apply to variables acceptance sampling situations. They considered the following costs:

(1) inspection cost,
(2) acceptance cost including the cost of accepting some defective items in a lot based upon a sample inspection,
(3) rejection cost due to screening and the decision to repair or replace the defective items,
(4) the rejection cost due to scrapping each item in a rejected lot plus the scrapping operation cost.

The measurement error distribution (often normal) has a known mean (bias)\(^1\) and a standard deviation (imprecision)\(^2\). Here, the lot standard deviation was assumed to be known and constant. Also, the authors assumed that the lot distribution of the true mean dimensions from lot to lot was best expressed by a continuous distribution.

\(^1\)Bias is the difference between the true dimension of a product and the average of a long series of repeated measurements made on that product.
\(^2\)Imprecision is the inability to repeat results when measurements on the same unit of product are taken.
David, Fay and Walsh (1959) assumed that the ratio of population and error standard deviations has a known lower limit greater than zero. The underlying probability distribution was assumed to be normal. Their paper considered a one sided acceptance inspection criteria which is optimum, when the producer's and consumer's risks have specified upper bounds.

Hahn (1982) presented a method to determine whether a sufficient proportion of a product meets its specification when the available test data are subject to reading errors.

Following the methodologies of the above researchers, Owen and Chou (1983) also investigated measurement errors. Their studies focused on error effects in relation to the operating characteristics of one sided variables sampling plans. They gave exact solutions to two aspects of the problem:

1. If the producer's and consumer's risk are assumed to be fixed quantities, the effect on the OC curve is obtained for the proportion nonconforming.

2. If the proportion nonconforming is assumed to be a fixed quantity, the resulting consumer's and producer's risks are computed for known data.

1.2.7. Process Control

Control charts are analytical tools to maintain current process control. Employing these charts requires knowledge on how to select

(1) the optimal sample size

(2) the interval between successive samples

(3) the control limits.

Shewart (1939) used $3\sigma$ control limits and sample sizes of four or five. The interval between successive samples was left to be determined by the practitioner.
Duncan (1956) analyzed a cost based method to determine the sample size. The interval between samples and the control limits were determined for $\bar{X}$ control chart. The underlying process was assumed to have a single assignable cause of failure occurring randomly and with known effect distribution.

Duncan (1971) extended the method of a single assignable cause to include the occurrence of several causes. The probability distributions for the assignable causes were assumed to be known.

Knapperberger and Grandage (1969) examined the economic design of $\bar{X}$ charts with several assignable causes. A two stage numerical procedure was developed to calculate the sample size, the number of units produced between samples, and the optimal control limits.

Montgomery and Klatt (1972) developed a cost model for a Hotelling $T^2$ control chart. They considered a process described by $P$ quality characteristics. A method was developed to determine the $T^2$ chart's optimal sample size, the interval between samples, and the critical region. Their model included several cost coefficients. They studied the model's sensitivity to the cost coefficients, the shift parameter, the sample correlation coefficients, and sample covariance matrix.

Schmidt and Taylor (1973) determined whether an operational facility produces items with undesirable quality characteristics. This type of failure is differentiated from the types which cause a shut down of a production facility. In their study they developed a sampling plan which minimizes the expected total annual cost of quality control. Their plan specified the sample size, the acceptance number, and the inspection lot time period. Under normal conditions the facility produces items at a constant rate with a relatively constant proportion defective, $P_0$. They assumed that the time until process fails as measured from the last inspection during the period required to produce a lot
of size $N$ is exponentially distributed. The proportion of defectives when the system is in failed state increases. Inspection was assumed to be nondestructive. The cost based quality control model was optimized with a multivariable search technique.

Latimar's Bennett's and Schmidt's (1973) economic model employs lot-by-lot sampling procedure to control the lot mean and determine the lot disposition. They present a method to limit the number of mutually independent characteristics. Each lot mean is approximated by a random variable and the behavior of each mean is captured by a marker process. One value of the mean $\mu$ is the lot mean in-control value. The remaining values are out of control. With constant lot size, the model incorporates the effect of assignable causes into product quality. The main disadvantage of this model is that it requires the specification of true values for its parameters. A sensitivity analysis must be performed to determine the error effects in these parameters.

Control charts proposed by Shewart (1939) have been used for a long time. As processes became more complex and costly, more efficient analytical tools were needed to improve product quality. Page (1954), developed charts based on the sums of observations rather than individual observations (CUSUM). Bernard (1959), Bissel (1969), Brown (1971), Duncan (1956), Ewan and Kemp (1960), Guel, Jain and Wu (1968) and Goel (1968) studied these procedures and their application to industrial and management problems. CUSUM charts control the mean of the process. CUSUM chart parameters are determined from the average run lengths and the acceptable and rejectable quality levels.

Ewan and Kemp (1960), Goel and Wu (1971) employ run monograms to evaluate CUSUM chart parameters. However, their procedures do not consider the costs of the process. Taylor (1968) discovered a method to obtain the
V-mask parameters from the long-run average cost. All these models require prior knowledge of the sample size and sampling interval. Goel and Wu (1973) developed CUSUM charts based on minimum cost criteria. They followed the steps of Duncan (1956) and Goel, Jain and Wu (1968). Here, the long-run average cost is given as a function of the chart's design parameters, the process cost and risk factors. A pattern search technique is used to provide optimal values for sample size, sampling interval, and decision limits.

Taylor (1968) considered the same case. Chiu and Wetherill (1974) studied a very simple semi-economic scheme for a control plan based on an $X$ chart. For this plan the control engineer must select a consumer's risk point on an $OC$ curve for protection. They provided a way to identify: the sample size, control limit coefficients, and sampling interval. This models has three features:

1. The parameters chosen minimize the average loss-cost subject to a constraint.
2. The method has practical applications to Duncan's single cost and multiple cause models and the modified Taylor's model.
3. The solution provides a good initial starting point via direct search for an exact optimum.

Gibra (1975) examined CUSUM, Economic $X$ chart, and multicharacteristic charts. Chiu (1976) considered the effect of errors in the estimation of critical parameters. He suggested a method of estimation to provide an optimum solution. The results are based on robustness.

Chiu and Cheung (1977) compared the minimum cost design for $X$ charts with and without warning limits. They found no significant difference between the two cases.

All the researchers following Duncan's school of thought investigated production processes with a fixed and known initial setting. This defines the
in-control state. The out-of-control state occurs at a later stage of production.

Chiu and Leung (1980) realized the practicality and viability of assuming a prior distribution for the in-control level. This assumption, introduced by Bather (1982) generalized the basic process model. In addition, a Bayesian method could be used to design $\bar{X}$ charts with low-cost functions. They concluded that complete knowledge about the shape of the prior distribution is not vital for the optimum solution, as the mean variance can be determined accurately.

The effects of inspection errors on various aspects of attributes acceptance sampling plans have been studied extensively. Papers dealing with the probability of acceptance, average outgoing quality, average total inspection and total cost per lot are available in the literature. In recent years there has been increasing work on variables acceptance sampling plans with measurement errors. However, there is little work on control charts with errors. Bovas (1977) studied $\bar{X}$, $R$ and CUSUM variable charts under the assumption of inspection errors.

Case (1980) examined mathematically and graphically, the effects of the inspection errors on $p$-charts. His paper discussed the specification of control limits based on a target value and error prone data. OC curves based on a $p$-charts under variable error rates were developed.

1.3. Technical Approach, Assumptions and Parameters

A review of the literature about acceptance sampling plans, reveals that in recent years much attention has been devoted to the development of single sampling economic acceptance sampling plans for attributes and variables. The researchers who dealt with methods for constructing and evaluating sampling plans based on statistical or economic considerations considered a limited
number of quality characteristics. Most of their methods examine the statistical characteristics separately from the economic ones. It has been a tradition for researchers to design plans which accept good items with a high fraction of \((1 - \alpha)\%\) of the time and to accept poor lots only \(\beta\%\) of the time. An acceptance sampling plan based upon cost or economic parameters eliminates the need to define \(\alpha\) or \(\beta\).

Another assumption made by most researchers is that all measurements are error free. However, the effects of inspection errors cause the OC curve to be translated along the horizontal axis and distorts of the sampling plan results. Studies show that attributes inspection error rates cause unexpectedly high increases in cost. Inspection errors in variables sampling plans (classical or Bayesian) have received little or no attention up to date.

Previous research suggested the disposition of rejected lots in a homogeneous manner. This means that if inspection is destructive, the rejected lots are scrapped or reworked, while if the inspection is non-destructive the rejected lots are screened. However little effort is made to study the quality of the units in the rejected lots.

In this dissertation a Bayesian variables sampling plan for fraction defective based upon economic considerations and inspection errors is developed. An economic cost model for a quality control inspection process is formulated for the items or dimensions of interest. The cost model consists of four components; the inspection cost, the acceptance cost, the rejection cost (cost of screening and the rejection cost (cost of scrapping). Most of the statistical models studied are based on the expected values of the parameters. This study considers the expected values of the parameters, their variability and distribution functions.
Any acceptance sampling plan by variables or by attributes requires the choice of prior distributions for the parameters involved. In this dissertation a relationship between the lot mean $\mu$ and the process fraction defective $p$ is developed such that defining a distribution for either $\mu$ or $p$ is sufficient to derive a prior distribution for the other. An economic cost model based on the prior and posterior probability density function is developed which leads to a set of decision points to accept or reject the lot. If the lot is rejected steps to dispose it are listed in Figure 1. The conditional expectations of the process fraction defective $p$ given the number of defectives in a sample or the sample mean are also derived.

One major factor in reducing the cost of production and improving the quality of products is to keep a production process under control. Most of the research dealing with production processes assumes the process starts in a state of statistical control at time $t = 0$. A quality characteristic of the units produced is defined and the mean and variance of its distribution are assumed to be constant. The initial setting of the process parameters define the quality level of the in-control state. The out of control state develops with time. It is not possible for the producer to set the machine at a precise value due to inaccuracy of humans and machines.

In this dissertation expressions of the loss-cost per hour are derived. The model is updated by assuming a prior distribution for the in-control level when a single or multiple quality characteristics are considered.

In estimating the cost of keeping a process under control, an expression for the loss cost per hour is formulated for a process in and out of control during a production cycle. Optimization of the cost yield values of the optimal sample sizes, the interval between samples and control limits. The process starts in a state of statistical control at a time zero and continues operating until a shift
Rejected Lots

Sorted (non destructive sampling)  Not sorted

---

Scrapped  Sold at reduced price

---

Effective items  Defective items

---

Not repaired  Repaired or replaced by effectives

---

Scrapped  Sold at reduced price

Figure 1. Disposition of a Rejected Lot.
Figure 2. Diagrammatic Representation of an In-Control Out-of-Control Production Process.
in the process mean is detected. The process standard deviation is assumed to be constant. A step by step procedure of searching for the assignable causes of failure are listed in Figure 2. Expression of the loss-costs per hour when a multiplicity of causes of failure occur or when several quality characteristics are involved are also considered.

1.4. Organization

In chapter two of this dissertation a family of prior distributions is defined and the posterior distributions are determined from the priors.

In Chapter three of this study an expression for the cost equation based on Bayesian variables sampling plans for fraction defective is formulated as a function of the sample size, the upper and lower limits of the sample mean for a sample drawn from an inspection lot, and the standard deviation and variances of the variables induced.

The sample size is incremented from a minimum value each time by 1. Then the cost equations are monitored to determine their minimum values. These values define the optimal starting point for the evaluation of the economic parameters. Numerical values of the cost are obtained by employing the upper and lower limits of \( \bar{X} \) and all possible values of the sample sizes. The optimum value of \( n \) corresponds to a minimum value for the cost. The minimum cost per lot yields a value of the mean of the quality characteristic \( X \) symmetrical about the value \( \frac{L+U}{2} \), where \( L \) and \( U \) are the lower and upper limits of \( X \). Also, in this chapter an expression for the sampling cost per unit is formulated and computed.

In Chapter four a cost model is developed for Bayesian variable sampling plans for fraction defective based on a set of economical parameters. An expression of the total cost with 4 components, namely the cost of inspection,
the cost of acceptance, the cost of screening and the cost of scrapping is formulated. The total cost is optimized relative to a certain set of variables to obtain a set of decision points based on the sample mean of a sample drawn from an inspection lot. The decision points are implemented to draw conclusions about the disposition of the lot, whether accepted, screened or scrapped. Both destructive and nondestructive sampling are considered.

Evaluating an acceptance sampling plan either by attributes or variables requires information on the fraction of items defective $p$. The conditional expectations of $p$ given the number of defectives in a sample $x$ or the sample mean $\bar{x}$ is employed in the analysis of costs and profits. Optimization of the costs or profits generates values for the decision points to dispose the inspection lot. In Chapter 5 an expression of the conditional expectation $E(p|x)$ and $E(p|\bar{x})$ as a function on the $K^{th}$ moment of $p$ is derived.

In Chapter 6 inspection error is assumed. The impact of inspection errors on the different parameters and distributions is significant. A set of new values for the parameters and variances are calculated and implemented for estimating the expected costs, consumer's and producer's risk and other expected qualities. The modified values for the quantities obtained are compared with those calculated under the assumption of error free systems to draw conclusions about the lot.

Chapter 7 of this paper deals with the cost of keeping a production process under control. A single or multiple characteristics is selected and defined as a measure of product quality. A production process starts in a state of statistical equilibrium with mean $\mu_0$ and variance $\sigma^2$. The process is out of control if a shift in the process mean is detected. The time before the assignable cause occurs is assumed to be exponentially distributed with parameter $\lambda$. Samples of size $n$ are taken every $h$ hours. The sample mean is plotted on an $\bar{X}$ chart
with control limits $\bar{x} \pm \frac{c}{\sqrt{n}}$ and if a point falls outside the control limits a search for the assignable cause begins. During the search for the cause of failure the production facility either continues in operation or shut down. The production cycle is defined as the length of time the process operates in control - out of control until the cause of failure is removed. An expression for the expected loss-cost per hour of operation based on a set of cost parameters and probabilities is formulated and optimized to obtain values of the sample sizes, the interval between successive samples and the control limits.

In testing for several quality characteristics a process is out of control if one or more points fall outside the control limits. In testing the quality control characteristics that shift the process out of control two techniques can be employed:

The principal component analysis or simultaneous confidence intervals procedure. The components of the cost here are: the expected cost of sampling per unit, the expected cost for searching for the assignable causes and the penalty cost due to the production process operating out of control.
CHAPTER 2

INSPECTION AND SELECTION TECHNIQUES

2.1. Background

Most of the research since 1960 on the economic development of acceptance sampling plans is devoted to acceptance sampling plans by attributes. Acceptance sampling plans by variables for fraction defective have received little attention. In attributes or variables acceptance sampling plans for fraction defective the items of a sample are classified as defective or non defective relative to a measurable quality characteristic ($X$). Frequently such measurements provide additional information about the quality of an item. This information may be incorporated into the sampling plan to improve the sampling procedure and minimize the sampling cost.

2.2. Variable Sampling for Fraction Defective

A sampling plan based upon a measurable quality characteristic is called a variable sampling plan. The following terms are defined and employed in all computations involved in this work:

- $X$ – The product quality characteristic
- $L$ – The lower specification limit of $X$
- $U$ – The upper specification limit of $X$
- $N$ – The lot size
- $n$ – The sample size
- $\bar{X}$ – The sample mean
- $\mu$ – The mean of the quality characteristic $X$
- $\sigma^2$ – The variance of the quality characteristic $X$
- $m$ – The mean of the mean $\mu$
\[ \sigma^2_\mu - \text{The variance of the mean } \mu \text{ (in this work } \sigma_\mu \text{ is assumed known)} \]

\[ f(x|\mu) - \text{The conditional probability density function of the quality characteristic (X) given } \mu \]

\[ h(\mu) - \text{The probability density function of the mean } \mu \]

\[ T(\bar{x}|\mu) - \text{The conditional probability of } \bar{x} \text{ given } \mu \]

\[ t(x|\bar{x}, \mu) - \text{The conditional probability density function of individual observations given the sample mean } \bar{x} \text{ and population mean } \mu \]

2.3. Distribution of \( \omega(p|x) \) Given \( f(x|\mu) \) and \( h(\mu) \)

In this chapter models are assumed where cost parameters are based upon lot fraction defective \( P \) and the distributional information must be based upon the quality characteristic \( X \) with p.d.f. \( f(x|\mu) \) or upon \( p \). Transformations are derived from \( X \) to \( p \) and \( p \) to \( X \).

In this study both lower, upper and double specification limits for \( X \) will be considered. For a lower specification limit an item is considered defective if \( X < L \), effective if \( X > L \). For an upper specification limit an item is defective if \( X > U \) and effective if \( X < U \). For a double specification limit an item is considered effective if \( L \leq X \leq U \), and defective if \( X < L \) or \( X > U \). In order to derive a sampling procedure it is necessary to assume a distribution of \( X \) that can depend on known or unknown parameters. In this chapter a prior p.d.f. of \( X \), \( f(x|\mu) \) is assumed with parameters \( \mu \) which are random variables. The parameters \( \mu \) are assumed to be random variables governed by a certain distribution. The distribution of the fraction of items defective \( p \) will be derived from the distribution of \( (\mu) \) via a transformation \( \mu = Q(p) \). The conditional p.d.f. of \( p \) given \( X \) can be evaluated as follows:

\[ \omega(p|x) = \frac{f(x|p)\omega(p)}{\int_0^1 f(x|p)\omega(p)dp} \] (2.1)
The distribution of $p$ in terms $h(\mu|m, \sigma^2_\mu)$ is:

$$\omega(p) = h(Q(p)|m, \sigma^2_\mu) \left| \frac{d\mu}{dp} \right|_{\mu=Q(p)}.$$  \hfill (2.2)

Substituting expression (2.2) into (2.1) yields

$$\omega(p|x) = \frac{f(x|p) \cdot h(Q(p)|m, \sigma^2_\mu) \left| \frac{d\mu}{dp} \right|_{\mu=Q(p)}}{\int_0^1 f(x|p)h(Q(p)|m, \sigma^2_\mu) \left| \frac{d\mu}{dp} \right|_{\mu=Q(p)} dp}.$$  \hfill (2.3)

An expression of $f(x|p)$ can be obtained by developing an expression of $\mu$ in terms of $p$ and employing expression (2.2). A set of prior probability density functions for $f(x|\mu)$ and $h(\mu)$ can be selected. Distributions for $p$ and $p|x$ are formulated for each case. The selection techniques of the priors is restricted to a set of conditions that are stated and discussed briefly.

To find a specific distribution start by selecting some family of distributions ($F$) defined by a mathematical formula with a number of adjustable parameters. Then select member of this family that meets the quantitative requirements of the decision maker. Assign proper numerical values of the parameters. If $\hat{\Theta}$ is a process fraction defective, then $\hat{\Theta}$ has values in the interval $[0, 1]$. The decision maker will be satisfied with a member of the Beta family that restricts the domain of $\Theta$ to that interval. The following are some requirements for selecting $F$.

1. $F$ should be tractable.
   
   (a) There should be no problem in determining the posterior distribution from the prior and a given sample.

   (b) The cost functions should be easily expressed relative to any member of the family

   (c) If the prior is a member of $F$ then the posterior should also be a member of $F$. 
(2) There should exist a member of $F$, that the decision maker can express his prior beliefs and include all the information available about a process. A member of $F$ satisfying this property is called a rich member.

(3) $F$ should be in close agreement with the decision maker prior judgment about the state space $\Theta$. The last property is referred to as interpretable.

The conditional expectation of $E(p|x)$ is derived by assuming a Beta prior probability density function for $p$, $\omega(p)$ given by the attribute sampling plan. The distribution of $h(\mu)$ is obtained from the distribution of $p$ through the same transformation given in expression (2.2). A mathematical form of $E(p|x)$ is obtained by employing the probability distributions of $p$ and $\mu$. The parameters involved and the prior probability density functions are reevaluated for sampling plans. The adjusted set of parameters and prior probabilities are employed for the reevaluation of the posterior probabilities. The posterior probabilities are in turn used to evaluate the adjusted values of the cost components and a set of decision points. The decision points provide information about the inspection lot and help to determine whether to accept screen or scrap the lot.

2.4. Distribution of the Fraction of Items Defective Given the Distribution of $\mu$

Consider a measurable quality characteristic $X$ having the distribution $f(x|\mu)$. For the lower specification limit $(L)$ if $X < L$ the item is regarded as defective. The fraction of items defective is given by:

$$p(\mu) = \int_{-\infty}^{L} f(x|\mu)dx = p_r(X \leq L|\mu). \quad (2.4)$$

Assuming $X$ follows a normal distribution with mean $\mu$ and variance $\sigma^2$, then:

$$f(x|\mu) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} \quad -\infty < x < +\infty \quad (2.5)$$

where mean $\mu$ is distributed with a mean $m$ and variance $\sigma^2_\mu$, $\mu \sim f(\mu|m,\sigma^2_\mu)$ and $\sigma^2$ is a known quantity. The distribution of the fraction of items defective
for a lower fraction limit is:

\[ p(\mu) = \int_{-\infty}^{L} \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{1}{2} \left( \frac{x - \mu}{\sigma} \right)^2} \, dx \]  
\[ \text{(2.6)} \]

or

\[ p(\mu) = \Phi \left( \frac{L - \mu}{\sigma} \right). \]  
\[ \text{(2.7)} \]

Define a new variable \( z \) as:

\[ z = \mu - L. \]  
\[ \text{(2.8)} \]

By expanding \( \Phi \left( \frac{L - \mu}{\sigma} \right) \) by a McLaurin series an expression for \( p(\mu) \) is obtained

\[ p(\mu) = \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k! (2k + 1)} \left( \frac{\mu - L}{\sigma} \right)^{2k+1} \quad -\infty < \mu < +\infty. \]  
\[ \text{(2.9)} \]

The above expression is based on the expansion:

\[ e^{-\frac{t^2}{2}} = \sum_{k=0}^{\infty} \frac{(-t^2/2)^k}{k!} \]  
\[ \text{(2.10)} \]

or

\[ e^{-\frac{t^2}{2}} = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{2^k k!}. \]  
\[ \text{(2.11)} \]

Transferring (2.9) to a function of \( z \) by employing expression (2.8) yields:

\[ h(z) = \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k! (2k + 1)} \left( \frac{z}{\sigma} \right)^{2k+1}. \]  
\[ \text{(2.12)} \]

Letting

\[ v = h(z) - \frac{1}{2} \]  
\[ \text{(2.13)} \]

then

\[ v = p(\mu) - \frac{1}{2} \]  
\[ \text{(2.14)} \]
or
\[ v = f(z) \quad (2.15) \]

then:
\[ z = f^{-1}(v). \quad (2.16) \]

If
\[ z = 0 \quad v = 0 \quad (2.17) \]

then
\[ p(\mu) = \frac{1}{2} \quad \mu = L. \quad (2.18) \]

By definition, the \( k \)th derivative of \( z(v) \) relative to \( v \) is:
\[ z^k(v) = \frac{d^k(z)}{dv^k}. \quad (2.19) \]

Employing expression (2.19), \( z(v) \) can be expanded into a series as follows:
\[ z(v) = \sum_{k=0}^{\infty} \frac{z^k(0)}{k!} v^k. \quad (2.20) \]

Expression (2.20) can be written as
\[ z(v) = -\sigma \left\{ \frac{\Phi_v}{1!} + \frac{(\Phi_v)^3}{3!} + \frac{7(\Phi_v)^5}{5!} + \frac{127(\Phi_v)^7}{7!} \right. \\
+ \frac{4369(\Phi_v)^9}{9!} + \frac{318493(\Phi_v)^{11}}{11!} + \cdots + R(v) \right\} \quad (2.21) \]

where \( \phi = \sqrt{2\pi} \). The above expression is derived in Appendix B.

Employing expression (2.14) expression (2.21) can be written as:
\[ z(p) = -\sigma l(p) \quad (2.22) \]

where \( l(p) \) is defined by the quantity within paranthesis of expression (2.21).

Employing expression (2.22) expression (2.8) yields:
\[ \mu = L - \sigma l(p). \quad (2.23) \]
The p.d.f. of $\mu, h(\mu)$ is
\[
h(\mu) = \frac{1}{\sqrt{2\pi}\sigma_\mu} e^{-\frac{1}{2} \left( \frac{\mu - m}{\sigma_\mu} \right)^2} \quad -\infty < \mu < +\infty. \tag{2.24}
\]

The p.d.f. of $p$ in terms of $h(\mu)$ is:
\[
\omega(p) = h(\mu(p)) \left| \frac{d\mu(p)}{dp} \right|
\tag{2.25}
\]

where:
\[
h(\mu(p)) = \frac{1}{\sqrt{2\pi}\sigma_\mu} e^{-\frac{1}{2} \left( \frac{L - \mu(p) - m}{\sigma_\mu} \right)^2} \tag{2.26}
\]

Expression (2.23) yields:
\[
\frac{d\mu(p)}{dp} = -\sigma l'(p). \tag{2.27}
\]

Substituting expressions (2.26) and (2.27) into expression (2.25) yield
\[
\omega(p) = \frac{1}{\sqrt{2\pi}\sigma_\mu} e^{-\frac{(L - m - \mu(p))^2}{2\sigma_\mu^2}} \cdot (\sigma l'(p))
\tag{2.28}
\]
\[
= \frac{l'(p)}{\sqrt{2\pi}\sigma_\mu} e^{-\frac{(L - m - l(p))^2}{2\sigma^2}} \quad 0 < p < 1. \tag{2.29}
\]

For $\omega(p)$ to be a p.d.f. the following condition must be satisfied:
\[
\int_0^1 \omega(p)dp = 1
\]
or
\[
\int_0^1 \frac{l'(p)}{\sqrt{2\pi}\sigma_\mu} e^{-\frac{1}{2} \left( l(p) - \frac{L - m}{\sigma} \right)^2 / (\sigma^2)} dp = 1. \tag{2.30}
\]

Letting
\[
\frac{l(p) - \left( \frac{L - m}{\sigma} \right)}{\sigma_\mu} = Q_1
\]
then
\[
\frac{l'(p)}{\sigma_\mu} dp = dQ_1 \implies l'(p)dp = \frac{\sigma_\mu}{\sigma} dQ_1.
\]
If
\[ p = 0 \quad Q_1 \to -\infty \]
and if
\[ p = 1 \quad Q_1 \to +\infty \]
expression (2.30) will be reduced to the form:
\[
\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}Q_1^2} dQ_1 = 1. \quad (2.31)
\]

2.5 Distributions of \( \mu \) Based Upon a Prior Distribution of \( p \)

Assuming that the distribution of \( p \), \( \omega(p) \) is beta then
\[
\omega(p) = \frac{\Gamma(c + \beta)}{\Gamma(c) \cdot \Gamma(\beta)} \cdot (1 - p)^{c-1} \cdot p^{\beta-1}. \quad (2.32)
\]

Define a function \( v_1(\mu) \) as
\[
v_1(\mu) = \frac{1 - p(\mu)}{p(\mu)} \quad (2.33)
\]
where \( p(\mu) \) is given by expression (2.6). Then it can be shown that the p.d.f. of \( v(\mu) \) is:
\[
S(v_1) = \frac{\Gamma(\beta + c)}{\Gamma(\beta) \Gamma(c) (v_1 + 1)^{c+\beta}} v_1^{c-1} \quad 0 < v_1 < \infty. \quad (2.34)
\]

To find the distribution of \( h(\mu(v_1)) \) use the transformation:
\[
S(v_1) = h(\mu(v_1)) \left| \frac{d\mu(v_1)}{dv_1} \right|. \quad (2.35)
\]

Employing expressions (2.34) and (2.35) yield an expression for \( h(\mu(v_1)) \):
\[
h(\mu(v_1)) = \frac{\Gamma(\beta + c)}{\Gamma(\beta) \Gamma(c) |p'(\mu)|} \frac{v_1^{c-1}}{(v_1 + 1)^{c+\beta-2}}. \quad (2.36)
\]

In terms of \( p(\mu) \), \( h(\mu) \) can be written as
\[
h(\mu) = \frac{\Gamma(\beta + c)}{\Gamma(\beta) \Gamma(c) |p'(\mu)|} \frac{[(1 - p(\mu))/p(\mu)]^{c-1}}{[1 + (1 - p(\mu))/p(\mu)]^{c+\beta-2}}. \quad (2.37)
\]
Arrange the terms in expression (2.37) and write it as:

\[ h(\mu) = \frac{\Gamma(\beta + c)}{\Gamma(\beta) \Gamma(c)} |p'(\mu)| [1 - p(\mu)]^{c-1} p(\mu)^{\beta-1}. \]  

(2.38)

A necessary condition for \( h(\mu) \) to be a p. d. f. is \( \int_{-\infty}^{+\infty} h(\mu) d\mu = 1 \) or

\[ \int_{-\infty}^{+\infty} \frac{\Gamma(\beta + c)}{\Gamma(\beta) \Gamma(c)} |p'(\mu)| [1 - p(\mu)]^{c-1} (p(\mu))^{\beta-1} d\mu. \]  

(2.39)

Setting \( p(\mu) = w \) implies \( p'(\mu) d\mu = dw. \)

If:

\[ \mu \to -\infty \quad \omega \to 1 \]
\[ \mu \to +\infty \quad \omega \to 0. \]

In terms of \( w \) expression (2.39) can be written as:

\[ \int_{0}^{1} \frac{\Gamma(\beta + c)}{\Gamma(\beta) \Gamma(c)} dw (1 - w)^{c-1} w^{\beta-1} = 1. \]  

(2.40)

Plotting \( p(\mu) \) vs \( \mu \) it is possible to verify that for a lower specification case \( p'(\mu) < 0 \). This condition can be written in the form:

\[ \frac{|p'(\mu)|}{p'(\mu)} = -1. \]  

(2.41)

For an upper specification \( p'(\mu) > 0 \), hence use the following notation in reference to this case:

\[ \frac{|p'(\mu)|}{p'(\mu)} = 1. \]  

(2.42)

Employing expressions (2.2) and (2.38) yield an expression for \( \omega(p) \) of the form:

\[ \omega(p) = \frac{\Gamma(c + \beta)}{\Gamma(c) \cdot \Gamma(\beta)} (1 - p)^{c-1} \cdot p^{\beta-1} \]

which is the same expression as suggested in expression (2.32).
2.6. Double Specification for the Measurable Quality Characteristic

Letting \( L \) and \( U \) be the lower and upper specifications of \( X \) respectively.

The fraction of material defective can be written in the form:

\[
p(\mu) = 1 - \int_L^U f(x|\mu)dx = \int_{-\infty}^L f(x|\mu)dx + \int_U^\infty f(x|\mu)dx = pL(\mu) + pU(\mu).
\]  \hfill (2.43)

Then there exists a value \( \mu' \) such that

\[
\frac{|p'(\mu)|}{p'(\mu)} = \begin{cases} 
1 & \text{if } \mu = \mu'_1 > \mu' \\
-1 & \text{if } \mu = \mu'_2 < \mu' \\
\text{undetermined} & \text{if } \mu = \mu'.
\end{cases}
\]  \hfill (2.44)

Limit cases

If \( U \to +\infty \)

\[
p(\mu) = pL(\mu)
\]  \hfill (2.45)

of \( L \to -\infty \)

\[
p(\mu) = pU(\mu).
\]  \hfill (2.46)

Since

\[
v'_1(\mu) = \frac{dv_1}{d\mu} = -\frac{p'(\mu)}{|p(\mu)|^2}.
\]  \hfill (2.47)

The p. d. f. of \( p, \omega(p) \) is obtained by using the transformation:

\[
\omega(p) = S(p)\left|\frac{dv_1}{dp}\right|.
\]  \hfill (2.48)

In this case for \( h(\mu) \) to be a p.d.f. the following condition must be true:

\[
\int_\mu h(\mu)d\mu = \int_{-\infty}^{\mu'_1} \frac{\Gamma(\beta + c)}{\Gamma(\beta)\Gamma(c)} (p'(\mu))[1 - p(\mu)]^{\beta-1}d\mu
\]

\[
+ \int_{\mu'_2}^\infty \frac{\Gamma(\beta + c)}{\Gamma(\beta)\Gamma(c)} (p'(\mu))[1 - p(\mu)]^{c-1}[\beta(\mu)]^{\beta-1}d\mu.
\]  \hfill (2.49)
Setting $\omega = p(\mu)$ implying $d\omega = p'(\mu)d\mu$. The limits of integration will be:

if $\mu = \mu'_1$ then $\omega = p(\mu'_1)$
if $\mu \to -\infty$ then $p(\mu) \to 1$
if $\mu = \mu'_2$ then $p(\mu) = p(\mu'_2)$
if $\mu \to \infty$ then $p(\mu) \to 1$.

Due to symmetry:

$$p'_L(\mu) = -p'_U(\mu).$$ (2.50)

Employing expression (2.50) expression (2.49) provides:

$$\int_{\mu} h(\mu) d\mu = -\int_{p(\mu'_1)}^{1} \frac{\Gamma(\beta + c)}{\Gamma(\beta)\Gamma(c)} \cdot (1 - w)^{c-1} w^{\beta-1} dw + \int_{p(\mu'_2)}^{1} \frac{\Gamma(\beta + c)}{\Gamma(\beta)\Gamma(c)} (1 - w)^{c-1} w^{\beta-1} dw$$

$$= \frac{2\Gamma(\beta + c)}{\Gamma(\beta)\Gamma(c)} \int_{p(\mu'_1)}^{1} (1 - w)^{c-1} w^{\beta-1} dw.$$ (2.51)

For $h(\mu)$ to be a p. d. f., a constant $k$ must be defined as:

$$k = \frac{2\Gamma(\beta + c)}{\Gamma(\beta)\Gamma(c) \int_{p(\mu'_1)}^{1} (1 - w)^{c-1} w^{\beta-1} dw}.$$ (2.52)

The probability density function of $h(\mu)$ is then:

$$h(\mu) = \frac{\Gamma(\beta + c)}{k\Gamma(\beta)\Gamma(c)} |p'(\mu)p(\mu)^{\beta-1}(1 - p(\mu))^{c-1} - \infty < \mu < \infty.$$ (2.53)

2.5.1. Limits of $v_1$ and $p$

Case 1. $\mu = \mu'_1$ then

$$0 < v_1 < \frac{1 - p(\mu'_1)}{p(\mu'_1)}.$$ (2.54)

Case 2. $\mu = \mu'_2$ then

$$0 < v_1 < \frac{1 - p(\mu'_2)}{p(\mu'_2)}.$$ (2.55)
In this case expression (2.34) will be reduced to the form:

\[ S(v_1) = \frac{2 \Gamma(\beta + c)}{k \Gamma(\beta)\Gamma(c)} \frac{v_1^{\alpha-1}}{1 + v_1^{\beta+1}} \]  

(2.56)

with

\[ 0 < v_1 < \frac{1 - p(\mu_1)}{p(\mu)} \quad 0 < v_1 < \frac{1 - p(\mu_2)}{p(\mu)} \]  

(2.57)

\[ \omega(p) = \frac{2 \Gamma(\beta + c)}{k \Gamma(\beta)\Gamma(c)} p^{\beta-1}(1-p)^{\beta-1}, \quad p(\mu_1) \leq p(\mu) \leq 1 \quad -\infty < \mu < \infty. \]  

(2.58)

The conditional probability of \( E(p|\mu) \) is:

\[ E(p|\mu) = \frac{\Gamma(\beta + c)}{\Gamma(\beta)\Gamma(c)} \int_{-\infty}^{+\infty} p'(\mu)p(\mu)^{(\beta+1)-1}[1 - p(\mu)]^{\beta-1} d\mu. \]  

(2.59)

Setting \( w = p(\mu) \) implies \( dw = p'(\mu)d\mu \), expression (2.59) yields:

\[ E(p|\mu) = \frac{\Gamma(\beta + c)2}{\Gamma(\beta)\Gamma(c)k} \int_{p(\mu_1)}^{p(\mu_2)} w^{(\beta+1)-1}(1-w)dw. \]  

(2.60)

2.7. General Approach

Consider a random variable that represents a product quality based on a random sample \( x_1, x_2, \ldots, x_n \), let the function \( \Phi(x_1, x_2, \ldots, x_n) \) be an estimate of an unknown parameter \( \mu \).

For accepting an inspection lot let \( \delta_L \) and \( \delta_U \) be the lower and upper acceptance limits of \( \phi \).

\[ \delta_L < \phi_n < \delta_U \]  

(2.61)

In terms of \( \mu \) define the following probabilities:

\[ P_R(\delta_L \leq \phi_n(x_1, \ldots, x_n) \leq \delta_U | \mu = \mu_1) = P_1 \]  

(2.62)

\[ P_R(\delta_L \leq \phi_n(x_1, \ldots, x_n) \leq \delta_U | \mu = \mu_2) = P_2. \]  

(2.63)
The posterior distribution of $\mu$ can be written in the form:

$$h(\mu|\phi_n) = \frac{T(\phi_n|\mu)h(\mu)}{T(\phi_n)}$$  \hfill (2.64)

where $T(\phi_n|\mu)$ is the conditional probability of $\Phi_n$ given $\mu$. Now let $k'_1 = k'_1(\mu)$, be the cost of accepting a lot with posterior lot quality represented by $\mu$. Also define $k'_2 = k'_2(\mu)$, cost of rejecting a lot, with posterior lot quality represented by $\mu$. Then the conditional expectations of $k'_1$ and $k'_2$ are:

$$E(k'_1|\phi_n) = \int_{-\infty}^{+\infty} k'_1(\mu)h(\mu|\phi_n)d\mu$$  \hfill (2.65)

$$E(k'_2|\phi_n) = \int_{-\infty}^{+\infty} k'_2(\mu)h(\mu|\phi_n)d\mu.$$  \hfill (2.66)

Let the values of $\Phi_n$ for which

$$E(k'_2|\phi_n) = E(k'_1|\phi_n)$$  \hfill (2.67)

be $\Phi_{n1}^{01}$ and $\Phi_{n2}^{02}$, respectively. If

$$\Phi_{n1}^{01} \leq \Phi_n \leq \Phi_{n2}^{02}$$  \hfill (2.68)

accept the lot, otherwise reject it.

Assume that the quality characteristic $X$ is normally distributed with mean $\mu$ and variance $\sigma^2$ known. Also assuming $\mu$ is normally distributed with mean $m$ and variance $\sigma^2_n$ then

$$h(\mu) = \frac{1}{\sqrt{2\pi}\sigma}\ e^{-\frac{1}{2} \frac{(\mu - m)^2}{\sigma^2}}$$  \hfill (2.69)

Setting $\phi(x_1, x_2, \ldots, x_n) = \bar{x}_n$, then $T(\bar{x}_n|\mu)$ is given by

$$T(\bar{x}_n|\mu) = \frac{1}{\sqrt{2\pi} \frac{\sigma}{\sqrt{n}}} \ e^{-\frac{1}{2} \frac{(\bar{x}_n - m)^2}{\sigma^2_n}}.$$  \hfill (2.70)
In estimating \( h(\mu|\phi_n) \) given by expression (2.64) the number of this expression can be obtained by employing expressions (2.70) and (2.69), thus

\[
T(\bar{x}_n|\mu)h(\mu) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{1}{2} (\frac{\bar{x}_n - \mu}{\sigma})^2} \cdot \frac{1}{\sqrt{2\pi} \sigma_{\mu}} e^{-\frac{1}{2} \frac{(\mu - m)^2}{\sigma_{\mu}^2}}. \tag{2.71}
\]

A more desirable form of (2.71) can be obtained by using the following transformation equation:

\[
\frac{(a - b)^2}{c^2} + \frac{(d - a)^2}{f^2} = \frac{(a - m)^2}{\delta^2} + \frac{(b - d)^2}{\sigma^2}
\]

where

\[
m = \frac{bf^2 + c^2d}{c^2 + f^2} \tag{2.73}
\]

\[
\delta^2 = \frac{c^2f^2}{c^2 + f^2} \tag{2.74}
\]

\[
\sigma^2 = c^2 + f^2. \tag{2.75}
\]

Expression (2.71) can be reduced to:

\[
T(\bar{x}_n|\mu)h(\mu) = \frac{1}{\sqrt{2\pi} \delta_n \sigma_{\mu}} e^{-\frac{1}{2} \frac{(\bar{x}_n - m)^2}{\sigma_n^2}} \cdot e^{-\frac{1}{2} \frac{(\bar{x}_n - m)^2}{\sigma_{\mu}^2}}. \tag{2.76}
\]

where

\[
\delta_n^2 = \frac{\sigma^2}{n}, \quad m_n = \frac{m\delta_n^2 + \sigma_n^2\bar{x}_n}{\delta_n^2 + \sigma_n^2}, \quad \sigma_n^2 = \frac{\sigma_{\mu}^2 \delta_n^2}{\delta_n^2 + \sigma_{\mu}^2}. \tag{2.77}
\]

Knowing the expression for \( T(\bar{x}_n|\mu)h(\mu) \) provides a distribution for \( T(\bar{x}_n) \):

\[
T(\bar{x}_n) = \int_{-\infty}^{+\infty} T(\bar{x}_n|\mu)h(\mu)d\mu = +\frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\delta_n^2 + \sigma_{\mu}^2}} e^{-\frac{1}{2} \frac{(\bar{x}_n - m)^2}{\delta_n^2 + \sigma_{\mu}^2}}. \tag{2.78}
\]

Expression (2.78) shows the expected value of \( \bar{x}_n \), \( E(\bar{x}_n) \) is \( m \). From (2.64), (2.75), and (2.78)

\[
h(\mu|\bar{x}_n) = \frac{1}{\sqrt{2\pi} \delta_n \sigma_{\mu}} e^{-\frac{1}{2} \frac{(\mu - mn)^2}{\sigma_n^2}} \cdot e^{-\frac{1}{2} \frac{(\bar{x}_n - m)^2}{\delta_n^2 + \sigma_{\mu}^2}} / \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\delta_n^2 + \sigma_{\mu}^2}} e^{-\frac{1}{2} \frac{(\bar{x}_n - m)^2}{\delta_n^2 + \sigma_{\mu}^2}}. \tag{2.79}
\]
Simplifying expression (2.79) yields:

\[
    h(\mu | \bar{x}_n) = \frac{1}{\sqrt{2\pi} \sigma_n} e^{-\frac{(\mu - m_n)^2}{2\sigma_n^2}} - \infty < \mu < +\infty.
\] (2.80)

Expression (2.80) shows that \( h(\mu | \bar{x}_n) \) is normal with mean \( m_n \) and variance \( \sigma_n^2 \), where \( m_n \) and \( \sigma_n^2 \) are given by expression (2.77).

Letting \( p(\mu) \) be the density representing the fraction defective for the lower and upper bound specifications of \( X \). The conditional expectation of \( p(\mu) \) given \( \bar{x}_n \) is

\[
    E(p(\mu) | \bar{x}_n) = \int_{-\infty}^{+\infty} p(\mu) h(\mu | \bar{x}_n) d\mu
\] (2.81)

where \( p(\mu) \) is given by expression (2.43). Substituting (2.43) into (2.81) provides

\[
    E(p(\mu) | \bar{x}_n) = \int_{-\infty}^{+\infty} [1 - \int_{L}^{U} f(x | \mu) dx] h(\mu | \bar{x}_n) d\mu
\] (2.82)

or

\[
    E(p(\mu) | \bar{x}_n) = 1 - \int_{-\infty}^{+\infty} \int_{L}^{U} f(x | \mu) h(\mu | \bar{x}_n) d\mu dx.
\] (2.83)

Employing expressions (2.5) and (2.80) expression (2.83) yields:

\[
    E(p(\mu) | \bar{x}_n) = 1 - \int_{L}^{U} \frac{1}{\sqrt{2\pi} \sqrt{\sigma^2 + \sigma_n^2}} e^{-\frac{1}{2} (\frac{x - m_n}{\sigma^2 + \sigma_n^2})^2} dx.
\] (2.84)

For purposes of clarity let

\[
    z = \frac{x - m_n}{\sqrt{\sigma^2 + \sigma_n^2}} \Rightarrow z_L = \frac{L - m_n}{\sqrt{\sigma^2 + \sigma_n^2}}, \quad z_u = \frac{U - m_n}{\sqrt{\sigma^2 + \sigma_n^2}}. \tag{2.85}
\]

Now (2.81) can be rewritten as:

\[
    E(p(\mu) | \bar{x}_n) = 1 - \int_{(L - m_n)/\sqrt{\sigma^2 + \sigma_n^2}}^{(U - m_n)/\sqrt{\sigma^2 + \sigma_n^2}} \left( e^{-\frac{1}{2} z^2 / \sqrt{2\pi} \sqrt{\sigma^2 + \sigma_n^2}} \right) \cdot \sqrt{\sigma^2 + \sigma_n^2} dz
\]

or

\[
    E(p(\mu) | \bar{x}_n) = \Phi(L - m_n/\sqrt{\sigma^2 + \sigma_n^2}) + 1 - \phi(u - m_n/\sqrt{\sigma^2 + \sigma_n^2}). \tag{2.86}
\]

The concepts introduced in this chapter will be employed in estimating the costs, the upper and lower limits of material fraction defectives, and its moments up to the \( k^{th} \) power \( k \geq 1 \) and the decision points. The above concepts are also needed for estimating the consumer's and producer's risks.
CHAPTER 3

OPTIMAL ECONOMIC COST MODELS

DESTRUCTIVE TESTING

3.1. Introduction

In the classical approach to acceptance sampling, the user specifies two quality levels, the acceptable quality level (AQL) and the rejectable quality level (RQL), and two probability parameters \( \alpha \) and \( \beta \). The consumer requires that the probability of acceptance be at least \((1 - \alpha)\) if the lot is of acceptable quality and the probability of rejection be at least \((1 - \beta)\) if the lot is of rejectable quality.

In this chapter cost equations are formulated for destructive testing in terms of economic cost parameters, the upper and lower limits of the quality characteristic \( X \), the sample size \( n \), the lot size \( N \) and the standard deviations of the quality characteristic parameters of of the distribution of \( \mu \) of the mean of \( X \) the quality characteristic. Expressions of the lower and upper limits of the sample mean for lot acceptance are derived, evaluated and implemented in estimating the costs and other quantities.

The expression of the total cost consists of three components, the cost of sampling, the cost of acceptance and the cost of rejection. An expression of the total cost is evaluated by averaging each component of the cost overall possible sample means for samples drawn from an inspection lot. In estimating the optimum cost the sample size is incremented each time by 1, starting from a minimum acceptable value of the sample size. A minimum value of the expected total cost is reached for a certain value of the sample size \( n_0 \). For this sample size the expected sampling cost per unit and the expected fraction defective
given the results of a sample are evaluated. All estimates in this chapter are carried out by assuming, the distributions are normal, the acceptance sampling plans by variables for fraction defective are destructive, the variance of the quality characteristic is known and its mean is unknown and normally distributed.

The economic evaluation of the acceptance procedure requires the identification of all possible actions (decisions) for which the producer's and consumer's prior and posterior cost functions are evaluated. In destructive testing the following actions are considered for the model proposed by Dietrich (1979). A. Prior Sampling (Decision Without Sampling)

1. Accept outright
2. Reject outright and scrap.

B. After Sampling Decisions

1. Accept the lot
2. Reject the lot and scrap.

3.2. Assumptions and Economic Parameters

A sample is randomly drawn from an inspection lot. The decision to accept or reject the lot is based on the results of the sample. In destructive sampling the following assumptions are made in developing the cost function.

1. Testing a sample for one or more quality characteristic renders it unfit for further use.
2. If the sample results require rejecting the lot, the lot must be scrapped.

Information on the following economic parameters is required to compute the prior and posterior costs and profits.

$K_R$ – sales price of an item

$K_p$ – production cost of an item

$K_j$ – junk value of a scrapping item = 0 in this model
$K_A$ – cost of accepting a defective item delivered to the consumer

$p$ – fraction of items defective

$C_1$ – prior cost function associated with the decision to accept outright

$C_2$ – prior cost function associated with the decision to reject outright and scrap

$K'_1$ – the posterior cost function associated with acceptance

$K'_2$ – the posterior cost function associated with rejection

$E(K'_1|\Phi)$ – expected posterior cost associated with acceptance

$E(K'_2|\Phi)$ – expected posterior cost associated with rejection

$K(n, \Phi_n^{01}, \Phi_n^{02})$ – cost equation in terms of the sample size $n$ and an upper and lower limits of the parameter $\Phi_n$.

### 3.3. Prior Cost Analysis

The following are the prior costs per unit from the producer’s point of view relative to each decision without sampling:

#### 3.3.1. Accept Outright

The cost per item $C_1$ for a decision to accept the lot without sampling is given by:

$$C_1 = K_A p$$

or

$$p = \frac{C_1}{K_A}$$

where $p$ is the fraction of items which are defective. The prior p.d.f. of $p$ given by expression (2.30)

$$\omega(p) = \frac{f'(p)}{\sqrt{2\pi} \sigma} e^{-\frac{1}{2} \left[ \frac{p-m}{\sigma} - f(p) \right]^2} \quad 0 < p < 1$$

(3.3)
The expression of \( l(p) \) is given by (2.23). An expression of the p.d.f. of \( C_1 \) can be obtained by employing expressions (3.2) and (3.3) as:

\[
f(C_1) = \frac{1}{\sqrt{2\pi} \sigma_n} e^{-\frac{\left( \frac{C_1 - m}{\sigma_n} - l(C_1) \right)^2}{\sigma_n^2}} \quad 0 < C_1 < K_A.
\] (3.4)

The expected value of \( C_1 \), \( E(C_1) \) is:

\[
E(C_1) = \int_0^{K_A} C_1 f(C_1) dC_1.
\] (3.5)

The variance of \( (C_1) \) can be obtained by employing the expression:

\[
V(C_1) = E(C_1^2) - [E(C_1)]^2.
\] (3.6)

Computing the expected value of \( C_1 \) and its variance, requires the application of numerical integration techniques since no closed forms are known. The profit per item \( P_1 \), for a decision to accept the lot without sampling is given by:

\[
P_1 = K_R - K_p - K_{AP}.
\] (3.7)

Expression (3.7) shows that \( P_1 \) is a random variable whose p.d.f. can be obtained from that of \( p \) given by expression (3.3) by the same procedure as above.

### 3.3.2. Reject Outright and Scrap.

The cost per unit \( C_2 \) for a decision to reject outright and to scrap the whole lot is:

\[
C_2 = K_R.
\] (3.8)

For this decision the cost per unit is not a random variable. The profit per item \( P_2 \), for a decision to reject outright and to scrap the whole lot is

\[
P_2 = -K_p.
\] (3.9)
For this decision the profit per unit is not a random variable.

**3.4. Posterior Cost Analysis**

Letting $\phi_n$ defined in expression (2.61) be equal to $\bar{x}_n$, referring to expressions (2.65) and (2.66) the decision to accept the lot yields:

$$E(k_1'|\bar{x}_n) = k_A E(p(\mu)|\bar{x}_n). \quad (3.10)$$

The decision to reject the lot is given by the expression

$$E(k_2'|\bar{x}_n) = k_r. \quad (3.11)$$

The point $p_0$ where

$$E(k_1'|\bar{x}_n) = E(k_2'|\bar{x}_n). \quad (3.12)$$

is the value of $p$ where the expected cost of the decision to accept equals the expected cost of the decision to reject, thus the decisions are equivalent.

Employing expressions (3.10), (3.11) (3.12) and (2.86) yield

$$E(p(\mu)|\bar{x}_n) = \Phi(L - m_n/\sqrt{\sigma^2 + \sigma^2_n}) + 1 - \Phi(U - m_n/\sqrt{\sigma^2 + \sigma^2_n}). \quad (3.13)$$

The expressions of $m_n$, $\sigma^2_n$ and $\delta^2_n$ are given by expression (2.77). It is possible to show that there exists two values of $m_n$ satisfying equation (3.12) given by:

$$m^1_n = \frac{m\delta_n^2 + \sigma^2 \bar{x}_n}{\delta_n^2 + \sigma^2} \quad (3.14)$$

$$m^2_n = \frac{m\delta_n^2 + \sigma^2 \bar{x}_n}{\delta_n^2 + \sigma^2} \quad (3.15)$$

where $m$ is the mean of $\mu$. To verify whether $m^1_n$ and $m^2_n$ are different requires solving equations (3.14) and (3.15). Leting $y = n$ and $x = n\bar{x}_n$, then (3.14) and (3.15) will yield two equations with two unknowns in $x$ and $y$, these are:

$$-\sigma^2 x + \sigma^2 m^1_n y = \sigma^2 (m - m^1_n) \quad (3.16)$$

$$-\sigma^2 x + \sigma^2 m^2_n y = \sigma^2 (m - m^2_n). \quad (3.17)$$
For a solution to exist the determinant of the coefficients of equations (3.16) and (3.17) must be different from zero which yields:

\[
\begin{bmatrix}
-\sigma^2_{\mu} & \sigma^2_{\mu} m_n^1 \\
-\sigma^2_{\mu} & \sigma^2_{\mu} m_n^2
\end{bmatrix} \neq 0.
\]

This implies that \(-\sigma^2_{\mu} \sigma^2_{\mu} m_n^2 + \sigma^2_{\mu} \sigma^2_{\mu} m_n^1 \neq 0\) or \(m_n^1 \neq m_n^2\). However the value of \(n\) is

\[
n\bar{x}_n = \frac{m(m_n^2 - m_n^1)\sigma^2_{\mu}}{(m_n^1 - m_n^2)\sigma^2_{\mu}}
\]

\[
= -\frac{m}{\sigma^2_{\mu}}
\]

and

\[
y = -\frac{\sigma^2}{\sigma^2_{\mu}}.
\]

The value of \(n\) obtained is not acceptable because it’s negative. Now the expressions for \(m_n^1\) and \(m_n^2\) yield two values for \(\bar{x}_n\). The values are:

\[
\bar{x}_n^1 = \frac{m_n^1(\sigma^2 + n\sigma^2_{\mu}) - m\sigma^2}{n\sigma^2_{\mu}}
\]

and

\[
\bar{x}_n^2 = \frac{m_n^2(\sigma^2 + n\sigma^2_{\mu}) - m\sigma^2}{n\sigma^2_{\mu}}.
\]

For a sample mean \(\bar{x}\), if

\[
\bar{x}_n^1 \leq \bar{x} \leq \bar{x}_n^2
\]

accept the lot otherwise reject it.

Defining a function \(f(m)\) by the expression

\[
f(m) = \Phi(L - m)/\sqrt{\sigma^2 + \sigma^2_n} + 1 - \Phi(U - m)/\sqrt{\sigma^2 + \sigma^2_n}.
\]

For \(f(m)\) to be a min, set \(f'(m) = 0\). This yields:

\[
e^{-\frac{1}{2} \frac{(L-m)^2}{\sigma^2 + \sigma^2_n}} - e^{-\frac{1}{2} \frac{(U-m)^2}{\sigma^2 + \sigma^2_n}} = 0.
\]
Therefore \((L - m)^2 = (U - m)^2\) or \(L - m = U - m\) implies \(m = \frac{L+U}{2}\). The expression of \(f(m)\) is reduced to the form:

\[
f(\frac{L+U}{2}) = \Phi\left(\frac{L - \frac{L+U}{2}}{\sqrt{\sigma^2 + \sigma_n^2}}\right) - \Phi\left(\frac{U - \frac{L+U}{2}}{\sqrt{\sigma^2 + \sigma_n^2}}\right) + 1 \tag{3.26}
\]

or

\[
f(\frac{L+U}{2}) = \Phi(L - U)/2\sqrt{\sigma^2 + \sigma_n^2} - \Phi(U - L)/2\sqrt{\sigma^2 + \sigma_n^2} + 1. \tag{3.27}
\]

Letting \(z = (U - L)/2\sqrt{\sigma^2 + \sigma_n^2}\), then

\[
f(\frac{L+U}{2}) = 2(1 - \Phi(z)) = 2\Phi(z). \tag{3.28}
\]

It is easy to verify that \(f(\frac{L+U}{2})\) is less than \(f(m_n^1)\) and \(f(m_n^2)\) where \(m_n^1\) and \(m_n^2\) are defined by expressions (3.14) and (3.15) respectively. Employing expression (3.28) expression (3.13), (3.11) and (3.12) yield:

\[
2\Phi(-z) \leq \frac{k_r}{k_A} < 1. \tag{3.29}
\]

As \(n \to \infty, \sigma_n^2 \to 0\). In this case \(z\) approach \(\frac{U-L}{2\sigma}\), and 100% inspection is required to satisfy the acceptance procedure. An additional constraint involving the variance of \(z\) is:

\[
\sigma^2 + \sigma_n^2 = \sigma^2 + \frac{\sigma_n^2}{\sigma^2 + n\sigma_n^2} = \sum^2 = \sigma^2\left(1 + \frac{\sigma_n^2}{\sigma^2 + n\sigma_n^2}\right) \tag{3.30}
\]

where \(\sum^2\) is defined to be equal to \(\sigma^2 + \sigma_n^2\). If

\[
\sigma_n^2 \to 0 \quad \sum^2 \to \sigma^2
\]

if

\[
\sigma_n^2 \to \infty \quad \sum^2 \to \sigma^2\left(1 + \frac{1}{n}\right). \tag{3.31}
\]
then

\[ \sigma^2 < \sum_{i=2}^{2} \sigma^2 (1 + \frac{1}{n}) \]. \quad (3.32)

### 3.5. Cost Equations for Destructive Sampling Plan

In formulating an expression of the cost for a destructive sampling plan by variables for fraction defective the following assumptions are made:

1) The distributions of the quality characteristic \( X \) and the lot mean \( \mu \) are normal.

2) Inspection of each lot is conducted relative to a single quality characteristic.

3) The items inspected and tested relative to a certain quality characteristic are assumed to be unfit for further testing.

4) In rejecting an inspection lot the items of the lot must be scrapped.

The expected cost is given by:

\[
k(n, m^1_n, m^2_n) = \frac{n(k_x + k_r)}{N - n} + \int_{m^1_n}^{m^2_n} \mathbb{E}(k'_1|m_n)k(m_n)dm_n
\]

\[
+ \int_{-\infty}^{m^1_n} \mathbb{E}(k'_2|m_n)k(m_n)dm_n
\]

\[
+ \int_{m^2_n}^{\infty} \mathbb{E}(k'_2|m_n)k(m_n)dm_n
\] 

where \( m^1_n \) and \( m^2_n \) are given by expressions (3.21) and (3.22). The last two integrals of (3.33) can be converted to the form:

\[
\int_{-\infty}^{m^1_n} \mathbb{E}(k'_2|m_n)k(m_n)dm_n + \int_{m^1_n}^{\infty} \mathbb{E}(k'_2|m_n)k(m_n)dm_n
\]

\[
= \int_{-\infty}^{m^1_n} k_r k(m_n)dm_n + \int_{m^1_n}^{\infty} k_r k(m_n)dm_n \quad (3.34)
\]

\[
= k_r \left[ 1 - \int_{m^1_n}^{m^2_n} k(m_n)dm_n \right].
\]
Substituting expression (3.34) into expression (3.33) yields:

\[ k(n, m_n^1, m_n^2) = \frac{n(k_s + k_r)}{N - n} + \int_{m_n^1}^{m_n^2} E(k'_1|m_n) k(m_n) dm_n \]

\[ + k_r \left[ 1 - \int_{m_n^1}^{m_n^2} k(m_n) dm_n \right] \]

\[ = \frac{n(k_s + k_r)}{N - n} + k_r + \int_{m_n^1}^{m_n^2} \left[ E(k'_1|m_n) - k_r \right] k(m_n) dm_n \]

\[ = \frac{n(k_s + k_r) + (N - n)k_r}{N - n} + \int_{m_n^1}^{m_n^2} \left[ E(k'_1|m_n) - k_r \right] k(m_n) dm_n \]

\[ = \frac{nk_s + Nk_r}{N - n} + \int_{m_n^1}^{m_n^2} \left[ E(k'_1|m_n) - k_r \right] k(m_n) dm_n. \]  

(3.35)

The p.d.f. for \( m_n \) can be obtained by using the transformation

\[ k(m_n) = T(\bar{x}_n) \left| \frac{d(\bar{x}_n)}{dm_n} \right|. \]  

(3.36)

The quantities \( \frac{d\bar{x}_n}{dm_n} \) and \( \bar{x}_n - m \) are evaluated from (3.21) and (3.22) as follows:

\[ \frac{d\bar{x}_n}{dm_n} = \frac{\sigma^2 + n\sigma^2_\mu}{n\sigma^2_\mu} = \frac{\delta^2_n + \sigma^2_\mu}{\sigma^2_\mu} \]  

(3.37)

and

\[ \bar{x}_n - m = \frac{m_n(\delta^2_n + \sigma^2_\mu)}{\sigma^2_\mu} - \frac{m\delta^2_n}{\sigma^2_\mu} - m. \]  

(3.38)

After simplification expression (3.38) can be written as

\[ \bar{x}_n - m = \frac{(\delta^2_n + \sigma^2_\mu)(m_n - m)}{\sigma^2_\mu}. \]  

(3.39)

employing (3.36), (3.37) and (3.38) yields the p.d.f. of \( m_n \)

\[ k(m_n) = \frac{1}{\sqrt{2\pi}\sqrt{\sigma^2_\mu + \delta^2_n}} e^{-\frac{(m_n-m)^2}{2\left(\sigma^2_\mu + \delta^2_n\right)}} \cdot \frac{\delta^2_n + \sigma^2_\mu}{\sigma^2_\mu}. \]  

(3.40)
Expression (3.40) can be written in the form

\[ k(m_n) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma^p_2} e^{-\frac{1}{2} \frac{(m_n - m)^2}{\sigma^2 + \sigma^2_p}}. \] (3.41)

Expression (3.41) shows that \( m_n \) is normally distributed with mean \( m \) and variance \( \frac{\sigma^4_p}{\delta^2_n + \sigma^2_n} \).

Setting

\[ \frac{\sigma^4_p}{\delta^2_n + \sigma^2_n} = \sum_n^2 \] (3.42)

then

\[ \int_{m_1^m}^{m_n^2} k(m_n)dm_n = \int_{m_1^m}^{m_n^2} \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma^p_2} e^{-\frac{1}{2} \frac{(m_n - m)^2}{\sum_n^2}} dm_n \]

\[ = \phi \left( \frac{m_n^2 - m}{\sum_n^2} \right) - \phi \left( \frac{m_n^1 - m}{\sum_n} \right). \] (3.43)

The expected posterior cost associated with acceptance given \( m_n \) is:

\[ \int_{m_1^m}^{m_n^2} E(k_1 | m_n)dm_n = k_A \int_{m_1^m}^{m_n^2} f(m_n)k(m_n)dm_n \] (3.44)

where

\[ f(m_n) = 1 + \phi \left( L - m_n / \sqrt{\sigma^2 + \sigma^2_n} \right) - \phi \left( U - m_n / \sqrt{\sigma^2 + \sigma^2_n} \right) \]

\[ = 1 - \int_L^U \frac{1}{\sqrt{2\pi} \sqrt{\sigma^2 + \sigma^2_n}} e^{-\frac{1}{2} \left( x - m_n / \sqrt{\sigma^2 + \sigma^2_n} \right)^2} dx. \] (3.45)

Substituting (3.45) into (3.44) yields:

\[ k_A \int_{m_1^m}^{m_n^2} \left[ 1 - \int_L^U 1/\sqrt{2\pi} \sqrt{\sigma^2 + \sigma^2_n} e^{-\frac{1}{2} \left( x - m_n / \sqrt{\sigma^2 + \sigma^2_n} \right)^2} \right] k(m_n)dx dm_n \]

or

\[ k_A \int_{m_1^m}^{m_n^2} \left[ 1 - \int_L^U L(x)dx \right] k(m_n)dm_n = k_A \int_{m_1^m}^{m_n^2} k(m_n)dm_n \]

\[ - k_A \int_{m_1^m}^{m_n^2} \int_L^U L(x)k(m_n)dx dm_n \] (3.46)
where
\[
L(x) = \frac{1}{\sqrt{2\pi} \sqrt{\sigma^2 + \sigma_n^2}} e^{-\frac{1}{2} \left( \frac{(x-m_n)^2}{\sigma^2 + \sigma_n^2} \right)}. \tag{3.47}
\]

A simplified form for expression (3.46) is given by
\[
k_A \left[ \phi \left( \frac{m_n^2 - m}{\sigma_n} \right) - \Phi \left( \frac{m_n^1 - m}{\sigma_n} \right) \right] - k_A \int_{m_n^1}^{m_n^2} L(x) k(m_n) dm_n. \tag{3.48}
\]

Employing expressions (3.40) and (3.47) to evaluate expression (3.48) yield:
\[
L(x) k(m_n) = \frac{1}{2\pi \Sigma_n} \sqrt{\sigma^2 + \sigma_n^2} e^{-\frac{1}{2} \left[ (x-m_n)^2 / (\sigma^2 + \sigma_n^2) \right]}. \tag{3.49}
\]

It is easy to verify that expression (3.48) is in the form of a bivariate normal distribution. Now the cost equation given by expression (3.33) can be written as:
\[
k(n, m_n^1, m_n^2) = \frac{n k_s + N k_r}{N-n} + k_A \left[ \phi \left( \frac{m_n^2 - m}{\Sigma_n} \right) - \Phi \left( \frac{m_n^1 - m}{\Sigma_n} \right) \right] \\
- k_r \left[ \phi \left( \frac{m_n^2 - m}{\Sigma_n} \right) - \Phi \left( \frac{m_n^1 - m}{\Sigma_n} \right) \right] \\
- k_A \int_{m_n^1}^{m_n^2} \int_L^U L(x) k(m_n) dx dm_n. \tag{3.50}
\]

All the parameters identified in (3.50) are defined in this chapter.

If the lot size is infinite then \( n \to \infty \). If the lot size is finite then \( n \to N - 1 \). In all cases there exists an \( n_{\text{min}} \) such that
\[
1 \leq n_{\text{min}} \leq N - 1. \tag{3.51}
\]

Expression (2.108) yields:
\[
\sigma_{N-1}^2 = \frac{\sigma_s^2 \sigma_n^2}{\sigma^2 + (N-1) \sigma_n^2}. \tag{3.52}
\]
Multiplying both sides of the inequality defined by expression (3.29) by \( k_A \) and employing the definition of \( Z \) given by expression (3.28) yield:

\[
2 \left[ 1 - \phi(U - L)/2\sqrt{\sigma^2 + \sigma_n^2} \right] k_A < k_r < k_A. \tag{3.53}
\]

Expression (3.53) can be written as

\[
\Phi\left( \frac{(U - L)}{2\sqrt{\sigma^2 + \sigma_n^2}} \right) \geq 1 - \frac{k_r}{2k_A}. \tag{3.54}
\]

To estimate a minimum value for the sample size \( n \), define the quantity by the right hand side of expression (3.54) by \( P_{zn} \), that is

\[
1 - \frac{k_r}{2k_A} = P_{zn}, \tag{3.55}
\]

where \( P_{zn} \) is the probability corresponding to a value of \( z, \ z_n \) then expression (3.54) yields

\[
(U - L)/2\sqrt{\sigma^2 + \sigma_n^2} \geq z_n. \tag{3.56}
\]

Expression (3.56) yields:

\[
\sigma^2 + \sigma_n^2 \leq \frac{(U - L)^2}{4P_{zn}^2}. \tag{3.57}
\]

Substituting \( \sigma_n^2 \) by its value given in (2.108) provides:

\[
\sigma^2 + \frac{\sigma^2 \sigma_n^2}{\sigma^2 + \sigma_n^2} = \left( \frac{U - L}{2P_{zn}} \right)^2. \tag{3.58}
\]

Expression (3.58) yields a minimum value of \( n \) given by:

\[
n \geq \frac{\sigma^2}{(U - L)/2z_n} - \sigma^2 \sigma_n^2 \tag{3.59}
\]

where

\[
\Phi(z_n) = 1 - \frac{k_r}{2k_A}. \tag{3.60}
\]

3.4.1. Step by step procedure for cost estimation.
(1) Select a value for \( n = n_{\text{min}} + 1 \) and find the points \( m_n^1 \) and \( m_n^2 \).

(2) Calculate \( k(n, m_n^1, m_n^2) \).

(3) Increase \( n \) by 1 and find another set of points \( m_{n+1}^1 \) and \( m_{n+1}^2 \), then find \( k(n+1, m_{n+1}^1, m_{n+1}^2) \).

(4) Continue this procedure until a maximum value for \( n \), \( n_{\text{max}} = n_0 \) is found such that:

\[
k(n_0 - 1, m_{n_0-1}^1, m_{n_0-1}^2) < k(n_0, m_{n_0}^1, m_{n_0}^2). \tag{3.61}
\]

The first and second derivative \( f'(m) \) and \( f''(m) \) of \( f(m) \) given by expression (3.24) are, respectively:

\[
f'(m) = \frac{1}{\sqrt{2\pi} (\sigma^2 + \sigma_n^2)} \left[ e^{-\frac{1}{2} \frac{(m - m_n)^2}{\sigma^2 + \sigma_n^2}} - e^{-\frac{1}{2} \frac{(m - m_n)^2}{\sigma^2 + \sigma_n^2}} \right] \tag{3.62}
\]

\[
f''(m) = -\frac{1}{\sqrt{2\pi} (\sigma^2 + \sigma_n^2)^{5/2}} \left[(L - m_n)e^{-\frac{1}{2} (L - m_n)^2 \sigma^2 + \sigma_n^2} - (U - m_n)e^{-\frac{1}{2} (U - m_n)^2 \sigma^2 + \sigma_n^2}\right]. \tag{3.63}
\]

It is verified that the minimum value of \( m_n, m_0 \) is \( \frac{L+U}{2} \). Substituting the value of \( m_n = \frac{L+U}{2} \) into (3.63) yields a positive value for \( f''(m) \). As a conclusion this value of \( m_n \) minimizes (3.24).

In general the two values of \( m_n, m_n^1 \) and \( m_n^2 \) symmetrical about \( m_0 \).

To verify the validity of this condition consider the following transformations

\[
m_n^1 = m_0 - v \quad m_n^2 = m_0 + v \tag{3.64}
\]

where \( m_0 = \frac{L+U}{2} \). Let \( v \) be the absolute difference between \( m_n \) and \( m_0 \). Substituting (3.64 into (3.24) yields:

\[
f(m_0 - v) = f(-v) = 1 + \Phi \left( \frac{L - U}{2\sqrt{\sigma^2 + \sigma_n^2}} + \frac{v}{\sqrt{\sigma^2 + \sigma_n^2}} \right) - \Phi \left( \frac{U - L}{2\sqrt{\sigma^2 + \sigma_n^2}} + \frac{v}{\sqrt{\sigma^2 + \sigma_n^2}} \right) \tag{3.65}
\]
and
\[ f(m_0 + v) = f(v) = 1 + \Phi \left( \frac{L - U}{2\sqrt{\sigma^2 + \sigma_n^2}} - \frac{v}{\sqrt{\sigma^2 + \sigma_n^2}} \right) - \Phi \left( \frac{U - L}{2\sqrt{\sigma^2 + \sigma_n^2}} - \frac{v}{\sqrt{\sigma^2 + \sigma_n^2}} \right). \] (3.66)

Letting
\[ z = \frac{U - L}{2\sqrt{\sigma^2 + \sigma_n^2}} \text{ and } g(v) = \frac{v}{\sqrt{\sigma^2 + \sigma_n^2}}, \] (3.67)

then expression (3.65) will be reduced to:
\[ f(-v) = 1 + \Phi(-z + g(v)) - \Phi(z + g(v)) \]
or
\[ f(-v) = 1 + 1 - \Phi(z - g(v)) - \Phi(z + g(v)) \] (3.68)
\[ f(-v) = 2 - \Phi(z - g(v)) - \Phi(z + g(v)). \]

Similarly (3.66) becomes
\[ f(v) = 1 + \Phi(-z - g(v)) - \Phi(z - g(v)) \]
\[ f(v) = 1 + 1 - \Phi(z + g(v)) - \Phi(z - g(v)) \] (3.69)
\[ f(v) = 2 - \Phi(z - g(v)) - \Phi(z + g(v)). \]

Comparing expression (3.68) and (3.69) leads to the conclusion that:
\[ f(v) = f(-v). \] (3.70)

Stated differently there are two values of \( m_n \) symmetric about the mean \( m_0 \).

3.6. Expected Sampling Cost Per Unit

The expected sampling cost consists of the following components: The cost of sampling and inspecting \( n \) units, the cost of rejecting effective units in a sample of size \( n \), and the cost of rejecting defective units in a sample of size...
The expected sampling cost per unit $C_{pu}$, assuming destructive sampling is then

$$C_{pu} = \frac{nk_s + nk_r}{N - n}$$  \hspace{1cm} (3.71)

or

$$C_{pu} = \frac{n}{N - n}(k_s + k_r).$$  \hspace{1cm} (3.72)

### 3.7. Operating Characteristic Curve (OC)

The following procedures allow the construction of the OC curve.

1. Employ $\bar{x}_{n}^{1*}$, $\bar{x}_{n}^{2*}$ given in (3.21) and (3.22) and also $n$ corresponding to the minimum posterior cost.

2. Use the distribution of $T(\bar{x}_n|\mu)$ given in (2.70).

3. Specify two constants $\alpha_0$, $\beta_0$ defined as:

   for $H_0: \mu = \mu_1$ accept $H_0$ at least $(1 - \alpha_0\%)$ of the time

   for $H_1: \mu = \mu_2 > \mu_1$ accept $H_0$ no more than $\beta\%$ of the time.  \hspace{1cm} (3.73)

The above null and alternative hypotheses lead to the requirements:

$$p(\bar{x}_n^{1*} < \bar{x}_n < \bar{x}_n^{2*}|\mu_1) = \int_{\bar{x}_n^{1*}}^{\bar{x}_n^{2*}} T(\bar{x}_n|\mu_1) d\bar{x} \geq 1 - \alpha_0$$  \hspace{1cm} (3.74)

$$p(\bar{x}_n^{1*} < \bar{x}_n < \bar{x}_n^{2*}|\mu_2) = \int_{\bar{x}_n^{1*}}^{\bar{x}_n^{2*}} T(\bar{x}_n|\mu_2) d\bar{x} \leq \beta_0. \hspace{1cm} (3.75)$$

The condition given in (3.74) and (3.75) can be written as

$$\Phi\left(\frac{\bar{x}_n^{2*} - \mu_1}{\sigma/\sqrt{n}}\right) - \Phi\left(\frac{\bar{x}_n^{1*} - \mu_1}{\sigma/\sqrt{n}}\right) \geq 1 - \alpha_0$$  \hspace{1cm} (3.76)

and

$$\Phi\left(\frac{\bar{x}_n^{2*} - \mu_2}{\sigma/\sqrt{n}}\right) - \Phi\left(\frac{\bar{x}_n^{1*} - \mu_2}{\sigma/\sqrt{n}}\right) \leq \beta_0. \hspace{1cm} (3.77)$$
The parameters $\mu_1$ and $\mu_2$ can be obtained by solving (3.76) and (3.77) respectively. Two options are possible for plotting the OC curve.

(a) Plot $\mu_1$ and $\mu_2$ vs $1 - \alpha_0$ and $\beta_0$ respectively.

(b) Plot $p(\mu_1) = p_1$ and $p(\mu_2) = p_2$ vs $(1 - \alpha_0)$ and $\beta_0$ respectively.

In the first option the means are plotted against the acceptance probabilities. In the second option the probabilities of material function defectives are plotted against the acceptance probabilities. This permits interpolation between principal values of the OC curve.

Interpolating more OC curve points requires selection of different $\mu$ values greater than $\mu_1$. These are then used to evaluate the corresponding probabilities.

The cost model developed in this chapter assumed all statistical and economic parameters are error free. A computer program written in FORTRAN 77 (Appendix C) computes the decision points, the optimal sample size for which the total cost is optimal, the total cost per unit, the sampling cost per unit and the conditional expectation of the material fraction defective given the results of a sample.

The following is an example of the computer output.
### TABLE 1: PRIOR COSTS FOR A VARIABLE SAMPLING PLAN

**INPUT**: MODEL SPECIFICATIONS

<table>
<thead>
<tr>
<th></th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>UPPER LIMIT OF THE OC.X</td>
<td>11.00000</td>
</tr>
<tr>
<td>LOWER LIMIT OF THE OC.X</td>
<td>6.00000</td>
</tr>
<tr>
<td>VARIANCE OF X</td>
<td>2.00000</td>
</tr>
<tr>
<td>VARIANCE OF THE MEAN OF X</td>
<td>1.00000</td>
</tr>
<tr>
<td>THE COST OF REJECTING A UNIT</td>
<td>0.7</td>
</tr>
<tr>
<td>THE COST OF ACCEPTING A DEFECTIVE UNIT</td>
<td>2.00000</td>
</tr>
<tr>
<td>COST OF SAMPLING PER UNIT</td>
<td>2.00000</td>
</tr>
<tr>
<td>LOT SIZE</td>
<td>1000</td>
</tr>
<tr>
<td>SAMPLING ERROR (TYPE1) KE1</td>
<td>0.60000</td>
</tr>
<tr>
<td>SAMPLING ERROR (TYPE2) KE2</td>
<td>0.50000</td>
</tr>
<tr>
<td>SALE PRICE PER UNIT</td>
<td>2.00000</td>
</tr>
<tr>
<td>RATE OF REWORKING DEFECTIVE ITEMS WITH SUCCESS KC</td>
<td>0.00000</td>
</tr>
<tr>
<td>PROPORTION OF DEFECTIVE ITEMS REWORKED WITHOUT SUCCESS KY</td>
<td>0.00000</td>
</tr>
</tbody>
</table>

**OUTPUT**: EXPECTED PRIOR COSTS PER UNIT

- **EXPECTED COST PER UNIT-ACCEPT OUTRIGHT WITHOUT SAMPLING E(C1)**: 0.25466
- **COST PER UNIT-REJECT OUTRIGHT AND SCRAP C2**: 1.30000
- **E(C1) = C2 AT P01**: 0.26500
### TABLE 1: A VARIABLE PLAN FOR THE COST MODEL

**INPUT: MODEL SPECIFICATIONS**

<table>
<thead>
<tr>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Upper limit of the O.C.X:</td>
<td>11.00000</td>
</tr>
<tr>
<td>Lower limit of the O.C.X:</td>
<td>6.00000</td>
</tr>
<tr>
<td>Variance of X:</td>
<td>2.00000</td>
</tr>
<tr>
<td>Variance of the mean of X:</td>
<td>1.00000</td>
</tr>
<tr>
<td>Unit cost of rejection of the remainder of the lot:</td>
<td>0.70000</td>
</tr>
<tr>
<td>Unit cost of acceptance:</td>
<td>2.00000</td>
</tr>
<tr>
<td>Unit cost of sampling:</td>
<td>2.00000</td>
</tr>
<tr>
<td>Lot size:</td>
<td>1000</td>
</tr>
<tr>
<td>Acceptance cost:</td>
<td></td>
</tr>
</tbody>
</table>

**OUTPUT: SAMPLE SIZE, ROOTS OF THE COST FUNCTION,**

**POSTERIOR AND SAMPLING COSTS PER UNIT**

<table>
<thead>
<tr>
<th>COL. No.</th>
<th>DESCRIPTION</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>SAMPLE SIZE</td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

\[
\frac{(a \cdot b)}{(b + n \cdot a)} \quad \text{WHERE:}
\]

- \( a \): Variance of the mean of X
- \( b \): Variance of X
- \( n \): Sample size

| 3        | 1st ROOT OF THE COST EQ. |
| 4        | 2nd ROOT OF THE COST EQ. |
| 5        | SAMPLING COST PER UNIT  |
| 6        | POSTERIOR COST PER UNIT |
| 7        | ACCEPTANCE COST PER UNIT|
| 8        | REJECTION COST PER UNIT |

<table>
<thead>
<tr>
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<th>Col.1</th>
<th>Col.2</th>
<th>Col.3</th>
<th>Col.4</th>
<th>Col.5</th>
<th>Col.6</th>
<th>Col.7</th>
<th>Col.8</th>
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<tr>
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<td>6.57466</td>
<td>10.42534</td>
<td>0.02452</td>
<td>2.00032</td>
<td>0.01939</td>
<td>1.98096</td>
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</tr>
<tr>
<td>4</td>
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<td>6.57245</td>
<td>10.42755</td>
<td>0.02727</td>
<td>2.00034</td>
<td>0.02158</td>
<td>1.97876</td>
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</tr>
<tr>
<td>5</td>
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<td>0.03003</td>
<td>2.00066</td>
<td>0.02360</td>
<td>1.97705</td>
<td></td>
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</tbody>
</table>
### OUTPUT2: AVERAGE PROCESS FRACTION DEFECTIVES AND THE SAMPLE MEANS

<table>
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<tr>
<th>COL. No.</th>
<th>DESCRIPTION</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>SAMPLE SIZE</td>
</tr>
<tr>
<td>2</td>
<td>LOWER LIMIT OF THE SAMPLE MEAN</td>
</tr>
<tr>
<td>3</td>
<td>UPPER LIMIT OF THE SAMPLE MEAN</td>
</tr>
<tr>
<td>4</td>
<td>AVERAGE FRACTION DEFECTIVE OF THE PROCESS GIVEN THE LOWER LIMIT OF THE SAMPLE MEAN</td>
</tr>
<tr>
<td>5</td>
<td>AVERAGE FRACTION DEFECTIVE OF THE PROCESS GIVEN THE UPPER LIMIT OF THE SAMPLE MEAN</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>No.</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
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<tbody>
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<tr>
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<td>6.16694</td>
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<td>6.21978</td>
<td>10.78022</td>
<td>0.65001</td>
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</tbody>
</table>
Figure 3. The Sampling Cost Per Unit Vs. the Sample Size for a Destructive Variable Sampling Plan for Fraction Defective for a Lot Size of 1000.
Figure 4. The Sampling Cost Per Unit Vs. the Sample Size for a Destructive Variable Sampling Plan for Fraction Defective for a Lot of Size 35.
Figure 5. The Acceptance Cost Per Unit vs. the Sample Size for a Defective Variable Sampling Plan for Fraction Defective.
Figure 6. The Rejection Cost Per Unit vs. the Sample Size for a Destructive Variable Sampling Plan for Fraction Defective.
Figure 7. The Posterior Cost Per Unit Vs. the Sample Size for a Destructive Variable Sampling Plan for Fraction Defective.
CHAPTER 4

OPTIMUM ECONOMIC COST MODEL
NON DESTRUCTIVE TESTING

4.1. Introduction

In non destructive as in destructive testing the economic evaluation of the acceptance procedure requires the identification of all possible actions (decisions) for which the producer’s prior and posterior costs functions are evaluated. In this chapter the following actions are considered.

A. Prior to Sampling (Decision Without Sampling)
   1. Accept outright
   2. Reject outright
      a. Scrap
      b. 100 percent screening (inspect the lot and eliminate defective items found)

B. After Sampling Decisions
   1. Accept the lot
   2. Reject the lot

4.2. Prior Cost Analysis

Information on the following economic parameters is required to compute the prior and posterior costs and profits.

\[ K_R \] – sales price of an item
\[ K_p \] – production cost of an item
The following are the prior costs per unit from the producer's point of view relative to each decision without sampling.
A. Accept Outright

Cost per item, $C_1$ for a decision to accept the lot without sampling is given by:

$$C_1 = K_A p$$

where $p$ is the fraction of items which are defective. The p.d.f. of $C_1$ is derived from the prior p.d.f. of $p$ given by expression (2.30)

$$W(p) = \frac{l'(p)}{\sqrt{2\pi} \sigma} e^{-\frac{1}{2} \left( \frac{L_m - l(p)}{\sigma} \right)^2} \quad 0 < p < 1.$$  \hspace{1cm} (4.2)

The expression of $l(p)$ is given by (2.23). Expression (4.1) yields

$$p = \frac{C_1}{K_A}.$$  \hspace{1cm} (4.3)

An expression of the p.d.f. of $C_1$ can be obtained by employing expressions (4.2) and (4.3) as:

$$f(C_1) = \frac{1}{K_A \sqrt{2\pi \sigma} e} \frac{l'(\frac{C_1}{K_A})}{\sqrt{2\pi} \sigma} e^{-\frac{1}{2} \left( \frac{L_m - l(\frac{C_1}{K_A})}{\sigma} \right)^2} \quad C_1 < K_A.$$  \hspace{1cm} (4.4)

The expected value of $C_1$, $E(C_1)$ is:

$$E(C_1) = \int_0^{K_A} f(C_1) dC_1.$$  \hspace{1cm} (4.5)

The variance of $(C_1)$ can be obtained by employing the expression:

$$V(C_1) = E(C_1^2) - [E(C_1)]^2.$$  \hspace{1cm} (4.6)

Computing the expected value of $C_1$ and its variance, requires the application of numerical integration since no closed forms are possible. The profit per item $P_1$, for a decision to accept the lot without sampling is given by:

$$P_1 = K_R - K_p - K_A p.$$  \hspace{1cm} (4.7)
Expression (4.7) shows that $P_1$ is a random variable whose p.d.f. can be obtained from that of $p$ given by expression (4.2) by following the same procedure as above.

B. Reject Outright and Scrap

The cost per unit $C_2$ for a decision to reject outright and to scrap the whole lot is:

$$C_2 = K_R - K_J. \quad (4.8)$$

For this decision the cost per unit is not a random variable. The profit per item $P_2$, for a decision to reject outright and to scrap the whole lot is

$$P_2 = -K_p + K_J. \quad (4.9)$$

For this decision the profit per unit is not a random variable.

C. Reject Outright and Screen

The cost per lot for the decision to reject outright and screen the entire lot consists of four parts:

1. The cost due to the defective items found during inspection and then scrapped is:

$$Np(1 - K_{E2})(K_R - K_J). \quad (4.10)$$

The profit per lot in this case is

$$Np(1 - K_{E2})(K_J - K_p) \quad (4.11)$$

2. The cost per lot resulting from defective items classified as good during inspection is:

$$NpK_{E2}(K_A). \quad (4.12)$$

The profit in this case

$$NpK_{E2}(K_R - K_p - K_A). \quad (4.13)$$
3. The cost per lot resulting from good items classified as defective during inspection and then scrapped is:

\[ N(1 - p)K_{E1}(K_R - K_J). \]  \hspace{1cm} (4.14)

The profit per lot in this case is

\[ N(1 - p)K_{E1}(K_J - K_p). \]  \hspace{1cm} (4.15)

In this case no cost per lot resulted from good items found during inspection while the profit per lot is

\[ N(1 - p)(1 - K_{E1})(K_R - K_p). \]  \hspace{1cm} (4.16)

4. Cost of inspection is

\[ NK_I. \]  \hspace{1cm} (4.17)

The total cost per lot is obtained by adding expressions (4.10), (4.12), (4.14) and (4.17). Dividing by \( N \) and arranging the terms leads to the following expression for the producer's cost per unit for this decision:

\[
C_3 = p\left[(1 - K_{E1})(K_R - K_J) + K_{E2}(K_A - K_R + K_J) + K_{E1}(K_R - K_J) + K_2\right].
\]  \hspace{1cm} (4.18)

Solving the above equation for \( P \) yields

\[
p = \frac{C_3 - [K_{E1}(K_R - K_J) + K_I]}{(1 - K_{E1})(K_R - K_J) + K_{E2}(K_A - K_R + K_J)^*}.
\]  \hspace{1cm} (4.19)

The profit per unit for this decision is obtained by subtracting expression (4.17) from the sum of expressions (4.11), (4.13), (4.15) and (4.16), which yields:

\[
P_3 = K_R(1 - K_{E1}) + K_J \cdot K_{E1} - K_p - K_I
\]
\[
+ [(1 - K_{E1} - K_{E2})(K_J - K_R) - K_A K_{E2}]p.
\]  \hspace{1cm} (4.20)
The p.d.f. of $C_3$ can be derived by employing expression (4.19) and the p.d.f. of $p$ given by expression (4.2). Hence:

$$F(C_3) = \frac{1}{(1 - K E_1)(K_R - K_J) + K E_2(K_A - K_R + K_J)} \cdot \frac{-1}{2} [\frac{K}{p^2} - l(p)]^2$$

$$K E_1(K_R - K_J) + K_I < C_3 < (K_R - K_J)(1 - K E_2) + K E_2K_A + K_I$$

(4.21)

where $p$ is given by expression (4.19). A similar expression of (4.21) for the p.d.f. of $P_3$ can be obtained easily.

