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Numerical study of the onset of instability in the flow past a sphere

Kim, Inchul, Ph.D.
The University of Arizona, 1989
NUMERICAL STUDY OF THE ONSET OF INSTABILITY
IN THE FLOW PAST A SPHERE

by
Inchul Kim

A Dissertation Submitted to the Faculty of the
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In the Graduate College
THE UNIVERSITY OF ARIZONA

1989
As members of the Final Examination Committee, we certify that we have read
the dissertation prepared by Inchul Kim
entitled NUMERICAL STUDY OF THE ONSET OF INSTABILITY
IN THE FLOW PAST A SPHERE

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for the Degree of Doctor of Philosophy

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Bruce Bailey 17 July 1989

Final approval and acceptance of this dissertation is contingent upon the
candidate's submission of the final copy of the dissertation to the Graduate
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I hereby certify that I have read this dissertation prepared under my
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Anne J. Pearlstein 8/7/89
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SIGNED:  Inchul Kim
DEDICATION

To my parents, wife Seunghee, and son Sungjun
ACKNOWLEDGMENTS

I am deeply grateful to my advisor, Professor Arne J. Pearlstein, for his advice and guidance in performing this work. Without his profound knowledge and help, this work would not have been completed. I also thank my other committee members, Professors C. F. Chen, Edward Kerschen, Wilfred Greenlee, and Bruce Bayly for their interest and suggestions concerning the dissertation. The willingness of Professor Chen and Professor Bayly to serve on my committee with relatively short notice is greatly appreciated.

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Finally, I would like to thank my wife, Seunghee, for her endless love and support, and our son, Sungjun, who always brings joy to the family.

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ABSTRACT

Experiment shows that the steady axisymmetric flow past a sphere becomes unstable in the range \(120 < \text{Re} < 300\). The resulting time-dependent nonaxisymmetric flow gives rise to nonaxisymmetric vortex shedding at higher Reynolds numbers. The present work reports a computational investigation of the linear stability of the axisymmetric base flow.

When the sphere is towed, fixed, or otherwise constrained, stability is determined solely by the Reynolds number. On the other hand, when the sphere falls due to gravity, the present work shows that an additional parameter, the ratio of fluid density to sphere density \((\beta = \rho_f/\rho_s)\) is involved.

We use a spectral technique to compute the steady axisymmetric flow, which is in closer agreement with experiment than previous calculations. We then perform a linear stability analysis of the base flow with respect to axisymmetric and nonaxisymmetric disturbances. A spectral technique similar to that employed in the base flow calculation is used to solve the linear disturbance equations in streamfunction form for axisymmetric disturbances, and in a modified primitive variable form for nonaxisymmetric disturbances.

For the density ratio \(\beta = 0\), which corresponds to a fixed sphere, the analysis shows that the axisymmetric base flow undergoes a Hopf bifurcation at \(\text{Re} = 175.1\), with the critical disturbance having azimuthal wavenumber \(m = 1\). The results are favorably compared to previous experimental work.
CHAPTER 1

INTRODUCTION

1.1 Motivation

Flow past a sphere is a classical problem in hydrodynamics, and is among other things, the simplest example of flow past a three-dimensional bluff body. When the Reynolds number is small enough, the flow past a sphere is laminar, steady, attached, and axisymmetric. As the Reynolds number increases, the flow becomes separated from the rear part of the sphere, and a secondary flow appears in the form of a standing eddy. Experiment shows that the laminar, steady, separated, axisymmetric wake behind a fixed sphere in a uniform flow becomes unstable at a Reynolds number (based on diameter) somewhere above 120, and eventually gives rise to temporally periodic nonaxisymmetric vortex shedding at a higher Reynolds number.

However, as discussed in §1.2, the details of the transition are not at all clear, with different investigators reporting different transition Reynolds numbers. For freely falling spheres, where one might expect the transition Reynolds number to depend on the fluid/sphere density ratio, no real conclusion can be drawn due to the limited amount of data and the fact that most previous investigators have been unaware of the potential importance of the density ratio, and so have reported neither the density ratio nor enough information to allow its calculation.

Current knowledge of transition in flow past a sphere is limited to the conclusions obtained from trajectory measurements for freely falling spheres, and flow
visualization experiments for fixed and falling spheres. As stated by Nakamura (1976) in reference to the previous analytical and numerical calculations of the steady axisymmetric solution, "None of the theoretical workers, however, has predicted the critical Reynolds number at which the wake behind a sphere becomes unstable."

In addition to the fundamental fluid mechanical interest of the present work, there are many applications in which the development of the more complicated unsteady, nonaxisymmetric velocity field can have an important influence on the rates of heat and mass transfer to and from the sphere, as well as a profound effect on the trajectory of the sphere in the event that it is not fixed. Among such applications are a number in which the range of relevant Re overlaps the transition range $100 < Re < 300$, including

- a) the dissolution and motion of dye crystals in flow visualization experiments (Harms & Wiegel 1979)
- b) heat and mass transfer to particles entrained in a high temperature gas in the production of molybdenum, zirconium, and titanium from their ores or concentrates, as well as in the production of ferro-alloys such as ferrovanadium and ferrocolumbium. The short contacting times require accurate knowledge of heat and mass transfer to and from the particles in the design, operation, and optimization of these electrometallurgical process (Sayegh & Gauvin 1979)
- c) the motion of metal particles added to solid propellant rocket fuels to increase the specific impulse (Zarin & Nicholls 1971)
- d) the motion of sedimenting particles in natural waters, for which the settling rate "is one of the most important factors affecting the erodibility and movement of earth material" (Stringham, Simons & Guy 1969)
The motion of small particles in numerous gas-solid fluidization processes (Cheremisinoff & Cheremisinoff 1984) is one of the applications of fluid dynamics. Another application is the settling of particles in clarifiers, sedimentation processes, and other unit operations in the chemical and process industries.

There are also many situations involving the motion of small liquid droplets in a gas for which the deformation and internal circulation of the drop are unimportant, and the motion of the drop, as well as the wake behind it do not differ significantly from the rigid sphere case (Beard 1976). The velocity field in the wake region and the effect of instability on droplet motion are important factors in the

- "rotation, oscillation, spinning, and swerving of falling burning drops" (Natarajan 1974)
- spray deposition of refractory and erosion resistant coatings (Vardelle et al. 1983)
- determination of collision frequencies, coalescence and breakup, rates of scavenging of pollutant gases (such as oxides of sulfur in acid rain formation), evaporation/condensation, and particulate capture for both cloud and precipitation drops, as well as related problems in the spray scrubbing of effluent gases (Wang & Pruppacher 1977, Suneja 1973)
- analysis of heat and mass transfer in spray column heat exchangers (Letan and Kehat 1968)
- study of transport phenomena and motion of "Significant portions of the droplets in sprays to quench fires, nuclear reactor emergency core cooling sprays, and high temperature sprays (which) can have Reynolds numbers on the order of 100" (Harpole 1977).
1.2 Previous Work

At sufficiently small Reynolds numbers, flow about bluff bodies is known to be attached and steady. At higher Reynolds numbers, the flow becomes separated, unsteady, and ultimately turbulent.

In recent years, the process by which this transition occurs has been extensively studied for two-dimensional bodies, both experimentally (Provansal, Mathis & Boyer 1987) and theoretically (Monkewitz & Nguyen 1987, Monkewitz 1988a). In the case of a circular cylinder (Thoman & Szewczyk 1969, Lin, Pepper & Lee 1976, Jackson 1987, Zebib 1987), the first stage in the transition