D. Decision Points

These are the points obtained when the cost (or profit) relative to the decision of acceptance is equivalent to the cost (or profit) relative to the decision of rejection.

1. For the decision of accepting outright or rejecting outright and scrapping the inspection lot, expressions (4.1) and (4.8) are equated yielding:

$$p_{01} = \frac{K_R - K_J}{K_A}.$$  \hspace{1cm} (4.22)

Accept outright if $p < p_{01}$ and reject outright and scrap if $p > p_{01}$.

Equating expressions (4.7) and (4.9) yield exactly the same value for $p_0$ given by expression (4.22).

2. For the decision of accepting outright or rejecting outright and screening the inspection lot, expressions (4.1) and (4.18) are equated yielding:

$$p_{02} = \frac{K E_1(K_R - K_J) + K_I}{K_A(1 - K E_2) - (1 - K E_1 - K E_2)(K_R - K_J)}.$$  \hspace{1cm} (4.23)

The same expression for $p_{02}$ can be obtained by equating expressions (4.7) and (4.20).
4.3. Posterior Cost Analysis

In this approach variables acceptance sampling plans are employed to evaluate optimum cost parameters. An expression is formulated to compute the optimal total cost. Also expressions are derived for the expected cost of lot acceptance, expected cost of lot screening and expected cost of lot scrapping.

The following assumptions are made

1. The distributions of the quality characteristic and the lot mean are normal.
2. The items tested can be used for their intended purpose after testing them, or can be repaired or reworked.
3. Rejected lots are either screened or scrapped.
4. The process can exist in one statistical state.

Information on the following economic parameters and distributions in addition to those defined in Chapter 3 is required to compute the optimum costs, namely: \( x, n, N, L, U, \bar{x}, \alpha, \mu, \sigma^2, f(x|\mu), T(\bar{X}_n|\mu), t(x|\bar{x}, \mu), h(\alpha), L_A, U_A, L_{sn}, U_{sn}, K_f, K_{sn}, K_{sp}, K_a, K_p \). Employing a given acceptance sampling plan, leads to two categories of the cost.

1. The cost of reaching a decision. This is the cost of selecting and inspecting a random sample from a given lot.
2. The cost of implementing the decision that depends on the action taken to dispose the lot.

The disposition procedure is shown in Chart 1. The decision to accept or reject a lot, depends on the sample mean \( \bar{x} \). The sample mean fails in some cases to provide perfect information necessary for lot disposition. Random errors affects can be reduced by increasing the sample size. Large samples cause an increase in cost. In this work different sample sizes are selected to compute the cost for each sample. By comparing the costs it is possible to discover the fluctuations
Chart 1. A Disposition Chart for Non Destructive Variable Sampling Plan for Fraction Defective.
if any in the model due to errors.

4.3.1. Disposition Chart

A disposition chart based on an actual mean product dimension (μ) is considered for illustration. It is logical to assume that the mean μ can take four different values depending on the action taken to dispose of the lot.Procedurely, select at random a sample of size n from an inspection lot. Based on the sample it is possible to draw the following conclusions.

1. If

\[ L_A < \bar{x} < U_A \]  \hspace{1cm} (4.24)

accept the lot. The cost in this case is a function of the number of defective units in the accepted lot and their replacement cost.

2. If

\[ L_{sn} < \bar{x} < L_A \quad \text{or} \quad U_A < \bar{x} < U_{sn} \]  \hspace{1cm} (4.25)

screen the lot. Here the lot should be screened to isolate the defective units. In this case the cost consists of three portions:
(a) the cost of inspecting each unit in the uninspected portion of the lot;
(b) the cost of replacing defective units in the inspection process;
(c) the cost of replacing the defective units in the initial sample.

3. If

\[ \bar{x} < L_{sn} \quad \text{or} \quad \bar{x} > U_{sn} \]  \hspace{1cm} (4.26)

the lot is scrapped. This cost consists of the following
(a) the cost of each unit scrapped
(b) the cost of the scrapping process reduced by revenue of the salvaged material. From the disposition chart it is possible to deduce that the
following cases exist: If

$$\mu < \mu_1 \quad K_A > K_{SN} > K_{sp}$$  \hspace{1cm} (4.27)

If

$$\mu_1 \leq \mu \leq \mu_2 \quad K_A > K_{SN} < K_{sp}$$  \hspace{1cm} (4.28)

If

$$\mu_2 \leq \mu < \mu_3 \quad K_A < K_{SN} < K_{sp}$$  \hspace{1cm} (4.29)

If

$$\mu_3 \leq \mu \leq \mu_4 \quad K_A > K_{SN} < K_{sp}$$  \hspace{1cm} (4.30)

If

$$\mu > \mu_4 \quad K_A > K_{SN} > K_{sp}.$$  \hspace{1cm} (4.31)

4.3.2. Economic Parameters and Probabilities

The terms listed are required for cost formulation:

$K_I$ cost of sampling inspection per unit

$K_P$ cost of replacing a defective unit detected in a sample drawn from the inspected lot

$K_a$ cost of a defective unit detected in the uninspected portion of the inspection lot

$K_{a'}$ cost of a defective unit detected in the uninspected portion of the inspection lot

$K_A(X, \mu)$ cost of acceptance as a function of the sample mean $\bar{x}$ and a actual mean of product dimension $\mu$

$K_{Sn}(X, \mu)$ cost of screening as a function of $\bar{x}$ and an actual mean of product $\mu$. 
The following probability density functions are employed for estimating the total cost in addition to those listed in Chapter 3.

1. \( h(\mu) \) given in expression (2.24). In Chapter 3 one state was assumed for the statistical process. For a statistical process that can exist in more than one state, \( i = 1, 2, 3, \ldots \), define the following

- \( \sigma^2_{\mu_i} \) - variance for the lot when the process is in the state \( i \)
- \( m_i \) - the mean of the lot mean when the process is in state \( i \)
- \( p_i \) - fraction of the time the process is in state \( i \), where \( \sum_{i=1}^{s} p_i = 1 \) when a statistical process can exist in more than one state then:

\[
h(\mu) = \sum_{i=1}^{s} p_i \frac{1}{\sigma_{\mu_i} \sqrt{2\pi}} e^{-\frac{1}{2} \frac{(\mu_i - \mu)^2}{\sigma^2_{\mu_i}}} - \infty < \mu_i < +\infty \quad 0 < p_i \leq 1.
\]

2. \( T(\bar{x}|\mu) \) given in expression (2.70).

3. \( t(x|\bar{x}, \mu) \). This material is treated in detail by Lehman and Scheffe (1950).

Let \( x_1, x_2, \ldots, x_n \) be a set of independent random variables which are normally distributed. Define the attribute estimate of the fraction defective by \( \hat{p} \). \( \hat{p} \) satisfies the following properties:

(i) It is the ratio of the number of defective items in the sample to the sample size.

(ii) A more precise definition of \( \hat{p} \) is that it is the sum of identically distributed random variables with values of 0 and 1.

(iii) It is an unbiased estimator of the true proportion \( p \).

Let \( T \) be a sufficient statistic for the normal distribution of interest. Blackwell (19-) showed that

\[
\hat{P}^* = E(\hat{p}|T)
\]
is the unique uniformly minimum variance unbiased estimate of $p$. Define $\hat{p}_1$ such that for any $x_1'$ which is any one of the observations $(x_1, x_2, \cdots, x_n)$ say $x_1$ the following is true:

$$\hat{p}_1(x_1', x_2, x_3, \cdots, x_n) = 0 \quad L \leq x_1' \leq U \quad (4.34)$$

and

$$\hat{p}_1(x_1', x_2, x_3, \cdots, x_n) = 1 \quad \text{otherwise.} \quad (4.35)$$

Under the above conditions expression (4.33) is identical to:

$$\hat{p}^* = E(\hat{p}_1 | T). \quad (4.36)$$

For a normally distributed population with a known mean and variance, $T$ will be equal to $\bar{x}$ where $\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$. Now expression (4.33) can be written in the form:

$$E(\hat{p}_1 | \bar{x}) = 1 - p[L \leq x_1' \leq U | \bar{x}]. \quad (4.37)$$

Let $g(x_1', \bar{x})$ be the joint p.d.f. of $x_1'$ and $\bar{x}$. The uniformly minimum variance unbiased estimate of $p$ is $\hat{p}^*$ given by

$$\hat{p}^*(\bar{x}) = 1 - \int_{L}^{U} g(x_1', \bar{x}) dx_1'$$

or

$$\hat{p}^*(\bar{x}) = 1 - \int_{L}^{U} \frac{g(x_1' | \bar{x})}{n(\bar{x})} dx_1'. \quad (4.38)$$

Let $\bar{x}'$ be the mean of the observations other than $x_1'$ then

$$\bar{x}' = \frac{1}{n-1} \sum_{i=2}^{n} \frac{x_i}{n-1}. \quad (4.39)$$

The joint p.d.f. $g_1(x_1', \bar{x}')$ of $x_1'$ and $\bar{x}'$ is then:

$$g_1(x_1', \bar{x}') = \frac{\sqrt{n-1}}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}(x_1'-\mu)^2 + (x_1'-\mu)^2}. \quad (4.40)$$
Expression (4.39) can be reduced to

\[(n - 1)\bar{x}' = \sum_{i=1}^{n} x_i - x_1'\]

or

\[\bar{x}' = \frac{n\bar{x} - x_1'}{n - 1}.\]  \hspace{1cm} (4.41)

Employing the transformations given in (4.41) and (4.40) provides the joint p.d.f. of \(x_1'\) and \(\bar{x}\)

\[g(x_1', \bar{x}) = \frac{\sqrt{n - 1}}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}(x_1' - \mu)^2 + \frac{1}{2(n-1)}(\frac{n\bar{x} - x_1'}{n - 1} - \mu)^2}.\]  \hspace{1cm} (4.42)

Dividing (4.42) by \(h(\bar{x})\) where

\[h(\bar{x}) = \frac{\sqrt{n}}{\sqrt{2\pi}\sigma} e^{-n(\frac{\bar{x} - \mu}{\sigma})^2}\]  \hspace{1cm} (4.43)

yields

\[t(x_1' | \bar{x}, \mu) = \sqrt{\frac{n}{n - 1}} \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(n^{-1})(x_1' - \bar{x})^2}.\]  \hspace{1cm} (4.44)

Expression (4.44) shows that

\[t(x_1' | \bar{x}, \mu) = \frac{g(x_1', \bar{x})}{h(\bar{x})}\]  \hspace{1cm} (4.45)

is normally distributed with mean \(\bar{x}\), and variance \(\sigma^2(\frac{n-1}{n})\). Since \(x_1'\) can be any of the values \(x_1, x_2, \ldots x_n\), then set \(x_1' = x\), \(t(x_1' | \bar{x}, \mu)\) reduces to \(t(x | \bar{x}, \mu)\).

4.3.3. Expected Cost of Acceptance

\[K_A(\bar{x}, \mu) = K_pE(ND\bar{S}) + K_AE(ND\bar{R}) + K_{\bar{S}}n.\]  \hspace{1cm} (4.46)

The expressions for \(E(ND\bar{S})\) and \(E(ND\bar{R})\) are respectively:

\[E(ND\bar{S}) = n[1 - \int_{L}^{U} t(x | \bar{x}, \mu)dx]\]  \hspace{1cm} (4.47)
and
\[ E(NDR) = N[1 - \int_L^U f(x|\mu)dx] - n[1 - \int_L^U t(x|\overline{x}, \mu)dx]. \quad (4.48) \]

An expression for \( K_A(\overline{x}, \mu) \) is obtained by employing (4.47) and (4.48) as:
\[ K_A(\overline{x}, \mu) = K_p n[1 - \int_L^U t(x|\overline{x}, \mu)dx] \]
\[ + K_A \{ N(1 - \int_L^U f(x|\mu)dx) - n(1 - \int_L^U t(x|\overline{x}, \mu)dx) \}. \quad (4.49) \]

After simplification (4.49) is reduced to
\[ n(K_A - K_p) \int_L^U f(x|\overline{x}, \mu)dx - K_A N \int_L^U t(x|\mu)dx \]
\[ + K_A (N - n) + K_p n. \quad (4.50) \]

The total expected cost of acceptance per lot is obtained by summing over all sample means and \( \mu \). This yields
\[ K_A = \int_{-\infty}^{+\infty} \int_{L_A}^{U_A} K_A(\overline{x}, \mu) T(\overline{X}_n|\mu)h(\mu)d\overline{x}d\mu \quad L_A < \overline{x} < U_A \]
\[ K_A = 0 \quad \text{if} \quad \overline{x} \leq L_A \quad \text{or} \quad \overline{x} \geq U_A. \quad (4.51) \]

From the model discussed above it is possible to deduce the following:

1. \( E \) (number of effective units in the sample \( [\overline{x}, \mu] \))
\[ = n \int_L^U t(x|\overline{x}, \mu)dx. \quad (4.52) \]

2. \( E \) (number of effective units in the uninspected portion of the lot \( [\overline{x}, \mu] \))
\[ = N \int_L^U f(x|\mu)dx - n \int_L^U f(x|\mu)dx. \quad (4.53) \]

3. \( E \) (number of effective units in the sample and the uninspected portion of lot \( [\overline{x}, \mu] \))
\[ = n \int_L^U f(x|\overline{x}, \mu)dx + N \int_L^U f(x|\mu)dx - n \int_L^U f(x|\mu)dx \]
\[ = N \int_L^U f(x|\mu)dx. \quad (4.54) \]
The price of an effective unit is $K_R$. The revenue from selling $N \int_{L}^{U} f(x|\mu)dx$ effective units can then be given by:

$$C_R(\mu) = K_R N \int_{L}^{U} f(x|\mu)dx. \quad (4.55)$$

Now the expected net profit will be:

$$E[\text{net profit}|\bar{x}, \mu] = \int_{-\infty}^{+\infty} \int_{L}^{U} C_R(\mu) T(\bar{X}_n|\mu) h(\mu) d\bar{x} d\mu$$

$$- \int_{-\infty}^{+\infty} \int_{L}^{U} K_A(\bar{x}, \mu) T(\bar{X}_n|\mu) h(\mu) d\bar{x} d\mu$$

or

$$E[\text{net profit}|\bar{x}, \mu] = \int_{-\infty}^{+\infty} \int_{L}^{U} [K_R(\bar{x}, \mu)$$

$$- K_A(\bar{x}, \mu)] T(\bar{X}_n|\mu) h(\mu) d\bar{x} d\mu. \quad (4.57)$$

### 4.3.4. Expected Cost of Screening

The cost of screening consists of two parts:

(a) The cost of screening the uninspected portion of the inspection lot.

(b) The cost of replacing the defective items in the lot. The number of units screened is $N - n$, with a cost of $K_{sn}(N - n)$. Therefore the expected number of defectives in the lot is:

$$N[1 - \int_{L}^{U} f(x|\mu)dx].$$

The cost of screening as a function of $\bar{x}$ and $\mu$ is then

$$K_{SN}(\bar{x}, \mu) = K_{sn}(N - n) + K_p N[1 - \int_{U}^{L} f(x|\mu)dx]. \quad (4.58)$$

Summing overall means and sample means yields:

$$E(\text{total screening cost}) = \int_{-\infty}^{+\infty} \int_{L}^{U} K_{sn}(\bar{x}, \mu) t(\bar{X}_n|\mu) d\bar{x}$$

$$+ \int_{U}^{L} K_{sn}(\bar{x}, \mu) t(\bar{X}_n|\mu) d\bar{x} h(\mu) d\mu. \quad (4.59)$$
The expected number of effective items in the lot is

\[ N - N[1 - \int_L^U f(x|\mu)dx] = N \int_L^U f(x|\mu)dx. \]

The revenue from selling \( N \int_L^U f(x|\mu)dx \) effective items is

\[ K_R(x, \mu) = K_R N \int_L^U f(x|\mu)dx. \] (4.60)

The expected net profit

\[
E(\text{net profit}) = \int_{-\infty}^{+\infty} \left[ K_R N \int_L^U f(x|\mu)dx \right] h(\mu) d\mu
- \int_{-\infty}^{+\infty} \left[ \int_{L_{sn}}^{L_A} K_{sN}(\bar{x}, \mu) t(\bar{X}_n|\mu) d\bar{x} \right] h(\mu) d\mu
+ \int_{U_A}^{U_{sn}} K_{sN}(\bar{x}, \mu) t(\bar{X}_n|\mu) d\bar{x} \right] h(\mu) d\mu.
\] (4.61)

\[ L_{sn} < \bar{x} \leq L_A \quad \text{or} \quad U_A \leq \bar{x} \leq U_{sn}. \]

4.3.5. Expected Cost of Scrapping

Each unit in the sample or uninspected portion of the lot is rejected at a cost of \( K_{sp} \). The expected number of defectives in the sample for a given \( \bar{x} \) and \( \mu \) is given in (4.47). The number of units in the uninspected portion of the inspection lot is \( N - n \). Now the expected cost of scrapping per inspection lot for a given \( \bar{x} \) and \( \mu \) is \( K_{sp}(\bar{x}, \mu) \). Mathematically therefore

\[
K_{sp}(\bar{X}, \mu) = K_{sp}(N - n) + K_{sp} n \left[ 1 - \int_L^U t(x|\bar{x}, \mu) dx \right]
= K_{sp}[N - n] \int_L^U t(x|\bar{x}, \mu)] dx.
\] (4.62)
The expected cost of scrapping averaged over all samples and means, is obtained by removing the restrictions over $\bar{x}$ and $\mu$. This results in:

$$K_{sp} = \int_{-\infty}^{+\infty} \int_{-\infty}^{L_{sn}} K_{sp}(\bar{x}, \mu) t(\bar{X}_n | \mu) d\bar{x}$$

$$+ \int_{U_{sn}}^{\infty} K_{sp}(\bar{x}, \mu) t(\bar{X}_n | \mu) d\bar{x} h(\mu) d\mu$$

or

$$\bar{x} \leq L_{sn} \quad \text{or} \quad \bar{x} \geq U_{sn}.$$  (4.63)

The expected number of effective units in the sample and the uninspected portion of the lot is:

$$N \int_{L}^{U} f(x | \mu) dx.$$  (4.64)

The revenue is:

$$K_R N \int_{L}^{U} f(x | \mu) dx.$$  (4.65)

The expected net profit is

$$E(\text{net profit}) = \int_{-\infty}^{+\infty} [K_R N \int_{L}^{U} f(x | \mu) dx] d\mu$$

$$- \int_{-\infty}^{+\infty} [\int_{-\infty}^{L_{sn}} K_{sp}(\bar{X}, \mu) t(\bar{X}_n | \mu) d\bar{x}$$

$$+ \int_{U_{sn}}^{\infty} K_{sp}(\bar{X}, \mu) t(\bar{X}_n | \mu) d\bar{x} h(\mu) d\mu$$

$$\bar{x} \leq L_{sn} \quad \text{or} \quad \bar{x} \geq U_{sn}.$$  (4.66)

The total cost is then the sum of the inspection, the acceptance, screening and scrapping costs. If $K_T$ is the total cost then

$$K_T = K_T n + K_A + K_{sn} + K_{sp}.$$  (4.67)
4.4. Model Optimization Algorithm

Step 1. a) Compute the partial derivatives of the total cost relative to $U_A$, $L_A$, $U_{sn}$ and $L_{sn}$.

b) Equate these values to zero.

Step 2. Sum the partial derivatives relative to a particular disposition limit (using expression (4.67)).

Example

\[ \frac{\partial K_T}{\partial U_A} = \frac{\partial K_A}{\partial U_A} + \frac{\partial K_{sn}}{\partial U_A} + \frac{\partial K_{sp}}{\partial U_A}. \]  

(4.68)

The following formulas are derived for the algorithms above:

\[ \frac{\partial K_T}{\partial U_A} = \int_{-\infty}^{+\infty} K_A(U_A, \mu) T(U_A|\mu) h(\mu) d\mu \]

\[ - \int_{-\infty}^{+\infty} K_{sn}(U_A, \mu) T(U_A|\mu) h(\mu) d\mu \]

(4.69)

where

\[ K_A(U_A, \mu) = K_A(\bar{x}, \mu)|_{x} = U_A \]

given in (4.49) and

\[ K_{sn}(U_A, \mu) = K_{sn}(\bar{x}, \mu)|_{x} = U_A \]

given in (4.58). Substituting $K_A(U_A, \mu)$ and $K_{sn}(U_A, \mu)$ by their values in (4.69) yields

\[ \frac{\partial K_T}{\partial U_A} = \int_{-\infty}^{+\infty} \left[ K_p n + K_a (N - n) - K_a N \int_{L}^{U} f(x|\mu) dx \right. \]

\[ + (K_a - K_p) \int_{L}^{U} t(x|\bar{x}, \mu) dx] T(U_A|\mu) h(\mu) d\mu \]

\[ - \int_{-\infty}^{+\infty} (K_{sn} (N - n) + K_p N) \]

\[ \cdot \left[ 1 - \int_{L}^{U} f(x|\mu) dx \right] T(U_A)|\mu) h(\mu) d\mu. \]  

(4.70)
Using $P'(\mu)$ and $Q'(\bar{u}, \mu)$ given by:

$$P'(\mu) = \int_L^U f(x|\mu)dx$$  \hspace{1cm} (4.71)$$

$$Q'(\bar{u}, \mu) = \int_L^U t(x|\bar{u}, \mu)dx$$  \hspace{1cm} (4.72)$$

will reduce expression (4.70). Therefore

$$\frac{\partial K_T}{\partial U_A} = [K_P n + K_a (N - n) - K_{sn} (N - n) - K_p N] \cdot \int_{-\infty}^{+\infty} T(U_A|\mu)h(\mu)d\mu$$

$$- (K_a - K_p)N \int_{-\infty}^{+\infty} P'(\mu)T(U_A|\mu)h(\mu)d\mu$$

$$+ (K_a - K_p)N \int_{-\infty}^{+\infty} Q'(U_A, \mu)T(U_A|\mu)h(\mu)d\mu.$$  \hspace{1cm} (4.73)$$

Equating $\frac{\partial K_T}{\partial U_A}$ to zero results in

$$\int_{-\infty}^{+\infty} P'(\mu)T(U_A|\mu)h(\mu)d\mu - \frac{n}{N} \int_{-\infty}^{+\infty} Q'(U_A, \mu)T(U_A|\mu)h(\mu)d\mu$$

$$= \left[ 1 - \frac{(N - n)K_{sn} + (K_a - K_p)n}{(K_a - K_p)N} \right] \cdot \int_{-\infty}^{+\infty} T(U_A|\mu)h(\mu)d\mu.$$  \hspace{1cm} (4.74)$$

Dividing both sides of equation (4.94) by $\int_{-\infty}^{+\infty} T(U_A|\mu)h(\mu)d\mu$ provides $Q_1(U_A, n)$ and $Q_2(U_A, n)$ as:

$$Q_1(U_A, n) = \frac{\int_{-\infty}^{+\infty} P'(\mu)T(U_A|\mu)h(\mu)d\mu}{\int_{-\infty}^{+\infty} T(U_A|\mu)h(\mu)d\mu}$$  \hspace{1cm} (4.75)$$

$$Q_2(U_A, n) = \frac{\int_{-\infty}^{+\infty} Q'(U_A, \mu)T(U_A|\mu)h(\mu)d\mu}{\int_{-\infty}^{+\infty} T(U_A|\mu)h(\mu)d\mu}.$$  \hspace{1cm} (4.76)$$
The expression of $Q_1(U_A, n)$ given in (4.75) can be written in the form:

$$Q_1(U_A, n) = \int_{-\infty}^{+\infty} P'(\mu) \frac{T(U_A|\mu)h(\mu)}{T(U_A)} \, d\mu$$

(4.77)

or

$$Q_1(U_A, n) = \int_{-\infty}^{+\infty} P'(\mu)h(U_A|\mu) \, d\mu.$$  

(4.78)

Expression (4.77) and (4.78) are identical to expression (2.81). Hence it is possible to write (4.77) and (4.78) as

$$Q_1(U_A, n) = \int_{L}^{U} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\sigma^2 + \sigma_n^2}} e^{-\frac{1}{2} \left( \frac{x-m_n}{\sigma^2 + \sigma_n^2} \right)^2} \, dx.$$  

(4.79)

Where

$$\delta^2_n = \frac{\sigma^2_n}{n}, \quad m_n = \frac{m^2 + \sigma^2 U_A}{\delta_n^2 + \sigma^2}, \quad \sigma^2_n = \frac{\sigma^2 - \delta^2_n}{\delta_n^2 + \sigma^2}.$$  

(4.80)

Now expression (4.74) is reduced to

$$Q_1(U_A, n) = \Phi \left( \frac{U - m_n}{\sqrt{\sigma^2 + \sigma_n^2}} \right) - \Phi \left( \frac{L - m_n}{\sqrt{\sigma^2 + \sigma_n^2}} \right).$$  

(4.81)

If the statistical process can exist in $s$ states, the p.d.f. of $\mu$ is given in (4.32). Define $D_i$ by the ratio $D_i = \frac{\sigma^2}{\sigma^2_{\mu_i}}$. Expression (4.79) for statistical process in state $i$ can be written as

$$Q_1'(U_A, n, D_i, P_i) = \frac{(n + D_i)^\frac{1}{2}}{(n + D_i + 1)} \cdot \frac{p_i}{\sqrt{2\pi}} \int_{L}^{U} +e^{-\frac{(x - nU_A + D_i\mu_i)^2}{2[n + D_i + 1]\sigma^2}} \, dx.$$  

(4.82)

In general for a statistical process than exist in $i = 1, 2, \cdots, s$ states the following is true:

$$Q_1(U_A, n, D, P) = Q_1'(U_A, n, D_1, P_1) + Q_1'(U_A, n, D_2, P_2) + \cdots + Q_1'(U_A, n, D_s, P_s).$$  

(4.83)
To evaluate $Q_2(U_A, n)$, the following expression

$$\frac{Q'(U_A, \mu)T(U_A|\mu)h(\mu)}{T(U_A)}$$  \hspace{1cm} (4.84)

can be written as:

$$= \frac{\sqrt{\sigma_\mu^2 + \delta_n^2}}{2\pi \sigma_{n-1} \delta_n \cdot \sigma_n \cdot \exp[-\frac{1}{2}(\frac{(U_A - \mu)^2}{\sigma_\mu^2 + \delta_n^2})]} \cdot e^{-\frac{1}{2}[(\frac{U_A - \mu}{\delta_n})^2 + (\frac{\mu - m}{\sigma_\mu^2 + \delta_n})^2]}$$  \hspace{1cm} (4.85)

where:

$$\sigma_{n-1} = \frac{\sigma \sqrt{n - 1}}{\sqrt{n}}, \quad \delta_n = \frac{\sigma^2}{n}. \quad \hspace{1cm} (4.86)$$

The sum of the last two terms of the exponential of (4.85) can be written as:

$$\frac{(U_A - \mu)^2}{\delta_n^2} + \frac{(\mu - m)^2}{\sigma_\mu^2} = \frac{(\mu - m_1)^2}{\delta_1^2} + \frac{(m - U_A)^2}{\sigma_\mu^2}$$  \hspace{1cm} (4.87)

where

$$\delta_1 = \frac{\sigma_\mu^2 \cdot \delta_n^2}{\sigma_\mu^2 + \delta_n^2}, \quad \sigma_1^2 = \sigma_\mu^2 + \delta_n^2, \quad m_1 = \frac{m \delta_n^2 + \sigma_\mu^2 \cdot U_A}{\sigma_\mu^2 + \delta_n^2}. \quad \hspace{1cm} (4.88)$$

Substituting (4.87) into (4.85) and integrating relative to $\mu$ yields:

$$\frac{Q_1(U_A, \mu)T(\mu_A|\mu)h(\mu)}{T(U_A)} = \frac{\sqrt{\sigma_\mu^2 + \delta_n^2 \cdot \delta_1}}{\sqrt{2\pi \sigma_{n-1} \cdot \delta_n \cdot \sigma_\mu}} \cdot e^{-\frac{1}{2}[(\frac{U_A - \mu}{\delta_1})^2 + (\frac{\mu - m}{\sigma_\mu^2 + \delta_n})^2]} \cdot \exp[-\frac{1}{2}[(\frac{U_A - \mu}{\delta_n})^2 + (\frac{\mu - m}{\sigma_\mu^2 + \delta_n})^2]]$$  \hspace{1cm} (4.89)

since $\sigma_1^2 = \sigma_\mu^2 + \delta_n^2$, the terms $e^{-\frac{1}{2}[(\frac{U_A - \mu}{\delta_1})^2]}$ and $e^{-\frac{1}{2}[(\frac{U_A - \mu}{\delta_n})^2]}$ are identical. Then (4.89) is reduced to:

$$= \frac{\sigma_\mu^2 \cdot \delta_n^2}{\sqrt{\sigma_\mu^2 + \delta_n^2 \cdot \sqrt{2\pi \cdot \delta_n \cdot \sigma_\mu}}} \cdot e^{-\frac{1}{2}[(\frac{U_A - \mu}{\delta_n})^2]}.$$  \hspace{1cm} (4.90)
Integrating (4.90) relative to \( x \) provides the expression for \( Q_2(U_A, n) \):

\[
Q_2(U_A, n) = \int_L^U \frac{\sigma_\mu \cdot \delta_n}{\sqrt{2 \pi \sigma_\mu^2 + \delta_n^2}} e^{-\frac{1}{2} \left(\frac{x - U_A}{\sigma_\mu^2 + \delta_n^2}\right)^2} \, dx
\]

\[
= \frac{\sigma_\mu \sigma_\nu^2 \sqrt{n-1}}{n \sqrt{\sigma_\mu^2 + \delta_n^2}} \left[ \Phi \left( \frac{U - U_A}{\sqrt{n-1}} \right) - \Phi \left( \frac{L - U_A}{\sqrt{n-1}} \right) \right].
\]  

\( (4.91) \)

For a statistical process that can exist in a single state \( i \), the expression of \( Q_2 \) is:

\[
Q_2(U_A, n, D_i, p_i) = \frac{\sigma_\mu \sigma_\nu^2 \sqrt{n(n-1)}}{\sqrt{n + D_i}} \cdot P_i \left[ \Phi \left( \frac{U - U_A}{\sqrt{n-1}} \right) - \Phi \left( \frac{L - U_A}{\sqrt{n-1}} \right) \right]
\]

\( (4.92) \)

where \( D_i = \frac{\sigma_\mu^2}{\sigma_\nu^2} \). For multiple states then

\[
Q_2(U_A, n, D, P) = Q_{21}(U_A, n, D_1, P_1) + Q_{22}(U_A, n, D_2, P_2) + \cdots + Q_{2s}(U_A, n, D_s, P_s).
\]

\( (4.93) \)

Expression (4.74) can be written in the form:

\[
\frac{\int_{-\infty}^{+\infty} P'(\mu)T(U_A|\mu)h(\mu)d\mu}{\int_{-\infty}^{+\infty} T(U_A|\mu)h(\mu)d\mu} = \frac{n}{N} \left[ \frac{\int_{-\infty}^{+\infty} Q'(U_A, \mu)T(U_A|\mu)h(\mu)d\mu}{\int_{-\infty}^{+\infty} T(U_A|\mu)h(\mu)d\mu} \right] = \left[ 1 - \frac{(N - n)K_{sn} + (K_a - K_p)n}{(K_a - K_p)n} \right].
\]

\( (4.94) \)

Substituting \( Q_1 \) and \( Q_2 \) by their values in (4.94) yields:

\[
\Phi \left( \frac{U - m_n}{\sqrt{\sigma^2 + \sigma_n^2}} \right) - \Phi \left( \frac{L - m_n}{\sqrt{\sigma^2 + \sigma_n^2}} \right) - \frac{n}{N} \frac{\sigma_\mu \cdot \sigma_\nu^2 \sqrt{n-1}}{n \sqrt{\sigma_\mu^2 + \delta_n^2}} \left[ \Phi \left( \frac{U - U_A}{\sqrt{n-1}} \right) - \Phi \left( \frac{L - U_A}{\sqrt{n-1}} \right) \right]
\]

\[
= 1 - \frac{(N - n)K_{sn} + (K_a - K_p)n}{(K_a - K_p)n}.
\]

\( (4.95) \)
In general for a statistical process that can exist in $s$ state, a similar expression of (4.81) is:

$$Q'_1(U_A,n,D_1,P_1) + Q'_2(U_A,n,D_2,P_2) + \cdots + Q'_s(U_A,n,D_s,P_s)$$

$$- \frac{n}{N}(Q'_1(U_A,n_1,D_1,P_1) + Q'_2(U_A,n,D_2,P_2)$$

$$+ \cdots + Q'_s(U_A,n,D_s,P_s))$$

$$= 1 - \frac{(N-n)K_{sn} + (K_a - K_p)n}{(K_a - K_p)N}.$$  (4.96)

The partial derivative of the total cost relative to the upper limit of screening disposition yields:

$$\frac{\partial K_T}{\partial U_{sn}} = [K_{sn}(N-n) + K_pN - K_{sp}N] \cdot \int_{-\infty}^{+\infty} T(U_{sn}|\mu)h(\mu)d\mu$$

$$- K_pN \int_{-\infty}^{+\infty} P'(\mu)T(U_{sn}|\mu)h(\mu)d\mu$$

$$+ K_{sp}n \int_{-\infty}^{+\infty} Q'(U_{sn},\mu)T(U_{sn}|\mu)h(\mu)d\mu.$$  (4.97)

Arranging the terms of (4.97) and setting them equal to zero provides:

$$\int_{-\infty}^{+\infty} P'(\mu)T(U_{sn}|\mu)h(\mu)d\mu - \frac{K_{sp}n}{K_pN} \int_{-\infty}^{+\infty} Q'(U_{sn},\mu)T(U_{sn}|\mu)h(\mu)d\mu$$

$$= \left[\frac{K_{sn}(N-n) + C_pN - C_{sp}n}{K_pN}\right]$$

$$\cdot \int_{-\infty}^{+\infty} T(U_{sn}|\mu)h(\mu)d\mu.$$  (4.98)

Dividing each term of expression (4.98) by $\int_{-\infty}^{+\infty} T(U_{sn}|\mu)h(\mu)d\mu$ yields:

$$\frac{\int_{-\infty}^{+\infty} P'(\mu)T(U_{sn}|\mu)h(\mu)d\mu}{\int_{-\infty}^{+\infty} T(U_{sn}|\mu)h(\mu)d\mu} - \frac{K_{sp}n}{K_pN} \frac{\int_{-\infty}^{+\infty} Q'(U_{sn},\mu)T(U_{sn}|\mu)h(\mu)d\mu}{\int_{-\infty}^{+\infty} T(U_{sn}|\mu)h(\mu)d\mu}$$

$$= \frac{K_{sn}(N-n) + C_pN - C_{sp}n}{K_pN}.$$  (4.99)
Expression (4.99) can be written as:

\[ Q_3(U_{sn}, n) - \frac{K_{sp}n}{K_pN} Q_4(U_A, n) = \frac{K_{sn}(N - n) + C_pN - C_{sp}n}{K_pN} \]  

(4.100)

where

\[ Q_3(U_{sn}, n) = Q_1(U_A, n)|_{U_A = U_{sn}} \]

\[ Q_4(U_{sn}, n) = Q_2(U_A, n)|_{U_A = U_{sn}} \]  

(4.101)

For a statistical process that can exist in a single state, (4.99) can be written as:

\[
\Phi\left(\frac{U - m_n}{\sqrt{\sigma^2 + \sigma_n^2}}\right) - \Phi\left(\frac{L - m_n}{\sqrt{\sigma^2 + \sigma_n^2}}\right) - \frac{K_{sp}n \sigma \cdot \sigma^2 \sqrt{n - 1}}{K_pN \sqrt{n} (\sigma_n^2 + \delta_n^2)} \left[ \Phi\left(\frac{U - U_{sn}}{\sigma \sqrt{n - 1}}\right) - \Phi\left(\frac{L - U_{sn}}{\sigma \sqrt{n - 1}}\right) \right] \\
= \frac{K_{sn}(N - n) + K_pN - K_{sp}N}{K_pN}
\]

(4.102)

where:

\[ \delta_n^2 = \frac{\sigma^2}{n}, \quad m_n = \frac{m \delta_n^2 + \sigma_n^2 U_{sn}}{\delta_n^2 + \sigma_n^2}, \quad \sigma_n^2 = \frac{\sigma^2 \cdot \delta_n^2}{\delta_n^2 + \sigma_n^2} \]  

(4.103)

Employing the same technique developed in Section 4.3, it is possible to verify the following relation:

\[ \frac{\partial K_T}{\partial U_A}|_{L_A} = -\frac{\partial K_T}{\partial L_A}. \]  

(4.104)

A similar expression to (4.104) involving \( U_{sn} \) and \( L_{sn} \) for screening process is given below:

\[ \frac{\partial K_T}{\partial U_{sn}}|_{L_{sn}} = -\frac{\partial K_T}{\partial L_{sn}}. \]  

(4.105)

or

\[ \frac{\partial}{\partial U_{sn}}(K_I + K_{sn} + K_{sp} + K_A)|_{L_{sn}} = -\frac{\partial}{\partial L_{sn}}(K_I + K_{sn} + K_{sp} + K_A) \cdots. \]  

(4.106)
Expressions (4.123) and (4.124) are used for finding the decision points for lot disposition and to estimate the components of the total cost.

4.5. Calculation Procedures

a) Disposition limits for a single state statistical process.

(1) $U_A$ is obtained by employing expression (4.95) where:

$$m_n = \frac{\delta_n^2 + \sigma_u^2 U_A}{\delta_n^2 + \sigma_u^2}, \quad \sigma_n^2 = \frac{\sigma_u^2}{\delta_n^2 + \sigma_u^2}.$$

Values of $U_A$ are assumed between the limits $L$ and $U$. The iteration procedure continues until the difference of the right and left hand side of equation (4.114) is of order $10^{-6}$.

(2) $L_A$ can be estimated by two procedures

(i) $U_A$ and $L_A$ are symmetrical about the mean $m_0 = \frac{L + U}{2}$. If $U_A = \frac{L + U}{2} + v$ ($v$ is a constant), then $L_A = \frac{L + U}{2} - v$.

(ii) $L_A$ is evaluated directly from expression (4.123) and employing the same procedure as in a(1).

(3) (a) $U_{sn}$ is obtained by employing expression (4.121). The same values of $m_n$, $\delta_n^2$, $\sigma_n^2$ as in case a(1) still hold true. $L_{sn}$ is obtained by two procedures. These procedures are identical to (i) and (ii) in case a(2).

(b) Disposition limits for multiple state statistical process.

(1) Expression (4.83) provides an estimate for $U_A$.

(2) Expression (4.104) provides an estimate for $L_A$.

(3) Expression (4.96) provides an estimate for $U_{sn}$.

(4) Expression (4.105) provides an estimate for $L_{sn}$.

(c) Cost estimation for single or multiple state statistical process.

(1) Expression (4.51) provides an estimate for the $E$ (cost of lot acceptance).
(2) Expression (4.59) provides an estimate for the $E$ (cost of lot screening).
(3) Expression (4.63) provides an estimate for $E$ (cost of lot scrapping).
(4) Expression (4.67) provides an estimate for $E$ (total cost per lot).

The computed upper and lower disposition limits used for $\bar{x}$ are used to estimate, the three costs and the total cost.

(d) Model approximations.

(a) Assume $f(x|\mu)$ and $t(x|\bar{x}, \mu)$ are equal then

$$\left[ \Phi \left( \frac{U - m_n}{\sqrt{\sigma^2 + \sigma_n^2}} \right) - \Phi \left( \frac{L - m_n}{\sqrt{\sigma^2 + \sigma_n^2}} \right) \right] \left[ 1 - \frac{n}{N} \right]$$

$$= 1 - \frac{(N - n)Ksn + (K_a - K_p)n}{(K_a - K_p)N}.$$

It was shown in Chapter 3 there exist two values of $m_n$, $m_n^1$ and $m_n^2$ symmetrical about $\frac{L + U}{2}$, such that $m_n^1 = m_0 - \nu$ and $m_n^2 = m_0 + \nu$. These $m_n$ is given by

$$m_n = \frac{\delta_n^2 + \sigma_n^2 U_A}{\delta_n^2 + \sigma_\mu^2},$$

then

$$U_A = \frac{m_n^2(\delta_n^2 + \sigma_\mu^2) - \delta_n^2}{\sigma_\mu^2}$$

and

$$L_A = \frac{m_n^1(\delta_n^2 + \sigma_\mu^2) - \delta_n^2}{\sigma_\mu^2}.$$

Expression (4.102) will reduce to

$$\Phi \left[ \left( \frac{U - m_n}{\sqrt{\sigma^2 + \sigma_n^2}} \right) - \Phi \left( \frac{L - m_n}{\sqrt{\sigma^2 + \sigma_n^2}} \right) \right] \left[ 1 - \frac{K_{sp}n}{K_pN} \right] = \frac{K_{sn}(N - n) + K_{p}N - K_{sp}N}{K_pN}.$$

The same technique employed for case (d(a)) holds true for different cost parameter values.
4.6. Cost of Rejecting and Reworking Defective Items

Expression of the costs prior to sampling and after sampling are derived under the assumption that rejecting a lot requires reworking the defective items.

4.6.1. Prior Cost for the Decision to Reject Outright and Reworking the Defective Items

The producer's cost from a decision to reject the lot and rework the defective items found during inspection consists of four parts:

1. The components of the cost per lot resulting from defective items found during inspection and reworked with and without success are respectively

\[ Np(1 - K_{E2})[+K_C]. \quad (4.107) \]

and

\[ Np(1 - K_{E2})[K_R - K_J](1 - K_Y)]. \quad (4.108) \]

The cost per unit is obtained by employing expressions (4.107) and (4.108)

\[ C_{R1} = p(1 - K_{E2})[K_R - K_J](1 - K_Y) + K_C]. \quad (4.109) \]

The producer's profit per unit in this case is:

\[ p(1 - K_{E2})[-K_C + K_Y(K_R - K_p) + (1 - K_Y)(K_s - K_p)]. \quad (4.110) \]

2. The cost per lot resulting from defective items classified as good during inspection is:

\[ NpK_{E2}(+K_A). \quad (4.111) \]

The cost per unit is

\[ C_{R2} = pK_{E2}(+K_A). \quad (4.112) \]
The profit per unit is

\[ p(K_{E2})(K_R - K_p + K_A). \]  \hspace{1cm} (4.113)

3. The producer’s cost per lot resulting from good items classified as defective during inspection and reworked is

\[ N(1 - p)K_{E1}[(K_R - K_J)(1 - K_Y) - K_C]. \]  \hspace{1cm} (4.114)

The cost per unit is:

\[ C_{R3} = (1 - p)K_{E1}[K_R - K_J](1 - K_y) + K_C. \]  \hspace{1cm} (4.115)

The profit per unit in this case is:

\[ (1 - p)K_{E1}[-K_C + K_Y(K_R - K_p) + (1 - K_Y)(K_s + K_p)]. \]  \hspace{1cm} (4.116)

In this case no cost per lot resulted from good items found during inspection from the producer’s standpoint. The producer’s profit per unit resulting from good items found during inspection is

\[ (1 - p)(1 - K_{E1})(K_R - K_p). \]  \hspace{1cm} (4.117)

4. The cost of inspection is:

\[ NK_I. \]  \hspace{1cm} (4.118)

The cost per unit for the decision to reject outright and rework is:

\[ C_4 = C_{R1} + C_{R2} + C_{R3} + NK_I. \]  \hspace{1cm} (4.119)

Expression (4.119) can be simplified to the form

\[ C_4 = p[K_{E2}K_A + (1 - K_{E2} - K_{E1})[K_R - K_J](1 - K_Y) + K_C]] + K_{E1}[(K_R - K_J)(1 - K_Y) + K_C]. \]  \hspace{1cm} (4.120)
Setting
\[ g_1 = p(1 - K_E2 - K_E1)(K_R - K_J)(1 - K_Y) + K_C \] (4.121)
\[ g_2 = K_E1[K_R - K_J](1 - K_Y) + K_C \] (4.122)

In terms of \((g_1)\) and \((g_2)\) \(C_4\) is:
\[ C_4 = g_1p + g_2. \] (4.123)

Expression (4.123) yields the following value for \(p\)
\[ p = \frac{C_4 - g_2}{g_1} \]

The p.d.f. of \(C_4\) can be obtained from that of \(p\) and expression (4.123), hence
\[ f(C_4) = \frac{1}{g_1 \sqrt{2\pi \sigma^2}} e^{-\frac{(C_4 - g_2)^2}{2g_1^2}} \] (4.125)
\[ g_2 < C_4 < g_1 + g_2. \]

An expression of the profit per unit for this decision can be obtained by subtracting expression (4.118) from the sum of expressions (4.110), (4.113), (4.116), (4.117) which leads to
\[ P_4 = -K_1 - K_p - K_{E1}(K_C) + [1 - K_{E1} + K_{E1}(K_Y)][K_R] + K_{E1}(1 - K_Y)(K_J) + P[(K_3 - K_R)(1 - K_J) - K_C] \] \cdot (1 - K_{E1} - K_{E2}) - K_A K_R). \] (4.126)

An expression for the p.d.f. of \(p_4\) can be obtained by following the same procedure in obtaining a p.d.f. for \(C_4\). For the decision of accepting outright or rejecting outright and reworking the defective items found with success or non success, expressions (4.1) and (4.33) are equated yielding
\[ P_{03} = K_{E1}[K_C + (K_R - K_J)(1 - K_Y)] + K_I \] \[ K_A(1 - K_{E2}) - (1 - K_{E1} - K_{E2})[(K_R - K_J)(1 - K_Y) + K_C] \] (4.127)
The same expression for \( P_{03} \) can be obtained by equating expressions (4.7) and (4.125).

4.6.2. The Posterior Cost of Rejecting and Reworking Defectives

In estimating expected cost of rejecting and reworking defective items the following assumptions are made:

1. The probability that an individual measurement is above or below the upper and lower specification limits for both the lot and the sample are considered.

2. The costs of accepting and repairing items with dimensions above or below the specification limits for both the lot and the sample are considered.

3. The screening errors of types I and II are negligible.

4. The process can exist in one statistical state model development.

The cost in this case consists of the following components:

1. The cost per lot resulting from defective items found during inspection and reworked with and without success. Denoting this component of the cost by \( K_{W1}(\bar{x}, \mu) \), then:

\[
K_{W1}(\bar{x}, \mu) = K_{C1} n P_{3s} + K_{C2} n P_{4s}
+ [(K_R - K_J)(1 - K_Y)] P_{3s} + n[(K_R - K_J)(1 - K_Y)] P_{4s}.
\]

(4.128)

The terms \( K_{C1}, K_{C2}, P_{3s}, P_{4s}, K_R, K_J, K_Y \) are all defined in Appendix A. For the remainder of the lot the cost is \( K_{W2}(\bar{x}, \mu) \) given as

\[
K_{W2}(\bar{x}, \mu) = K_{C1}(N - n) P_{1u} + K_{C2}(N - n) P_{2L}
+ (N - n) P_{1u}[(K_R - K_J)(1 - K_Y)]
+ (N - n) P_{1u}[(K_R - K_J)(1 - K_Y)]
\]

(4.129)

The probabilities \( P_{1u} \) and \( P_{2L} \) are defined in Appendix A.
Assuming $K_{C1} = K_{C2} = K_C$ and $P_{1u} = P_{2L}$, expression (3.78) can be written as:

$$K_{W1}(\bar{x}, \mu) = n[K_C + (K_R - K_J)(1 - K_Y)][P_{3s} + P_{4s}]. \quad (4.130)$$

Defining $K_{R1}$ as:

$$K_{R1} = [K_C + (K_R - K_J)(1 - K_Y)]. \quad (4.131)$$

Employing expression (4.131), expressions (4.130) and (3.78) can be written respectively as:

$$K_{W1}(\bar{x}, \mu) = nK_{R1}\int_{U}^{\infty} t(x|\bar{x}, \mu) + \int_{-\infty}^{L} t(x|\bar{x}, \mu)dx \quad (4.132)$$

and

$$K_{W2}(\bar{x}, \mu) = (N - n)K_{R1}\int_{U}^{\infty} f(x|\mu)dx + \int_{-\infty}^{L} f(x|\mu)dx. \quad (4.133)$$

The expected cost $K_W(\bar{x}, \mu)$ of reworking defective items with and without success can be obtained by adding expressions (4.132) and (4.133), thus

$$K_W(\bar{x}, \mu) = NK_{R1} - nK_{R1}Q_{1D}(\bar{x}, \mu) - (N - n)K_{R1}P_{1D}(\mu). \quad (4.134)$$

where the two probabilities $P_{1D}(\mu)$ and $Q_{1D}(\mu)$ are defined as:

$$P_{1D}(\mu) = \int_{L}^{U} f(x|\mu)dx \quad (4.135)$$

$$Q_{1D}(\bar{x}, \mu) = \int_{L}^{U} t(x|\bar{x}, \mu)dx. \quad (4.136)$$

Under the assumption that the rejecting lots are screened and reworked.
The total expected cost can be written as:

\[ K_T = \int_{-\infty}^{+\infty} \left[ \int_{U_A}^{L_A} n(K_A - K_P)Q_1D(\bar{x}, \mu)T(\bar{x}_n|\mu)d\bar{x} \right] h(\mu)d\mu \\
- K_An \int_{-\infty}^{+\infty} \left[ \int_{L_A}^{U_A} P_1D(\mu)T(\bar{x}_n|\mu)d\bar{x} \right] h(\mu)d\mu \\
+ (K_A(N - n) + K_Pn) \int_{-\infty}^{+\infty} \left[ \int_{L_A}^{U_A} T(\bar{x}_n|\mu)d\bar{x} \right] h(\mu)d\mu \\
\] (4.137)

\[ + \int_{-\infty}^{+\infty} \left[ \int_{L_n}^{L_A} K_W(\bar{x}, \mu)T(\bar{x}_n|\mu)d\bar{x} \right] h(\mu)d\mu \\
+ \int_{-\infty}^{+\infty} \left[ \int_{U_n}^{U_A} K_W(\bar{x}, \mu)T(\bar{x}_n|\mu)d\bar{x} \right] h(\mu)d\mu + K_Rn. \]

The decision points \( L_A, U_A, L_{sn}, U_{sn} \) are all defined in this chapter. Taking the partial derivative of \( K_T \) relative to \( U_A \) yields:

\[ \frac{\partial K_T}{\partial U_A} = n(K_A - K_P) \int_{-\infty}^{+\infty} Q_1D(U_A, \mu)T(U_A|\mu)h(\mu)d\mu \\
- K_An \int_{-\infty}^{+\infty} P_1D(\mu)T(U_A|\mu)h(\mu)d\mu \\
+ [K_A(N - n) + K_Pn] \int_{-\infty}^{+\infty} T(U_A|\mu)h(\mu)d\mu \\
- NKR1 \int_{-\infty}^{+\infty} T(U_A|\mu)h(\mu)d\mu \\
+ nKR1 \int_{-\infty}^{+\infty} Q_1D(U_A, \mu)T(U_A|\mu)h(\mu)d\mu \\
+ (N - n)KR1 \int_{-\infty}^{+\infty} P_1D(\mu)T(U_A|\mu)h(\mu)d\mu. \] (4.138)

Arranging the terms in expression (4.138) yields:

\[ \frac{\partial K_T}{\partial U_A} = n[(K_A - K_P) + KR1] \int_{-\infty}^{+\infty} Q_1D(U_A, \mu)T(U_A|\mu)h(\mu)d\mu \\
+ [(N - n)KR1 - K_A n] \int_{-\infty}^{+\infty} P_1D(\mu)T(U_A|\mu)h(\mu)d\mu \\
+ [K_A(N - n) + K_Pn - nKR1 - NKR1 + nKR1] \int_{-\infty}^{+\infty} T(U_A|\mu)h(\mu)d\mu. \] (4.139)
Setting $\frac{\partial K_F}{\partial U_A} = 0$, then dividing each term of expression (4.139) by

$$\int_{-\infty}^{\infty} T(U_A|\mu) h(\mu) d\mu$$

and simplifying yields:

$$\frac{\int_{-\infty}^{\infty} P_{1D}(\mu) T(U_A|\mu) h(\mu) d\mu}{\int_{-\infty}^{\infty} T(U_A|\mu) h(\mu) d\mu} + n[(K_A - K_P) + K_{R1}] = \frac{\int_{-\infty}^{\infty} Q_{1D}(U_A, \mu) T(U_A|\mu) h(\mu) d\mu}{\int_{-\infty}^{\infty} T(U_A|\mu) h(\mu) d\mu}.\frac{\int_{-\infty}^{\infty} Q_{1D}(U_A, \mu) T(U_A|\mu) h(\mu) d\mu}{\int_{-\infty}^{\infty} T(U_A|\mu) h(\mu) d\mu} = \frac{N K_{R1} - [K_A(N - n) + K_P n]}{(N - n) K_{R1} - K_A N}$$

(4.140)

Define the following qualities $Q_1(U_A, n)$ and $Q_2(U_A, n)$ as:

$$Q_1(U_A, n) = \frac{\int_{-\infty}^{\infty} P_{1D}(\mu) T(U_A|\mu) h(\mu) d\mu}{\int_{-\infty}^{\infty} T(U_A|\mu) h(\mu) d\mu}$$

(4.141)

and

$$Q_2(U_A, n) = \frac{\int_{-\infty}^{\infty} Q_{1D}(U_A, \mu) T(U_A|\mu) h(\mu) d\mu}{\int_{-\infty}^{\infty} T(U_A|\mu) h(\mu) d\mu}.$$  

(4.142)

Employing expressions (2.64), (2.71) and (2.81) expressions (4.141) and (4.142) can be written as:

$$Q_1(U_A, n) = \frac{1}{\sqrt{2\pi} \sqrt{\sigma^2 + \sigma_n^2}} \int_L^U e^{-\frac{(x - m_n)^2}{2(\sigma^2 + \sigma_n^2)}} dx$$

(4.143)

where

$$\delta_n^2 = \frac{\sigma^2}{n}, \quad m_n^2 = \frac{m \delta_n^2 + \sigma_n^2 U_A}{\delta_n^2 + \sigma_n^2}, \quad \sigma_n^2 = \frac{\sigma^2 \cdot \delta_n^2}{\delta_n^2 + \sigma_n^2}.$$  

(4.144)

An expression for $U_A$ can be obtained by employing expression (4.144) as:

$$U_A = \frac{m_n^2 (\delta_n^2 + \sigma_n^2) - m \delta_n^2}{\sigma_n^2}.$$  

(4.145)
Expression (4.142) can be written as:

$$Q_2(U_A, n) = \frac{\sigma_\mu \cdot \delta_n}{\sqrt{2\pi} \sqrt{\sigma_\mu^2 + \delta_n^2}} \int_{L}^{U} e^{-\frac{(x-U_A)^2}{2(\sigma_\mu^2 + \delta_n^2)}} dx. \quad (4.146)$$

Employing expressions (4.143), (4.144) and (4.146), expression (4.140) can be written as:

$$\Phi\left(\frac{U - m_n}{\sqrt{\sigma^2 + \sigma_n^2}}\right) - \Phi\left(\frac{L - m_n}{\sqrt{\sigma^2 + \sigma_n^2}}\right) + \frac{n \left[ (K_A - K_P) + K_{R1} \right]}{(N-n)K_{R1} - K_A N}$$

$$\frac{\sigma_\mu \cdot \sigma^2 \sqrt{n-1}}{n \sqrt{\sigma_\mu^2 + \delta_n^2}} \left[ \Phi\left(\frac{U - U_A}{\sigma} \right) - \Phi\left(\frac{L - U_A}{\sigma} \right) \right] \quad (4.147)$$

$$= \frac{NK_{R1} - [K_A(N-n) + K_P n]}{(N-n)K_{R1} - K_A N}.$$ 

Taking the partial derivative of $K_T$ relative to $U_{sn}$ yields:

$$\frac{\partial K_T}{\partial U_{sn}} = \int_{-\infty}^{+\infty} K_W(U_{sn}, \mu) T(U_{sn}|\mu) h(\mu) d\mu. \quad (4.148)$$

Expression (3.97) can be written as:

$$\frac{\partial K_T}{\partial U_{sn}} = NK_{R1} \int_{-\infty}^{+\infty} T(U_{sn}|\mu) h(\mu) d\mu$$

$$- (N-n)K_{R1} \int_{-\infty}^{+\infty} P_{1D}(\mu) T(U_{sn}|\mu) h(\mu) d\mu$$

$$- nK_{R1} \int_{-\infty}^{+\infty} Q_{1D}(U_{sn}, \mu) T(U_{sn}|\mu) h(\mu) d\mu. \quad (4.149)$$

Setting $\frac{\partial K_T}{\partial U_{sn}} = 0$, then dividing the above expression by $\int_{-\infty}^{+\infty} T(U_{sn}|\mu) h(\mu) d\mu$ and simplifying yields:

$$\frac{\int_{-\infty}^{+\infty} P_{1D}(\mu) T(U_{sn}|\mu) h(\mu) d\mu}{\int_{-\infty}^{+\infty} T(U_{sn}|\mu) h(\mu) d\mu} + \frac{n}{N-n}$$

$$\cdot \frac{\int_{-\infty}^{+\infty} Q_{1D}(U_{sn}, \mu) T(U_{sn}|\mu) h(\mu) d\mu}{\int_{-\infty}^{+\infty} T(U_{sn}|\mu) h(\mu) d\mu} = \frac{N}{N-n} \quad (4.150)$$
Expression (4.150) can be written as

\[
\phi\left(\frac{U - m_n}{\sqrt{\sigma^2 + \sigma_n^2}}\right) - \Phi\left(\frac{L - m_n}{\sqrt{\sigma^2 + \sigma_n^2}}\right) + \frac{n}{N - n} \cdot \frac{[\sigma_\mu \cdot \sigma^2 \sqrt{n - 1}]}{\sqrt{\sigma^2_\mu + \sigma_n^2}} \left[ \Phi\left(\frac{U - U_{sn}}{\sigma \sqrt{\frac{n-1}{n}}}\right) - \Phi\left(\frac{L - U_{sn}}{\sigma \sqrt{\frac{n-1}{n}}}\right) \right] = \frac{N}{N - n}.
\]

where

\[
\delta_n^2 = \frac{\sigma^2}{n}, \quad m_n^2 = \frac{m \sigma_n^2 + \sigma^2 U_{sn}}{\delta_n^2 + \sigma^2_\mu}, \quad \sigma_n^2 = \frac{\sigma_\mu^2 \cdot \delta_n^2}{\delta_n^2 + \sigma^2_\mu}.
\]

The above analysis can be extended to a \(s\)-states statistical process. For any possible state \(i\) define the following:

\(P_i\) = fraction of time the process is in state \(i\) where \(\sum_i P_i = 1\)

\(m_i\) = mean of the distribution of the lot mean \(i = 1, 2, \cdots, s\)

\(\sigma_{m_i}^2\) = variance of the mean lot means defined in Appendix A

\(h(\mu)\) = the p.d.f. of the mean of the lot given by expression (4.32)

Under the above assumptions \(Q_1(U_A, n)\) given in (4.141) will be the sum of \(s\) terms

\[
Q_1(U_A, n) = Q_{11} + Q_{12} + \cdots + Q_{1s} = \sum_{i=1}^{s} Q_{1i}.
\]

A similar expression holds for \(Q_2(U_A, n)\) given in (4.142)

\[
Q_2(U_A, n) = Q_{11} + Q_{12} + \cdots + Q_{1s} = \sum_{i=1}^{s} Q_{2i}.
\]
Expression (4.147) can be modified to:

\[
\sum_{i=1}^{s} Q_{1i} + \frac{n[(K_A - K_P) + K_{R1}]}{(N - n)K_{R1} - K_A N} \\
\cdot \frac{\sigma_{\mu} \cdot \sigma^2 \sqrt{n - 1}}{n \sqrt{\sigma^2_{\mu} + \delta^2_n}} \sum_{i=1}^{s} Q_{2i} \\
= \left( \frac{nK_{R1} - [K_A(N - n) + K_P n]}{(N - n)K_{R1} - K_A N} \right) \sum_{i=1}^{s} Q_{1i}.
\]

Expression (4.151) can also be modified by employing expression (4.153) and (4.154), hence:

\[
\sum_{i=1}^{s} Q_{1i} + \frac{n}{N - n} \sum_{i=1}^{s} Q_{2i} = \frac{2N - n}{N - n}.
\]

In single or multistate statistical process the following expressions are true due to symmetry about the minimum value of the true mean: are

\[
\frac{\partial K_T}{\partial U_A} = -\frac{\partial K_T}{\partial L_A} \quad \text{and} \quad \frac{\partial K_T}{\partial U_{sn}} = -\frac{\partial K_T}{\partial L_{sn}}.
\]

For a multistate statistical process the partial derivatives are functions of \( U_A, L_A, U_{sn}, L_{sn}, m_i \) and \( \sigma^2_{mi} \). Hence

\[
\sum_{i=1}^{s} (K'_T)|_{U_A}(U_A, m_i, \sigma^2_{mi}) = -\sum_{i=1}^{s} K'_T(L_A, m_i, \sigma^2_{mi} \]

and

\[
\sum_{i=1}^{s} (K'_T)|_{U_{sn}}(U_{sn}, m_i, \sigma^2_{mi}) = -\sum_{i=1}^{s} (K'_T)|_{L_{sn}}(L_{sn}, m_i, \sigma^2_{mi}).
\]

Expressions (4.147), (4.150) and (4.151) allow the determination of the decision points for a single state statistical process. Expressions (4.155), (4.156), (4.158) and (4.159) are the determination of the decision points for a multistate statistical process. The decision points are employed to determine the components of the total cost and the optimal value of the total cost.
4.7. A General Model for Rejection and Acceptance Costs

In this model lot inspection is conducted relative to $q$ quality characteristics $q > 1$, therefore $i = 1, 2, \cdots, q$ represents the $i^{th}$ characteristic. In this section destructive and nondestructive sampling plans are considered. In both plans it is assumed that when rejection occurs on one or more quality characteristics the lot is scrapped. The total cost in this case is the cost of disposing and replacing items destroyed during destructive sampling inspection, the cost of accepting defective units on scrappable characteristics, and the cost of reworking items found during nondestructive testing if the lot is accepted. For a nondestructive sampling plan the cost relative to the decision of screening the rejected lot, also the cost of screening a portion and scrapping a portion of the rejection lots are considered.

4.7.1. Assumptions

(a) Each quality characteristic variable is governed by a certain distribution. These distributions are assumed to be independent.

(b) For $q_d$ quality characteristics $q_d \leq q$ the sampling is destructive. The entire lot is scrapped if rejection occurs on one or more characteristics.

(c) For $(q - q_d)$ quality characteristics the sampling inspection is non destructive.

(c.1) The lot is scrapped if rejection occurs on one or more characteristics.

(c.2) The lot is screened if rejection occurs on one or more characteristics.

4.7.2. Expected Inspection Cost Per Lot

Samples of sizes $n_1, n_2 \cdots n_q$ are drawn from lots of size $N$ each. The samples are inspected relative to quality characteristics $i = 1, 2, \cdots, q$. The
expected inspection cost per lot is:

\[ E(\text{inspection cost per lot}) = \sum_{i=1}^{q} K_i n_i. \] (4.160)

4.7.3. Expected Cost of Rejection when Sampling is Destructive and the Rejected Lots are Scrapped.

The probability of acceptance relative to variable \( i \) is

\[ P_{ai} = p(\text{acceptance on variable}|m_i). \] (4.161)

Expression (4.161) can be written as:

\[ P_{ai} = 1 - \delta_{qi} \left[ 1 - \int_{L_{ADi}}^{U_{ADi}} T(\bar{X}_{ni}|\mu_i) \right] dx_i \] (4.162)

where

\[ \delta_{qi} = \begin{cases} 0 & n_i = 0 \\ 1 & n_i > 0 \end{cases} \] (4.163)

and \( L_{ADi}, U_{ADi} \) are the lower and upper limits of the mean of the \( i^{th} \) variable.

The expected cost of scrapping per lot is then:

\[ E(\text{scrapping cost per lot}) = K_{DN} - K_{DN} \prod_{i=1}^{q} \int_{-\infty}^{+\infty} P_{ai} h(\mu_i) d\mu_i = K_{DN} \]

\[ - K_{DN} \prod_{i=1}^{q} \int_{-\infty}^{+\infty} [1 - \delta_{qi} \left[ 1 - \int_{L_{ADi}}^{U_{ADi}} T(\bar{X}_{ni}|\mu_i) dx_i \cdot h(\mu_i) d\mu_i. \right] \] (4.164)

Setting \( \delta_{qi} = 1 \), reduces (4.164) to:

\[ E(\text{scrapping cost per lot}) = K_{DN} \]

\[ - K_{DN} \prod_{i=1}^{q} \int_{-\infty}^{+\infty} \int_{L_{ADi}}^{U_{ADi}} T(\bar{X}_{ni}|\mu_i) dx_i \cdot h(\mu_i) d\mu_i. \] (4.165)
Expression (4.165) is based on the decision of rejecting lots where sampling inspection is destructive.

4.7.4. Expected Cost of Rejection When Sampling is Nondestructive and the Rejected Lots are Scrapped

An expression similar to expression (4.165) for the expected cost of rejection on nondestructive variables can be derived. The basic assumptions here are: The good items found during sampling inspection are not scrapped, the lot is accepted on all destructive variables and is rejected on one or more nondestructive variables.

An expression for the probability of acceptance can be written as

\[ p_{ai}' = 1 - \delta_{qi} [1 - \int_{L_i}^{U_i} t(x_i|\bar{x}_i, \mu_i)dx_i] \]  

(4.166)

where \( \delta_{qi} \) is defined by expression (4.163). The expected cost of rejection is then

\[ E(\text{cost of rejection}) = K_{sp} \left\{ N - n_i \prod_{i=q_d+1}^{q} p_{ai}' \right\} \cdot \left\{ \prod_{i=1}^{q_d} p_{ai} \cdot (1 - \prod_{i=q_d+1}^{q} p_{ai}') \right\}. \]

(4.167)

Expression (4.167) can be written as

\[ E(\text{cost of rejection}) = K_{sp}(\bar{x}_i, \mu_i). \]

(4.168)

The unconditional expected cost of rejection is obtained by averaging overall sample and lot means, hence the expected cost of rejection is:

\[ E(CRNDV) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \int_{L_{AD_1}}^{U_{AD_1}} \cdots \int_{L_{AD_q}}^{U_{AD_q}} \int_{L_{AD_q}}^{U_{AD_q}} K_{sp}(\bar{x}_i, \mu_i)T(\bar{X}_1|\mu_1) \cdots T(\bar{X}_q|\mu_q) \cdots d\bar{x}_1 \cdots d\bar{x}_q \cdots h(\mu_1) \cdots h(\mu_q). \]

(4.169)
4.8. Expected Cost of Acceptance

The cost of acceptance consists of the following components:

(a) The cost of replacing units destroyed in sampling inspection relative to the characteristics $i = 1, 2, \cdots, q_d$.

(b) The cost of accepting units with dimensions below or above the specification limits relative to $q$ variables of the remaining lots.

(c) The cost of accepting units with dimensions below or above the specification limits relative to each variable for samples $n_1, n_2, \cdots, n_q$.

(d) The cost of accepting units in samples $n_{q_d+1}, \cdots, n_{q_d}$ with dimensions below or above the specification limits.

(e) The cost of repairing items for samples $n_{q_d+1}, \cdots, n_q$ with dimensions below or above specification limits. The cost acceptance as a function of $q$ sample means and lot means is

\[
K_A(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_{q_d}, \bar{x}_{q_d+1}, \cdots, \bar{x}_q, \mu_1, \mu_2, \cdots, \mu_{q_d+1}, \cdots, \mu_q) = K_A(\bar{x}, \mu).
\]  

Expression (4.170) is given by:
\[ K_A(\overline{x}, \mu) = K_{BD} \sum_{i=1}^{q_d} n_i \]

\[ + \sum_{i=1}^{q} [(N - n_i)K_{PL_i} \int_{-\infty}^{L_i} f(x_i | \mu_i) dx_i] \]

\[ + (N - n_i)K_{PU_i} \int_{U_i}^{\infty} f(x_i | \mu_i) dx_i \]

\[ + \sum_{i=1}^{q} [n_iK_{PL_i} \int_{-\infty}^{L_i} t(x_i | \overline{x}_i, \mu_i) dx_i] \]

\[ + n_iK_{PU_i} \int_{U_i}^{\infty} t(x_i | \overline{x}_i, \mu_i) dx_i \]

\[ - \sum_{i=q+1}^{q_d} [K_{RL_i} n_i \int_{-\infty}^{L_i} t(x_i | \overline{x}_i, \mu_i) dx_i] \]

\[ + K_{PU_i} n_i \int_{U_i}^{\infty} t(x_i | \overline{x}_i, \mu_i) dx_i \]

\[ + \sum_{i=q+1}^{q} [K_{RL_i} n_i \int_{-\infty}^{L_i} t(x_i | \overline{x}_i, \mu_i) dx_i] \]

\[ + K_{RU_i} n_i \int_{U_i}^{\infty} t(x_i | \overline{x}_i, \mu_i) dx_i \].

To simplify the problem assume

\[ K_{PL_i} = K_{PU_i} = K_{P1} \quad \text{and} \quad K_{RL_i} = K_{RU_i} = K_{R1}. \]
Under the assumptions stated in (4.172) permits rewriting (4.171) as

\[ K_A(\overline{x}, \mu) = K_{BD} \sum_{i=1}^{q_d} n_i \]

\[ + \sum_{i=1}^{q} [(N - n_i)K_{P1}(1 - \int_{L_i}^{U_i} f(x_i|\mu_i)dx_i)] \]

\[ + n_iK_{P1}(1 - \int_{L_i}^{U_i} t(x_i|\overline{x}_i, \mu_i)dx_i] \]

\[ + \sum_{i=q_{d+1}}^{q} [(K_{R1} - K_{P1})n_i(1 - \int_{L_i}^{U_i} t(x_i|\overline{x}_i, \mu_i)dx_i]. \]...

Therefore expression (4.173) can be written as

\[ K_A(\overline{x}, \mu) = \sum_{i=1}^{q_d} K_A(\overline{x}_{q_d}, \mu_{q_d}) + \sum_{i=q_{d+1}}^{q} K_A(\overline{x}_{q_{d+1}}, \mu_{q_{d+1}}). \]...

The unconditional expected cost of scrapping is obtained by removing the conditions on \( \overline{x} \) and \( \mu \). Hence the expected acceptance cost per lot is:
In practical situations sampling is conducted relative to a certain variable using the techniques developed earlier in this chapter. The parameters involved are estimated following the same procedures presented previously. If the lot is rejected proper action is taken to dispose of the lot. However the disposition limits for the sample mean vary from lot to lot. Some special cases are:

1. Assume that all the variables are destructive scrapable, then \((q \sum_{i=qd+1}^{q} K(\bar{x}_{q}, \mu_{q}))\) of expression (4.174) will vanish. The acceptance cost will be reduced to:

\[
K_{A}(\bar{x}, \mu) = \sum_{i=1}^{q} K(\bar{x}_{q}, \mu_{q}).
\]  

(4.176)

2. Assume the acceptance cost of a unit with dimensions above the upper
specification limit is the same as the cost of acceptance of a unit below the lower specification limit, that is \( K_{PU} = K_{PL} \).

(3) Under the assumption that \( P_{1D}(\mu), Q_{1D}(\overline{x}, \mu) \) given in (4.71) and (4.72) are identical and special cases 1 and 2 hold true then:

\[
K_A(\overline{x}, \mu) = K_{BD} \sum_{i=1}^{q} n_i + \sum_{i=1}^{q} (N - n_i)K_{P1}(1 - P_{1D}(\mu_i)) + n_iK_{P1}(1 - P_{1D}(\mu_i)).
\]  

(4.175)

\( K_A(\overline{x}, \mu) \) can be written as:

\[
K_A(\overline{x}, \mu) = \sum_{i=1}^{q} n_iK_{BD} + K_{P1}N(1 - P_{1D}(\mu_i))
\]  

(4.176)

Computer programs written in FORTRAN 77 (Appendices D, E, F) are employed to estimate the expected costs of acceptance, screening, and scrapping. The decision points and other quantities are also calculated. The following is an example.
TABLE 1: PRIOR COST AND RELATED SPECIFICATIONS

INPUT: MODEL SPECIFICATIONS

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>LOWER LIMIT OF THE O.C.X</td>
<td>6.500000</td>
</tr>
<tr>
<td>VARIANCE OF X_i</td>
<td>0.062500</td>
</tr>
<tr>
<td>VARIANCE OF THE MEAN OF X_i</td>
<td>0.004200</td>
</tr>
<tr>
<td>UNIT COST OF SCREENING CSN</td>
<td>0.300000</td>
</tr>
<tr>
<td>UNIT COST OF ACCEPTANCE KA</td>
<td>8.000000</td>
</tr>
<tr>
<td>MEAN1</td>
<td>7.100000</td>
</tr>
<tr>
<td>SAMPLING ERROR (TYPE1) KE1</td>
<td>0.800000</td>
</tr>
<tr>
<td>SAMPLING ERROR (TYPE2) KE2</td>
<td>0.400000</td>
</tr>
<tr>
<td>SALE PRICE KR</td>
<td>2.000000</td>
</tr>
<tr>
<td>JUNK VALUE KJ</td>
<td>0.000000</td>
</tr>
<tr>
<td>RATE OF REWORKING DEFECTIVES</td>
<td></td>
</tr>
<tr>
<td>ITEMS RE WITH SUCCESS KC</td>
<td>0.500000</td>
</tr>
<tr>
<td>PROPORTION OF ITEMS RE: WITHOUT SUCCESS KV</td>
<td>0.500000</td>
</tr>
</tbody>
</table>

OUTPUT: PRIOR AND EXPECTED COSTS PER UNIT

<table>
<thead>
<tr>
<th>Expected Cost</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>EXPECTED COST PER UNIT-ACCEPT</td>
<td></td>
</tr>
<tr>
<td>OUTRIGHT WITHOUT SAMPLING E(C1)</td>
<td>0.43511</td>
</tr>
<tr>
<td>E(C1)=C2 AT POI</td>
<td>0.28000</td>
</tr>
<tr>
<td>COST PER UNIT-REJECT OUTRIGHT AND SCRAP C2:</td>
<td>1.40000</td>
</tr>
<tr>
<td>EXPECTED COST PER UNIT-REJECT OUTRIGHT AND SCREEN E(C3):</td>
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<tr>
<td>E(C1)+E(C3) AT PO2</td>
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### TABLE 2: ACCEPTANCE COST AND RELATED SPECIFICATIONS

**INPUT: MODEL SPECIFICATIONS**

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<thead>
<tr>
<th>Description</th>
<th>Value</th>
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<td>7.50000</td>
</tr>
<tr>
<td>Lower limit of the Q.C. X₁</td>
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</tr>
<tr>
<td>Variance of X</td>
<td>0.06250</td>
</tr>
<tr>
<td>Variance of the mean of X</td>
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</tr>
<tr>
<td>Unit cost of screening</td>
<td>0.30000</td>
</tr>
<tr>
<td>Unit cost of acceptance</td>
<td>5.00000</td>
</tr>
<tr>
<td>Cost of scraping or replacing a defective unit found during sampling or screening inspection</td>
<td>0.60000</td>
</tr>
<tr>
<td>Unit cost of scraping</td>
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</tr>
<tr>
<td>Lot size</td>
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**OUTPUT: SAMPLE SIZE, ROOTS OF THE COST FUNCTION, POSTERIOR AND SAMPLING COSTS PER UNIT**

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<td>a: Variance of the mean of X</td>
</tr>
<tr>
<td></td>
<td>b: Variance of X</td>
</tr>
<tr>
<td></td>
<td>n: Sample size</td>
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<td>Lower disposition limit for lot acceptance</td>
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<td>Upper disposition limit for lot acceptance</td>
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<tr>
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<td>Upper disposition limit of the sample mean</td>
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<td>Acceptance cost per lot</td>
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</tr>
<tr>
<td>3</td>
<td>THE ACCEPTANCE COST PER LOT.</td>
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<tr>
<td>5</td>
<td>THE EXPECTED VALUE OF THE COST OBTAINED BY SUMMING OVER ALL SAMPLE MEANS AND LOT MEANS.</td>
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<th>ET-COST</th>
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DENSITY PRODUCT FACTOR = 0.859623758426818
DENSITY PRODUCT FACTOR = 0.8559841011386036
DENSITY PRODUCT FACTOR = 0.8524189462828511
DENSITY PRODUCT FACTOR = 0.8489265767021979
Figure 8. The Lower Limit of the Sample Mean for Lot Acceptance Vs. the Sample Size for a Nondestructive Variable Sampling Plan for Fraction Defective.
Figure 9. The Upper Limit of the Sample Mean for lot Acceptance Vs. the Sample Size for Nondestructive Variable Sampling Plan for Fraction Defective.
Figure 10. The Cost of Acceptance Per Lot Vs. the Sample Size for Nondestructive Variables Sampling Plan for Fraction Defective.
### TABLE 2: SCREENING COST AND RELATED SPECIFICATIONS

**INPUT: MODEL SPECIFICATIONS**

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<tr>
<td>LOWER LIMIT OF THE O.C. X</td>
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</tr>
<tr>
<td>VARIANCE OF X</td>
<td>0.06250</td>
</tr>
<tr>
<td>VARIANCE OF THE MEAN OF X</td>
<td>0.00420</td>
</tr>
<tr>
<td>UNIT COST OF SCREENING</td>
<td>0.30000</td>
</tr>
<tr>
<td>UNIT COST OF ACCEPTANCE</td>
<td>0.60000</td>
</tr>
<tr>
<td>COST OF SCRAPING OR REPLACING A DEFECTIVE UNIT FOUND DURING SAMPLING OR SCREENING INSPECTION</td>
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**OUTPUT: SAMPLE SIZE, ROOTS OF THE COST FUNCTION, POSTERIOR AND SAMPLING COSTS PER UNIT**

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<td>7</td>
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<td>(P_1) IS THE FRACTION DEFECTIVE AT WHICH THE COSTS OF ACCEPTANCE AND SCREENING ARE EQUAL</td>
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<tr>
<td>5</td>
<td>THE EXPECTED VALUE OF THE COST OBTAINED BY SUMMING OVER ALL SAMPLE MEANS AND LOT MEANS.</td>
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Figure 11. The Cost of Screening Per Lot Vs. the Sample Size for a Nondestructive Variables Sampling Plan for Fraction Defective for a Lot of Size 1000.
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<td></td>
</tr>
<tr>
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DENSITY PRODUCT FACTOR = 0.676659442639973
DENSITY PRODUCT FACTOR = 0.677777776626595
DENSITY PRODUCT FACTOR = 0.680599993591675
DENSITY PRODUCT FACTOR = 0.684952506970624
DENSITY PRODUCT FACTOR = 0.68048666061698284

OUTPUT OF PROGSA:

1- THE SAMPLE SIZE.
3- THE TOTAL EXPECTED COST.

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<th>PGU</th>
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<tr>
<td>53</td>
<td>0.4197E+05</td>
<td>0.3443E+04</td>
<td></td>
<td></td>
</tr>
<tr>
<td>54</td>
<td>0.4252E+05</td>
<td>0.3438E+04</td>
<td></td>
<td></td>
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<tr>
<td>55</td>
<td>0.4304E+05</td>
<td>0.3437E+04</td>
<td></td>
<td></td>
</tr>
<tr>
<td>56</td>
<td>0.4355E+05</td>
<td>0.3439E+04</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Figure 12. Total Scrapping Cost Per Lot for the Data Given for a Nondestructive Variables Sampling Plan for Fraction Defective for a Lot of Size 1000.
Figure 13. The Expected Total Cost vs. the Sample Size for a Nondestructive Variable Sampling Plan for Fraction Defective for a Lot of Size 1000.
CHAPTER 5

CONDITIONAL EXPECTATIONS AND MOMENTS OF FRACTION DEFECTIVE $p$ RAISED TO THE $k^{TH}$ POWER

5.1. Introduction

Variable sampling permits the specification of a particular quality parameter in statistical terms (e.g., its mean and variance). Using this information, the analyst can then assess the relative changes in a unit's fraction defective for an ongoing process. This is done herein by examining the expected fraction defective raised to the $k^{th}$ power, $E(p^k)$. These moments along with their corresponding upper and lower limits provide a control chart type measure of quality. The main advantage of this analysis technique over the classical control chart approach is the availability of all the $k$ moments. In a modern factory the QC processes are automated at different production stages. This permits the collection and storage of variable sampling data quickly and accurately. The QC processes are for the most part loop-type control structures. Each process employs automatic test equipment such as sensors and cameras connected to a centrally located micro-computer. This micro-computer monitors the levels of quality at all production stages in the factory. Since variable sampling tests can now be automated, the Bayesian approach provides the overall methodology for a particular QC process.

It should be emphasized that finding the $k$ moment, $E(p^k)$ leads to derivation of the conditional moments $E(p|x)$ and $E(p|x)$. These allow updating of fraction defective $p$ with respect to the sample mean $\bar{x}$ and QC test result, $x$. 
5.2. Model Development

5.2.1. Expected Value of the Fraction Defective—p

Expression (2.7) provides a measure of the fraction defective, p, in terms of the lot mean, \( \mu(p) \) and variance, \( \sigma^2 \). Namely,

\[
\frac{dp}{d\mu(p)} = -\frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{1}{2} \left( \frac{L-\mu(p)}{\sigma^2} \right)^2}
\]  

(5.1)

where \( L \) = lower limit of the quality characteristic. For analysis purposes, the lot mean is assumed to be normally distributed with mean \( m \) and variance, \( \sigma^2 \) (i.e., \( \mu(p) \sim N(m, \sigma^2) \)). The weighting probability distribution function (p.d.f.) \( p \) is

\[
w(p) = h(\mu|m, \sigma^2) \left| \frac{d\mu(p)}{dp} \right|.
\]  

(5.2)

Hence

\[
w(p) = \frac{1}{\sigma \mu} e^{-\frac{1}{2} \left( \frac{\mu(p) - m}{\sigma} \right)^2} \cdot \sigma e^{-\frac{1}{2} \left( \frac{L-\mu(p)}{\sigma^2} \right)^2}.
\]  

(5.3)

The number of defectives, \( x \), in a sample conditional p.d.f. with respect to \( p \) is given by the binomial density

\[
f(x|p) = \binom{n}{x} p^x (1 - p)^{n-x}.
\]  

(5.4)

Combining expressions (5.3) and (5.4) with Bayes formula yields the joint p.d.f. of \( x \) and \( p \). Therefore,

\[
l(x,p) = f(x|p)w(p)
\]

\[
= \binom{n}{x} p^x (1 - p)^{n-x} \frac{\sigma}{\sigma \mu} e^{-\frac{1}{2} \left( \frac{\mu(p) - m}{\sigma} \right)^2 - \frac{1}{2} \left( \frac{L-\mu(p)}{\sigma^2} \right)^2}.
\]  

(5.5)

The marginal p.d.f. for the number of defectives is now represented by

\[
l(x) = \int_0^1 l(x,p)dp
\]

\[
= \int_0^1 \binom{n}{x} p^x (1 - p)^{n-x} \frac{\sigma}{\sigma \mu} e^{-\frac{1}{2} \left( \frac{\mu(p) - m}{\sigma} \right)^2 - \frac{1}{2} \left( \frac{L-\mu(p)}{\sigma^2} \right)^2}.
\]  

(5.6)
Employing the binomial theorem yields an expression for $p^x(1 - p)^{n-x}$ as

$$p^x(1 - p)^{n-x} = \sum_{y=0}^{n-x} \binom{n-x}{y} (-p)^y \cdot p^x$$

$$= \sum_{y=0}^{n-x} \binom{n-x}{y} (-1)^y p^{x+y}. \tag{5.7}$$

If $\alpha = x + y$ then

$$p^x(1 - p)^{n-x} = \sum_{\alpha=x}^{n} \binom{n-x}{\alpha-x} (-1)^{\alpha-x} p^\alpha. \tag{5.8}$$

Substituting expression (5.8) into (5.6) results in

$$l(x) = \int_0^1 \sum_{\alpha=x}^{n} \binom{n}{x} \binom{n-x}{\alpha-x} (-1)^{\alpha-x} p^\alpha \frac{\sigma}{\sigma^2} e^{-\frac{1}{2} \left( \frac{(\mu(x)-m)^2}{\sigma^2} + \frac{(x-v)^2}{\sigma^2} \right)} dp. \tag{5.9}$$

This expression can be simplified by setting $v = \mu(p)$ and $p = p(v)$. Now with $dv = \mu'(p)dp$ and $dp = p'(v)dv$ expression (5.9) transforms to:

$$l(x) = \sum_{\alpha=x}^{n} \binom{n}{x} \binom{n-x}{\alpha-x} \int_{-\infty}^{\infty} \frac{(-1)^{\alpha-x} p^\alpha(v) e^{-\frac{1}{2} \left( \frac{(v-m)^2}{\sigma^2} \right)}}{\sqrt{2\pi}\sigma^2} dv. \tag{5.10}$$

Letting

$$\beta_\alpha(v) = \frac{1}{\sqrt{2\pi}\sigma^2} p^\alpha(v)e^{-\frac{1}{2} \left( \frac{(v-m)^2}{\sigma^2} \right)} \tag{5.11}$$

yields an expression for $l(x)$. Therefore

$$l(x) = \sum_{\alpha=x}^{n} \binom{n}{x} \binom{n-x}{\alpha-x} (-1)^{\alpha-x} \int_{-\infty}^{\infty} \beta_\alpha(v)dv. \tag{5.12}$$

The conditional p.d.f. of $p$ given $x$ is found from relations (5.5) and (5.12) as:

$$w(p|x) = \frac{l(x,p)}{l(x)} = \frac{\frac{l(x,p)}{\sum_{\alpha=x}^{n} \binom{n}{x} \binom{n-x}{\alpha-x} (-1)^{\alpha-x} \int_{-\infty}^{\infty} \beta_\alpha(v)dv}}. \tag{5.13}$$
Now the expected value of the fraction defective given the number of defective in the sample is:

\[ E(p|x) = \int_0^1 pw(p|x)dp = \int_0^1 \frac{p(l(x,p)dp}{l(x)}. \quad (5.14) \]

Substituting expressions (5.5) and (5.7) into (5.14) yields:

\[ E(p|x) = \int_0^1 p^{(n)}p^{z}(1-p)^{n-x} \frac{e^{-\frac{1}{2}\frac{(p(x)-m)^2}{\sigma^2}}}{(\alpha-1)^{\alpha-z}} \sum_{\alpha=x}^{\infty} \beta_\alpha(v)dv. \quad (5.15) \]

Again by setting \( v = \mu(p) \) and \( p = p(v) \) results in \( dv = \mu'(p)dp \) and \( dp = p'(v)dv \). Transforming relation (5.15) therefore provides:

\[ E(p|x) = \int_0^\infty \frac{p_{\alpha+1}(v)[1-p(v)]^{n-x}}{\alpha-1} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2}\frac{(v-m)^2}{\sigma^2}} \sum_{\alpha=x}^{\infty} \beta_\alpha(v)dv. \quad (5.16) \]

Since

\[ \left(\begin{array}{c} n \\ x \end{array}\right) p^{x+1}(v)[1-p(v)]^{n-x} = \sum_{\alpha=x}^{\infty} \left(\begin{array}{c} n \\ x \end{array}\right) \left(\begin{array}{c} n-x \\ \alpha-x \end{array}\right)(-1)^{\alpha-x} p^{\alpha+1}. \quad (5.17) \]

\( E(p|x) \) can be rewritten as:

\[ E(p|x) = \frac{\int_0^\infty \sum_{\alpha=x}^{\infty} \left(\begin{array}{c} n \\ x \end{array}\right) \left(\begin{array}{c} n-x \\ \alpha-x \end{array}\right)(-1)^{\alpha-x} p^{\alpha+1} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2}\frac{(v-m)^2}{\sigma^2}} \sum_{\alpha=x}^{\infty} \beta_\alpha(v)dv. \quad (5.18) \]

Finally if

\[ \beta_{\alpha+1}(v) = p^{\alpha+1} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2}\frac{(v-m)^2}{\sigma^2}} \quad (5.19) \]

then expression (5.18) can be shown to be

\[ E(p|x) = \frac{\sum_{\alpha=x}^{\infty} \left(\begin{array}{c} n \\ x \end{array}\right) \left(\begin{array}{c} n-x \\ \alpha-x \end{array}\right)(-1)^{\alpha-x} \int_0^\infty \beta_{\alpha+1}(v)dv \sum_{\alpha=x}^{\infty} \left(\begin{array}{c} n \\ x \end{array}\right) \left(\begin{array}{c} n-x \\ \alpha-x \end{array}\right)(-1)^{\alpha-x} \int_0^\infty \beta_\alpha(v)dv. \quad (5.20) \]
Examining relations (5.11), (5.19) and (5.20) indicate that

\[ E(p^\alpha) = \int_{-\infty}^{\infty} \beta_\alpha(v) dv. \]  

(5.21)

Table 1 presents the results of \( \int_{-\infty}^{\infty} \beta_\alpha(v) dv \) for the special cases of \( \alpha = 0 \) and \( \alpha = 1 \), respectively. Again if \( dv = \mu'(p) dp \) and \( dp = \mu'(v) dv \) then

\[ dp = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} \frac{(v-m)^2}{\sigma^2}} dv. \]  

(5.22)

Using this transformation for the case where \( \alpha = 1 \) (see Table 2) results in:

\[ E(p) = \int_{-\infty}^{\infty} \beta_1(v) dv \]

\[ = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} \frac{(u-v)^2}{\sigma^2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} \frac{(v-m)^2}{\sigma^2}} dv. \]  

(5.23)

Table 1. Special Cases for \( \beta_\alpha(v) \).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Factor ( \beta_\alpha(v) )</th>
<th>( E(p^\alpha) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha = 0 )</td>
<td>( \beta_0(v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} \frac{(u-v)^2}{\sigma^2}} )</td>
<td>( E(p^0) = \int_{-\infty}^{\infty} \beta_0(v) dv = 1 )</td>
</tr>
<tr>
<td>( \alpha = 1 )</td>
<td>( \beta_1(v) = \frac{p(v)}{\sqrt{2\pi}} e^{-\frac{1}{2} \frac{(u-v)^2}{\sigma^2}} )</td>
<td>( E(p^1) = \int_{-\infty}^{\infty} \beta_1(v) dv )</td>
</tr>
</tbody>
</table>

Combining the exponential terms and rearranging the integrals in expression (5.23) provides:

\[ E(p) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} \frac{(z-v)^2}{\sigma^2} + \frac{(u-m)^2}{\sigma^2}} dz dv. \]  

(5.24)

The sum of the exponential terms can be written as:

\[ \frac{(z-v)^2}{\sigma^2} + \frac{(u-m)^2}{\sigma^2} \frac{1}{\sigma^2 + \sigma^2}. \]  

(5.25)
where
\[ m' = \frac{m\sigma^2 + z\sigma^2_\mu}{\sigma^2 + \sigma^2_\mu} \]
\[ \sigma'^2 = \frac{\sigma^2 \sigma^2_\mu}{\sigma^2 + \sigma^2_\mu}. \tag{5.26} \]

Substituting expression (5.25) into (5.24) yields:
\[ E(p) = \int_{-\infty}^{\infty} \int_{-\infty}^{L} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2} \left[ \frac{(v-m')^2}{\sigma^2} + \frac{(v-m')^2}{\sigma^2 + \sigma^2_\mu} \right]} dv \, dz. \tag{5.27} \]

Multiplying expression (5.27) by the identity \( \left( \frac{\sigma^2 + \sigma^2_\mu}{\sigma^2 + \sigma^2_\mu} \right)^{\frac{1}{2}} \) results in:
\[ E(p) = \int_{-\infty}^{\infty} \int_{-\infty}^{L} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi\sigma}} \frac{1}{\sigma^2 + \sigma^2_\mu} e^{-\frac{1}{2} \left( \frac{(v-m')^2}{\sigma^2} + \frac{(v-m')^2}{\sigma^2 + \sigma^2_\mu} \right)} dv \, dz. \tag{5.28} \]

But \( \sigma' = \frac{\sigma \sigma_\mu}{(\sigma^2 + \sigma^2_\mu)^{\frac{1}{2}}} \) and rearranging the terms in this relation provides:
\[ E(p) = \int_{-\infty}^{L} \left[ \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma'} e^{-\frac{1}{2} \left( \frac{v-m'}{\sigma'} \right)^2} dv \right] \frac{1}{\sqrt{2\pi}\sigma} \frac{1}{\sigma^2 + \sigma^2_\mu} e^{-\frac{1}{2} \left( \frac{v-m'}{\sigma^2 + \sigma^2_\mu} \right)^2} dv \, dz. \tag{5.29} \]

Finally after simplification \( E(p) \) is restricted to:
\[ E(p) = \int_{-\infty}^{L} \frac{1}{\sqrt{2\pi} \left( \sigma^2 + \sigma^2_\mu \right)^{\frac{3}{2}}} e^{-\frac{1}{2} \left( \frac{v-m'}{\sigma^2 + \sigma^2_\mu} \right)^2} dv \, \Phi \left\{ \frac{(L-m)}{(\sigma^2 + \sigma^2_\mu)^{\frac{1}{2}}} \right\}. \tag{5.30} \]

The term on the far right represents the unit normal distribution evaluated at \( \frac{(L-m)}{(\sigma^2 + \sigma^2_\mu)^{\frac{1}{2}}}. \)

5.2.2. Covariance of \( \bar{x}_n \) and \( \mu - \text{Cov}(\bar{x}_n, \mu) \)

In general the covariance of \( \bar{x}_n \) and \( \mu \) is given by:
\[ \text{Cov}(\bar{x}_n, \mu) = E\{[\bar{x}_n - E(x_n)][\mu - E(\mu)]\} \tag{5.31} \]
or

\[ \operatorname{Cov}(\bar{x}_n, \mu) = E(\bar{x}_n, \mu) - E(\bar{x}_n)E(\mu). \]  \hfill (5.32)

Employing expression (2.112) namely

\[ T(\bar{x}_n) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\delta_n^2 + \sigma_\mu^2}} e^{-\frac{1}{2} \left( \frac{(\bar{x}_n - m)^2}{\delta_n^2 + \sigma_\mu^2} \right)} \]  \hfill (5.33)

where \( \delta_n^2 = \frac{\sigma^2}{n} \). The expected value of \( \bar{x}_n \) is then

\[ E(\bar{x}_n) = \int_{-\infty}^{\infty} \bar{x}_n T(\bar{x}_n) d\bar{x}_n \]
\[ = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\delta_n^2 + \sigma_\mu^2}} e^{-\frac{1}{2} \left( \frac{(\bar{x}_n - m)^2}{\delta_n^2 + \sigma_\mu^2} \right)} d\bar{x}_n. \]  \hfill (5.34)

this results in

\[ E(\bar{x}_n) = m \]  \hfill (5.35)

where \( m = \text{mean of the lot mean } \mu \). Finally, the expected value of \( \bar{x}_n \cdot \mu \) is given by the relation

\[ E(\bar{x}_n \cdot \mu) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{x}_n \cdot \mu T(\bar{x}_n | \mu) h(\mu) d\mu d\bar{x}_n \]  \hfill (5.36)

where \( T(\bar{x}_n, \mu) = T(\bar{x}_n | \mu) h(\mu) \). Now substituting the appropriate expressions for \( T(\bar{x}_n | \mu) \) and \( h(\mu) \) into expression (5.36) provides:

\[ E(\bar{x}_n \cdot \mu) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{x}_n \mu e^{-\frac{1}{2} \left( \frac{(\bar{x}_n - m)^2}{\delta_n^2 + \sigma_\mu^2 + \sigma^2} + \frac{(\bar{x}_n - m)^2}{\sigma^2 + \sigma_\mu^2} \right)} d\mu d\bar{x}_n \]  \hfill (5.37)

where \( \delta_n = \frac{\sigma}{\sqrt{n}} \). Multiplying expression (5.37) by \( \frac{\sigma_n}{\sigma} \) and rearranging terms yields

\[ E(\bar{x}_n \cdot \mu) = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} \frac{\mu}{\sqrt{2\pi} \sigma_n} e^{-\frac{1}{2} \left( \frac{(\bar{x}_n - m)^2}{\sigma_n^2} \right)} d\mu \right] \frac{\bar{x}_n \sigma_n}{\sqrt{2\pi} \delta_n \sigma_\mu} e^{-\frac{1}{2} \left( \frac{(\bar{x}_n - m)^2}{\sigma_n^2 + \delta_n^2 \sigma_\mu^2} \right)} d\bar{x}_n. \]  \hfill (5.38)
The term within the brackets whoever is \( m_n \), therefore expression (5.38) becomes

\[
E(\overline{x}_n \cdot \mu) = \frac{\sigma_n}{\sqrt{2\pi} \delta_n \sigma_\mu} \int_{-\infty}^{\infty} m_n \overline{x}_n e^{-\frac{1}{2} \left( \frac{(\overline{x}_n - m)^2}{\sigma_n^2 + \delta_n^2} \right)} d\overline{x}_n. \tag{5.39}
\]

Expression (2.77) shows that:

\[
m_n = \frac{m \delta_n^2 + \overline{x}_n \sigma_\mu^2}{\sigma_\mu^2 + \delta_n^2}. \tag{5.40}
\]

Substituting this relation into expression (5.39) and performing some algebra yields:

\[
E(\overline{x}_n \cdot \mu) = \frac{\sigma_n}{\delta_n \sigma_\mu (\sigma_\mu^2 + \delta_n^2)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \left[ \frac{m \delta_n^2 \overline{x}_n}{\sqrt{2\pi} (\sigma_\mu^2 + \delta_n^2)^{\frac{1}{2}}} \right.
+ \left. \frac{\sigma_\mu^2 \overline{x}_n^2}{\sqrt{2\pi} (\sigma_\mu^2 + \delta_n^2)^{\frac{1}{2}}} \right] e^{-\frac{1}{2} \left( \frac{(\overline{x}_n - m)^2}{\sigma_\mu^2 + \delta_n^2} \right)} d\overline{x}_n. \tag{5.41}
\]

Rewriting expression (5.41) in a more general form yields:

\[
E(\overline{x}_n \cdot \mu) = \frac{\sigma_n}{\delta_n \sigma_\mu (\sigma_\mu^2 + \delta_n^2)^{\frac{1}{2}}} \left[ m \delta_n^2 E(\overline{x}_n) + \sigma_\mu^2 E(\overline{x}_n^2) \right]. \tag{5.42}
\]

Finally with \( \sigma_n = \frac{\sigma_n \delta_n}{(\sigma_\mu^2 + \delta_n^2)^{\frac{1}{2}}} \), \( E(\overline{x}_n) = m \) and \( E(\overline{x}_n^2) = \sigma_\mu^2 + \delta_n^2 + m^2 \) expression (5.42) becomes

\[
E(\overline{x}_n \cdot \mu) = \frac{1}{\sigma_\mu^2 + \delta_n^2} \cdot [m^2 \delta_n^2 + \sigma_\mu^2 (\sigma_\mu^2 + \delta_n^2 + m^2)]. \tag{5.43}
\]

Substituting this quantity into expression (5.32) along with the fact that \( E(\overline{x}_n) \) \( E(\mu) = m^2 \) yields after simplification

\[
Cov(\overline{x}_n, \mu) = \sigma_\mu^2. \tag{5.44}
\]

It can be shown that the correlation coefficient between \( \overline{x}_n \) and \( \mu \) is

\[
\rho_{\overline{x}_n, \mu} = \frac{\sigma_\mu}{\sqrt{\sigma_\mu^2 + \delta_n^2}}. \tag{5.45}
\]
Here it is evident that as \( n \to \infty \), \( \delta_n^2 \to 0 \) and \( \rho_{\bar{x}_n, \mu} \to 1 \).

Now the objective is to generalize the covariance function as:

\[
Cov(\bar{x}_n, \alpha) = E(\bar{x}_n \cdot \alpha) - E(\bar{x}_n)E(\alpha). \tag{5.46}
\]

Earlier it was shown that \( E(\mu) = m \). The p.d.f. of \( \bar{x}_n \) may be written as:

\[
g(\bar{x}_n) = \int_{-\infty}^{\infty} T(\bar{x}_n, \mu) d\mu = \int_{-\infty}^{\infty} \frac{1}{2\pi \delta_n \sigma_{\mu}} e^{-\frac{1}{2} \left[ \frac{(\bar{x}_n - m)^2}{\sigma_n^2 + \delta_n^2} \right]} d\mu. \tag{5.47}
\]

Multiplying the above expression by \( \frac{\bar{x}_n}{\sigma_n} \) and letting \( \alpha = \mu \) yields after integration the following:

\[
g(\bar{x}_n) = \frac{1}{\sqrt{2\pi(\sigma_\alpha^2 + \delta_n^2)^{\frac{3}{2}}}} e^{-\frac{1}{2} \left( \frac{\bar{x}_n - m}{\sigma_\alpha^2 + \delta_n^2} \right)^2} \tag{5.48}
\]

where \( \sigma_n = \frac{\sigma_\alpha \delta_n}{(\delta_n^2 + \sigma_\alpha^2)^{\frac{1}{2}}} \). It is evident from expression (5.48) that \( E(\bar{x}_n) = m \).

Therefore from expression (5.46)

\[
Cov(\bar{x}_n, \alpha) = E(\bar{x}_n \cdot \alpha) - m^2. \tag{5.49}
\]

Following the previous derivations for \( E(\bar{x}_n \cdot \mu) \) it can be shown that

\[
E(\bar{x}_n \cdot \alpha) = m^2 + \sigma_\alpha^2. \tag{5.50}
\]

Substituting expression (5.50) into (5.49) yields

\[
Cov(\bar{x}_n, \alpha) = \sigma_\alpha^2. \tag{5.51}
\]

It is instructive to note that the correlation coefficient

\[
\rho_{\bar{x}_n, \alpha} = \frac{\sigma_\alpha}{(\sigma_\alpha^2 + \delta_n^2)^{\frac{1}{2}}}. \tag{5.52}
\]

Table 2 lists the special convergence cases for \( \rho_{\bar{x}_n, \alpha} \).
Table 2. Special cases for \( \rho_{\bar{x}_n}, \alpha \).

<table>
<thead>
<tr>
<th>Condition</th>
<th>( \rho_{\bar{x}_n}, \alpha ) Convergence</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n \to \infty )</td>
<td>( \rho_{\bar{x}_n}, \alpha = 1 )</td>
</tr>
<tr>
<td>( n \to N - 1 )</td>
<td>( \rho_{\bar{x}_n}, \alpha = \frac{\sigma^2}{\sqrt{\sigma_x^2 + \sigma^2}/N-1} )</td>
</tr>
</tbody>
</table>

5.3. Expected Value of \( p^k (k \geq 1) \)

For any \( \alpha = k \) expression (5.21) can be written as:

\[
E(p^k) = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{L} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(z-v)^2} dz \right]^K \frac{1}{\sqrt{2\pi}\sigma_{\mu}} e^{-\frac{1}{2} \frac{(v-\mu)^2}{\sigma_{\mu}^2}} dv
\]

where \( v = \mu(p) \). If the function \( T_K(v) \) is set equal to the \( dv \) integrand, that is

\[
T_K(v) = \left[ \int_{-\infty}^{L} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(z-v)^2} dz \right] \frac{1}{\sqrt{2\pi}\sigma_{\mu}} e^{-\frac{1}{2} \frac{(v-\mu)^2}{\sigma_{\mu}^2}}.
\]

Now expression (5.53) becomes

\[
E(p^k) = \int_{-\infty}^{\infty} T_k(v) dv.
\]

Relations (5.54) and (5.55) permit the development of a computer algorithm to find \( E(p^k) \). This algorithm employs an estimate for the first integral in expression (5.54).

5.3.1. Upper and Lower Limits for \( E(p^k) \)

The objective now is to determine the upper and lower limits for \( E(p^k) \).

This is done by considering the error in estimating the probability of acceptance for a particular lot (this is the probability of fraction defective \( p \) is less than an optimum value \( p_L \), \( p < p_L \)). For analysis purposes let \( F_1(v) \) represent the estimate of this probability. To have a good estimate it is required that

\[
\left| \int_{-\infty}^{L} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2} \frac{(z-v)^2}{\sigma^2}} dz - F_1(v) \right| < \epsilon(v)
\]
where $\epsilon(v) =$ error in estimating the probability of acceptance (i.e., very very small value). Now assume that $F_1(v)$ is of exponential form, that is expression (5.56) may be rewritten as:

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2} \left(\frac{v-v_m}{\sigma_v}\right)^2} dz - e^{g(v)} < \epsilon(v)$$

(5.57)

where $g(v) =$ function of $v$. As a first approximation the function $g(v)$ may take the following form:

$$g(v) = -\frac{1}{2} \frac{(v-v_m)^2}{\sigma_v^2}$$

(5.58)

where $v_m =$ mean of $v$; $\sigma_v^2 =$ variance of $v$. Substituting (5.58) into (5.57) and rearranging terms yields:

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2} \left(\frac{v-v_m}{\sigma_v}\right)^2} dz < \epsilon(v) + e^{-\frac{1}{2} \frac{(v-v_m)^2}{\sigma_v^2}}$$

(5.59)

and

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2} \left(\frac{v-v_m}{\sigma_v}\right)^2} dz > e^{-\frac{1}{2} \frac{(v-v_m)^2}{\sigma_v^2}} - \epsilon(v).$$

(5.60)

The right hand side quantifies in the above inequalities define the upper and lower limits, respectively for the integral. Raising each term in expressions (5.59) and (5.60) to the $k^{th}$ power yields:

$$\left[ e^{-\frac{1}{2} \frac{(v-v_m)^2}{\sigma_v^2}} - \epsilon(v) \right]^k < \left[ \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2} \left(\frac{v-v_m}{\sigma_v}\right)^2} dz \right]^k$$

(5.61)

$$< \left[ \epsilon(v) + e^{-\frac{1}{2} \frac{(v-v_m)^2}{\sigma_v^2}} \right]^k.$$

Employing the binomial theorem provides expressions for the left and right hand side inequalities as:

$$\left[ e^{-\frac{1}{2} \frac{(v-v_m)^2}{\sigma_v^2}} \pm \epsilon(v) \right]^k = \sum_{j=0}^{k} \binom{k}{j} e^{-\frac{1}{2} \frac{(v-v_m)^2}{\sigma_v^2}} \{\pm\epsilon(v)\}^{k-j}.$$

(5.62)
Therefore integrating these quantities over all possible \( v \) values yields:

\[
E(p^k_{\hat{U}}) = \int_{-\infty}^{\infty} \sum_{j=0}^{k} \begin{pmatrix} k \\ j \end{pmatrix} e^{-j \frac{(v-v_m)^2}{\sigma^2_v}} \{e(v)\}^{k-j} e^{-\frac{1}{2} \frac{(v-m)^2}{\sigma^2_\mu}} \ dv \tag{5.63}
\]

\[
E(p^k_L) = \int_{-\infty}^{\infty} \sum_{j=0}^{k} \begin{pmatrix} k \\ j \end{pmatrix} e^{-j \frac{(v-v_m)^2}{\sigma^2_v}} \{-e(v)\}^{k-j} e^{-\frac{1}{2} \frac{(v-m)^2}{\sigma^2_\mu}} \ dv. \tag{5.64}
\]

It can be shown that

\[
\frac{(v - v_m)^2}{\sigma^2_v/j} + \frac{(v - m)^2}{\sigma^2_\mu} = \frac{(v - m')^2}{\sigma'^2} + \frac{(v_m - m)^2}{\sigma^2_\mu + \sigma^2_v/j} \tag{5.65}
\]

where

\[
m' = \frac{m \sigma^2_\mu + v_m \sigma^2_\mu}{\sigma^2_\mu + \sigma^2_v/j}, \tag{5.66}
\]

\[
\sigma'^2 = \frac{\sigma^2_\mu}{\sigma^2_\mu + \sigma^2_v/j}. \tag{5.67}
\]

Expressions (5.63) and (5.64) can be simplified by letting

\[
I_v = \int_{-\infty}^{\infty} e^{-\frac{1}{2} \frac{(v-v_m)^2}{\sigma^2_v/j} + \frac{(v-m)^2}{\sigma^2_\mu}} \ dv. \tag{5.68}
\]

Substituting expression (5.65) in (5.67) and rearranging the integral where only a function of \( v \) appears yields

\[
I_v = e^{-\frac{1}{2} \frac{(v_m - m)^2}{\sigma^2_\mu + \sigma^2_v/j}} \int_{-\infty}^{\infty} e^{-\frac{1}{2} \frac{(v-m')^2}{\sigma'^2}} \ dv. \tag{5.69}
\]

Multiplying and dividing the right hand side of expression (5.68) by \( \sqrt{2\pi} \sigma' \) and performing the integration results in

\[
I_v = e^{-\frac{1}{2} \frac{(v_m - m)^2}{\sigma^2_\mu + \sigma^2_v/j}} \cdot \sqrt{2\pi} \sigma' \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi} \sigma'} e^{-\frac{1}{2} \frac{(v-m')^2}{\sigma'^2}} \ dv. \tag{5.69}
\]
Therefore replacing $\epsilon(v)$ with $\epsilon$ in expressions (5.63) and (5.64) provide estimates of the upper and lower limits for $E(p^k)$. That is,

$$E(p_U^k) = \epsilon^k + \sum_{j=1}^{k} \binom{k}{j} \epsilon^{k-j} \left[ \frac{\sigma_v^2}{\sigma_v^2 + j\sigma_{\mu}^2} \right]^{1/2} e^{-\frac{1}{2} \frac{(v_m-m)^2}{\sigma_v^2 + j\sigma_{\mu}^2}}$$

(5.70)

and

$$E(p_L^k) = (-1)^k \epsilon^k + \sum_{j=1}^{k} \binom{k}{j} \epsilon^{k-j} \left[ \frac{\sigma_v^2}{\sigma_v^2 + j\sigma_{\mu}^2} \right]^{1/2} e^{-\frac{1}{2} \frac{(v_m-m)^2}{\sigma_v^2 + j\sigma_{\mu}^2}}.$$  

(5.71)

The expected value of $p$ raised to the $K^{th}$ power is then:

$$E(p_L^k) < E(p^k) < E(p_U^k).$$

(5.72)

5.4. The Expected Value of $p^k$, an Alternative Approach

Employing the normal deviates $z_1, z_2, \ldots, z_n$, an expression for the expected value of $p$ raised to the $k^{th}$ power can be written as:

$$E(p^k) = \int_{-\infty}^{+\infty} \left[ \int_{-\infty}^{L} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2} (z_1 - v)^2 / \sigma^2} dz_1 \right]
\cdot \left[ \int_{-\infty}^{L} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2} (z_2 - v)^2 / \sigma^2} dz_2 \right] \ldots \left[ \int_{-\infty}^{L} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2} (z_n - v)^2 / \sigma^2} dz_n \right]
\cdot \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2} (v-m)^2 / \sigma^2} dv.$$  

(5.73)

Letting $s = v - m$ imply $ds = dv$ and $v = s + m$, then $z_1 - v = z_i - s - m$. Setting $t_i = z_i - m$ yields $dt_i = dz_i$. Consider the following expression:

$$\int_{-\infty}^{L} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2} (z_1 - v)^2 / \sigma^2} dz_1 \cdot \int_{-\infty}^{L} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2} (z_2 - v)^2 / \sigma^2} dz_2 \cdot \ldots
\cdot \int_{-\infty}^{L} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2} (z_k - v)^2 / \sigma^2} dz_k.$$  

(5.74)
Expression (5.74) can be written as
\[
\int_{-\infty}^{L} \int_{-\infty}^{L} \cdots \int_{-\infty}^{L} \frac{1}{(2\pi)^{k/2} \sigma^k} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^{k} (z_i - v)^2} dz_1 \cdots dz_k. \tag{5.75}
\]
Expression (5.75) can be reduced to:
\[
= \int_{-\infty}^{L-m} \int_{-\infty}^{L-m} \cdots \int_{-\infty}^{L-m} \frac{1}{(2\pi)^{k/2} \sigma^k} e^{-\frac{1}{2\sigma^2} \sum (t_i - s)^2} dt_1 \cdot dt_k. \tag{5.76}
\]
Substituting expression (5.76) into (5.73) yields
\[
E[p^k] = \frac{1}{(2\pi)^{k+1/2} \sigma^k} \int_{-\infty}^{+\infty} \left[ \int_{-\infty}^{L-m} \int_{-\infty}^{L-m} \cdots \int_{-\infty}^{L-m} e^{-\frac{1}{2} \sum (t_i - s)^2 / \sigma^2} dt_1 \right.
\]
\[
\cdots \cdot dt_k 1 \cdot e^{-\frac{1}{2} (v - m)^2 / \sigma^2} dv] . \tag{5.77}
\]
Consider the summations
\[
\sum_{i=1}^{k} \frac{(t_i - s)^2}{\sigma^2} + \frac{s^2}{\sigma^2}. \tag{5.78}
\]
This sum can be written in the following form:
\[
\frac{\sum t_i^2}{\sigma^2} - \frac{2s}{\sigma^2} \sum t_i + s^2 \left[ \frac{k \sigma^2_\mu + \sigma^2}{\sigma^2 \sigma^2_\mu} \right]. \tag{5.79}
\]
Expression (5.79) can be reduced to:
\[
\frac{A}{\sigma^2} - \frac{2s}{\sigma^2} B + s^2 \left[ \frac{k \sigma^2_\mu + \sigma^2}{\sigma^2 \sigma^2_\mu} \right] \tag{5.80}
\]
where \( A = \sum t_i^2, \) \( B = \sum t_i. \) Expression (5.80) can be written as:
\[
\frac{A}{\sigma^2} - \frac{\sigma^2_\mu B^2}{(k \sigma^2_\mu + \sigma^2) \sigma^2} + \frac{(k \sigma^2_\mu + \sigma^2)}{\sigma^2 \sigma^2_\mu} \left[ s - \frac{\sigma^2_\mu B}{k \sigma^2_\mu + \sigma^2} \right]^2. \tag{5.81}
\]
Now define \( \sigma^2_k \) and \( m_k \) as:
\[
\sigma^2_k = \frac{\sigma^2_\mu}{\sigma^2 + k \sigma^2_\mu} \tag{5.82}
\]
\[
m_k = \frac{\sigma^2_\mu B}{k \sigma^2_\mu + \sigma^2} = \sigma^2_k B. \]
Expression (5.81) can be transformed to:

\[
\frac{A}{\sigma^2} - \frac{\sigma^2 \mu^2}{(k\sigma^2 + \sigma^2)\sigma^2} + \frac{(k\sigma^2 + \sigma^2)}{\sigma^2 \sigma_k^2} [s - \sigma_k^2 B]^2. \tag{5.83}
\]

Substituting \(\sigma_k^2\) given in (5.82) into (5.83) yields:

\[
\frac{A}{\sigma^2} = \frac{\sigma^2 \mu^2}{(k\sigma^2 + \sigma^2)\sigma^2} + \frac{1}{\sigma^2 \sigma_k^2} [s - \sigma_k^2 B]^2. \tag{5.84}
\]

Using \(B = \sum_{i=1}^{k} t_i\), results in

\[
B^2 = \left[ \sum_{i=1}^{k} t_i \right]^2 = (t_1 + t_2 + t_3 + \cdots + t_k)^2
\]

\[
= t_1^2 + t_2^2 + t_3^2 + \cdots + t_k^2 + 2t_1t_2 + 2t_2t_3 + \cdots + 2t_k t_k
\]

\[
= \sum_{i=1}^{k} t_i^2 + 2 \sum_{i \neq j} t_i t_j. \tag{5.85}
\]

The sum \(\frac{k(k-1)}{2}\) is obtained by noticing that there are \(k\) numbers chosen two at a time. Hence \(\binom{k}{2} = \frac{k!}{2!(k-2)!} = \frac{k(k-1)}{2}\). Since \(A\) is defined to be \(\sum t_i^2\), then (5.85) can be written as:

\[
B^2 = A + 2 \sum_{i \neq j} t_i t_j. \tag{5.86}
\]

Setting

\[
\sum_{i \neq j} t_i t_j = C. \tag{5.87}
\]

Then expression (5.86) results in:

\[
B^2 = A + 2C. \tag{5.88}
\]

Employing expression (5.88), expression (5.83) can be expressed as:

\[
\frac{A}{\sigma^2} \left(1 - \sigma_k^2 \right) - 2C \frac{\sigma_k^2}{\sigma^2} + \frac{1}{\sigma^2 \sigma_k^2} (s - \sigma_k^2 B)^2. \tag{5.89}
\]
Combining expressions (5.73), (5.72) and (5.89) yields:

\[
E[p^k] = \frac{1}{(2\pi)^{k+1/2}\sigma^k\sigma_\mu} \int_{-\infty}^{+\infty} \int_{-\infty}^{L-m} \int_{-\infty}^{L-m} \ldots \int_{-\infty}^{L-m} e^{-\frac{1}{2} \sum (t_i - s)^2 / \sigma^2 - \frac{1}{2} s^2} \, dt_1 \ldots \, dt_k \cdot ds
\] (5.90)

or

\[
E[p^k] = \frac{1}{(2\pi)^{k/2}\sigma^k\sigma_\mu} \int_{-\infty}^{+\infty} \int_{-\infty}^{L-m} \int_{-\infty}^{L-m} \ldots \int_{-\infty}^{L-m} e^{-\frac{1}{2} [A/\sigma^2(1-\sigma^2_\mu)]} \cdot e^{-2C\sigma^2_k/\sigma^2 + 1/2\sigma^2_\mu(s-\sigma^2_kB)^2} \, dt_1 \ldots \, ds.
\] (5.91)

The integral

\[
\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma_k} e^{-\frac{1}{2} \left( \frac{s-\sigma^2_k}{\sigma^2_k} \right)^2} \, ds = 1.
\]

Then \(E[p^k]\) will be

\[
E[p^k] = \frac{1}{(2\pi)^{k/2}\sigma^k\sigma_\mu} \cdot \frac{\sigma_k}{\sqrt{2\pi}\sigma_k} \int_{-\infty}^{+\infty} \int_{-\infty}^{L-m} \int_{-\infty}^{L-m} \ldots \int_{-\infty}^{L-m} \ldots \int_{-\infty}^{L-m} \int_{-\infty}^{L-m} \ldots \int_{-\infty}^{L-m} e^{-\frac{1}{2} [A/\sigma^2(1-\sigma^2_\mu) - 2C\sigma^2_k/\sigma^2 + 1/2\sigma^2_\mu(s-\sigma^2_kB)^2]} \, dt_1 \cdot \, dt_2 \ldots \, dt_k \cdot ds.
\] (5.92)

Expression (5.92) can be written as

\[
E(p^k) = \frac{1}{(2\pi)^{k/2}\sigma^k\sigma_\mu} \cdot \frac{\sigma_k}{\sqrt{2\pi}\sigma_k} \int_{-\infty}^{L-m} \int_{-\infty}^{L-m} \ldots \int_{-\infty}^{L-m} \ldots \int_{-\infty}^{L-m} \ldots \int_{-\infty}^{L-m} \ldots \int_{-\infty}^{L-m} e^{-\frac{1}{2} [A/\sigma^2(1-\sigma^2_\mu) - 2C\sigma^2_k/\sigma^2]} \, dt_1 \cdot \, dt_2 \ldots \, dt_k
\] (5.93)

or

\[
E(p^k) = \frac{\sigma_k}{(2\pi)^{k/2}\sigma^k\sigma_\mu} \int_{-\infty}^{L-m} \int_{-\infty}^{L-m} \ldots \int_{-\infty}^{L-m} \ldots \int_{-\infty}^{L-m} \ldots \int_{-\infty}^{L-m} \ldots \int_{-\infty}^{L-m} \ldots \int_{-\infty}^{L-m} e^{-\frac{1}{2} (1-\sigma^2_\mu)/\sigma^2[A - 2C\sigma^2_k/(1-\sigma^2_k)]} \, dt_1 \ldots \, dt_k
\]

where

\[
\frac{1 - \sigma^2_k}{\sigma^2} = \frac{1 - \sigma^2_\mu/(\sigma^2 + k\sigma^2_\mu)}{\sigma^2} = \frac{\sigma^2 + k\sigma^2_\mu - \sigma^2_\mu}{\sigma^2(\sigma^2 + k\sigma^2_\mu)} = \frac{\sigma^2 + \sigma^2_\mu(k - 1)}{\sigma^2(\sigma^2 + k\sigma^2_\mu)}
\] (5.94)
and

\[ e^{-\frac{1}{2}(1-\sigma^2/\sigma^2_k)A - 2C\sigma^2_k/1-\sigma^2_k} \]

\[ = e^{-\frac{1}{2}[A(\sigma^2 + \sigma^2_k(k-1))/\sigma^2 + \sigma^2_k(k-1)]} - 2C\sigma^2_k(\sigma^2 + \sigma^2_k(k-1))/\sigma^2 + \sigma^2_k(k-1)]. \] (5.95)

Expressions (5.80) provides a value for \( A \). This expression when multiplied by \( \sigma^2 + \sigma^2_k(k-1) \) yields

\[ A(\sigma^2 + \sigma^2_k(k-1)) = \sum_{i=1}^{k} t_i^2(\sigma^2 + \sigma^2_k(k-1)) \]

\[ = \sum_{i=1}^{k} [t_i(\sigma^2 + \sigma^2_k(k-1))]^2. \] (5.96)

Setting

\[ z_i = \frac{t_i}{[\sigma^2 + \sigma^2_k(k-1)]^\frac{1}{2}}. \] (5.97)

The expression of \( A \) given in (5.80) in terms of \( z \) is:

\[ A = \sum_{i=1}^{k} t_i^2 = \sum_{i=1}^{k} [\sigma^2 + \sigma^2_k(k-1)]z_i^2 \]

or

\[ A = [\sigma^2 + \sigma^2_k(k-1)] \sum_{i=1}^{k} z_i^2. \] (5.98)

Expression (5.87) in terms of \( z \) is

\[ C = [\sigma^2 + \sigma^2_k(k-1)] \sum_{i \neq j}^{k(k-1)/2} z_iz_j. \] (5.99)

The following terms \( A', C', l_k, \sigma_0^2 \) are defined as follows:

\[ A' = \sum_{i=1}^{k} z_i^2, \quad C' = \sum_{i \neq j}^{k(k-1)/2} z_iz_j, \]

\[ l_k = \frac{\sigma_0^2}{\sigma^2 + (k-1)\sigma_0^2}, \quad \sigma_0^2 = \sigma^2 + k\sigma_0^2. \] (5.100)
It is easy to verify that:

\[ \sigma_0^2 = \frac{\sigma^2 + (k - 1)\sigma_0^2}{1 - \sigma_k^2} \]  (5.101)

where \( \sigma_k^2 \) is given in (5.82). Expression (5.101) yields

\[ \sigma^2 + (k - 1)\sigma_0^2 = \sigma_0^2(1 - \sigma_k^2). \]  (5.102)

Here \( \sigma_\mu^2 \) can be expressed in terms of \( l_k \) as

\[ \sigma_\mu^2 = \frac{l_k \sigma^2}{[1 - l_k(k - 1)]}. \]  (5.103)

The expression for \( \sigma_0^2 \) in terms of \( l_k \) is:

\[ \sigma_0^2 = \sigma^2 \left[ \frac{1 + l_k}{1 - l_k(k - 1)} \right]. \]  (5.104)

The expression of \( \sigma_k^2 \) given in (5.82) in terms of \( l_k \) defined in (5.100) is:

\[ \sigma_k^2 = \frac{l_k}{1 + l_k}. \]  (5.105)

Expression (5.102) can be written in terms of \( l_k \) as:

\[ \sigma^2 + (k - 1)\sigma_\mu^2 = \frac{\sigma^2}{1 - l_k(k - 1)}. \]  (5.106)

Substituting \( \sigma_\mu^2, \sigma_k^2 \) given in (5.103) and (5.105) respectively expression (5.93) yields:

\[ E[p^k] = \frac{1}{(2\pi)^{k/2}(1 + l_k)^{1/2}(1 - (k - 1)l_k)(k - 1)^{1/2}} \int_{-\infty}^{l_k} \cdots \int_{-\infty}^{l_k} e^{-\frac{1}{2} [A' - 2l_kc'/(1 + l_k)(1 - (k - 1)l_k)]z_1 \cdots dz_k}. \]  (5.107)

Example. Consider a special case for \( k = 1 \). Expression (5.107) yields

\[ E(p) = \frac{1}{\sqrt{2\pi}(1 + l_1)^{1/2}} \int_{-\infty}^{l_1} e^{-\frac{1}{2} [A' - 2l_1c'/(1 + l_1)]z_1}. \]
The values of \( l_1, A', C' \) given in Expression (5.100) will be:

\[
l_1 = \frac{\sigma^2_\mu}{\sigma^2_2}, \quad A' = \sum_{i=1}^{k=1} z_i^2 = z_1^2, \quad C' = \sum_{i \neq j} z_i \cdot z_j = 0.
\]

The expected value of \( p \) is then:

\[
E(p) = \frac{1}{\sqrt{2\pi(1 + \sigma^2_\mu/\sigma^2_2)^{1/2}}} \int_{-\infty}^{L-m} e^{-\frac{1}{2}z_i^2/(1+\sigma^2_\mu/\sigma^2_2)} dz_1 = \Phi\left(\frac{L - m}{\sqrt{\sigma^2 + \sigma^2_\mu}}\right).
\]

This result is verified earlier by a different computation technique. A more sophisticated technique is employed for evaluating \( E(p^k) \) for \( k \geq 2 \). The procedure of computation is given below.

Define a vector \( \hat{Z}_k \) by

\[
\begin{pmatrix}
  z_1 \\
  z_2 \\
  \vdots \\
  z_k
\end{pmatrix}
\]

(5.108)

letting

\[
a_{ii} = \frac{1}{(1 + l_k)(1 - (k - 1)l_k)} \quad i = 1, 2, \ldots, k
\]

(5.109)

and

\[
a_{ij} = \frac{-l_k}{(l + l_k)(1 - (k - 1)l_k)} = -a_{ii}l_k.
\]

(5.110)

For \( k = 1 \), \( A_1 = a_{11} \). For \( k = 2 \), define the matrix \( A \) by:

\[
A_2 = \begin{pmatrix}
  a_{11} & -a_{11}l_1 \\
  -a_{11}l_1 & a_{11}
\end{pmatrix}
\]

(5.111)

For \( k = 3 \), the matrix \( A \) is defined by:

\[
A_3 = \begin{pmatrix}
a_{11} & -a_{11}l_1 & -a_{11}l_3 \\
-a_{11}l_1 & a_{11} & -a_{11}l_3 \\
-a_{11}l_3 & -a_{11}l_3 & a_{11}l_3
\end{pmatrix}
\]

(5.112)

It is easy to verify that

\[
\frac{A' - 2l_k C'}{(1 + l_k)(1 - (k - 1)l_k)} = \bar{z}_k A_k \bar{z}_k
\]

(5.113)
where $A', C', l_k, z_k$ are given in (5.100) and (5.108). The vector $z'_k$ given in expression (5.113) is the transpose of $\hat{z}_k$. If $A_k$ is a positive definite matrix then:

$$\hat{z}_k^t A_k \hat{z}_k > 0 \text{ for all } \hat{z}_k \neq 0$$

there exists a matrix $p_k$ such that:

$$P_k^t A_k P_k = D_k. \quad (5.114)$$

The matrix $D_k$ has the diagonal elements which are eigenvalues of $A_k$. That is

$$D_k = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \quad (5.115)$$

Let $\tau_k$ be a matrix whose elements are defined as follows:

$$\tau_{ij} = \begin{cases} \frac{1}{\sqrt{\lambda_i}} & i = j \quad i = 1, 2, 3, \ldots, k \\ 0 & i \neq j \quad i = 1, 2, 3, \ldots, k. \end{cases} \quad (5.116)$$

A unit matrix $I_k$ is defined as

$$I_k = \tau_k^t D_k \tau_k \quad (5.117)$$

where $\tau_k^t$ is the transpose of $\tau_k$. Using (5.114), (5.117) can be written as

$$\tau_k^t P_k^t A_k P_k \tau_k = \tau_k^t D_k \tau_k = I_k. \quad (5.118)$$

Setting

$$C_k = P_k \tau_k \implies C_k^t = \tau_k^t P_k. \quad (5.119)$$

Define a vector $\tilde{X}_k$ in terms $\tilde{Z}_k$ such that

$$\tilde{Z}_k = C_k \tilde{X}_k. \quad (5.120)$$
Expression (5.113) will be reduced to:

\[ \tilde{Z}_k A_k Z_k = \tilde{X}_k^t \tilde{X}_k. \]  \hspace{1cm} (5.121)

Expressing \( Z_i \) in terms of \( X_i \) leads to:

\[ Z_i = \sum_{j=1}^{k} C_{ij} X^j. \]  \hspace{1cm} (5.122)

The Jacobian of the transformation consists of the following elements:

\[ \frac{\partial Z^j}{\partial x^i} = C_{ij}. \]  \hspace{1cm} (5.123)

The Jacobian is then:

\[ J = \det C_k. \]  \hspace{1cm} (5.124)

Using the definition of matrix \( C_k \) given in (5.119), expression (5.118) yields:

\[ \det I_k = [\det (C_k)]^2 \cdot \det A_k = 1. \]  \hspace{1cm} (5.125)

Expression (5.125) can be rewritten as:

\[ \det C_k = \frac{1}{\sqrt{\det A_k}}. \]  \hspace{1cm} (5.126)

**Example 2.** Find an expression for \( E^k(p) \) for

(a) \( k = 1 \)

(b) \( k = 2 \).

a) Expression (5.109) yields

\[ A = a_{11} = \frac{1}{1 + l_1}. \]

Expressions (5.126) and (5.100) yield:

\[ \det C_1 = \frac{1}{\sqrt{\det A_1}} = \frac{1}{\sqrt{a_{11}}} = \frac{1}{\sqrt{1/1 + l_1}} = \sqrt{1 + l_1} = \sqrt{1 + \sigma^2 / \sigma^2}. \]
Since $C_1$ is a $1 \times 1$ matrix then,

$$C_1^{-1} = \frac{\sigma}{\sqrt{\sigma^2 + \sigma_\mu^2}} \quad \text{and} \quad \tilde{x}_1 = C_1^{-1} z_1 = \frac{\sigma}{\sqrt{\sigma^2 + \sigma_\mu^2}} z_1.$$  

Using expressions (5.107), (5.113) and (5.121) leads to

$$E(p^1) = \frac{1}{\sqrt{2\pi(1 + l_1)}} \int_{-\infty}^{l_1} e^{-\frac{1}{2} \xi_1 A_1 \xi_1} d\xi_1 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{L - m / \sqrt{\sigma^2 + \sigma_\mu^2}} e^{-\frac{1}{2} \xi^2} d\xi_1$$

or

$$E(p^1) = \Phi\left(\frac{L - m}{\sqrt{\sigma^2 + \sigma_\mu^2}}\right).$$

The expression of $E(p^1)$ is the same as that reported in Example 1.

b) For $k = 2$, expression (5.110) yields:

$$A_2 = \begin{pmatrix} a_{11} & -a_{11} l_2 \\ -a_{11} l_2 & a_{11} \end{pmatrix}.$$  

The eigenvalues of matrix $A_2$, $\lambda_1$ and $\lambda_2$ can be obtained as follows. For a unit matrix $I$ set:

$$A_2 = I \lambda = 0 \implies \begin{pmatrix} a_{11} & -a_{11} l_2 \\ -a_{11} l_2 & a_{11} \end{pmatrix} = \lambda I$$

or

$$\begin{pmatrix} a_{11} - \lambda & a_{11} l_2 \\ -a_{11} l_2 & a_{11} - \lambda \end{pmatrix} = 0.$$  

The above determinant leads to the following equation:

$$(a_{11} - \lambda)^2 - a_{11}^2 l_2^2 = 0.$$  

Solving the above equation provides values for $\lambda_1$ and $\lambda_2$ respectively: $\lambda_1 = a_{11}(1 - l_2)$; $\lambda_2 = a_{11}(1 + l_2)$. Now the eigenvalues of matrix $A_2$ can be found by the following procedure:

i) $\lambda_1 = a_{11}(1 - l_2)$:

$$\begin{pmatrix} a_{11} & -a_{11} l_2 \\ -a_{11} l_2 & a_{11} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = a_{11}(1 - l_2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
or

\[ a_{11}x_1 - a_{11}l_2 x_2 = a_{11}(1 - l_2)x_1 \]
\[ -a_{11}l_2 x_1 + a_{11} x_2 = a_{11}(1 - l_2)x_2 \]

(1)

\[ (2) \]

ii) \( \lambda_2 = a_{11}(1 + l_2) \) or

\[ a_{11}x_1 - a_{11}l_2 x_2 = a_{11}(1 + l_2)x_1 \]
\[ -a_{11}l_2 x_1 + a_{11} x_2 = a_{11}(1 + l_2)x_2 \]

\[ (1') \]

\[ (2') \]

The first set yields \( x_1 = x_1, \ x_1 = x_2 \). The second set yields \( x_1 = x_1, \ x_2 = -x_1 \).

To normalize the eigenvectors set \( x_1^2 + x_2^2 = 1 \). This yields \( x_1 = \frac{1}{\sqrt{2}}, \ x_2 = \frac{1}{\sqrt{2}} \) for the first set and \( x_1 = \frac{1}{\sqrt{2}}, \ x_2 = -\frac{1}{\sqrt{2}} \) for the second set. Then the two eigenvectors are:

\[ \tilde{x}_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \quad \tilde{x}_2 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \]

The matrix \( P_2 \) is:

\[ P_2 = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \]

The matrix \( r_2 \) is:

\[ r_2 = \begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & 1/\sqrt{\lambda_2} \end{pmatrix} \]

The matrix \( C_2 \) is: \( C_2 = P_2 r_2 \)

\[ C_2 = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{\lambda_1} & 0 \\ 0 & 1/\sqrt{\lambda_2} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2}\lambda_1 & 1/\sqrt{2}\lambda_2 \\ 1/\sqrt{2}\lambda_1 & -1/\sqrt{2}\lambda_2 \end{pmatrix} \]

Then

\[ \det C_2^{-1} = -\frac{1}{2\sqrt{\lambda_1\lambda_2}} - \frac{1}{2\sqrt{\lambda_1\lambda_2}} = -\frac{1}{\sqrt{\lambda_1\lambda_2}} \]

and

\[ C_2^{-1} = -\sqrt{\lambda_1\lambda_2} \begin{pmatrix} -1/2\sqrt{2} & -1/\sqrt{2}\lambda_1 \\ -1/2\sqrt{2}\lambda_2 & 1/\sqrt{2}\lambda_1 \end{pmatrix}' = \begin{pmatrix} \sqrt{\lambda_1} & \sqrt{\lambda_1}/2 \\ \sqrt{\lambda_2} & -\sqrt{\lambda_2}/2 \end{pmatrix} \]
The matrix $\tilde{x}_2$ is:

$$\tilde{x}_2 = C_2^{-1} \tilde{z}_2 \quad \text{or} \quad \tilde{x}_2 = \begin{pmatrix} \sqrt{\lambda_2} / 2 - \sqrt{\lambda_1} / 2 \\ \sqrt{\lambda_2} / 2 + \sqrt{\lambda_1} / 2 \end{pmatrix} \begin{pmatrix} z_1 \\ \tilde{z}_2 \end{pmatrix}.$$

In terms of $z$ the matrix $\tilde{x}_2$ can be written as:

$$\tilde{x}_2 = \frac{\sqrt{\lambda_1}}{2} z_1 - \frac{\sqrt{\lambda_2}}{2} \tilde{z}_2, \quad \tilde{x}_2^2 = \frac{\sqrt{\lambda_2}}{2} z_1 + \frac{\sqrt{\lambda_2}}{2} \tilde{z}_2.$$

Here

$$\det C_2 = \frac{1}{\sqrt{\det A_2}} = \frac{1}{\sqrt{a_{11}^2 - a_{11} l_2^2}} \quad \text{or} \quad \det C_2 = (1 - l_2)\sqrt{2}.$$

The components of the matrix $\tilde{z}_k$ given in expression (5.108) are:

$$z_1 = \frac{1}{\sqrt{2 \lambda_1}} x_1 + \frac{1}{\sqrt{2 \lambda_2}} x_2, \quad z_2 = -\frac{1}{\sqrt{2 \lambda_2}} x_1 + \frac{1}{\sqrt{2 \lambda_2}} x_2.$$

For $k = 2$, the product $\tilde{z}_k A_2 \tilde{z}_k$ is reduced to $\tilde{z}_2 A_2 \tilde{z}_2$. The matrix $A_2$ is given by expression (5.111), then

$$\tilde{z}_2' A_2 \tilde{z}_2 = a_{11} \left( \frac{1}{\sqrt{2 \lambda_1}} x_1 + \frac{1}{\sqrt{2 \lambda_2}} x_2, - \frac{1}{\sqrt{2 \lambda_1}} x_1 + \frac{1}{\sqrt{2 \lambda_2}} x_2 \right)$$

$$+ \left( \frac{1}{\sqrt{2 \lambda_2}} x_2 \right) \begin{pmatrix} 1 & -l_2 \\ -l_2 & 1 \end{pmatrix} \left( \frac{1}{\sqrt{2 \lambda_1}} x_1 + \frac{1}{\sqrt{2 \lambda_2}} x_2 \right).$$

Substituting $\lambda_1 = a_{11}(1 - l_2)$ and $\lambda_2 = a_{11}(1 + l_2)$ into the above expression leads to:

$$\tilde{z}_2' A_2 \tilde{z}_2 = \left( \frac{\lambda_2}{\sqrt{2 \lambda_1}} x_1 + \frac{\lambda_1}{\sqrt{2 \lambda_2}} x_2 \right) \left( \frac{1}{\sqrt{2 \lambda_1}} x_1 + \frac{1}{\sqrt{2 \lambda_2}} x_2 \right)$$

$$+ \left( -\frac{\lambda_2}{\sqrt{2 \lambda_1}} x_1 + \frac{\lambda_1}{\sqrt{2 \lambda_2}} x_2 \right) \left( -\frac{1}{\sqrt{2 \lambda_1}} x_1 + \frac{1}{\sqrt{2 \lambda_2}} x_2 \right).$$

or

$$\tilde{z}_2' A_2 \tilde{z}_2 = \frac{\lambda_2}{\lambda_1} x_1^2 + \frac{\lambda_1}{\lambda_2} x_2^2.$$
In transforming \((z_1, z_2)\) into \(x_1\) and \(x_2\) it is necessary to determine the new limits of integration. Let \(D_z\), \(D_x\) be the domains of definition \((z_1, z_2)\) and \((x_1, x_2)\) respectively, then

\[
D_z\{z_1, z_2| -\infty \leq z_1 \leq l_2, -\infty \leq z_2 \leq l_2\}
\]

is transformed to:

\[
D_x\{x_1, x_2\mid \sqrt{2\lambda_1}l_2 - \frac{\sqrt{\lambda_1}}{\lambda_2}x_2 \leq x_1 \leq -\sqrt{2\lambda_1}l_2 + \frac{\sqrt{\lambda_1}}{\lambda_2}x_2, -\infty < x_2 \leq l_2\sqrt{2\lambda_2}\}.
\]

For \(k = 2\), expressions (5.107) and (5.113) yield:

\[
E(p^2) = \frac{(1 - l_2^2)^{1/2}}{2\pi(1 - l_2^2)} \int_{-\infty}^{l_2\sqrt{2\lambda_2}} \int_{\sqrt{2\lambda_1}l_2 - \frac{\sqrt{\lambda_1}}{\lambda_2}x_2}^{\sqrt{2\lambda_1}l_2 + \frac{\sqrt{\lambda_1}}{\lambda_2}x_2} e^{-\frac{1}{2} \frac{x_1^2}{\lambda_1} - \frac{1}{2} \frac{x_2^2}{\lambda_2}} dx_1 dx_2.
\]

Setting \(\frac{\lambda_2}{\lambda_1} = u^2\) and expanding \(e^{-\frac{1}{2} \frac{x_1^2}{\lambda_1}}\) into a series yields:

\[
e^{-\frac{u^2 x_1^2}{2\lambda_1}} = \sum_{k=0}^{\infty} \frac{(-\frac{u^2 x_1^2}{2})^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k u^{2k} x_1^{2k}}{2^k k!}.
\]

The second integral to the right of the expression of \(E(p^2)\) leads to:

\[
\sum_{k=0}^{\infty} \frac{(-1)^k u^{2k} x_1^{2k+1}}{2^k k!(2k + 1)} \left[ -\sqrt{2\lambda_1}l_2 \left(1 - \frac{1}{\sqrt{2\lambda_2}}x_2\right) \right]^{2k+1}
\]

where \(u^2 = \frac{\lambda_2}{\lambda_1}\), \(F(2l_2^2 \lambda_2) = \int_0^{2l_2^2 \lambda_2} f_\gamma(x) dx\) and \(f_\gamma\) is the p.d.f. for a gamma distribution. Rewrite the above expression as

\[
\sum_{k=0}^{\infty} \frac{(-1)^k u^{2k}}{2^k k!(2k + 1)} \left[ -\sqrt{2\lambda_1}l_2 \left(1 - \frac{1}{\sqrt{2\lambda_2}}x_2\right) \right]^{2k+1} - \sum_{k=0}^{\infty} \frac{(-1)^k u^{2k}}{2^k k!(2k + 1)} \left[ \sqrt{2\lambda_1}l_2 \left(1 - \frac{1}{\sqrt{2\lambda_2}}x_2\right) \right]^{2k+1}.
\]
The above expression can be written in the following form:

\[ \sum_{k=0}^{\infty} \frac{(-1)^k u^{2k}}{2^k k!(2k + 1)} \left( -\sqrt{2\lambda_1 l_2} \right)^{2k+1} - \left( \sqrt{2\lambda_1 l_2} \right)^{2k+1} \left( 1 - \frac{1}{\sqrt{2\lambda_2}} x_2 \right)^{2k+1}. \quad (E) \]

Since \((2k + 1)\) is a positive integer, expand \(1 - \frac{1}{\sqrt{2\lambda_2}} x_2\)^{2k+1} by employing Newton's binomial formula where

\[ (1 - \frac{1}{\sqrt{2\lambda_2}} x_2)^{2k+1} = \sum_{j=0}^{2k+1} \binom{2k+1}{j} \left( -\frac{1}{\sqrt{2\lambda_2}} x_2 \right)^j, \quad j = 0, 1, 2, \ldots, 2k + 1. \]

Substituting the above expression into equation (E) yields:

\[ \sum_{k=0}^{\infty} \sum_{j=0}^{2k+1} \binom{2k+1}{j} \left( -\frac{1}{\sqrt{2\lambda_2}} \right)^j \frac{(-1)^k u^{2k}}{2^k k!(2k + 1)} \left( -\sqrt{2\lambda_1 l_2} \right)^{2k+1} \left( 1 - \frac{1}{\sqrt{2\lambda_2}} x_2 \right)^{2k+1} x_2^j, \quad j = 0, 1, \ldots, 2k + 1. \]

Setting \( y = x_2^2 \) and computing \( E(p^2) \) by integrating over \( x_2 \) yields:

\[ E(p^2) = \frac{1}{2\pi \sqrt{1 - l_2^2}} \sum_{k=0}^{\infty} \sum_{j=0}^{2k+1} \binom{2k+1}{j} \left( -\frac{1}{\sqrt{2\lambda_2}} \right)^j \frac{(-1)^k \cdot u^{2k+j+1} \Gamma(\frac{j+1}{2})}{2^{k+j/2-(j-1/2)k!}(2k + 1)} \cdot F(2l_2^2 \lambda_2), \quad j = 0, 1, \ldots, 2k + 1. \]

5.5 Expected Value of \( p \) Given the Results of a Sample

The expression \( p(\mu) = 1 - \int_{\mu}^{U} f(x|\mu)dx \) yields a minimum value of \( p, p' \) at \( \mu = \frac{L+U}{2} \). In attribute sampling the distribution of \( p \) is assumed to be a \( \beta, \beta(A, B) \). The existence of a minimum value of \( p(\mu) \) defined above leads to the conclusion that \( \beta(A, B) \) is governed by a truncated \( \beta(A, B) \) distribution of the form

\[ \omega(p) = \frac{k\Gamma(A + B)}{\Gamma(A)\Gamma(B)} [p(\mu)]^{A-1} [1 - p(\mu)]^{B-1} \quad p' \leq p \leq 1 \quad A, \quad B > 0 \quad (5.127) \]
where
\[
\frac{1}{k} = \int_{p'(\mu)}^{1} \frac{\Gamma(A + B)}{\Gamma(A)\Gamma(B)} [p(\mu)]^{A-1} [1 - p(\mu)]^{B-1} dp. \tag{5.128}
\]

The corresponding distribution of $\mu$ is obtained through the transformation
\[
h(\mu) = \omega(p(\mu)) \left| \frac{dP(\mu)}{d\mu} \right|. \tag{5.129}
\]

Expression (5.20) yields:
\[
h(\mu) = \frac{k \Gamma(A + B)}{2 \Gamma(A)\Gamma(B)} [p(\mu)]^{A-1} [1 - p(\mu)]^{B-1} |p'(\mu)|. \tag{5.130}
\]

For a lower specification limit for the quality characteristic $x$
\[
h(\mu) = \frac{\Gamma(A + B)}{\Gamma(A)\Gamma(B)} [p(\mu)]^{A-1} [1 - p(\mu)]^{B-1} |p'(\mu)|. \tag{5.131}
\]

The expected value of $p$ given the result of the sample is then:
\[
E[p(\mu)|\bar{x}] = \int_{-\infty}^{+\infty} p(\mu) h(\mu|\bar{x}) d\mu \tag{5.132}
\]

where
\[
h(\mu|\bar{x}) = \frac{T(\bar{x}|\mu)h(\mu)}{\int_{-\infty}^{+\infty} T(\bar{x}|\mu)h(\mu) d\mu}. \tag{5.133}
\]

The posterior distribution of $T(\bar{x}|\mu)$ is given in expression (2.70). The distribution of $h(\mu)$ is given in expression (5.131). Substituting expressions (2.70) and (5.131) into expression (5.133) yields:
\[
\begin{align*}
    h(\mu|\bar{x}) &= \frac{1}{\sqrt{2\pi}\delta_n} e^{-\frac{1}{2} \frac{(\bar{x} - \mu)^2}{\delta_n^2}} \cdot \frac{\Gamma(A + B)}{\Gamma(A)\Gamma(B)} [p(\mu)]^{A-1} [1 - p(\mu)]^{B-1} |p'(\mu)| \\
    &\quad \frac{1}{\sqrt{2\pi}\delta_n} \int_{-\infty}^{+\infty} e^{-\frac{1}{2} \frac{(\bar{x} - \mu)^2}{\delta_n^2}} \frac{\Gamma(A + B)}{\Gamma(A)\Gamma(B)} [p(\mu)]^{A-1} [1 - p(\mu)]^{B-1} |p'(\mu)| d\mu. \tag{5.134}
\end{align*}
\]

It is possible to expand the term $[1 - p(\mu)]^{B-1}$ into a series through the application of the Newton’s binomial formula. Two cases will be considered herein.
Case 1. Assume $B - 1$ is a positive integer then:

$$[1 - p(\mu)]^{B-1} = \sum_{j=0}^{B-1} \binom{B-1}{j} (-1)^j [p(\mu)]^j. \quad (5.135)$$

All right hand terms in expression (5.135) containing powers higher than $[p(\mu)]^{B-1}$ vanish. Employing expression (5.135) it is possible to write the product $[p(\mu)]^{A-1}[1 - p(\mu)]^{B-1}$ in the form:

$$[p(\mu)]^{A-1}[1 - p(\mu)]^{B-1} = \sum_{j=0}^{B-1} \binom{B-1}{j} (-1)^j [p(\mu)]^j [p(\mu)]^{A-1}. \quad (5.136)$$

Substituting expression (5.136) into the denominator of expression (5.134) yields:

$$\frac{1}{\sqrt{2\pi\delta_n}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2} \frac{(x-\mu)^2}{\delta_n^2}} [p(\mu)]^{A+j-1} [p'(\mu)] d\mu. \quad (5.137)$$

Substituting the expression for $p'(\mu)$ into (5.137) yields:

$$\frac{1}{\sqrt{2\pi\delta_n \sigma}} \sum_{j=0}^{B-1} \binom{B-1}{j} (-1)^j \int_{-\infty}^{+\infty} e^{-\frac{1}{2} \frac{(x-\mu)^2}{\delta_n^2} - \frac{1}{2} \frac{(x-L)^2}{\sigma^2}} [p(\mu)]^{A+j-1} d\mu. \quad (5.138)$$

The sum of the exponentials given in expression (5.138) can be written as:

$$\frac{(\bar{x} - \mu)^2}{\delta_n^2} + \frac{(L - \mu)^2}{\sigma^2} = \frac{(\mu - m)^2}{\delta_1^2} + \frac{(\bar{x} - L)^2}{\sigma_1^2} \quad (1.139)$$

where

$$\delta_n^2 = \frac{\sigma^2}{n}, \quad \delta_1^2 = \frac{\delta_n^2 \cdot \sigma^2}{\delta_1^2 + \sigma^2}, \quad \sigma_1^2 = \delta_1^2 + \sigma^2, \quad m = \frac{\bar{x} \sigma^2 + \delta_n^2 L}{\delta_n^2 + \sigma^2}. \quad (5.140)$$

Under the transformation given in (5.139), expression (5.138) is reduced to:

$$\frac{1}{\sqrt{2\pi\delta_n \sigma}} \sum_{j=0}^{B-1} \binom{B-1}{j} (-1)^j e^{-\frac{1}{2} \frac{(x-\mu)^2}{\delta_n^2}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2} \frac{(x-L)^2}{\sigma_1^2}} [p(\mu)]^{A+j-1} d\mu. \quad (5.141)$$
Substituting expression (5.141) into (5.134) and employing expression (5.132), through the transformation given in (5.139) expression (5.132) can be written as:

\[
E(p(\mu)|\bar{x}) = \frac{\sum_{j=0}^{B-1} \binom{B-1}{j} (-1)^j E[p(\mu)]^A + j}{\sum_{j=0}^{B-1} \binom{B-1}{j} (-1)^j E[p(\mu)]^{A+j-1}}
\]  

(5.142)

where

\[
E[p(\mu)]^q = \frac{1}{\sqrt{2\pi} \delta_1} \int_{-\infty}^{+\infty} e^{-\frac{1}{2} \frac{(\mu-m)^2}{\epsilon^2}} [p(\mu)]^q d\mu.
\]

The expression for \(\delta_1\) and \(m\) are given in expression (5.138).

**Case 2.** Assume \(B - 1\) is not a positive integer, then \([1 - p(\mu)]^{B-1}\) can be expressed in terms of an infinite series given below:

\[
[1 - p(\mu)]^{B-1} = \sum_{j=0}^{\infty} \binom{B-1}{j} (-1)^j [p(\mu)]^j.
\]  

(5.143)

An expression identical to that given in (5.142) can be obtained for \(E[p(\mu)|\bar{x}]\). The only difference is that \(\sum_{j=0}^{B-1}\) is infinite.

The following is an example.
### INPUT FILE REQUIREMENTS

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QC RESULTS

K VALUE = 0.1000E+01
L VALUE = 0.6000E+00
VARIANCE OF X VALUE = 0.1000E+01
VARIANCE OF Y VALUE = 0.2500E+00
MEAN - W VALUE = 0.2000E+01
E(P**K) = 0.3037E-01

P(L) ESTIMATE COEFFICIENTS FOR:
PEST(L)=EXP[C(1)+C(2)*V+C(3)*V**2+...]

C(1) = -0.7306E+00
C(2) = -0.8147E+00
C(3) = -0.2849E+00
C(4) = -0.3135E-01
*** QC RESULTS ***

<table>
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</thead>
<tbody>
<tr>
<td>K VALUE</td>
<td>0.2600E+01</td>
</tr>
<tr>
<td>L VALUE</td>
<td>0.0000E+00</td>
</tr>
<tr>
<td>VARIANCE OF X VALUE</td>
<td>0.1000E+01</td>
</tr>
<tr>
<td>VARIANCE OF W VALUE</td>
<td>0.2500E+00</td>
</tr>
<tr>
<td>MEAN - W VALUE</td>
<td>0.2000E+01</td>
</tr>
<tr>
<td>E(P*K)</td>
<td>0.3130E-02</td>
</tr>
</tbody>
</table>

**P(L) ESTIMATE COEFFICIENTS FOR:**

\[
\text{PEST}(L) = \exp[C(1) + C(2) \cdot V + C(3) \cdot V^2 + \ldots]
\]

- \( C(1) = -0.1461E+01 \)
- \( C(2) = -0.1629E+01 \)
- \( C(3) = -0.5698E+00 \)
- \( C(4) = -0.6269E-01 \)
QC RESULTS

\[ K \text{ VALUE} = 0.3000 \times 10^1 \]
\[ L \text{ VALUE} = 0.0000 \times 10^0 \]
\[ \text{VARIANCE OF X VALUE} = 0.1000 \times 10^1 \]
\[ \text{VARIANCE OF Y VALUE} = 0.2500 \times 10^0 \]
\[ \text{MEAN - W VALUE} = 0.2000 \times 10^1 \]
\[ E(K) = 0.4500 \times 10^{-3} \]

P(L) ESTIMATE COEFFICIENTS FOR:
\[ \text{PEST(L)} = \exp[C(1) + C(2) + C(3) + C(4) + \ldots] \]
\[ C(1) = -0.2192 \times 10^1 \]
\[ C(2) = -0.2444 \times 10^1 \]
\[ C(3) = -0.8547 \times 10^0 \]
\[ C(4) = -0.9404 \times 10^{-1} \]
*** QC RESULTS ***

**K VALUE** = 0.4000E+01

**L VALUE** = 0.0000E+00

**VARIANCE OF X VALUE** = 0.1000E+01

**VARIANCE OF MU VALUE** = 0.2500E+00

**MEAN - MU VALUE** = 0.2000E+01

**E(Po*K)** = 0.0096E-04

**P(L) ESTIMATE COEFFICIENTS FOR:**

PEST(L) = EXP[C(1) + C(2)*V + C(3)*V**2 + ...]

C(1) = -0.2922E+01
C(2) = -0.3259E+01
C(3) = -0.1140E+01
C(4) = -0.1254E+00
QC RESULTS

K VALUE = 1.6000E+01
L VALUE = 0.0000E+00
VARIANCE OF X VALUE = 0.1000E+01
VARIANCE OF MU VALUE = 0.2600E+00
MEAN - M VALUE = 0.2000E+01
E(P**K) = 0.2140E-04

P(L) ESTIMATE COEFFICIENTS FOR:
PEST(L) = EXP{C(1)+C(2)*V+C(3)*V**2+...}

C(1) = -0.3653E+01
C(2) = -0.4073E+01
C(3) = -0.1425E+01
C(4) = -0.1567E+00
Figure 14. The V Values Vs. the First and Second Factors of the Kernel of the Expected Moment of the Fraction Defective Raised to the K-th Power.
Figure 15. The V Values Vs. the Combined Values of the First and Second Factors of the Kernel of the Expected Value of Fraction Defective p.
CHAPTER 6

INSPECTION ERRORS IN ATTRIBUTE AND VARIABLE SAMPLING PLANS

In recent years the possibility of inspection errors is considered by researchers in various quality control procedures. The presence of these errors lead to changes in the OC curves, the average outgoing quality, the average total inspection, costs and many other terms.

6.1. Attribute Sampling Plans

In attribute sampling plans the errors are of two kinds:

Type I error: A good item is classified as bad, with a probability $e_1$

Type II error: A bad item is classified as good, with a probability $e_2$

Information on the following parameters defined in Appendix A is required to understand the foregoing discussion: $p, p_e, e_1, e_2, N, X, x, n, P_a, c, i, P_{ae}, y_e, AOQ, AOQ_e, ATI, ATI_e, OCC, OCC_e$. Collins and Case [1976] derived an expression for the probability of acceptance under inspection error. An expression is derived for the marginal distribution of the observed defectives. Case, Bennett and Schmidt [1978] developed formulas for calculating the average outgoing quality ($AOQ$) when attribute inspection is subject to Type I and Type II inspection errors. Nine different rectification inspection policies are considered. These policies were first introduced by Wortham and Mogg [1970]. Case and Beainy [1981] generalized the model of Case, Bennett and Schmidt. They developed nine different sample/rest-of-lot disposition policies for a single and double sampling along explicitly developed $AOQ$ and $ATI$ expressions for each. Both are considered with and without inspection errors. In this chapter both attribute and variable sampling plans are considered and a Bayesian technique is
developed to estimate the different parameters and the probability distribution studied in Chapter 2 through 5.

6.2. Probability of Acceptance

Assumptions

(1) The number of defectives $X$ in a lot of size $N$ is binomially distributed with a p.d.f:

$$f_N(X) = \binom{N}{X} p^X (1-p)^{N-X} \quad X = 0, 1, \ldots, N$$  \hspace{1em} (6.1)

where $p$ is the process fraction defective.

(2) The number of defectives $x$ in a sample of size $n$ given $X$ is hypergeometric that is:

$$f(x|X) = \binom{n}{x} \binom{N-n}{X-x} \binom{N}{X}$$  \hspace{1em} (6.2)

Under the assumptions 1 and 2 it is easy to verify that the marginal distribution of $x$ is also binomial

$$g_n(x) = \binom{n}{x} p^x (1-p)^{n-x} \quad x = 0, 1, \ldots, n.$$

This result is described by Hald [1976] by saying the binomial distribution is reproduced by hypergeometric sampling.

For a type $B$ OC curve, the probability of lot acceptance is given by:

$$p_a = \sum_{x=0}^{c} g_n(x) = \sum_{x=0}^{c} \binom{n}{x} p^x (1-p)^{n-x} \quad x = 0, 1, \ldots, n \quad 0 \leq p \leq 1$$  \hspace{1em} (6.3)

where $c$ is the acceptance number. For the inspection error analysis the observed defectives from a sample is replaced by observed number of defectives $y_e$. The probability of lot acceptance given in (6.3) will be reduced to

$$P_{ae} = \sum_{y_e=0}^{c} g_n(y_e)$$  \hspace{1em} (6.4)
where
\[ g_n(y_e) = \binom{n}{y_e} p_e^{y_e} (1 - p_e)^{n-y_e} \quad y_e = 0, 1, \ldots, n. \] (6.5)

### 6.3. The Conditional and Marginal Probabilities of the Observed Number of Defectives

The distribution of actual defectives correctly classified \( i \) can be defined in terms of the number of true defectives in the sample as:
\[ g_x(i) = \binom{x}{i} (1 - e_2)^i e_2^{x-i} \] (6.6)
where \( i \leq x \). The number of good items in the sample is:
\[ (n - x). \] (6.7)

The number of good items classified as bad is \( (y_e - i) \). The distribution of \( (y_e - i) \) is then:
\[ g_{n-x}(y_e - i) = \binom{n-x}{y_e - i} e_1^{y_e-i} (1 - e_1)^{n-x-y_e+i} \quad y_e = 0, 1, \ldots, n. \] (6.8)

The conditional probability of \( (y_e| x) \) is the joint probability of \( i \) items classified as bad and \( y_e - i \) good items classified as bad, that is
\[ f(y_e|x) = \sum_{i=\max\left\{ y_e-(n-x), 0 \right\}}^{\min\left\{ y_e, x \right\}} \binom{x}{i} (1 - e_2)^i e_2^{x-i} \binom{n-x}{y_e-i} e_1^{y_e-i} (1 - e_1)^{n-x-y_e+i}. \] (6.9)

The marginal distribution of the observed defectives is:
\[ g_n(y_e) = \sum_{x=0}^{n} f(y_e|x) g_n(x) \]
\[ = \sum_{x=0}^{n} \sum_{i=\max\left\{ y_e-(n-x), 0 \right\}}^{\min\left\{ y_e, x \right\}} \binom{x}{i} (1 - e_2)^i e_2^{x-i} \binom{n-x}{y_e-i} e_1^{y_e-i} (1 - e_1)^{n-x-y_e+i} \]
\[ \cdot \binom{n}{x} p^x (1 - p)^{n-x}. \] (6.10)
Expression (6.10) can be reduced to:

\[
\sum_{x=0}^{n} \sum_{i=\min\left[\frac{\left[x,y_{e}\right]}{\left[y_{e}-(n-x),0\right]}\right]}^{\min\left[\frac{\left[x,y_{e}\right]}{\left[y_{e}-(n-x),0\right]}\right]} \cdot \left(\begin{array}{c}
\frac{x}{i}
\end{array}\right) [p(1-e_{2})]^{i}p^{x-i}[(1-p)]^{y_{e}-i}e_{1}[1-p^{n-x-y_{e}-i}]
\]

Writing the following product in the form:

\[
\left(n-x\right)\left(n-x\right)\left(y_{e}-1\right)(1-e_{1})^{n-x-y_{e}+i}.
\]

Substituting expression (6.12) into (6.11) leads to:

\[
\sum\left(n-x\right)\left(n-x\right)\left(y_{e}-1\right)(1-e_{1})^{n-x-y_{e}+i}.
\]

Arranging the terms within expression (6.13) and rewriting it yields

\[
g_{n}(y_{e}) = \left(\begin{array}{c}
n
\end{array}\right) \sum_{i=\min\left[\frac{\left[x,y_{e}\right]}{\left[y_{e}-1\right]}\right]}^{\min\left[\frac{\left[x,y_{e}\right]}{\left[y_{e}-1\right]}\right]} [p(1-e_{2})]^{i}[(1-p)e_{1}]^{y_{e}-i}[1-p(1-e_{1})]^{n-x-y_{e}+i}
\]

The summation over \(x\) leads to

\[
[p(1-e_{2}) + (1-p)(1-e_{1})]^{n-y_{e}}.
\]

The summation over \(i\) leads to:

\[
[p(1-e_{2}) + (1-p)e_{1}]^{y_{e}}.
\]
Since
\[ p(1 - e_2) + (1 - p)e_1 = p_e \quad \text{and} \quad p_{e2} + (1 - p)(1 - e_1) = 1 - p_e \] (6.17)
then expression (6.14) can be written as:
\[
g_n(y_e) = \binom{n}{y_e} p_e^{y_e} (1 - p_e)^{n - y_e}. \tag{6.18}
\]
Expression (6.18) shows that the observed number of defectives follow a binomial distribution with parameters \( n, p_e \).

6.3.1. Mixed Binomial Prior

Define the following terms
\[ p_{e,1} = \text{the fraction of items defective for process 1.} \]
\[ p_{e,2} = \text{the fraction of items defective for process 2.} \]
\[ m_1 = \text{the frequency with which } p_{e,1} \text{ occurs.} \]
\[ m_2 = \text{the frequency with which } p_{e,2} \text{ occurs.} \]

Expression (6.18) can be written as:
\[
g_n(y_e) = m_1 \binom{n}{y_e} p_{e,1}^{y_e}(1 - p_{e,1})^{n - y_e} + m_2 \binom{n}{y_e} p_{e,2}^{y_e}(1 - p_{e,2})^{n - y_e}. \tag{6.19}
\]
For \( k \) weighted components \( m_1, m_2, \ldots, m_k \):
\[
g_n(y_e) = m_1 \binom{n}{y_e} p_{e,1}^{y_e}(1 - p_{e,1})^{n - y_e} + m_2 \binom{n}{y_e} p_{e,2}^{y_e}(1 - p_{e,2})^{n - y_e} \\
+ \cdots + m_k \binom{n}{y_e} (p_{e,k})^{y_e}(1 - p_{e,k})^{n - y_e}
\]
where
\[
\sum_{i=1}^{k} m_i = 1. \tag{6.20}
\]
Example 1. A beta prior probability density function for \( p \) is given by:

\[
\omega(p) = \frac{\Gamma(A + B)}{\Gamma(A)\Gamma(B)} p^{A-1}(1 - p)^{B-1}.
\]

The joint distribution of \((p, y_e)\) is:

\[
f(p, y_e) = \frac{\Gamma(A + B)}{\Gamma(A)\Gamma(B)} p^{A-1}(1 - p)^{B-1} \left( \begin{array}{c} n \\ y_e \end{array} \right) p_e^{y_e} (1 - p_e)^{n-y_e}.
\]

The marginal distribution of \((y_e)\) is

\[
g_n(y_e) = \left( \begin{array}{c} n \\ y_e \end{array} \right) \int_0^1 \frac{\Gamma(A + B)}{\Gamma(A)\Gamma(B)} p^{A-1}(1 - p)^{B-1} p_e^{y_e} (1 - p_e)^{n-y_e} dp
\]

where \( p_e \) is given in expression (6.17).

6.4. The Effect of Inspection Error on AOQ and ATI

For perfect inspection the probability of lot acceptance is given by expression (6.3). The probability of lot acceptance when inspection errors are present, \( p_{ae} \), obtained form (6.3) by replacing the true \((p)\) fraction defective by the apparent fraction defective \((p_e)\). This can be represented as:

\[
p_{ae} = \sum_{y_e=0}^c \left( \begin{array}{c} n \\ y_e \end{array} \right) p_e^{y_e} (1 - p_e)^{n-y_e}.
\]

An expression for the AOQ is derived when sampling with replacement and without replacement.

6.4.1. AOQ Sampling With Replacement

The average outgoing quality is defined by the ratio:

\[
AOQ = \frac{\text{expected number of defective items remaining after inspection}}{\text{total number of items in the lot}}
\]

\[
= \frac{(N - n)p_{Pa}}{N}.
\]

(6.22)
An expression for AOQ can be derived by introducing the following terms:

\[ p(N - n) \] The number of defectives in the uninspected portion of an accepted lot.
\[ p(N - n)e_2 \] the number of defective items classified as being good in the screened portion of a rejected lot.
\[ n\rho e_2 \] The number of defective items classified as good in the sample.
\[ DITR \] The number of defective items introduced through replacement into the lot.

For an accepted lot, the expected number of defective items replaced in the lot is

\[ y = n\rho e. \quad (6.23) \]

The probability that an item is classified as being good is then:

\[ P_g = (1 - p)(1 - e_1) + p e_2. \quad (6.24) \]

A set of \( n_1 \) items are selected at random, tested and classified as good or bad. A total of \( n\rho e \) items were needed to replace the defective items in the accepted lot.

This procedure of sampling defines a negative binomial process. The expected number of items tested to obtain \( n\rho e \) items which are good is then:

\[ \frac{y}{P_g}. \quad (6.25) \]

The expected number of defective items replaced in an accepted lot is then:

\[ DITR_a = Pe_2 \left( \frac{y}{P_g} \right). \quad (6.26) \]

The expected number of defective items replaced in a rejected lot which is screened is:

\[ DITR_s = pe_2 \frac{(N - n)\rho e}{P_g}. \quad (6.27) \]
The expected number of items to be replaced is:

\[ DITR = pe_2 \left( \frac{y}{P_g} \right) + P e_2 \left( \frac{N - n}{P_g} \right) p_e (1 - P_{ae}) \]

\[ = \frac{pe_2}{P_g} [y + (N - n)p_e(1 - P_{ae})]. \] (6.28)

The expression \( AOQ \) is then:

\[ AOQ = \frac{p(N - n)P_{ae} + p(N - n)(1 - P_{ae})e_2 + npe_2 + DITR}{N}. \] (6.29)

Expression (6.28) can be reduced to the form:

\[ AOQ = \frac{npe_2 + p(N - n)(1 - p_e)P_{ae} + p(N - n)(1 - P_{ae})e_2}{N(1 - p_e)}. \] (6.30)

6.4.2. \( AOQ \) Sampling Without Replacement

In this case no defectives are introduced through replacement

\[ AOQ = \frac{p(N - n)P_{ae} + p(N - n)(1 - P_{ae})e_2 + npe_2}{N - nP_e - (1 - P_{ae})(N - n)p_e}. \] (6.31)

6.4.3. \( ATI \) Sampling With Replacement

For a rejected lot the expected number of items to be screened is:

\[ (N - n)(1 - P_{ae}). \] (6.32)

The expected number of items to be replaced in a screening process is:

\[ \frac{(N - n)p_e(1 - P_{ae})}{P_g} \] (6.33)

where the expression for \( P_g \) is given in (6.24). The expression for \( ATI \) is then:

\[ ATI = n + (N - n)(1 - P_{ae}) + \frac{npe_e}{P_g} + \frac{(N - n)p_e}{P_g}(1 - P_{ae}). \] (6.34)
Substituting $P_g$ given in (6.24) into (6.34) yields:

$$ATI = \frac{n + (N - n)(1 - P_{ae})(1 - p)(1 - e_1) + pe_2}{(1 - p)(1 - e_1) + pe_2} + \frac{np_e + (N - n)p_c(1 - P_{ae})}{(1 - p)(1 - e_1) + pe_2}.$$  

(6.35)

Expression (6.35) can be simplified to:

$$ATI = \frac{n + (1 - P_{ae})(N - n)}{1 - p_c}.$$  

(6.36)

6.4.4. ATI Sampling Without Replacement

$$ATI = \frac{n + (N - n)(1 - P_{ae})(1 - p)(1 - e_1) + pe_2}{(1 - p)(1 - e_1) + pe_2} + \frac{np_e + (N - n)p_c(1 - P_{ae})}{(1 - p)(1 - e_1) + pe_2}.$$  

(6.37a)

which can be reduced to:

$$ATI = n + (N - n)(1 - P_{ae}).$$  

(6.38b)

Wortham and Mogg [1970] provided formulas for calculating the AOQ for nine different ways. The sampling conditions assumed are single sampling by attributes. The equations of AOQ for various rectifying inspection policies are listed in Table.

Case and Beainy [1981] extended the model of Wortham and Mogg to deal with ATI. They developed expressions for AOQ and ATI for single and double sampling. Both cases with and without inspection errors are considered.
6.4.5. AOQ Under Both Error Free and Error Prone Sampling

The following terms are needed to describe the single sampling AOQ expressions. Terms subscripted with a means perfect inspection is assumed while those subscripted with ae means error-prone inspection is assumed.

\[ A_1 = p(N - n)P_a \]
\[ A_2 = p(N - n)P_{ae} \] ⇒ The expected number of defectives in the uninspected portion of an accepted lot.

\[ B_1 = 0 \] and \[ B_2 = P(N - n)e_2(1 - P_{ae}) \] ⇒ The expected number of defective items classified as good when a lot is screened.

\[ C_1 = 0 \] and \[ C_2 = npe_2 \] ⇒ The expected number of defective items classed as good in a sample.

\[ D_1 = 0 \] and \[ D_2 = (N - n)p_a e_2\left(\frac{1-P_{ae}}{1-P_e}\right) \] ⇒ The expected number of defective items are introduced through replacement to replenish the screened portion of rejected lot.

\[ E_1 = 0 \] and \[ E_2 = \frac{nP_a P_e}{(1-P_e)} \] ⇒ The expected number of defectives introduced through replacement items used to replenish the sample.

\[ F_1 = np \] and \[ F_2 = np_e \] ⇒ The expected number of apparent defectives removed from the sample.

\[ G_1 = p(N - n)(1 - p_a) \] and \[ G_2 = p_e(N - n)(1 - p_{ae}) \] ⇒ The expected number of defectives removed from the screened portion of a rejected lot.

\[ H_1 = (N - n)P_a \] and \[ H_2 = (N - n)P_{ae} \] ⇒ The expected number of items in the uninspected portion of an accepted lot.

6.4.6. ATI Under Error Free and Error Prone Sampling

ATI is the long run number of items inspected per lot including

(1) the sample,
(2) the screened portion of a rejected lot,
(3) items to replace defective items discarded from the sample,
(4) items to replace defective items discarded from the screened portion of a rejected lot.

The following are some terms needed to derive equations for ATI:

\[ J_1 = \frac{n}{1 - p} \]
\[ J_2 = \frac{n}{1 - P_e} \]
\[ K_1 = (N - n)(1 - P_a) \]
\[ K_2 = (N - n)(1 - P_{ae}) \]
\[ L_1 = \frac{(N-n)(1-P_a)}{1-P} \]
\[ L_2 = \frac{(N-n)(1-P_{ae})}{1-P_e} \]

The various results reported in this section are listed in Table 3.

6.5. Variables Sampling Plans for Fraction Defective

Product quality characteristics are classified according to an upper, lower or upper and lower specifications. A random sample is taken and each item is tested against a quality characteristic of interest. The measurements generate values of the sample means relative to each quality. The most significant errors are due to bias and imprecision.

a) Bias is the difference between the true product quality characteristic and the average of a long series of repeated measurements, made on that characteristic.

Definitions:
\[ \hat{\zeta} \] the observed product quality characteristic
\[ \zeta \] the true product quality characteristic
\[ \mu_e \] the mean of the measurement error distribution
\[ x_0 \] the observed quality characteristic of mean \( \mu_0 \)
Table 3. Single Sample AOQ (Top) and ATI (Bottom) Equations for Various Rectifying Inspection Policies.

Substituting the values of A, B, C, D, E, F, G, H for perfect inspection for the nine different policies listed yields the AOQ of Wortham and Hogg.

<table>
<thead>
<tr>
<th>Discard Rejected Lots Completely</th>
<th>Inspect 100% and Discard Defective Units but do not Replace Defective Units with Good Units</th>
<th>Inspect 100% Replace Defective Units with Good Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>p</td>
<td>$\frac{A+B}{N-n-G}$</td>
<td>$A+B+D$</td>
</tr>
<tr>
<td>n</td>
<td>$n + k$</td>
<td>$n + L$</td>
</tr>
<tr>
<td>$\frac{A+C}{n-P+H}$</td>
<td>$\frac{A+B+C}{N-P-G}$</td>
<td>$\frac{A+B+C+D}{N-P}$</td>
</tr>
<tr>
<td>n</td>
<td>$n + K$</td>
<td>$n + L$</td>
</tr>
<tr>
<td>$\frac{A+C+B}{n+H}$</td>
<td>$\frac{A+B+C+B}{N-G}$</td>
<td>$\frac{A+B+C+D+B}{N}$</td>
</tr>
<tr>
<td>J</td>
<td>$J + K$</td>
<td>$J + L$</td>
</tr>
</tbody>
</table>
$x$ the true quality characteristic of mean $\mu$

$x_e$ the difference between the observed and true quality characteristics.

Now, the parameter $\mu_e$ is given by:

$$\mu_e = \mathbb{E}(\hat{\zeta}) - \zeta. \quad (6.38)$$

b) Imprecision occurs when measurements on the same unit of a product fail to generate the same results. Imprecision can be expressed mathematically by:

$$\sigma_e = \sqrt{\text{Var} \hat{\zeta}}. \quad (6.39)$$

Some of the sources of inspection errors are:

1. Human inspection error is caused by human inspectors. These inaccuracies are defined by bias $\mu_h$ and imprecision $\sigma_h$.

2. Instrument test error is caused by faulty instruments. The inaccuracies are defined by bias $\mu_i$ and imprecision $\sigma_i$.

3. Measurement error is caused either by humans or instruments. These errors are defined by

$$\mu_e = \mu_h + \mu_i$$

$$\sigma_e = (\sigma_h^2 + \sigma_i^2 + 2\rho \sigma_h \sigma_i)^{\frac{1}{2}} \quad (6.40)$$

where $\rho$ is the correlation coefficient between human and instrument inspection errors.

Assume the lot distribution and measurement error distribution are independent. The distribution of the observed quality characteristic $x_0$ is the convolution of the lot distribution and the measurement error distribution:

$$f(x_0) = f_1(x) * f_2(x_e). \quad (6.41)$$
Expression (6.41) yields the following p.d.f. for $x_0$:

$$f(x_0) = \frac{1}{\sqrt{2\pi} \sqrt{\sigma_{\mu_0}^2 + \sigma^2 + \sigma_e^2}} e^{-\frac{1}{2} \frac{(x_0 - \mu_0)^2}{\sigma_{\mu_0}^2 + \sigma^2 + \sigma_e^2}} \quad -\infty < x_0 < +\infty. \quad (6.42)$$

The conditional p.d.f. of $x_0$ given $\mu_0$ is

$$f(x_0|\mu_0) = \frac{1}{\sqrt{2\pi} \sqrt{\sigma^2 + \sigma_e^2}} e^{-\frac{1}{2} \frac{(x_0 - \mu_0 + \mu_0)^2}{\sigma^2 + \sigma_e^2}}. \quad (6.43)$$

The p.d.f. of $\mu_0$ is given by:

$$h(\mu_0) = \frac{1}{\sqrt{2\pi} \sigma_{\mu_0}} e^{-\frac{1}{2} \frac{(\mu_0 - m_0)^2}{\sigma_{\mu_0}^2}} \quad -\infty < \mu_0 < +\infty. \quad (6.44)$$

The distribution of the sample mean given $(\mu_0)$ is:

$$T(\bar{X}_{0n}|\mu_0) = \frac{1}{\sqrt{2\pi} \sqrt{\frac{\sigma^2 + \sigma_e^2}{n}}} e^{-\frac{1}{2} \frac{(\bar{X}_{0n} - \mu_0)^2}{\frac{\sigma^2 + \sigma_e^2}{n}}} \quad \sigma_{\mu_0} = \frac{\sigma^2}{\sqrt{n}} + \sigma_e^2. \quad (6.45)$$

Expression (6.44) is identical to expression (2.70) derived in Chapter 2. The marginal distribution of $(\bar{X}_{0n})$ is obtained by employing expressions (6.45) and (6.44) as follows:

$$T(\bar{X}_{0n}) = \int_{-\infty}^{+\infty} T(\bar{X}_{0n}|\mu_0) h(\mu_0) \, d\mu_0. \quad (6.46)$$

Rewriting expression (6.46) yields:

$$T(\bar{X}_{0n}) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi} \sqrt{\frac{\sigma^2 + \sigma_e^2}{n}}} e^{-\frac{1}{2} \frac{(\bar{X}_{0n} - \mu_0)^2}{\frac{\sigma^2 + \sigma_e^2}{n}}} \cdot \frac{1}{\sqrt{2\pi} \sigma_{\mu_0}} e^{-\frac{1}{2} \frac{(\mu_0 - m_0)^2}{\sigma_{\mu_0}^2}} d\mu_0. \quad (6.47)$$

The sum of the exponentials given in expression (6.47) can be transformed to:

$$\frac{(\mu_0 - m_0)^2}{\sigma_{\mu_0}^2} + \frac{(\bar{X}_{0n} - \mu_0)^2}{\frac{\sigma^2 + \sigma_e^2}{n}} = \frac{(\mu_0 - m_0)^2}{\sigma_{\mu_0}^2} + \frac{(m_0 - \bar{X}_{0n})^2}{\sigma_{\bar{X}_{0n}}^2} \quad (6.48)$$
Expression (6.47) can be written as:

$$T(\bar{X}_{0n}) = \frac{1}{2\pi \sqrt{\sigma_{\mu_0}^2 + \delta_{0n}^2}} \cdot \frac{1}{\sqrt{2\pi \sigma_{\mu_0}}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2} \left[ \frac{(m_0 - m_\mu_0)^2 + (m_0 - \bar{X}_{0n})^2}{\sigma_{\mu_0}^2 + \delta_{0n}^2} \right]} d\mu_0. \quad (6.50)$$

Expression (6.50) can be simplified to the form:

$$T(\bar{X}_{0n}) = \frac{1}{\sqrt{2\pi \sigma_{\mu_0}^2 + \delta_{0n}^2}} e^{-\frac{1}{2} \left( \frac{m_0 - \bar{X}_{0n}}{\sqrt{\sigma_{\mu_0}^2 + \delta_{0n}^2}} \right)^2}. \quad (6.51)$$

Expression (6.51) shows that $\bar{X}_{0n}$ is normally distributed with mean $m_0$ and variance $\sigma_{\mu_0}^2 + \delta_{0n}^2$. The expression for $\delta_{0n}^2$ is defined in (6.49). The conditional distribution of $\mu_0$ given $\bar{X}_{0n}$ is given by:

$$h(\mu_0 | \bar{X}_{0n}) = \frac{T(\bar{X}_{0n} | \mu_0) h(\mu_0)}{T(\bar{X}_{0n})}. \quad (6.52)$$

Substituting expression (6.45), (6.44) and (6.46) into expression (6.52) yields:

$$h(\mu_0 | \bar{X}_{0n}) = \frac{1}{\sqrt{2\pi \delta_{0n}^2 \sigma_{\mu_0}^2}} e^{-\frac{1}{2} \left( \frac{m_0 - m_\mu_0}{\delta_{0n}^2 \sigma_{\mu_0} + \delta_{0n}^2} \right)^2}. \quad (6.53)$$

The above expression shows that $h(\mu_0 | \bar{X}_{0n})$ is normally distributed with mean $m_3$ and variance $\delta_1^2$, given in (6.49).
An expression for the conditional expectation of \( p(\mu_0) \) given \( \overline{X}_{0n} \) can be obtained by following the technique developed in Chapter 2. Employing expressions (6.43) and (6.53), yield:

\[
E(p(\mu_0)|\overline{X}_{0n}) = 1 - \frac{1}{2\pi\sqrt{\sigma^2 + \sigma^2_e}\delta_1} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-\frac{1}{2} \left( \frac{(\mu_0 - \mu_0)^2}{\sigma^2} + \frac{X_0 - \mu_0)^2}{\sigma^2_e} \right)} \, dX_0 \, d\mu_0. 
\]

(6.54)

The exponential given in expression (6.54) can be reduced to:

\[
\frac{(\mu_0 - m_3)^2}{\delta^2_1} + \frac{(X_0 - \mu_0)^2}{\sigma^2 + \sigma^2_e} = \frac{(\mu_0 - m_4)^2}{\delta^2_2} + \frac{(m_3 - X_0)^2}{\sigma^2_e} 
\]

(6.55)

where

\[
\delta^2_2 = \frac{\delta^2_1(\sigma^2 + \sigma^2_e)}{\delta^2_1 + \sigma^2 + \sigma^2_e} \\
\sigma^2_2 = \frac{\delta^2_1 + \sigma^2 + \sigma^2_e}{\delta^2_1 + \sigma^2 + \sigma^2_e} \\
m_4 = \frac{m_3(\sigma^2 + \sigma^2_e) + \delta^2_2 X_0}{\delta^2_1 + \sigma^2 + \sigma^2_e}. 
\]

Substituting expression (6.55) into (6.54) yields:

\[
E(P(\mu_0)|\overline{X}_{0n}) = 1 - \frac{1}{2\pi\sigma_2} \int_{L}^{U} e^{-\frac{1}{2} \left( \frac{m_3 - X_0)^2}{\sigma^2_2} \right)} \, dX_0 
\]

(6.57)

or

\[
E(p(\mu_0)|\overline{X}_{0n}) = 1 - \left[ \Phi \left( \frac{U - m_3}{\sigma_2} \right) - \Phi \left( \frac{L - m_3}{\sigma_2} \right) \right]. 
\]

(6.58)

The expressions for \( m_3 \) and \( \sigma^2_2 \) are given in (6.49) and (6.56) respectively. An expression similar to (3.13) can be written as:

\[
\Phi \left( \frac{U - m_3}{\sigma_2} \right) - \Phi \left( \frac{L - m_3}{\sigma_2} \right) = k_r \frac{k_r}{k_E}. 
\]

(6.59)

Expression (6.59) provides an expression for \( \overline{X}_{0n} \) given by:

\[
\overline{X}_{0n} = \frac{m_3(\sigma^2_{\mu_0} + \delta^2_{0n})}{\sigma^2_{\mu_0}} - \frac{m_0\delta^2_{0n}}{\sigma^2_{\mu_0}}. 
\]

(6.60)
6.5.1. Distribution Of $m_3$

A probability density function for $m_3$ can be obtained by employing expression (6.51). The Jacobian of the transformation is:

$$\frac{d\overline{X}_{0n}}{dm_3} = \frac{\sigma^2_{\mu_0} + \sigma^2_{\theta_0}}{\sigma^2_{\mu_0}}.$$

(6.61)

Using expression (6.60) the difference $\overline{X}_{0n} - m_0$ is:

$$\overline{X}_{0n} - m_0 = \frac{m_3(\sigma^2_{\mu_0} + \sigma^2_{\theta_0})}{\sigma^2_{\mu_0}} - \frac{m_0\sigma^2_{\theta_0}}{\sigma^2_{\mu_0}} - m_0$$

$$= (m_3 - m_0) \left[ \frac{\sigma^2_{\mu_0} + \sigma^2_{\theta_0}}{\sigma^2_{\mu_0}} \right].$$

(6.62)

The distribution of $K(m_3)$ is obtained from the distribution of $T(\overline{X}_{0n})$ through the transformation

$$K(m_3) = T(\overline{X}_{0n}) \bigg|_{\overline{X}_{0n} = m_3 = m_0} \left[ \frac{\sigma^2_{\mu_0} + \sigma^2_{\theta_0}}{\sigma^2_{\mu_0}} \right].$$

(6.63)

Expression (6.63) can be written as

$$K(m_3) = \frac{1}{\sqrt{2\pi} \sqrt{\sigma^2_{\mu_0} + \sigma^2_{\theta_0}}} e^{-\frac{1}{2} (m_3 - m_0)^2 \left[ \frac{\sigma^2_{\mu_0} + \sigma^2_{\theta_0}}{\sigma^2_{\mu_0}} \right]}. \left( \frac{\sigma^2_{\mu_0} + \sigma^2_{\theta_0}}{\sigma^2_{\mu_0}} \right).$$

(6.64)

If

$$\sigma_{m_3} = \frac{\sigma^2_{\mu_0}}{\sqrt{\sigma^2_{\mu_0} + \sigma^2_{\theta_0}}}$$

(6.65)

then the distribution of $K(m_3)$ is:

$$K(m_3) = \frac{1}{\sqrt{2\pi} \sigma_{m_3}} e^{-\frac{1}{2} \frac{(m_3 - m_0)^2}{\sigma^2_{m_3}}}.$$

(6.66)

Expression (6.66) shows that $m_3$ is normally distributed with mean $m_0$ on standard deviation given in (6.65).
6.6. Cost Equation and the Total Cost Per Unit

An expression for the cost equation can be obtained by following the technique developed in Chapter 3. The expected cost equation can be written in the form

\[ K(n, m_1^3, m_3^2) = \frac{n k_s + N k_r}{N - n} + k_E \int_{m_3^1}^{m_3^2} K(m_3) dm_3 \]

\[ - k_r \int_{m_3^1}^{m_3^2} K(m_3) dm_3. \]  

(6.67)

The expression of \( f(m_3) \) is given by:

\[ f(m_3) = \frac{1}{\sqrt{2\pi}\sigma_2} \int_{L}^{U} \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{1}{2}(m_3 - x_0)^2/\sigma_2^2} \]  

(6.68)

where \( m_1^3 \) and \( m_3^2 \) are the solution of equation (6.59), and \( \sigma_2^2 \) is given in (6.56).

Substituting expression (6.64) into the last integral of (6.66) yields:

\[ \int_{m_3^1}^{m_3^2} K(m_3) dm_3 = \sqrt{\frac{\sigma_0^2 + \delta_{0n}}{2\pi}} \int_{m_3^1}^{m_3^2} e^{-\frac{1}{2}(m_3 - m_0)^2/(\sigma_0^2 + \delta_{0n})} dm_3. \]  

(6.69)

Expression (6.69) can be written as:

\[ \int_{m_3^1}^{m_3^2} K(m_3) dm_3 = \Phi\left( \frac{m_3^2 - m_0}{\sigma_0^2 + \delta_{0n}} \right) - \Phi\left( \frac{m_3^1 - m_0}{\sigma_0^2 + \delta_{0n}} \right). \]  

(6.70)

The second integral of expression (6.70) can be evaluated by substituting expressions (6.64) and (6.67) for \( K(m_3) \) and \( f(m_3) \) respectively

\[ k_E \int_{m_3^1}^{m_3^2} K(m_3) f(m_3) dm_3 = k_E \int_{L}^{U} \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{1}{2}(m_3 - x_0)^2/\sigma_2^2} dx_0 \]

\[ \cdot \int_{m_3^1}^{m_3^2} \frac{1}{\sqrt{2\pi}\sigma_{m_3}} e^{-\frac{1}{2}(m_3 - m_0)^2/\sigma_{m_3}^2} dm_3 \]  

(6.71)
where \( \sigma_{m_a} \) is given in (6.65). The expression for the cost equation is then the sum of the following terms

\[
K(n, m_3^1, m_3^2) = \frac{nk_a + Nk_r}{N - n} \quad \text{Expression (6.70)} + \quad \text{Expression (6.71).} \quad (6.72)
\]

The following is a step by step algorithm to estimate the optimum cost.

1. Select a value for \( n \geq N_{\text{min}} + 1 \) and find the optimal values of \( m_3, m_3^1 \) and \( m_3^2 \) by solving equation (6.59).
2. Calculate \( K(n, m_3^1, m_3^2) \) given by expression (6.72).
3. Increase \( n \) by 1 and find another set of values for \( m_3, m_3^1', m_3^2' \) and estimate

\[
K(n + 1, m_3', m_3^1') = K(n + 1, m_3^1, m_3^2|n' = n + 1).
\]

4. Obtain a maximum value for \( n, n_{\text{max}} = n_0 \), such that:

\[
K(n_0 - 1, m_3^1, m_3^2|n = n_0 - 1) < K(n_0, m_3^1, m_3^2|n = n_0).
\]

The total cost relative to a sample of size \( n_0 - 1 \) is then the desired optimum cost.

6.6.1. Expected Fraction Defective, \( p \)

The expected value of \( p \), \( E(P|\mu_0) \) is:

\[
E[p(\mu_0)] = 1 - \int_{-\infty}^{+\infty} \int_{L}^{U} f(x_0|\mu_0)h(\mu_0)dx_0d\mu_0. \quad (6.73)
\]

Substituting \( f(x_0|\mu_0) \) and \( h(\mu_0) \) given in expressions (6.43) and (6.44) respectively into expression (6.73) yield:

\[
E[p(\mu_0)] = 1 - \int_{-\infty}^{+\infty} \int_{L}^{U} \frac{1}{\sqrt{2\pi} \sqrt{\sigma^2 + \sigma_e^2}} e^{-\frac{(x_0 - \mu_0)^2}{2(\sigma^2 + \sigma_e^2)}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(\mu_0 - m_0)^2}{2\sigma_0^2}} dx_0d\mu_0. \quad (6.74)
\]
The sum of the exponentials given in (6.74) can be transformed to the following form:

\[
\frac{(\mu_0 - m_0)^2}{\sigma_{\mu_0}^2} + \frac{(x_0 - \mu_0)^2}{\sigma^2 + \sigma_e^2} = \frac{(\mu_0 - m_4)^2}{\delta^2_4} + \frac{(m_0 - x_0)^2}{\sigma_e^2} \tag{6.75}
\]

where

\[
m_4 = \frac{m_0(\sigma^2_e + \sigma_\mu^2) + \sigma_\mu^2 \cdot x_0}{\sigma_\mu^2 + \sigma^2 + \sigma_e^2}
\]

\[
\sigma_e^2 = \sigma_\mu^2 + \sigma^2 + \sigma_e^2
\]

\[
\delta_4^2 = \frac{\sigma_\mu^2 \cdot (\sigma^2 + \sigma_e^2)}{\sigma_\mu^2 + \sigma^2 + \sigma_e^2}.
\]

Employing expression (6.75) expression (6.74) yields:

\[
E[p(\mu_0)] = 1 - \frac{1}{\sqrt{2\pi} \sqrt{\sigma_{\mu_0}^2 + \sigma^2 + \sigma_e^2}} \int_L^U e^{-\frac{1}{2} \frac{(m_0 - x_0)^2}{\sigma_{\mu_0}^2 + \sigma^2 + \sigma_e^2}} dx_0. \tag{6.77}
\]

The above expression can be written in the form:

\[
1 - \left[ \Phi\left(\frac{U - m_0}{\sqrt{\sigma_{\mu_0}^2 + \sigma^2 + \sigma_e^2}}\right) - \Phi\left(\frac{L - m_0}{\sqrt{\sigma_{\mu_0}^2 + \sigma^2 + \sigma_e^2}}\right) \right]. \tag{6.78}
\]

6.7. Optimum Economic Cost Model

In Chapter 4 a model is developed for the non destructive sampling optimum economic cost model. Error free statistical and economic characteristics were considered. In this chapter error prone sampling inspection techniques are reconsidered. The distributions employed for both error free and error prone cases are listed in Table 4. Also in the same table the means and variances for the different variables involved are provided. Employing Table 5, expressions for the expected costs and other economic quantities can be obtained by a direct transformation from one case to another.

The following is an algorithm for optimum cost evaluation.
(1) Employ the distributions for an error prone sampling inspection plan derived in the chapter and listed in Table 4.

(2) Obtain an expression for the optimal total cost. Both nondestructive and destructive sampling type plans can be considered.

(a) Nondestructive Sampling. Expressions for the expected costs of acceptance, screening, scrapping, and reworking can be derived. Follow the techniques employed in Sections 4.3.3, 4.3.4, 4.3.5, and 4.6.2 respectively.

(b) Destructive Sampling. Expressions for expected costs of acceptance, rejection and inspection can be obtained. Follow the techniques developed in Section 3.5.

(3) Obtain expressions for disposition limits for both destructive and nondestructive type plans.

(a) Nondestructive Sampling. The disposition limits for acceptance, screening and scrapping can be obtained in a similar technique followed to derive expressions (4.95), (4.102), (4.137) and (4.147).

(b) Destructive Sampling. The disposition limits for acceptance and rejection can be obtained in a technique similar of that followed to obtain expressions (3.21) and (3.22).

(4) For statistical process that can exist in more than one state with different probabilities, cases 1, 2, 3 can be generalized by following the techniques of Chapter 4.
Table 4. Some Probability Density Functions (All Normally Distributed) Employed in Chapters 1-4 When Sampling is Error Free or Error Prone.

<table>
<thead>
<tr>
<th>Error-Free Sampling Distribution</th>
<th>Error Prone Sampling Distributions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability Density Function</td>
<td>Mean</td>
</tr>
<tr>
<td>$f(x</td>
<td>\mu)$</td>
</tr>
<tr>
<td>$f(x)$</td>
<td>$m$</td>
</tr>
<tr>
<td>$T(\overline{X}_n</td>
<td>\mu)$</td>
</tr>
<tr>
<td>$T(\overline{X}_n)$</td>
<td>$m$</td>
</tr>
<tr>
<td>$t(x</td>
<td>\overline{X}, \mu)$</td>
</tr>
</tbody>
</table>

Single State Statistical Process $h(\mu)$

- $p_i =$ the fraction of time the process is in state $i$, $i = 1,2,\cdots, s$
- $\mu_i =$ mean of the distribution of the lot when the process is in state $i$, $i = 1,2,\cdots, s$
- $\sigma^2_{\mu_i} =$ variance of the lot mean when the process is in state $i$, $i = 1,2,\cdots, s$
- $\mu_{0i} =$ mean of the distribution of the lot when the process is in state $i$, $i = 1,2,\cdots, s$
- $\sigma^2_{\mu_{0i}} =$ variance of the lot mean when the process is in state $i$, $i = 1,2,\cdots, s$

Multistate Statistical Process $h(\mu)$

- $h(\mu|\overline{X}_n)$
- $K(m_n)$
6.8. Conditional Distribution of Material Fraction Defective

6.8.1. Marginal Distribution of $y_e$

The joint probability density function of the observed number of defectives from the sample and $p$ is:

$$f(y_e, p) = f(y_e | p)w(p), \quad 0 \leq p \leq 1.$$ (6.79)

The marginal p.d.f. of $y_e$ is:

$$g_n(y_e) = \int_0^1 f(y_e | p)w(p) dp$$

$$= \left(\begin{array}{c} n \\ y_e \end{array}\right) \int_0^1 p_e^{y_e} (1 - p_e)^{n-y_e}w(p) dp$$ (6.80)

where $p_e$ is the apparent fraction defective given by:

$$p_e = p(1 - e_2) + (1 - p)e_1$$ (6.81)

where $e_1$ and $e_2$ are defined earlier in the chapter. The distribution of $p$, $w(p)$ is obtained from that of $h(p)$ through the transformation:

$$w(p) = h(\mu_0(p)) \left| \frac{d\mu_0(p)}{dp} \right|.$$ (6.82)

The Jacobian of the transformation $\frac{d\mu_0(p)}{dp}$ is

$$\frac{d\mu_0(p)}{dp} = \frac{1}{\frac{dp(\mu_0)}{d\mu_0}}$$

where

$$\frac{dp(\mu_0)}{d\mu_0} = -\frac{1}{\sigma_{\mu_0}} \Phi\left(\frac{L - \mu_0}{\sigma_{\mu_0}}\right).$$ (6.83)

Substituting expression (6.44) and (6.83) into (6.82) yields

$$w(p) = \frac{\sqrt{\sigma_e^2 + \sigma_{\mu_0}^2}}{\sigma_{\mu_0}} e^{-\frac{1}{2}\left(\frac{\mu_0 - \mu_0}{\sigma_{\mu_0}^2} - \frac{(L - \mu_0)^2}{\sigma_e^2}\right)}.$$ (6.84)
Substituting expressions (6.81) and (6.84) into (6.79) yields:

\[
g_n(y_e) = \binom{n}{y_e} \int_0^1 \left[ p(1-e_2) + (1-p)e_1 \right]^{y_e} \left[ pc_2 + (1-p)(1-e_1) \right]^{n-y_e} \cdot \frac{\sqrt{\sigma^2 + \sigma^2_e}}{\sigma_{\mu_0}} \cdot \frac{1}{e^{\frac{1}{2} \left( \frac{\mu_e - \mu_{\mu_0}}{\sigma_{\mu_0}} \right)^2 - \frac{(L-e_0)^2}{\sigma^2 + \sigma^2_e}}} dp \quad 0 < p < 1.
\]  

(6.85)

Employing the binomial theorem to write:

\[
(1 - p_e)^{n-y_e} = \sum_{y=0}^{n-y_e} \binom{n-y_e}{y} (-p_e)^y.
\]  

(6.86)

Employing expression (6.86) the following product can be written as

\[
p_e^{y_e} (1 - p_e)^{n-y_e} = \sum_{y=0}^{n-y_e} \binom{n-y_e}{y} (-p_e)^y p_e^{y_e} = \sum_{y=0}^{n-y_e} \binom{n-y_e}{y} (-1)^y p_e^{y_e+y}.
\]  

(6.87)

Letting

\[
Q_e = y_e + y
\]  

(6.88)

then

\[
p_e^{y_e} (1 - p_e)^{n-y_e} = \sum_{\alpha_y = y_e}^{n} \binom{n-y_e}{\alpha_y - y_e} (-1)^{\alpha_y - y_e} p_e^{\alpha_y}.
\]  

(6.89)

Employing expression (6.89) the marginal distribution of \(y_e\) will be:

\[
g_n(y_e) = \int_0^1 \sum_{\alpha_y = y_e}^{n} \binom{n}{\alpha_y} \binom{n-y_e}{\alpha_y - y_e} (-1)^{\alpha_y - y_e} p_e^{\alpha_y} \cdot \frac{\sqrt{\sigma^2 + \sigma^2_e}}{\sigma_{\mu_0}} \cdot \frac{1}{e^{\frac{1}{2} \left( \frac{\mu_e - \mu_{\mu_0}}{\sigma_{\mu_0}} \right)^2 - \frac{(L-e_0)^2}{\sigma^2 + \sigma^2_e}}} dp \quad 0 < p < 1.
\]  

(6.90)
Expression (6.90) can be written as:

\[
g_n(y_e) = \sum_{\alpha_e = y_e}^{n} \binom{n}{y_e} \binom{n - y_e}{\alpha_e - y_e} \int_0^1 (-1)^{\alpha_e - y_e} p_e^{\alpha_e} \sqrt{\sigma^2 + \sigma_e^2} \frac{1}{\sigma_{\mu_0}} e^{-\frac{1}{2} \frac{(\alpha_e - m_0)^2}{\sigma_{\mu_0}^2}} \cdot e^{\frac{1}{2} \frac{(L - \mu_0)^2}{\sigma_{\mu_0}^2}} dp.
\] (6.91)

Letting \( v = \mu_0(p) \) then

\[
dp = p'(v) dv.
\] (6.92)

Employing expression (6.92) expression (6.91) can be written as:

\[
g_n(y_e) = \frac{1}{\sqrt{2\pi \sigma_{\mu_0}}} \sum_{\alpha_e = y_e}^{n} \binom{n}{y_e} \binom{n - y_e}{\alpha_e - y_e} \int_{-\infty}^{+\infty} (-1)^{\alpha_e - y_e} [p_e(v)]^{\alpha_e} \cdot e^{-\frac{1}{2} \frac{(v - m_0)^2}{\sigma_{\mu_0}^2}} dv.
\] (6.93)

Define \( B_{\alpha_e}(v) \) as

\[
B_{\alpha_e}(v) = \frac{1}{\sqrt{2\pi \sigma_{\mu_0}}} [p_e(v)]^{\alpha_e} e^{-\frac{1}{2} \frac{(v - m_0)^2}{\sigma_{\mu_0}^2}}.
\] (6.94)

Substituting expression (6.94) into (6.93) yields:

\[
g_n(y_e) = \sum_{\alpha_e = y_e}^{n} \binom{n}{y_e} \binom{n - y_e}{\alpha_e - y_e} (-1)^{\alpha_e - y_e} \int_{-\infty}^{+\infty} B_{\alpha_e}(v) dv.
\] (6.95)

6.8.2. The Conditional Probability of \( w(p|y_e) \)

The conditional Probability density function of \( p \) given \( y_e \) is:

\[
w(p|y_e) = \frac{f(y_e|p)w(p)}{\int_0^1 f(y_e|p)w(p)dp}.
\] (6.96)

Expression (6.96) can be written as:

\[
w(p|y_e) = \frac{\left[ \sum_{\alpha_e = y_e}^{n} \binom{n - y_e}{\alpha_e - y_e} (-1)^{\alpha_e - y_e} p_e^{\alpha_e} \right] \cdot \frac{1}{\sqrt{2\pi \sigma_{\mu_0}^2 + \sigma_e^2}} e^{-\frac{1}{2} \frac{(v - m_0)^2}{\sigma_{\mu_0}^2} + \frac{1}{2} \frac{(L - \mu_0)^2}{\sigma_{\mu_0}^2}}}{\sum_{\alpha_e = y_e}^{n} \binom{n - y_e}{\alpha_e - y_e} (-1)^{\alpha_e - y_e} \int_{-\infty}^{+\infty} B_{\alpha_e}(v) dv}.
\] (6.97)
The expected value of $p$ given an observed number of defectives from a sample is:

$$E(p|y_e) = \frac{\int_0^1 pf(y_e|p)w(p)dp}{\sum_{\alpha_e = y_e}^n \binom{n}{y_e} \binom{n-y_e}{\alpha_e-y_e} (-1)^{\alpha_e} \int_0^{+\infty} B_{\alpha_e}(v)dv}.$$  

(6.98)

Expression (6.98) can be written in the form:

$$E(p|y_e) = \frac{\int_0^{+\infty} \sum_{\alpha_e = y_e}^n \binom{n}{y_e} \binom{n-y_e}{\alpha_e-y_e} (-1)^{\alpha_e-y_e} pp_{p_{\alpha_e}} \frac{1}{\sqrt{2\pi}\mu_0} e^{-\frac{1}{2} \frac{(v-m_0)^2}{\mu_0}} \cdot dv}{\sum_{\alpha_e = y_e}^n \binom{n}{y_e} \binom{n-y_e}{\alpha_e-y_e} (-1)^{\alpha_e-y_e} \int_0^{+\infty} B_{\alpha_e}(v)dv}.$$  

(6.99)

Letting

$$B_{p_{\alpha_e}} = \frac{1}{\sqrt{2\pi}\mu_0} pp_{p_{\alpha_e}} e^{-\frac{1}{2} \frac{(v-m_0)^2}{\mu_0}}.$$  

(6.100)

Then, expression (6.100) will be reduced to:

$$E(p|y_e) = \frac{\sum_{\alpha_e = y_e}^n \binom{n}{y_e} \binom{n-y_e}{\alpha_e-y_e} (-1)^{\alpha_e-y_e} \int_0^{+\infty} B_{p_{\alpha_e}}(v)dv}{\sum_{\alpha_e = y_e}^n \binom{n}{y_e} \binom{n-y_e}{\alpha_e-y_e} (-1)^{\alpha_e-y_e} \int_0^{+\infty} B_{\alpha_e}(v)dv}.$$  

(6.101)

Expressions (6.94) and (6.100) show respectively that:

$$E[p_{\alpha_e}(v)] \alpha_e = \int_0^{+\infty} B_{\alpha_e}(v)dv.$$  

(6.102)

and

$$E[pp_{p_{\alpha_e}}] = \int_0^{+\infty} B_{p_{\alpha_e}}(v)dv.$$  

(6.103)

Expression (6.101) can be written in terms of(6.102) as

$$E(p|y_e) = \frac{\sum_{\alpha_e = y_e}^n \binom{n}{y_e} \binom{n-y_e}{\alpha_e-y_e} (-1)^{\alpha_e-y_e} E[pp_{p_{\alpha_e}}]}{\sum_{\alpha_e = y_e}^n \binom{n}{y_e} \binom{n-y_e}{\alpha_e-y_e} (-1)^{\alpha_e-y_e} E[p_{\alpha_e}(v)] \alpha_e}.$$  

(6.104)
For evaluating $E(p|\gamma_e)$, the expected values of $E[pp_e^\alpha]$ and $E[p_e(v)]^\alpha$ must be calculated. A technique similar to that developed in Chapter 5 will be discussed later for evaluating the moments of $p$ raised to a power $k, k \geq 1$.

**Example.** Evaluate $E[p_e(v)]^\alpha$ and $E[pp_e^\alpha]$ given in expressions (6.102) and (6.103) respectively assuming $\alpha_e = 0$ and $\epsilon_1 = \epsilon_2 = 0$.

Under assumption $\alpha_e = 0$ and $\epsilon_1 = \epsilon_2 = 0$, $E[p_e(v)]^\alpha = 1$, and expression (6.94) is reduced to

$$
\frac{1}{\sqrt{2\pi}\sigma_{\mu_0}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}(v-m_0)^2/\sigma_{\mu_0}^2} \, dv = 1.
$$

In this case expression (6.101) is verified. Expression (6.102) yields:

$$
E(p) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma_{\mu_0}} e^{-\frac{1}{2}(v-m_0)^2/\sigma_{\mu_0}^2} \, dv.
$$

Substituting expression (2.60) given for $p$ into the above expression yields:

$$
E(p) = \frac{1}{2\pi \sqrt{\sigma^2 + \sigma_e^2}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left[(z-v)^2 + (v-m_0)^2\right]/\sigma^2} \, dxdv.
$$

The above expression can be written in the form:

$$
E(p) = \frac{1}{2\pi \sqrt{\sigma^2 + \sigma_e^2}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left[(z-v)^2 + (v-m_0)^2\right]/\sigma^2} \, dxdv
$$

where

$$
\delta_1^2 = \frac{\sigma_{\mu_0}^2 \left(\sigma^2 + \sigma_e^2\right)}{\sigma_{\mu_0}^2 + \sigma^2 + \sigma_e^2} \quad \text{and} \quad \sigma_1^2 = \sigma_{\mu_0}^2 + \sigma^2 + \sigma_e^2.
$$

Then $E(p)$ will be:

$$
E(p) = \Phi\left(\frac{L - m_0}{\sqrt{\sigma_{\mu_0}^2 + \sigma^2 + \sigma_e^2}}\right).
$$

For an error free sampling inspection process $\sigma_{\mu_0}^2 = \sigma_\mu^2, m_0 = m$ and $\sigma_e^2 = 0$. $E(p)$ will be reduced to $E(p) = \Phi\left(\frac{L - m}{\sqrt{\sigma_{\mu}^2 + \sigma_e^2}}\right)$, which is given in expression (5.31).
For an error prone sampling inspection plan the expected value of \( P^K \), \( K \geq 1 \) can be written as:

\[
E[P^K_0] = \int_{-\infty}^{+\infty} \left[ \int_{-\infty}^{L} \frac{1}{\sqrt{2\pi} \sigma_0} \frac{1}{\sqrt{2\pi} \sigma_\mu + \sigma_\sigma^2} \right]^k \frac{1}{\sqrt{2\pi} \sigma_0} e^{-\frac{1}{2} \frac{(x_0-m_0)^2}{\sigma_0^2}} dX_0 \frac{1}{\sqrt{2\pi} \sigma_\mu + \sigma_\sigma^2} d\mu_0.
\]

(6.105)

An expression similar to (4.55) is

\[
\int_{-\infty}^{L} \frac{1}{\sqrt{2\pi} \sigma_0} e^{-\frac{1}{2} \frac{(x_0-m_0)^2}{\sigma_0^2}} dX_0 < \epsilon(v_0) \quad (6.106)
\]

For finding the lower and upper limits of \( E[p^K] \) consider the following cases:

**Case 1.** It is verified numerically and graphically that expression \( g_1(v_0) \) can be written as:

\[
g_1(v_0) = e^{-\frac{1}{2\sigma_v^2}(v_0-v_{m_0})^2}.
\]

(6.107)

Employing expression (6.107) expression (6.105) will be:

\[
\int_{-\infty}^{L} \frac{1}{\sqrt{2\pi} \sigma_0} e^{-\frac{1}{2} \frac{(x_0-m_0)^2}{\sigma_0^2}} dX_0 < \epsilon(v_0) + e^{-\frac{1}{2\sigma_v^2}(v_0-v_{m_0})^2}
\]

(6.108)

or

\[
\int_{-\infty}^{L} \frac{1}{\sqrt{2\pi} \sigma_0} e^{-\frac{1}{2} \frac{(x_0-m_0)^2}{\sigma_0^2}} dX_0 > -\epsilon(v_0) + e^{-\frac{1}{2\sigma_v^2}(v_0-v_{m_0})^2}.
\]

(6.109)

Employing expression (6.108) and (6.109) an estimate for \( E[p^K] \) similar to (5.68) can be obtained:

\[
(-1)^k \epsilon(v_0)^k + \sum_{j=1}^{k} \binom{k}{j} \left[ \epsilon(v_0) \right]^{k-j} \left[ \frac{\sigma_{v_0}^2}{\sigma_{v_0}^2 + j(\sigma_{\mu}^2 + \sigma_{\sigma}^2)} \right]^{\frac{1}{2}} e^{-\frac{1}{2} \frac{(v_{m_0}-m_0)^2}{\sigma_\mu^2 + \sigma_\sigma^2}}
\]

\[
< E(P^K) < \left[ \epsilon(v_0) \right]^{k}
\]

\[
+ \sum_{j=1}^{k} \binom{k}{j} \left[ \epsilon(v_0) \right]^{k-j} \left[ \frac{\sigma_{v_0}^2}{\sigma_{v_0}^2 + j(\sigma_{\mu}^2 + \sigma_{\sigma}^2)} \right]^{\frac{1}{2}} e^{-\frac{1}{2} \frac{(v_{m_0}-m_0)^2}{\sigma_\mu^2 + \sigma_\sigma^2}}
\]

(6.110)
where \( \mu_0 \) and \( \sigma_0^2 \) are the mean and variance respectively of \( x_0 \) defined in Section 5, \( \mu_e \) and \( \sigma_e \) are defined in expressions (6.37) and (6.38) respectively.

**Case 2.** A slightly different expression for \( E(P^k) \) other than (6.110) can be obtained by expanding \( e^{g_1(v)} \) into a series as:

\[
g_1(v_0) = C_1 + C_1 v_0 + c_2 v_0^2 + \cdots + C_s v_0^s = \sum_{i=0}^{s} C_i v_0^i. \tag{6.111}
\]

Employing expression (6.111) expressions (6.108) and (6.109) yield

\[
\left[ \int_{-\infty}^{L} \frac{1}{\sqrt{2\pi\sigma_0}} e^{-\frac{(x_0-\mu_0)^2}{2\sigma_0^2}} dx_0 \right]^k < \left[ e(v_0) + e \sum_{i=1}^{s} C_i v_0^i \right]^k \tag{6.112}
\]

and

\[
\left[ \int_{-\infty}^{L} \frac{1}{\sqrt{2\pi\sigma_0}} e^{-\frac{(x_0-\mu_0)^2}{2\sigma_0^2}} dx_0 \right]^k > \left[ -e(v_0) + e \sum_{i=1}^{s} C_i v_0^i \right]^k. \tag{6.113}
\]

The righthand side of expression (6.112) can be written as

\[
\sum_{j=0}^{k} \binom{k}{j} \left[ e \sum_{i=1}^{s} C_i v_0^i \right]^j \left[ e(v_0) \right]^{k-j}. \tag{6.114}
\]

A similar form for the righthand side of (6.113) is

\[
\sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \left[ e \sum_{i=1}^{s} C_i v_0^i \right]^j \left[ e(v_0) \right]^{k-j}. \tag{6.115}
\]

The lower and upper limits for \( E(P^k) \) are

\[
\frac{1}{\sqrt{2\pi \sqrt{\sigma_{\mu_0}^2 + \sigma_e^2}}} \cdot \int_{-\infty}^{+\infty} \sum_{j=0}^{k} \binom{k}{j} \cdot e^{j \sum_{i=1}^{s} C_i v_0^i} \left[ e(v_0) \right]^{k-j} \cdot e^{-\frac{(v_0-\mu_0)^2}{2\sigma_{\mu_0}^2 + \sigma_e^2}} dv_0 < E(P^k) < \frac{1}{\sqrt{2\pi \sqrt{\sigma_{\mu_0}^2 + \sigma_e^2}}} \int_{-\infty}^{+\infty} \sum_{j=0}^{k} \binom{k}{j} \left[ -1 \right]^{k-j} \cdot \left[ e^{j \sum_{i=1}^{s} C_i v_0^i} \left[ e(v_0) \right]^{k-j} dv_0. \tag{6.116}
\]
In all cases computing the upper and lower limits of \( E(p^K) \) requires numerical integration. A computer program written in FORTRAN 77 to compute the limits is listed in Appendix F.

6.9. Conditional Expectation of \( p \) given \( \bar{X} \)

Informations on most of the distributions needed for the foregoing discussion are listed in Table 6. In this section an intensive use of \( E(p^K) \) is required for conditional probabilities estimation. The variables employed in Chapter 5 will be modified to

\[
S = v_0 - m_0 \quad \text{and} \quad t_i = z_i - m_0. \tag{6.117}
\]

The terms \( A, B, \sigma_K^2, m_K, B^2, z_i, C, A', C', \rho_K, \sigma_0^2 \), defined in expressions (5.76), (5.78), (5.81), (5.83), (5.93), (5.95), (5.96), respectively, will be modified as follows. The expressions for \( A \) and \( B \) are

\[
A = \sum_{i=1}^{k} t_i^2 \quad \text{and} \quad B = \sum_{i=1}^{k} t_i. \tag{6.118}
\]

Employing Table 6 of this chapter the expression of \( \sigma_k^2 \) will be

\[
\sigma_k^2 = \frac{\sigma_{\mu_0}^2}{(\sigma^2 + \sigma_e^2) + k(\sigma_{\mu_0}^2)}. \tag{6.119}
\]

The expression for \( m_k \) will be

\[
m_k = \frac{(\sigma_\mu^2 + \sigma_e^2)B}{k(\sigma_{\mu_0}^2) + \sigma^2 + \sigma_e^2}. \tag{6.120}
\]

The above expression can be written as:

\[
m_k = \sigma_k^2 \cdot B. \tag{6.121}
\]
Using expression (6.118) an expression for $\sigma^2$ can be written as

$$B^2 = \sum_{i=1}^{k} t_i^2 + 2 \sum_{i \neq j}^{k(k-1)/2} t_i t_j.$$  \hspace{1cm} (6.122)

Setting $\sum_{i \neq j}^{k(k-1)/2} t_i t_j = C$, expression (6.122) is given by:

$$B^2 = \sum_{i=1}^{k} t_i^2 + 2C.$$  \hspace{1cm} (6.123)

The expression for $z_i$ is:

$$z_i = \frac{t_i}{[\sigma^2 + \sigma_z^2 + \sigma^2_{\mu_0}(k-1)]^{1/2}}.$$  \hspace{1cm} (6.124)

In terms of $z_i$, the expression for $A$ given in (6.121) can be written as

$$A = \sum_{i=1}^{k_1} [(\sigma^2 + \sigma_z^2 + \sigma^2_{\mu_0}(k-1)] z_i^2.$$  \hspace{1cm} (6.125)

Expression (6.125) will be:

$$A = [(\sigma^2 + \sigma_z^2 + \sigma^2_{\mu_0}(k-1)] \sum_{i=1}^{k} z_i^2.$$  \hspace{1cm} (6.126)

The expression for $C$ defined in (5.106) can be defined in terms of $z$ as:

$$C = [(\sigma^2 + \sigma_z^2 + \sigma^2_{\mu_0}(k-1)] \sum_{i \neq j}^{k(k-1)/2} z_i z_j.$$  \hspace{1cm} (6.127)

The expression for $A'$, $C'$, $l_K$ and $\sigma_0^2$ can be written as:

$$A' = \sum_{i=1}^{k} z_i^2.$$  \hspace{1cm} (6.128)
where
\[ C' = \frac{k(k-1)/2}{\sum_{i \neq j} \tilde{z}_i \tilde{z}_j} \quad (6.129) \]
\[ l_k = \frac{\sigma_{\mu_0}^2}{(\sigma^2 + \sigma_z^2) + (k-1)\sigma_{\mu_0}^2} \quad (6.130) \]

The expected value of $p$ raised to a power $k \geq 1$ can be evaluated easily by employing expression (5.103). In evaluating expression (5.103) the modified parameters derived in this section must be employed to determine the conditional expectations. An explicit relationship between $p$ and $p_0$ can be obtained by employing the following expressions:
\[ dp(\mu) = \frac{1}{\sigma} e^{-\frac{1}{2} \left( \frac{L-\mu}{\sigma} \right)^2} \]
and
\[ dp_0(\mu_0) = -\frac{1}{\sqrt{\sigma^2 + \sigma_z^2}} e^{-\frac{1}{2} \left( \frac{L-\mu_0}{\sigma^2 + \sigma_z^2} \right)^2}. \quad (6.131) \]

From the above expression it is possible to deduce that:
\[ \frac{dp(\mu)}{dp_0(\mu_0)} = \frac{\sqrt{\sigma^2 + \sigma_z^2} \Phi_1}{\sigma \Phi_2} \quad (6.132) \]
where
\[ \Phi_1 = e^{-\frac{1}{2} \left( \frac{L-\mu}{\sigma_0} \right)^2} \quad \text{and} \quad \Phi_2 = e^{-\frac{1}{2} \left( \frac{L-\mu_0}{\sigma^2 + \sigma_z^2} \right)^2}. \]

Solving equation (6.132) for $p(\mu)$ yields:
\[ p(\mu) = \int \frac{\sqrt{\sigma^2 + \sigma_z^2}}{\sigma} \frac{\Phi_1}{\Phi_2} dp_0 + C'. \quad (6.133) \]
The constant of integration \( C' \) can be evaluated by using the following boundary conditions. For \( x_e = 0, \mu = \mu_0, \sigma^2_e = 0 \) and \( \Phi_1 = \Phi_2 \), then expression (6.116) yields

\[
p(\mu) = p(\mu) + C' \quad \text{or} \quad C' = 0.
\]

Rewriting expression (6.133) into the form:

\[
p(\mu) = \int \frac{\sqrt{\sigma^2 + \sigma^2_e \Phi_1}}{\Phi_2} dp_0. \quad (6.134)
\]

Since \( \Phi_1 \) and \( \Phi_2 \) are functions of \( \mu_0(p_0) \), it is not possible to find a unique solution for \( p(\mu) \) in terms of \( p_0(\mu_0) \). Expression (6.134) can be simplified by setting

\[
p(\mu) = F(p_0(\mu_0)). \quad (6.135)
\]

The distribution of \( \mu_0 \) can be obtained through the transformation:

\[
h(\mu_0) = w(p_0) \left| \frac{dp_0(\mu_0)}{d\mu_0} \right|. \quad (6.136)
\]

The distribution of \( p_0 \) can be derived from the distribution of \( (p) \) given in expression (5.123) which yields:

\[
w(p_0) = \frac{k_0 \Gamma(A + B)}{\Gamma(A) \Gamma(B)} [F(p_0)]^{A-1}[1 - F(p_0)]^{B-1} \cdot \left| \frac{dp(\mu)}{dp_0(\mu_0)} \right|. \quad (6.137)
\]

Substituting expressions (6.135) and (6.132) into (6.137) yields:

\[
w(p_0(\mu_0)) = \frac{k_0 \Gamma(A + B)}{\Gamma(A) \Gamma(B)} [F(p_0)]^{A-1}[1 - F(p_0)]^{B-1} \cdot \frac{\sqrt{\sigma^2 + \sigma^2_e \Phi_1}}{\Phi_2}. \quad (6.138)
\]

The constant \( K_0 \) is introduced into equation (6.137) because the distribution is a truncated \( \beta \)-distribution as proved in Chapter 5.

Employing expressions (6.138) and (6.132) the distribution of \( \mu_0 \) is:

\[
h(\mu_0) = \frac{k_0}{2} \frac{\Gamma(A + B)}{\Gamma(A) \Gamma(B)} [F(p_0)]^{A-1}[1 - F(p_0)]^{B-1} \cdot \frac{\sqrt{\sigma^2 + \sigma^2_e \Phi_1}}{\Phi_2} p_0(\mu_0) \quad (6.139)
\]
where

\[ p'_0(\mu_0) = \frac{dp_0(\mu_0)}{d\mu_0}. \]

### 6.10. Conditional Expectation of \( \mu_0 \) Given \( \bar{X} \)

Expression (5.129) can be written in the form:

\[
h(\mu_0 | \bar{X}_{0n}) = \frac{T(\bar{X}_{0n}|\mu_0)h(\mu_0)}{\int_{-\infty}^{+\infty} T(\bar{X}_{0n}|\mu_0)h(\mu_0) d\mu_0}. \tag{6.140}
\]

Substituting expression (6.44) for \( T(\bar{X}_{0n}|\mu_0) \) and expression (6.23) for \( h(\mu_0) \) into (6.140) yields:

\[
h(\mu_0 | \bar{X}_{0n}) = \frac{c}{\sqrt{2\pi} \sigma^2_{0n}} \left[ F(p_0) \right]^{A-1} \left[ 1 - F(p_0) \right]^{B-1} \frac{\Phi_1}{\Phi_2} \frac{|p'_0(\mu_0)|}{d\mu_0} \]

where \( F(p_0), \Phi_1 \) and \( \Phi_2, p'_0(\mu_0) \) are given in expressions (6.135), (6.132) respectively.

To evaluate expression (6.141) consider the following two cases.

**Case 1.** Assume \( B - 1 \) is a positive integer then:

\[
[1 - F(P_0)]^{B-1} = \sum_{j=0}^{B-1} \binom{B-1}{j} (-1)^j [F(p_0)]^j. \tag{6.142}
\]

All right hand terms in expression (6.137) containing powers higher than \([p_0(\mu_0)]^{B-1}\) Vanish. Employing expression (6.140) it is possible to write the product \([p_0(\mu_0)]^{A-1}[1 - P_0(\mu_0)]^{B-1}\) as:

\[
[F(p_0)]^{A-1}[1 - F(p_0)]^{B-1} = \sum_{j=0}^{B-1} \binom{B-1}{j} (-1)^j [F(P_0)]^{A+j-1}. \tag{6.143}
\]
The denominator of expression (6.141) can be written as:

$$
\int_{-\infty}^{+\infty} e^{-\frac{1}{2} \frac{(X-\mu_0)^2}{\delta_0^2}} \cdot \sum_{j=0}^{B-1} \binom{B-1}{j} (-1)^j [F(p_0)]^{A+j-1} \cdot \frac{1}{\Phi_1 \sqrt{\sigma^2 + \sigma_e^2}} e^{-\frac{1}{2} \frac{(L-\mu_0)^2}{\delta_0^2}} d\mu_0.
$$

The expression for $\delta^2_{0n}$ is given in (6.49). The sum of the exponentials given in (6.144) can be written in the form:

$$
\frac{(\mu_0 - \overline{X})^2}{\delta^2_{0n}} + \frac{(L - \mu_0)^2}{\sigma^2 + \sigma_e^2} = \frac{(\mu_0 - \overline{m}_5)^2}{\delta_5^2} + \frac{(\overline{X} - L)^2}{\sigma}
$$

where

$$
\delta_5^2 = \frac{\delta_{0n}^2 (\sigma^2 + \sigma_e^2)}{\delta_{0n}^2 + \sigma^2 + \sigma_e^2}, \quad \sigma^2 = \delta_{0n}^2 + \sigma^2 + \sigma_e^2
$$

$$
m_5 = \frac{\overline{X} (\sigma^2 + \sigma_e^2) + \delta_{0n}^2 L}{\delta_{0n}^2 + \sigma^2 + \sigma_e^2}.
$$

Employing the terms given in (6.144), expression (6.144) can be written as:

$$
\frac{1}{\sqrt{\sigma^2 + \sigma_e^2}} \cdot \sum_{j=0}^{B-1} \binom{B-1}{j} (-1)^j e^{-\frac{1}{2} \frac{(X-L)^2}{\sigma_5^2}} \cdot \int_{-\infty}^{+\infty} e^{-\frac{1}{2} \frac{(\mu_0 - m_5)^2}{\delta_5^2}} [F(p_0)]^{A+j-1} \frac{\Phi_1}{\Phi_2} d\mu_0.
$$

Expression (6.145) can be written in the form:

$$
\frac{\delta_5 \sqrt{2\pi}}{\sqrt{\sigma^2 + \sigma_e^2}} \sum_{j=0}^{B-1} \binom{B-1}{j} (-1)^j e^{-\frac{1}{2} \frac{(X-L)^2}{\sigma_5^2}} E\left\{ [F(p_0)]^{A+j-1} \frac{\Phi_1}{\Phi_2} \right\}
$$

where

$$
E\left\{ [F(p_0)]^{A+j-1} \frac{\Phi_1}{\Phi_2} \right\} = \frac{1}{\sqrt{2\pi} \delta_5} \int_{-\infty}^{+\infty} [F(p_0)]^{A+j-1} \frac{\Phi_1}{\Phi_2} e^{-\frac{1}{2} \frac{(\mu_0 - m_5)^2}{\delta_5^2}} d\mu_0.
$$
Substituting expressions (6.143) and (6.148) into (6.139) yields:

\[ h(\mu_0 | \bar{X}_{on}) = \frac{\sum_{j=0}^{B-1} \binom{B-1}{j} (-1)^j e^{-\frac{1}{2} \frac{(\mu_0 - m_0)^2}{\sigma_0^2}} F(p_0)^{A+j-1} \Phi_1}{\sqrt{2\pi \delta_0} \sum_{j=0}^{B-1} \binom{B-1}{j} (-1)^j E\{[F(p_0)]^{A+j-1} \Phi_1 \}}. \] 

(6.149)

6.11. Conditional Expectation of \( P_0(\mu_0) \) given \( \bar{X} \)

The expression of \( E(p_0(\mu_0) | \bar{X}_{on}) \) is given by:

\[ E(p_0(\mu_0) | \bar{X}_{on}) = \int_{-\infty}^{+\infty} p_0(\mu_0) h(\mu_0 | \bar{X}_{on}) d\mu_0. \] 

(6.150)

Employing expression (6.149), expression (6.150) can be written as:

\[ E(p_0(\mu_0) | \bar{X}_{on}) = \frac{\sum_{j=0}^{B-1} \binom{B-1}{j} (-1)^j E\{[p_0(\mu_0)] F(p_0)^{A+j-1} \Phi_1 \}}{\sum_{j=0}^{B-1} \binom{B-1}{j} (-1)^j E\{[F(p_0)]^{A+j-1} \Phi_1 \}} \] 

(6.151)

where

\[ E\{[p_0(\mu_0)] F(p_0)^{A+j-1} \Phi_1 \} = \frac{1}{\sqrt{2\pi \delta_0}} \int_{-\infty}^{+\infty} [p_0] F(p_0)^{A+j-1} \Phi_1 e^{-\frac{1}{2} \frac{(\mu_0 - m_0)^2}{\sigma_0^2}} d\mu_0. \] 

(6.152)

**Special case.** For an error free sampling plan \( x_0 = x, x_e = 0, \sigma_e^2 = 0, \)
\( p = p_0, \) \( F(p_0) = p_0 \) and \( \phi_1 = \Phi_2, \) expression (6.139) will be reduced to:

\[ E(p_0(\mu_0) | \bar{X}_{on}) = E(p(\mu_0) | \bar{X}) \]

\[ = \frac{\sum_{j=0}^{B-1} \binom{B-1}{j} (-1)^j E[p(\mu)^{A+j}] }{\sum_{j=0}^{B-1} \binom{B-1}{j} (-1)^j E[p^{A+j-1}]} \] 

(6.153)
It is obvious that expression (6.153) is identical to expression (6.137).

**Case 2.** Assume $B - 1$ is not a positive integer, then $[1 - p_0(\mu_0)]^{B-1}$ can be expressed in terms of an infinite series given below:

$$[1 - p_0(\mu_0)]^{B-1} = \sum_{j=0}^{\infty} \binom{B-1}{j}(-1)^j [p_0(\mu_0)]^j. \quad (6.154)$$

An expression identical to that given in (6.153) can be obtained for $E[p_0(\mu_0)|\bar{X}_{on}]$. The only difference is that $\sum_{j=0}^{B-1}$ is infinite.

**Example**

Assuming that the individual observations of the values of the quality characteristic $X$ are shifted by an average value of 1 percent due to the functioning of a production process under inspection errors; estimate the following quantities

1- The variance of the quality characteristic $X$
2- The mean of the quality characteristic $X$
3- The decision points
4- The rejection cost per unit (cost of scrapping per unit)
5- The acceptance cost per unit
6- The expected total cost per unit
7- Plot the sample means versus the sample size for both error free and error prone sampling inspection
8- Plot the different costs per unit for both error free and error prone sampling inspection
### TABLE 1: A VARIABLE PLAN FOR THE COST MODEL

**INPUT: MODEL SPECIFICATIONS**

<table>
<thead>
<tr>
<th></th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Upper Limit of U.C.X</td>
<td>11.01100</td>
</tr>
<tr>
<td>Lower Limit of U.C.X</td>
<td>6.00600</td>
</tr>
<tr>
<td>Variance of X</td>
<td>2.00010</td>
</tr>
<tr>
<td>Variance of Mean of X</td>
<td>1.00010</td>
</tr>
<tr>
<td>Unit Cost of Rejection of the Remainder of the Lot</td>
<td>0.70000</td>
</tr>
<tr>
<td>Unit Cost of Acceptance</td>
<td>2.00000</td>
</tr>
<tr>
<td>Unit Cost of Sampling</td>
<td>2.00000</td>
</tr>
<tr>
<td>LOT SIZE</td>
<td>1000</td>
</tr>
</tbody>
</table>

**OUTPUT1: SAMPLE SIZE, ROOTS OF THE COST FUNCTION, POSTERIOR AND SAMPLING COSTS PER UNIT**

<table>
<thead>
<tr>
<th>COL. No.</th>
<th>DESCRIPTION</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>SAMPLE SIZE</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{(a + b)}{(n + n_a)} ) WHERE:</td>
</tr>
<tr>
<td></td>
<td>( a ): Variance of the Mean of X</td>
</tr>
<tr>
<td></td>
<td>( b ): Variance of X</td>
</tr>
<tr>
<td></td>
<td>( n ): Sample Size</td>
</tr>
<tr>
<td>3</td>
<td>1st ROOT of the COST EQ.</td>
</tr>
<tr>
<td>4</td>
<td>2nd ROOT of the COST EQ.</td>
</tr>
<tr>
<td>5</td>
<td>SAMPLING COST PER UNIT</td>
</tr>
<tr>
<td>6</td>
<td>POSTERIOR COST PER UNIT</td>
</tr>
<tr>
<td>7</td>
<td>ACCEPTANCE COST PER UNIT</td>
</tr>
<tr>
<td>8</td>
<td>REJECTION COST PER UNIT</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
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<th>DESCRIPTION</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>SAMPLE SIZE</td>
</tr>
<tr>
<td>2</td>
<td>LOWER LIMIT OF THE SAMPLE MEAN</td>
</tr>
<tr>
<td>3</td>
<td>UPPER LIMIT OF THE SAMPLE MEAN</td>
</tr>
<tr>
<td>4</td>
<td>AVERAGE FRACTION DEFECTIVE OF THE PROCESS GIVEN THE LOWER LIMIT OF THE SAMPLE MEAN</td>
</tr>
<tr>
<td>5</td>
<td>AVERAGE FRACTION DEFECTIVE OF THE PROCESS GIVEN THE UPPER LIMIT OF THE SAMPLE MEAN</td>
</tr>
</tbody>
</table>

### OUTPUT2: AVERAGE PROCESS FRACTION DEFECTIVES AND THE SAMPLE MEANS

<table>
<thead>
<tr>
<th>COL. No.</th>
<th>DESCRIPTION</th>
</tr>
</thead>
<tbody>
<tr>
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</tr>
<tr>
<td>5</td>
<td>AVERAGE FRACTION DEFECTIVE OF THE PROCESS GIVEN THE UPPER LIMIT OF THE SAMPLE MEAN</td>
</tr>
</tbody>
</table>

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</thead>
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<tr>
<td>2</td>
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<tr>
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<td>UPPER LIMIT OF THE SAMPLE MEAN</td>
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<tr>
<td>4</td>
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Figure 16. The First Root of the Cost Equation for a Destructive Variable Sampling Plan for Fraction Defective for Both Error Free and Error Prone Inspection for a Lot of Size 1000.
Figure 17. The Second Root of the Cost Equation for a Destructive Variable Sampling Plan for Fraction Defective for Both Error Free and Error Prone Inspection for a Lot of Size 1000.
Figure 18. The Acceptance Cost Per Unit for a Destructive Variable Sampling Plan for Fraction Defective for Both Error Free and Error Prone Inspection for a Lot of Size 1000.
Figure 19. The Rejection Cost Per Unit for a Destructive Variable Sampling Plan for Fraction Defective for Both Error Free and Error Prone Inspection for a Lot of Size 1000.
Figure 20. The Total Expected Cost per Unit for a Destructive Variable Sampling Plan for Fraction Defective for Both Error Free and Error Prone Inspection for a Lot of Size 1000.
CHAPTER 7

EXPECTED COST OF KEEPING A PRODUCTION PROCESS UNDER CONTROL

7.1. Background

Implementing a reliable, precise manufacturing line with high speed computers, reduces the rate of defective items produced. The cost of constant maintenance and updating equipment add significantly to the cost of items produced. In the long run a low fraction of defective items satisfies the producer and the consumer who are interested in producing and purchasing lots of high quality with minimum costs. The cost of an item is then due to the cost incurred by sampling and inspecting lots of finished products as treated in Chapters 2 through 6 and the cost of keeping a production process under control which is the subject of this chapter.

The use of the control charts require the optimal selection of the sample size \( n \), an interval between the successive samples \( (h) \) and the control limit \( (k) \). Duncan [1956] first proposed a model to determine \( n, h, k \). In this model an \( \bar{X} \) chart is employed. The loss function is minimized when a si958] Gibra [1968, 1975], Taylor [1968], Goal, Wu [1973] and an algorithm for computing CUSUM charts.


Many authors use control c purposes. Knappenberger and Grandage [1971], Duncan [1971] developed models for determination of the optimal test procedure when several causes of failure exist. A Bayesian approach to quality control charts is proposed by Chiu [1980]. Chiu assumed that at the beginning of
production or after each search the process mean is reset. The prior probability
density function for this initial resetting is \( f(\mu) \). Montgomery and Klatt [1972]
considered a process characterised by several quality characteristics where the
joint effect determined the product quality characteristic. A Hotelings \( T^2 \) control
chart was employed. They recommend that for a process out of control two
procedures be used to determine the subset of quality characteristics that cause
the failure these are:

(a) Principal component techniques suggested by Jackson [1959];

(b) Simultaneous confidence interval suggested by Montgomery and
Wadsworth [1972].

Jones and Case [1981] developed a model that assumes that the process
is either in control or out of control. The in control state is defined as that when
the process parameters (mean and variance) are in control. The out of control
state is defined when either the mean or the variance or both are out of control.
Several different states in which the production process can exist are considered.

In this work the following points are studied.

(1) A computer program is developed in FORTRAN 77 to estimate the
probability of type I error, the power of the chart, the interval between
samples in hours, the width of the control limits of the chart, the aver­
age time the process is operating in the presence of an assignable cause
and the optimal expected cost per hour incurred by the process. In esti­
mating the above probabilities and quantities the following assumptions
are made:

i. A production process is out of control due to a single cause of failure.

ii. A production process is out of control due to multiple causes of
failure.

iii. A production facility continues in operation during the search for a
single or multiple causes of failure.

iv. A production facility is shut down during the search for the cause or causes of failure.

(2) A Bayesian expression for the probability for a point falling outside the control limit when the process is in control.

(3) Bayesian expressions to extend the models of Duncan, Montgomery and Klatt et al.

7.2. Single Assignable Cause of Failure Production Cycle

7.2.1. Model A

A process is allowed to continue in operation during the search for an assignable cause of failure. The process starts in a state of statistical control with mean $\mu_0$ and a standard deviation $\sigma$. If an assignable cause of failure occurs the mean will be shifted from $\mu_0$ to $\mu_0 + \delta\sigma$, where $\delta$ is known.

The production cycle consists of the following steps:

(1) The in control period;
(2) The out of control period;
(3) The time of selecting a sample and interpreting the results;
(4) The time elapsed for looking for the assignable cause.

The time before the assignable cause occurs is assumed to be exponentially distributed. Duncan [1956] shows that the average time of occurrence of an assignable cause between the $n^{th}$ and the $n+1^{st}$ sample is:

$$\tau = \frac{\int_{nh}^{(n+1)h} e^{-\lambda t} \lambda(t - nh)dt}{\int_{nh}^{(n+1)h} e^{-\lambda t} \lambda dt} = \frac{1 - (1 + \lambda h)e^{-\lambda h}}{\lambda(1 - e^{-\lambda h})}. \quad (7.1)$$

Expression (7.1) can be written as:

$$\tau = \frac{h}{2} - \frac{\lambda h^2}{12} + \text{terms of } \lambda^3h^4 \text{ or higher.} \quad (7.2)$$
Define the following probabilities:

\( P = \) probability that assignable cause will be detected.

\( Q = \) probability that an assignable cause will not be detected.

For a shift of the process mean from \( \mu_0 \) to \( \mu_0 + \delta \sigma \) \( P \) will be:

\[
P = \int_{-\infty}^{-k-\delta\sqrt{n}} \frac{e^{-\frac{k^2}{2}}}{\sqrt{2\pi}} dz + \int_{k-\delta\sqrt{n}}^{\infty} \frac{e^{-\frac{k^2}{2}}}{\sqrt{2\pi}} dz
\]

or

\[
P = \Phi(\delta\sqrt{n} - k) + \Phi(-\delta\sqrt{n} - k). \tag{7.3}
\]

When the process is under control the probability of a sample point falling outside the control limit is:

\[
\alpha = 2 \int_{k}^{\infty} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz. \tag{7.4}
\]

The average time for occurrence of the assignment cause is \( \frac{1}{\lambda} \). The probability of an assignable cause will be caught in the \( r^{th} \) sample is \( Q^{r-1}P \). The mean number of samples is \( \frac{1}{P} \). The average time for an out of control process is then

\[
\frac{h}{P} - \tau. \tag{7.5}
\]

Here \( \tau \) is given in expression (7.1). The average cycle of length for a process in control–out of control is:

\[
\frac{1}{\lambda} + \left( \frac{1}{P} - \frac{1}{2} + \frac{\lambda h}{12} \right)h + en + D. \tag{7.6}
\]

Duncan shows that the expected loss cost per hour is

\[
L = \frac{\lambda B a_4 + \alpha a_3}{1 + \lambda B} + \frac{(a_1 + a_2 n)}{h} \tag{7.7}
\]

where \( B = \left( \frac{1}{P} - \frac{1}{2} + \frac{\lambda h}{12} \right)h + en + D \). Assuming \( \lambda B << 1 \), expression (7.7) will be reduced to

\[
L' = \lambda B a_4 + \frac{\alpha a_3}{n} + \frac{(a_1 + a_2 n)}{h}. \tag{7.8}
\]
Taking the derivatives of $L'$ relative to $n, h, k$ yields:

\[
\lambda \frac{\partial B}{\partial n} (a_4 - \frac{aa_3'}{h} - \lambda a_3) + a_2(1 + \lambda B)^2 = 0 \tag{7.9}
\]

\[
\lambda h^2 \frac{\partial B}{\partial h} (a_4 - \frac{aa_3'}{h} - \lambda a_3) - \alpha a_3' (1 + \lambda B) - (a_1 + a_2 n)(1 + \lambda B) = 0 \tag{7.10}
\]

\[
\lambda \frac{\partial B}{\partial k} (a_4 - \frac{aa_3'}{h - \lambda a_3}) + \frac{a_3'}{h} \frac{\partial \alpha}{\partial k} (1 + \lambda B) = 0 \tag{7.11}
\]

where

\[
\frac{\partial B}{\partial n} = \frac{h \frac{\partial P}{\partial n}}{P^2} + e + \frac{\partial B}{\partial h} = \frac{1}{P} - \frac{1}{2} + \delta h \tag{7.12}
\]

\[
\frac{\partial B}{\partial k} = -\frac{h \frac{\partial P}{\partial k}}{P^2}. \tag{7.13}
\]

Employing expressions (7.10), (7.11), (7.13) and (7.14) yield:

\[
2(a_2 + \lambda g a_4 h) - \delta^2 a_3' \frac{\partial \alpha}{\partial k} = 0 \tag{7.15}
\]

and

\[
\lambda a_4 (\frac{1}{P} - 0.5) h^2 - \alpha a_3' - (a_1 + a_2 n) = 0 \tag{7.16}
\]

where

\[
\frac{\partial \alpha}{\partial k} = -\frac{2e^{\frac{k^2}{2}}}{\sqrt{2\pi}} \tag{7.17}
\]

with

\[
\frac{e^{-\frac{k^2}{2}}}{\sqrt{2\pi}} = \Phi(k) \quad \text{and} \quad \delta \sqrt{n} - k = z. \tag{7.18}
\]

Then expressions (7.17), (7.18), (7.15) and (7.16) yield respectively:

\[
\frac{z + k}{\Phi(k)} = \frac{\delta^2 a_3'}{a_2 + \lambda a_4 g h} \tag{7.19}
\]

and

\[
h = \left[\frac{aa_3' + (a_1 + a_2 n)}{\lambda a_4 (\frac{1}{P} - 0.5)}\right]^{1/2}. \tag{7.20}
\]
Setting

\[ A = \frac{\delta^2 a_3'}{a_2 + \lambda a_4 g h}. \]  

(7.21)

Then expression (7.19) can be written as \( A = \frac{r + k}{\phi(k)} \). A use of this term will be discussed later.

7.2.2. Model 2A

In this model the process is stopped during the search for the assignable cause of failure. The foregoing discussion requires a definition of each of the following terms.

\( k_s \): The average cost of searching for an assignable cause of failure.

\( k_r \): The average cost of repairing the process.

\( \tau_s \): The duration of time the process is shut down when an actual cause of failure is detected.

\( \tau_r \): The additional duration of time required to repair the process when an actual cause of failure occurs.

\( v_0 \): Profit per hour when the process is in control.

\( v_1 \): Profit per hour for a process functioning out of control.

The average length of a production cycle is:

\[ \frac{1}{\lambda} + B + \left\{ 1 + \frac{\alpha}{\lambda h} \right\} \tau_s + \tau_r \]  

(7.22)

where

\[ B = \left( \frac{1}{P} - \frac{1}{2} + \frac{\lambda h}{12} \right) h. \]  

(7.23)

Chiu and Wetherill [1974] derived an expression for the loss cost function given below

\[ L = \frac{\lambda B a_4 + a_1 a_3 + \lambda a_3 + \frac{(a_1 + a_2 \lambda)(1 + \lambda B)}{h}}{1 + \lambda B + \frac{\lambda u}{h} + \lambda v} \]  

(7.24)
where
\[ a_4 = v_0 - v_1, \quad a'_3 = k_s + v_0 \tau_s, \quad a_3 = k_r + k_s + v_0(\tau_s + \tau_r) \]
\[ u = \tau_s, \quad v = \tau_r + \tau_s. \]  
(7.25)

Comparing expression (7.24) to (7.7) shows that the two expressions are identical if \( u - v = 0 \).

An optimal solution for (7.24) can be obtained by equating the partial derivatives of \( L \) relative to \( n, k, h \) respectively to zero. Chiu and Wetherill [1974] suggested that the same procedure followed in model A can be used. The only difference is that the expression of \( A \) given in (7.21) is approximated by \( A^* \) given as:
\[ A^* = \frac{\delta^2 a'_3}{a_2}. \]  
(7.26)

7.3. Multiplicity of Assignable Causes

Duncan [1971] extended the single cause model, to the case of a multiplicity of causes of failure. An assignable cause \( j, j = 1, 2, \ldots, n \) shifts a process mean \( \bar{x}'' \) to \( \bar{x}'' + \delta_j \sigma'' \), where \( \sigma'' \) is the process standard deviation. The occurrence times of various assignable causes of failure are assumed to be independent and exponentially distributed with mean times \( \lambda_j^{-1}, j = 1, 2, \ldots, s \). In the work of Duncan the following two models are considered.

7.3.1. Model 1B

Once an assignable cause of failure occurs, the process is assumed to be free from the occurrence of other assignable causes. The time at which the process goes out of control when \( s \) assignable causes occur is a negative exponential distribution with a mean \( \frac{1}{\lambda} \) and
\[ \lambda = \sum_{i=1}^{s} \lambda_i. \]  
(7.27)
Duncan showed that the probability $p_j$ that a point falls outside the control limits after the occurrence of cause $j$ is:

$$P_j = \int_{-\infty}^{-k-\delta_j \sqrt{n}} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \, dx + \int_{k-\delta_j \sqrt{n}}^{\infty} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \, dx. \quad (7.28)$$

The average time of occurrence of an assignable cause between the $m^{th}$ and $m+1^{st}$ samples is:

$$\tau_j = \frac{\int_{mh}^{(m+1)h} e^{-\lambda_j t} \lambda_j (t - mh) \, dt}{\int_{mh}^{(m+1)h} e^{-\lambda_j t} \lambda_j \, dt}. \quad (7.29)$$

Expression (7.29) can be reduced to:

$$\tau_j = \frac{1 - (1 + \lambda_j h) e^{-\lambda_j h}}{\lambda_j (1 - e^{-\lambda_j h})}. \quad (7.30)$$

Ignoring terms of order $\lambda^3 h^4$ or higher, expression (7.30) will be reduced to:

$$\tau_j = \frac{h}{2} - \frac{\lambda_j h^2}{12}. \quad (7.30)$$

The expression for $B_j$ can be obtained by modifying $B$ defined in expression (7.7), thus:

$$B_j = \frac{h}{P_j} - \tau_j + en + D_j. \quad (7.31)$$

The expected additional loss per hour of operation is:

$$L_1 = \frac{(\sum \lambda_j B_j a_{4j})/\lambda}{(1 + \sum \lambda_j B_j)/\lambda} = \frac{\sum \lambda_j B_j a_{4j}}{1 + \sum \lambda_j B_j}. \quad (7.32)$$

The expected number of false alarms ($A$) can be obtained by employing expression (7.4) that is

$$A = \alpha \sum_{i=0}^{\infty} \int_{i}^{(i+1)h} i \lambda e^{-\lambda t} \, dt$$

$$= \frac{\alpha e^{-\lambda h}}{1 - e^{-\lambda h}}. \quad (7.33)$$
The expected cost per hour when looking for assignable causes of failure and non exists is:

$$L'_2 = \frac{\lambda \alpha e^{-\lambda h} a'_3}{1 - e^{-\lambda h}}.$$  \hfill (7.34)

The expected cost per hour when an assignable cause of failure exists is:

$$L'_3 = \frac{\sum \lambda_j a_{3j}}{1 + \sum \lambda_j B_j}.$$  \hfill (7.35)

The expression of $B_j$ is given in (7.31). The sampling and inspection cost per unit is:

$$L'_4 = \frac{a_1 + a_2 n}{h}.$$  \hfill (7.36)

The total expected loss cost per hour of operation is:

$$L' = \frac{\sum \lambda_j B_j a_{4j} + \lambda A a'_3 + \sum \lambda_j a_{3j} + a_1 + a_2 n}{1 + \sum \lambda_j B_j}.$$  \hfill (7.37)

The partial derivative of $L'$ relative to $k$ is:

$$\frac{\partial L'}{\partial k} = -h \sum \lambda_j a_{4j} \left[ \frac{\partial P_j}{\partial k} / P_j^2 + \lambda A a'_3 \left( \frac{\partial \alpha}{\partial k} \right) / \alpha \right] (1 + \sum \lambda_j B_j)$$

$$+ \frac{h [\sum \lambda_j a_{4j} + \lambda A a'_3 + \sum \lambda_j a_{3j}][\lambda_j (\frac{\partial P_j}{\partial k}) / P_j^2]}{(1 + \sum \lambda_j B_j)^2}$$

where

$$\frac{\partial P_j}{\partial k} = \frac{e^{-k - \delta_j \sqrt{n}}/2}{\sqrt{2\pi}}.$$  \hfill (7.39)

Expression (7.39) is equivalent to the expression:

$$\Phi(-k - \delta_j \sqrt{n}) - \Phi(k - \delta_j \sqrt{n}) < 0.$$  \hfill (7.40)

It can be verified easily that:

$$\frac{\partial \alpha}{\partial k} = \frac{2e^{-k^2}}{\sqrt{2\pi}} = -2\phi(k).$$  \hfill (7.41)
The partial derivative of $L'$ relative to $h$ is:

$$
\frac{\partial L'}{\partial h} = \frac{\sum \lambda_j a_{4j}/P_j - \sum \lambda_j a_{4j} \tau_j^* + \lambda T A^*}{1 + \sum \lambda_j B_j}
- \frac{[\sum \lambda_j B_j a_{4j} + \lambda A T + \sum \lambda_j a_{3j}]\left[\frac{\chi_j}{P_j} - \sum \lambda_j \tau_j^*\right]}{[1 + \sum \lambda_j B_j]^2}
- \frac{a_1 + a_2 n}{h^2}
$$

where

$$
\tau_j^* = \frac{\partial \tau_1}{\partial n} \quad \text{and} \quad A^* = \frac{\partial A}{\partial n}.
$$

7.3.1.1. Computational Procedures. The time parameters $a_{4j}, A_{3j}, D_j$ are dependent on $j$ (assume $j = 1$ for a single cause of failure). The other parameters $a_{4j}, a_1, a_2, g$ are the same as for a single cause of failure. Finding an exact optimum solution for $n, k, h$ is possible but tedious. Duncan suggested a matched single cause model for estimating $n, k, h$ which will be discussed below.

**Case 1.** Single cause models. Chiu and Wetherill [1974] in their semi-economic scheme suggested a solution for expression (7.19) as follows:

1. Assuming a positive shift for the mean only $\delta > 0$. The expression for $p$ given in (7.3) is reduced to $p = \Phi(\delta \sqrt{n} - k)$.

2. Assuming two values for $p$, 0.90 and 0.95, the assumption is equivalent to specifying the consumer's risk to 0.10 or 0.05. The $p$-values yield two values of $z$, 1.2827 and 1.7449 respectively.

3. Approximating $\frac{z+k}{\Phi(k)}$ by $\frac{\delta^2 a_{4j}}{a_2 + \lambda a_{4j}}$ and ignoring the value of $h$ given by expression (7.19), leads to the evaluation of $n, k, h$.

For values of $1.2827 \leq z \leq 1.7449$ and also values for $\delta$ solve equations (7.18) and (7.19) for $n$. The value of $n$ obtained is used to set upper and lower limits for the search for the optimal sample size. Computer programs listed in Appendices
G and H evaluates the optimum values of $n$, $h$, $k$ based on this logic. In general if the process is stopped during the search for the cause of failure, a slight modification is required. The expressions of $A$ given in (7.21), $B$ given in (7.7) and $L$ given in (7.7) are replaced by $A^*$, $B^*$, $L^*$ given by (7.26) (7.23) and (7.24) respectively.

**Case 2. Multiple cause models. a) Matched single cause model.**

a.1. The cause of failures are assumed to shift the process mean $\mu_0$ to $\mu_0 \pm \delta_s \sigma''$. Where $\sigma''$ is the process standard deviation and $\delta_s$ is given by

$$\delta_s = \frac{\sum_{j=1}^{s} \lambda_j \delta_j}{\sum_{j=1}^{s} \lambda_j}, \quad j = 1, 2, \cdots, s. \tag{7.44}$$

The expression for the increased loss per hour of operation due to the presence of several causes of failure is:

$$a_{4s} = \frac{\sum_{i=1}^{s} \lambda_j a_{4j}}{\sum_{j=1}^{s} \lambda_j}. \tag{7.45}$$

Employing expressions (7.44) and (7.45) together with $\lambda = \sum_{j=1}^{s} \lambda_j$ and treating the model as a single cost model as in Case 1 it is possible to get estimates for $n$, $h$, and $k$.

a.2 Consider each cause separately and estimate in each case $B_j$, $j = 1, \cdots, s$. Obtain an estimate for $h$ by employing the technique mentioned in Case a.1. The expressions for the loss-cost per hour of operation given in expressions (7.7) and (7.24) are calculated.
7.3.2. Model 2B

The following discussions and derivations were initiated by Duncan[1971].

For this model the process is assumed to exist in one of the following states:

1. The process starts in a state of statistical control with mean $\mu_0$ at time $t = 0$.
2. The process is disturbed by the occurrence of an assignable cause $j$, that shifts its mean by $\delta_j \sigma$.
3. The process is disturbed by the occurrence of a second assignable cause.
   The joint effect of the two causes shift the process mean by $\Delta \sigma$.

7.3.2.1. Assumptions

1. The assignable causes are assumed to occur independently and at random.
2. The time at which the process goes out of control from state 1 to state 2 has a negative exponential distribution with a mean time $\frac{1}{\lambda}$.
3. When the process is in state 2, no further disturbance occurs until the first sample is taken after the process enters state 2.
4. A second cause can occur at random after the first sample. Its time of occurrence is a negative exponential distribution with a mean time $\frac{1}{\lambda'}$. $\lambda'$ may or may not be equal to $\lambda$.
5. As long as the first assignable cause continues undetected, at most a second assignable cause is assumed to occur.

7.3.2.2. Probabilities.

1. $p_j$ - is the probability of detecting an assignable cause on the first sample after the process entered state 2.
2. $e^{-\lambda' h}$ - is the probability that a second assignable cause will not occur between 0 and $h$ or the process mean remains constant at $\mu \pm \delta_j \sigma$. 
(3) \( e^{-\lambda' h}(1 - p_j)p_j \) - is the probability that the process is in state 2 at the time a second sample is taken and the first cause is detected.

(4) \( e^{-\lambda' h}(1 - p_j)^2 \) - is the probability that the first assignable cause is undetected.

(5) \( p' \) - is the probability of detecting a shift \( \Delta \) in the process mean.

(7) \( (1 - e^{-\lambda' h})(1 - p_j)p' \) - is the probability that the process is in state 3 before taking a second sample. The occurrence of the two assignable causes is detected.

(7) \( (1 - e^{-\lambda' h})(1 - p_j)(1 - p') \) - is the probability that the process is in state 3 and the occurrence of the two assignable causes goes undetected.

### 7.3.2.3. Matrix of probabilities of detection

Let \( a_{uv} \) be an element for the matrix, where:

- \( u \) = sample number taken as the ordinate;
- \( v \) = sample number taken as the abscissa.

If \( v = 0 \)

\[
a_{u0} = (1 - p_j)^{u-1}(e^{-(u-1)\lambda' h})p_j. \tag{7.46}
\]

If \( v \neq 0 \), then

\[
a_{uv} = (1 - p_j)^v(e^{-(u-1)\lambda' h})(1 - e^{-\lambda' h})(1 - p')^{u-v-1}p' \tag{7.47}
\]

\[
0 < v \leq u - 1 \quad a_{uv} = 0 \quad \text{for} \quad v > u - 1.
\]

Assume \( \tau' = \tau \) and \( \lambda' = \lambda_j \) then

\[
\tau_j = (1 - p_j)e^{-\lambda' h}. \tag{7.48}
\]
The average length of runs in state 2 due to the initial assignable cause is

\[ R_j = p_j \sum_{s=0}^{\infty} [(s + 1)h - \tau_j + g_n + D_j]r_j^s \]

\[ + (1 - p_j)(1 - e^{-\lambda'h})p' \cdot \left( \sum_{s=0}^{\infty} (1 - p')^s \right) \cdot \left( \sum_{s=0}^{\infty} [(s + 1)h - \tau_j + \tau']r_j^s \right). \tag{7.49} \]

Expression (7.49) can be reduced to:

\[ R_j = \frac{h}{1 - r_j} - \tau_j + \frac{p_j(g_n + D)}{1 - r_j} + (1 - \frac{p_j}{1 - r_j})\tau'. \tag{7.50} \]

The average runs in state 3 due to initial assignable cause is:

\[ R'_j = \frac{h}{p'} - \tau' + gn + D' \left( 1 - \frac{p_j}{1 - r_j} \right). \tag{7.51} \]

The overall mean time of a cycle is:

\[ \frac{1}{\lambda} + \frac{\sum \lambda_j(R_j + R'_j)}{\lambda}. \tag{7.52} \]

The expected loss per hour of operation out of control is

\[ \left[ \frac{\sum \lambda_jR_j\mu_j}{\lambda} + \frac{M'\sum \lambda_jR'_j}{\lambda} \right] / \left[ 1 + \frac{\sum \lambda_j(R_j + R'_j)}{\lambda} \right] = \frac{\sum \lambda_jR_jM_j + M'\sum \lambda_jR'_j}{1 + \sum \lambda_j(R_j + R'_j)}. \tag{7.53} \]

The loss per hour due to false alarms is

\[ \frac{\lambda AT}{1 + \sum \lambda_j(R_j + R'_j)}. \tag{7.54} \]

where \( A \) is defined as the expected number of false alarms per cycle given by:

\[ A = \alpha \sum_{i=0}^{\infty} \int_{i}^{(i+1)h} i\lambda e^{-\lambda t} dt. \tag{7.55} \]
The loss per hour when causes of failure exist is:

\[
\sum \frac{\lambda_i W_i E_i}{1-r_j} + W' \sum \frac{\lambda_j [1 - \frac{E_j}{1-r_j}]}{1 + \sum \lambda_j (R_j + R'_j)}.
\]  

(7.56)

The expected loss is then

\[
L'' = \text{loss given in expression (7.53)} + \text{loss given in expression (7.54)} + \text{loss given in expression (7.56)} + \frac{a_1 + a_2 n}{h}.
\]  

(7.57)

7.4. Multivariate Statistical Process

Consider a quality control process that can be described by \( c \)-quality characteristics. The column vector \( X \) is a \((e \times 1)\) random vector distributed according to the \( c \)-variate normal that is:

\[
f(X) = \frac{1}{(2\pi)^{c/2} \cdot |\Sigma|^{1/2}} \exp\left[-\frac{1}{2}(X - \mu)' \Sigma^{-1} (X - \mu)\right]
\]  

(7.58)

where \( E(X) = \mu \) is a \((e \times 1)\) vector of the means. If the process is a state of statistical control then \( E(X) = \mu_0 \). \( \Sigma = \text{cov}(X) \) is a \((c \times c)\) covariance matrix of \( X \). A set of \( p \) samples each of size \( n \) are selected when the process is a state of statistical control. Letting \( X_{1q}, X_{2q}, \cdots, X_{nq} \) be the \( q^{th} \) sample where \( q = 1, 2, \cdots, c \), then the sample covariance matrix will be:

\[
s = \frac{1}{n - 1} \sum_{i=1}^{n} \sum_{q=1}^{c} (X_{iq} - \bar{X}_q)(X_{1q} - \bar{X}_q)'.
\]  

(7.59)

The sample mean is

\[
\bar{X}_q = \frac{1}{n} \sum_{i=1}^{n} X_{iq}.
\]  

(7.60)
A Hotelling's $T^2$ statistic with $c$ and $(n - 1)$ degrees of freedom is defined as

$$T^2 = n(\overline{X} - \mu_0)'s^{-1}(\overline{X} - \mu_0).$$  \hspace{1cm} (7.61)

The probability of a type 1 error is then

$$p_r[T^2 > T^2_\alpha, c, n - 1] = \alpha.$$  \hspace{1cm} (7.62)

$T^2$ can be defined in terms of the $F$-distribution as follows

$$F = \frac{n - c}{c(n - 1)}T^2 \quad \text{or} \quad T^2 = \frac{c(n - 1)}{n - c}F.$$  \hspace{1cm} (7.63)

Consider the following two cases:

**Case 1.** If

$$\mu = \mu_0, \quad \text{then} \quad T^2_{\alpha, c, n - 1} = \frac{c(n - 1)}{n - c}F_{\alpha, c, n - 1}.$$  \hspace{1cm} (7.64)

The tables of the $F$-distribution can be used to estimate the critical value of $T^2$.

**Case 2.** If

$$\mu \neq \mu_0, \quad \text{then} \quad T^2*_{\alpha, c, n - 1} = \frac{c(n - 1)}{n - c}F^*_{\alpha, c, n - 1}.$$  \hspace{1cm} (7.65)

In this case $F$ is governed by a non-central $F$-distribution with a non-centrality parameter

$$\tau_{nc} = n(\mu - \mu_0)' \sum^{-1}(\mu - \mu_0)'.$$  \hspace{1cm} (7.66)

Tables for non-central $F$ distributions are listed in Anderson [1958].

A statistical process is considered out of control if one or more quality characteristics cause a point to fall outside the control limits of the $T^2$-chart. Two techniques can be employed to check for the subsets of quality control characteristics shifting the process out of control:

a) principal component analysis technique
b) simultaneous confidence intervals technique.

7.5. Components of the Total Expected Cost Per Unit

For a Hotellings $T^2$ control chart, the components of the total cost per unit are relative to:

a) Sampling. The expected cost of sampling per unit is given by

$$E(C_1) = \frac{a_1 + a_2 n}{h}$$  \hspace{1cm} (7.67)

where $a_1$, $a_2$ are defined in Appendix A and $h$ is the number of units produced between successive samples.

b) A process operating out of control. The cost arises from searching for the cause or causes of failure and removing these causes. The expected value of this cost is given by

$$E(C_2) = \frac{a_3 P_r(z = 1)}{h}$$  \hspace{1cm} (7.68)

where $a_3$ is defined in Appendix A and $z$ is a variable defined as follows:

$$z = 0 \quad \text{if} \quad T^2 < T^2_{\alpha,c,n-1}$$

$$z = 1 \quad \text{if} \quad T^2 > T^2_{\alpha,c,n-1}.$$  \hspace{1cm} (7.69)

Employing expression (7.69) expression (7.68) is reduced to:

$$E(C_2) = \frac{a_3 P_r(z = 1)}{h}.$$  \hspace{1cm} (7.70)

Expression (7.70) can be evaluated by considering the column vectors of probabilities: $q$, $\alpha$ defined below. An element of a column vector $q$, $q_i$ is defined as

$q_i$ - probability that the process is operating out of control given that a process is in state $\mu_i$.

An element of the vector $\alpha$, $\alpha_i$ is defined as
$\alpha_i$ - the steady state probability that the process is in state $\mu_i$ at the time when the test is performed or a sample is selected.

Employing the above definitions expression (7.70) can be written as:

$$E(C_2) = \frac{q_3}{h} q' \alpha$$  \hspace{1cm} (7.71)

where $q'$ is the transpose of the column vector $q$.

c) Penalty cost of producing defective units. The expression for this cost is

$$E(C_3) = a_4 \Omega$$  \hspace{1cm} (7.72)

where $a_4$ is defined in Appendix A and $\Omega$ is either 0 if the unit is not defective and 1 if the unit is defective. An expression for $E(C_3)$ will be given in terms of the column vectors $f$, $\gamma$ defined below. An element of the column vector $f$, $f_i$ is defined as:

$f_i$ - probability of producing defective units given the process is in state $\mu_i$.

An element of the column vector $\gamma$, $\gamma_i$ is defined as:

$\gamma_i$ - probability that the process is in state $\mu_i$ at any point in time.

Employing the above definitions, expression (7.72) can be written as:

$$E(C_3) = a_4 f' \gamma.$$  \hspace{1cm} (7.73)

where $f'$ is the transpose of the column matrix $f$.

7.6. Transition Matrix of Probabilities

The following are the elements of the transition matrix.

$g_{ij}$ - The probability of the process shifting from state $\mu_i$ to state $\mu_j$ during the production of $k$ units.

$g_{00}p_0$ - is the probability that the process is in control at $(d+1)^{st}$ sample given the process was in control at the $d^{st}$ sample.
The above definition is the same as saying $p_0$ is the probability of remaining in state $\mu_0$, for $t$ hours during the production of $k$ units or

$$p_0 = 1 - \int_0^t \lambda e^{-\lambda z} dz = e^{-\lambda t}.$$  \hfill (7.74)

If $R$ is the number of units produced per hour, then

$$p_0 = e^{-\frac{k}{R}}.$$  \hfill (7.75)

g_{01} = p_1$ is the probability assigned to the out-of-control state. Hence

$$p_1 = 1 - e^{\frac{k}{R}}.$$  \hfill (7.76)

The probabilities $q_i$ and $\alpha_i$ are defined and given above. Expressions for the other elements of the matrix are derived below. Consider a statistical process that can exist in $s$ possible states with process means $\mu_1, \mu_2, \cdots, \mu_s$. The probability of $i$ cases in $s$ trials is governed by a binomial of distribution of the form:

$$p_i^* = \frac{s!}{i!(s-i)!} \pi^i (1-\pi)^{s-i}, \quad 0 < \pi < 1.$$  \hfill (7.77)

Consider the following summation

$$\sum_{i=1}^s p_i^* = 1 - p_0^* = 1 - (1-\pi)^s.$$  \hfill (7.78)

Substituting expression (7.75) into expression (7.78) yields:

$$\sum_{i=1}^s p_i^* = 1 - e^{-\lambda'k}$$  \hfill (7.79)

where

$$\lambda' = \frac{\lambda}{R}$$  \hfill (7.80)
letting
\[ p_i = \frac{(1 - e^{-\lambda'k})s!\pi^i(1 - \pi)^{s-i}}{[1 - (1 - \pi)^s]i!(s - i)!}. \] (7.81)

Then
\[ \sum_{i=0}^{s} p_i = e^{-\lambda'k} + \frac{1 - e^{-\lambda'k}}{1 - (1 - \pi)^s} \sum_{i=1}^{s} \frac{s!\pi^i(1 - \pi)^{s-i}}{i!(s - i)!}. \] (7.82)

Expression (7.82) can be reduced to
\[ \sum_{i=0}^{s} p_i = e^{-\lambda'k} + \frac{1 - e^{-\lambda'k}}{1 - (1 - \pi)^s} [1 - (1 - \pi)^s] = e^{-\lambda'k} + 1 - e^{-\lambda'k} = 1. \]

7.6.1. Assumptions

1. The time for a statistical process under control is an exponential random variable with mean \( \frac{1}{\lambda} \) hours.

2. When the process goes out of control it stays out of control until detected.

3. When a process goes out of control it will not improve but it may get worse.

The elements of the transition matrix \( G \) are defined as follows:
\[ b_{ij} = \begin{cases} 
\text{the process is in state } \mu_i \text{ at time } t \\
\text{the process is in state } \mu_j \text{ at time } t + \frac{k}{\lambda_i}.
\end{cases} \] (7.83)

Consider the following two cases:

Case 1. \( j < i \) then
\[ b_{ij} = q_i^j p_j \] (7.84)

and
\[ g_{ij} = p_j \quad 0 < b_{ij} < 1 \] (7.85)

where \( p_i \) is defined before and \( p_j \) is the probability of shifting to state \( \mu_j \) during the production of \( k \) units.
Case 2. $j > i$ then

\[ b_{ij} = q_i p_j + \frac{(1 - q_i) p_j}{1 - p_0} \]  

(7.86)

and

\[ b_{ii} = q_i p_i + (1 - q_i) p_{ii} \]  

(7.87)

where

\[ p_{ii} = \frac{\sum_{j=1}^{i} p_j}{1 - p_0} \quad i \neq 0. \]  

(7.88)

7.6.2. Two States Statistical Process

Consider a statistical process that can exist in two states only. An in control state $\mu_0$ and the out of control state $\mu_1$. The elements of the matrix $G$ are $g_{00}$, $g_{01}$, $g_{10}$, $g_{11}$. The matrix $G$ is then:

\[ G = \begin{bmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{bmatrix}. \]  

(7.89)

When $i = 0$, expression (7.85) yields

\[ b_{00} = p_0, \quad b_{01} = p_1. \]  

(7.90)

When $i = 1, g = 0$, expression (7.84) yields

\[ b_{10} = q_1 p_0. \]  

(7.91)

$b_{10}$ is the probability that the process is in control at the $(d + 1)^{st}$ sample given that it was out of control at the $d^{th}$ sample. The process is assumed to remain in control for the production of $k$ units.

When $i = j = 1$, expression (7.87) yields:

\[ b_{11} = q_1 p_1 + (1 - q_1) \frac{p_1}{1 - p_0} \]
or

\[ b_{11} = q_1 p_1 + (1 - q_1). \]  \hspace{1cm} (7.92)

\( b_{11} \) is the probability that the process is out of control at the \((d + 1)^{st}\) sample given it was out of control at the \(d^{th}\) sample. The components of this probability are

i) \( q_1 p_1 \) - is the probability of going out of control during the production of \( k \) units.

ii) \( 1 - q_1 \) - is the probability of not detecting the out of control state at the \(d^{th}\) sample.

The transition matrix \( G \) is then

\[ G = \begin{bmatrix} p_0 & p_1 \\ q_1 p_0 & q_1 p_1 + (1 - q_1) \end{bmatrix}. \]  \hspace{1cm} (7.93)

The expressions for the probabilities \( q_0 \) and \( q_1 \) are

\[ q_0 = p_r[T^2 > T^2_{\alpha,c,n-1}] = \int_{T^2_{\alpha,c,n-1}}^{\infty} F(T^2) dT^2 \]  \hspace{1cm} (7.94)

and

\[ q_1 = p_r[T^{2*} > T^2_{\alpha,c,n-1}] = \int_{T^2_{\alpha,c,n-1}}^{\infty} f(T^{2*}) dt^{2*} \]  \hspace{1cm} (7.95)

where \( T^2 \) and \( T^{2*} \) are defined in expressions (7.64) and (7.65) respectively. The matrix \( G \) is an irreducible, aperiodic and positive recurrent Markov chain. It is possible to verify that a vector \( \alpha \) exists such that

\[ \alpha' G = \alpha. \]  \hspace{1cm} (7.96)

Solving expression (7.96) yields two values of \( \alpha \), \( \alpha_0 \) and \( \alpha_1 \) relative to states \( \mu_0 \) and \( \mu_1 \) given by:

\[ \alpha_0 = \frac{q_1 p_0}{p_1 + q_1 p_0} + [1 - (p_1 + q_1 p_0)]n[\alpha_0 - \frac{q_1 p_0}{p_1 + q_1 p_0}] \]  \hspace{1cm} (7.97)
and

\[ \alpha_1 = \frac{p_1}{p_1 + q_1 p_0} + [1 - p_1 - q_1(1 - p_0)]^n [\alpha_1 - \frac{p_1}{p_1 + q_1 p_0}] \]  \hspace{1cm} (7.98)

where \( p_0, p_1 \) and \( q_1 \) are defined by expressions (7.75), (7.76) and (7.95) respectively. The limits as \( n \) approaches infinity of \( \alpha_0 \) and \( \alpha_1 \) are:

\[ \alpha_0 = \frac{q_1 p_0}{p_1 + q_1 p_0} \]  \hspace{1cm} (7.99)

\[ \alpha_1 = \frac{p_1}{p_1 + q_1 p_0}. \]  \hspace{1cm} (7.100)

### 7.7. Elements of Vector \( \gamma_i \)

The probability \( \gamma_i \) is defined by expression (7.22). The components of this vector are:

1. The probability of the event that the process is in state \( \mu_i \) and remains there during the production of \( K \) units. The expression of this probability is

\[ \alpha_i p_{ij}. \]  \hspace{1cm} (7.101)

Substituting \( p_{ii} \) by its value given in expression (7.88) in terms of \( p_j \) and \( p_0 \) yields

\[ \alpha_i p_{ii} = \alpha_i \sum_{j=1}^{i} \frac{p_j}{1 - p_0} \]  \hspace{1cm} (7.102)

where \( p_0 \) is given in expression (7.74).

2. The probability of the event that the process is in state \( \mu_0 \) and shifts to state \( \mu_i \) during the production of \( K \) units is:

\[ \alpha_0 (1 - \tau) p_i \]  \hspace{1cm} (7.103)

where \( \alpha_0 \) and \( \tau \) are given in expressions (7.99) and (7.1) respectively.
(3) The probability of a process is in a lower state \( v, v < i \) and shifts to a state \( \mu_i \) during the production of \( K \) units is:

\[
\sum_{v=1}^{i-1} \alpha_v \frac{p_i}{1 - p_0} (1 - \tau).
\]

(7.104)

The above probability is zero if \( i = 1 \).

(4) The probability of the event the process is in state \( \mu_i \) at the time the sample is taken and shifted to \( y, y > i \) during the production of \( K \) units is

\[
\frac{\alpha_i \tau}{1 - p_0} \sum_{y=i+1}^{s} p_y
\]

(7.105)

this probability is zero if \( i = s \).

In all the above expressions of the probability, the following assumption is being made: the fraction of time a process stays in state \( \mu_0 \) and shifts to the state \( \mu_1 \) is the same as the fraction of time a process stays in state \( v_1 \) before shifting to a state \( v_2 \) where \( v_1 < v_2 \).

The probability expressions defined in cases 1-4 lead to the following expression of \( \gamma_i \)

\[
\gamma_i = \frac{\alpha_i}{1 - p_0} \sum_{j=1}^{i} p_j + \alpha_0 (1 - \tau) p_i + \sum_{v=1}^{i-1} \alpha_v \frac{p_i}{1 - p_0} (1 - \tau)
\]

\[
+ \frac{\alpha_i \tau}{1 - p_0} \sum_{y=i+1}^{s} p_y.
\]

(7.106)

For a statistical process that can exist in two states only:

\[
\gamma_0 = \alpha_0 p_0 + \alpha_0 \tau (1 - p_0)
\]

(7.107)

\[
\gamma_1 = \alpha_1 + \alpha_0 (1 - \tau) p_1.
\]

(7.108)
7.8. Probability Vector \( f \)

An element of this vector \( f \) is defined as the probability of producing defective units given the process is in state \( \mu_i \).

Define the following terms:

\( X \): The vector of \( C \)-quality characteristics, their joint distributions is governed by the \( c \)-variate normal.

\( L \): The vector of the lower specifications of \( X \).

\( U \): The vector of the upper specifications of \( X \).

\( \mu_i \): the vector of the means relative to a state \( i \).

\( S \): The unbiased estimate of the unknown covariance matrix \( \Sigma \).

Then

\[
    f_i = \frac{1}{(2\pi)^{\frac{c}{2}} |S|^{\frac{1}{2}}} \int_{l_1}^{u_1} \int_{l_2}^{u_2} \cdots \int_{l_p}^{u_p} \exp\left\{ -\frac{1}{2} (X - \mu_i)'S^{-1}(X - \mu_i) \right\} dX_1 \cdots dX_p
\]

for a two state statistical process (in or out of control)

\[
    \mu_i = \begin{cases} 
        \mu_0 & \text{or } f_i = f_0 \quad \text{process starts in a state of statistical control} \\
        \mu_1 & \text{or } f_i = f_1 \quad \text{process is out of control.}
    \end{cases}
\]

Expected total cost per unit. This is obtained by adding expressions (7.67), (7.71) and (7.73) respectively. The expected cost \( E(C) \) is:

\[
    E(C) = \frac{a_1 + a_2n}{h} + \frac{a_3}{h} (q_0\alpha_0 + q_1\alpha_1) + q_4(f_0\gamma_0 + f_1\gamma_1).
\]

All the terms given above are defined in this chapter and in Appendix A.

7.9. Expected Length of Time of a Production Process

The probability \( (\alpha) \) of a point falling outside the control limits of a production process under control with mean \( \mu_0 \) is given by expression (7.4). If the mean of the process \( \mu \) is assumed to be a random variable then:

\[
    \alpha(\mu) = \Phi(\mu\sqrt{n} - k) + \Phi(-\mu\sqrt{n} - k).
\]
The expression for the probability that an assignable cause will be detected is given by expression (7.3). An expression for the probability $P_C$ that the process is under control during any period of time can be written in the form:

$$P_C = P_c g_{00} (1 - \alpha) + g_{00} (\alpha)$$  \hspace{1cm} (7.113)$$

where $g_{00}$ is defined by expression (7.74). Simplifying expression (7.113) yields:

$$P_C = \frac{g_{00} \alpha(\mu)}{1 - P_{00} (1 - \alpha(\mu))}. \hspace{1cm} (7.114)$$

The expression for the probability that the process is out of control at any period $P_{0c}$ can be derived either directly or by employing expression (7.114). Hence

$$P_{0c} = \frac{1 - g_{00}}{1 - g_{00} (1 - \alpha(\mu))}. \hspace{1cm} (7.115)$$

### 7.9.1. Expected Time Process is in Control

Let $n_{ij}$ be the number of sampling intervals until a production process switches from state $i$ to state $j$. An expression of $E(n_{ij})$ can be derived in terms of the following two probabilities. The probability of a process switching from state $i$ to state $j$, $g_{ij}$ and the probability of the process staying in state $i$, $P_{ii}$. For an upper limit $m$ for $n_{ij}$, the expected number of sampling intervals until a process switches from state $i$ to state $j$ is

$$E(n_{ij}) = \sum_{n_{ij}=1}^{m} n_{ij} g_{ij} P_{ii}^{n_{ij}-1} \hspace{1cm} (7.116)$$

or

$$E(n_{ij}) = \frac{P_{ij} [1 - (m + 1) P_{ii}^{m} + m P_{ii}^{m+1}]}{(1 - P_{ii})^2}. \hspace{1cm} (7.117)$$

Since $P_{ii} < 1$, in the limit case as $m \to \infty$ $E(n_{ij})$ yields:

$$E(n_{ij}) = \frac{P_{ij}}{(1 - P_{ii})^2}. \hspace{1cm} (7.118)$$
Consider a production process that can exist in $s$-states. The expected time $E(T_{00})$ for a process starting in control until a shift occurs can be obtained by employing expressions (7.115) and (7.118). Hence

$$E(T_{00}) = \sum_{j=1}^{s} \frac{P_{0j}P_{0c}(\mu)h}{(1 - P_{00})^2} \quad (7.119)$$

where $h$ is the interval between samples. Expression (7.119) can be written as:

$$E(T_{00}) = \sum_{j=1}^{s} \left( \frac{q_{0j}q_{00}\alpha(\mu)}{(1 - q_{00})^2[1 - q_{00}(1 - \alpha(\mu))]} \right). \quad (7.120)$$

For a two states statistical process, $s = 2$ the states are 0 for a process in control and 1 out of control. In this case expression (7.120) is reduced to:

$$E(T_{00}) = \frac{h}{1 - q_{00}(1 - \alpha(\mu))} \quad (7.121)$$

In switching from state $i$ to state $j$, the length of time the process in is in state $j$ before the next sample is taken is:

$$E(T_j) = (h - \tau)\left( \frac{1 - q_{00}}{1 - q_{00}(1 - \alpha(\mu))} \right) \quad (7.122)$$

where $\tau$ is defined in expression (7.1). Employing expressions (7.119) and (7.122) an expression for the expected time for a process under control $E(T_{PC})$ is:

$$E(T_{PC}) = E(T_{00}) - E(T_j) \quad (7.123)$$

Expression (7.122) can be written in the form:

$$E(T_{PC}) = \frac{h - (h - \tau)(1 - q_{00})}{1 - q_{00}(1 - \alpha(\mu))} \quad (7.124)$$
7.9.2. Expected Time for a Process Out of Control

The average time for an out of control process is given in expression (7.5). An expression for the expected time for a process out of control $E(T_{0c})$ can be obtained by employing expressions (7.118) and (7.115). Hence:

$$E(T_{0c}) = \frac{q_{01} \left( \frac{a}{p} - \tau \right)}{(1 - q_{00})^2} \cdot \frac{1 - q_{00}}{[1 - q_{00}(1 - \alpha(\mu))]}.$$  \hspace{1cm} (7.125)

Substituting $P_{01}$ by its value in terms of $q_{00}$, expression (7.125) yields

$$E(T_{0c}) = \frac{\left( \frac{a}{p} - \tau \right)}{1 - P_{00}(1 - \alpha(\mu)).}$$  \hspace{1cm} (7.126)

7.9.3. Expected Time for a Production Cycle

Define the following terms:

$T_1$ - The duration of time of shutting a production process down during a search for an assignable cause(s) of failure and non exists.

$T_r$ - The time of repairing a process when causes of failure does exist.

The expected time for a production cycle for $\sigma$, given process mean $\mu$, $E(\text{cycle time } | \mu)$ is:

$$E(\text{cycle time } | \mu) = E(T_{PC} + E(T_{0C}) + P_CT_1 + P_{0C}(T_1 + T_r)).$$  \hspace{1cm} (7.127)

The mathematical formulas for $E(T_{PC})$, $E(T_{0C})$, $P_C$, $P_{0C}$ are given in expressions (7.124), (7.126), (7.114) and (7.115) respectively.

7.9.4. Expected Loss-cost Per Hour

Assuming that the cost of looking for an assignable cause(s) and none exists is $K_1$ and the cost of repairing the process when the assignable causes does exist is $K_r$. The expected cost of keeping the process under control is $E(\text{cost of } P_{uc})$ is:

$$E(\text{cost of } P_{uc}) = P_eK_1 + P_{0C}(K_1 + K_r) + [E(T_{PC}) + E(T_{0C})\left[\frac{a_1 + a_2n}{h}\right]]$$  \hspace{1cm} (7.128)
where \( P_C, P_{0C}, E(T_{PC}), E(T_{0C}), q_1, q_2, h \) are defined in expressions (7.114), (7.115), (7.124), (7.126), and Appendix A respectively.

An expression for the loss-cost per hour (LCP) can be obtained by taking the ratio of expressions (7.128) to (7.127) hence

\[
LCP = \frac{E(\text{cost of } P_{uC})}{E(\text{cycle time } \mu)}. \tag{7.129}
\]

Since \( \mu \) is assumed to be a random variable then an integration over \( \mu \) is necessary. The expected loss per hour of operation is then:

\[
E(LCP) = \int_{-\infty}^{+\infty} \frac{E(\text{cost of } P_{uC})}{E(\text{cycle time } \mu)} \cdot f(\mu) \, d\mu. \tag{7.130}
\]

Keeping a production process under control reduces the expected length of time the process operates out of control. In this case the economic losses are diminished due to the reduction in the number of defective items produced. In this chapter it is assumed that a production process is out of control if the mean of the process is shifted in either direction by a certain quantity \( \delta \). A significant variation in the product outgoing quality requires a measure other than the mean like the variance, the coefficient of variation and others to monitor the variation in the product outgoing quality. Jones, L. and Case, K. (1981) suggested an economic design based on a joint \( \bar{X} \) and \( R \) control chart to study an in-out of control production process. For this process define the following terms.

- \( \mu_{ic} \): The process mean is under control
- \( \mu_{oc} \): The process mean is out of control
- \( \sigma_{ic}^2 \): The process variance is under control
- \( \sigma_{oc}^2 \): The process variance is under control
- \( O_{cn} \): An out of control condition has not been detected
An out of control condition has been detected.

The techniques developed in this chapter can be extended to deal with a production process that can exist in one of the following states.

1. \((\mu_{ic}, \sigma_{ic}^2)\)
2. \((\mu_{oc}, \sigma_{oc}^2, \sigma_{cn})\)
3. \((\mu_{ic}, \sigma_{oc}^2, \sigma_{cn})\)
4. \((\mu_{oc}, \sigma_{oc}^2, \sigma_{cn})\)
5. \((\mu_{oc}, \sigma_{ic}^2, \sigma_{cd})\)
6. \((\mu_{ic}, \sigma_{oc}^2, \sigma_{cd})\)

The following is an example for estimating the limits for the control chart, the interval of time of taking and checking the samples, the type I error rate, the power of the chart, the cost, the testing factor, the average time the process operates under an assignable cause or causes of failure and the average time of occurrence of an assignable cause or causes of failure. The input parameters are defined in appendices H and I.

Computer programs written in FORTRAN V to compute \(n, h, k\) are listed in Appendices G and H.

The following example estimates \(n, h, k\) to be employed to estimate the components of the cost given by expressions (7.32), (7.34), (7.35), (7.36), and (7.37) when several causes of failure occur. Expressions (7.44) and (7.45) yield respectively \(\delta_s = 0.87597, a_{4s} = 150,\) and \(\lambda = 0.077.\) Running the program multrun employing the matched single cause of failure technique with parameters \(\lambda, \delta_s, a_{4s}, G, a_1, a_2,\) and \(a_3'\) yield values for \(n, h, k,\) and \(\alpha\) given below.
### Running the Icecess, Individual Lambda

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*** PRIMARY INPUT VARIABLES ***

NUMBER OF CAUSES = 2

SAMPLING COST PARAMETERS
A1 = 1.000000
A2 = 0.1000000

COST OF FINDING ASSIGNABLE CAUSE NONE EXISTS
A3P(1) = 25.0000
A3P(2) = 27.0000

VALUES OF LAMBDA - XLAM
XLAM(1) = 0.040000
XLAM(2) = 0.037000

VALUES OF A4
A4(1) = 15.0000
A4(2) = 150.0000

VALUES OF A3
A3(1) = 50.0000
A3(2) = 60.0000

VALUES OF DEL
DEL(1) = 0.9000
DEL(2) = 0.8500

VALUES OF U
U(1) = 0.1000
U(2) = 0.1000

VALUES OF V
V(1) = 0.3000
V(2) = 0.3700

VALUES OF KR
KR(1) = 15.0000
KR(2) = 17.0000

VALUES OF KS
KS(1) = 10.0000
KS(2) = 8.0000

VALUES OF TAUR
TAUR(1) = 0.2000
TAUR(2) = 0.1000

VALUES OF TAUS
256

TAlI~( I )
TAUS(2)

(,AUSE
N

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2
3
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12
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19
20
21
22

"

OPllMUM K
1.45
1.72
1.83
1.91
1. 96
2.01
2.06
2.10
2. 13
2.17
2.21
2.24
2.28
2.31
2.35
2.38
2.41
2.45
2.48
2.51
2.54
2.58

23

2.61

CAUSE
N

1
2
3
4
5
6
7
8
9
10
11
12
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15
16
17

18
19
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21
22

23
24

n.lUUu
(1.12UO

"

OPTIMUM H
0.52
0.47
0.49
0.53
0.57
0.61
0.64
0.67
0.71
0.74
0.77
0.1l0
0.83
0.85
0.87
0.90
0.92
0.94
0.96
0.98
1.00
1.02
1.04

ALPHA
0.1471
0.0854
0.0672
0.0561
0.0500
0.0444
0.0394
0.0357
0.0332
0.0300
0.0271
0.0251
0.0226
0.0209
0.01B8
0.0173
0.0160
0.0143
0.0131
0.0121
0.0111
0.0099
0.0091

POWER
0.291'2
0.3274
0.3931
0.4562
0.5209
0.5771
0.6260
0.6721
0.7157
0.7505
0.7808
0.8099
0.8327
0.8549
0.8720
0.8888
0.9033
0.9144
0.9255
0.9351
0.9434
0.9496
0.9560

COST
18.82
15.44
13.47
12.17
11.26
10.59
10.08
9.69
9.39
9.15
8.97
8.82
8.71
8.63
8.57
8.53
8.51
8.50
8.50
8.51
8.53
8.56
'8.59

TEST FAC 2
0.65E.Ol
0.14E·02
0.21E.02
0.28E+02
0.:J4E+02
0.42E+02
0.50E+02
0.58E+02
0.65E+02
0.75E+02
0.8fiE+02
0.96E+02
0.11E·03
0.12E-03
0.14E+03
0.15E+03
0.17E+03
0.19E+03
0.2IE+Oj
0.24E+03
0.26E-03
0.30E+03
0.33E+03

B
0.260
0.233
0.246
0.262
0.283
0.301
0.318
0.336
0.354
0.369
0.383
0.398
0.410
0.424
0.435
0.447
0.458
0.468
0.478
0.488
0.498
0.506
0.515

lAU
1.529
1. Hl3
1.008
0.890
0.808
0.748
0.702
0.667
0.641
0.620
0.6U3
0.5!!1
0.581
0.574
0.568
0.565
0.5b3
0.562
0.562
0.563
0.565
0.567
0.570

OPTIMUM H
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0.47
0.50
0.53
0.57
0.61
0.65
0.69
0.72
0.75
0.79
0.82
0.85
0.87
0.90
0.93
0.95
0.97
0.99
1.01
1.03
1.05
1.07
1.09

ALPHA
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0.0672
0.0574
0.0512
0.0455
0.0424
0.0385
0.0349
0.0324
0.0300
0.0278
0.0257
0.0238
0.0220
0.0203
0.0188
0.0173
0.0160
0.0147
0.0135
0.0124
0.0114
0.0105

POWER
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0.3023
0.3603
0.4207
0.4803
0.5327
0.5866
0.6309
0.6700
0.7081
0.7419
0.7717
0.7981
0.8213
0.8418
0.8599
0.8759
0.8899
0.9024
0.9133
0.9229
0.9315
0.9390
0.9457

COST
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16.05
14.07
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11.10
10.57
10.15
9.81
9.55
9.34
9.17
9.03
8.93
8.85
8.78
B.74
8.71
8.69
8.69
8.69
8.70
8.72
8.74

TEST FAC 2
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0.13E+02
0.20E+02
0.26E+02
0.32E+02
0.39E+02
0.44E+02
0.51E-02
0.59E+02
0.67E+02
0.74E+02
0.83E+02
0.92E+02
O.10E+03
0.IIE+03
0.13E+03
0.14E+03
0.15E+03
0.17E+03
0.19E+03
0.21E+03
0.23E+03
0.25E+03
0.28E-03

8
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0.236
0.248
0.265
0.285
0.303
0.325
0.342
0.358
0.375
0.391
0.407
0.421
0.435
0.448
0.460
0.472
0.483
0.494
0.504
0.514
0.523
0.533
U.542

TAU
1.697
1.333
1. 131
1.000
0.907
0.839
0.787
0.746
0.714
0.689
0.669
0.653
0.640
0.630
0.622
0.616
0.612
0.6U9
0.607
0.6U7
0.607
0.608
0.610
0.612

2

OPTIMUM K
1.44
1.72
1.83
1. 90
1.95
2.00
2.03
2.07
2.11
2.14
2.17
2.20
2.23
2.26
2.29
2.32
2.35
2.38
2.41
2.44
2.47
2.50
2.53
2.56


## STOP THE PROCESS FOR INDIVIDUAL LAMBDA

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| CAUSE # 2 | TR1= 7.155038 | TR2= 4.558399 | TR3= 4.7700256E-02 |
| CAUSE # 2 | TR1= 6.050993 | TR2= 3.387717 | TR3= 4.0339954E-02 |
| CAUSE # 2 | TR1= 5.340255 | TR2= 2.647385 | TR3= 3.560166E-02 |
| CAUSE # 2 | TR1= 4.846275 | TR2= 2.175201 | TR3= 3.2305006E-02 |
| CAUSE # 2 | TR1= 4.485046 | TR2= 1.810134 | TR3= 2.9900309E-02 |
| CAUSE # 2 | TR1= 4.213050 | TR2= 1.518629 | TR3= 2.808999E-02 |
| CAUSE # 2 | TR1= 4.04721 | TR2= 1.301879 | TR3= 2.698142E-02 |
| CAUSE # 2 | TR1= 3.843425 | TR2= 1.141818 | TR3= 2.562235E-02 |
| CAUSE # 2 | TR1= 3.717266 | TR2= 0.9892686 | TR3= 2.471771E-02 |
| CAUSE # 2 | TR1= 3.618945 | TR2= 0.8596356 | TR3= 2.412629E-02 |
| CAUSE # 2 | TR1= 3.543036 | TR2= 0.7637829 | TR3= 2.362042E-02 |
| CAUSE # 2 | TR1= 3.484769 | TR2= 0.6667299 | TR3= 2.3231795E-02 |
| CAUSE # 2 | TR1= 3.441425 | TR2= 0.5953825 | TR3= 2.294235E-02 |
| CAUSE # 2 | TR1= 3.410038 | TR2= 0.5209882 | TR3= 2.273586E-02 |
| CAUSE # 2 | TR1= 3.388892 | TR2= 0.4667207 | TR3= 2.2593213E-02 |
| CAUSE # 2 | TR1= 3.376455 | TR2= 0.4186954 | TR3= 2.250901E-02 |
| CAUSE # 2 | TR1= 3.371035 | TR2= 0.3669406 | TR3= 2.2473566E-02 |
| CAUSE # 2 | TR1= 3.371874 | TR2= 0.3296414 | TR3= 2.2471606E-02 |
| CAUSE # 2 | TR1= 3.377955 | TR2= 0.2963341 | TR3= 2.2519700E-02 |
| CAUSE # 2 | TR1= 3.38514 | TR2= 0.2665233 | TR3= 2.259091E-02 |
| CAUSE # 2 | TR1= 3.40755 | TR2= 0.2331608 | TR3= 2.268536E-02 |
| CAUSE # 2 | TR1= 3.420416 | TR2= 0.2059841 | TR3= 2.280272E-02 |

| SS= 2.000000 |

| CAUSE # 2 | TR1= 9.418499 | TR2= 7.354369 | TR3= 6.2789984E-02 |
| CAUSE # 2 | TR1= 7.396949 | TR2= 6.624443 | TR3= 4.9312994E-02 |
| CAUSE # 2 | TR1= 6.548718 | TR2= 5.834737 | TR3= 4.1895576E-02 |
| CAUSE # 2 | TR1= 5.503523 | TR2= 4.635217 | TR3= 3.6991455E-02 |
| CAUSE # 2 | TR1= 4.860466 | TR2= 3.893249 | TR3= 3.4180376E-02 |
| CAUSE # 2 | TR1= 4.567539 | TR2= 1.972666 | TR3= 2.9116930E-02 |
| CAUSE # 2 | TR1= 4.142396 | TR2= 1.860899 | TR3= 2.7615976E-02 |
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| CAUSE # 2 | TR1= 3.379142 | TR2= 0.4651869 | TR3= 2.2527611E-02 |
| CAUSE # 2 | TR1= 3.369688 | TR2= 0.4187614 | TR3= 2.2466457E-02 |
| CAUSE # 2 | TR1= 3.366503 | TR2= 0.3771769 | TR3= 2.2443350E-02 |
| CAUSE # 2 | TR1= 3.367666 | TR2= 0.3398543 | TR3= 2.2453105E-02 |
| CAUSE # 2 | TR1= 3.373703 | TR2= 0.3063031 | TR3= 2.2491354E-02 |
| CAUSE # 2 | TR1= 3.363153 | TR2= 0.2761020 | TR3= 2.2554357E-02 |
| CAUSE # 2 | TR1= 3.395834 | TR2= 0.2468879 | TR3= 2.263893E-02 |
| CAUSE # 2 | TR1= 3.411328 | TR2= 0.2243448 | TR3= 2.2742184E-02 |

| SS= 2.220000 |
Example

The following example estimates \( n, h, k \) to be employed to estimate the components of the cost given by expressions (7.32), (7.34), (7.35), (7.36), and (7.37) when several causes of failure occur. Expressions (7.44) and (7.45) yield respectively \( \delta = 0.87597 \), \( a_4 = 150 \), and \( \lambda = 0.077 \). Running the program multrun employing the matched single cause of failure technique with parameters \( \lambda, \delta, a_4, G, a_1, a_2, \) and \( a_3' \) yield values for \( n, h, k, \) and \( \alpha \) given below.

-----
INPUT:

\( A_1 \) : COST PER SAMPLE OF SAMPLING AND PLOTTING.
\( A_2 \) : IS THE RATE AT WHICH A SAMPLE IS TAKEN AND PLOTTED.
\( A_3P \) : COST OF LOOKING FOR ANAssignable CAUSE AND NONE EXISTS.
\( G \) : TIME REQUIRED TO SAMPLE ONE ITEM.
\( D \) : TIME REQUIRED TO FIND ANAssignable CAUSE OF FAILURE.
\( A_3 \) : COST OF FINDING ANAssignable CAUSE WHEN ONE EXISTS.
\( \text{LAMBDAs} \) : IS THE SUM OF THE TWO LAMBDAs ITS VALUE IS 0.077.
\( A_4S \) : IS THE PENALTY COST DUE THE OPERATION OF A PRODUCTION PROCESS OUT OF CONTROL, THIS VALUE IS OBTAINED BY USING THE VALUES OF \( A_4 \) RELATIVE FOR EACH CAUSE.
\( \text{DEL} \) : IS AN APPROXIMATE VALUE OF THE SHIFT OF THE PROCESS MEAN DUE TO THE OCCURANCE OF SEVERAL CAUSES OF FAILURE.

RUNNING THE PROCESS, INDIVIDUAL LAMDA

*** PRIMARY INPUT VARIABLES ***

NUMBER OF CAUSES = 1

SAMPLING COST PARAMETERS
\( A_1 = 1.000000 \)
\( A_2 = 0.1000000 \)
\( A_3P = 25.00000 \)
\( G = 2.0000000E-02 \)

VALUES OF LAMDA - XLAM
\( XLAM(1) = 0.077000 \)

VALUES OF D
\( D(1) = 1.0000 \)

VALUES OF \( A_4 \)
\( A_4(1) = 150.0000 \)

VALUES OF \( A_3 \)
\( A_3(1) = 50.0000 \)

VALUES OF \( \text{DEL} \)
\( \text{DEL}(1) = 0.8760 \)
OUTPUT:
1- SAMPLE SIZE = 12
2- OPTIMAL VALUE OF K = 2.18
3- INTERVAL BETWEEN SAMPLES MEASURED IN HOURS = 0.58
4- PROBABILITY OF TYPE 1 ERROR = 0.0293
5- PROBABILITY OF TYPE 2 ERROR = 0.1964
6- LOSS COST PER HOUR OF OPERATION = $25.44
CHAPTER 8

SUMMARY AND RECOMMENDATIONS

8.1. Summary

In this study a variety of statistical and economic characteristics were
developed for acceptance sampling plans by variables for fraction defective. The
Bayesian approach was employed to develop statistical measures and economic
models. Cost models were considered for both destructive and non destructive
sampling plans. The conditional probability density function of the quality charac-
teristic \( X \) relative to an inspection lot with mean \( \mu \), the probability density
function of \( \mu \) and the conditional probability density of the individual measure-
ments in a sample with a given sample mean \( \bar{x} \) and a lot mean \( \mu \) are assumed
to be normally distributed. The distribution of the material fraction defective
\( p \) is derived from that of \( \mu \). An expression for \( \mu \) in terms of \( p \) is formulated and
employed to derive an expression for probability density function of \( p \). In the
development of an economic acceptance sampling plan it seems inappropriate
to dispose of the rejected lots in a homogeneous manner without investigating
the quality of the items rejected. This means that if the inspection of an item
or a product is destructive then the rejected lots are scrapped. If however, the
inspection of an item is nondestructive the rejected lots are either scrapped,
screened, or reworked. In this work the total cost is a function of the sample
size, economic cost parameters, and the upper and lower limits of the sample
mean, \( \bar{x} \), for the lot to be accepted, when the inspection is destructive. In non-
destructive sampling inspection the cost is a function of the sample size, the
economic cost parameters, and the upper and lower limits of \( \bar{x} \) for the lot to
be accepted, and the upper and lower limits of, \( \bar{x} \), for the lot to be screened or
scrapped. In destructive sampling a set of decision points is calculated directly. For non destructive sampling plans by variables the total cost is optimized relative to those limits. A set of integral equations are generated and solved to obtain a set of decision points that are employed to obtain the components of the total cost namely the cost of acceptance, screening and scrapping. These costs are verified to be optimal by studying the sign of the second derivative relative to those points.

A computer program written in FORTRAN 77 provides values of the neutral points of the material fraction defective $p$ for the decision to accept, screen, scrap or rework the rejected lots prior to sampling. The program also estimates the prior and posterior costs for destructive and non destructive sampling plans by variables for fraction defective. The values of the second derivatives for every possible sample size are also provided to verify the validity of the optimal costs.

An expression of the conditional expectation of $p$ given the number of defectives in a sample $x$ showed that $E(p|x)$ is a function of the $K^{th}$ moment of $p$, $E(p^k)$. In estimating $E(p|x)$ the distribution of the number of defectives in the sample is assumed to be governed by a binomial distribution with parameters $n$ and $p$. The distribution of $p$ is derived from the one for $\mu$. An expression of $E(p^k)$ is assumed to be of the form:

$$E(p^k) = g(s) = e^{c_1 s + c_2 s^2 + \cdots + c_n s^n}$$

where $s$ is a function of $p$.

Graphical and numerical techniques verified the validity of the above assumption. A computer program written in FORTRAN 77 computes the constants $c_1, c_2, \cdots, c_n$ up to $K = 5$ permitting estimation of $E(p^k)$. 
In estimating $E(p|x)$ a gamma p.d.f. is assumed for $p$. It is verified that a minimum value of $p$ namely $p'$ exist. Due to the existence of $p'$ a truncated gamma p.d.f. for $p$ is assumed. The expression of $E(p|x)$ shows that it is also a function of $E(p^k)$. Here another technique is required to evaluate $E(p^k)$. This technique employs the integrands of normal variates $z_1, z_2, \cdots, z_n$. The integrands product is transformed into a matrix of the form $\tilde{z}_k^t A_k \tilde{z}_k$. The elements of the matrix $A_k$ are functions of the lower and upper specification limits of the quality characteristic $X$, its variance and the variance of the mean of the inspection lot $\mu$. Other matrices are defined for computational purposes namely $D_k$, $C_k$ and $X_k$. The matrix $D_k$ is an orthogonal matrix whose elements are the eigenvalues of $A_k$. The matrix $C_k$ is a non singular matrix that can be transformed to an identity matrix. The matrix $X_k$ is a product of the inverse of the matrix $C_k$ and $\tilde{z}_k$. An expression for $E(p^k)$ is derived as a function of the determinant of $C_k$ and an integrand as a product of $\tilde{X}_k^t \tilde{X}_k$. The techniques to integrate exponential functions of exponents of the $\tilde{X}_k^t \tilde{X}_k$ are provided.

Expressions of ATI and AOQ were also developed. The ATI enable the user to compare minimum sample sizes for both attribute and variables sampling plans for fraction defective that minimize the costs. Error prone variables sampling plans for fraction defective are also considered. Expressions of the quality characteristic $X$, the lot mean $\mu$, the sample mean $\bar{x}$ and the material fraction defective $p$ are assumed. With the above values, a new set of probability density functions and variances permit estimating the decision points, the components of the cost, the conditional expectations and the moments.

Two types of costs were considered, the first is the cost based on sampling and inspecting items of finished products, and the second is the cost of keeping a production process under control.
A computer program written in FROTRAN 77 is provided to estimate the interval between samples, the limits of the control chart, the optimal cost, the average time of occurrence of an assignable cause within an interval between samples when the process is under control, and the average time the process is operating in the presence of an assignable cause. The control chart techniques are generalized to deal with the case when several quality characteristics are involved. For a multivariate production process the following Bayesian expressions are derived; the expected time the process is in control, the expected time the process is out of control, and the expected loss-cost per hour for an in-out of control production process.

8.2. Recommendations

The following are some areas for further study.

8.2.1. Producer's and Consumer's Costs

Considering the producer's and consumer's costs in constructing an acceptance sampling plan is important, since both the producer and the consumer are important partners in applying the acceptance procedure. The set of requirements that must be used in constructing an acceptance sampling plan must meet the needs of the producer and consumer. The following are factors affecting the set of requirements:

a. The number of lots of the same items produced by the producer.

b. The number of lots of the same items demanded by the consumer.

c. Available data.
   i. Prior distribution of lot quality from the producer viewpoint.
   ii. Prior distribution of lot quality from the consumer viewpoint.
   iii. Economic parameters for the producer.
   iv. Economic parameters for the consumer.
In general an acceptance procedure that meets the needs of the producer and consumer must provide maximum profit for the producer and minimum cost for the consumer. However extending this model to deal with profits is significant from the standpoint of the producer.

8.2.2. Producer’s and Consumer’s Risks

A risk expresses the probability of a wrong decision in the sampling procedure. The types of risks associated with the procedure and consumer are: the classical, average and posterior.

8.2.2.1. Producer’s Risk. The producer’s risk is expressed by the probability of rejecting a good lot.

Classical risk of type B, $\alpha_B$. The classical producer’s risk is the probability of rejecting lots of acceptable quality level (AQL). The producer may consider a stream of lots each of different quality level and the average fraction defective $\bar{p}$ is evaluated. The OC curves associated with such sampling plans is called an OC curve of type B and the risk is given by:

$$\alpha_B = P_r(\text{rejecting a lot} | \bar{p} = \text{AQL}).$$

Two estimates of $\alpha_B$ can be obtained depending on whether rejection calls for screening or scrapping the entire lot.

Classical risk of type A, $\alpha_A$. The producer may consider sampling from an isolated lot with fraction defective $p$. The producer’s risk is given by:

$$\alpha_A = P_r(\text{rejecting a lot} | p = \text{AQL}).$$

Two estimates of $\alpha_A$ can be obtained depending on whether rejection calls for screening or scrapping the entire lot. However if sampling inspection is destructive, rejecting a lot calls for scrapping it entirely.
Average risk $\alpha$. The producer rejects lot(s) of quality level(s) falling in the acceptance region. This type of producer's risk is suggested by Easterling (1970) which is defined as:

$$\bar{\alpha} = p_r(\text{rejecting a lot}|p \leq \text{AQL})$$

In estimating this risk, the value of risk depends on the inspecting lot and the sampling inspection whether it is destructive or non-destructive.

Posterior risk, $\alpha^*$, is the probability of rejecting lots with a proportion $p$ of them of acceptable quality:

$$\alpha^* = p_r[p \leq \text{AQL}|\text{rejecting the lot}].$$

8.2.2.2. Consumer's Risks. The consumer's risks expresses the probability of accepting a lot with bad quality. In general the consumer's risks are functions of the quality of the remainder of the lot. Classical risk, $\beta$, is the probability of accepting a lot at the lower quality limit (lot tolerance percent defective LTPD):

$$\beta = p_r(\text{accepting a lot}|p = \text{LTPD}).$$

Average risk $\overline{\beta}$ is the probability of accepting a lot with quality level in the rejection region. Easterling (1970) suggested the use of the average consumer's risk $\beta$ defined as

$$\overline{\beta} = p_r(\text{accepting a lot}|p \geq \text{LTPD}).$$

Posterior risk $\beta^*$ is the probability of accepting lots with a proportion $p$ of them as bad:

$$\beta^* = p_r(p \geq \text{LTPD}|\text{accepting the lot}).$$
Optimizing each of the above expressions yield a set of decision points for the sample mean $X$. Employing the decision points to evaluate the risks, the profits, the costs, the probabilities and the ATI. A set that yield values for the above quantities that does not meet the requirement of the producer and consumer is not acceptable. The best solution is that decided upon by the producer and the consumer depending on the quality, quantity, manufacturing facilities, (maintenance and depreciation rates of manufacturing equipment and high speed computers) and costs of items produced or purchased.

8.2.3. Workstations and a Series of Transfer Lines

An area of future research is the application of Bayesian variable sampling to series transfer lines (Figure 4). Here, sampling provides a form of feedback control to the various workstations along the line. A general model employing Markov chain and stochastic control methods is under development to analyze this type of hybrid system. This model has modified entries in the basic transition matrix. In this case each entry reflects the in-control or out-of-control state as dynamically defined by the Bayesian variable sampling plan. The transition matrix when reduced provides a measure of the production rate for the line. Tracking these rates and combining them with the demand rate permit a dynamic assessment of a manufacturing facility's output.

The main advantages of the Markov chain methods to analyze feedback control transfer lines (FCTL's) are:

1. solutions are tractable using geometric $z$-transforms
2. computer algorithms can be developed through the application of vector-matrix methods.

Presently, the material in Chapter 5 only is under consideration as the primary control factor for a transfer line. This is due in large part to the principal
Figure 21. A Transfer Line for a Series of Workstations.
parameter analyzed, namely, fraction defective \((p)\). The moments of \(p\) allow the development of a control chart density to predict in-control or out-of-control conditions as a function of time. These conditions can then be synchronized to the movement of product along the line.

8.2.4. Variances of \(X\), OQ, TI

A sampling plan may exist which has adequate values of the quality characteristic \(X\) whose standard deviation is relatively large. In this particular case a decision based on the sample mean \(\bar{x}\) alone will be a misleading measure of the effectiveness of the plan. The variances of the quality characteristic of a stream of lots produced by a random process must be treated as a random variable with a certain distribution. A set of decision points based either on the material fraction defective or the sample means can be obtained. The decision points are employed in drawing a final conclusion about lots disposition.

Employing a variable sampling plan for fraction defective and considering two other random variables of strong economic implications namely the outgoing quality (OQ) and the total inspection (TI) of each lot. The average values of (OQ), AOQ and the maximum value of AOQ, AOQL also of (TI), ATI do not reflect the lot to lot fluctuations in the outgoing quality. It is recommended that the variances of OQ and TI must be considered. If the variances of OQ and of TI are large, then AOQ and ATI are not satisfactory measures of the effectiveness of the plan. Employing a variables sampling plan for fraction defective and find a set of decision points for lot disposition based on the sample means and variance of \(X\). An economic model can be developed for both \(\bar{x}\) and \(R\) charts. A production process is assumed under control if the mean and variance of \(X\) are within the acceptance region. The possibility of false alarms or both or either the mean and variance of \(X\) are out of control and the out of
control conditions are either detected or undetected must be considered. The model can be generalized by dealing with a Bayesian Hotelling’s $T^2$-chart, where a stream of lots will be inspected relative to several quality characteristics. A set of decision points can be derived for every characteristic inspected for each lot.

8.2.5. Computer Programs

The programs listed in the appendices can be combined into one program to perform the following additional tasks:

1. The program can be adjusted to sort the upper and lower limits, and the variance of a quality characteristic $X$. The values of $X$ are generated by a production process producing lots to be inspected relative to $X$. The program can be extended to estimate the mean and the variance of the lot means. The quantities estimated can be stored in a common area to be used in estimating costs and other quantities.

2. The program can be adjusted to deal with priors of different families of distributions.

3. The programs can be extended to provide producer profits and consumer costs.

4. The program can be extended to estimate the risks from the producer’s and consumer’s points of view.

5. The program can be extended to estimate the average total inspection for both the attribute and variables sampling plans for fraction defective.

6. The program can be extended to deal with statistical process that can exist in more than one state with different probabilities.
7. The program "TKEVAL" can be used as a subprogram where the conditional expectations $E(p|x)$ and $E(p|x)$ can be called by the main program. The conditional expectations can be employed to estimate the sampling and posterior costs per unit. The capability of the program can also be adjusted to employ the moments of $p$ to the $k^{th}$ power to approximate the prior probabilities and to use them whenever needed.

8. The programs MULSTOP and MULTRUN can be employed as subprograms to generate values for the expected loss cost per hour for keeping the production process under control. This cost adds to the cost of acceptance or rejection, whether the rejection calls for scrapping, screening, or reworking the entire lot.
APPENDIX A

ABBREVIATIONS AND NOTATIONS

Abbreviations
AOQ - average outgoing quality
AOQL - average outgoing quality limit
AQL - acceptable quality level
ASN - average sample number
ATI - average total inspection
BOC - Bayesian operating characteristic
c.d.f. - cumulative density function
GERT - graphic evaluation and review technique
IQL - indifferent quality level
LTPD - lot tolerance percent defective
OC - operating characteristic
OQ - outgoing quality
p.d.f. - probability density function
VOQL - variance outgoing quality limit

Mathematical Notations
$X$ - the value of the measurable quality characteristic in variables sampling plans, for fraction defective.
$x_i$ - the $i^{th}$ measurement for an observed sample of size $n$.
$\bar{x}$ - the sample mean
$L$ - lower specification limit of the quality characteristic.
$U$ - upper specification limit of the quality characteristic.
$\mu$ - mean of the quality characteristic.
\( \sigma \) - standard deviation of the quality characteristic.

\( \sigma_\mu \) - standard deviation of the mean \( \mu \).

\( s \) - the unbiased estimate of \( \sigma \) if \( \sigma \) is unknown.

\( p_0 \) - the fraction of material defective such that the plans criteria must accept that fraction at least \((1 - \alpha)\%\) of the time.

\( p_1 \) - the fraction of material defective such that the plans criteria must accept that fraction at most \( \beta\% \) of the time.

\( p \) - fraction of items defective.

\( f(x|\alpha) \) - the conditional probability density function \( x \) given a parameter \( \alpha \).

\( h(\alpha) \) - the distribution of the parameter \( \alpha \).

\( h(\mu) \) - the distribution of the lot mean \( \mu \).

\( T(X|\alpha) \) - the conditional distribution of \( X \) given \( \alpha \).

\( t(x|\bar{x},\mu) \) - the conditional probability density function of individual measurements \( x \) in a sample with mean \( \bar{x} \) drawn from a lot of mean \( \mu \).

\( L_A \) - the lower disposition limit of \( \bar{x} \) for accepting the lot.

\( U_A \) - the upper disposition limit of \( \bar{x} \) for accepting the lot.

\( L_{Sn} \) - the lower disposition limit for \( \bar{x} \) for screening inspection.

\( U_{Sn} \) - the upper disposition limit for \( \bar{x} \) for screening inspection.

\( P_{a_0} \) - probability of acceptance on variable \( i \) when a lot is inspected relative to \( q \ (q > 1) \) quality characteristics.

\( P_a \) - probability of acceptance for a single variable acceptance sampling plan for fraction defective.

\( L_{ADI} \) - lower disposition limit for a sample mean for acceptance relative to a variable \( i \) in variables sampling plan for fraction defective.

\( L_{AD} \) - lower disposition limit for a sample mean for lot acceptance relative to a single variable for variables sampling plan for fraction defective.
\(U_{AD_i}\) - upper disposition limit for a sample mean for lot acceptance relative to a variable \(i\) for variables sampling plan for fraction defective.

\(U_{AD}\) - upper disposition limit for a sample mean for lot acceptance relative to a single variable for variables sampling plan for fraction defective.

\(C_D\) - cost of scrapping on a unit of product when sampling is destructive.

\(C_{BD}\) - cost of replacing a unit destroyed during destructive sampling inspection.

\(C_{pL}\) - cost of a unit of a product in an acceptance lot having a dimension below the lower specification limit.

\(c_{pU}\) - cost of a unit of a product in an accepted lot having a dimension above the upper specification limit.

\(e_1\) - type I inspection error

\(e_2\) - type II inspection error

\(E(X_1)\) - expectation of a random variable \(X_1\)

\(E(p^k)\) - \(k^{th}\) moment of \(p\)

\(K_{E_1}\) - screening error type 1

\(K_{E_2}\) - screening error type 2

\(K_I\) - Inspection cost per item inspected

\(K_J\) - junk value of a scrapped item

\(K_P\) - production cost of an item

\(K_R\) - sales price of an item

\(K_y\) - rework yield rate

\(C_1\) - the cost per item for the decision to accept the lot without sampling

\(P_1\) - the profit per item for a decision to accept the lot without sampling for variables sampling plans for fraction defective

\(C_2\) - the cost per unit for a decision to reject outright and scrap the whole lot for nondestructive sampling plans for fraction defective
$P_2$ - The profit per item for the decision to reject outright for non destructive variables sampling plans for fraction defective

$C_3$ - the producer's cost per unit for rejecting a lot outright and screen for nondestructive variables sampling plans for fraction defective.

$P_3$ - the profit per unit for the producer's decision to reject outright and screen for nondestructive variables sampling plans for fraction defective

$C_R$ - the cost per unit for the decision to reject outright and rework for nondestructive variables sampling plans for fraction defective

$P_y$ - the profit per unit for the producer's decision to reject outright and rework for nondestructive variables sampling plans for fraction defective

$k_A(x, \mu)$ - cost of acceptance as a function of the sample mean $x$ per lot and an actual mean product dimension $\mu$

$K_{sn}(\overline{x}, \mu)$ - cost of screening per lot as a function of $\overline{x}$ and an actual mean product $\mu$

$k_{sp}(\overline{x}, \mu)$ - cost of scrapping per lot as a function of $\overline{x}$ and an actual mean product $\mu$

$A_1$ - The expected number of defectives in the uninspected portion of an accepted lot assuming no inspection errors

$A_2$ - The expected number of defectives in the uninspected portion of an accepted lot assuming error-prone inspection

$B_1$ - the expected number of defective items classified as good when a lot is screened assuming no inspection errors

$B_2$ - The expected number of defective items classified as good when a lot is screened assuming error-prone inspection

$C_1$ - the expected number of defective items in a sample classified as good assuming no inspection errors
$C_2$ - the expected number of defective items in a sample classified as good assuming error-prone inspection.

$D_1$ - the expected number of defective items introduced through replacement to replenish the screened portion of a rejected lot assuming no inspection error

$D_2$ - The expected number of defective items introduced through replacement to replenish the screened portion of a rejected lot assuming error-prone inspection

$E_1$ - the expected number of items introduced through replacement to replenish the sample assuming no inspection errors

$E_2$ - the expected number of items introduced through replacement to replenish the sample assuming error-prone inspection

$F_1$ - the expected number of apparent defectives removed from the sample assuming no inspection errors.

$F_2$ - the expected number of apparent defectives removed from the sample assuming error-prone inspection

$H_1$ - the expected number of items in the uninspected portion of an accepted lot assuming no inspection errors

$H_2$ - the expected number of items in the uninspected portion of an accepted lot assuming error-prone inspection

$J_1$ - the sample size plus the expected number of items inspected to replenish the sample assuming no inspection errors

$J_2$ - the sample size plus the expected number of items inspected to replenish the sample assuming error-prone inspection

$K_1$ - the number of items screened in a rejected lot assuming no inspection errors

$K_2$ - the number of items screened in a rejected lot assuming error-prone inspection
$L_1$ - the number of items screened plus the expected number of items inspected to replenish the lot assuming no inspection error

$L_2$ - the number of items screened plus the expected number of items inspected to replenish the lot assuming error-prone inspection

$P_{3i}$ - probability that an individual measurement $x_i$ $i = 1, 2, \cdots, s$ in a sample with a sample mean $\bar{x}_i$ drawn for a lot of mean $\mu_i$ is above the upper specification limit.

$P_{4i}$ - probability that an individual measurement $x_i$ $i = 1, 2, \cdots, s$ in a sample with a sample mean $\bar{x}_i$ drawn from a lot of mean $\mu_i$ is below the lower specification limit.

$P_{1u_i}$ - probability that an individual measurement $x_i$ in a lot of mean $\mu_i$ is above the upper specification limit.

$P_{2L_i}$ - probability that an individual measurement $x_i$ in a lot of mean $\mu_i$ is below the lower specification limit.

$P_{3s}$ - probability that an individual measurement in a sample drawn from a lot is above the upper specification limit in a single variable acceptance sampling plan.

$P_{4s}$ - probability that an individual measurement in a sample drawn from a lot is below the lower specification limit in a single variable acceptance sampling plan.

$P_{1u}$ - probability that an individual measurement in a lot of mean $\mu$ is above the upper specification limit when a single variable is involved.

$P_{2L}$ - probability that an individual measurement in a lot of mean $\mu$ is below the lower specification when a single variable is involved.
$K_{Rui}$ - cost of repairing an item with a dimension above the upper specification limit when several variables are involved.

$k_{Ru}$ - cost of repairing an item when a dimension above the upper specification limit when a single variable is involved.

$k_{RLi}$ - cost of repairing an item with a dimension below the lower specification limit when several variables are involved.

$k_{RL}$ - cost of repairing an item with a dimension below the lower specification limit when a single variable is involved.

$\lambda_j$ - average rate of occurrence of cause $j$

$\lambda$ - rate of occurrence of a second assignable cause

$\delta_j$ - shift in the mean of the process due to cause $j$

$\delta_c$ - shift in the process mean due to double occurrence of assignable causes

$D_j$ - average time to find an assignable cause $j$

$a_{3j}$ - average cost of finding an assignable cause $j$ when it occurs

$a_{4j}$ - loss per hour of a process operating out of control due to cause $j$

$\tau_j$ - average time of occurrence of an assignable cause $j$ within an interval between samples when the process is in control

$B_j$ - average time the process is operating in the presence of an assignable cause $j$

$\lambda_x$ - expected cost per cycle of discovering the assignable causes

$T_{R1}$ - expected additional loss per cycle arising from out of control conditions

$T_{R2}$ - expected cost per cycle when looking for an assignable cause when none exists

$T_{R3}$ - average time the process is operating in the presence of an assignable cause $j$

$\tau_s$ - expected length of time during which the process is shut down

$\tau_r$ - expected length of time to repair the machine if the alarm is true

$U = \tau_s$
\( v = \tau_r + \tau_s \)

- \( k_s \) - average cost of sampling and searching for an assignable cause
- \( k_r \) - average cost of repairing the process
- \( A \) - the expected number of false alarms
- \( R_j \) - the average run length in state 2 due to the initial assignable cause \( j \)
- \( R'_j \) - the average runs in state 3 due to the initial assignable cause
- \( E(n_{ij}) \) - the expected number of sampling intervals until the process switches from state \( i \) to state \( j \)
- \( E(T_{00}) \) - the expected time the process starts in control until a shift occurs
- \( E(T_{ij}) \) - the expected length of time for a process starting in state \( j \) and remaining in state \( j \)
- \( E(T_{pC}) \) - the expected time that a process is under control
- \( E(T_{0C}) \) - the expected time that a process is out of control
- \( P \) - the probability that an assignable cause will be detected
- \( Q \) - the probability that an assignable cause will not be detected
- \( \alpha \) - the probability of a point falling outside the control limits when the process is under control
- \( P_j \) - the probability that a point falls outside the control limits after the occurrence of an assignable cause of failure \( j \)
- \( p_j \) - the probability of detecting an assignable cause on the first sample after the process entered state 2
- \( e^{-\lambda' h} \) - the probability that a second assignable cause will not occur between 0 and \( h \) or the process mean remains constant
- \( p' \) - the probability of detecting a shift \( \delta \) in the process mean
- \( q_i \) - the probability that the process is operating out of control given that the process is in state \( \mu_i \)
\( \alpha_i \) - the steady state probability that the process is in state \( \mu_i \) at the time when the test is performed or a sample is selected

\( f_i \) - the probability of producing defective units given the process is in state \( \mu_i \)

\( \gamma_i \) - the probability that the process is in state \( \mu_i \) at any point in time

\( g_{ij} \) - the probability of the process shifting from state \( \mu_i \) to \( \mu_j \) during the production of \( k \) units

\( g_{00} \) - the probability of the process in control at the production of the first \((d+1)\) samples given that the process is in control when the \( d \) first samples are produced

\( b_{ij} \) - the probability that the process is in state \( \mu_i \) at time \( t \) and in state \( \mu_j \) after the production of \( k \) units

\( p_c \) - the probability that the process is under control during any period of time

\( p_{0c} \) - the probability that the process is out of control at any period of time

\( E(C_1) \) - the expected cost of sampling per unit

\( E(C_2) \) - the expected cost of searching for a cause or causes of failure and removing this/these causes

\( E(C_3) \) - penalty cost due to a process producing defective units

\( E(L_{C_p}) \) - the expected loss per hour of operation
APPENDIX B

AN EXPRESSION OF THE MEAN OF A LOT QUALITY CHARACTERISTIC

In this appendix an expression for \( z \) which is defined as the difference of the lot mean \( \mu \) and the lower specification limit \( L \) of the quality characteristic \( x \) is derived. The fraction of items defective is given by expression (2.8) as:

\[
p(\mu) = \int_{-\infty}^{L} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(z-\mu)^2} dx.
\]  \( \text{(B.1)} \)

Translating parameters to about the origin by defining

\[
z = \mu - L \quad v = p(\mu) - \frac{1}{2}.
\]  \( \text{(B.2)} \)

Employing expression (B.2), expression (B.1) can be written as:

\[
p(\mu) = p(z + L) = \Phi\left(-\frac{z}{\sigma}\right) = v + \frac{1}{2}.
\]  \( \text{(B.3)} \)

Set \( \Phi(-\frac{z}{\sigma}) = h(z) \) and define:

\[
h^{(k)}(z) = \frac{d^{(k)}h(z)}{dz^{k}}
\]  \( \text{(B.4)} \)

where

\[
k \geq 1, \text{ for } k = 0, \quad h^{(0)}(z) = h(z).
\]  \( \text{(B.5)} \)

Employing expression (B.4) leads to:

\[
h^{(1)}(z) = -\frac{1}{\sqrt{2\pi}\sigma} e^{-z^2/\sigma^2} \text{ where } h^{(1)}(0) = -\frac{1}{\sqrt{2\pi}\sigma}.
\]  \( \text{(B.6)} \)
The expression for $h^{(2)}(z)$ can be derived from expression (B.6). Hence:

$$h^{(2)}(z) = -\frac{1}{\sqrt{2\pi} \sigma} \left( -\frac{z}{\sigma^2} e^{-z^2/2\sigma^2} \right) \text{ where } h^{(2)}(0) = 0. \quad \text{(B.7)}$$

Similarly the expressions of $h^{(3)}(z), h^{(4)}(z), h^{(5)}(z), h^{(6)}(z)$ are:

$$h^{(3)}(z) = e^{-z^2/2\sigma^2} - \frac{z^2 e^{-z^2/2\sigma^2}}{(2\pi)^{1/2} \sigma^3} - \frac{z^2 e^{-z^2/2\sigma^2}}{(2\pi)^{1/2} \sigma^5} \quad \text{(B.8)}$$

$$h^{(4)}(z) = \left( \frac{z^2}{\sigma^4} - \frac{1}{\sigma^2} \right) h^{(2)}(z) + \frac{2z}{\sigma^4} h^{(1)}(z) \quad \text{(B.9)}$$

$$h^{(5)}(z) = \left( \frac{z^2}{\sigma^4} - \frac{1}{\sigma^2} \right) h^{(3)}(z) + \frac{4z}{\sigma^4} h^{(2)}(z) + 2 \frac{h^{(1)}(z)}{\sigma^4} \quad \text{(B.10)}$$

$$h^{(6)}(z) = \left( \frac{z^2}{\sigma^4} - \frac{1}{\sigma^2} \right) h^{(4)}(z) + \frac{6h^{(2)}(z)}{\sigma^4} + \frac{6z h^{(3)}(z)}{\sigma^4} \quad \text{(B.11)}$$

Expressions for $h^{(k)}(z)$ for any positive integer values of $k$ can be derived. However in this work expressions of $h^{(k)}(z)$ up to $k = 12$ are derived. The values of $h^k(z)$ for $z = 0$ up to $k = 12$ are listed in Table B1.

Expression (B.2) can be written as:

$$v = f(z) = h(z) - \frac{1}{2}. \quad \text{(B.12)}$$

Expression (B.12) yields:

$$z = f^{-1}(z). \quad \text{(B.13)}$$

Employing expressions (B.12) and (B.13) it is possible to write:

$$\frac{dz}{dv} = \frac{1}{h^{(1)}(z)}. \quad \text{(B.14)}$$

An expression for $z^{(k)}(v)$ similar to expression (B.4) can be defined as:

$$z^{(k)}(v) = \frac{d^{(k)}(z)}{dv^k} \quad \text{(B.15)}$$
where:

\[ z^0(v) = z(v) \]
\[ z^0(0) = z(0) = 0. \]

If
\[ v = 0 \] then \( p = \frac{1}{2} \) and \( z = 0. \) \hspace{1cm} (B.16)

Employing expression (B.15), \( z^{(2)}(v) \) can be written as:
\[ z^{(2)}(v) = \frac{d^{(2)}z}{dv^2} = -\frac{h^{(2)}(z)}{[h^{(1)}(z)]^3}. \] \hspace{1cm} (B.17)

To simplify the expressions that will follow define the following relations:
\[ E_k = h^{(k)}(z) \] \hspace{1cm} (B.18)
\[ E = \frac{1}{h^{(1)}(z)} \] \hspace{1cm} (B.19)

Employing \( E_k, E \) given by expressions (B.18) and (B.19), \( z^{(1)}(v) \) and \( z^{(2)}(v) \) can be written as:
\[ z^{(1)}(v) = E \] \hspace{1cm} (B.20)
and
\[ z^{(2)}(v) = -E_2E^3 = E' \] \hspace{1cm} (B.21)

where \( E' \) is the derivative of \( E \) relative to \( v. \)

Taking the derivative of \( E_k \) relative to \( v \) yields:
\[ \frac{dE_k}{dv} = \frac{dh^{(k)}(z)}{dv} = \frac{dh^{(k)}(z)}{dz} \cdot \frac{dz}{dv} = E_{k+1} \cdot E \] \hspace{1cm} ((B.22))

where
\[ E(0) = \frac{1}{h^{(1)}(0)} = -\sqrt{2\pi}\sigma. \] \hspace{1cm} (B.23)
Employing expression (B.21) to derive an expression for \( z^{(3)}(v) \) yields:

\[
z^{(3)}(v) = -E_3 E^4 + 3E_2^2 E^5. \tag{B.24}
\]

The following are the expressions of \( z^{(4)}(v) \), \( z^{(5)}(v) \), \( z^{(6)}(v) \) derived in the same way as \( z^{(3)}(v) \):

\[
z^{(4)}(v) = -E^4 \frac{dE_3}{dv} - 4E^3 E_3 E' + 6E_2 E_3 E E^5 \\
+ 3E_2^2 \cdot 5E^4 \cdot (-E_2 E_3). \tag{B.25}
\]

Expression (B.25) can be simplified to:

\[
z^{(5)}(v) = -5E^4 \cdot (-E_2 E_3^3)E_4 - E^5 E_5 E + 10E_4 E_2 E^6 \cdot E \\
+ 10E_3 \cdot E_3 E^6 E + 10E_2 \cdot E_3 \cdot 6E^5 \cdot (-E_2 E^3) \\
- 15(z)(E_2^3) \cdot E_3 \cdot E \cdot E^7 - 15E_2^3 \cdot 7E^6 \cdot (-E_2 E^3). \tag{B.26}
\]

Expression (B.27) can be simplified to:

\[
z^{(5)}(v) = 5E^7 E_2 E_4 - E^6 E_5 - 60E^8 E_2^2 E_3 + 10E_4 E_2 E^7 \\
+ 10E_3^2 E^7 - 45E_2^2 E_3 E^8 + 105E_2^4 E^9. \tag{B.27}
\]

An expression for \( z^{(6)}(v) \) is given by:

\[
z^{(6)}(v) = -E^7 E_6 + 21E^8 E_2 E_5 + 35E^8 E_3 E_4 - 210E^9 E_2^2 E_4 \\
- 280E^9 E_2 E_3^2 + 1260E^{10} E_2^2 E_3 - 945E^{11} E_2^5. \tag{B.28}
\]

Expressions for \( z^{(k)}(v) \) for positive integer values of \( k \) can be derived by following the above procedure. However values of \( z^{(k)}(v) \) up to \( k = 12 \) are listed in Table B1.
Table B.1. Values of $z^{(k)}(0)$ and $h^{(k)}(0)$ for $k = 1, 2, \cdots, 12$.

| $k$ | $z^{(k)}(v) \bigg|_{v=0}$ | $\frac{d^{(k)}(3)}{d\sigma^3} \bigg|_{v=0}$ | $h^{(k)}(z) \bigg|_{z=0}$ | $\frac{dh^{(k)}(z)}{dz^k} \bigg|_{z=0}$ |
|-----|-----------------|-----------------|-----------------|-----------------|
| 0   | 0               | 1/2             | 0               | 0               |
| 1   | $-\sqrt{2}\pi\sigma$ | $-\frac{1}{\sqrt{2}\pi\sigma}$ | 0               | 0               |
| 2   | 0               | 0               | 0               | 0               |
| 3   | $-(2\pi)^{3/2}\sigma$ | $\frac{1}{\sqrt{2}\pi\sigma^3}$ | 0               | 0               |
| 4   | 0               | 0               | 0               | 0               |
| 5   | $-7(2\pi)^{5/2}\sigma$ | $-\frac{3}{\sqrt{2}\pi\sigma^5}$ | 0               | 0               |
| 6   | 0               | 0               | 0               | 0               |
| 7   | $-127(2\pi)^{7/2}\sigma$ | $\frac{15}{\sqrt{2}\pi\sigma^7}$ | 0               | 0               |
| 8   | 0               | 0               | 0               | 0               |
| 9   | $-4369(2\pi)^{9/2}\sigma$ | $-\frac{105}{\sqrt{2}\pi\sigma^9}$ | 0               | 0               |
| 10  | 0               | 0               | 0               | 0               |
| 11  | $-318493(2\pi)^{11/2}\sigma$ | $\frac{945}{\sqrt{2}\pi\sigma^{11}}$ | 0               | 0               |
| 12  | 0               | 0               | 0               | 0               |

Note: $z^{(k)}(v) \bigg|_{v=0} = 0$ for even values of $k$

$h^{(k)}(z) \bigg|_{z=0} = 0$ for even values of $k$. 
Employing the expressions of $z^k(0)$ and $h^k(0)$ listed in Table B1 and developing $z(v)$ by a maclaurin series of the form:

$$z(v) = \sum_{k=0}^{\infty} \frac{z^k(0)}{k!} v^k$$

(B.30)
yield:

$$z(v) = \frac{z^0(0)}{0!} v^0 + \frac{z(1)(0)}{1!} v^1 + \cdots + \frac{z^{12}(0)}{12!} v^{12} + \cdots + R(v)$$

(B.31)
where $R(v)$ is the remainder defined by the series. Expression (B.31) can be written as:

$$z(v) = -\sigma \{ \sqrt{2\pi} v + \frac{(\sqrt{2\pi} v)^3}{3!} + \frac{7(\sqrt{2\pi} v)^5}{5!} + 127(\sqrt{2\pi} v)^7}{7!} + 4369 \frac{(\sqrt{2\pi} v)^9}{9!} + 318493 \frac{(\sqrt{2\pi} v)^{11}}{11!} + 20493907 \frac{(\sqrt{2\pi} v)^{13}}{13!} + R(v) \}.$$

(B.32)
Employing expression (B.2), expression (B.32) can be written in terms of $p(\mu)$ as:

$$z(p(\mu)) = -\sigma \left\{ \sqrt{2\pi} \left( p - \frac{1}{2} \right) + \frac{\sqrt{2\pi} \left( p - \frac{1}{2} \right)^3}{3!} + \frac{127 \sqrt{2\pi} \left( p - \frac{1}{2} \right)^7}{7!} + 4369 \frac{\sqrt{2\pi} \left( p - \frac{1}{2} \right)^9}{9!} + 318493 \frac{\sqrt{2\pi} \left( p - \frac{1}{2} \right)^{11}}{11!} + 20493907 \frac{\sqrt{2\pi} \left( p - \frac{1}{2} \right)^{13}}{13!} + R(p) \right\}.$$

(B.33)
Denoting the quantity within brackets of expression (B.33) by $l(p)$, then expression (B.33) can be written as:

$$z(p(\mu)) = -\sigma l(p).$$

Employing expressions (B.2) and (B.34) an expression for $\mu(p)$ can be written as:

$$\mu(p) = L - \sigma l(p).$$

(B.35)
Assuming a normal distribution for \( \mu(p) \), \( g(\mu(p)) \) of mean \( m_0 \) and variance \( \sigma_\mu^2 \), \( g(\mu(p)) \) can be written as:

\[
g(\mu(p)) = \frac{1}{\sqrt{2\pi}\sigma_\mu} e^{-\frac{1}{2} \frac{(\mu - \sigma_l(p) - m_0)^2}{\sigma_\mu^2}}. \tag{B.36}
\]

The probability distribution of the material fraction defective \( p \), \( w(p) \) can be obtained from \( \mu(p) \) through the transformation:

\[
w(p) = g(\mu(p)) \left| \frac{d\mu(p)}{dp} \right|
= \frac{\sigma}{\sqrt{2\pi}\sigma_\mu} e^{-\frac{1}{2} \frac{(\mu - \sigma_l(p) - m_0)^2}{\sigma_\mu^2}} \cdot l'(p) \quad 0 \leq p < 1. \tag{B.37}
\]

The distribution of \( \mu(p) \) and \( p \) are used in Chapters 2 through 6 of this work.
APPENDIX C

PROGRAM FINALI

Input:
Upper limit $U$ of the quality characteristic $X$
Lower limit $L$ of the quality characteristic $X$
Variance $\sigma^2_\mu$ of the mean of the quality characteristic $X$
Variance $\sigma^2$ of the quality characteristic $X$
Cost of scrapping a rejected unit
Cost of accepting a defective unit
Cost of sampling and inspecting per unit
Lot size

Output 1:
The first root of the cost equation
The second root of the cost equation
The prior sampling cost per unit
The Bay’s estimate of the posterior cost per unit
The Bay’s estimate of the acceptance cost per unit
The Bay’s estimate of the rejection cost per unit

Output 2:
The upper limit of $\bar{X}$, $\bar{X}_{BU}$ for lot acceptance
The lower limit of $\bar{X}$, $\bar{X}_{BL}$ for lot acceptance
The conditional expectations of $p$ given $X_{BU}$ and $X_{BL}$ evaluated at the value of $N$ relative to the minimum posterior cost per unit.
PROGRAM MAIN
INCLUDE 'COM,VAR'

*** TO RUN THIS PROGRAM THE FOLLOWING FILES MUST BE LINKED: ***
C   MAIN+REAP1+HEAD1+SUB1+SUB2+SIMP+UNITCOST+DBINTG
C   THE FILE "SUB2.FOR" CONTAINS TWO SUBROUTINES: SUB2 AND SUB3
C   THE 'COM,VAR' INCLUDES ALL THE VARIABLES AND CONSTANTS
C

CALL READ1
CALL HEAD1
CALL SUB1
END

SUBROUTINE HEAD1
INCLUDE 'COM,VAR'
WRITE (99.1)
& FORMAT(/BX,'TABLE 1: A VARIABLE PLAN FOR THE COST MODEL'/
& BX,'------------------------------------------'/)
& WRITE (99.2)U,L,VAR,VARA,CR,CE,CS,LS
& FORMAT(/BX,'------------------------------------------'/)
& WRITE(99.3)BX,'COL. No. DESCRIPTION '
& & BX,'1' SAMPLE SIZE '
& & BX,'2' (a * b) / (b + n*a) WHERE: '
& & BX,'a: VARIANCE OF THE MEAN OF X'/
& & BX,'b: VARIANCE OF X'
& & BX,'n: SAMPLE SIZE '
& & BX,'3' 1st ROOT OF THE COST EQ. '/
& & BX,'4' 2nd ROOT OF THE COST EQ. '/
& & BX,'5' SAMPLING COST PER UNIT'/
& & BX,'6' POSTERIOR COST PER UNIT'/
& & BX,'7' ACCEPTANCE COST PER UNIT'/
& & BX,'8' REJECTION COST PER UNIT'/
& WRITE(99.4)BX,T(75(' '-')/)
& T12,'1',T18,'2',T28,'3',T38,'4',T48,'5',T58,'6',
& T68,'7',T78,'8'/
& BX,75(' '-')/
& WRITE(99.5)BX,'OUTPUT2: AVERAGE PROCESS FRACTION DEFECTIVES'/
& & BX,'AND THE SAMPLE MEANS'/

WRITE(99.1)
}& FORMAT(/BX,'TABLE 1: A VARIABLE PLAN FOR THE COST MODEL'/
& BX,'------------------------------------------'/)
& WRITE (99.2)U,L,VAR,VARA,CR,CE,CS,LS
& FORMAT(/BX,'------------------------------------------'/)
& WRITE(99.3)BX,'COL. No. DESCRIPTION '
& & BX,'1' SAMPLE SIZE '
& & BX,'2' (a * b) / (b + n*a) WHERE: '
& & BX,'a: VARIANCE OF THE MEAN OF X'/
& & BX,'b: VARIANCE OF X'
& & BX,'n: SAMPLE SIZE '
& & BX,'3' 1st ROOT OF THE COST EQ. '/
& & BX,'4' 2nd ROOT OF THE COST EQ. '/
& & BX,'5' SAMPLING COST PER UNIT'/
& & BX,'6' POSTERIOR COST PER UNIT'/
& & BX,'7' ACCEPTANCE COST PER UNIT'/
& & BX,'8' REJECTION COST PER UNIT'/
& WRITE(99.4)BX,T(75(' '-')/)
& T12,'1',T18,'2',T28,'3',T38,'4',T48,'5',T58,'6',
& T68,'7',T78,'8'/
& BX,75(' '-')/
& WRITE(99.5)BX,'OUTPUT2: AVERAGE PROCESS FRACTION DEFECTIVES'/
& & BX,'AND THE SAMPLE MEANS'/
& BX, -----------------------------' '//
& BX, COL. NO. DESCRIPTION '/
& BX, ' 1 ' SAMPLE SIZE '/
& BX, ' 2 ' LOWER LIMIT OF THE SAMPLE MEAN '/
& BX, ' 3 ' UPPER LIMIT OF THE SAMPLE MEAN '/
& BX, ' 4 ' AVERAGE FRACTION DEFECTIVE OF THE '/
& BX, ' 5 ' PROCESS GIVEN THE LOWER LIMIT OF '/
& BX, ' THE SAMPLE MEAN '/
& BX, ' 6 ' PROCESS GIVEN THE UPPER LIMIT OF '/
& BX, ' THE SAMPLE MEAN '/

END
SUBROUTINE READ
INCLUDE 'COM, VAR'
WRITE (6,1)
FORMAT ('1')
WRITE (6,*) 'IS THE VALUE IN THE FILE...ENTER 0 OR 1'
READ (5,*) ANS
IF (ANS .EQ. 1) GOTO 100

************************************************************
C * THIS PROGRAM IS AN INTERACTIVE PROGRAM, THE VALUES OF U,
C * THE MEAN OF THE QUALITY CHARACTERISTICS, AND THE COST PARAMETERS
C * CAN BE ENTERED EASILY.
************************************************************
WRITE(6,*) 'ENTER VALUES FOR THE FOLLOWING VARIABLES'
WRITE(6,*)
WRITE(6,2)
FORMAT (1X, 'U=?', £)
READ(S,*) U

WRITE(6,3)
FORMAT (1X, 'L=?', £)
READ(S,*) L

WRITE(6,4)
FORMAT (1X, 'VAR=', £)
READ(S,*) VAR

WRITE(6,5)
FORMAT (1X, 'VARA=', £)
READ(S,*) VARA

WRITE(6,6)
FORMAT (1X, 'CR=', £)
READ(S,*) CR

WRITE(6,7)
FORMAT (1X, 'CE=', £)
READ(S,*) CE

WRITE(6,8)
FORMAT (1X, 'CS=', £)
READ(5,*)CS
WRITE(6,0)
FORMAT(1X,'LS=? ','$')
READ(10,*)LS
WRITE (10,*)U,VARA,CR,CE,CS,LS
RETURN
100 
READ (10,*)U,VARA,CR,CE,CS,LS
RETURN
END

C********************************************************************
C THIS SUBROUTINE FINDS THE ROOTS OF f(MN).
C THERE ARE TWO ROOTS SATISFYING THIS EQUATION, MN1 & MN2.
C THE SUBROUTINE USES THE NEWTON-RAPHSON ROOT FINDING TECHNIQUE.
C MN1, & MN2 ARE SYMMETRICAL RELATIVE TO THE UPPER AND LOWER
C SPECIFICATIONS (L,U).
C MN1, MN2 ARE EQUAL TO THE SUM OF (L+U)/2 & OR- V, WHERE f(V)=f(-V).
C********************************************************************
SUBROUTINE SUB1
EXTERNAL G,GM
DOUBLE PRECISION G,GM,GGM,X1,X2,X3,MN,TOLL
INCLUDE 'COM,VAR'
REAL MO
PARAMETER (TOLL = .0001)
C TOLL: TOLERANCE
DO 10 N=7,3,0.5
VARN = (VARA * VAR)/(VAR + N * VARA)
C INITIAL VALUE OF MN
MN=U + 1
X1=MN
X2=1000000
I=0
C NEWTON-RAPHSON ROOT FINDING TECHNIQUE.
DU 20 WHILE (DABS(X1)-X3) .GT. TOLL)
I=I+1
X3=X1
GGM=G(X1)
C PRINT*, 'GGM = ',GGM
C PRINT*, 'GGM = ',GGM
X2=X1-GG/GGM
X1=X3
20 CONTINUE
C********************************************************************
C THE APPROXIMATE SOLUTION FOR MN
C THE VALUE OF MN IS OBTAINED BY THE NEWTON-RAPHSON ROOT FINDING
C THE INITIAL VALUE OF MN IS SET AT U+1.
C WE STARTED WITH THIS VALUE OF MN AND THEN AFTER A CERTAIN # OF
C ITERATIONS MN CONVERGES TO THE REAL ROOTS OF THE FUNCTION f(MN)
C WITHIN 0.0 OR 0.001.
C THESE ROOTS WERE VERIFIED BY USING THE SECANT ROOT FINDING TECHNIQUE.
C THE VALUES OF THE ROOTS OBTAINED BY THE TWO TECHNIQUES ARE THE SAME
C UP TO THE FORTH DECIMAL PLACE.
C********************************************************************

MN=X2
MO=(U-L)/2
V=ABS(MN-MO)
C PRINT*, 'V = ',V
MN1=MI1*v
MN2=MII+v
CALL SUB2
CALL UNITCOST
CALL SUB3
WRITE(6,*)(N,VAR,N,MN1,MN2,PRCOST,CPPOST,CREJ,CACCP)
WRITE(99,3,U1N,VAR,N,MN1,MN2,PRCOST,CPPOST,CREJ,CACCP)
30 FORMAT(I1,111,7(F10.5))
10 CONTINUE
END
C *****************************************************
C * THE FUNCTION "G" IS ASSUMED TO BE "O". THIS FUNCTION IS THEN
C * EQUAL TO "f(MN) - (CR/CE)". IF MN IS THE EXACT ROOT OF THE
C * FUNCTION f(MN) THEN "G" CONVERGES TO "O".
C *****************************************************
DOUBLE PRECISION FUNCTION G(X)
INCLUDE 'COM,VAR'
DOUBLE PRECISION X,A,B,E,TERM
TERM=DSQRT(VAR+VARN)
A=(L-X)/TERM
B=(U-X)/TERM
CALL SIMP(A,B,E)
G=-1-(CR/CE)
RETURN
END
C *****************************************************
C * THE FUNCTION "GM" IS THE FIRST DERIVATIVE OF THE FUNCTION "G".
C * THIS FUNCTION IS ASSUMED TO BE DEFINED AND TO BE CONTINUOUS.
C *****************************************************
DOUBLE PRECISION FUNCTION GM(X)
INCLUDE 'COM,VAR'
DOUBLE PRECISION X,A,B,E,TERM
PRINT *,TERM=TERM
A=(L-X)/TERM
B=(U-X)/TERM
CALL SIMP(A,B,E)
PRINT*,E=E,E
GM=(1/DSQRT(VAR+VARN))*E
RETURN
END
SUBROUTINE SUB2
INCLUDE 'COM,VAR'
DOUBLE PRECISION A,B,E1,E2
AV=(L+U)/2.
VN=VAR/N
SIGN=DSQRT(VARA**4)/(VN+VARA**2)
A=(MN1-AV)/SIGN
B=(MN2-AV)/SIGN
CALL SIMP(A,B,E1)
A=MN1
B=MN2
CALL DBINTG(A,B,E2)
CPPOST= (N*CS + LS*CR)/((REAL (LS-N))+ ((CE - CR) • E1) -CE*E2
CREJ=N*CR/((REAL (LS-N))+CR*CR*E1
CACCP=CPPOST-CREJ
END
SUBROUTINE SIMP(A,B,E)

* FUNCTION: APPROXIMATES THE INTEGRAL OF f(x) OVER THE
 INTERVAL [A,B] BY SIMPSON'S RULE

* USAGE:
 CALL SEQUENCE: CALL SIMP(A,B,M,E)

* EXTERNAL FUNCTIONS/SUBROUTINES: FUNCTION F(X)

* PARAMETERS:
 INPUT:
 A = INTERVAL LEFT ENDPOINT
 B = INTERVAL RIGHT ENDPOINT
 M = NUMBER OF SUBINTERVALS (POSITIVE EVEN INTEGER)

OUTPUT:
 E = ESTIMATE OF THE INTEGRAL

DOUBLE PRECISION A,B,E

*** INITIALIZATION ***
 M=I(M+1)/2
 H=(B-A)/M
 X=A-H

*** COMPUTE THE SUM OF ODD INDEX TERMS ***
 DO 1 I=1,M-1,2
  X=X+2.0*H
  E=E+4.0*F(X)
  CONTINUE
1

*** COMPUTE THE SUM OF EVEN INDEX TERMS ***
 X=A
 DO 2 I=2,M-2,2
  X=X+2.0*H
  E=E+2.0*F(X)
  CONTINUE
2

END

* STANDARD NORMAL DENSITY FUNCTION

DOUBLE PRECISION FUNCTION F(T)
DOUBLE PRECISION T,VALX
SUBROUTINE UNICOST
INCLUD 'CUM.VAR'
DOUBLE PRECISION A,B,E,AVE,VARGAM
AVE = (L+U)/2.
VARGAM=(VAR*VAR)/(VAR+VAR)
B=(U-AVE)/(DSORT(VAR+VARGAM))
A=(L-AVE)/(DSORT(VAR+VARGAM))
CALL SIMP(A,B,E)
PRCOST=(N*(CS+CR))/(REAL(LS-N))
END

SUBROUTINE DBINTG(A,B,E)
DOUBLE PRECISION A,B,E
REAL J1,J2,J3,K1,K2,K3,L
A=.1
B=.5
M=50
N=50
H=(B-A)/(2*N)
J1=0
J2=0
J3=0
DO 1 I=0,2*N
  X=A+I*M
  HX=(D(X)-C(X))/(2*M)
  K1=F1(X,V)
  IF (J2(J/2),EQ,J) THEN
    K2=K2+Z
  ELSE
    K3=K3+Z
  ENDIF
  CONTINUE
  L=(K1+2*K2+4*K3)*HX/3
  IF (1,EQ,0,OR,1,EQ,2*M) THEN
    J1=J1+L
  ELSEIF (2*(1/2),EQ,1) THEN
    J2=J2+L
  ELSE
    J3=J3+L
  ENDF
1 CONTINUE
E=(J1+2*J2+4*J3)*H/3
PRINT *,E=E
END

DOUBLE PRECISION FUNCTION F1(X,Y)
INCLUD 'COM.VAR'
DOUBLE PRECISION TERM1,H,X,Y
PI=3.1416
TERM1= ((X-Y)**2)/(2*(VAR+VARGAM))
IF (TERM1 .LT. 90) THEN
  H = 1. / (DSQRT(2*PI*(VAR+VARN)) * (DEXP(TERM1)))
ELSE
  H = 0
ENDIF
TERM2 = ((V-AV)**2) / (2*SIGN)
IF (TERM2 .LT. 90) THEN
  Q = 1. / (SORT(2*PI*SIGN) * EXP(TERM2))
ELSE
  Q = 0
ENDIF
F = H
END

DOUBLE PRECISION FUNCTION D(X)
INCLUDE 'COM.VAR'
D = U
END

DOUBLE PRECISION FUNCTION C(X)
INCLUDE 'COM.VAR'
C = L
END

SUBROUTINE COM_VAR
*****************************************************************************
THIS SUBROUTINE INCLUDES ALL THE CONSTANT AND VARIABLES
THAT ARE EMPLOYED IN THIS PROGRAM
*****************************************************************************

DOUBLE PRECISION U, L, VAR, VARA, VARN, CR, CE, MN1, MN2, XBARN1, XBARN2
& "RFJ, CALCP
COMMON /BLOCK1/ U, L, VAR, VARA, VARN, CR, CE, CS, LS, MN1, MN2, N
& VNSIGN, PRCUST, CPPOST, AV, CREJ, CACCP
PROGRAM JALLT.ONE

This program is employed for plotting only.

Enter the quantities defined in the WRITE statements only.

For the set of data available in a file, enter the file name.

In running this program follow the following steps:

1. Start by typing setup display.
2. Define the execution file.
3. Follow the same format as that of the program that you are copying from.
4. Specify the range of values for the X and Y correctly.
5. Delete each time you run this program, the file STDOUGB.

Note: In case of an error in the data, the data will be plotted according to the std file and no correction will be considered by the computer unless this file is regenerated.

In running the following statements:

1. Plot the following given in Form98.fio

REAL (X,12.), XRN(100), XN(100), XN2,100), XBU(100),XBL(100)

DATA (M11,1, 1,1,1, 1,1,1, 1,1, 1, 1, 1, 1,1, 1,1)

DATA (L1,1,1, 1,1,1, 1,1, 1,1,1, 1,1, 1,1,1, 1,1)

CALL IPRU(11,0, 10)

CALL PAGE (10,0, 11,0)

CALL PLOUTN(,ARN,MN1,MN2, XBU, XBL, ECSTAC, XOR, YOR, IBUF, XBL, VL, CREJ, CALP)

CALL ENDPL(0)

CALL DUNEPJ(0)

END

SUBROUTINE PLOUTN(ARN,MN1,MN2, XBU, XBL, ECSTAC, XOR, YOR, IBUF

CALL PLOUTN(,ARN,MN1,MN2, XBU, XBL, ECSTAC, XOR, YOR, IBUF

CALL PLOUTN(,ARN,MN1,MN2, XBU, XBL, ECSTAC, XOR, YOR, IBUF

REAL ECSTAC(100), CREJ(100), CALP(100)
DIMENSION BUF(10)
CALL RESET('ALL')
CALL NORDR
CALL BSARLE(1.0,1.0)
CALL PHYSOR(XGR,YOR)
CALL GRACE(1.0)
CALL AREASD(XLONG,YLONG)
CALL FRAME

C *************************************************************
C * TITLES FOR THE X AND Y AXES.
C *************************************************************
CALL XNAME(' SAMPLING COST PER UNIT Ь.100)
CALL VNAME(' SAMPLE SIZE Ь.100)
CALL GRAF(0.08,0.08,1.09,1.1,1.1,1.1)
DO 10 I=1,15
CALL MARKER(I)
IF (I .EQ. 1) THEN
  CALL DOT
END IF
C *************************************************************
C * THIS SUBROUTINE SPECIFIES THE VARIABLES TO BE PLOTTED
C * XBU IS ON THE X AXIS AND N IS ON THE Y AXIS.
C *************************************************************
CALL CURVE(XE,YE,10,1)
CALL RESET(MARKER)
CALL RESET(DOT)
END IF
C CALL DOT
CALL CURVE(YT,PEF,11,-1)
CALL RESET(MARKER)
CALL RESET(DOT)
END IF
C CALL DOT
CALL CURVE(VT,PEF,11,-1)
CALL RESET(MARKER)
CALL RESET(DOT)
END IF
C CALL DOT
CALL CURVE(VE,DE,101,0)
END IF
C CALL DOT
CALL CURVE(VE,TE,101,0)
C CALL DOT
CALL CURVE(VE,EE,101,0)
C CALL DOT
CALL CURVE(VE,EE,101,0)
C CONTINUE
C CALL ENDGR(0)
RETURN
END
APPENDIX D

PROGRAM 2AA

The structure of the program listed in Appendix D is similar to that given in Appendix C, where the sampling is assumed to be nondestructive.

Input Data:
\[ L, U, \sigma^2, \sigma^2_{\mu}, K_s, K_A, K_p, K_I, K_{sp}, N. \]
All the above parameters are defined in Appendix A.

Output 1:
Sample size, variance of \( X \) given the results of the sample and the lot mean, the roots of the cost equation, the upper and lower limits of the sample for lot acceptance (decision points for lot acceptance) and the acceptance cost per lot.

Output 2:
The sample size, the second derivative of the total cost relative to the decision points for lot acceptance, an estimate of the cost of acceptance, an estimate of \( PGU = \int f(x|x, \mu)dx \) and the exact value of the cost of acceptance.
**PROGRAM PROG2**

- **THIS PROGRAM COMPUTES THE ACCEPTANCE COST PER LOT**
- **ALSO IT PROVIDES VALUES FOR THE UPPER AND LOWER DISPOSITION LIMITS FOR LOT ACCEPTANCE. THIS COST IS VERIFIED TO BE A MINIMUM BY STUDYING THE SIGN OF THE SECOND DERIVATIVE OF THE TOTAL COST RELATIVE TO THE UPPER AND LOWER DISPOSITION LIMITS.**
- **TO RUN THIS PROGRAM THE FOLLOWING FILES MUST BE LINKED:**
  1) PROG2AA
  2) SUBROUTINE READ2
  3) SUBROUTINE HEAD2
  4) SUBROUTINE HEAD2A

**ASSUMPTIONS**

- **a) THE DISTRIBUTIONS ARE NORMAL**
- **b) THE INTEGRAL OF X GIVEN THE MEAN MU IS EQUAL TO THE INTEGRAL OF T GIVEN XBAR AND THE MEAN MU,**
- **c) THE STATISTICAL PROCESS CAN EXIST IN ONE STATE.**

**CALCULATION PROCEDURES**

1. AN EXPRESSION OF THE TOTAL COST IS DERIVED
2. THE FIRST DERIVATIVE OF THE TOTAL COST IS TAKEN RELATIVE TO THE FOLLOWING PARAMETERS:
   a) UPPER DISPOSITION LIMIT OF LOT ACCEPTANCE
   b) LOWER DISPOSITION LIMIT OF LOT ACCEPTANCE
   c) UPPER DISPOSITION LIMIT OF LOT SCREENING
   d) LOWER DISPOSITION LIMIT OF LOT SCREENING.
3. THE PARTIAL DERIVATIVES ARE EQUATED TO ZERO. THE FOUR EQUATIONS OBTAINED ARE SOLVED FOR UA, LA, USN, LSN.
4. THE COST OF ACCEPTANCE RELATIVE TO A CERTAIN SAMPLE MEAN XBAR AND A MEAN MU IS EQUIVALENT TO THE COST OF SCREENING AT XBAR=LA OR XBAR=UA. SOLVING THIS EXPRESSION YIELDS A VALUE FOR THE INTEGRAL OF X GIVEN MU.
5. THE TOTAL COST OF ACCEPTANCE IS OBTAINED BY SUMMING OVER ALL SAMPLE MEANS AND POPULATION MEANS.

**INPUT PARAMETERS**

- L = LOWER SPECIFICATION LIMIT OF QUALITY CHARACTERISTICS X
- U = UPPER SPECIFICATION LIMIT OF QUALITY CHARACTERISTICS X
- LSS = LOT SIZE
- N = SAMPLE SIZE
- X = A PRODUCT DIMENSION OR A QUALITY CHARACTERISTIC
- XBAR = A SAMPLE MEAN
- MU = THE MEAN OF THE VARIABLE X RELATIVE TO A CERTAIN LOT
- M1 = THE MEAN OF THE MEAN MU
- VAR = LOT VARIANCE
- CI = COST OF INSPECTION PER UNIT OF A PRODUCT
- CSSN = COST OF SCREENING INSPECTION PER UNIT OF A PRODUCT
- CSSP = COST OF SCRAPPING PER UNIT OF A PRODUCT
- CA = COST OF ACCEPTANCE PER UNIT OF A PRODUCT
- CP = COST OF SCRAPPING AND REPLACING A DEFECTIVE UNIT DURING SAMPLING OR SCREENING INSPECTION.

**OUTPUT PARAMETERS**

- **1. SAMPLE N**
2. Variance of the joint distribution of the mean $\mu$ and
the distribution of $\bar{X}$ given $\mu$

3. Lower limit generated by taking the derivative of the
total cost relative to $L_c$

4. Upper limit generated by taking the derivative of the
total cost relative to $U_a$

5. Upper disposition limit of lot acceptance

6. Lower disposition limit of lot acceptance

7. The approximate cost of acceptance per lot

8. The exact value of the cost given in 7

9. The value of the second derivative relative to each $N$

10. The value of $PGU$ which is the integral of the probability

density function of $X$ given $\mu$ between the limits $L_u$,

EXTERNAL G_FF1, FF2

INCLUDE 'COM2, VAR'

DOUBLE PRECISION XBU, PGU, M1, Z

COMMON /BLOCK2/XBU, PGU, M1, Z

COMMON /BLOCK3/NO

DOUBLE PRECISION A, B, E, X, ERR, MN, ECSTAC, XLO, XUP, D2, D2A

DOUBLE PRECISION EVAL, ETCOST

COMMON /ULOCK4/ECSTAC, ETCOST

DOUBLE PRECISION KE1, KE2, KR, KJ, Kc, KY

DOUBLE PRECISION C1, ECC1, P01, C2, ECC2, P02, C3, ECC3, P03

COMMON /BLOCKS/ KE1, KE2, KR, KJ, Kc, KY

DOUBLE PRECISION CC1, ECC1, P01, CC2, ECC2, P02, CC3, ECC3, P03

COMMON /BLOCK6/CC1, ECC1, P01, CC2, ECC2, P02, CC3, ECC3, P03

REAL PIE, S2PIE

C ***********************************************

C SUBROUTINES CALLED: READ2, HEAD2, SECA, NDRM11, NDRM12, DBINTG2

C **************************************************

C FUNCTIONS CALLED:

REAL AA(3), BB(3), M(3), A, B, X, ERR

CALL READ2

CALL HEAD2

WRITE(6,*) 'COST OF INSPECTION?'
READ(5,*) CI

WRITE(6,*) 'STARTING SAMPLE SIZE?'
READ(5,*) ISAM

WRITE(6,*) 'ENTER MI VALUE?'
READ(5,*) MI

CALL PRIORCOST

CALL PRIORCOSTCOMPUTE

CALL PRIORCOSTDATINOUT

CALL PRIORCOSTRESULTS

CALL HEAD2A

C CALL D2HEAD

CALL D2AHEAD

DO 10 N=ISAM..3*LSS

VARN = (VARA * VAR)/(VAR + DFLOAT(N) * VARA)
**INITIAL VALUE OF MN**

```plaintext
erre=1.0E-09
a=1-2
b=u
CALL SECA(A,B,X,ERR)
MN=x
MU=(U+L)/2
V=ABS(MN-MO)
MN1=MO-V
MN2=MO+V
XBU=(MN2*(VAR+N*VARA)-(L+U)*VAR/2.)/(N*VARA)
WRITE(66,*) XBU
XBL=(MN1*(VAR+N*VARA)-(L+U)*VAR/2.)/(N*VARA)
PGU=1-(CSN/(CA-CP))
```

```plaintext
NITR=20
NQ=N
CALL SIMP3(XBL,XBU,NITR,EVAL)
WRITE(30,*) 'DENSITY PRODUCT FACTOR = ', EVAL
```

**THE FOLLOWING RELATES TO TRIPLE INTEGRATION.**

```plaintext
NN=3
AL=1.5
EE=1.E-01
M(1)=.6
H(2)=.6
AA(1)=L
AA(2)=L-2
AA(3)=MN1
BB(1)=U
BB(2)=U-2
BB(3)=MN2
AA2=AA(2)
AA2=L
AA3=AA(3)
AA3=L
BB2=BB(2)
BB2=U
BB3=BB(3)
BB3=U
```

**THE FOLLOWING SUBROUTINE COMPUTES AN APPROXIMATION TO THE N-DIMENSIONAL INTEGRAL OVER A PARALLELEPİPED A ROMBERG TYPE METHOD BASED ON THE MIDPOINT RULE IS EMPLOYED. EACH INTERVAL IS SUBDIVIDED INTO [I/H(I)] SUBINTERVALS. O<H(I)<1.1,....,N,M IS THE MAXIMUM NUMBER OF ITERATIONS ALLOWED AND SHOULD DECREASE WITH INCREASING N SINCE THE**
NUMBER OF FUNCTIONAL EVALUATIONS IN THE jth ITERATION IS AT
LEAST J**N. IF N IS SET TO 15, IT IS RESET TO 15.
AL IS A CONSTANT BETWEEN 1.5 AND 2, PREFERABLY 1.5 WHICH
REGULATES THE INCREASE IN THE NUMBER OF POINTS USED.
E IS A TOLERANCE WHICH SHOULD INCREASE AS N INCREASES
KEY IS SET EQUAL TO 1 IF THIS IS NOT ACHIEVED WITH
M ITERATIONS. KEY IS SET EQUAL TO -1 IF N = 1.
THE SUBROUTINE
EXITS WITH KEY = 0. A PROGRAM FUNCTION F(X, N) MUST SUPPLIED
BY THE USER WITH XD DECLARED BY THE STATEMENT DIMENSION X(N)
F MUST BE DECLARED EXTERNAL IN THE CALLING PROGRAM
TOL IN THE DATA STATEMENT IS A MACHINE DEPENDENT PARAMETER
WHICH SHOULD BE SET TO THE RELATIVE MACHINE ACCURACY.
IF E < TOL, IT IS RESET TO TOL.

********************************************************************************
* A = ARRAY OF LOWER LIMITS
* B = ARRAY OF UPPER LIMITS
* M = NUMBER BETWEEN 1.5 AND 2, PREFERABLY 1.5
* N = MAXIMUM NUMBER OF ITERATIONS SHOULD DECREASE AS
* E = TOLERANCE INCREASES WITH N
* AINT = RESULT
* F = FUNCTION ROUTINE
* KEV = INDICATOR SET BY SUBROUTINE
******************************************************************************

C CALL NDIMRI1(NN, AA, BB, M, LSS, EE, FF1, AINT1, KEV)
C CALL NDIMRI2(NN, AA, BB, M, LSS, EE, FF2, AINT2, KEV)
C CALL DBINTG2(AA2, BB2, AA3, BB3, AINT3)
C WRITE(21,50) AINT2
C50 FORMAT(1X, 'DBL INT TERM = ',15.6)

C T1 = SQRT(N/((2*PI)**VAR))
C T2 = 1/(VAR*SQRT(2*PI))
C VARMU = 0.1
C T3 = 1/(VARMU*SQRT(2*PI))
C C1 = (CA-CP)*T1*T2/SORT(N-1)*AINT1
C C2 = (CA*TT)*T2*T3*AINT2
C C3 = (CP*N+CA*(LSS-N))**T1*T2*AINT3
C COST = C1+C2+C3

C******************************************************************************

C WRITE(99,20)N,VARN,MN1,MN2,XBU,XBL,ECSTAC
20 FORMAT(1X, 11,7(F15.5))
C XLO=MO-. 1
C XUP=MO-. 1
C ND=N
C CALL DERV2(XLO,XUP,D2)
C CALL DERV2A(L,U,D2A)
C CONTINUE
C END

REAL FUNCTION G(X)
INCLUDE 'COM2.VAR'
DOUBLE PRECISION a,b,x,e1,term1,term2,term3,term4
C REAL X
M=100
print *, '** G(x) starts **'
TERM1 = SORT(VAR + VARV)
MO = (L + U) / 2.
write(*,'(n = ', n
write(*,'(x = ', x
TERM2 = (MO * VAR + N * VAR * X) / (VAR * N * VAR)
write(*,'(term1 = ', term1, ', term2 = ', term2
A = (L - TERM2) / TERM1
B = (U - TERM2) / TERM1

** ALL THE FOLLOWING TERMS ARE NECESSARY FOR TESTING 

A = L - 2
B = U
write(*,'(A = ' A, ', B = ' B CALL SIMP1(A, B, M, E)
print*,'G = ', G RETURN
END

REAL FUNCTION FF1(X, NN)
INCLUDE 'COM2, VAR'
DOBLE PRECISION X(NN)
FF1 = 1 / (EXP( (N*(X(1) - X(3))**2) / (2*(N-1)*VAR))
& 1 / (EXP( (N*(X(3) - X(2))**2) / (2*VAR))) RETURN
END

REAL FUNCTION FF2(X, NN)
INCLUDE 'COM2, VAR'
DOUBLE PRECISION X(NN)
FF2 = 1 / (EXP((X(1) - X(3))**2) / (2*VAR))
& 1 / (EXP((X(3) - X(2))**2) / (2*VAR)) RETURN
END

SUBROUTINE SIMP1(A, B, M, E)

* FUNCTION: APPROXIMATES THE INTEGRAL OF F(X) OVER THE INTERVAL [A,B] BY SIMPSON'S RULE *
FUNCTION: CALL SIMP(A,B,M,E)  
PARAMETERS:  
    A = INTERVAL LEFT ENDPOINT  
    B = INTERVAL RIGHT ENDPOINT  
    M = NUMBER OF SUBINTERVALS (POSITIVE EVEN INTEGER)  
OUTPUT:  
    E = ESTIMATE OF THE INTEGRAL

DOUBLE PRECISION A, B, M, E
*** INITIALIZATION ***
IF (B .LT. -4 .OR. A .GT. 4) THEN
   E=0.00000000000001
RETURN
ENDIF
E=F(A)+F(B)
H=(B-A)/M
X=A-M
C ••••••••••••••••••••••••••••••••••••••••••••••••••••••••••••
DO 1 I=1, M-1
   X=I*M+2.0*H
   E=E+4.0*F(X)
1 CONTINUE
C ••••••••••••••••••••••••••••••••••••••••••••••••••••••••••••
DO 2 I=2, M-2, 2
   X=I*M+2.0*H
   E=E+2.0*F(X)
2 CONTINUE
E=E*H/3.0

SUBROUTINE SECA(A,B,X,EPS)

FUNCTION: THIS SUBROUTINE COMPUTES THE APPROXIMATE ROOT OF F(X)=0 USING THE SECANT METHOD
USAGE:
CALL SEQUENCE: CALL SECA(A,B,X,EPS)
EXTERNAL FUNCTIONS/SUBROUTINES: FUNCTION G(X)
PARAMETERS:
    A = INITIAL APPROXIMATION X(0)
    B = INITIAL APPROXIMATION X(1)
    EPS = ERROR BOUND
OUTPUT:

DOUBLE PRECISION function F(T)
DOUBLE PRECISION T
F=1.0/((SQRT(2*3.1416)) * EXP(+.5 * T**2))
END

*** STANDARD NORMAL DENSITY FUNCTION ***

DOUBLE PRECISION FUNCTION F(T)
DOUBLE PRECISION T
F=1.0/((SQRT(2*3.1416)) * EXP(+.5 * T**2))
END

303
* X-SECANT APPROXIMATION OF THE ROOT *

```c
* DOUBLE PRECISION a,b,r,u,v,eps
** INITIALIZATION **
print*: 'SECA Starts-----------------
I=I+1
v=G(b)
print*, 'I,B,G(B) ',I,B,v
u=G(a)
print*, 'A,G(A) ',A,u
x=B
if (abs(u-v) .lt. .000000000001) return
** COMPUTE APPROXIMATE ROOT ITERATIVELY **
do while(dabs(x-a),gt,eps)
p=-int(u=v, v)
X=B-v0(b-a)/(v-u)
A=B
B=x
v=G(x)
print*, 'G IN SECA= ',v
end do
print*, 'SECA Ends ---------------
return
end
SUBROUTINE D2HEAD
write(20,10)
10 format(5x,'N',9x,'XLO',14x,'XUP',12x,'2-ND DERV',12x,'M1')
return
end
SUBROUTINE DERV2(XLO,XUP,D2)
includc 'COM2,VAR'
DOUBLE PRECISION XB,PGU,M1,Z
COMMON /BLQCK2/XB,PGU,M1,Z
COMMON /BLOCK3/NO
DOUBLE PRECISION XLO,XUP,TEMP1,TEMP2,TEMP3,CI,C2,C3,VARAI,D2
DOUBLE PRECISION XNO
NINT=20
write(6,*) 'd2 =',d2
call SIMP2(XLO,XUP,NINT,D2)
XNO=XNO
TEMP1=-(XNO**(3./2.))*VARA)/(SQRT(VAR)*SQRT(2.*PI)*((XNO*VARA)+VARA))
TEMP1=Z/((VARA*VARA/XNO)*((SQRT(2.*PI)))
TEMP2=XB-U-((M1*VARA+XNO*VARA*XB))/(XNO*VARA+VARA))
C1=CA-CP-CPN
C2=PGU*(CA-CP)
write(6,*) 'C1 =',C1, ' C2 =',C2
```

304
I. \( V_{A_k} = V_{A_k} + (V_{A_k}/X_{NO}) \)

\[ \text{TEMP} = \exp\left(-0.5\left((M_1-X_{BU})^2/V_{AR}\right)\right) \]

\[ \text{TEMP} = \exp\left(-0.5\left(Z^2\right)\right) \]

\( C_1 = L_{55} - X_{NO} \)

\[ D_2 = C_3 \times \text{TEMP} \times \text{TEMP}^2 \times (C_1 - C_2) \times \text{TEMP} \]

\[ D_2 = C_3 \times \text{TEMP} \times \text{TEMP}^3 \times (C_1 - C_2) \]

\( M_1 = Z^2 \times (V_{AR} + CV_{AR}/X_{NO}) + X_{BU} \)

\( \text{WRITE}(20,20) \text{NU,XLO,XUP,D2,M1} \)

\( \text{FORMAT}(1X,1D,4X,E13.4,4X,E13.4,4X,E13.4,4X,E13.4,4X,E13.4) \)

\( \text{RETURN} \)

**SUBROUTINE SIMP2(A,B,M,E)**

**FUNCTION: APPROXIMATES THE INTEGRAL OF F(X) OVER THE INTERVAL [A,B] BY SIMPSON'S RULE**

**USAGE:**

- CALL SEQUENCE: CALL SIMP(A,B,M,E)

**EXTERNAL FUNCTIONS/SUBROUTINES: FUNCTION F(X)**

**PARAMETERS:**

- **INPUT:**
  - A = INTERVAL LEFT ENDPOINT
  - B = INTERVAL RIGHT ENDPOINT
  - M = NUMBER OF SUBINTERVALS (POSITIVE EVEN INTEGER)

- **OUTPUT:**
  - E = ESTIMATE OF THE INTEGRAL

**DOUBLE PRECISION A,B,E,X,H,F**

**INITIALIZATION**

\[ \text{IF} \ (B .LT. -4 .OR. A .GT. 4) \text{THEN} \]

\[ L = 0.0000000000001 \]

\( \text{RETURN} \)

\( \text{ENDIF} \)

\( E = F(A) + F(B) \)

\( H = (B - A)/M \)

\( X = A - H \)

**COMPUTE THE SUM OF ODD INDEX TERMS**

\[ \text{DO} \ 1 \ I = 1, M - 1, 2 \]

\[ X = X + 2.0 \times H \]

\[ E = E + 4.0 \times F(X) \]

\[ 1 \text{ CONTINUE} \]

**COMPUTE THE SUM OF EVEN INDEX TERMS**

\[ X = A \]

\[ \text{DO} \ 2 \ I = 2, M - 2, 2 \]

\[ X = X + 2.0 \times H \]

\[ E = E + 2.0 \times F(X) \]

\[ 2 \text{ CONTINUE} \]
E=E*H/3.0
RETURN
END

SUBROUTINE SIMP3(A,B,M,E)
WRITE(6,*) 'TEMP1=VARA+(VAR/XNO)', TEMP1
WRITE(6,*) 'TEMP2 = ', TEMP2
WRITE(6,*) 'FL FUNCTION = ', FL
RETURN
END

DOUBLE PRECISION FUNCTION FR(UU)
INCLUDE 'COM2,VAR'
DOUBLE PRECISION XBU,PGU,M1
COMMON /BLOCK2/XBU,PGU,M1
COMMON /BLOCK3/NO
DOUBLE PRECISION UU,C1,C2,C3,C4,TEMP1,TEMP2,TEMP3,HU,v1,v2
N=NO
C1=N**(3.0/2.0)
C2=XBU-UU
C3=N
C4=C3*((C2**2)/(2*VAR))
WRITE(6,*) 'c1 = ',c1
WRITE(6,*) 'c2 = ',c2
WRITE(6,*) 'c3 = ',c3
WRITE(6,*) 'c4 = ',c4
WRITE(6,*) 'xbu = ',xbu
WRITE(6,*) 'HU=(EXP(-((UU-MO)**2)/(2*VAR)))/((VAR**.5)*SQRT(2*PI))
FR=TEMP3*HU*((CA-CP-C5N)-(PGU*(CA-CP)))
v1=CA-CP-C5N
v2=PGU*(CA-CP)
RETURN
END

SUBROUTINE D2AHEAD
WRITE(45,10)
10 FORMAT(5X,'N',7X,'2-ND DERIV',9X,'ECSTAC',13X,'PGU',13X,'ETCOST')
RETURN
END

SUBROUTINE DERV2A(XLO,XUP,D2A)
INCLUDE 'COM2,VAR'
DOUBLE PRECISION XBU,PGU,M1,Z
COMMON /BLOCK2/XBU,PGU,M1,Z
COMMON /BLOCK3/NO
DOUBLE PRECISION ECSTAC,ETCOST
COMMON /BLOCK4/ECSTAC,ETCOST
DOUBLE PRECISION XLO,XUP,TEMP1,TEMP2,TEMP3,TEMP4,D2A
DOUBLE PRECISION XNO,TEMP5,TEMP6,TEMP7

NINT=20
WRITE(6,'(d2 =',d2
CALL SIMP4(XLO,XUP,NINT,D2A)

XNO=XNO
TEMP1=(VARA*(VAR/XNO))/(VARA+(VAR/XNO))**.5
TEMP2=((TEMP1*VARA/((VAR+TEMP1)**.5)
TEMP3=(XNO**.5)*TEMP2/((VAR**2)*(VARA**.5)*(2*PI))
TEMP4=EXP(-.5*(M1-XBUI)**2)/((VARA*(VAR/XNO))
TEMP5=((2*PI)**.5)*((VARA+(VAR/XNO))**.5)
TEMP6=EXP(-.5*(M1-XBUI)**2)/((VARA+(VAR/XNO))
TEMP7=((LSS-XNO)*(CA-CSN-CP)*(XBU-M1))/TEMP5*TEMP6
D2A=TEMP7*(LSS-XNO)*(CA-CP)*TEMP3*TEMP4*D2A

WRITE(45,20) NO,D2A,ECSTAC,PGU,ETCOST
20 FORMAT(1X,15,4X,E13.4,4X,E13.4,4X,E13.4,4X,E13.4,4X,E13.4)
WRITE(62,22)NO,D2A,ETCOST
22 FORMAT(1X,15,4X,E13.4,1X,E13.4)
CALL PRIORCOSTCOMPUTE
CALL PRIORCOSTDATINOUT
CALL PRIORCOSTRESULTS
RETURN
END

SUBROUTINE SIMP4(A,B,M,E)

FUNCTION: APPROXIMATES THE INTEGRAL OF F(X) OVER THE
INTERVAL [A,B] BY SIMPSON'S RULE
USAGE:
CALL SEQUENCE: CALL SIMP(A,B,M,E)
EXTERNAL FUNCTIONS/SUBROUTINES: FUNCTION F(X)
PARAMETERS:
INPUT: 
A=INTERVAL LEFT ENDPOINT
B=INTERVAL RIGHT ENDPOINT
M=NUMBER OF SUBINTERVALS(POSITIVE EVEN INTEGER)
OUTPUT: 
E=ESTIMATE OF THE INTEGRAL

DOUBLE PRECISION A,B,E,X,M,F
*** INITIALIZATION ***
IF (B .LT. 4 .OR. B .GT. 4) THEN
-0.0000000000000
RETURN
ENDIF
E=FB(A)**FB(B)
H=(B-A)/M
X=A-M
*** COMPUTE THE SUM OF ODD INDEX TERMS ***
DO 1 I=1,M-1,2
X=X+2.0*M
E=E+4.0*FB(X)
1 CONTINUE
DO 1 CONTINUE
   CONTINUE
   *** COMPUTE THE SUM OF EVEN INDEX TERMS ***
   X = A
   DO 2 I = 2, M - 2, 2
      X = X + 2.0 * M
      E = E + 2.0 * FB(X)
   2 CONTINUE
   E = E + M / 3.0
END

DOUBLE PRECISION FUNCTION FB(UU)
INCLUDE 'COM2.VAR'
DOUBLE PRECISION XBU, PGU, M1
COMMON /BLOCK2/XBU, PGU, M1
COMMON /BLOCK3/NO
DOUBLE PRECISION UU, C1, C2, C3, C4, TEMP1, TEMP2, TEMP3, HU, V1, V2
DOUBLE PRECISION TEMP4, TEMPS
XNO = NO
TEMP1 = (VARA * VAR) * (XNO * VARA) + VARA + VAR
TEMP2 = (VARA + (VAR/XNO))
TEMP3 = (M1 * (VAR/XNO) * (VARA * XBU)) / TEMP2
TEMP4 = ((VARA * VAR) * XNO) / TEMP2
TEMP = ((TEMP3 * VAR) * (TEMP5 * UU)) / (TEMP5 * VAR)
HU = (XBU - TEMP4) * DEXP(-((TEMP3 - UU) ** 2) / (2 * (TEMP5 * VAR)))

FB = HU
WRITE(6, *) 'FB VALUE = ', FB
RETURN
END

SUBROUTINE PRIORCOST
INCLUDE 'COM2.VAR'
DOUBLE PRECISION XBU, PGU, M1, Z
COMMON /BLOCK2/XBU, PGU, M1, Z
DOUBLE PRECISION KE1, KE2, KR, KJ, KC, KV
DOUBLE PRECISION C1, ECC, PO, C2, ECC, PO, C3, ECC, PO, C4, ECC, PO
DOUBLE PRECISION CC1, ECC1, PO, CC2, ECC2, CC3, ECC3, CC4, ECC, PO
COMMON /BLOCK6/C1, ECC, PO, C2, ECC, PO, C3, ECC, PO, C4, ECC, PO
COMMON /BLOCK6/CC1, ECC1, PO, CC2, ECC2, CC3, ECC3, CC4, ECC, PO
REAL PIE, S2PIE
COMMON /BLOCK7/ PIE, S2PIE
CALL PRIORCOSTDATIN
PIE = 3.14159
S2PIE = SQRT(2 * 3.14159)
CALL PRIORCOSTCOMPUTE
CALL PRIORCOSTDATOUT
CALL PRIORCOSTRESULTS
RETURN
END

SUBROUTINE PRIORCOSTDATIN
DOUBLE PRECISION KE1, KE2, KR, KJ, KC, KV
DOUBLE PRECISION C1, ECC, PO, C2, ECC, PO, C3, ECC, PO, C4, ECC, PO
DOUBLE PRECISION CC1, ECC1, PO, CC2, ECC2, CC3, ECC3, PO, CC4, ECC, PO
COMMON C

COMMON /BLOCKS/ KE1, KE2, KR, KJ, KC, KV
COMMON /BLOCKS/ C1, ECI, PU1, C2, ECI2, C3, ECI3, PO1, ECI4, PO2, ECI5, PO3
COMMON /BLOCKS/ CC1, ECC1, PO1, CC2, ECC2, CC3, ECC3, PO2, CC4, ECC4, PO3

WRITE(6,*) 'ENTER SAMPLING ERROR-TYPE1 KE1 ?'
READ(5,*) KE1

WRITE(6,*) 'ENTER SAMPLING ERROR-TYPE2 KE2 ?'
READ(5,*) KE2

WRITE(6,*) 'ENTER SALES PRICE KR ?'
READ(5,*) KR

WRITE(6,*) 'ENTER JUNK VALUE KJ ?'
READ(5,*) KJ

WRITE(6,*) 'ENTER RATE REWORKING WITH SUCCESS KC ?'
READ(5,*) KC

WRITE(6,*) 'ENTER PROPORTION OF ITEMS REWORKED WITHOUT SUCCESS KV ?'
READ(5,*) KV

RETURN

END

DOUBLE PRECISION FUNCTION FAC(N)
DOUBLE PRECISION NN
IF (N.EQ.0) THEN
FAC=1
ELSE
NN=1.0
I=N
DO WHILE (I.GT.1)
NN=NN*I
I=I-1
END DO
ENDIF
FAC=NN
RETURN
END

DOUBLE PRECISION FUNCTION LL(P)
REAL PIE, S2PIE
COMMON /BLOCK7/ PIE, S2PIE
DOUBLE PRECISION P, FXP

FXP=P-0.5
N1=3
N2=5
N3=7
N4=9
N5=11
LL=S2PIE*FXP+((S2PIE**3)*(FXP**3)/FAC(N1))
&+(7*(S2PIE**5)*(FXP**5)/FAC(N2))
&+(127*(S2PIE**7)*(FXP**7)/FAC(N3))
&+(4369*(S2PIE**9)*(FXP**9)/FAC(N4))
&+(318493*(S2PIE**11)*(FXP**11)/FAC(N5))
WRITE(6,*) 'funct LL = ', LL
RETURN
COMMON /BLOCK2/ XBU,PGU,M1,Z
DOUBLE PRECISION KE1,KE2,KR,KJ,KC,KV
DOUBLE PRECISION C1,EC1,P01,C2,EC2,C3,EC3,P02,C4,ECR,P03
DOUBLE PRECISION C1,EC1,P01,C2,EC2,C3,EC3,P02,C4,ECR,P03
COMMON /BLOCK5/ KE1,KE2,KR,KJ,KC,KV
COMMON /BLOCK6/ C1,EC1,P01,C2,EC2,C3,EC3,P02,C4,ECR,P03
COMMON /BLOCK6/ C1,EC1,P01,C2,EC2,C3,EC3,P02,C4,ECR,P03
REAL PIE,S2PIE
COMMON /BLOCK7/ PIE,S2PIE
DOUBLE PRECISION A1,G1,G2
DOUBLE PRECISION LL,LP

A1=(CC3/G2)/G1
A2=VARA/VAR

EC3KRN=(C3/G2)/(LP(A1))/(S2PIE*DSORT(A2)))
& *(EXP(-(0.5/A2)*((L-M1)/DSORT(VAR))-(LL(A1))))**2)))
& *G2
RETURN
END

C
DOUBLE PRECISION FUNCTION EC3KRN(CC3)
DOUBLE PRECISION FUNCTION EC3KRN(C3)
DOUBLE PRECISION FUNCTION EC3KRN(C3)
DOUBLE PRECISION FUNCTION EC3KRN(C3)
DOUBLE PRECISION FUNCTION EC3KRN(C3)
DOUBLE PRECISION FUNCTION EC3KRN(C3)
DOUBLE PRECISION FUNCTION EC3KRN(C3)
DOUBLE PRECISION FUNCTION EC3KRN(C3)

SUBROUTINE PRIORCOSTCOMPUTE
INCLUDE 'COM2,VAR'
DOUBLE PRECISION XBU,PGU,M1,Z
DOUBLE PRECISION KE1,KE2,KR,KJ,KC,KV
DOUBLE PRECISION C1,EC1,P01,C2,EC2,C3,EC3,P02,C4,ECR,P03
DOUBLE PRECISION C1,EC1,P01,C2,EC2,C3,EC3,P02,C4,ECR,P03
DOUBLE PRECISION C1,EC1,P01,C2,EC2,C3,EC3,P02,C4,ECR,P03
DOUBLE PRECISION C1,EC1,P01,C2,EC2,C3,EC3,P02,C4,ECR,P03
DOUBLE PRECISION C1,EC1,P01,C2,EC2,C3,EC3,P02,C4,ECR,P03
DOUBLE PRECISION C1,EC1,P01,C2,EC2,C3,EC3,P02,C4,ECR,P03
DOUBLE PRECISION C1,EC1,P01,C2,EC2,C3,EC3,P02,C4,ECR,P03
DOUBLE PRECISION C1,EC1,P01,C2,EC2,C3,EC3,P02,C4,ECR,P03
DOUBLE PRECISION C1,EC1,P01,C2,EC2,C3,EC3,P02,C4,ECR,P03
DOUBLE PRECISION C1,EC1,P01,C2,EC2,C3,EC3,P02,C4,ECR,P03
DOUBLE PRECISION C1,EC1,P01,C2,EC2,C3,EC3,P02,C4,ECR,P03
DOUBLE PRECISION C1,EC1,P01,C2,EC2,C3,EC3,P02,C4,ECR,P03
DOUBLE PRECISION C1,EC1,P01,C2,EC2,C3,EC3,P02,C4,ECR,P03
DOUBLE PRECISION C1,EC1,P01,C2,EC2,C3,EC3,P02,C4,ECR,P03
DOUBLE PRECISION C1,EC1,P01,C2,EC2,C3,EC3,P02,C4,ECR,P03
DOUBLE PRECISION C1,EC1,P01,C2,EC2,C3,EC3,P02,C4,ECR,P03
DOUBLE PRECISION C1,EC1,P01,C2,EC2,C3,EC3,P02,C4,ECR,P03
DOUBLE PRECISION C1,EC1,P01,C2,EC2,C3,EC3,P02,C4,ECR,P03
DOUBLE PRECISION C1,EC1,P01,C2,EC2,C3,EC3,P02,C4,ECR,P03
DOUBLE PRECISION C1,EC1,P01,C2,EC2,C3,EC3,P02,C4,ECR,P03
DOUBLE PRECISION C1,EC1,P01,C2,EC2,C3,EC3,P02,C4,ECR,P03
DOUBLE PRECISION C1,EC1,P01,C2,EC2,C3,EC3,P02,C4,ECR,P03
DOUBLE PRECISION C1,EC1,P01,C2,EC2,C3,EC3,P02,C4,ECR,P03
DOUBLE PRECISION C1,EC1,P01,C2,EC2,C3,EC3,P02,C4,ECR,P03
DOUBLE PRECISION C1,EC1,P01,C2,EC2,C3,EC3,P02,C4,ECR,P03
DOUBLE PRECISION C1,EC1,P01,C2,EC2,C3,EC3,P02,C4,ECR,P03
DOUBLE PRECISION C1,EC1,P01,C2,EC2,C3,EC3,P02,C4,ECR,P03
DOUBLE PRECISION C1,EC1,P01,C2,EC2,C3,EC3,P02,C4,ECR,P03
DOUBLE PRECISION C1,EC1,P01,C2,EC2,C3,EC3,P02,C4,ECR,P03
DOUBLE PRECISION KBU, PGU, MI, Z

COMMON /BLOCK2/ KBU, PGU, MI, Z
DOUBLE PRECISION KE1, KE2, KR, KJ, KC, KV

C COMMON /BLOCK2/ C1, EC1, PO1, C2, EC2, C3, EC3, PO2, C4, ECR, PO3
DOUBLE PRECISION C1, EC1, PO1, C2, EC2, C3, EC3, PO2, C4, ECR, PO3

CALL SIMPS(LOW, UP, MINT, ECC1)
CC1 = CA * P

CALL SIMPS(LOW, UP, MINT, ECC1)
CC1 = CA * P

CALL SIMPS(LOW, UP, MINT, ECC3)
CC3 = CA * P

CALL SIMPS(LOW, UP, MINT, ECR)
CC4 = GI * P + G2
RETURN
END
WRITE(99,3) FORMAT(1H1)
WRITE(99,1)
FORMAT(/BX,'*******************************')/
& RX, 'TABLE I: PRIOR COST AND RELATED SPECIFICATIONS'/
& RX, '*******************************')/
WRITE(99,2) L,VAR,VARA,CSN,CA,KE1,KE2,KR,KJ,KC,KY
FORMAT(/BX, 'INPUT: MODEL SPECIFICATIONS'//
& RX, '-------------------------------------------'//
& RX, 'LOWER LIMIT OF THE Q.C.X: ',F15.5/
& RX, 'VARIANCE OF X: ',F15.5/
& RX, 'VARIANCE OF THE MEAN OF X: ',F15.5/
& RX, 'UNIT COST OF SCREENING CSN: ',F15.5/
& RX, 'UNIT COST OF ACCEPTANCE KA: ',F15.5/
& RX, 'MEAN: ',F15.5/
& RX, 'SAMPLING ERROR (TYPE1) KE1: ',F15.5/
& RX, 'SAMPLING ERROR (TYPE2) KE2: ',F15.5/
& RX, 'SALE PRICE KR: ',F15.5/
& RX, 'JUNK VALUE KJ: ',F15.5/
& RX, 'RATIO OF REWORKING DEFECTIVES: /
& RX, 'ITEMS RE WITH SUCCESS KC: ',F15.5/
& RX, 'PROPORTION OF ITEMS RE: /
& RX, 'WITHOUT SUCCESS KY: ',F15.5/) RETURN
END

SUBROUTINE PRIORCOSTRESULTS
INCLUDE 'I.\02_VAR'
DOUBLE PRECISION XBU, P01, I, Z
COMMON /BLOCK2/ XBU, P01, I, Z
DOUBLE PRECISION KE1, KE2, KR, KJ, KC, KY
C DOUBLE PRECISION CI, EC1, POI, C2, EC2, C3, EC3, PO2, C4, ECR, P03
DOUBLE PRECISION CCI, ECC1, POI, C22, ECC2, C33, ECC3, PO2, C44, ECR, P03
COMMON /BLOCK5/ KE1, KE2, KR, KJ, KC, KY
C COMMON /BLOCK6/ C1, ECC1, PO1, C2, EC2, C3, EC3, PO2, C4, ECR, P03
COMMON /BLOCK7/ PI, PIE, S2PIE
REAL PI, S2PIE
C WRITE(99,1) CI, ECC1, PO1, C2, EC2, C3, EC3, PO2, C4, ECR, P03
C WRITE(99,1) CCI, ECC1, PO1, C22, ECC2, C33, ECC3, PO2, C44, ECR, P03
WRITE(99,1) EC1, PO1, C22, ECC2, C33, PO2, C44, ECR, P03
FORMAT(/BX,'OUTPUT:PRIOR AND EXPECTED COSTS PER UNIT'//
& BX, 'COST PER UNIT-ACCEPT OUTRIGHT' '//
& BX, 'COST PER UNIT-REJECT OUTRIGHT AND SCRAP C2:' '/
& BX, 'COST PER UNIT-REJECT OUTRIGHT' '//
& BX, 'COST PER UNIT-REJECT OUTRIGHT AND SCREEN C3:' '/
& BX, 'COST PER UNIT-REJECT OUTRIGHT' '//
& BX, 'COST PER UNIT-ACCEPT OUTRIGHT'//
& BX, 'COST PER UNIT-ACCEPT OUTRIGHT E(C1): ',F15.5/
& BX, 'E(C1)=E(C2) AT PO1: ',F15.5/
& BX, 'COST PER UNIT-ACCEPT OUTRIGHT'//
& BX, 'COST PER UNIT-ACCEPT OUTRIGHT E(C3): ',F15.5/
& BX, 'COST PER UNIT-ACCEPT OUTRIGHT E(C1)=E(C3) AT PO2: ',F15.5/
**SUBROUTINE SIMP5(A,B,M,E)**

*FUNCTION: APPROXIMATES THE INTEGRAL OF F(X) OVER THE INTERVAL [A,B] BY SIMPSON'S RULE*  

**PARAMETERS:**  
- **A** = INTERVAL LEFT ENDPOINT 
- **B** = INTERVAL RIGHT ENDPOINT 
- **M** = NUMBER OF SUBINTERVALS (POSITIVE EVEN INTEGER) 

**EXTERNAL FUNCTIONS/SUBROUTINES:** FUNCTION F(X)

**OUTPUT:** 
- **E** = ESTIMATE OF THE INTEGRAL

**DOUBLE PRECISION** A,B,E,X,H,ECKRN  

**INITIALIZATION** 

```vbnet
IF (B .LT. -4.0 OR. A .GT. 4.0) THEN  
  E=0.0000000000001  
  RETURN
ENDIF
```

**COMPUTE THE SUM OF ODD INDEX TERMS**

```vbnet
DO 1 I=1,M-1,2  
  X=X+2.0*M  
  E=E+4.0*ECKRN(X)  
  WRITE(6,9) 'simp5 odd terms e =',E  
1 CONTINUE
```

**COMPUTE THE SUM OF EVEN INDEX TERMS**

```vbnet
X=A  
DO 2 J=2,M-2,2  
  X=X+2.0*M  
  E=E+2.0*ECKRN(X)  
  WRITE(6,9) 'simp5 even terms e =',E  
2 CONTINUE
```

```vbnet
E=E*M/3.0  
RETURN
END
```

**SUBROUTINE SIMP6(A,B,M,E)**
C FUNCTION: APPROXIMATES THE INTEGRAL OF F(X) OVER THE
C INTERVAL [A,B] BY SIMPSON'S RULE
C USAGE:
C CALL SEQUENCE: CALL SIMP(A,B,M,E)
C EXTERNAL FUNCTIONS/SUBROUTINES: FUNCTION F(X)
C PARAMETERS:
C INPUT: 
A=INTERVAL LEFT ENDPOINT
B=INTERVAL RIGHT ENDPOINT
M=NUMBER OF SUBINTERVALS(POSITIVE EVEN INTEGER)
C OUTPUT: 
E=ESTIMATE OF THE INTEGRAL

C DOUBLE PRECISION A,B,E,X,M,EC3KRN
C *** INITIALIZATION ***
C IF (B .LT.-4 .OR. A .GT. 4)THEN
C E=0.0000000000001
C RETURN
C ENDIF
C E=EC3KRN(A)+EC3KRN(B)
C M=(B-A)/M
C X=A
C *** COMPUTE THE SUM OF ODD INDEX TERMS ***
C DO 1 I=1,M-1,2
C X=X+2.0*M
C E=E+4.0*EC3KRN(X)
C 1 CONTINUE
C *** COMPUTE THE SUM OF EVEN INDEX TERMS ***
C X=A
C DO 2 I=2,M-2,2
C X=X+2.0*M
C 2 CONTINUE
C E=E*M/3.0
C RETURN
C END

SUBROUTINE SIMP7(A,B,M,E)
C FUNCTION: APPROXIMATES THE INTEGRAL OF F(X) OVER THE
C INTERVAL [A,B] BY SIMPSON'S RULE
C USAGE:
C CALL SEQUENCE: CALL SIMP(A,B,M,E)
C EXTERNAL FUNCTIONS/SUBROUTINES: FUNCTION F(X)
C PARAMETERS:
C INPUT: 
A=INTERVAL LEFT ENDPOINT
B=INTERVAL RIGHT ENDPOINT
M=NUMBER OF SUBINTERVALS(POSITIVE EVEN INTEGER)
C OUTPUT: 
E=ESTIMATE OF THE INTEGRAL
C  DOUBLE PRECISION A,B,E,X,M,ECRKN
C  *** INITIALIZATION ***
C  IF (B .LT. -4 .OR. A .GT. 4) THEN
C     E=0.0000000000001
C  RETURN
C  ENDIF
C     E=ECRKN(A)+ECRKN(B)
C     M=(B-A)/M
C     X=A-H
C  *** COMPUTE THE SUM OF ODD INDEX TERMS ***
C  DO 1 I=1,M-1,2
C     X=X+2.0*H
C     E=E+4.0*ECRKN(X)
C  CONTINUE
C  *** COMPUTE THE SUM OF EVEN INDEX TERMS ***
C  DO 2 I=2,M-2,2
C     X=X+2.0*H
C     E=E+2.0*ECRKN(X)
C  CONTINUE
C     E=E*M/3.0
C  RETURN
C  END
SUBROUTINE READ2
INCLUDE 'COM2.VAR'

WRITE (6,1)
FORMAT('I')
WRITE(6,*) 'IS THE VALUE IN THE FILE...ENTER 0 OR 1'
READ(S,*) ANS
ANs=1
IF (ANS .EQ. 1) GOTO 100
WRITE(6,*) 'ENTER VALUES FOR THE FOLLOWING VARIABLES'
WRITE(6,*)
2 FORMAT(1X, 'U=', S)
READ(S,1U)
WRITE(6,3)
FORMAT(1X, 'L=', S)
READ(S,1L)
WRITE(6,4)
FORMAT(1X, 'VAR=', S)
READ(S,VAR)
WRITE(6,5)
FORMAT(1X, 'VARA=', S)
READ(S,VARA)
WRITE(6,6)
FORMAT(1X, 'CSN=', S)
READ(S,CSN)
WRITE(6,7)
FORMAT(1X, 'CA=', S)
READ(S,CA)
WRITE(6,8)
FORMAT(1X, 'CP=', S)
READ(S,CP)
WRITE(6,9)
FORMAT(1X, 'LSS=', S)
READ(S,LSS)
WRITE(6,10)
FORMAT(1X, 'CSSP=', S)
READ(S,CSSP)
WRITE (10,*) U, L, VAR, VARA, CSN, CA, CP, LSS, CSSP
RETURN
100 READ (10,*) U, L, VAR, VARA, CSN, CA, CP, LSS, CSSP
RETURN
END
SUBROUTINE HEAD?
INITLUE '((UM2,VAR'
WRITE (99,1)
1 FORMAT(//Bx, '*******************************************************/
 & Bx, 'TABLE 2: ACCEPTANCE COST AND RELATED SPECIFICATIONS'
 & Bx, '*******************************************************/
WRITE (99,2)U.L,VAR,VARA,CSN,CA,CP,CS5,-LSS
2 FORMAT(//Bx, 'INPUT: MODEL SPECIFICATIONS'/
 & Bx, '--------------------------------------------------'/
 & Bx, 'UPPER LIMIT OF THE Q.C.X: ',F15.5/
 & Bx, 'LOWER LIMIT OF THE Q.C.X: ',F15.5/
 & Bx, 'VARIANCE OF X: ',F15.5/
 & Bx, 'VARIANCE OF THE MEAN OF X: ',F15.5/
 & Bx, 'UNIT COST OF SCREENING: ',F15.5/
 & Bx, 'UNIT COST OF ACCEPTANCE: ',F15.5/
 & Bx, 'COST OF SCRAPING OR REPLACING: '
 & Bx, 'A DEFECTIVE UNIT FOUND DURING: '
 & Bx, 'UNIT COST OF SCRAPPING: ',F15.5/
 & Bx, 'LOT SIZE: ',19/
WRITE(99,3)
3 FORMAT(//Bx, 'OUTPUT: SAMPLE SIZE, ROOTS OF THE COST FUNCTION.'/
 & Bx, 'POSTERIOR AND SAMPLING COSTS PER UNIT'/
 & Bx, '--------------------------------------------------'/
 & Bx, '--------
 & Bx, '1 SAMPLE SIZE /
 & Bx, '2 (a + b) / (b + n a) WHERE: a: VARIANCE OF THE MEAN OF X'/
 & Bx, '3: b: VARIANCE OF X /
 & Bx, '4: n: SAMPLE SIZE /
 & Bx, '5 LOWER DISPOSITION LIMIT FOR LOT ACCEPTANCE /
 & Bx, '6 LOWER DISPOSITION LIMIT OF THE SAMPLE MEAN /
 & Bx, '7 ACCEPTANCE COST PER LOT /
WRITE(99,4)
C FORMAT(Bx,95('-.')/}
C & T12,'1',T23,'2',T38,'3',T53,'4',T68,'5',T63,'6',T98,'7'/
C & BX,95('-.')/}
END
SUBROUTINE HCA2A
INCLUDE 'COM2,VAR'
WRITE(99,4)
FORMAT(8X,95('-')/
& T12,'1',T23,'2',T38,'3',T53,'4',T68,'5',T83,'6',T9A,'7'/
& 8X,95('-')/)
END

C SUBROUTINE COM,VAR
******************************************************************************
C THIS SUBROUTINE INCLUDES ALL THE CONSTANT AND VARIABLES
C THAT ARE EMPLOYED IN THIS PROGRAM
******************************************************************************

DOUBLE PRECISION U,L,VAR,VARA,VARNM,CR,CE,MN1,MN2,XBN1,XBNN2,
& CSN,CA,CP,CSSP,NS,MO
COMMON /BLOCK1/ U,L,VAR,VARA,VARNM,CR,CE,CS,LS,MN1,MN2,
& VN,SIGN,CSN,CA,CP,LSS,CSSP,N,NS,MO
PARAMETER (PI=3.1459)
SUBROUTINE NDMRI1(N,A,B,H,AL,M,E,F01,AINT,KEY)
REAL*4 FF1
REAL*8 A(N),B(N),C(9),K(15),M(N),G(9),NN(9),P(9),AA(15)
* .V(15),X(9),D(9)
data k(1),k(2)/1.2/
data tol /1.E-07/
KEY=0
write(6,*), ' Test Pt NDMRI1 1'
IF(N.LT.1.OR.N.GT.9.OR.M.LT.1.OR.AL.LT.1.5.OR.AL.GT.2.) RETURN
write(6,*), ' Test Pt NDMRI1 2'
DO 1 I=1,N
write(6,*), ' Test Pt NDMRI1 3'
write(6,*), ' I = ',I,' M(I) = ',M(I)
IF(M(I).LE.0.,OR.,M(I).GT.1.) RETURN
write(6,*), ' Test Pt NDMRI1 4'
2 D(I)=A(I)
EE=AMAX1(E,TOL)
MM=MIN(M,15)
DO 1 I=1,8
P(I+1)=0.
NN(I+1)=1
1 D(I+1)=0.
10 L=1
DO 20 I=1,N
20 C(I)=B(I)-D(I)
21 U=0.
KT=0
DO 22 I=1,N
22 G(I)=M(I)/K(L)
23 NN(I)=1./G(I)+.5
38 P(I)=C(I)*G(I)
39 NNB=NN(9)
DO 30 I=1,NNB
X(I)=D(I)+P(I)*(19.-.5)
30 NN9=NN(8)
DO 30 I=1,NN9
X(I)=D(I)+P(I)*(18.-.5)
35 NN7=NN(7)
DO 30 I=1,NN7
X(I)=D(I)+P(I)*(17.-.5)
36 NN6=NN(6)
DO 30 I=1,NN6
X(I)=D(I)+P(I)*(16.-.5)
34 NN5=NN(5)
DO 30 I=1,NN5
X(I)=D(I)+P(I)*(15.-.5)
33 NN4=NN(4)
DO 30 I=1,NN4
**Fortran Program**

```fortran
32  X(4)=I(4)*P(4)*(14-.5)
31  NNI=NN(3)
   DO 30 13=1,NN3
30  X(3)=D(3)+P(3)*(13-.5)
31  NNI=NN(2)
   DO 30 12=1,NN2
30  X(2)=D(2)+P(2)*(12-.5)
31  NNI=NN(1)
   DO 30 11=1,NN1
30  X(1)=D(1)+P(1)*(11-.5)
   WRITE(6,*) 'U = ',U
30  U=U*F1(X,N)
   DO 40 1'=1,N
40  U=U*P(I)
   V(L)=U
   IF(L-1) 43,43,44
43  AA(1)=V(1)
   L=L+1
   GO TO 21
44  EN=K(L)
   DO45 LL=2,L
   J=L-1-LL
45  V(I)=V(I-1)+(V(I-1)-V(I))/((EN/K(I))*2-1.)
   AINT=V(I)
   PRINT, ' AINT = ', AINT
   KEY=1
   IF(ABS(AINT-AA(L-1)).LT.ABS(AINT*EE)) RETURN
   KEY=1
   IF(L.EQ.MM) RETURN
   AA(L)=AINT
   L=L+1
   K(L)=AL*K(L-1)
   GO TO 21
ENDE
SUBROUTINE NDMR12(N,A,B,H,AL,M,E,FF2,AINT,KEY)
   REAL*8 FF2
   REAL*8 A(n),b(n),c(9),h(n),g(9),nn(9),p(9),aa(15),V(15)
   * .x(9),d(9)
   data k(1),k(2)/1,2/
   data tol '/1.E-07/
   KEY=0
   WRITE(6,*) ' Test Pt NDMR12 1'
   IF(N.LT.1.OR.N.GT.9,OR.M.LT.1.OR.AL.LT.1.5,OR.AL.GT.2.) RETURN
   WRITE(6,*) ' Test Pt NDMR12 2'
   DO 2 I=1,N
   WRITE(6,*) ' Test Pt NDMR12 3'
   IF(H(I).LE.0..OR..H(I).GT.1.) RETURN
   WRITE(6,*) ' Test Pt NDMR12 4'
   D(I)=A(I)
   EE=AMAX1(E,TOL)
   M=M/N(N,15)
   DO 1 I=1,N,8
   P(I+1)=0.
   NN(I+1)=1
```

\[ D(1+1) = 0. \]

DO 20 J = 1, N

10 \[ C(1) = B(1) - D(1) \]

DO 20 J = 1, N

20 \[ U = U \cdot \times 0 \]

DO 22 J = 1, N

\[ G(1) = M(1) / K(L) \]

DO 30 J = 1, N

30 \[ X(I) = O(1) \]

DO 30 J = 1, N

32 \[ X(I) = P(I) + Q(I) \]

DO 40 J = 1, N

40 \[ U = U + F(z, N) \]

DO 40 J = 1, N

40 \[ U = U \cdot P(I) \]

V(L) = U

IF(L - 1) = 43, 43, 44

AA(1) = V(1)

L = L + 1

GO TO 21

EN = K(L)

DO 45 L = 1, 2, L

45 \[ V(1) = V(I) + (V(I+1) - V(I)) \times ((E/N(K(1))) ** 2 - 1.) \]

KEY = 1

IF(AINT = AA(L - 1)) LT, ABS(AINT * EE)) RETURN

KEY = -1
IF(L.EQ.MM) RETURN
AA(L)=AIRT
L=L+1
K(L)=AL*K(L-1)
GO TO 21
END
THIS PROGRAM IS EMPLOYED FOR PLOTTING ONLY
IN RUNNING THIS PROGRAM THE FOLLOWING STEPS MUST BE FOLLOWED
1. START BY TYPING YOUR DATA
2. DEFINE THE EXECUTION FILE AND INSERT IT IN THE PROGRAM TO
   OPEN UNIT 15 SEE BELOW:
   OPEN UNIT 15, FILE = 'FILE NAME', STATUS = 'OLD')
3. FOLLOW THE SAME FORMAT OF OF THE WRITE STATEMENT OF THE
   PROGRAM THAT YOU COPYING FROM
4. SPECIFY THE RANGE OF VALUES FOR X AND Y CORRECTLY
5. IN MODIFYING THE VALUES OF X AND Y DELETE THE EXECUTION
   FILE CREATED BEFORE RUNNING THE PROGRAM WITH A NEW
   SET OF DATA

REAL N0(100), D2A(100), ETCOST(100), PGU(100), ECSTAC(100)
REAL X'GK101, LACCP101)
DIMENSION IBUF(10)
OPEN UNIT = 15, FILE = 'KUMU.A', STATUS = OLD)
DATA (NH(1), 11.4, 11.1, 13.1, 14.1)
DATA (NH(111), 11.6, 6.747, 6.70, 6.77, 6.787, 6.793)/

DEFINES THE TOTAL NUMBER OF POINTS TO BE PLOTTED**

* READ (15, *, N=1:1111, Mn=1111, MN2(II), XBU(II), XBL(II)

* THIS REAL STATEMENT IS THE SAME DEFINED IN THE MAIN PROGRAM

* THE FORMAT STATEMENT IS THE SAME DEFINED IN THE MAIN PROGRAM

V=(NPTS) V=1.0 VORX=1.0

VF=0

CALL PLOT(X, Y, MN1, MN2, XBU, XBL, ECSTAC, XOR1, VOR1, IBUF, VL, VREJ, ECF)
CALL ENDPL(0)
CALL DUNEPL(0)
STOP
END

SUBROUTINE PLOTIN, VARN, MN1, MN2, XBU, XBL, ECSTAC, XOR, YOR, IBUF
SUBROUTINE PLOTIN(MN1, MN2, XBU, XBL, ECSTAC, XOR, YOR, IBUF

REAL NO(100), DA(100), PGU(100)
REAL ECSTAC(100), ETCOST(100)
DIMENSION N(100), VARN(100), MN1(100), MN2(100), XBU(100), XBL(100)
DIMENSION NO(100), DA(100), ECSTAC(100), PGU(100), ETCOST(100)
REAL ECSTAC(100), ETCOST(100), PGU(100)
DIMENSION IBUF(16)

CALL RESET('ALL')
CALL NOBDR
CALL BSSCALE(1.0, 1.0)
CALL PLOTAREA(XLONG, YLONG)
CALL FRAME

CALL PNAME; ACCEPTANCE COST PER LOT $ .100
CALL NAME('SAMPLE SIZE $ .100')

IN THE SUBROUTINE PLOT 252 IS THE LOWER LIMIT OF ETCOST
ETCOST IS INCREASED BY NINE UNITS EACH TIME
'10 IS THE MINIMUM SAMPLE SIZE TO BE SELECTED
THE SAMPLE SIZE IS INCREASED BY FIVE UNITS EACH TIME
THE MAXIMUM SAMPLE SIZE IS THE MAXIMUM SAMPLE SIZE

CALL PLOTFILE(252, 0.0, 0.15, 10.5, 100)
CALL MARKER(15)
CALL CURVE(11)
CALL CURVE(11)
CALL RESET('MARKER')
CALL DOT

THE CURVE TO BE PLOTTED IN THIS CASE IS ETCOST VS. SAMPLE
SIZE NO
MAXIMUM NUMBER OF POINTS TO BE PLOTTED IS 90 SINCE THE
MAXIMUM NUMBER OF POINT IS 100 AND THE MINIMUM VALUE IS 10
THE VALUES OF N ARE: 10, 15, 20, ... ETC (N IS INCREASED BY 5

CALL CURVE('DOT', NO, 90.5)
CALL RESET('MARKER')
CALL RESET('DOT')
END IF
CALL DOUT
CALL DOT
CALL CURVE('DOT', 11)
CALL RESET('MARKER')
CALL DOUT
END IF
CALL DOUT
CALL CURVE('DOT', 11)
CALL RESET('MARKER')
CALL RESULT('DOT')
END IF

IF (I .EQ. 4) THEN
  CALL DOT
  CALL CURVE(VT, PEF, 11, -1)
  CALL RESET('MARKER')
  CALL RESET('DOT')
END IF

IF (I .EQ. 5) THEN
  CALL CURVE(VE, GE, 101, 0)
END IF

IF (I .EQ. 6) THEN
  CALL CURVE(VE, TE, 101, 0)
END IF

IF (I .EQ. 7) THEN
  CALL CURVE(VE, EE, 101, 0)
END IF

CONTINUE

CALL ENDGR(0)
RETURN
END
Input Data:

Same as in Appendix D, where the program 2AA is employed as a subprogram.

Output 1:

Same as in Appendix D, except that the decision points are those for lot screening and the cost is that of screening per lot, also the value of $p$ at which the costs of acceptance and screening is evaluated.

Output 2:

Same as in Appendix D, the estimates are based on the decision points for lot screening where the exact value of the cost of screening per unit is evaluated.
PROGRAM PROG3

**FUNCTION:**
This program computes the screening cost per lot. Also, it provides values for the upper and lower disposition limits for lot acceptance. This cost is verified to be a minimum by studying the sign of the second derivative of the total cost relative to the upper and lower disposition limits.

**USAGE:**
Call sequence: RUN PROG3AA (PRESS) RETURN

**EXTERNAL FUNCTIONS:**
- G(X)
- FF1(X,NN)
- FF2(X,NN)
- F(T)
- FL(XBAR)
- FR(UU)
- FB(UU)

**EXTERNAL SUBROUTINES:**
- READ2
- HEAD2
- SEC2
- DRMII
- DRMI2
- OSINTG2

**ASSUMPTIONS:**
1. The distributions are normal.
2. The integral of X given the mean MU is equal to the integral of T given XBAR and the mean MU.
3. The statistical process can exist in one state.

**PARAMETERS:**
**INPUT:**
- L=lower specification limit of quality characteristics X
- U=upper specification limit of quality characteristics X
- LSS=lot size
- N=sample size
- X=product dimension or a quality characteristic
- XBAR=a sample mean
- MU=the mean of the variable X relative to a certain lot
- MU1=the mean of the mean MU
- VAR=lot variance
- CI=cost of inspection per unit of a product
- CSN=cost of screening inspection per unit of a product
- CSSP=cost of scrapping per unit of a product
- CA=cost of acceptance per unit of a product
- CP=cost of scrapping and replacing a defective unit during sampling or screening inspection.

**OUTPUT:**
1. Sample n
2. Variance of the joint distribution of the mean MU and the distribution of XBAR given MU
3. Lower limit generated by taking the derivative of the total cost relative to LA
4. Upper limit generated by taking the derivative of the total cost relative to UA
5. Upper disposition limit of lot acceptance
6. Lower disposition limit of lot acceptance
7. The approximate cost of acceptance per lot
8. The exact value of the cost given in 7
9. The value of the second derivative relative to each N
10. The value of pgw which is the integral of
   the probability density function of x given
   mu between the limits l.u.

Calculation procedures:
1. An expression for the total cost is derived
2. The first derivative of the total cost is taken relative to the following parameters:
   a) Upper disposition limit of lot acceptance
   b) Lower disposition limit of lot acceptance
3. The partial derivatives are equated to zero.
4. The four equations obtained are solved for ua, la, usn, lsn.
5. The cost of acceptance relative to a certain sample mean xbar and mean mu is equated
   with the cost of screening at xbar=la, or xbar=ua. Solving this expression yields a
   value for the integral of x given mu.
6. The total cost of acceptance is obtained by summing over all sample means and population means.

Global declarations

**main program - begin**

real aa(3),bb(3),m(3),a,b,x,err
call read2
call head2
write(6,*),'cost of inspection?'
read(5,*),ci
c
write(6,*),'starting sample size?'
read(5,*),isam
c
write(6,*),'enter m value?'
read(5,*),m1
c
call d2head
call d2ahead
DO 10 N=ISAM+3*LSS
VARN = (VARA * VAR) / (VAR + N * VARA)

*** COMPUTE INITIAL VALUE OF MN ***

ERR=1.0E-09
A=L-2
B=U
CALL SECA(A,B,X,ERR)
MN=X
MO=(U+L)/2
V=ABS(MN-MO)
MN1=MO-V
MN2=MO+V

READ(66,*) XBU2AA

XBU=(MN2*(VARA+N*VARA)-(L+U)*VAR/2.)/(N*VARA)
XBL=(MN1*(VARA+N*VARA)-(L+U)*VAR/2.)/(N*VARA)
PGU=(LSS*(CSN+CP-CSSP)-N*CSN)/(LSS*CP-N*CSP)
PGUi=CSN/(CA-CP)
NITR=20
NO=N
CALL SIMP3(XBU2AA,XBU,NITR,EVAL)

WRITE(30,*) 'DENSITY PRODUCT FACTOR' , EVAL

*** COMPUTE COSTS ***

ECSTAC=CI*N + CSN*(LSS-N) + CP*LSS*(1-PGU)
ETNOST=ECSTAC*EVAL

******************************************************************************

* NOTE: THE FOLLOWING PORTION RELATES TO TRIPLE INTEGRATION*

******************************************************************************

NN=3
AL=1.5
EE=1.0E-01
H(1)=.6
H(2)=.6
H(3)=.6
AA(1)=L
AA(2)=L-2
AA(3)=L
BB(1)=U
BB(2)=U+2
BB(3)=U
AA2=AA(2)
AA2=L
AA3=AA(3)
AA3=L
BB2=BB(2)
BB2=U
BB3=BB(3)
BB3=U
NOTE: THE FOLLOWING SUBROUTINE COMPUTES AN APPROXIMATION TO THE N-DIMENSIONAL INTEGRAL OVER A PARALLELEPIPED A ROMBERG TYPE METHOD BASED ON THE MID-POINT RULE IS EMPLOYED, EACH INTERVAL IS SUBDIVIDED INTO [1/(H(I))] SUBINTERVALS, D*(H(I))<1.1=1,...,N. M IS THE MAXIMUM NUMBER OF ITERATIONS ALLOWED AND SHOULD DECREASE WITH INCREASING N SINCE THE NUMBER OF FUNCTIONAL EVALUATIONS IN THE JTH ITERATION IS AT LEAST J**2IF M IS SET >15, IT IS RESET TO 15. AL IS A CONSTANT BETWEEN 1.5 AND 2, PREFERABLY 1.5 WHICH REGULATES THE INCREASE IN THE NUMBER OF POINTS USED. E IS A TOLERANCE WHICH SHOULD INCREASE AS N INCREASES KEY IS SET EQUAL TO 1, IF THIS IS NOT ACHIVED WITHIN M ITERATIONS, KEY IS SET EQUAL TO -1, IF N<1, THE SUBROUTINE EXITS WITH KEY=0, A PROGRAM FUNCTION F(X,N) MUST BE SUPPLIED BY THE USER WITH X DECLARED BY THE STATEMENT DIMENSION X(N). TOL IN THE DATA STATEMENT IS A MACHINE-DEPENDENT PARAMETER WHICH SHOULD BE SET TO THE RELATIVE MACHINERY ACCURACY. IF E < TOL IT IS RESET TO TOL.

A=ARRAY OF LOWER LIMITS
B=ARRAY OF UPPER LIMITS
H=ARRAY UI FRACTIONS OF FORM 1/N.
N=POSITIVE INTEGER
AL=NUMBER BETWEEN 1.5 AND 2, PREFERABLY 1.5
M=MAXIMUM NUMBER OF ITERATIONS SHOULD DECREASE AS N INCREASES
E=TOLERANCE INCREASES WITH N
AINT=RESULT
F=FUNCTION ROUTINE
KEY=INDICATOR SET BY SUBROUTINE

CALL NDIMRI(1,NN,AA,BB,H,AL,LSS,EE,FF1,AINT1,KEY)
CALL NDIMRI(2,NN,AA,BB,H,AL,LSS,EE,FF2,AINT2,KEY)
CALL DBINTG2(AA2,BB2,AA3,BB3,AINT3)
WRITE(21,50) AINT2
      FORMAT(1X,'DBL INT TERM = ',F15.6)
      T1=SQRT(N/(2*PI)*VAR)
      T2=1/((VARA*SQRT(2*PI)))
      VARMU=0.1
      T3=1/(VARMU*SQRT(2*PI))
      C1=(CA-CP)*T1/T2*SQRT(N-1.)*AINT1
      C2=CA*T1*T2*T3*LSS*AINT2
      C3=(CP*N*(CA*(LSS-N))*T1*T2*AINT3
      COST=C1+C2+C3
      WRITE(99,20)N,VARN,MM1,MM2,KBU,KBL,ECSTAC,PGU1
      FORMAT(1X,111,7(F15.5))
      XLO=MO-.1
      XUP=MO+.1
      NO=N
      CALL DERV2(XLO,XUP,O2)
      CALL DERV2A(L,U,O2A)
      CONTINUE
MAIN PROGRAM - END

REAL FUNCTION G(X)
INCLUDE 'COM2,VAR'
DOUBLE PRECISION A,B,X,E1,TERM1,TERM2,TERM3,TERM4
REAL X
M=100

PRINT*, '** G(X) STARTS **'
TERM1=SQRT(VAR+VARN)
MO=(L+U)/2.
WRITE(6,*) ' N = ',N
WRITE(6,*) ' X = ',X
TERM2= (MO * VAR + N* VARA*(VAR+N*VARA))
WRITE(6,*) 'TERM1 = ',TERM1, ' TERM2 = ',TERM2
A= (L - TERM2)/TERM1
B= (U - TERM2)/TERM1

A=L-2
B=U
WRITE(6,*) 'A = ',A, ' B = ',B
CALL SIMP1(A,B,M,E1)
PRINT*, 'E1 = ',E1

TERM3=VARA+VAR
TERM6: 1,-1./(REALCN»
WRITE(6,*) 'TERM3 = ',TERM3

A= (L-(XOTERM3 + VAROTERM4.MO»/(VAROTERM4+TERM3
B= (U-(XOTERM3+VAROTERM4.MO»/(VAROTERM4+TERM3
CALL SIMP1(A,B,M,E2)
G=I-EI*(I-CSSP/(CPOLSS»-(CSSP"LSS-CSN0(LSS-N»/(CPOLSS)
PRINT*, 'G = ',G
PRINT*, '** END G(X) **'
RETURN

REAL FUNCTION FF1(X,NN)
INCLUDE 'COM2,VAR'
DOUBLE PRECISION X(NN)
FF1=1/( EXP( (N*(X(1)-X(2))**2)/(2*(N-1)*VARA) ) + 1/( EXP( (X(2)-MO)**2/(2*VARA) ) ) )
& 1/( EXP( (N*(X(3)-X(2))**2)/(2*VARA) ) )
RETURN

REAL FUNCTION FF2(X,NN)
INCLUDE 'COM2,VAR'
DOUBLE PRECISION X(NN)
FF2= 1/ EXP( ((X(1)-X(2))**2)/(2*VARA) )
& 1/ EXP( ((X(2)-MO)**2)/(2*VARA) )
& 1/ EXP( ((X(3)-X(2))**2)/(2*VARA/N) )
RETURN
SUBROUTINE SIMP(A,B,M,E)

**FUNCTION:** Approximates the integral of f(x) over the interval [A,B] by Simpson's rule

**USAGE:**
- Call sequence: CALL SIMP(A,B,M,E)
- External functions/subroutines: Function f(x)
- Parameters:
  - Input: A=Interval Left Endpoint B=Interval Right Endpoint M=Number of Subintervals (Positive Even Integer)
  - Output: E=Estimate of the integral

```c
DOUBLE PRECISION A,B,E,X,H,F

**INITIALIZATION**

IF (B .LT. -4 .OR. A .GT. 4) THEN
  E=0.000000000001
  RETURN
ENDIF

E=F(A)+F(B)
H=(B-A)/M
X=A-H

**COMPUTE THE SUM OF ODD INDEX TERMS**

DO 1 I=1,M-1,2
  X=X+2.0*H
  E=E+4.0*F(X)
  CONTINUE
1

**COMPUTE THE SUM OF EVEN INDEX TERMS**

X=A

DO 2 I=2,M-2,2
  X=X+2.0*H
  E=E+2.0*F(X)
  CONTINUE
2

E=E*H/3.0
RETURN
END
```

**STANDARD NORMAL DENSITY FUNCTION**

```c
DOUBLE PRECISION FUNCTION F(T)
DOUBLE PRECISION T
F=1./((SORT(2*3.1416)) * DEXP(+.5 * T**2))
END
```

SUBROUTINE SECA(A,B,X,EPS)

**FUNCTION:** This subroutine computes the approximate root of f(x)=0 using the secant method

**USAGE:**
- Call sequence: CALL SECA(A,B,X,EPS)
EXTERNAL FUNCTIONS/SUBROUTINES: FUNCTION GeX)

PARAMETERS:

INPUT:
A=INITIAL APPROXIMATION X(0)
B=INITIAL APPROXIMATION X(1)
EPS=ERROR BOUND

OUTPUT:
X=SECANT APPROXIMATION OF THE ROOT

DOUBLE PRECISION a,b,v,eps

INITIALIZATION

I.=1
V=G(B)
PRINT*, 'I,B,G(B)', I,B,V
U=G(A)
PRINT*, 'A,G(A)', A,U
X=B
if (abs(u-v) .lt. .00000000001) return

DO WHILE(abs(X-B).GT.EPS)

PRINT*, 'U= ',U,' V= ',V
X=B-V*(B-A)/(V-U)
A=B
B=X
U=v
g=v(G(x))

PRINT*, 'G IN SECA= ',V
END DO

PRINT*, 'SECA ENDS _________'
RETURN

END

DOUBLE PRECISION a,b,a,v,eps

*** INITIALIZATION ***

PRINT*, 'SECA Starts______________'
I=1
V=G(B)
PRINT*, 'I,B,G(B)', I,B,V
U=G(A)
PRINT*, 'A,G(A)', A,U
X=B
if (abs(u-v) .lt. .00000000001) return

DO WHILE(abs(X-B).GT.EPS)

PRINT*, 'U= ',U,' V= ',V
X=B-V*(B-A)/(V-U)
A=B
B=X
U=v
g=v(G(x))

PRINT*, 'G IN SECA= ',V
END DO

PRINT*, 'SECA ENDS _________'
RETURN

END

DOUBLE PRECISION xlo,xup,d2)

INCLUDE 'CM2.VAR'

DOUBLE PRECISION XLO,XUP,TEMP1,TEMP2,TEMP3,C1,C2,C3,VARA1,D2

DOUBLE PRECISION XNO

NINT=20

WRITE(6,*), 'D2 = ',D2

CALL SIMP2(XLO,XUP,NINT,D2)

XNO=XNO

TEMP1=-(XNO**((3./2.))*VARA)/(SORT(VAR)*SORT(2.*PI)*((XNO*
&VARA)+VARA))

TEMP1=-Z/((VARA*(VARA/XNO))*(SORT(2.*PI)))
TEM$XBU$ = -((M1 * VAR) + (XNO * VARA * XBU)) / ((XNU * VARA) + VAR)
C1 = CA - CP - CSN
C2 = PGU * (CA - CP)
WRITE(6, *) 'C1 = ', C1, ' ', C2
VARA1 = VARA + (VAR/XNO)
TEMP3 = EXP(-0.5(* ((M1 - XBU)**2) / VARA1))
TEMP3 = EXP(-0.5*Z**2)
C3 = LSS - XNO
D2 = C3 * TEMP1 * TEMP2 * (C1 - C2) * TEMP3
D2 = C3 * TEMP1 * TEMP3 * (C1 - C2)
M = Z * (VARA + (VAR/XNO)) * XBU
WRITE(20, 20) NO, XLO, XUP, D2, M
FORMAT(1X, I5, 4X, E13.4, 4X, E13.4, 4X, E13.4, 4X, E13.4)
RETURN
END
SUBROUTINE SIMP2(A,B,M,E)

FUNCTION: APPROXIMATES THE INTEGRAL OF F(X) OVER THE
INTERVAL [A,B] BY SIMPSON'S RULE
USAGE:
EXTERNAL FUNCTIONS/SUBROUTINES: FUNCTION F(X)
PARAMETERS:
INPUT:
A = INTERVAL LEFT ENDPOINT
B = INTERVAL RIGHT ENDPOINT
M = NUMBER OF SUBINTERVALS (POSITIVE EVEN INTEGER)
OUTPUT:
E = ESTIMATE OF THE INTEGRAL

DOUBLE PRECISION A, B, E, K, M, F
*** INITIALIZATION ***
IF (B .LT. -4.0 OR. A .GT. 4) THEN
E = 0.00000000D0
RETURN
ENDF
E = F(A) + F(B)
M = (B-A)/M
X = A
*** COMPUTE THE SUM OF ODD INDEX TERMS ***
DO 1 I=1,M-1,2
X = X + 2.0*H
E = E + 4.0*FR(X)
1 CONTINUE
*** COMPUTE THE SUM OF EVEN INDEX TERMS ***
SUBROUTINE SIMP3(A,B,M,E)
C
DOUBLE PRECISION A,B,E,X,H,F

*** INITIALIZATION ***
IF (B .LT. -4 .OR. A .GT. 4) THEN
   E=0.000000000001
RETURN
ENDIF
E=FL(A)*FL(B)
H=(B-A)/M
X=A-H

*** COMPUTE THE SUM OF ODD INDEX TERMS ***
DO 1 I=1,M-1,2
   X=X+2.0*H
   E=E+4.0*FL(X)
1 CONTINUE

*** COMPUTE THE SUM OF EVEN INDEX TERMS ***
X=A
DO 2 I=2,M-2,2
   X=X+2.0*H
   E=E+2.0*FL(X)
2 CONTINUE
E=E*H/3.0
RETURN
END

DOUBLE PRECISION FUNCTION FL(XBAR)
INCLUDE 'COM2.VAR'
DOUBLE PRECISION XBU,PGU,M1
COMMON /BLOCK2/XBU,PGU,M1
COMMON /BLOCK3/NO
DOUBLE PRECISION XBAR,TEMP1,TEMP2,XNO

XNO=XNU
WRITE(6,*) 'TEST PT 1'
DOUBLE PRECISION FUNCTION FR(UU)
INCLUDE 'COM, AR'
DOUBLE PRECISION XBU,PGU,M1
COMMON /BLOCK2/XBU,PGU,M1
COMMON /BLOCK3/NO
DOUBLE PRECISION UU,C1,C2,C3,C4,TEMP1,TEMP2,TEMP3,HU,v1,v2
N=N**((3.0/2.0))
C2=XBU-UU
C3=U
C4=C3*((C2**2)/(2*VAR))
WRITE(6,*) 'C1 = ',C1
WRITE(6,*) 'C2 = ',C2
WRITE(6,*) 'C3 = ',C3
WRITE(6,*) 'C4 = ',C4
TEMP1=-(C1*C2*(DEXP(-1.0*C4)))
TEMP2=((VAR**((1.5))*((2*PI)**(2.5))))
WRITE(6,*) 'TEMP1 = ',TEMP1,' TEMP2 = ',TEMP2
TEMP3=TEMP1/TEMP2
WRITE(6,*) 'XBU = ',XBU
MU=(EXP(-(UU-MO)**2)/(2*VARA))>((VARA**.5)*SQR(2*PI))
FR=TEMP3*MU*(((CA-CP-CSN)-(PGU*(CA-CP)))
WRITE(6,*) 'V1 = ',V1,' ',V2 = ',V2
RETURN
END
SUBROUTINE DERVZA(XLO,KUP,DZA)
INCLUDE 'COM2,VAR'
DOUBLE PRECISION XBU,PGU,M1,Z
COMMON /BLOCK2/XBU,PGU,M1,Z
COMMON /BLOCK3/NO
DOUBLE PRECISION ECSTAC,ETCOST
COMMON /BLOCK4/ECSTAC,ETCOST
WRITE(6,10)
FORMAT(5X,'N',7X,'2-ND DERIV',8X,'ECSTSN',13X,'PGU',13X,'ETCOST')
RETURN
END
DOUBLE PRECISION XLO, XUP, TEMPS, TEMPS, TEMPS, TEMPS, D2A
DOUBLE PRECISION XNO, TEMPS, TEMPS, TEMPS

NINT = 20
WRITE(*,*) 'D2 = ', D2
CALL SIMP4(XLO, XUP, NINT, D2A)

XNO = XNO

TEMP1 = (VARAO(VAR/XNO)/(VARA+(VAR/XNO)))
TEMP2 = ((VARA+VAR)/(VAR+VAR))
TEMP3 = (XNO*VARA/(VARA+VARA))
TEMP4 = (VARA+VAR)/(VARA+VARA)
TEMP5 = (VARA+VAR)/(VARA+VARA)
TEMP6 = (VARA+VAR)/(VARA+VARA)
TEMP7 = (VARA+VAR)/(VARA+VARA)
D2A = TEMPS + CSSP*CSP*CSS*CSS/

WRITE(4,20) NO, D2A, ECSTAC, PGU, ETCOST
20 FORMAT(IX,IS,4X,E13.4,4X,E13.4,4X,E13.4)
WRITE(6,22)
22 FORMAT(IS,IX,E13.4,1X,E13.4)

SUBROUTINE SIMP4(A, B, M, E)

FUNCTION: APPROXIMATES THE INTEGRAL OF F(X) OVER THE
INTERVAL [A, B] BY SIMPSON'S RULE
USAGE:
CALL SEQUENCE: CALL SIMP(A, B, M, E)
EXTERNAL FUNCTIONS/SUBROUTINES: FUNCTION F(X)
PARAMETERS:
INPUT:
A = INTERVAL LEFT ENDPOINT
B = INTERVAL RIGHT ENDPOINT
M = NUMBER OF SUBINTERVALS (POSITIVE EVEN INTEGER)
OUTPUT:
E = ESTIMATE OF THE INTEGRAL

DOUBLE PRECISION A, B, E, X, M, F
*** INITIALIZATION ***
IF (B .LT. 4 OR A .GT. 4) THEN
E = 0.0
RETURN
ENDIF
E = FB(A) + FB(B)
M = B-A)
X = A + M

*** COMPUTE THE SUM OF ODD INDEX TERMS ***
DO 1 I = 1, M - 1
X = X + 2.0*M
E = E + 4.0*FB(X)
1 CONTINUE

*** COMPUTE THE SUM OF EVEN INDEX TERMS ***
X = A
DO 2 I = 2, M - 2
X = X + 2.0*M
2 CONTINUE
DOUBLE PRECISION FUNCTION FB(UU)
INCLUDE 'COM2.VAR'
DOUBLE PRECISION XBU, GU, M1
COMMON /BLOCK2/XBU, PGU, M1
COMMON /BLOCK3/NO
DOUBLE PRECISION UU, C1, C2, C3, C4, TEMP1, TEMP2, TEMP3, HU, v1, v2
DOUBLE PRECISION TEMP4, TEMP5

XNO=NO
TEMP1=(((VARA*VAR)/((XNO*VARA)+VAR))+VAR)
TEMP2=(VARA+(VAR/XNO))
TEMP3=(((M1*(VAR/XNO))+(VARA*XBU))/TEMP2)
TEMP5=((VARA*VAR)/XNO)/TEMP2
TEMP4=((TEMP3*VAR)+(TEMP5*UU))/(TEMP5+VAR)
HU=(XBU-TEMP4)*DEXP(-(TEMP3-UU)**2)/(2*(TEMP5+VAR))

FB=HU

WRITE(6,*) 'FB VALUE = ', FB
RETURN
END
SUBROUTINE READ
  INCLUDE 'COM2.VAR'

  WRITE (6,1)
  FORMAT('I')
  WRITE(6,*)'IS THE VALUE IN THE FILE...ENTER 0 OR 1'
  READ(5,*)ANS

  ANS = 1
  IF (ANS .EQ. 1) GOTO 100
  WRITE(6,*)'ENTER VALUES FOR THE FOLLOWING VARIABLES'
  WRITE(6,*)

  IF (ANS .NE. 1) GOTO 100

  WRITE(6,2)
  FORMAT(1X,'U=? ',S)
  READ(5,*)U

  WRITE(6,3)
  FORMAT(1X,'L=? ',S)
  READ(5,*)L

  WRITE(6,4)
  FORMAT(1X,'VAR= ',S)
  READ(5,*)VAR

  WRITE(6,5)
  FORMAT(1X,'VARA= ',S)
  READ(5,*)VARA

  WRITE(6,6)
  FORMAT(1X,'CSN= ',S)
  READ(5,*)CSN

  WRITE(6,7)
  FORMAT(1X,'CA= ',S)
  READ(5,*)CA

  WRITE(6,8)
  FORMAT(1X,'CP= ',S)
  READ(5,*)CP

  WRITE(6,9)
  FORMAT(1X,'LSS= ',S)
  READ(5,*)LSS
  WRITE(6,10)
  FORMAT(1X,'CSSP= ',S)
  READ(5,*)CSSP

  WRITE (10,*)U,L,VAR,VARA,CSN,CA,CP,LSS,CSSP
  RETURN

  READ (10,*)U,L,VAR,VARA,CSN,CA,CP,LSS,CSSP
  RETURN

END
SUBROUTINE HEAD2
INCLUDE 'LOM2,VAR'
WRITE (99,1)
1 FORMAT(/B8,'******************************************************************************'/
& B8,'TABLE 2: SCREENING COST AND RELATED SPECIFICATIONS'/
& B8,'******************************************************************************'/
WRITE (99,2)U,L,VAR,VARA,CSN,CA,CP,CSSP,LSS
2 FORMAT(/B8,'INPUT: MODEL SPECIFICATIONS'/
& B8,'-------------------------------------------'11
& B8,'UPPER LIMIT OF THE Q.C.X: ',F15.5/
& B8,'LOWER LIMIT OF THE Q.C.X: ',F15.5/
& B8,'VARIANCE OF X: ',F15.5/
& B8,'VARIANCE OF THE MEAN OF X: ',F15.5/
& B8,'UNIT COST OF SCREENING: ',F15.5/
& B8,'UNIT COST OF ACCEPTANCE: ',F15.5/
& B8,'COST OF SCRAPING OR REPLACING'/
& B8,'A DEFECTIVE UNIT FOUND DURING'/
& B8,'SAMPLING OR SCREENING'/
& B8,'INSPECTION: ',F15.5/
& B8,'UNIT COST OF SCRAPING: ',F15.5/
& B8,'LOT SIZE: ',191)
WRITE(99,3)
3 FORMAT(/B8,'OUTPUT: SAMPLE SIZE, ROOTS OF THE COST FUNCTION,/'
& B8,'----------------------------------------'11
& B8,'COL. NO. DESCRIPTION'/
& B8,'-------- ---------------------------------------'11
& B8,'1 SAMPLE SIZE /'
& B8,'2 (a * b) / (b + n*a) WHERE: a /
& B8,'3 b: VARIANCE OF THE MEAN OF X/'
& B8,'4 b: VARIANCE OF X /
& B8,'n: SAMPLE SIZE /
& B8,'5 LOWER DISPOSITION LIMIT FOR LOT SCREENING:/
& B8,'6 UPPER DISPOSITION LIMIT FOR LOT SCREENING:/
& B8,'7 LOWER DISPOSITION LIMIT OF THE SAMPLE MEAN:/
& B8,'8 SCREENING COST PER LOT:/
& B8,'9 P: IS THE FRACTION DEFECTIVE AT WHICH THE'/
& B8,'COSTS OF ACCEPTANCE AND SCREENING ARE 1/EXP10)/
WRITE(99,4)
4 FORMAT(B8,75('--'))/
& B8,'T10,'1,'T16,'2','T26,'3','T36,'4','T46,'5','T56,'6','T57,'/
& 7,'T76','8'1/
& 8,B8,75('-'))/
END
SUBROUTINE COM_VAR
************************************************************
This subroutine includes all the constants and variables
that are employed in this program. The input and output
constants and variables are defined as double precision.
************************************************************

DOUBLE PRECISION U,L,VAR,VARA,VARN,VARMU,CR,CE,MN1,MN2,XBAR1,XBAR2,
& LSN,CA,LP,CSSP,NS,MO
COMMON /BLOCK1/ U,L,VAR,VARA,VARN,VARMU,CR,CE,CS,LS,MN1,MN2,
& VN,STUN,LSN,CA,LP,LS2,CSSP,N,NS,MO
PARAMETER (PI=3.1459)
THE SUBROUTINE HEA02A IS EMPLOYED TO PRINT THE PRIOR COSTS AND PROBABILITIES IN A SEPARATE TABLE.

SUBROUTINE HEA02A
INCLUDE 'COM2.VAR'
WRITE(99,4)
4 FORMAT(8X,95(' '))
& 8X,95(' ') )
END
SUBROUTINE NDMR11(H,A,B,H,AL,M,E,FF1,AIN7,KEY)
  REAL*4 A(N),B(N),C(9),K(15),M(N),G(9),NN(9),P(9),AA(15)
  REAL*4 V(15),X(9),D(9)
  CkAL.B Frl REAL.U A(N),B(N),C(9),K(15),M(N),G(9),NN(9),P(9),AA(15)
  Init tu 1/1.E-07/
  KEY=0
  WRITE(6,*) ' Test Pt NDMR11 1'
  IF(N.LT.1.OR.N.GT.9.OR.M.LT.1.OR.AL.LT.15.OR.AL.GT.2.)
     RETURN
  WRITE(6,*) ' Test Pt NDMR11 2'
  DU 2 I=1,N
  WRITE(6,*) ' Test Pt NDMR11 3'
  WRITE(6,*) ' I = ',I,' M(I) = ',M(I)
  IF(M(I).LE.0..OR..M(I).GT.1.) RETURN
  WRITE(6,*) ' Test Pt NDMR11 4'
  DO 2 I=1,N
  D(I)=A(I)
  EF=AMAX1(E,TOL)
  MM=MIN(M,15)
  DU 1 I=N,8
  P(I-1)=0.
  NN(I+1)=1
  10 L=1
  DU 20 I=1,N
  20 C(I)=B(I)-D(I)
  21 U=0.
  KT=0
  DU 22 I=1,N
  G(I)=M(I)/K(L)
  NN(I)=1./G(I)+.5
  38 P(I)=C(I)*G(I)
  22 P(I)=C(I)*G(I)
  NN9=NN(9)
  DO 30 I9=1,NN9
  X(9)=D(9)+P(9)*(I9-.5)
  30 NN9=NN(9)
  DO 39 I9=1,NN9
  X(8)=D(8)+P(8)*(I8-.5)
  39 NN6=NN(6)
  DO 35 I6=1,NN6
  X(6)=D(6)+P(6)*(I6-.5)
  35 NN6=NN(6)
  DO 30 I6=1,NN6
  X(6)=D(6)+P(6)*(I6-.5)
  34 NN5=NN(5)
  DO 30 I5=1,NN5
  X(5)=D(5)+P(5)*(I5-.5)
  33 NN4=NN(4)
  DO 30 I4=1,NN4
  35
X(4)=D(4)*P(4)*(14-.5)
N=NN(J)
DO 30 I=1,NN
X(I)=X(I-1)+P(I)*((I-1)-.5)

30 CONTINUE

X(2)=U(2)*P(2)*(12-.5)
N=NN(1)
DO 30 I=1,NN
X(I)=D(I)+P(I)*((I-1)-.5)
WRITE(6,*) U = ',U
30 CONTINUE

U=FF1(X,N)
DO 40 I=1,N
40 (U=U+P(I))
V(I)=U
43 IF(L-1) 43,44
44 AA(1)=V(1)
L=L+1
GO TO 21
EN=K(L)
DO 45 LL=2,L
I=L+1-LL
V(I)=V(I)+(V(I+1)-V(I))/((EN/K(I))*2-1.)
AINT=V(I)
PRINT*, 'AINT = ', AINT
KEY=1
IF(ABS(AINT-AA(L-1)).LT.ABS(AINT*EE)) RETURN
KEY=1
IF(L.EQ.MM) RETURN
AA(L)=AINT
L=L+1
K(L)=AL*K(L-1)
GO TO 21
END

SUBROUTINE NDMRI2(N,A,B,H,AL,M,E,FF2,AINT,KEY)
REAL*8 FF2
REAL*8 A(n),b(r),c(9),k(15),h(n),g(9),nn(9),p(9),aa(15),V(15)
*
data k(1),k(2)/1,2/
data tol /1.E-07/
KEY=0
WRITE(6,*) ' Test Pt NDMRI2 1
IF(N.LT.1.OR.N.GT.9.0I<,M.LE.1.0R,AL,LT.1.5.OR.AL.GT.2.) RETURN
WRITE(6,*) ' Test Pt NDMRI2 2
DO 2 I=1,N
WRITE(6,*) ' Test Pt NDMRI2 3
IF(H(I),LE.0..OR.H(I).GT.1.) RETURN
WRITE(6,*) ' Test Pt NDMRI2 4
D(I)=A(I)
E=MAX1(E,TOL)
MM=MIN(M,15)
DO 1 I=MM,8
P(I+1)=0.
nn(I+1)=1
1 D(I+1)=0.
10 L=1
20 DO 20 I=1,N
21 C(I)=B(I)-D(I)
22 U=0.
23 K=0
24 DO 22 I=1,N
25 G(I)=H(I)/K(I)
26 NN(I)=G(I)+.5
27 P(I)=C(I)*G(I)
28 NN=NN(5)
29 DO 30 I=1,NN9
30 X(9)=D(9)+P(9)*(I9-.5)
31 X(8)=D(8)+P(8)*(I8-.5)
32 NN8=NN(6)
33 DO 30 I=1,NN8
34 X(7)=D(7)+P(7)*(I7-.5)
35 NN7=NN(7)
36 DO 30 I=1,NN7
37 X(6)=D(6)+P(6)*(I6-.5)
38 NN6=NN(6)
39 DO 30 I=1,NN6
40 X(5)=D(5)+P(5)*(I5-.5)
41 NN5=NN(5)
42 DO 30 I=1,NN5
43 X(4)=D(4)+P(4)*(I4-.5)
44 NN4=NN(4)
45 DO 30 I=1,NN4
46 X(3)=D(3)+P(3)*(I3-.5)
47 NN3=NN(3)
48 DO 30 I=1,NN3
49 X(2)=D(2)+P(2)*(I2-.5)
50 NN2=NN(2)
51 DO 30 I=1,NN2
52 X(1)=D(1)+P(1)*(I1-.5)
53 NN1=NN(1)
54 DO 30 I=1,NN1
55 X(0)=D(0)+P(0)*(I0-.5)
56 V(I)=X(I)
57 IF(L-1) 43,43,44
58 AA(1)=V(1)
59 L=L+1
60 GO TO 21
61 AA(K)=V(1)
62 EN=K(L)
63 DO 45 LL=2,L
64 I=I+L-LL
65 V(I)=V(I+1)*(V(I+1)-V(I))/((EN/K(I))*2-1.)
66 AINT=V(I)
67 K(I)=1
68 IF(AINT=AA(L-1)),LT,AINT=AA(L-EE)) RETURN
69 KE=-1
IF(L .EQ. MM) RETURN
AA(L) = INT
L = L + 1
K(L) = AL + K(L - 1)
GO TO 21
END
APPENDIX F

PROGRAM 4AA AND 5AA

Input Data:
Same as in Appendix D

Output 1 of Program 4AA:
Same as in Appendix D, the same decision points and the cost estimated is that of scrapping per lot. Also the value of $p$ at which the costs of screening and scrapping are equal is evaluated.

Output 2 of Program 4AA:
Same as in appendix D, the exact cost of scrapping per lot is estimated.

Output of Program 5AA:
1- Sample size.
2- The second derivative of the total cost relative to the variables involved.
3- The expected total cost per lot.
PROGRAM PR0G4

FUNCTION: THIS PROGRAM COMPUTES THE SCRAPPING COST PER LOT, ALSO IT PROVIDES VALUES FOR THE UPPER AND LOWER DISPOSITION LIMITS FOR LOT ACCEPTANCE.

THE COST IS VERIFIED TO BE A MINIMUM BY STUDYING THE SIGN OF THE SECOND DERIVATIVE OF THE TOTAL COST RELATIVE TO THE UPPER AND LOWER DISPOSITION LIMITS.

USAGE:

CALL SEQUENCE: RUN PR0G2A (PRESS) RETURN

EXTERNAL FUNCTIONS: G(x), FF1(x,NN), FF2(x,NN)

F(T), FL(XBAR), FR(UU), FB(UU)

EXTERNAL SUBROUTINES: READ2, HEAD2, SECA, NORM11, NORM12, DBINTG2

ASSUMPTIONS:

1. THE DISTRIBUTIONS ARE NORMAL.
2. THE INTEGRAL OF X GIVEN THE MEAN MU IS EQUAL TO THE INTEGRAL OF T GIVEN XBAR AND THE MEAN MU.
3. THE STATISTICAL PROCESS CAN EXIST IN ONE STATE.

PARAMETERS:

INPUT:

L=LOWER SPECIFICATION LIMIT OF QUALITY CHARACTERISTICS X

U=UPPER SPECIFICATION LIMIT OF QUALITY CHARACTERISTICS X

LSS=LOT SIZE

N=SAMPLE SIZE

X=x PRODUCT DIMENSION OR A QUALITY CHARACTERISTIC

XBAR=x SAMPLE MEAN

MU=THE MEAN OF THE VARIABLE X RELATIVE TO A CERTAIN LOT

M1=THE MEAN OF THE MEAN MU

VAR=LOT VARIANCE

CI=cost of inspection per unit of a product

CSN=cost of screening inspection per unit of a product

CSSP=cost of scrapping per unit of a product

CA=cost of acceptance per unit of a product

CP=cost of scrapping and replacing a defective unit during sampling or screening inspection.

OUTPUT:

1. SAMPLE N

2. VARIANCE OF THE JOINT DISTRIBUTION OF THE MEAN MU AND THE DISTRIBUTION OF XBAR

3. LOWER LIMIT GENERATED BY TAKING THE DERIVATIVE OF THE TOTAL COST RELATIVE TO LA

4. UPPER LIMIT GENERATED BY TAKING THE DERIVATIVE OF THE TOTAL COST RELATIVE TO UB

5. UPPER DISPOSITION LIMIT OF LOT ACCEPTANCE

6. LOWER DISPOSITION LIMIT OF LOT ACCEPTANCE

7. THE APPROXIMATE COST OF ACCEPTANCE PER LOT

8. THE EXACT VALUE OF THE COST GIVEN IN 7
9. The value of the second derivative relative to each n
10. The value of pg which is the integral of the probability density function of x given mu between the limits l,u.

**CALCULATION PROCEDURES:**

1. An expression for the total cost is derived.
2. The first derivative of the total cost is taken relative to the following parameters:
   a) Upper disposition limit of lot acceptance
   b) Lower disposition limit of lot acceptance
   c) Upper disposition limit of lot screening
   d) Lower disposition limit of lot screening.
3. The partial derivatives are equated to zero.
4. The four equations obtained are solved for ua, la, usn, lsn.
5. The cost of acceptance relative to a certain sample mean xbar and a mean mu is equated with the cost of screening at xbar=la, or xbar=ua. Solving this expression yields a value for the integral of x given mu.
6. The total cost of acceptance is obtained by summing over all sample means and population means.

***GLOBAL DECLARATIONS***

EXTERNAL G,FF1,FF2
INCLUDE 'COM2,VAR'
DOUBLE PRECISION XBU,PGU,M1,Z
COMMON /BLOCK2/XBU,PGU,M1,Z
COMMON /BLOCK3/NO
DOUBLE PRECISION A,B,E,X,ERR,MN,ECSTAC,XLO,XUP,D2,D2A,XU
DOUBLE PRECISION EVAL,ETCOST
COMMON /BLOCK4/ECSTAC,ETCOST
REAL AA(J),BB(J),H(3),A,B,X,ERR

***MAIN PROGRAM-BEGIN***

CALL READ2
CALL HEAD2

WRITE(6,'(*)'COST OF INSPECTION?'
READ(5,*)C1

WRITE(6,'(*)'STARTING SAMPLE SIZE?'
READ(5,*)ISAM

WRITE(6,'(*)'ENTER M1 VALUE?'
READ(5,*) M1

CALL D2HEAD
CALL D2AHEAD

DO 10 N=ISAM,.3*ISAM
VARN = (VARA * VAR)/(VAR + N * VARA)

***COMPUTE INITIAL VALUE OF MN***

ERR=1.0E-09
A=L-2
CALL SECA(A,B,X,ERR)
MN=UL/2
V=ABS(MN-MO)
MN=MO-V
XBI=(MN**2*(VAR+N*VARA)-(L+U)*VAR/2.)/(N*VARA)
XBL=(MN**2*(VAR+N*VARA)-(L+U)*VAR/2.)/(N*VARA)
PGU=(LSS*(CSN+CP-CSSP)-N*CSN)/(LSS*CP-N*CSP)
PWU=1-PGU
NITR=20
NO=N
XI=XU+.7
CALL SIMP3(XBU,XU+4,*NITR,EVAL)

WRITE(30,*) 'DENSITY PRODUCT FACTOR = ',EVAL

*** COMPUTE COSTS ***
ECSTAC=2.0*(CI*N*CSP*(LSS-N*PGU))
ETCOST=2.0*ECSTAC*EVAL

*******************************************************************************************

NN=3
AL=1.5
EE=1.E-01
H(1)=.6
H(2)=.6
H(3)=.6
AA(1)=L
AA(2)=L-2
AA(3)=MN1
BB(1)=U
BB(2)=U+2
BB(3)=MN2
AA2=AA(2)
AA2=L
AA2=U
AA2=AA(3)
AA3=L
BB2=BB(2)
BB2=U
BB2=BB(3)
BB3=U

*******************************************************************************************

* NOTE: THE FOLLOWING SUBROUTINE Computes AFA, AN APPROX. IMATION TO THE N DIMENSIONAL INTEGRAL OVER A PARA-
* LEL-PIPED & ROMBERG TYPE METHOD BASED ON THE MID-
* POINT RULE IS EMPLOYED. EACH INTERVAL IS SUBDIVIDED INTO [1/(H(I))] SUBINTERVALS. 0xM(I)<1.I=1,...,N. 
* M IS THE MAXIMUM NUMBER OF ITERATIONS ALLOWED AND SHOULD DECREASE WITH INCREASING N SINCE THE NUMBER 
* OF FUNCTIONAL EVALUATIONS IN THE Jth ITERATION IS AT LEAST J**N.IF M IS SET >15, IT IS RESET TO 2. 
* AL IS A CONSTANT BETWEEN 1.5 AND 2, PREFERABLY 1.5 

* NOTE: THE FOLLOWING PORTION RELATES TO TRIPLE INTEGRATION
WHICH REGULATES THE INCREASE IN THE NUMBER OF POINTS USED. E IS A TOLERANCE WHICH SHOULD INCREASE AS N INCREASES; KEY IS SET EQUAL TO 1, IF THIS IS NOT ACHIEVED WITHIN M ITERATIONS, KEY IS SET EQUAL TO -1. IF N > 1, THE SUBROUTINE EXITS WITH KEY = 0. A PROGRAM FUNCTION F(X, N) MUST BE SUPPLIED BY THE USER WITH X DECLARED BY THE STATEMENT DIMENSION X(N). If F MUST BE DECLARED EXTERNAL IN THE CALLING PROGRAM. TOL IN THE DATA STATEMENT IS A MACHINE-DEPENDENT PARAMETER WHICH SHOULD BE SET TO THE RELATIVE MACHINE ACCURACY. IF E < TOL, IT IS RESET TO TOL.

A = ARRAY OF LOWER LIMITS
B = ARRAY OF UPPER LIMITS
H = ARRAY OF FRACTIONS OF FORM 1/N,
AL = NUMBER BETWEEN 1.5 AND 2, PREFERABLY 1.5
M = MAXIMUM NUMBER OF ITERATIONS SHOULD DECREASE AS N INCREASES
E = TOLERANCE INCREASES WITH N
AINT = RESULT
KEY = INDICATOR SET BY SUBROUTINE

CALL NDIMRII(NN, AA, BB, AL, LSS, EE, FF1, AINT1, KEY)
CALL NDIMRI2(NN, AA, BB, AL, LSS, EE, FF2, AINT2, KEY)
CALL DBINTG2(AA2, BB2, AA3, BB3, AINT3)
WRITE(21, 50) AINT2
C50 FORMAT(IX, 25H DBL INT TERM = , E15.6)

T1 = SQRT(N/(2 * PI)) * VAR
T2 = 1/(VAR * SQRT(2 * PI))
VARMU = U / 1
T3 = 1/(VARMU * SQRT(2 * PI))
C1 = (CA - C~) * N * (T2) * T1 / SQRT(N - 1) * AINT1
C2 = -CA * T1 * T2 * T3 * LSS * AINT2
C3 = (CP - CA) * (LSS - N) * T1 * T2 * AINT3
COST = C1 + C2 + C3
WRITE(99, 20) N, VAR, MN1, MN2, XBU, XBL, ECSTAC, PGU2
FORMAT(IX, 111, 7(F15.5))
C
XLO = MO - .1
XUP = MO + .1
N = N
CALL NERV2(XLO, XUP, D2)
C
CALL DERV2A(L, U, D2A)
CONTINUE
C
*** MAIN PROGRAM - END ***
END

REAL FUNCTION G(X)
INCLUDE 'COM2, VAR'
DOUBLE PRECISION A, B, X, E1, TERM1, TERM2, TERM3, TERM4
REAL X
M = 100
**FUNCTION**  
**NAME:** SIMP1  
**PURPOSE:** Approximates the integral of \( f(x) \) over the interval \( [a, b] \) by Simpson's rule.

**USAGE:** 
- CALL SEQUENCE: CALL SIMP1(A, B, M, E)
- **EXTERNAL FUNCTIONS/SUBROUTINES:** FUNCTION F(X)
- **PARAMETERS:**
  - A: \( a \)
  - B: \( b \)
  - M: Subdivision integer
  - E: Output error

**FUNCTION:** FF1(X, NN)  
**DOUBLE PRECISION X(NN)**

**FUNCTION:** FF2(X, NN)  
**DOUBLE PRECISION X(NN)**

**SUBROUTINE** SIMP1(A, B, M, E)

**FUNCTION:** F(X)
INPUT:
A=INTERVAL LEFT ENDPOINT
B=INTERVAL RIGHT ENDPOINT
M=NUMBER OF SUBINTERVALS (POSITIVE EVEN INTEGER)

OUTPUT:
E=ESTIMATE OF THE INTEGRAL

DOUBLE PRECISION A,B,E,X,H,F

*** INITIALIZATION ***
IF (B .LT.-4 .OR. A .GT. 4) THEN
E=0.0000000000001
RETURN
ENDIF
E=F(A)+F(B)
M=(B-A)/M
X=A-M
*** COMPUTE THE SUM OF ODD INDEX TERMS ***
DO 1 I=1,M-1,Z
X=X+2.0*I
E=E+4.0*F(X)
CONTINUE

*** COMPUTE THE SUM OF EVEN INDEX TERMS ***
X=A
DO 2 1=2,M-.Z
X=X+2.0*I
E=E+2.0*F(X)
CONTINUE
E=E*M/3.0
RETURN
END

SUBROUTINE SECA(A,B,X,EPS)

FUNCTION: THIS SUBROUTINE COMPUTES THE APPROXIMATE ROOT OF F(X)=0 USING THE SECANT METHOD
USAGE:
CALL SEQUENCE: CALL SECA(A,B,X,EPS)
PARAMETERS:
INPUT:
A=INITIAL APPROXIMATION X(0)
B=INITIAL APPROXIMATION X(1)
EPS=ERROR BOUND
OUTPUT:
X=SECANT APPROXIMATION OF THE ROOT

DOUBLE PRECISION A,B,E,U,V,EPS
I = I + 1
V = G(B)
PRINT*, 'I, B, G(B) ', I, B, V
U = G(A)
PRINT*, 'A, G(A) ', A, U
x = B
IF (ABS(U - V) .LT. .00000000001) RETURN
*** COMPUTE APPROXIMATE ROOT ITERATIVELY ***
DO WHILE (ABS(x - A) .GT. EPS)
PRINT*, 'U = ', U, ' V = ', V
x = B - V *(B - A)/(V - U)
A = B
B = x
I = V
PRINT *, 'G IN SECA = ', V
END DO
PRINT*, ' SECA Ends '...
RETURN
END
SUBROUTINE D2HEAV
WRITE(6, 10)
RETURN
END
SUBROUTINE DERV2(XLO, XUP, D2)
INCLUDE 'COM2, VAR'
DOUBLE PRECISION X8U, PRU, MI, Z
COMMON /BLOCK2/ X8U, PRU, MI, Z
COMMON /BLOCK3/ NO
DOUBLE PRECISION XLO, XUP, TEMPI, TEMP2, TEMP3, C1, C2, C3, VARA1, D2
DOUBLE PRECISION XNO
NINT = 2D
WRITE(6, *) 'D2 = ', D2
CALL SIMP2(XLO, XUP, NINT, D2)
XNO = NO
TEMP1 = -(XNO**((3./2.))*VARA)/(SQR(T(VAR)*SQR(T(2.*PI)))*((XNO* &VARA)+VAR))
TEMP2 = Z/(VARA*(VAR/XNO))*(SQR(T(2.*PI)))
TEMP3 = XBU - (((M1*VARA)*(XNO*VARA*XBU))/((XNO*VARA)+VAR))
C1 = CA - CP - CSN
C2 = PGU*(CA - CP)
WRITE(6, *) 'C1 = ', C1, ' C2 = ', C2
VARA1 = VARA + (VAR/XNO)
TEMP3 = EXP(-.5*(((M1 - XBU)**2)/VARA1))
C TEMP3 = EXP(-.5*(Z**2))
SUBROUTINE SIMP2(A,B,M,E)

FUNCTION: APPROXIMATES THE INTEGRAL OF F(X) OVER THE INTERVAL [A,B] BY SIMPSON'S RULE

USAGE:
CALL SEQUENCE: CALL SIMP(A,B,M,E)

EXTERNAL FUNCTIONS/SUBROUTINES: FUNCTION F(X)

PARAMETERS:
INPUT:
A=INTERVAL LEFT ENDPOINT
B=INTERVAL RIGHT ENDPOINT
M=NUMBER OF SUBINTERVALS (POSITIVE EVEN INTEGER)

OUTPUT:
E=ESTIMATE OF THE INTEGRAL

DOUBLE PRECISION A,B,E,X,H,F

** INITIALIZATION **
IF (B .LT.-4 .OR. A .GT. 4) THEN
E=0.00000000000001
RETURN
ENDIF
E=F(A)+F(B)
X=(B-A)/M

*** COMPUTE THE SUM OF ODD INDEX TERMS ***
DO 1 I=1,M-1,2
X=X+2.0*M
E=E+4.0*FRC(X)
1 CONTINUE

*** COMPUTE THE SUM OF EVEN INDEX TERMS ***
X=A
DO 2 I=2,M-2,2
X=X+2.0*M
E=E+2.0*FRC(X)
2 CONTINUE
E=E*M/3.0
RETURN
END

SUBROUTINE SIMP3(A,B,M,E)

* FUNCTION: APPROXIMATES THE INTEGRAL OF F(X) OVER THE *
* INTERVAL \([A,B]\) BY SIMPSON'S RULE *

** CALL SEQUENCE:** CALL SIMP(A,B,M,E)  
** EXTERNAL FUNCTIONS/SUBROUTINES:** FUNCTION F(X)  
** PARAMETERS:**  
A = INTERVAL LEFT ENDPOINT  
B = INTERVAL RIGHT ENDPOINT  
M = NUMBER OF SUBINTERVALS(POSITIVE EVEN INTEGER)  
E = ESTIMATE OF THE INTEGRAL  

** DOUBLE PRECISION A,B,E,X,H,F **  
*** INITIALIZATION ***  
IF (B .LT.-4 .OR. A .GT. 4) THEN  
E=0.0000000000001  
RETURN  
ENDIF  
E=F(A)+F(B)  
H=(B-A)/M  
X=A-H  

*** COMPUTE THE SUM OF ODD INDEX TERMS ***  
DO 1 I=1,M-1,2  
X=X+2,0*M  
E=E+4,0*F(X)  
1 CONTINUE  

*** COMPUTE THE SUM OF EVEN INDEX TERMS ***  
X=A  
DO 2 I=2,M-2,2  
X=X+2,0*M  
E=E+2,0*F(X)  
2 CONTINUE  
E=E+H/3,0  
RETURN  
END

** DOUBLE PRECISION FUNCTION F(L(XBAR)) **  
INCLUDE 'COM2,VAR'  
DOUBLE PRECISION XBU,PGU,M1  
COMMON /BLOCK2/XBU,PGU,M1  
COMMON /BLOCK3/NO  
DOUBLE PRECISION XBAR,TEMP1,TEMP2,XNO  
XNO=NO  
WRITE(6,*) 'TEST PT 1'  
TEMP1=VARA*(VAR/XNO)  
WRITE(6,*) 'TEMP1=VARA*(VAR/XNO)',TEMP1  
TEMP2=1./(((2*PI)**.5)*(TEMP1**.5))  
WRITE(6,*) 'TEMP2 =',TEMP2  
FL=TEMP2*DEXP(-.5*((M1-XBAR)**2)/TEMP1)  
WRITE(6,*) 'FL FUNCTION = ',FL  
RETURN  
END

** DOUBLE PRECISION FUNCTION F(R(U)) **
INCLUDE 'COM2.VAR'
DOUBLE PRECISION XBU, PGU, M1
COMMON /BLOCK2/XBU, PGU, M1
COMMON /BLOCK3/NO
DOUBLE PRECISION UU, C1, C2, C3, C4, TEMPI, TEMP2, TEMP3, HU, v1, v2

N=NO
C1=NO*(3.0/2.0)
C2=XBU/uu
C3=C4=((C2**2)/(2*VAR))

WRITE(6,*) 'C1 = ', C1
WRITE(6,*) 'C2 = ', C2
WRITE(6,*) 'C3 = ', C3
WRITE(6,*) 'C4 = ', C4

TEMPI=-(C1*C2*(exp(-1.0*C4)))
TEMP2=(((VAR**1.5)**((2*PI)**.5))
WRITE(6,*) 'TEMPI = ', TEMPI, ' TEMP2 = ', TEMP2
TEMPI=TEMPI/TEMP2

WRITE(6,*) 'XBU = ', XBU
HU=exp(-((UU+MO)**2)/(2*VAR))/((VAR**.5)*SQRT(2*PI))
FR=TEMPI*HU*(((CA-CP-CSN)-(PGU*(CA-CP))))

V1=CA-CP-CSN
V2=PGU*(CA-CP)

WRITE(6,*) 'V1 = ', V1, ' V2 = ', V2
RETURN
END

SUBROUTINE D2AHEAD
WRITE(45,10)
10 FORMAT(5X,'N',7X,'2-ND DERIV',9X,'ECSTSP',13X,'PGU',13X,'ETCOST')
RETURN
END

SUBROUTINE DERV2A(XLO,XUP,D2A)
INCLUDE 'COM2.VAR'
DOUBLE PRECISION XBU, PGU, M1, Z
COMMON /BLOCK2/XBU, PGU, M1, Z
COMMON /BLOCK3/NO
DOUBLE PRECISION ECSTAC, ETCOST
COMMON /BLOCK4/ECSTAC, ETCOST

DOUBLE PRECISION XLO, XUP, TEMPI, TEMP2, TEMP3, TEMP4, D2A
DOUBLE PRECISION XNO, TEMPS, TEMPP, TEMPE, TEMP7

NINT=20
WRITE(6,*) 'D2 = ', D2
CALL SIMP4(XLO, XUP, NINT, D2A)

XNO=NO
TEMPI=(VAR*(VAR/XNO))/(VAR+(VAR/XNO))
TEMP2=(((TEMP1*VAR)/(VAR+TEMP1))**.5)
TEMP3=(XNO**((3/2))**TEMP2/((VAR**2)*(VAR**.5)*(2*PI))
TEMP4=exp(-.5*((M1-XBU)**2)/(VAR+(VAR/XNO)))
DOUBLE PRECISION A,B,E,X,M,F

*** INITIALIZE ***
IF (B .LT. -4 .OR. A .GT. 4) THEN
E=0.00000000000001
RETURN
ENDIF

E=F(A)+F(B)
M=(B-A)/M
X=A-M

*** COMPUTE THE SUM OF ODD INDEX TERMS ***
DO 1 I=1,M-1,2
X=X+2.0*M
E=E+4.0*F(X)
1 CONTINUE

*** COMPUTE THE SUM OF EVEN INDEX TERMS ***
X=A
DO 2 I=2,M-2,2
X=X+2.0*M
E=E+2.0*F(X)
2 CONTINUE
E=E*H/3.0
RETURN
END

DOUBLE PRECISION FUNCTION F(A)
INCLUDE 'CM2.Var'
DOUBLE PRECISION XBU,PGU,M1
COMMON /BLOCK2/XBU,PGU,M1
COMMON /BLOCK3/NO
DOUBLE PRECISION UU,C1,C2,C3,C4,TEMP1,TEMP2,TEMP3,HH,1,2
DOUBLE PRECISION TEMP4,TEMP5
C

XNU=NO

TEMP1=((VARA*VAR)/(XNO*VARA)+VAR)

TEMP2=(VARA*(VAR/XNO))

TEMP3=((M1*(VAR/XNO))*(VARA*XBU))/TEMP2

TEMP5=((VARA*XBU)/XNO)/TEMP2

TEMP4=((TEMP3+VAR)*(TEMP5+VAR))/TEMP2

HU=(XBU-TEMP4)*DEXP(-((TEMP3+U)*2)/(2*(TEMP5+VAR)))

C

FB=HU

C

WRITE(6,'') 'FB VALUE = ',FB
RETURN
END
SUBROUTINE READ2
INCLUDE 'COM2.VAR'

WRITE (6,1)
FORMAT('1')
WRITE(6,*)'IS THE VALUE IN THE FILE....ENTER 0 OR 1'
READ(5,*)ANS

ANS=1
IF (ANS .EQ. 1) GOTO 100
WRITE(6,*)'ENTER VALUES FOR THE FOLLOWING VARIABLES'
WRITE(6,2)
READ(S,*)U
WRITE(6,3)
FORMAT(1X,'U=? ',$)
READ(S,*)U
WRITE(6,4)
FORMAT(1X,'L=? ',$)
READ(S,*)L
WRITE(6,5)
FORMAT(1X,'VAR=? ',$)
READ(S,*)VAR
WRITE(6,6)
FORMAT(1X,'VARA=? ',$)
READ(S,*)VARA
WRITE(6,7)
FORMAT(1X,'CSN=? ',$)
READ(S,*)CSN
WRITE(6,8)
FORMAT(1X,'CA=? ',$)
READ(S,*)CA
WRITE(6,9)
FORMAT(1X,'CP=? ',$)
READ(S,*)CP
WRITE(6,10)
FORMAT(1X,'LSS=? ',$)
READ(S,*)LSS
WRITE(6,11)
FORMAT(1X,'CSSP=? ',$)
READ(S,*)CSSP

WRITE (10,*)U,L,VAR,VARA,CSN,CA,CP,LSS,CSSP
RETURN

READ (10,*)U,L,VAR,VARA,CSN,CA,CP,LSS,CSSP
RETURN
END
SUBROUTINE HEAD2
INCLUDE '(UM2,VAR'
WRITE (99,1)
1 FORMAT(/&
&write (99,1), 'TABLE 2: SCRAPING COST AND RELATED SPECIFICATIONS'/
&write (99,2), 'INPUT: MODEL SPECIFICATIONS'/
&write (99,2), '-------------------------------------------'/
&write (99,2), 'UPPER LIMIT OF THE O.C.X: ',F15.5/
&write (99,2), 'LOWER LIMIT OF THE O.C.X: ',F15.5/
&write (99,2), 'VARIANCE OF X: ',F15.5/
&write (99,2), 'VARIANCE OF THE MEAN OF X: ',F15.5/
&write (99,2), 'UNIT COST OF SCREENING: ',F15.5/
&write (99,2), 'UNIT COST OF ACCEPTANCE: ',F15.5/
&write (99,2), 'COST OF SCRAPING OR REPLACING: '
&write (99,2), 'A DEFECTIVE UNIT FOUND DURING: '/
&write (99,2), 'SAMPLING OR SCREENING: '
&write (99,2), 'INSPCTION: ',F15.5/
&write (99,2), 'UNIT COST OF SCRAPING: ',F15.5/
&write (99,2), 'LOT SIZE: ',F15.5/
WRITE(99,3)
3 FORMAT(/&
&write (99,3), 'OUTPUT: SAMPLE SIZE, ROOTS OF THE COST FUNCTION, '/
&write (99,3), 'POSTERIOR AND SAMPLING COSTS PER UNIT'/
&write (99,3), 'COL. NO. DESCRIPTION '/
&write (99,3), '1 SAMPLE SIZE '/
&write (99,3), '2 (a * b) / (b * n*b) WHERE: '/
&write (99,3), 'b: VARIANCE OF THE MEAN OF X'/
&write (99,3), 'u: VARIANCE OF X '/
&write (99,3), 'n: SAMPLE SIZE '/
&write (99,3), '3 LOWER DISPOSITION LIMIT FOR LOT SCREENING'/
&write (99,3), '4 UPPER DISPOSITION LIMIT FOR LOT SCREENING'/
&write (99,3), '5 UPPER DISPOSITION LIMIT OF THE SAMPLE MEAN'/
&write (99,3), '6 LOWER DISPOSITION LIMIT OF THE SAMPLE MEAN'/
&write (99,3), '7 SCREENING COST PER LOT'/
&write (99,3), 'P2 IS THE FRACTION DEFECTIVE AT WHICH THE '/
&write (99,3), 'COSTS OF SCREENING AND SCRAPING ARE '/
&write (99,3), 'EQUAL'/
WRITE(99,4)
4 FORMAT(/&
&write (99,4), 'T10,'T16,'T26,'T36,'T46,'T56,'T66','T76,'
&write (99,4), '7,'T76,'B',/}
&write (99,4), '== T5(''-')/)
END
DOUBLE PRECISION U,L,VAR,VARA,VARN,VARMU,CR,CE,MN1,MN2,XBARN1,XBARN2,
& CSH,CA,CP,CSSP,NS,MO
COMMON /BLOCK1/ U,L,VAR,VARA,VARN,VARMU,CR,CE,CS,LS,MN1,MN2,
& VN,SIGN,CSH,CA,CP,LSS,CSSP,N,NS,MO
PARAMETER (PI=3.1459)
SUBROUTINE NOIMR11(N,A,B,H,M,E,FF1,AINT,KEY)
REAL*8 FF1
* A = A(i), B = B(i), H(i), M(i), E(i), FF1, AINT, KEY
* RETURN
* IF(N,L.T.1.OR.N.GT.9,OR,M.LT.1.OR.A.LT.1.5.OR.A.GT.2.
* RETURN
* DO 2 I=1,N
* IF(H(I).LE.0. OR. H(I).GT.1.) RETURN
* D(I)=A(I)
* E = AMAX1(E,TOL)
* M = MIN(M,15)
* DO 1 J=N,B
* P(I+1)=D(I) X(I)=0.
* N(I+1)=1
* D(I+1)=0.
* 10 L=1
* DO 20 I=1,N
* C(I)=D(I)-D(I)
* U=0.
* K=0.
* DO 22 I=1,N
* G(I)=H(I)/K(I)
* NH(I)=1./G(I)+.5
* 22 P(I)=C(I)*G(I)
* NH(I)=NH(I)
* DO 3U I=1,NN9
* X(I)=0(9)+P(9)*O(I9-.S)
* 3U=1,NNB=NN9
* X(B)=0(B)+P(B)*O(B-.S)
* 3U=1,NN7=NN9
* X(7)=0(7)+P(7)*O(7-.S)
* 3U=1,NN6=NN7
* X(S)=0(5)+P(5)*O(5-.S)
* 3U=1,NN5=NN6
* X(4)=0(4)+P(4)*O(4-.S)
* 3U=1,NN3=NN5
* X(3)=D(3)+P(3)*O(3-.S)
* 3U=1,NN2=NN3
* X(2)=D(2)+P(2)*O(2-.S)
* 3U=1,NN1=NN2
* X(1)=D(1)+P(1)*O(1-.S)
SUBROUTINE NDIMR12(N,A,B,H,AL,M,E,FF2,AINT,KEV)
REAL*B
REAL*8 A(n),B(n),C(9),K(15),H(n),G(n),NN(9),P(9),E(15),V(15)
REAL*8 FF2
REAL*8 A(n),B(n),C(9),K(15),H(n),G(n),NN(9),P(9),E(15),V(15)
REAL*8 FF2
REAL*8 A(n),B(n),C(9),K(15),H(n),G(n),NN(9),P(9),E(15),V(15)
REAL*8 FF2
REAL*8 A(n),B(n),C(9),K(15),H(n),G(n),NN(9),P(9),E(15),V(15)
REAL*8 FF2
REAL*8 A(n),B(n),C(9),K(15),H(n),G(n),NN(9),P(9),E(15),V(15)
REAL*8 FF2
REAL*8 A(n),B(n),C(9),K(15),H(n),G(n),NN(9),P(9),E(15),V(15)
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REAL*8 FF2
REAL*8 A(n),B(n),C(9),K(15),H(n),G(n),NN(9),P(9),E(15),V(15)
REAL*8 FF2
REAL*8 A(n),B(n),C(9),K(15),H(n),G(n),NN(9),P(9),E(15),V(15)
PROGRAM PROGSAA

THIS PROGRAM READS THE DATA FROM FILES GENERATED BY PROGRAMS :PROG2AA, FOR, PROG3AA, FOR, PROG4AA, FOR.

INTEGER N2,N3,N4
DOUBLE PRECISION D2A2,D2A3,D2A4,D2A
DOUBLE PRECISION ETCOST2,ETCOST3,ETCOST4,ETCOST

DO 10 I=1,91
READ(62,5) N2,D2A2,ETCOST2
READ(63,5) N3,D2A3,ETCOST3
READ(64,5) N4,D2A4,ETCOST4
       5 FORMAT (15,1X,E13.4,1X,E13.4)
D2A=D2A2+D2A3+D2A4
ETCOST=ETCOST2+ETCOST3+ETCOST4
WRITE(65,6) N2,D2A,ETCOST
       6 FORMAT (1X,15,4X,E13.4,4X,E13.4)
CONTINUE
STOP
END

OUTPUT OF PROGSAA.

<table>
<thead>
<tr>
<th>COL. NO.</th>
<th>TERMS COMPUTED</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>SAMPLE SIZE</td>
</tr>
<tr>
<td>2.</td>
<td>THE SECOND DERIVATIVE OF THE TOTAL COST RELATIVE TO THE VARIABLES INVOLVED.</td>
</tr>
<tr>
<td>3.</td>
<td>THE EXPECTED VALUE OF THE TOTAL COST.</td>
</tr>
</tbody>
</table>
APPENDIX G

PROGRAM TKEVAL

This program computes the moments of the fraction defective $p$ up to the $k^{th}$ power $k \geq 1$.

Input:

The following two subroutines imput the required data to compute $E(p^k)$ and outputs are the user supplied input data for display respectively: DATIN and DATOUTIN.

Output:

The following subroutines generate the output data:

COMPUTTK, COMPTTINIT (INITIALIZE all variables for the subroutine COMPUTTK), COMPUTV, SORTAS(X,N), COMPUTPOFL, SIMP(A,B,M,E), COMPUTCOEF, FUNC(X,RESULT), CUNCTION FNORMK(Z,MEAN,VAR) [this function subroutine evaluates the normal density’s kernal value], COMPUTTKOFV, COMPLITKOFV, DATOUT, DATOUTPOFL,

DATOUTCOEF, DATOUTTTK, DATOUTTEPK, PLTDATOUT,

PLTTTRUDATOUT, PLTESTDATOUT, STRNUM (N,STR), ESTIMATE,

ESTPOFL(L,J), ERRPOFL(L,J), COMPUTEPK, SIMPMOD(U,FN,NPT,E),

DATOUTTESTPOFL, LSQM(N,M,X,F,C), GAUSI(N,M,ND,A,DILT).
PROGRAM TKEVAL

FUNCTION: THIS PROGRAM COMPUTES THE QC'S PROBLEM E(P**K) USING THE VAX COMPUTER

CALLS: DATIN, COMPUTTK, DATOUT, PLTDATOUT

PARAMETERS:
  INPUT: SEE DATIN SUBROUTINE
  OUTPUT: SEE DATOUT SUBROUTINE

PROGRAMMER: FOUAD JALBOUT

COMPUTER: VAX

LANGUAGE: FORTRAN 77

*** SET GLOBAL VARIABLES DECLARATIONS ***
REAL V(31), TK(6,6,6,6,6,31), POFL(6,6,6,31), COEF(6,6,31)
REAL KVAL(6), VARX(6), MMEAN(6), VARMU(6), LVAL(6)
REAL VMAX, VMIN, DV
INTEGER KMAX, IMAX, VMAX, MNMAX, VMUMAX, LMAX
REAL NEGINF, FACTOR, PIE
CHARACTER INDV*1
REAL A1, A2
REAL POFLE1(6,6,6,31), CPFLE1(6,6,6,10)
REAL EPOFL(6,6,6,31), REPOFL(6,6,6,31)
INTEGER NOFIL(10)
INTEGER MPOFL
CHARACTER FILNAM*2
REAL EXPK(6,6,6,6)
COMMON /OCDATA/VMAX, VMIN, DV, TK, POFL, COEF, EXPK
COMMON /OCDATA2/KVAL, VARX, MMEAN, VARMU, LVAL
COMMON /OCDATA3/KMAX, IMAX, VMAX, MNMAX, VMUMAX, LMAX
COMMON /OCDATA5/NEGINF, FACTOR, PIE
COMMON /OCDATA6/INDV
COMMON /OCDATA7/A1, A2
COMMON /OCDATA8/POFLE1, MPOFL, CPFLE1
COMMON /OCDATA9/EPOFL, REPOFL
COMMON /ST1/NOFIL
COMMON /ST2/FILNAM

*** PROGRAM - BEGIN ***
*** INPUT DATA ***
CALL DATIN

*** COMPUTE TK(V) FOR ASSORTED CONDITIONS ***
CALL COMPUTTK

*** OUTPUT DATA ***
CALL DATOUT

*** STORE OUTPUT IN FILE FOR PLOTTING ***
CALL PLTDATOUT
STOP

END MAIN PROGRAM TKEVAL

END SUBROUTINE DATIN

FUNCTION: THIS SUBROUTINE INPUTS THE REQUIRED DATA TO COMPUTE E(P**K) FOR DIFFERENT VALUES OF K USING A VAX
COMPUTER (P=RATION DEFECTIVE)

USAGE:

CALL SEQUENCE: CALL DATIN

PARAMETERS:

INPUT: (STORED IN FILE='QC1IN1.DAT')

VMAX=MAXIMUM V VALUE FOR TK(V)

VMIN=MINIMUM V VALUE FOR TK(V)

KMAX=MAXIMUM NUMBER OF K VALUES

IMAX=MAXIMUM NUMBER OF V VALUES (ODD INTEGER)

VMAX=MAXIMUM NUMBER OF V VALUES

VMAX=MAXIMUM NUMBER OF VARIANCE OF X VALUES

MMAX=MAXIMUM NUMBER OF MEAN - M VALUES

VMUMAX=MAXIMUM NUMBER OF VARIANCE OF MU VALUES

LMAX=MAXIMUM NUMBER OF LOWER LIMIT - L VALUES

KVAL=KMAX BY 1 ARRAY OF K VALUES

MMEAN=MMAX BY 1 ARRAY OF MEAN - M VALUES

VARMU=VMAX BY 1 ARRAY OF VARIANCE OF MU VALUES

VARX=VMAX BY 1 ARRAY OF VARIANCE OF X VALUES

LVAL=LMAX BY 1 ARRAY OF LOWER LIMIT - L VALUES

V=IMAX BY 1 ARRAY OF V VALUES

OUTPUT:

TO PROGRAM SUBROUTINE COMPUTTK (VIA COMMON)

PROGRAMMER: FOUAD JALBOUT

COMPUTER: VAX

LANGUAGE: FORTRAN 77

*** SET GLOBAL VARIABLES DECLARATIONS ***

REAL V(31), TK(6,6,6,6,6,31), POFL(6,6,6,31), COEF(6,6,31)
REAL KVAL(6), VARX(5), MMEAN, VARMU, LVAL
REAL VMAX, VMIN, DV
INTEGER KMAX, IMAX, VMAX, VMUMAX, LMAX
REAL NEGINF, FACTOR, PIE
CHARACTER INDV*1
REAL A1, A2
REAL POFLE1(6,6,6,31), C(10)
INTEGER MPOFL
REAL EXPK(6,6,6,6)
COMMON /QCDAOT/V, VMAX, VMIN, DV, TK, POFL, COEF, EXPK
COMMON /QCDA2/KVAL, VARX, MMEAN, VARMU, LVAL
COMMON /QCDA3/KMAX, IMAX, VMAX, VMAX, VMUMAX, LMAX
COMMON /QCDA4/NEGINF, FACTOR, PIE
COMMON /QCDA5/INDV
COMMON /QCDA7/A1, A2
COMMON /QCDA8/POFLE1, MPOFL, C

*** SUBROUTINE - BEGIN ***

*** READ DATA FROM FILE='QC1IN1.DAT' ***

OPEN(UNIT=2, STATUS='OLD', FILE='QC1IN1,DAT')

READ(2,1) VMAX, VMIN, INDV
1 FORMAT(E15.4, E15.4, A1)
READ(2,2) KMAX, IMAX, VMAX, VMAX, VMUMAX, LMAX
2 FORMAT(I3, I3, I3, I3, I3)
READ(2,4) (KV, I=1, KMAX)
3 FORMAT(SE15.4)
READ(2,4) (MMIEAN, I=1, MMAX)
4 FORMAT(SE15.4)
READ(2,4) (VARX, I=1, VXMAX)
5 FORMAT(SE15.4)
READ(2,4) (VARX, I=1, VXMAX)
6 FORMAT(SE15.4)
READ(2,7) (LVAL(I),I=1,LMAX)
7 FORMAT(SE15.4)
IF (INDV.'Y'.AND.INDV.NE.'y') THEN
    READ(2,6) (V(I),I=1,IMAX)
ELSE
ENDIF
READ(2,9) MPOFL
9 FORMAT(I3)
CLOSE(UNIT=2,STATUS='SAVE')
RETURN

C *** COMPUTE DV FROM VMAX AND VMIN ***

C END SUBROUTINE DATIN

END SUBROUTINE COMPUTTK

C FUNCTION: THIS SUBROUTINE COMPUTES THE PROBABILITY P(L) GIVEN V AS THE FIRST PART OF THE TK(V) EQUATION

C USAGE:
C CALL SEQUENCE: CALL COMPUTTK
C EXTERNAL FUNCTIONS/SUBROUTINES:
C    COMPUTINIT, COMPUTV, COMPUTPOFL
C    COMPUTCOEF, COMPUTTKOFV
C PARAMETERS:
C    INPUT:
C    SEE SUBROUTINES COMPUTINIT, COMPUTV, COMPUTPOFL
C    COMPUTCOEF, COMPUTTKOFV
C    OUTPUT:
C    SEE SUBROUTINES COMPUTINIT, COMPUTV, COMPUTPOFL, COMPUTCOEF, COMPUTTKOFV
C PROGRAMMER: FOUAD JALBOUT
C COMPUTER: VAX
C LANGUAGE: FORTRAN 77

*** SET GLOBAL VARIABLES DECLARATIONS ***
REAL V(31),TK(6,6,6,6,31),POFL(6,6,6,31),COEF(6,6,31)
REAL KVAL(6),VARX(6),MMEAN(6),VARMU(6),LVAL(6)
REAL VMAX,VMIN,DV
INTEGER KMAX,IMAX,VXMAX,VMUMAX,LMAX
REAL NEGINF,FACTOR,PIE
CHARACTER INDV'*1
REAL A1,A2
REAL POFLE1(6,6,6,31),C(10)
INTEGER MPOFL
REAL EXPK(6,6,6,6)
COMMON /QCDAT0/V,VMAX,VMIN,DV,TK,POFL,COEF,EXPK
COMMON /QCDAT2/KVAL,VARX,MMEAN,VARMU,LVAL
COMMON /QCDAT3/KMAX,IMAX,VXMAX,VMUMAX,LMAX
COMMON /QCDAT5/NEGINF,FACTOR,PIE
COMMON /QCDAT6/INDV
COMMON /QCDAT7/A1,A2
COMMON /QCDATB/POFLE1,MPOFL,C

*** INITIALIZE CONSTANT VARIABLES ***

CALL COMPUTINIT
**SUBROUTINE - BEGIN**

**COMPUTE V VALUES**

CALL COMPUTV

**COMPUTE POFL(K,L,J,I)=FIRST FACTOR OF TK(V)**

CALL COMPUTPOFL

**COMPUTE COEF(M,N,I)=SECOND FACTOR OF TK(V)**

CALL COMPUTCOEF

**COMPUTE POFL1(K,L,J,I)=ESTIMATE**

CALL ESTIMATE

**COMPUTE E(P**K)**

CALL COMPUTEPK

RETURN

**END SUBROUTINE COMPUTTTK**

**FUNCTION: THIS SUBROUTINE INITIALIZES ALL VARIABLES FOR THE**

**SUBROUTINE - COMPUTTK**

**USAGE:**

**CALL SEQUENCE: CALL COMPUTINIT**

**PARAMETERS:**

**INPUT:**

NO

**OUTPUT:** (VIA COMMON)

NEGINF=-.IE+03

FACTOR=10.0

PIE=3.141596

PROGRAMMER: FOUAD JALBOUT

COMPUTER: VAX

LANGUAGE: FORTRAN

**SET GLOBAL VARIABLES DECLARATIONS**

REAL NEGINF,FACTOR,PIE

COMMON /QCDAT5/NEGINF,FACTOR,PIE

**SUBROUTINE - BEGIN**

NEGINF=-.IE+03

FACTOR=10.0

PIE=3.141596

RETURN

**END SUBROUTINE COMPUTINIT**

**FUNCTION: THIS SUBROUTINE COMPUTES THE V VALUES BASED ON THE**

**INPUT DATA VMAX AND VMIN OR ON USER SUPPLIED V VALUES**

**USAGE:**

**CALL SEQUENCE: CALL COMPUTV**

**PARAMETERS:**

**INPUT:** (VIA COMMON)

INDV=USER·RESPONSE (Y=YES OR N=NO) INDICATES WHETHER THE
DATA IS TO BE COMPUTED FROM VMAX AND VMIN OR THE
V VALUES ARE TO BE USED DIRECTLY
IMAX=MAXIMUM NUMBER OF V VALUES (ODD INTEGER)
VMAX=MAXIMUM V VALUE FOR TK(V)
VMIN=MINIMUM V VALUE FOR TK(V)
V=IMAX BY 1 ARRAY OF V VALUES (UNSORTED)
OUTPUT: (VIA COMMON)
VMAX=MAXIMUM V VALUE FOR TK(V)
VMIN=MINIMUM V VALUE FOR TK(V)
V=IMAX BY 1 ARRAY OF V VALUES (SORTED)
PROGRAMMER: FOUAD JALBOUT
COMPUTER: VAX
LANGUAGE: FORTRAN 77

*** SET GLOBAL VARIABLES DECLARATIONS ***
REAL V(31),TK(6,6,6,6,6,31),PDFL(6,6,6,31),COEF(6,6,31)
REAL KVAL(6),VARX(6),MMean(6),VARMU(6),LVAL(6)
REAL VMAX,VMIN,DV
INTEGER KMAX,IMAX,VXMAX,MNMAX,VMUMAX,LMAX
REAL NEGINF,FACTOR,PIE
CHARACTER INDV*1
REAL A1,A2
REAL EXPK(6,6,6,6)
COMMON /QCDAT0/V,VMAX,VMIN,DV,TK,PDFL,COEF,EXPK
COMMON /QCDAT1/KVAL,VARX,MMEAN,VARMU,LVAL
COMMON /QCDAT2/KMAX,IMAX,VXMAX,MNMAX,VMUMAX,LMAX
COMMON /QCDAT5/NEGINF,FACTOR,PIE
COMMON /QCDAT6/INDV
COMMON /QCDAT7/A1,A2

*** SUBROUTINE - BEGIN ***
*** COMPUTE V VALUES IF REQUIRED ***
IF (INDV.EQ.'V' .OR. INDV.EQ.'y') THEN
DV=(VMAX-VMIN)/FLOAT(IMAX-1)
V(1)=VMIN
V(IMAX)=VMAX
DO 1 I=2,IMAX-1
V(I)=V(I-1)+DV
1 CONTINUE
ELSE
*** ORDER SEQUENCE OF V VALUES IN ASCENDING ORDER ***
CALL SORTAS(V,IMAX)
*** SPECIFY MINIMUM AND MAXIMUM VALUES ***
VMIN=V(1)
VMAX=V(IMAX)
ENDIF
RETURN

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C
* NUMBER OF VALUES TO SORT
* X=N BY 1 ARRAY OF UNSORTED VALUES
* OUTPUT:
* X=N BY 1 ARRAY OF SORTED VALUES
* PROGRAMMER: THOMAS HASSETT
* COMPUTER: VAX
* LANGUAGE: FORTRAN 77
******************************************************************************
C
DIMENSION X(N)
C
*** INITIALIZATION ***
JTRPOS=N-1
C
*** SUBROUTINE - BEGIN ***
*** PERFORM SORT ***
DO 1 J=1,JTRPOS
   M=N-J
   DO 1 I=1,M
      IF (X(I).GT.X(I+1)) THEN
         TEMP=X(I)
         X(I)=X(I+1)
         X(I+1)=TEMP
      ELSE
         CONTINUE
      ENDIF
1 CONTINUE
RETURN
C
******************************************************************************
C
END SUBROUTINE SORTAS
C
******************************************************************************
C
C
FUNCTION: THIS SUBROUTINE COMPUTES THE PROBABILITY P(L) GIVEN AS THE FIRST PART
         OF THE TK(V) EQUATION
C
USAGE:
      CALL SEQUENCES: CALL COMPUTPOFL
C
EXTERNAL FUNCTIONS/SUBROUTINES: SUBROUTINE SIMP(A,B,M,E)
C
PARAMETERS:
C
INPUT:
      KMAX=MAXIMUM NUMBER OF K VALUES
      IMAX=MAXIMUM NUMBER OF V VALUES (ODD INTEGER)
      VXMAX=MAXIMUM NUMBER OF VARIANCE OF X VALUES
      LMAX=MAXIMUM NUMBER OF LOWER LIMIT - L VALUES
      KVAL=KMAX BY I ARRAY OF K VALUES
      VARX=VXMAX BY I ARRAY OF VARIANCE OF X VALUES
C
OUTPUT:
      POFL=KMAX BY LMAX BY VXMAX BY IMAX ARRAY OF P(L) VALUES
      (1ST FACTOR IN TK(V))
C
PROGRAMMER: FOUAD JALBOUT
C
COMPUTER: VAX
C
LANGUAGE: FORTRAN 77
******************************************************************************
C
*** SET GLOBAL VARIABLES DECLARATIONS ***
REAL V(31),TK(6,6,6,6,6,31),POFL(6,6,6,31),COEF(6,6,31)
REAL KVAL(6),VARX(6),MMEAN(6),VARMU(6),LVAL(6)
REAL VMAX, VMIN, DV
INTEGER KMAX, IMAX, VXMAX, MNMAX, VMUMAX, LMAX
REAL NNEGIF, FACTOR, PIE
CHARACTER INDV
REAL NEGINF, FACTOR, PIE
REAL AI, A2
REAL EXPK(6, 6, 6, 6, 6)
COMMON /QCDAT0/V, VMAX, VMIN, DV, TK, POFL, COEF, EXPK
COMMON /QCDAT2/KVAL, VARX, MMEAN, VARMU, LVAL
COMMON /QCDAT3/KMAX, IMAX, VXMAX, MNMAX, VMUMAX, LMAX
COMMON /QCDAT5/NEGINF, FACTOR, PIE
COMMON /QCDAT6/IND
COMMON /QCDAT7/A1, A2

*** SUBROUTINE - BEGIN ***
*** COMPUTE POFL ***
DO 1 K=1, KMAX
  DO 1 L=1, LMAX
    **COMPUTE THE NUMBER OF SUBINTERVALS**
    MM=FACTOR*(LVAL(L)-NEGINF)
    DO 1 J=1, VXMAX
      A2=VARX(J)
      DO 1 I=1, IMAX
        AI=V(I)
        CALL SIMP(NEGINF, LVAL(L), MM, POFL(K, L, J, I))
      1 CONTINUE
    POFL(K, L, J, I)=POFL(K, L, J, I)/(SQRT(2.0*PIE*VARX(J)))
  CONTINUE
RETURN

*** END SUBROUTINE COMPUTPOFL ***

END SUBROUTINE SIMP(A, B, M, E)

***************************************************************************
FUNCTION: APPROXIMATES THE INTEGRAL OF F(X) OVER THE
INTERVAL [A, B] BY SIMPSON'S RULE
USAGE:
CALL SEQUENCE: CALL SIMP(A, B, M, E)
EXTERNAL FUNCTIONS/SUBROUTINES: FUNCTION F(X)
PARAMETERS:
INPUT:
A=INTERVAL LEFT ENDPOINT
B=INTERVAL RIGHT ENDPOINT
M=NUMBER OF SUBINTERVALS (POSITIVE EVEN INTEGER)
OUTPUT:
E=ESTIMATE OF THE INTEGRAL
COMPUTER: VAX
LANGUAGE: FORTRAN 77
COMMENT: FROM 'AN INTRODUCTION FROM TO NUMERICAL COMPUTATIONS',
BY SIDNEY YAKOWITZ AND FERENC SZJAROVZKY,
MACMILLAN PUBL. CO., NEW YORK, NY, C.1986
***************************************************************************

*** INITIALIZATION ***
CALL FUNC(A, RESA)
CALL FUNC(B, RESB)
E=RESA-RESB
H=(B-A)/M
X=A-H
**SUBROUTINE SIMP**

```fortran
*** COMPUTE THE SUM OF ODD INDEX TERMS ***
DO 1 I=1,M-1.2
   X=X+2.0*H
   CALL FUNC(X,RESULT)
   E=E+4.0*RESULT
1 CONTINUE

*** COMPUTE THE SUM OF EVEN INDEX TERMS ***
DO 2 I=2,M-2.2
   X=X+2.0*H
   CALL FUNC(X,RESULT)
   E=E+2.0*RESULT
2 CONTINUE
E=E*H/3.0
RETURN
END
```

**FUNCTION: THJS SUBROUTINE COMPUTES THE COEFFICIENT VALUE GIVEN AS THE SECOND PART OF THE TK(V) EQUATION**

**USAGE:**
- CALL SEQUENCE: CALL COMPUTCOEF
- **EXTERNAL FUNCTIONS/SUBROUTINES:** FUNCTION NORMKER(Z,MEAN,VAR).

**PARAMETERS:**
- **INPUT:**
  - VMUMAX=MAXIMUM NUMBER OF VARIANCE OF MU VALUES
  - MNMAX=MAXIMUM NUMBER OF MEAN - M VALUES
  - IMAX=MAXIMUM NUMBER OF V VALUES (ODD INTEGER)
  - VARMU=VMUMAX BY 1 ARRAY OF VARIANCE OF MU VALUES
  - MMEAN=MNMAX BY 1 ARRAY OF MEAN - M VALUES
  - V=IMAX BY 1 ARRAY OF V VALUES
- **OUTPUT:**
  - COEF=VMUMAX BY MNMAX BY IMAX ARRAY OF COEFFICIENT VALUES
  - (2ND FACTOR IN TK(V))

**PROGRAMMER:** FOUDAL JALBOUT
**COMPUTER:** VAX
**LANGUAGE:** FORTRAN 77

**SET GLOBAL VARIABLES DECLARATIONS**

```fortran
REAL V(31),TK(6,6,6,6,6,31),POFL(6,6,6,31),COEF(6,6,31)
REAL KVAL(6),VARX(6),MMEAN(6),VARMU(6),LVAL(6)
REAL VMAX,VMIN,DV
INTEGER KMAX,IMAX,VXMAX,MNMAX,VMUMAX,LMAX
REAL NEGINF,FACTOR,PIE
CHARACTER INDV*1
REAL A1,A2
REAL EXPK(6,6,6,6)
COMMON /QCDATQ/V,VMAX,VMIN,DV,TK,POFL,COEF,EXPK
COMMON /QCDAT2/KVAL,VARA,MMEAN,VARMU,LVAL
COMMON /QCDAT3/KMAX,IMAX,VXMAX,MNMAX,VMUMAX,LMAX
COMMON /QCDAT5/NEGINF,FACTOR,PIE
COMMON /QCDAT6/INDV
COMMON/QCDAT7/A1,A2
```

**SUBROUTINE - BEGIN**
```fortran
C *** COMPUTE COEF ***
DO I = 1, VMUMAX
   DO J = 1, MMAX
      COEF(M, N, I) = FNORMK(V(I), MMEAN(N), VARMU(M)) / SQRT(2.0 * PI * VARMU(M))
   CONTINUE
END

SUBROUTINE CDMPUTCOEF
*** END SUBROUTINE COMPUTECOEF ***
END

SUBROUTINE FUNC(X, RESULT)
FUNCTION, THIS FUNCTION SUBROUTINE EVALUATES THE NORMAL DENSITY'S KERNEL VALUE USING A SINGLE ARGUMENT
USAGE:
CALL SEQUENCE: F(X)
PARAMETERS:
INPUT:
X = VALUE OF THE PRINCIPAL ARGUMENT
A1 = NORMAL DENSITY'S MEAN (VIA COMMON)
A2 = NORMAL DENSITY'S VARIANCE (VIA COMMON)
OUTPUT:
F = NORMAL DENSITY'S KERNEL VALUE AT X
PROGRAMMER: FOUAD JALBOUT
COMPUTER: VAX
LANGUAGE: FORTRAN 77

*** SET GLOBAL VARIABLES DECLARATIONS ***
REAL X, RESULT
REAL A1, A2
COMMON /QCDAT7/A1, A2

*** FUNCTION - BEGIN ***
*** EVALUATE NORMAL DENSITY'S KERNEL ***
RESULT = FNORMK(X, A1, A2)
RETURN

END FUNCTION SUBROUTINE F

SUBROUTINE FNORMK(Z, MEAN, VAR)
FUNCTION: THIS FUNCTION SUBROUTINE EVALUATES THE NORMAL DENSITY'S KERNEL VALUE
USAGE:
CALL SEQUENCE: NORMKER(Z, MEAN, VAR)
PARAMETERS:
INPUT:
Z = VALUE OF THE PRINCIPAL ARGUMENT
MEAN = NORMAL DENSITY'S MEAN
VAR = NORMAL DENSITY'S VARIANCE
OUTPUT:
NORMKER = NORMAL DENSITY'S KERNEL VALUE AT Z
PROGRAMMER: FOUAD JALBOUT
COMPUTER: VAX
```
* LANGUAGE: FORTRAN 77

*** SET VARIABLES DECLARATIONS ***
REAL Z, MEAN, VAR

*** FUNCTION - BEGIN ***
** EVALUATE NORMAL DENSITY'S KERNAL **
FNORMK = EXP(-(Z-MEAN)**2)/(2.0*VAR)
RETURN

*** END FUNCTION SUBROUTINE NORMKER ***

END SUBROUTINE COMPUTTKOFV

*** FUNCTION: THIS SUBROUTINE COMPUTES THE TK(V), UPPER BOUND T(K), AND LOWER BOUND T(K) VALUES UNDER VARIOUS CONDITIONS ***
** USAGE:**
** CALL SEQUENCE: CALL COMPUTTKOFV **
** PARAMETERS:**
** INPUT:**
  KMAX = MAXIMUM NUMBER OF K VALUES
  VMUMAX = MAXIMUM NUMBER OF VARIANCE OF MU VALUES
  MNMAX = MAXIMUM NUMBER OF MEAN - M VALUES
  IMAX = MAXIMUM NUMBER OF V VALUES (ODD INTEGER)
  LMN = MAXIMUM NUMBER OF LOWER LIMIT - L VALUES
  VXMAX = MAXIMUM NUMBER OF VARIANCE OF X VALUES
  POFL = MAX BY LMAX BY VXMAX BY VMUMAX BY MNMAX BY IMAX+1 ARRAY OF COEFFICIENT VALUES
** OUTPUT:**
  TK = MAX BY LMAX BY VXMAX BY VMUMAX BY MNMAX BY IMAX ARRAY CONTAINING THE COMPUTED TK(V) VALUES
   (COMBINED 1ST AND 2ND FACTORS IN TK(V))
** PROGRAMMER: FOUAD JALBOUT **
** COMPUTER: VAX **
** LANGUAGE: FORTRAN 77 **

*** SET GLOBAL VARIABLES DECLARATIONS ***
REAL V(31), TK(6,6,6,6,6,31), POFL(6,6,6,31), COEF(6,6,31)
REAL KVAL(6), VARX(6), MMEAN(6), VARMU(6), LVAL(6)
REAL VMAX, VMIN, DV
INTEGER KMAX, IMAX, LMAX, LMAX
REAL NEGINF, FACTOR, PIE
CHARACTER INDV*
REAL A1, A2
REAL EXPK(6,6,6,6)
COMMON / QCDATO/V, VMAX, VMIN, DV, TK, POFL, COEF, EXPK
COMMON / QCDAT2/KVAL, VARX, MMEAN, VARMU, LVAL
COMMON / QCDAT3/KMAX, IMAX, VXMAX, VMUMAX, LMAX
COMMON / QCDAT5/NEGINF, FACTOR, PIE
COMMON / QCDAT6/INDV
COMMON / QCDAT7/A1, A2

*** SUBROUTINE - BEGIN ***
*** CALL INDIVIDUAL SUBROUTINES ***
CALL COMPUTTRUTKOFV
CALL COMPUTUPTKOFV
CALL COMPUTTKOFV
RETURN

END SUBROUTINE COMPUTTKOFV

END SUBROUTINE COMPUTTRUTKOFV

FUNCTION: THIS SUBROUTINE COMPUTES THE TK(V) VALUES UNDER VARIOUS CONDITIONS

USAGE:
CALL SEQUENCE: CALL COMPUTTKOFV

PARAMETERS:

INPUT:
KMAX = MAXIMUM NUMBER OF K VALUES
VMUMAX = MAXIMUM NUMBER OF VARIANCE OF MU VALUES
MNMAX = MAXIMUM NUMBER OF MEAN - M VALUES
LMAX = MAXIMUM NUMBER OF V VALUES (ODD INTEGER)
VXMAX = MAXIMUM NUMBER OF VARIANCE OF X VALUES
POFL = KMAX BY LMAX BY VXMAX BY IMAX+1 ARRAY OF P(L) VALUES
COEF = VMUMAX BY MNMAX BY IMAX ARRAY OF COEFFICIENT VALUES

OUTPUT:
TK = KMAX BY LMAX BY VXMAX BY VMUMAX BY MNMAX BY IMAX ARRAY CONTAINING THE COMPUTED TK(V) VALUES (COMBINED 1ST AND 2ND FACTORS IN TK(V))

PROGRAMMER: FOUAD JALBOUT
COMPUTER: VAX
LANGUAGE: FORTRAN 77

SET GLOBAL VARIABLES DECLARATIONS
REAL V(31), TK(6, 6, 6, 6, 31), POFL(6, 6, 31), COEF(6, 6, 31)
REAL KVAL(6), VARX(6), MMEAN(6), VARMU(6), LVAL(6)
REAL VMAX, VMIN, DV
INTEGER KMAX, IMAX, VXMAX, MNMAX, VMUMAX, LMAX
REAL NEGINF, FACTOR, PIE
CHARACTER INDV, AI, A2
REAL EXPK(6, 6, 6, 6)
COMMON QCDAT0/V, VMAX, VMIN, DV, TK, POFL, COEF, EXPK
COMMON QCDAT2/KVAL, VARX, MMEAN, VARMU, LVAL
COMMON QCDAT3/KMAX, IMAX, VXMAX, MNMAX, VMUMAX, LMAX
COMMON QCDAT5/NEGINF, FACTOR, PIE
COMMON QCDAT6/INDV
COMMON QCDAT7/A1, A2

SUBROUTINE BEGIN

COMPUTE TK(V)...

CONTINUE
RETURN
SUBROUTINE COMPUTUPTKOFV

FUNCTION: THIS SUBROUTINE COMPUTES THE UPPER BOUND TK(V) VALUES UNDER VARIOUS CONDITIONS

USAGE:
CALL SEQUENCE: CALL COMPUTUPTKOFV

PARAMETERS:

INPUT:
KMAX=MAXIMUM NUMBER OF K VALUES
VMUMAX=MAXIMUM NUMBER OF VARIANCE OF MU VALUES
IMAX=MAXIMUM NUMBER OF V VALUES (ODD INTEGER)
LMAX=MAXIMUM NUMBER OF LOWER LIMIT - L VALUES
VXMAX=MAXIMUM NUMBER OF VARIANCE OF X VALUES
POFL=KMAX BY LMAX BY VXMAX BY IMAX+1 ARRAY OF P(L) VALUES

COEF=VMUMAX BY VMUMAX BY VMUMAX BY LMAX ARRAY OF COEFFICIENT VALUES

OUTPUT:
TK=KMAX BY LMAX BY VXMAX BY VMUMAX BY MNMAX BY IMAX ARRAY CONTAINING THE COMPUTED TK(V) VALUES (COMBINED 1ST AND 2ND FACTORS IN TK(V))

PROGRAMMER: FOUAD JALBOUT
COMPUTER: VAX
LANGUAGE: FORTRAN 77

* SET GLOBAL VARIABLES DECLARATIONS *
REAL V(31),TK(6,6,6,6,6,31),POFL(6,6,6,31),COEF(6,6,31)
REAL KVAL(6),VARX(6),MMEAN(6),VARMU(6),LVAL(6)
REAL VMAX,VMIN,DV
INTEGER KMAX,IMAX,VXMAX,MNMAX,VMUMAX,LMAX
REAL NEGINF,FACTOR,PIE
CHARACTER INDV,
REAL A1,A2
REAL EXPK(6,6,6,6)

COMMON /QCDAT0/V,VMAX,VMIN,DV,TK,POFL,COEF,EXPK
COMMON /QCDAT2/KVAL,VARX,MMEAN,VARMU,LVAL
COMMON /QCDAT3/KMAX,IMAX,VXMAX,MNMAX,VMUMAX,LMAX
COMMON /QCDAT5/NEGINF,FACTOR,PIE
COMMON /QCDAT6/INDV
COMMON /QCDAT7/A1,A2

** SUBROUTINE - BEGIN **
*** COMPUTE UPPER BOUND FOR TK(V) ***
RETURN

*** END SUBROUTINE COMPUTUPTKOFV ***

END SUBROUTINE COMPUTUPTKOFV

END SUBROUTINE COMPUTUPTKOFV

* FUNCTION: THIS SUBROUTINE COMPUTES THE LOWER BOUND TK(V) VALUES UNDER VARIOUS CONDITIONS

USAGE:
CALL SEQUENCE: CALL COMPUTLOTKOFV

PARAMETERS:
**INPUT:**

- KMAX = MAXIMUM NUMBER OF K VALUES
- VMUMAX = MAXIMUM NUMBER OF VARIANCE OF MU VALUES
- MNMAX = MAXIMUM NUMBER OF MEAN - M VALUES
- IMAX = MAXIMUM NUMBER OF V VALUES (ODD INTEGER)
- VXMAX = MAXIMUM NUMBER OF VARIANCE OF X VALUES
- POFL = KMAX BY LMAX BY VXMAX BY IMAX + 1 ARRAY OF P(L) VALUES
- COEF = VMUMAX BY MNMAX BY IMAX ARRAY OF COEFFICIENT VALUES

**OUTPUT:**

- TK = KMAX BY LMAX BY VXMAX BY VMUMAX BY MNMAX BY IMAX
  - ARRAY CONTAINING THE COMPUTED TK(V) VALUES
  - (COMBINED 1ST AND 2ND FACTORS IN TK(V))

**PROGRAMMER:** FOUAD JALBOUT

**COMPUTER:** VAX

**LANGUAGE:** FORTRAN 77

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**GLOBAL VARIABLES DECLARATIONS**

- REAL V(31), TK(6, 6, 6, 6, 6, 31), POFL(6, 6, 6, 31), COEF(6, 6, 31)
- REAL KVAL(6), VARX(6), MMEAN(6), VARMU(6), LVAL(6)
- REAL VMAX, VMIN, DV
- INTEGER KMAX, IMAX, VXMAX, MNMAX, VMUMAX, LMAX
- REAL NEGINF, FACTOR, PIE
- CHARACTER INOV, REAL AI, A2
- REAL EXPK(6, 6, 6, 6)
- COMMON /QCDA0/V, VMAX, VMIN, DV, TK, POFL, COEF, EXPK
- COMMON /QCDA2/KVAL, VARX, MMEAN, VARMU, LVAL
- COMMON /QCDA3/KMAX, IMAX, VXMAX, MNMAX, VMUMAX, LMAX
- COMMON /QCDA5/NEGINF, FACTOR, PIE
- COMMON /QCDA6/INDV
- COMMON /QCDA7/A1, A2

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**SUBROUTINE - BEGIN**

**COMPUTE LOWER BOUND FOR TK(V)**

**RETURN**

**END SUBROUTINE COMPUTLOTKOFV**

**FUNCTION:** This subroutine outputs the QC input data and results

**USAGE:**

- CALL SEQUENCE: CALL DATOUT
- EXTERNAL FUNCTIONS/SUBROUTINES:
  - DATOUTIN, DATOUTPOFL, DATOUTCOEF, DATOUTTK, DATOUTESTPOFL

**PARAMETERS:**

**INPUT:**

- SEE SUBROUTINES DATOUTIN, DATOUTPOFL, DATOUTCOEF
- DATOUTTK, DATOUTESTPOFL

**OUTPUT:**

- SEE SUBROUTINES DATOUTIN, DATOUTPOFL, DATOUTCOEF
- DATOUTTK, DATOUTESTPOFL

**PROGRAMMER:** FOUAD JALBOUT

**COMPUTER:** VAX

**LANGUAGE:** FORTRAN 77
*** SET GLOBAL VARIABLES DECLARATIONS ***
REAL V(31), TK(6,6,6,6,31), POFL(6,6,6,31), CDEF(6,6,31)
REAL KVAL(6), VARX(6), MMEAN(6), VARMU(6), LVAL(6)
REAL VMAX, VMIN, DV
INTEGER KMAX, IMAX, VXMAX, MNMAX, VMUMAX, LMAX
REAL NEGINF, FACTOR, PIE
CHARACTER INDV*1
REAL A1, A2
REAL POFLE1(6,6,6,31), C(1)
REAL EXPK(6,6,6,6)
COMMON /QCDATO/V, VMAX, VMIN, DV, TK, POFL, CDEF, EXPK
COMMON /QCDAT2/KVAL, VARX, MMEAN, VARMU, LVAL
COMMON /QCDAT3/KMAX, IMAX, VXMAX, MNMAX, VMUMAX, LMAX
COMMON /QCDAT5/NEGINF, FACTOR, PIE
COMMON /QCDAT6/INDV
COMMON /QCDAT7/A1, A2
COMMON /QCDAT8/POFLE1, C

*** SUBROUTINE - BEGIN ***
*** OUTPUT INPUT DATA ***
CALL DATOUTIN
*** OUTPUT 1ST FACTOR RESULTS ***
CALL DATOUTPOFL
*** OUTPUT 2ND FACTOR RESULTS ***
CALL DATOUTCDEF
*** OUTPUT COMBINED RESULTS ***
CALL DATOUTTK
CALL DATOUTTEPOFL
*** OUTPUT E(P**K) RESULTS ***
CALL DATOUTEPK
RETURN

*** DRAWING END SUBROUTINE DATOUT ***

END SUBROUTINE DATOUTIN

 FUNCTION: THIS SUBROUTINE OUTPUTS THE USER SUPPLIED INPUT DATA
     FOR DISPLAY
     CALL SEQUENCE: CALL DATOUTIN
     INPUT:
     OUTPUT:
     SEE SUBROUTINE DATIN (VIA COMMON)
     (STORED IN FILE='QC11D01.DAT')
     PROGRAMMER: FOAUD JALBOUT
     COMPUTER: VAX
     LANGUAGE: FORTRAN 77

*** SET GLOBAL VARIABLES DECLARATIONS ***
REAL V(31), TK(6,6,6,6,31), POFL(6,6,6,31), CDEF(6,6,31)
REAL KVAL(6), VARX(6), MMEAN(6), VARMU(6), LVAL(6)
REAL VMAX, VMIN, DV
INTEGER KMAX, IMAX, VXMAX, MNMAX, VMUMAX, LMAX
REAL NEGINF, FACTOR, PIE
CHARACTER INOV,
REAL AI.A2
REAL EXPK(6,6,6,6)
COMMON /QCDAT0/V, VMAX, VMIN, DV, TK, PDPL, COEF, EXPK
COMMON /QCDAT2/KVAL, VARX, MMMEAN, VARMU, LVAL
COMMON /QCDAT3/KMAX, IMAX, VXMAX, MNNMAX, VMUMAX, LMAX
COMMON /QCDAT5/NEGINF, FACTOR, PIE
COMMON /QCDAT6/INDV
COMMON /QCDAT7/A1, A2

*** SUBROUTINE - BEGIN ***
*** OUTPUT INPUT DATA ***
OPEN(UNIT=10, STATUS='NEW', FILE='QC1101.DAT')
WRITE(10,1)
1 FORMAT(11H1, //)
WRITE(10,2)
2 FORMAT(30X, '*** QC INPUT DATA ***', //)
WRITE(10,3)
3 FORMAT(10X, '-------------------------------------------------'
********, //)
WRITE(10,4) KMAX
4 FORMAT(14X, 'MAXIMUM NUMBER OF K VALUES = ', 15, //)
WRITE(10,5) LMAX
5 FORMAT(14X, 'MAXIMUM NUMBER OF LOWER LIMIT - L VALUES = ', 15, //)
WRITE(10,6) VXMAX
6 FORMAT(14X, 'MAXIMUM NUMBER OF VARIANCE OF X VALUES = ', 15, //)
WRITE(10,7) VARMU
7 FORMAT(14X, 'MAXIMUM NUMBER OF VARIANCE OF MU VALUES = ', 15, //)
WRITE(10,8) MNMAX
8 FORMAT(14X, 'MAXIMUM NUMBER OF OF MEAN - M VALUES = ', 15, //)
WRITE(10,9) IMAX
9 FORMAT(14X, 'MAXIMUM NUMBER OF V VALUES (ODD INTEGER) = ', 15, //)
WRITE(10,3)

*** OUTPUT K VALUES ***
WRITE(10,10)
10 FORMAT(14X, 'K VALUES', //)
WRITE(10,11) (KVAL(K), K=1, KMAX)
11 FORMAT(14X, 4E15.4, //)
WRITE(10,3)

*** OUTPUT L VALUES ***
WRITE(10,12)
12 FORMAT(14X, 'L VALUES', //)
WRITE(10,13) (LVAL(L), L=1, LMAX)
13 FORMAT(14X, 4E15.4, //)
WRITE(10,3)

*** OUTPUT VARIANCE OF X VALUES ***
WRITE(10,14)
14 FORMAT(14X, 'VARIANCE OF X VALUES', //)
WRITE(10,15) (VARX(J), J=1, VXMAX)
15 FORMAT(14X, 4E15.4, //)
WRITE(10,3)

*** OUTPUT VARIANCE OF MU VALUES ***
WRITE(10,16)
16 FORMAT(14X, 'VARIANCE OF MU VALUES', //)
WRITE(10,17) (VARMU(M), M=1, VMUMAX)
17 FORMAT(14X, 4E15.4, //)
WRITE(10,3)

*** OUTPUT MEAN - M VALUES ***
WRITE(10,18)
18 FORMAT(14X, 'MEAN - M VALUES', //)
WRITE(10,19) (MMEAN(N), N=1, MNMAX)
19 FORMAT(14X,4(E15.4,/)  
WRITE(10,1)  
WRITE(10,2)  
WRITE(10,3)

*** OUTPUT V VALUES ***
WRITE(10,20) VMIN
20 FORMAT(14X,'MINIMUM V VALUE FOR TK(V) = ',E15.4,/)  
WRITE(10,21) VMAX
21 FORMAT(14X,'MAXIMUM V VALUE FOR TK(V) = ',E15.4,/)  
WRITE(10,22) DV
22 FORMAT(14X,'DELTA V VALUE FOR TK(V) = ',E15.4,/)  
WRITE(10,23) V
23 FORMAT(14X,'V VALUES',/)  
DO 25 I=1, IMAX  
25 WRITE(10,24) VI
CONTINUE  
CLOSE(UNIT=10, STATUS='SAVE')
RETURN

END SUBROUTINE OATOUTPOFL

FUNCTION: THIS SUBROUTINE OUTPUTS THE 1ST FACTOR QC RESULTS
FOR DISPLAY
USAGE:   CALL SEQUENCE: CALL OATOUTPOFL
PARAMETERS:
  INPUT: (VIA COMMON)  
    KMAX=MAXIMUM NUMBER OF K VALUES  
    IMAX=MAXIMUM NUMBER OF V VALUES (INTEGER)  
    VXMAX=MAXIMUM NUMBER OF VARIANCE OF X VALUES  
    LMAX=MAXIMUM NUMBER OF LOWER LIMIT - L VALUES  
    KVAL(KMAX) BY 1 ARRAY OF K VALUES  
    VARX(VXMAX) BY 1 ARRAY OF VARIANCE OF X VALUES  
    LVAL(LMAX) BY 1 ARRAY OF LOWER LIMIT - L VALUES  
    POFL(KMAX) BY LMAX BY VXMAX BY IMAX ARRAY OF P(L) VALUES (1ST FACTOR IN TK(V))  
  OUTPUT:   SEE INPUT PARAMETERS (STORED IN FILE='OC1RES1.OAT')
PROGRAMMER: FOUAD JALBOUT  
COMPUTER: VAX  
LANGUAGE: FORTRAN 77

*** SET GLOBAL VARIABLES DECLARATIONS ***
REAL V(31), TK(6,6,6,6,31), POFL(6,6,6,31), COEF(6,6,31)
REAL KVAL(KMAX), VARX(VXMAX), MMEAN(MNMAX), VARMU(MNMAX), LVAL(LMAX)
REAL VMAX, VMIN, DV  
INTEGER KMAX, IMAX, VXMAX, MNMAX, VMUMAX, LMAX  
REAL NEGINF, FACTOR, PIE  
CHARACTER INDV*1  
REAL A1, A2  
REAL EXPK(6,6,6,6)
COMMON /QCDAT0/V, VMAX, VMIN, DV, TK, POFL, COEF, EXPK
**SUBROUTINE BEGIN**

*** OUTPUT 1ST FACTOR RESULTS ***

OPEN(UNIT=20,STATUS='NEW',FILE='QC1RES1.DAT')

DO 9 K=1,KMAX
   DO 9 J=1,VXMAX
      WRITE(20,1)
      WRITE(20,2)
      WRITE(20,3)
      WRITE(20,4)
      WRITE(20,5)
      WRITE(20,6)
      WRITE(20,7)
      WRITE(20,8)
      WRITE(20,9)
      CONTINUE
   CLOSE (UNIT=20,STATUS='SAVE')

RETURN

*** END SUBROUTINE DATOUTPFIL ***

**SUBROUTINE DATOUTCOEF**

FUNCTION: THIS SUBROUTINE OUTPUTS THE 2ND FACTOR QC RESULTS FOR DISPLAY

USAGE:

CALL SEQUENCE: CALL DATOUTCOEF

PARAMETERS:

INPUT:

- VMUMAX=MAXIMUM NUMBER OF VARIANCE OF MU VALUES
- MNMAX=MAXIMUM NUMBER OF MEAN - M VALUES
- JMAX=MAXIMUM NUMBER OF V VALUES (ODD INTEGER)
- VARMU=VMUMAX BY 1 ARRAY OF VARIANCE OF MU VALUES
- MMEAN=MNMAX BY 1 ARRAY OF MEAN - M VALUES
- V=JMAX BY 1 ARRAY OF V VALUES
- COEF=VMUMAX BY MNMAX BY JMAX ARRAY OF COEFFICIENT VALUES (2ND FACTOR IN TK(V))

OUTPUT:

SEE INPUT PARAMETERS (STORED IN FILE='QC1RES2.DAT')

PROGRAMMER: FOUAD JALBOUT
**COMPUTER**: VAX
**LANGUAGE**: FORTRAN 77

*** SET GLOBAL VARIABLES DECLARATIONS ***
REAL V(31), TK(6, 6, 6, 6, 6, 31), POFL(6, 6, 6, 31), COEF(6, 6, 31)
REAL KVAL(6), VARX(6), MMEAN(6), VARMU(6), LVAL(6)
REAL VMAX, VMIN, DV
INTEGER KMAX, IMAX, VMAX, MNMAX, VMUMAX, LMAX
REAL NEGINF, FACTOR, PIE
CHARACTER INDV*1
REAL A1, A2
REAL EXPK(6, 6, 6, 6)
COMMON/QCDATO/V, VMAX, VMIN, DV, TK, POFL, COEF, EXPK
COMMON/QCDAT2/KVAL, VARX, MMEAN, VARMU, LVAL
COMMON/QCDAT3/KMAX, IMAX, VMAX, MNMAX, VMUMAX, LMAX
COMMON/QCDAT5/NEGINF, FACTOR, PIE
COMMON/QCDAT6/INDV
COMMON/QCDAT7/A1, A2

*** SUBROUTINE - BEGIN ***
*** OUTPUT 2ND FACTOR RESULTS ***
OPEN(UNIT=21, STATUS='NEW', FILE='QCRES2.DAT')
DO B = 1, VMUMAX
   WRITE(21, 1)
   FORMAT(1H1, //)
   WRITE(21, 2)
   FORMAT(31X, 'QC RESULTS', //)
   WRITE(21, 3)
   FORMAT(10X, 'QC RESULTS', //)
   WRITE(21, 14)
   FORMAT(14X, 'VARIANCE OF MU VALUE = ', E15.4, //)
   WRITE(21, 15)
   FORMAT(14X, 'MEAN - M VALUE = ', E15.4, //)
   WRITE(21, 16)
   FORMAT(14X, 'V VALUES COEFFICIENT VALUES', //)
   WRITE(21, 20)
   FORMAT(23X, E15.4, 5X, E15.4)
   DO I = 1, IMAX
      WRITE(21, 22)
      WRITE(21, 27)
      FORMAT(23X, E15.4, 5X, E15.4)
      CONTINUE
   CLOSE (UNIT=21, STATUS='SAVE')
RETURN

*** END SUBROUTINE DATOUTCOEF ***

*** END SUBROUTINE DATOUTTK ***

**FUNCTION**: This subroutine outputs the combined TK(v) QC results
**USAGE**: Call sequence: CALL DATOUTTK
**PARAMETERS**: (VIA COMMON)
KMAX = MAXIMUM NUMBER OF K VALUES
VMAX = MAXIMUM NUMBER OF V VALUES (ODD INTEGER)
VMAX = MAXIMUM NUMBER OF VARIANCE OF X VALUES
LMAX = MAXIMUM NUMBER OF LOWER LIMIT - L VALUES
KVAL = KMAX BY 1 ARRAY OF K VALUES
VARX = VMAX BY 1 ARRAY OF VARIANCE OF X VALUES
MLVAL = LMAX BY 1 ARRAY OF LOWER LIMIT - L VALUES
VMUMAX = VMAX BY 1 ARRAY OF VARIANCE OF MU VALUES
MMEAN = MNMAX BY 1 ARRAY OF MEAN - M VALUES
VARMU = VMUMAX BY 1 ARRAY OF VARIANCE OF MU VALUES
V = IMAX BY 1 ARRAY OF V VALUES
TK = KMAX BY 1 ARRAY OF V VALUES
TKV = KMAX BY LMAX BY VXMAX BY VMUMAX BY MNMAX BY IMAX

* OUTPUT: *
* SEE INPUT PARAMETERS (STORED IN FILE = 'QC1RES3.DAT') *
* PROGRAMMER: FOUAD JALBOUT *
* COMPUTER: VAX *
* LANGUAGE: FORTRAN 77 *

*** SET GLOBAL VARIABLES DECLARATIONS ***
REAL V(31), TK(6, 6, 6, 6, 31), POFL(6, 6, 6, 31), COEF(6, 6, 31)
REAL KVAL(6), VARX(6), MMEAN(6), VARMU(6), LVAL(6)
REAL VMAX, VMIN, DV
INTEGER KMAX, IMAX, VXMAX, MNMAX, VMUMAX, LMAX
REAL NEGINF, FACTOR, PIE
CHARACTER INDV*, AI, A2
REAL EXPK(6, 6, 6, 6, 6)
COMMON /OCDAT0/V, VMAX, VMIN, DV, TK, POFL, COEF, EXPK
COMMON /OCDAT2/KMAX, IMAX, VXMAX, MNMAX, VMUMAX, LMAX
COMMON /OCDAT5/NEGINF, FACTOR, PIE
COMMON /OCDAT6/INDV
COMMON /OCDAT7/AI, A2

*** SUBROUTINE - BEGIN ***
*** OUTPUT COMBINED RESULTS ***
OPEN (UNIT=22, STATUS='NEW', FILE='QC1RES3.DAT')
DO 11 K = 1, KMAX
DO 11 M = 1, VMAX
DO 11 N = 1, MNMAX
WRITE(22, 1)
1 FORMAT(1HI, //)
WRITE(22, 2)
2 FORMAT(3IX, 'QC RESULTS ***', //)
WRITE(22, 3)
3 FORMAT(10X, '************************************')
WRITE(22, 4) KVAL(K)
4 FORMAT(14X, 'K VALUE = ', E15.4, //)
WRITE(22, 5) LVAL(L)
5 FORMAT(14X, 'L VALUE = ', E15.4, //)
WRITE(22, 6) VARX(J)
6 FORMAT(14X, 'VARIAHCE OF X VALUE = ', E15.4, //)
WRITE(22,7) VARMU(M)
FORMAT(14X,'VARIANCE OF MU VALUE = ','E15.4,/)
WRITE(22,8) MMEAN(N)
FORMAT(14X,'MEAN - M VALUE = ','E15.4,/)
WRITE(22,9)
FORMAT(23X,'V VALUES TK(V) VALUES',I,22X,'COMBINED FACTOR',I,22X,'-------------------------------------'/)
DO 11 I=1,IMAX
WRITE(22,10) V(I),TK(K,L,J,M,N,I)
10 FORMAT(23X,EI5,4,5X,EI5,4)
CONTINUE
CLOSE(UNIT=22,STATUS='SAVE')
RETURN
*** END SUBROUTINE DATOUTTK ***
END
SUBROUTINE DATOUTEPK
*** FUNCTION: THIS SUBROUTINE OUTPUTS THE E(P**K) QC RESULTS FOR DISPLAY (P=FRACTION DEFECTIVE) ***
USAGE:
CALL SEQUENCE: CALL DATOUTEPK
PARAMETERS:
INPUT: (VIA COMMON)
KMAX=MAXIMUM NUMBER OF K VALUES
IMAX=MAXIMUM NUMBER OF V VALUES (ODD INTEGER)
VXMAX=MAXIMUM NUMBER OF VARIANCE OF X VALUES
LMAX=MAXIMUM NUMBER OF LOWER LIMIT - L VALUES
KVAL=KMAX BY 1 ARRAY OF K VALUES
VARX=VXMAX BY 1 ARRAY OF VARIANCE OF X VALUES
LVAL=LMAX BY 1 ARRAY OF LOWER LIMIT - L VALUES
VMUMAX=MAXIMUM NUMBER OF VARIANCE OF MU VALUES
MNMAX=MAXIMUM NUMBER OF MEAN - M VALUES
VARMU=VMUMAX BY 1 ARRAY OF VARIANCE OF MU VALUES
MMEAN=MNMAX BY 1 ARRAY OF MEAN - M VALUES
V=IMAX BY 1 ARRAY OF V VALUES
EXPK=KMAX BY LMAX BY VXMAX BY VMUMAX BY MNMAX ARRAY CONTAINING THE COMPUTED E(P**K) VALUES FOR THE MANUFACTURING PROCESS (P=FRACTION DEFECTIVE)
OUTPUT:
SEE INPUT PARAMETERS (STORED IN FILE='QC1RES4.DAT'
PROGRAMMER: FOUD JALBOUT
COMPUTER: VAX
LANGUAGE: FORTRAN 77

*** SET GLOBAL VARIABLES DECLARATIONS ***
REAL V(31),TK(6,6,6,6,31),POFL(6,6,6,31),COEF(6,6,31)
REAL KVAL(6),VARX(6),MMEAN(6),VARMU(6),LVAL(6)
REAL VMAX,VMIN,DV
INTEGER KMAX,IMAX,VXMAX,MNMAX,VMUMAX,LMAX
REAL NEGINF,FACTOR,PIE
CHARACTER INDV-1
REAL A1,A2
REAL EXPK(6,6,6,6)
**SUBROUTINE - BEGIN**

*** OUTPUT COMBINED RESULTS ***

OPEN (UNIT=30, STATUS='NEW', FILE='QC1RES4.DAT')

DO 10 K=1,KMAX
  DO 10 L=1,LMAX
  WRITE(30,1)
  WRITE(30,2)

1  FORMAT(1H1, //)
  WRITE(30,2)

2  FORMAT(3IX, ' QC RESULTS ')
  WRITE(30,3)

3  FORMAT(10X, ' *********** ')
  WRITE(30,4) KVAL(K)
  WRITE(30,5) LVAL(L)

4  FORMAT(14X, ' K VALUE = ',E15.4, //)
  WRITE(30,6) VARX(J)

5  FORMAT(14X, ' L VALUE = ',E15.4, //)
  WRITE(30,7) VARMU(M)

6  FORMAT(14X, ' VARIANCE OF X VALUE = ',E15.4, //)
  WRITE(30,8) MMEAN(N)

7  FORMAT(14X, ' VARIANCE OF MU VALUE = ',E15.4, //)
  WRITE(30,9) EXPK(K,L,J,M,N)

8  FORMAT(14X, 'E(P**K) = ',E15.4, //)

9  CONTINUE

10 CLOSE(UNIT=30, STATUS='SAVE')

**FUNCTION:** THIS SUBROUTINE OUTPUTS THE 1ST FACTOR, 2ND FACTOR, COMBINED TK(V) QC RESULTS, AND P(L) ESTIMATES FOR PLOTTING

**USAGE:**

**CALL SEQUENCE:** CALL PLTDATOUT

**PARAMETERS:**

- **KMAX:** MAXIMUM NUMBER OF K VALUES
- **IMAX:** MAXIMUM NUMBER OF V VALUES (ODD INTEGER)
- **VMAX:** MAXIMUM NUMBER OF VARIANCE OF X VALUES
- **LMAX:** MAXIMUM NUMBER OF LOWER LIMIT - L VALUES
- **KVAL=KMAX BY 1 ARRAY OF K VALUES
- **VARX=VMAX BY 1 ARRAY OF VARIANCE OF X VALUES
- **VUMAX=MAXIMUM NUMBER OF VARIANCE OF MU VALUES
C ** MNMAX=MAXIMUM NUMBER OF MEAN - M VALUES
C ** VARMU=VMUMAX BY 1 ARRAY OF VARIANCE OF MU VALUES
C ** MMEAN=MNMAX BY 1 ARRAY OF MEAN - M VALUES
C ** VM=IMAX BY 1 ARRAY OF V VALUES
C ** POFL=KMAX BY LMAX BY VXMAX BY IMAX ARRAY OF P(L) VALUES
C ** (1ST FACTOR IN TK(V))
C ** COEF=VMUMAX BY MNMAX BY IMAX ARRAY OF COEFFICIENT VALUES
C ** (2ND FACTOR OF TK(V))
C ** TK=KMAX BY LMAX BY VXMAX BY VMUMAX BY MNMAX BY IMAX
C ** ARRAY CONTAINING THE COMPUTED TK(V) VALUES
C ** (COMBINED 1ST AND 2ND FACTORS IN TK(V))
C ** POFL=KMAX BY LMAX BY VXMAX BY IMAX ARRAY OF P(L) ESTIMATES
C ** OUTPUT:
C ** SEE INPUT PARAMETERS
C ** (STORED IN FILE='OC1PLTPL.(NO)')
C ** (STORED IN FILE='OC1PLTCF.(NO)')
C ** (STORED IN FILE='OC1PLTTK.(NO)')
C ** PROGRAMMER: FOUDA JALBOUT
C ** COMPUTER: VAX
C ** LANGUAGE: FORTRAN 77
C *******************************************
C *** SET GLOBAL VARIABLES DECLARATIONS ***
REAL V(31),TK(6,6,6,6,31),POFL(6,6,6,31),COEF(6,6,31)
REAL KVAL(6),VARX(6),MMEAN(6),VARMU(6),LVAL(6)
REAL VMAX,VMIN,DV
INTEGER KMAX,IMAX,VXMAX,MNMAX,VMUMAX,LMAX
REAL NEGINF,FACTOR,PIE
CHARACTER INOV.,REAL A1,A2
REAL POFLE1(6,6,6,31),CPFLE1(6,6,6,10)
INTEGER NOFIL(10)
CHARACTER FILNAM(12),FILE(9,EXT(3)
COMMON /QCDAT0/V,VMAX,VMIN,DV,TK,POFL,COEF,EXPK
COMMON /QCDAT0/KVLMAX,VARX,MMEAN,VARMU,LVAL
COMMON /QCDAT5/KMAMAX,VXMAX,MNMAX,VMUMAX,LMAX
COMMON /QCDAT5/NEGINF,FACTOR,PIE
COMMON /QCDAT6/INDV
COMMON /QCDAT7/A1,A2
COMMON /QCDAT6/POFLE1,MPDFL,CPFLE1
COMMON /ST1/NOFIL
COMMON /ST2/FILNAM
C *** SUBROUTINE - BEGIN ***
C *** CALL INDIVIDUAL PLOT FILE SUBROUTINES ***
CALL PLTRUDATOUT
CALL PLTESTDATOUT
RETURN
C ***************************************************************
C ***************************************************************
C ***************************************************************
C ***************************************************************
C ***************************************************************

C ** FUNCTION: THIS SUBROUTINE OUTPUTS THE 1ST FACTOR, 2ND FACTOR,
C AND COMBINED TK(V) QC RESULTS FOR PLOTTING
C ** USAGE:
CALL SEQUENCE: CALL PLTTRUDATOUT

PARAMETERS:

INPUT: (VIA COMMON)

KMAX=MAXIMUM NUMBER OF K VALUES

IMAX=MAXIMUM NUMBER OF V VALUES (ODD INTEGER)

VXMAX=MAXIMUM NUMBER OF VARIANCE OF X VALUES

LMAX=MAXIMUM NUMBER OF LOWER LIMIT - L VALUES

KVAL=KMAX BY 1 ARRAY OF K VALUES

VARX=VXMAX BY 1 ARRAY OF VARIANCE OF X VALUES

LVAL=LMAX BY 1 ARRAY OF LOWER LIMIT - L VALUES

VMUMAX=MAXIMUM NUMBER OF VARIANCE OF MU VALUES

MNMAX=MAXIMUM NUMBER OF MEAN - M VALUES

VARMU=VMUMAX BY 1 ARRAY OF VARIANCE OF MU VALUES

MMEAN=MNMAX BY 1 ARRAY OF MEAN - M VALUES

VMAX=VXMAX BY 1 ARRAY OF V VALUES

POFL=KMAX BY LMAX BY VXMAX BY VMAX ARRAY OF P(L) VALUES

(1ST FACTOR IN TK(V))

COEF=VMUMAX BY MNMAX BY IMAX ARRAY OF COEFFICIENT VALUES

(2ND FACTOR OF TK(V))

TK=KMAX BY LMAX BY VXMAX BY VMAX BY VMUMAX BY MNMAX BY IMAX

ARRAY CONTAINING THE COMPUTED TK(V) VALUES

(COMBINED 1ST AND 2ND FACTORS IN TK(V))

OUTPUT:

SEE INPUT PARAMETERS

(STORED IN FILE='QCIPPLTPL.(NO)')

(STORED IN FILE='QCIPPLTCF.(NO)')

(STORED IN FILE='QCIPPLTTK.(NO)')

PROGRAMMER: FOAUD JALBOUT

COMPUTER: VAX

LANGUAGE: FORTRAN 77

******************************************************************************

*** SET GLOBAL VARIABLES DECLARATIONS ***

REAL V(31),TK(6,6,6,6,6,31),POFL(6,6,6,6,6,31),COEF(6,6,31)

REAL KVAL(6),VARX(6),MMEAN(6),VARMU(6),LVAL(6)

REAL VMAX,VMIN,DV

INTEGER KMAX,IMAX,VXMAX,MNMAX,VMUMAX,LMAX

REAL NEGINF,FACTOR,PIE

CHARACTER INDV,, REAL AI,A2

INTEGER NOFIL(10)

CHARACTER FILNAM*12,FILE*9,EXT*3

REAL EXPK(6,6,6,6)

COMMON /QCDA/ V,VMAX,VMIN,DV,TK,POFL,COEF,EXPK

COMMON /QCDAT2/KVAL,VARX,MMEAN,VARMU,LVAL

COMMON /QCDA3/KMAX,IMAX,VXMAX,MNMAX,VMUMAX,LMAX

COMMON /QCDATS/NEGINF,FACTOR,PIE

COMMON /QCDATS/INDV

COMMON /QCDAT7/A1,A2

COMMON /ST1/NOFIL

COMMON /ST2/FILNAM

*** STORE OUTPUT IN SEPARATE FILES FOR PLOTTING ***

*** INITIALIZATION ***

DO 1 I=1,5

NOFIL(I)=0

1 CONTINUE

*** SUBROUTINE - BEGIN ***
DO 8 L=1,LMAX
DO 8 J=1,VXMAX

*** OUTPUT P(L) DATA TO A SEQUENTIAL FILE ***
*** FOR PLOTTING ***

FILE='QC1PLTPL.'
EXT='
NOFIL(1)=NOFIL(1)+1
CALL STRNUM(NOFIL(1),EXT)
FILNAM=FILE//EXT
OPEN(UNIT=50,STATUS='NEW',FILE=FILNAM)
DO 4 K=1,KMAX

*** LABEL DATA IN FILE ***
WRITE(50,2)
FORMAT(2X,A15)

*** WRITE DATA IN FILE ***
DO 4 I=1,IMAX
WRITE(50,3) V(I),POFL(K,L,J,I)
CONTINUE

CLOSE(UNIT=50,STATUS='SAVE')

*** END OUTPUT OF P(L) DATA ***
*** TO A SEQUENTIAL FILE ***

DO 8 M=1,VMUMAX
DO 8 N=1,MNMAX

*** OUTPUT COEFFICIENT DATA TO A SEQUENTIAL FILE ***
*** FOR PLOTTING ***

FILE='QC1PLTCF.'
EXT='
NOFIL(2)=NOFIL(2)+1
CALL STRNUM(NOFIL(2),EXT)
FILNAM=FILE//EXT
OPEN(UNIT=50,STATUS='NEW',FILE=FILNAM)

*** LABEL DATA IN FILE ***
WRITE(50,2)

*** WRITE DATA IN FILE ***
DO 6 I=1,IMAX
WRITE(50,3) V(I),COEF(M,N,I)
CONTINUE

CLOSE(UNIT=50,STATUS='SAVE')

*** END OUTPUT COEFFICIENT DATA ***
*** TO A SEQUENTIAL FILE ***

*** OUTPUT TK(V) DATA TO A SEQUENTIAL FILE ***
*** FOR PLOTTING ***

FILE='QC1PLTTK.'
EXT='
NOFIL(3)=NOFIL(3)+1
CALL STRNUM(NOFIL(3),EXT)
FILNAM=FILE//EXT
OPEN(UNIT=50,STATUS='NEW',FILE=FILNAM)
DO 7 K=1,KMAX

*** LABEL DATA IN FILE ***
C
WRITE(50,2)
*** WRITE DATA IN FILE ***
DO 7 I=1,IMAX
WRITE(50,3) V(I),TK(K,L,J,M,N,I)
7 CONTINUE
CLOSE(UNIT=50,STATUS='SAVE')
C
*** END OUTPUT OF TK(V) DATA ***
*** TO A SEQUENTIAL FILE ***
C
B CONTINUE
RETURN
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C
根据以下代码的内容，将其翻译成自然语言：

```fortran
* FUNCTION: THIS SUBROUTINE OUTPUTS THE 1ST FACTOR, P(L), ESTIMATE FOR PLOTTING
* USAGE: CALL SEQUENCE: CALL PLTESTDATOUT
* PARAMETERS:
* INPUT: (VIA COMMON) KMAX=MAXIMUM NUMBER OF K VALUES
* IMAX-MAXIMUM NUMBER OF V VALUES (ODD INTEGER)
* VMAX-MAXIMUM NUMBER OF VARIANCE OF X VALUES
* LMAX-MAXIMUM NUMBER OF LOWER LIMIT - L VALUES
* KVAL=KMAX BY 1 ARRAY OF K VALUES
* V=VMAX BY 1 ARRAY OF V VALUES
* POFLEI=KMAX BV LMAX BV VMAX BV IMAX ARRAY OF P(L) ESTIMATES
* OUTPUT:
* SEE INPUT PARAMETERS (STORED IN FILE='OC1PLTPE.(NO)')
* PROGRAMMER: FOUAD JALBOUT
* COMPUTER: VAX
* LANGUAGE: FORTRAN 77

* SET GLOBAL VARIABLES DECLARATIONS *
REAL V(31),TK(6,6,6,6,31),POFL(6,6,6,31),COEF(6,6,31)
REAL KVAL(6),VARX(6),MMEAN(6),VARMU(6),LVAL(6)
REAL VMAX,VMIN,DV
INTEGER KMAX,IMAX,VXMAX,MNMAX,VMUMAX,LMAX
REAL NEGINF,FACTOR,PIE
CHARACTER INOV.’
REAL A1,A2
REAL POFLEI(6,6,6,31),CPFLEI(6,6,6,10)
INTEGER NOFIL(10)
CHARACTER FILNAM*2,FILE*9,EXT*3
REAL EXPK(6,6,6,6,6)
COMMON /QCDAT0/V,VMAX,VMIN,DV,TK,POFL,COEF,EXPK
COMMON /QCDAT2/KVAL,VARX,MMEAN,VARMU,LVAL
COMMON /QCDAT3/KMAX,IMAX,VMAX,MNMAX,VMUMAX,LMAX
COMMON /QCDAT5/NEGINF,FACTOR,PIE
COMMON /QCDAT6/INDV
COMMON /QCDAT7/A1,A2
COMMON /QCDAT8/POFL,CPFLEI,MOPFL,CPFLEI
COMMON /ST1/NOFIL
COMMON /ST2/FILNAM
```
*** STORE OUTPUT IN SEPARATE FILES FOR PLOTTING ***

*** INITIALIZATION ***
DO 1 I=6,10
   NOFIL(I)=0
1 CONTINUE

*** SUBROUTINE - BEGIN ***
DO 5 L=1,LMAX
   DO 4 J=1,VXMAX
     *** OUTPUT P(L) ESTIMATE DATA TO A SEQUENTIAL FILE ***
     *** FOR PLOTTING ***
     FILE='QC1PLTPE.
     EXT=',
     NOFIL(6)+NOFIL(6)+1
     CALL STRNUM(NOFIL(6),EXT)
     FILNAM=FILE//EXT
     OPEN(UNIT=50,STATUS='NEW',FILE=FILNAM)
     DO 4 K=1,KMAX
       *** WRITE DATA IN FILE ***
       4 WRITE(50,2)
       *** WRITE LABEL DATA IN FILE ***
       WRITE(50,1)
     CONTINUE
     CLOSE(UNIT=50,STATUS='SAVE')
5 CONTINUE
RETURN

END SUBROUTINE PLTESTDATOUT

END SUBROUTINE STRNUM(N,STR)

* FUNCTION: THIS SUBROUTINE DISPLAYS THE ASCII STRING VERSION OF AN INTEGER BY USING THE FORTRAN CHAR FUNCTION *
* USAGE: CALL STRNUM(N,STR) *
* PARAMETERS: *
* INPUT: N=INTEGER NUMBER FOR ASCII CONVERSION *
* OUTPUT: STR=ASCII STRING EQUIVALENT OF N *
* PROGRAMMER: THOMAS HASSETT *
* COMPUTER: VAX *
* LANGUAGE: FORTRAN77 *

*** SET LOCAL VARIABLES DECLARATIONS ***
INTEGER N,O,T
CHARACTER STR*(1),SIGN*1

*** INITIALIZATION ***
STR=' ',

END
**FUNCTION: THIS SUBROUTINE COMPUTES THE VARIOUS ESTIMATES USING **
**   THE LEAST SQUARES ALGORITHM FOR POLYNOMIAL ESTIMATION**
**USAGE:**
**   CALL SEQUENCE: CALL ESTIMATE**
**EXTERNAL FUNCTIONS/SUBROUTINES: SUBROUTINE ESTPOFL(L,J)**
**PARAMETERS:**
**   INPUT: (VIA COMMON)**
**   POFL=KMAX BV LMAX BV VXMAX BV IMAX ARRAV OF pel) VALUES**
**   V=IMAX BV**
**OUTPUT:**
**   POFLE2=KMAX BV IMAX ARRAV OF pel) ESTIMATES**
**PROGRAMMER: THOMAS HASSETT**
**COMPUTER: VAX**
**LANGUAGE: FORTRAN 77**

**SET GLOBAL VARIABLES DECLARATIONS **
REAL V(31),TK(6,6,6,6.31),POFLE(6,6,6,31),COEF(6,6,31)
REAL KVAL(6),VARX(6),MMEAN(6),VARMU(6),LVAL(6)
REAL VMAX,VMIN,DV
INTEGER KMAX,IMAX,VXMAX,MNMAX,VMUMAX,LMAX
REAL NEGINF,FACTOR,PIE
CHARACTER INDV*1
REAL A1,A2
REAL POFLE(6,6,6,31),CPFLE(6,6,6,10)
REAL EPOFLE(6,6,6,31),REPDL(6,6,6,31)
INTEGER MPOFL
REAL EXPK(6,6,6,6,6)
COMMON /QCDATO/V,VMAX,VMIN,DV,TK,POFL,COEF,EXPK
COMMON /QCDAT2/KVAL,VARX,MMEAN,VARMU,LVAL
COMMON /QCDAT3/KMAX,IMAX,VMAX,MNMAX,VMUMAX,LMAX
COMMON /QCDAT5/NEGINF,FACTOR,PIE
COMMON /QCDAT6/INDV
COMMON /QCDAT7/A1,A2
COMMON /QCDATB/POFLE1,MPOFL,CPFLE1
COMMON /QCDAT9/EPOFL,REPOFL

C**** SUBROUTINE - BEGIN ***
DO 1 L=1,LMAX
    DO 1 J=1,VXMAX
    CALL ESTPOFL(L,J)
    CALL ERRPOFL(L,J)
1 CONTINUE
RETURN
C*************************************************************************
C*************************************************************************
END
SUBROUTINE ESTPOFL(L,J)
C
FUNCTION: THIS SUBROUTINE COMPUTES THE ESTIMATE FOR P(L) USING
THE LEAST SQUARES ALGORITHM FOR POLYNOMIAL ESTIMATION
USAGE:
CALL SEQUENCE: CALL ESTIMATE
EXTERNAL FUNCTIONS/SUBROUTINES: SUBROUTINE ESTPOFL(L,J)
PARAMETERS:
INPUT: (VIA COMMON)
POFL=KMAX BY LMAX BY VXMAX BY IMAX ARRAY OF P(L) VALUES
V=IMAX BY 1 ARRAY OF V VALUES
OUTPUT:
POFLE2=KMAX BY LMAX BY VXMAX BY IMAX ARRAY OF P(L) ESTIMATES
PROGRAMMER: THOMAS HASSETT
COMPUTER: VAX
LANGUAGE: FORTRAN 77

*************************************************************************
*************************************************************************
*** SET GLOBAL VARIABLES DECLARATIONS ***
REAL V(31),TK(6,6,6,6,6,31),POFLE(6,6,6,31),COEF(6,6,31)
REAL KVAL(6),VARX(6),MMEAN(6),VARMU(6),LVAL(6)
REAL VMAX,VMIN,DV
INTEGER KMAX,IMAX,VXMAX,MNMAX,VMUMAX,LMAX
REAL NEGINF,FACTOR,PIE
CHARACTER INDV=1
REAL A1,A2
REAL POFLE1(6,6,6,31),POFLE2(31),C(10),CPFLE1(6,6,6,10)
INTEGER MPOFL
REAL EXPK(6,6,6,6)
COMMON /QCDATO/V,VMAX,VMIN,DV,TK,POFL,COEF,EXPK
COMMON /QCDAT2/KVAL,VARX,MMEAN,VARMU,LVAL
COMMON /QCDAT3/KMAX,IMAX,VMAX,MNMAX,VMUMAX,LMAX
COMMON /QCDAT5/NEGINF,FACTOR,PIE
COMMON /QCDAT6/INDV
COMMON /QCDAT7/A1,A2
COMMON /QCDATB/POFLE1,MPOFL,CPFLE1
**SUBROUTINE - BEGIN**

DO 5 K=1,KMAX
   DO 1 I=1,IMAX
      POFlE2(I)=ALOG(POFl(K,L,J,I))
   CONTINUE

**COMPUTE LEAST SQUARES P(L) ESTIMATE COEFFICIENTS**

CALL LSQIMAX,MPOFl,V,POFlE2,C

**STORE P(L) COEFFICIENT ESTIMATES**

DO 2 II=1,MPOFl+1
   CPFLE1(K,L,J,II)=C(II)
2 CONTINUE

**COMPUTE P(L) ESTIMATES AS:**

DO 4 II=1,MPOFl
   SUM=C(II)
   DO 3 II=II+1,MPOFl+1
      SUM=SUM+(C(II)*(V(I)**(II-1)))
3 CONTINUE
   POFlEl(K,L,J)=EXP(SUM)
4 CONTINUE
5 CONTINUE
RETURN

**END SUBROUTINE ESTPOFl**

**-----------------------------------------------**

**SUBROUTINE ERRPOFl(L,J)**

* FUNCTION: THIS SUBROUTINE COMPUTES THE ERROR AND RELATIVE ERROR IN THE LEAST SQUARES P(L) ESTIMATE *

* USAGE: *

* CALL SEQUENCE: CALL ERRPOFl(L,J) *

* PARAMETERS: *

* INPUT: (VIA COMMON) *
   POFl=KMAX BY LMAX BY VXMAX BY IMAX ARRAY OF P(L) VALUES *
   POFlEl=KMAX BY LMAX BY VXMAX BY IMAX ARRAY OF P(L) ESTIMATES *

* OUTPUT: *
   EPOFl=KMAX BY LMAX BY VXMAX BY IMAX ARRAY OF ERRORS *
   REPOFl=KMAX BY LMAX BY VXMAX BY IMAX ARRAY OF RELATIVE ERRORS *

* PROGRAMMER: THOMAS HASSETT *

* COMPUTER: VAX *

* LANGUAGE: FORTRAN 77 *

**-----------------------------------------------**

**SET GLOBAL VARIABLES DECLARATIONS**

REAL V(31),TK(6,6,6,6,31),POFl(6,6,6,31),COEF(6,6,31)
REAL KVAl(6),VARX(6),MMEAN(6),VARMU(6),lVAL(6)
REAL VMAX,VMIN,DV
INTEGER KMAX,IMAX,VXMAX,VMAX,LMAX
REAL NEGINP,FACTOR,PIE
CHARACTER INDV1
REAL A1,A2
REAL POFlEl(6,6,6,31),CPFLE2(31),C(10),CPFLE1(6,6,6,10)
REAL EPOFl(6,6,6,31),REPOFl(6,6,6,31)
INTEGER MPOFl
REAL EXPK(6,6,6,6)
COMMON /OCDAT0/V,VMAX,VMIN,DV,TK,POFl,COEF,EXPK
COMMON /OCDAT2/KVAL,VARX,MMEAN,VARMU,LVAL
COMMON /OCDAT3/KMAX,IMAX,VXMAX,VMAX,VMUMAX,LMAX
*** SUBROUTINE - BEGIN ***

DO 1 K=1,KMAX
    DO 1 I=1,IMAX
        EPOFL(K,L,J,I)=POFL(K,L,J,I)-POFLEI(K,L,J,I)
        REPOFL(K,L,J,I)=ABS(EPOFL(K,L,J,I)/POFL(K,L,J,I))
    1 CONTINUE

RETURN

*** END SUBROUTINE ERRPOFL ***

END

SUBROUTINE COMPUTEPK

FUNCTION: THIS SUBROUTINE COMPUTES THE EXPECTED FRACTION DEFECTIVE RAISED TO THE K-TH POWER, E(P**K) USING A BANDLIMITED MODIFIED SIMPSON'S RULE ALGORITHM

USAGE:
CALL SEQUENCE: CALL COMPUTEPK

PARAMETERS:
INPUT: (VIA COMMON)
TK=KMAX BV lMAX BV VXMAX BV VMUMAX BV MNMAX BV IMAX
ARRAY OF

OUTPUT:
EPOFL=MAX BY LMAX BY VXMAX BY VMUMAX BY MNMAX ARRAY OF
EXPECTED FRACTION DEFECTIVE RAISED TO THE K-TH POWER

PROGRAMMER: THOMAS HASSETT
COMPUTER: VAX
LANGUAGE: FORTRAN 77

*** SET GLOBAL VARIABLES DECLARATIONS ***
REAL V(31),TK(6,6,6,6,31),POFL(6,6,6,31),COEF(6,6,31)
REAL KVAL(6),VARX(6),MMEAN(6),VARMU(6),LVAL(6)
REAL VMAX,VMIN,DV
INTEGER KMAX,IMAX,VXMAX,MNMAX,VMUMAX,LMAX
REAL NEGINF,FACTOR,PIE
CHARACTER INDV*,1
REAL A1,A2
REAL POFLEI(6,6,6,31),POFLE2(31),C(10),CPFLEI(6,6,6,10)
REAL EPOFL(6,6,6,31),REPOFL(6,6,6,31)
INTEGER MPOFL
REAL EXPK(6,6,6,6)

*** SET LOCAL VARIABLES DECLARATIONS ***
REAL U(301),FN(301)
COMMON /QCDAT0/V,VMAX,VMIN,DV,TK,POFL,COEF,EXPK
COMMON /QCDAT2/KVAL,VARX,MMEAN,VARMU,LVAL
COMMON /QCDAT3/KMAX,IMAX,VXMAX,MNMAX,VMUMAX,LMAX
COMMON /QCDAT5/NEGINF,FACTOR,PIE
COMMON /QCDAT6/INDV
COMMON /QCDAT7/A1,A2
COMMON /QCDAT8/POFLEI,MPOFL,CPFLEI
COMMON /QCDAT9/EPOFL,REPOFL
*** SUBROUTINE - BEGIN ***
DO 2 K=1,KMAX
DO 2 L=1,LMAX
DO 2 J=1,VMAX
DO 2 M=1,VMMAX
DO 1 I=1,IMAX
U(I)=V(I)
FN(I)=TK(K,L,J,M,N,I)
1 CONTINUE
CALL SIMPMOD(U,FN,IMAX,E)
EXPK(K,L,J,M,N)=E
2 CONTINUE
RETURN
*** END SUBROUTINE COMPUTE PK ***
***
END SUBROUTINE

FUNCTION: APPROXIMATES THE INTEGRAL OF A SET OF EQUALLY SPACED DATA POINTS FN OVER THE INTERVAL [U(I),U(NPT)] BY SIMPSON'S RULE

USAGE:
CALL SEQUENCE: CALL SIMPMOD(U,FN,NPT,E)

PARAMETERS:
INPUT:
U=NPT BY 1 ARRAY OF EQUALLY SPACED ABSCISSA DATA POINTS
FN=NPT BY 1 ARRAY OF EQUALLY SPACED ORDINATE DATA POINTS
NPT=NUMBER OF DATA POINTS

OUTPUT:
E=ESTIMATE OF THE INTEGRAL

PROGRAMMER: THOMAS HASSETT
COMPUTER: VAX
LANGUAGE: FORTRAN 77
COMMENT: ADAPTED FROM 'AN INTRODUCTION FROM TO NUMERICAL COMPUTATIONS', BY SIDNEY YAKOWITZ AND FERENC SZIDAROVSKY, MACMILLAN PUBL. CO., NEW YORK, NY., C.1986

*** SET LOCAL VARIABLES DECLARATIONS ***
REAL U(NPT),FN(NPT)
H=(U(NPT)-U(1))/(NPT-1)
E=FN(1)+FN(NPT)
*** SUBROUTINE - BEGIN ***
*** COMPUTE SUM OF EVEN POINTS ***
DO 1 I=2,NPT-1,2
E=E+4.0*FN(I)
1 CONTINUE
*** COMPUTE SUM OF ODD POINTS ***
DO 2 I=3,NPT-2,2
E=E+2.0*FN(I)
2 CONTINUE
E=E*H/3.0
RETURN
***
END SUBROUTINE SIMPMOD
**SUBROUTINE OATOUTESTPOFL**

**FUNCTION:** This subroutine outputs the 1st factor p(l) estimate with the true p(l) QC results.

**USAGE:**
- CALL sequence: CALL OATOUTESTPOFL

**PARAMETERS:**
- INPUT: (via COMMON)
  - \( \text{VMAX} \): Maximum number of \( \text{K} \) values
  - \( \text{LMAX} \): Maximum number of \( \text{L} \) values
  - \( \text{VXMAX} \): Maximum number of variance of \( \text{K} \) values
  - \( \text{MAX} \): Maximum number of \( \text{V} \) values (odd integer)
  - \( \text{POFLE1} \): \( \text{KMAX} \) by \( \text{LMAX} \) by \( \text{VXMAX} \) by \( \text{IMAX} \) array of \( \text{P} \) \( \text{L} \) values
  - \( \text{V} \): \( \text{IMAX} \) by 1 array of \( \text{V} \) values
  - \( \text{POFLE1} \): \( \text{KMAX} \) by \( \text{LMAX} \) by \( \text{VXMAX} \) by \( \text{IMAX} \) array of \( \text{P} \) \( \text{L} \) estimates

**OUTPUT:**
- See input parameters (stored in file='QC1RES1E.DAT')

**PROGRAMMER:** THOMAS HASSETT

**COMPUTER:** VAX

**LANGUAGE:** FORTRAN 77

```fortran
C** SET GLOBAL VARIABLES DECLARATIONS **
REAL V(31),TK(6,6,6,6,6,31),POFLE(6,6,6,31),COEF(6,6,31)
REAL KVAL(6),VARX(6),MMEAN(6),VARMU(6),LVAL(6)
REAL VMAX,VMIN,DV
INTEGER KMAX,IMAX,VXMAX,NMAX,VMUMAX,LMAX
REAL NEGINF,FACTOR,PIE
CHARACTER INDV’1
REAL A1,A2
REAL POFLE1(6,6,6,31),CPFLE1(6,6,6,10)
REAL EPOFL(6,6,6,31),RPOFL(6,6,6,31)
INTEGER MPOFL
REAL EXPK(6,6,6,6)
COMMON /QCDAT5/V,VMAX,VMIN,DV,TK,POFLE,COEF,EXPK
COMMON /QCDAT2/KVAL,VARX,MMEAN,VARMU,LVAL
COMMON /QCDAT3/KMAX,IMAX,VXMAX,NMAX,VMUMAX,LMAX
COMMON /QCDAT5/NEGINF,FACTOR,PIE
COMMON /QCDAT6/INDV
COMMON /QCDAT7/A1,A2
COMMON /QCDAT8/POFLE1,MPOFL,CPFLE1
COMMON /QCDAT9/EPOFL,RPOFL
C** SUBROUTINE - BEGIN **
C** OUTPUT 1ST FACTOR RESULTS AND ESTIMATES **
OPEN(UNIT=20,STATUS='NEW',FILE='QC1RES1E.DAT')
DO 12 K=1,KMAX
   DO 12 L=1,LMAX
      WRITE(20,1)
1 FORMAT(1H1,///)
      WRITE(20,2)
2 FORMAT(24X,'*** QC RESULTS AND ESTIMATES ***',///)
      WRITE(20,3)
3 FORMAT(10X,'*************************/')
      WRITE(20,4) KVAL(K)
```

401
5 FORMAT(14X,'K VALUE
WRITE(20,5) LVAL(L)
5 FORMAT(14X,'L VALUE
WRITE(20,6) VARX(J)
6 FORMAT(14X,'VARIANCE OF X VALUE
WRITE(20,3)
WRITE(20,7)
7 FORMAT(14X,'P(L) ESTIMATE COEFFICIENTS FOR:
1 17X,'PEST(L)=EXP[C(1)+C(2)*X+C(3)*X**2+...]
DO 9 II=1,IMPOFL+1
WRITE(20,8) II,CPFLEI(K,L,J,II)
8 FORMAT(30X,'C(II)='E15,4)
9 CONTINUE
WRITE(20,3)
WRITE(20, 10)
10 FORMAT(8X, 'V VALUES',5X, 'PEX VALUES',3X, 'P(L) ESTIMATE',
1 7X,'ERROR',7X,'REL ERROR',
2 22X,'IST FACTOR',.
3 1X,-----------------------------
11 DO 30 II=1,IMAX
WRITE(20, II) V(I),POFL(K,L,J,I),POFLEI(K,L,J,I),
30 FORMAT(2X,EI5,4,EI5,4,EI5,4,EI5,4,EI5,4)
31 CONTINUE
CLOSE(UNIT=20,STATUS='SAVE')
RETURN
C ***********************************************************************
END SUBROUTINE DATOUTESTPOFL
C ***********************************************************************
C SUBROUTINE LSQM(N,M,X,F,C)
C FUNCTION: THIS SUBROUTINE COMPUTES THE LEAST-SQUARES POLYNOMIAL
C P(X)=C(1)+C(2)*X+...+C(M+1)*X**M
C FOR A SET OF FUNCTIONAL VALUES F AND DOMAIN POINTS X
C USAGE:
C CALL SEQUENCE: CALL LSQM(N,M,X,F,C)
C EXTERNAL FUNCTIONS/SUBROUTINES:
C SUBROUTINE GAUSI
C PARAMETERS:
C INPUT:
C N=NUMBER OF POINTS (NOT GREATER THAN 1000)
C M=POLYNOMIAL DEGREE (LESS THAN 30)
C X=N BY 1 ARRAY OF DOMAIN POINTS
C F=N BY 1 ARRAY OF FUNCTIONAL VALUES
C OUTPUT:
C C=M+1 BY 1 ARRAY OF COEFFICIENTS FOR THE
C LEAST-SQUARES POLYNOMIAL
C COMPUTER: VAX
C LANGUAGE: FORTRAN 77
C COMMENTS: FROM AN INTRODUCTION TO NUMERICAL COMPUTATIONS BY
C SIDNEY YAKOWITZ AND FERENC SZIDAROVSKY, MACMILLAN
C PUB. CO. , C.1986
C DIMENSION X(N),F(N),C(M+1),U(1000),V(1000),A(30,31),S(100)
C *** INITIATION ***
C DO 1 I=1,N
U(I)=1.0
V(I)=F(I)
CONTINUE

C *** COMPUTE COEFFICIENTS OF THE NORMAL EQUATION ***
A(I,M+2)=0.0
DO 2 I=1,N
   A(I,M+2)=A(I,M+2)+V(I)
2 CONTINUE

S(I)=N
DO 3 I=2,M+1
   S(I)=0.0
   A(I,M+2)=0.0
   DO 3 J=1,N
      U(J)=U(J)*X(J)
      V(J)=V(J)*X(J)
      S(I)=S(I)+U(J)
      A(I,M+2)=A(I,M+2)+V(J)
3 CONTINUE

DO 4 I=M+2,(M+1)*N
   S(I)=0.0
4 CONTINUE

DO 5 I=1,M+1
   DO 5 J=1,M+1
      A(I,J)=S(I-J+1)
5 CONTINUE

C *** PERFORM GAUSSIAN ELIMINATION ***
N1=M+1
M1=1
ND=30
C *** USER MAY WANT TO RESET EPS ***
EPS=0.000001
CALL GAUSI(N1,M1,ND,A,EPN)
DO 6 I=1,M+1
   C(I)=A(I,M+2)
6 CONTINUE

RETURN

END

SUBROUTINE GAUSI(N,M,ND,A,DELT)

FUNCTION: THIS SUBROUTINE COMPUTES THE SOLUTIONS FOR M SYSTEMS
WITH N EQUATIONS AND N unknowns USING GAUSSIAN
ELIMINATION
USAGE:
CALL SEQUENCE: CALL GAUSI(N,M,ND,A,DELT)
PARAMETERS:
INPUT:
N=NUMBER OF EQUATIONS AND UNKNOWNs
M=NUMBER OF SYSTEMS (RIGHT HAND SIDE VECTORS)
ND=UPPER BOUND TO THE LINEAR EQUATION ORDER
A=N BY M+N ARRAY OF COEFFICIENTS AUGMENTED
WITH EACH RIGHT SIDE VECTOR
DELT=MACHINE ZERO (TOLERANCE)
OUTPUT:
DIMENSION A(ND,ND+M)

IQ(N,G.T.1) THEN
  DO 1 K=1,N-1
2  U=ABS(A(K,K))
   K=K+1
  END IF
  IF(U.LT.DELT) THEN
    WRITE(6,4)
    RETURN
  ELSE
    IF(ABS(A(I,I).LT.DELT) THEN
      WRITE(6,4)
      RETURN
    END IF
    A(I,N+J)=A(I,N+J)/A(I,I)
    1 CONTINUE
  END IF
  A(1,1)=A(1,1)/A(1,1)
  RETURN
  ELSE IF(ABS(A(1,1)).LT.DELT) THEN
    WRITE(6,4)
    RETURN
    ELSE
      A(1,1)=A(1,1)/A(1,1)
      A(1,N+J)=A(1,N+J)/A(1,1)
      END IF
B CONTINUE
    RETURN
C
C *** END SUBROUTINE GAUS1
C END
APPENDIX H

PROGRAM MULTSTOP

This program is employed when a production process continues operating while searching for a single or multiple causes of failure. The input and outputs of the program are listed.

PROGRAM MAIN

C- THIS PROGRAM IS USED FOR EITHER SINGLE OR MULTIPLE CAUSES OF
C- FAILURE. THE PROCESS IS SHUT DOWN FOR A RANDOM LENGTH OF TIME
C- DURING THE SEARCH FOR THE AssignABLE CAUSE OR CAUSES OF FAILURE
C

FUNCTION: THIS PROGRAM GENERATES THE OPTIMAL COST OF
KEEPING THE PROCESS UNDER CONTROL

PARAMETERS:

SINGLE CAUSES OF FAILURE

INPUT:

XLAM= AVERAGE RATE OF OCCURRENCE OF AN ASSIGNABLE CAUSE
DEL= SHIFT IN THE MEAN OF THE PROCESS RESULTING FROM
THE OCCURRENCE OF AN ASSIGNABLE CAUSE
A1=COST PER SAMPLE OF SAMPLING AND PLOTTING THAT IS
INDEPENDENT OF THE SAMPLE SIZE - N
A2= RATE AT WHICH THE TIME BETWEEN TAKING A SAMPLE
AND PLOTTING A POINT ON THE X-BAR CHART
INCREASES WITH THE SAMPLE SIZE
A3=COST OF FINDING AN ASSIGNABLE CAUSE WHEN IT
EXISTS
VB= RATE PER HOUR AT WHICH INCOME ACCRUES FROM THE
OPERATION OF THE PROCESS WHEN IT IS IN THE STATE
OF STATISTICAL CONTROL
V1= RATE PER HOUR AT WHICH INCOME ACCRUES FROM THE
OPERATION OF THE PROCESS WHEN IT IS IN A STATE
OF OUT OF CONTROL
A1eV1= V1
A3=P= COST OF LOOKING FOR AN ASSIGNABLE CAUSE WHEN
NONE EXISTS
TAUS= EXPECTED LENGTH OF TIME DURING WHICH THE PROCESS
IS SHUT DOWN
TAUR= EXPECTED LENGTH OF TIME TO REPAIR THE MACHINE IF
THE ALARM IS TRUE
U= TAUS
V= TAUR + TAUS
KS= AVERAGE COST OF SEARCHING FOR AN ASSIGNABLE
CAUSE
K= AVERAGE COST OR REPAIRING THE PROCESS
A3= KS + V + TAUS
A3P= KS + V + TAUS

OUTPUT:

N= SAMPLE SIZE
K= UPPER AND LOWER LIMITS ON THE X-BAR CHART
H= INTERVAL BETWEEN SAMPLES MEASURED IN HOURS
ALPHA= PROBABILITY OF TYPE 1 ERROR
1-BETA= POWER OF THE CHART
( WHEN 2= 0.202 1-BETA=0.90; 2= 1.845 1-BETA=0.95 )
SOLVE 2 EQUATIONS:

(1) (z+K)/PHI(K) = ((DEL**2)+(A3P)/(A2+(XLAM+4*O))

(2) DEL=SQR(N)-K = Z

TO OBTAIN AN ESTIMATE OF THE SAMPLE SIZE, THE
VALUE OF N OBTAINED FROM THESE EQUATIONS IS USED
TO SET AN UPPER AND LOWER LIMITS FOR THE SEARCH
OF THE OPTIMAL SAMPLE SIZE

TAU= AVERAGE TIME OF OCCURRENCE OF AN ASSIGNABLE CAUSE
WITHIN AN INTERVAL BETWEEN SAMPLES WHEN THE
PROCESS IS IN CONTROL
C

C COST=TOTAL COST PER UNIT FOR KEEPING THE PROCESS

C UNDER CONTROL UNDER THE ASSUMPTION THAT DURING

C THE SEARCH FOR THE ASSIGNABLE CAUSE THE PROCESS

C CONTINUES OPERATING

C TESFAC=((DEL**2)*A3P)/(A2+(XLAM*A4))

C THIS FACTOR IS COMPARED TO (2-K)/PHI(K), A

C SEARCH IN THE TABLES FOR A VALUE OF THIS FACTOR

C AS CLOSE AS POSSIBLE TO TESFAC GENERATES A VALUE

C FOR THE OPTIMAL COST

C B=AVERAGE TIME THE PROCESS IS OPERATING IN THE

C PRESENCE OF AN ASSIGNABLE CAUSE

MULTIPLE CAUSES OF FAILURE

INPUT:

LAMDAJ=AVERAGE RATE OF OCCURRENCE OF CAUSE J

LAMDAP=RATE OF OCCURRENCE OF A SECOND ASSIGNABLE CAUSE

DELJ=SHIFT IN THE MEAN OF THE PROCESS DUE TO CAUSE J

DELCL=SHIFT IN THE PROCESS MEAN DUE TO DOUBLE OCCURRENCE OF ASSIGNABLE CAUSES

A3J=AVERAGE COST OF FINDING AN ASSIGNABLE CAUSE J

WHEN IT OCCURS

A4J=LOSS PER HOUR OF A PROCESS OPERATING OUT OF

CONTROL DUE TO A CAUSE J

TAUSJ=EXPECTED LENGTH OF TIME DURING WHICH THE PROCESS

IS SHUT DOWN DURING THE SEARCH FOR AN ASSIGNABLE

CAUSE OF FAILURE J

TAURJ=EXPECTED LENGTH OF TIME TO REPAIR THE MACHINE

BY REMOVING THE ASSIGNABLE CAUSE OF FAILURE J

UJ=TAUSJ

VJ=TAURJ + TAUSJ

A3PJ=COST FOR SEARCHING FOR AN ASSIGNABLE CAUSE J

WHEN IT DOES NOT EXIST

OUTPUT:

TAUJ=AVERAGE TIME OF OCCURRENCE OF AN ASSIGNABLE

CAUSE J WITHIN AN INTERVAL BETWEEN SAMPLES WHEN

THE PROCESS IS IN CONTROL

BJ=AVERAGE TIME THE PROCESS IS OPERATING IN THE

PRESENCE OF AN ASSIGNABLE CAUSE J

SS=LAMDA*EXPECTED COST PER CYCLE OF DISCOVERING

THE ASSIGNABLE CAUSE OR CAUSES

TR1=LAMDA*EXPECTED ADDITIONAL LOSS PER CYCLE ARISING

FROM OUT OF CONTROL CONDITIONS

TR2=LAMDA*EXPECTED COST PER CYCLE WHEN LOOKING FOR

AN ASSIGNABLE CAUSE WHEN NONE EXISTS

TR3=LAMDA*B (B IS DEFINED ABOVE)

NOTE: THE OTHER PARAMETERS ARE THE SAME AS THE CASE OF A

SINGLE CAUSE

******************************************************************************

REAL A3(10),A3P(10),A4(10),XLAM(10),U(10),V(10),DEL(10),SS(10)
REAL KR(10),KS(10),TAUR(10),TAUS(10)
COMMON /BLOCK1/A1,A2,A3,A3P,A4,XLAM,DEL,DEL,DEL,U,V
COMMON /BLOCK2/KR,KS,TAUR,TAUS
WRITE(17,*)'STOP THE PROCESS FOR INDIVIDUAL LAMDA'

CALL READI(NUM)
\begin{verbatim}
S1=0
S2=0
S3=0

CALL OUTP(NUM)

DO J=1,NUM
  SS(J)=SS(J)+XLM(J)*A3(J)
  S1=S1+XLM(J)*DEL(J)
  S2=S2+XLM(J)
ENDO

DO 200 LL=1,NUM
  DELT=DEL(LL)

  WRITE(6,25) LL
  FORMAT(5X,' CAUSE # ',13)
  WRITE(16,3) LL
  WRITE(17,25) LL
  WRITE(17,8) LL
  WRITE(18,27) LL,SS(LL)

  S4=0
  S5=0
  DO 100 KK=1,NUM
  DELT=DELT+.25

  WRITE(6,3)
  WRITE(16,3)
  FORMAT(5X,' OPTIMUM K',5X,' OPTIMUM H',&
         ' ALPHA',5X,' POWER',5X,' COST',5X,' TEST FAC 2',&
         ' TAU')
  AA=DELT**2*A3P(LL)/(A2)
  XX=0.
  DO 4 I=1,10
    RHS=SIGN(+1.2B26+XX)/ORDN(X)
    IF(RHS.GT,AA)GO TO 5
    XX=XX+0.5
  4 XX=XX+0.5
  IF(XX.LT,0.) XX=0.
  N=SIGN((1.2B26+XX)/DELT)**2+0.5

  IF(NMIN.LE.0) NMIN=1
  NMIN=1
  NMAX=N+10
  IF (NMAX .LE. 20) NMAX=20

  DO 9 N=NMIN,NMAX
    XN=I
    XX=0.5

  9 CONTINUE
\end{verbatim}
STEP=0.5

DO B = 1,3

BESTFN=1.0*E+30
IF(J, EQ, 2) STEP=0.1
IF(J, EQ, 3) STEP=0.01
ARG=-1.0*XK
A=2.0*PNORM(ARG)
ARG=DELT*SORT(XN)-XK
P=PNORM(ARG)

H=SORT((A*A3P(LL)+A1*A2*XN)/(XLAM(LL)*A4(LL)*(1./P-0.5)))
B=H*(1./P-0.5)*XLAM(LL)*H/12.)
OBJFN=(XLAM(LL)*A4(LL)*B + A*X3P(LL)/H + XLAM(LL)*A3(LL)) +
(A1 + A2*XN)*((1 + XLAM(LL)*B)/H) /
& ( 1 + XLAM(LL)*B + A*U(LL)/XN + XLAM(LL)*V(LL))

IF(OBJFN.GT.BESTFN) GO TO 7
BESTFN=OBJFN
BESTA=A
BESTP=P
BESTH=XK
BESTI=XI
BESTH=H
XI=XI+STEP
GO TO 6

7 IF(J, EQ, 3) GO TO 8
XK=BESTK-STEP
CONTINUE
AA2=DELT*SORT(XN)/ORDN(BESTK)

TA= BESTH*(.5 - XLAM(LL)*BESTH/12.)
BB= BESTH/BESTP-TA
TR1=BB*XLAM(LL)*A4(LL)
S4=S4+TR1
TR3=BB*XLAM(LL)
S5=S5+TR3
TR2=(52*BESTA*(EXP(-S2*H)))*A3P(LL)/(1-EXP(-S2*H))

WRITE(17,*) 'TR1=',TR1,' TR2=',TR2,' TR3=',TR3
WRITE(16,10) I,BESTK,BESTH,BESTA,BESTP,BESTFN,AA2,TA,BB

9 WRITE(6,10) I,BESTK,BESTH,BESTA,BESTP,BESTFN,AA2,TA,BB
10 FORMAT(1H,I4,2(5X,F7.2),3X,2(3X,F7.4),4X,F7.2,4X,E10.2,2(2X,F7.3))

WRITE(17,*) '
WRITE(17,*) 'S5=',S5(LL)

200 CONTINUE
STOP
END

FUNCTION ORDN(Z)
ORDN=0.3989422B*(1/EXP(Z*Z/2))
RETURN
END

FUNCTION PNORM(X)
DIMENSION C(7)
DATA C/319361530.\$35663782.1.781477937.
& -1.821255978.1.330274429.2316419.2.506628725/
V=X
IF(X,L.0.,) V=-X.
T=1.\$(((C(5)\$T+C(4))\$T+C(3))\$T+C(2))\$T+C(1))\$T
PNORM=5\$1/(EXP(V\$Y/2))/C(7)
IF(X,G.0.,) PNorm = -PNORM
RETURN
END
SUBROUTINE READ1(NUM)
REAL A3(10), A3P(10), A4(10), XLAM(10), U(10), V(10), DEL(10)
REAL KR(10), KS(10), TAUR(10), TAUS(10)
COMMON /BLOCK1/ A1, A2, A3, A3P, A4, XLAM, DELT, DEL, U, V
COMMON /BLOCK2/ KR, KS, TAUR, TAUS
DO I=1,25
WRITE(6,*)
ENDDO
WRITE(6,*) 'ENTER THE NUMBER OF CAUSES'
READ(5,*) NUM
WRITE(6,*) 'ENTER THE FOLLOWING PARAMETERS SEPARATE'
& THEM BY A COMMA'
WRITE(6,*) 'A1, A2'
READ(5,*) A1, A2
WRITE(6,2) NUM
2 FORMAT (1X,'ENTER ',12,' VALUES OF A3P')
READ(5,*) (A3P(J), J=1,NUM)
WRITE(6,1) NUM
1 FORMAT (1X,'ENTER ',12,' VALUES OF XLAM')
READ(5,*) (XLAM(J), J=1,NUM)
WRITE(6,3) NUM
3 FORMAT (1X,'ENTER ',12,' VALUES OF A4')
READ(5,*) (A4(J), J=1,NUM)
WRITE(6,4) NUM
4 FORMAT (1X,'ENTER ',12,' VALUES OF A3')
READ(5,*) (A3(J), J=1,NUM)
WRITE(6,5) NUM
5 FORMAT (1X,'ENTER ',12,' VALUES OF DEL')
READ(5,*) (DEL(J), J=1,NUM)
WRITE(6,6) NUM
6 FORMAT (1X,'ENTER ',12,' VALUES OF U')
READ(5,*) (U(J), J=1,NUM)
WRITE(6,7) NUM
7 FORMAT (1X,'ENTER ',12,' VALUES OF V')
READ(5,*) (V(J), J=1,NUM)
WRITE(6,8) NUM
FORMAT (1X, 'ENTER', 12, 'VALUES OF KR')
READ(5, *) (KR(J), J=1, NUM)

WRITE(6, 9) NUM
FORMAT (1X, 'ENTER', 12, 'VALUES OF KS')
READ(5, *) (KS(J), J=1, NUM)

WRITE(6, 10) NUM
FORMAT (1X, 'ENTER', 12, 'VALUES OF TAU')
READ(5, *) (TAU(J), J=1, NUM)

WRITE(6, 11) NUM
FORMAT (1X, 'ENTER', 12, 'VALUES OF TAU')
READ(5, *) (TAUS(J), J=1, NUM)
RETURN
END

SUBROUTINE OUTP(NUM)
C
*** FUNCTION: OUTPUTS PRIMARY DATA ***
RFAL A3(10), A3P(10), A4(10), XLAM(10), U(10), V(10), DEL(10)
R= L, KR(10), KS(10), TAU(10), TAUS(10)
COMMON /BLOCK1/ A1, A2, A3, A3P, A4, XLAM, DEL, U, V
COMMON /BLOCK2/ KR, KS, TAU, TAUS
WRITE(16, *) *** PRIMARY INPUT VARIABLES ***
WRITE(16, 11) NUM
FORMAT (1X, 'NUMBER OF CAUSES = ', 13)
WRITE(16, 12) 'SAMPLING COST PARAMETERS:
WRITE(16, 13) A1 = ', A1
WRITE(16, 14) A2 = ', A2
WRITE(16, 15) COST OF FINDING ASSIGNABLE CAUSE NONE EXISTS
WRITE(16, *)
DO 14 J = 1, NUM
WRITE(16, 16) J, A3P(J)
14 CONTINUE
WRITE(16, 17) VALUES OF LAMDA = XLAM
WRITE(16, 18) DO 3 J = 1, NUM
WRITE(16, 19) J, XLAM(J)
1 FORMAT (1X, 'XLAM(', I, ') = ', F10.6)
3 CONTINUE
WRITE(16, 20) VALUES OF A4
WRITE(16, 21) DO 6 J = 1, NUM
WRITE(16, 22) J, A4(J)
5 FORMAT (1X, 'A4(', I, ') = ', F10.4)
6 CONTINUE
WRITE(16, *)
WRITE(16,*), VALUES OF A3
WRITE(16,*),
DO B J=1,NUM
   WRITE(16,7) J, A3(J)
   FORMAT(BX,'A3(',I1,') = ',F10.4)
END!
WRITE(16,*), VALUES OF DEL
WRITE(16,*),
DO 10 J=1,NUM
   WRITE(16,9) J, DEL(J)
   FORMAT(BX,'DEL(',I1,') = ',F10.4)
END!
WRITE(16,*), VALUES OF U
WRITE(16,*),
DO 18 J=1,NUM
   WRITE(16,17) J, U(J)
   FORMAT(BX,'U(',I1,') = ',F10.4)
END!
WRITE(16,*), VALUES OF V
WRITE(16,*),
DO 20 J=1,NUM
   WRITE(16,20) J, V(J)
   FORMAT(BX,'V(',I1,') = ',F10.4)
END!
WRITE(16,*), VALUES OF KR
WRITE(16,*),
DO 24 J=1,NUM
   WRITE(16,23) J, KR(J)
   FORMAT(BX,'KR(',I1,') = ',F10.4)
END!
WRITE(16,*), VALUES OF KS
WRITE(16,*),
DO 26 J=1,NUM
   WRITE(16,25) J, KS(J)
   FORMAT(BX,'KS(',I1,') = ',F10.4)
END!
WRITE(16,*), VALUES OF TAU
WRITE(16,*),
DO 27 J=1,NUM
   WRITE(16,26) J, TAU(J)
   FORMAT(BX,'TAU(',I1,') = ',F10.4)
END!
WRITE(16,*), VALUES OF TAUS
WRITE(16,*),
DO 30 J=1,NUM
   WRITE(16,29) J, TAUS(J)
END!
20 FORMAT(BX,'TAUS(',11,') = ',F10.4)
30 CONTINUE
      WRITE(16,*)
      RETURN
      END
APPENDIX I

PROGRAM MULTRUN

This program is employed when a production process is shutdown while searching for the assignable cause or causes of failure. The input and the outputs of the program are listed.
PROGRAM MAIN
C- THIS PROGRAM IS USED FOR EITHER SINGLE OR MULTIPLE CAUSES OF
C- FAILURE AND WHEN THE PROCESS CONTINUES OPERATING WHILE SEARCHING
C- FOR THE SINGLE OR MULTIPLE CAUSES OF FAILURE
C
FUNCTION: THIS PROGRAM GENERATES THE OPTIMAL COST OF
C- KEEPING THE PROCESS UNDER CONTROL
C
PARAMETERS:
C
SINGLE CAUSES OF FAILURE
C
INPUT:
C
XLAM=AVERAGE RATE OF OCCURENCE OF AN ASSIGNABLE CAUSE
C DEL=SHIFT IN THE MEAN OF THE PROCESS RESULTING FROM
C THE OCCURRENCE OF AN ASSIGNABLE CAUSE
C A1=COST PER SAMPLE OF SAMPLING AND PLOTTING THAT IS
C INDEPENDENT OF THE SAMPLE SIZE - N
C A2=RATE AT WHICH THE TIME BETWEEN TAKING A SAMPLE
C AND PLOTTING A POINT ON THE X-BAR CHART
C INCREASES WITH THE SAMPLE SIZE
C A3=COST OF FINDING AN ASSIGNABLE CAUSE WHEN IT
C EXISTS
C VO=RATE PER HOUR AT WHICH INCOME ACCRUES FROM THE
C OPERATION OF THE PROCESS WHEN IT IS IN THE STATE
C OF STATISTICAL CONTROL
C V1=RATE PER HOUR AT WHICH INCOME ACCRUES FROM THE
C OPERATION OF THE PROCESS WHEN IT IS IN A STATE
C OF OUT OF CONTROL
C A4=VO-V1
C A3P=COST OF LOOKING FOR AN ASSIGNABLE CAUSE WHEN
C NONE EXISTS
C G=TIME REQUIRED TO SAMPLE ONE
C ITEM AND INTERPRET
C THE RESULTS
C D=TIME REQUIRED TO FIND THE ASSIGNABLE CAUSE
C FOLLOWING AN ACTION SIGNAL
C
OUTPUT:
C N=SAMPLE SIZE
C K=UPPER AND LOWER LIMITS ON THE X-BAR CHART
C H=INTERVAL BETWEEN SAMPLES MEASURED IN HOURS
C ALPHA=PROBABILITY OF TYPE I ERROR
C 1-BETA=POWER OF THE CHART
C (WHEN Z=1.2826 1-BETA=0.90; Z=1.645 1-BETA=0.95)
C
SOLVE 2 EQUATIONS:
C (1) (Z+K)/PHI(K) =
C ((DEL**2)*A3P)/(A2+(XLAM*A4*G))
C
(2) DEL*SQRT(N)-K = Z
C TO OBTAIN AN ESTIMATE OF THE SAMPLE SIZE, THE
C VALUE OF N OBTAINED FROM THESE EQUATIONS IS USED
C TO SET AN UPPER AND LOWER LIMITS FOR THE SEARCH
C OF THE OPTIMAL SAMPLE SIZE
C TAU=AVERAGE TIME OF OCCURENCE OF AN ASSIGNABLE CAUSE
C WITHIN AN INTERVAL BETWEEN SAMPLES WHEN THE
C PROCESS IS IN CONTROL
C COST=TOTAL COST PER UNIT FOR KEEPING THE PROCESS
C UNDER CONTROL UNDER THE ASSUMPTION THAT DURING
C THE SEARCH FOR THE ASSIGNABLE CAUSE THE PROCESS
C CONTINUES OPERATING
C TESFAC=((DEL**2)*A3P)/(A2+(XLAM*A4*G))
C THIS FACTOR IS COMPARED TO (Z+K)/PHI(K). A
C SEARCH IN THE TABLES FOR A VALUE OF THIS FACTOR
C AS CLOSE AS POSSIBLE TO TESFAC GENERATES A VALUE*
FOR THE OPTIMAL COST
THE PROCESS IS OPERATING IN THE
PRESENCE OF AN ASSIGNABLE CAUSE

MULTIPLE CAUSES OF FAILURE

INPUT:
LAMDAJ=AVG RATE OF OCCURRENCE OF CAUSE J
LAMDAP=RATE OF OCCURRENCE OF A SECOND ASSIGNABLE CAUSE
DELJ=SHIFT IN THE MEAN OF THE PROCESS DUE TO CAUSE J
DEL=SHIFT IN THE PROCESS MEAN DUE TO DOUBLE OCCURRENCE OF ASSIGNABLE CAUSES
DJ=AVG TIME TO FIND AN ASSIGNABLE CAUSE J FOLLOWING A SIGNAL
A3J=AVG COST OF FINDING AN ASSIGNABLE CAUSE J WHEN IT OCCURS
A4J=LOSS PER HOUR OF A PROCESS OPERATING OUT OF CONTROL DUE TO A CAUSE J

OUTPUT:
TAUJ=AVG TIME OF OCCURRENCE OF AN ASSIGNABLE CAUSE J WITHIN AN INTERVAL BETWEEN SAMPLES WHEN THE PROCESS IS IN CONTROL
BJ=AVG TIME THE PROCESS IS OPERATING IN THE PRESENCE OF AN ASSIGNABLE CAUSE J
SS=LAMDA*EXPECTED COST PER CYCLE OF DISCOVERING THE ASSIGNABLE CAUSE OR CAUSES
TR1=LAMDA*EXPECTED ADDITIONAL LOSS PER CYCLE ARISING FROM OUT OF CONTROL CONDITIONS
TR2=LAMDA*EXPECTED LOSS PER CYCLE WHEN LOOKING FOR AN ASSIGNABLE CAUSE WHEN NONE EXISTS
TR3=LAMDA*B (B IS DEFINED ABOVE)

NOTE: THE OTHER PARAMETERS ARE THE SAME AS THE CASE OF A SINGLE CAUSE

REAL A3(10), A3P(10), A4(10), XLAM(10), D(10), DEL(10), SS(10)
COMMON /BLOCK/ A1, A2, A3, A3P, A4, XLAM, DEL, G, D, DEL

WRITE(16,*) 'RUNNING THE PROCESS, INDIVIDUAL LAMDA'

CALL READ1(NUM)
S1=0
S2=0

CALL OUTP(NUM)

DO J=1,NUM
S1=S1+XLAM(J)*DEL(J)
S2=S2+XLAM(J)
SS(J)=SS(J)+XLAM(J)*A3(J)
ENDDO

DO 200 LL=1,NUM
DEL=DEL(LL)
S4=0
SS=0

WRITE(6.25) LL FORMATA(5X,'CAUSE # ',13)
WRITE(16.25) LL
WRITE(18,25) LL
WRITE(18,*):
C
DU 100 KK=1,NUM
C
DELT=DEL(KK)
WRITE(6,3)
WRITE(16,3)
FORMAT(//,3X,'N',5X,'OPTIMUM K',3X,'OPTIMUM M'
&      ,6X,'ALPHA',5X,'POWER',4X,'COST',7X,'TEST FAC.'
&      ,6X,'TAU')
AA=DELT**2*A3P(LL)/(A2*XLAM(LL)*A4(LL)*G)
XK=0.
DO 4 I=1,10
RHS=(1.2B26+XK)/ORDN(XK)
IF(RHS.GT.AA)GO TO 5
4
XK=XK+0.5
5
XK=XK-0.5
IF(XK.LT.O.) XK=O.
C
*** N * EXPRESSION FOR THE SAMPLE SIZE ***
N=((1.2B26+XK)/DELT)**2+0.5
C
NMJN=N-10
C
IF(NMIN.LE.O) NMIN=1
NMJN=1
NMAX=N+10
IF (NMAX .LE.20) NMAX=20
DO 9 I=NMIN,NMAX
XN=I
XK=0.5
STEP=0.5
DO 6 J=1,3
BESTFN=1.0E+39
IF(J.EQ.2) STEP=0.1
IF(J.EQ.3) STEP=0.01
ARG=-1.0*XK
A=2.0*PNORM(ARG)
ARG=DELT*SQR(XN)-XK
P=PNORM(ARG)
C
*** H = INTERVAL BETWEEN SAMPLES MEASURED IN HOURS ***
H=SQR((A*A3P(LL)+A1*A2*XN)/(XLAM(LL)*A4(LL)*(1./P-0.5)))
C
*** B = MATHEMATICAL EXPRESSION EMPLOYED TO SIMPLIFY ***
C
*** THE LOSS COST FUNCTIONS ***
B=H*(1./P-0.5*XLAM(LL)*H/12.)*G*XN+D(LL)
OBJFN=(A4(LL)*XLAM(LL)*B**A3P(LL)/H*XLAM(LL)*A3(LL))/
       (XLAM(LL)*B+1.0)
& +(A1*A2*XN)/H
IF(OBJFN.GT.BESTFN) GO TO 7
BESTFN=OBJFN
BESTA=A
BESTP=P
BESTK=XK
BESTH=H
XK=XK+STEP
GO TO 6
7
IF(J.EQ.3) GO TO 8
XK=BESTK+STEP
CONTINUE
AA1=DEL*T*QR{T(XN)/ORDN(BESTK)}
TA= BEST*(.5- XLAM(LL)*BEST/12.)
BB= BEST/BESTP-TA=G*XN+D(LL)
TR1=BB*XLAM(LL)*A4(LL)
S4=S4-TR1
TR3=BB*XLAM(LL)
S5=S5-TR3
TR2 = (S2*BEST*EXP(-S2*M))/A3P(LL)/(1-EXP(-S2*M))
WRITE(18,*) 'TR1=',TR1,' TR2=',TR2,' TR3=',TR3
WRITE(16,1Q)
I,BESTK,BESTH,BESTA,BESTP,BESTF,M,AA1,TA,BB
C100 CONTINUE
C WRITE(18,*)'S4=',S4,' S5=',S5
C WRITE(18,*)'
C WRITE(18,*)' SS=',SS(LL)
D00 CONTINUE
STOP
FUNCTION ORDN(Z)
C FUNCTION: COMPUTES EXPONENTIAL DISTRIBUTION
ORDN = 0.39894228*(1/EXP(Z*Z/2))
RETURN
END
FUNCTION PNNM(X)
C FUNCTION: COMPUTES PHI(K)
DIMENSION C(7)
DATA C/.319381530,-.356563782,1.781477937,
& -.1821255978,1.330274429,.2316419,2.5066287251
Y=X
IF(X.LT.0.) Y=-X
T=1./(1.+C(6)*Y)
S=((-C(5)*T+C(4))*T+C(3))*T+C(2))*T+C(1))
PNORM=S*(1/(EXP(Y^2/2)))/C(7)
IF(X.GT.0.) PNORM=1.-PNORM
RETURN
END
SUBROUTINE READI(NUM)
C FUNCTION: INPUTS PRIMARY DATA
REAL A3(10),A3P(10),A4(10), XLAM(10), D(10),DEL(10)
COMMON /BLOCK1/A1,A2,A3,A3P,A4,XLAM,DEL,G,D,DEL
DO I=1,12,
WRITE(6,*)
ENDDO
WRITE(6,*)'ENTER THE NUMBER OF CAUSES'
READ(5,*)NUM
WRITE(6,*)'ENTER THE FOLLOWING PARAMETERS SEPARATE BY A COMMA'
WRITE(6,*)'A1,A2,G'
READ(5,*)A1,A2,G
WRITE(6,6)NUM
FORMAT(1X,'ENTER',I2,'VALUES OF A3P')
SUBROUTINE OUTP(NUM)

C *** FUNCTION : OUTPUTS PRIMARY DATA ***
REAL A3(10), A3P(10), A4(10), XLAM(10), D(10), DELT(10)
COMMON /BLOCK1/ A1, A2, A3, A3P, A4, XLAM, D, DELT, G, D, DEL

WRITE(16,*) ' *** PRIMARY INPUT VARIABLES ***
WRITE(16,*) ' NUMBER OF CAUSES = ', NUM
WRITE(16,11) NUM

11 FORMAT(' NUMBER OF CAUSES = ',I3)
WRITE(16,12) J, A3P(J)
12 FORMAT(BX,'A3P(',I1,')=',F10.4)
CONTINUE
WRITE(16,13) ' VALUES OF LAMDA = XLAM'
WRITE(16,14) 'XLAM(',I1,')=',F10.6)
CONTINUE
WRITE(16,15) ' VALUES OF D'
WRITE(16,16) 'D(',I1,')=',F10.6)
CONTINUE
WRITE(16,17) ' VALUES OF DEL'
WRITE(16,18) 'DEL(',I1,')=',F10.6)
CONTINUE
RETURN
END

READ(5,*) (A3(J), J=1, NUM)
WRITE(6,1) NUM
1 FORMAT (' ENTER ',I2, ' VALUES OF XLAM')
READ(5,*) (XLAM(J), J=1, NUM)
WRITE(6,2) NUM
2 FORMAT (' ENTER ',I2, ' VALUES OF D')
READ(5,*) (D(J), J=1, NUM)
WRITE(6,3) NUM
3 FORMAT (' ENTER ',I2, ' VALUES OF A4')
READ(5,*) (A4(J), J=1, NUM)
WRITE(6,4) NUM
4 FORMAT (' ENTER ',I2, ' VALUES OF A3')
READ(5,*) (A3(J), J=1, NUM)
WRITE(6,5) NUM
5 FORMAT (' ENTER ',I2, ' VALUES OF DEL')
READ(5,*) (DEL(J), J=1, NUM)
RETURN
END

SUBROUTINE OUTP(NUM)

C *** FUNCTION : OUTPUTS PRIMARY DATA ***
REAL A3(10), A3P(10), A4(10), XLAM(10), D(10), DELT(10)
COMMON /BLOCK1/ A1, A2, A3, A3P, A4, XLAM, D, DELT, G, D, DEL

WRITE(16,*) ' *** PRIMARY INPUT VARIABLES ***
WRITE(16,*) ' NUMBER OF CAUSES = ', NUM
WRITE(16,11) NUM

11 FORMAT(' NUMBER OF CAUSES = ',I3)
WRITE(16,12) J, A3P(J)
12 FORMAT(BX,'A3P(',I1,')=',F10.4)
CONTINUE
WRITE(16,13) ' VALUES OF LAMDA = XLAM'
WRITE(16,14) 'XLAM(',I1,')=',F10.6)
CONTINUE
WRITE(16,15) ' VALUES OF D'
WRITE(16,16) 'D(',I1,')=',F10.6)
CONTINUE
WRITE(16,17) ' VALUES OF DEL'
WRITE(16,18) 'DEL(',I1,')=',F10.6)
CONTINUE
RETURN
END

SUBROUTINE OUTP(NUM)

C *** FUNCTION : OUTPUTS PRIMARY DATA ***
REAL A3(10), A3P(10), A4(10), XLAM(10), D(10), DELT(10)
COMMON /BLOCK1/ A1, A2, A3, A3P, A4, XLAM, D, DELT, G, D, DEL

WRITE(16,*) ' *** PRIMARY INPUT VARIABLES ***
WRITE(16,*) ' NUMBER OF CAUSES = ', NUM
WRITE(16,11) NUM

11 FORMAT(' NUMBER OF CAUSES = ',I3)
WRITE(16,12) J, A3P(J)
12 FORMAT(BX,'A3P(',I1,')=',F10.4)
CONTINUE
WRITE(16,13) ' VALUES OF LAMDA = XLAM'
WRITE(16,14) 'XLAM(',I1,')=',F10.6)
CONTINUE
WRITE(16,15) ' VALUES OF D'
WRITE(16,16) 'D(',I1,')=',F10.6)
CONTINUE
WRITE(16,17) ' VALUES OF DEL'
WRITE(16,18) 'DEL(',I1,')=',F10.6)
CONTINUE
RETURN
END
WRITE(16,1) J, D(J)
FORMAT(BX,'D(''I1,'') = ''F10.4')
CONTINUE
WRITE(16,*)
WRITE(16,*) 'VALUES OF A4'
WRITE(16,*)
DO 6 J=1, NUM
WRITE(16,5) J, A4(J)
5 CONTINUE
WRITE(16,*)
WRITE(16,*) 'VALUES OF A3'
WRITE(16,*)
DO 8 J=1, NUM
WRITE(16,7) J, A3(J)
7 CONTINUE
WRITE(16,*)
WRITE(16,*) 'VALUES OF DEL'
WRITE(16,*)
DO 10 J=1, NUM
WRITE(16,9) J, DEL(J)
9 CONTINUE
WRITE(16,*)
RETURN
END
REFERENCES


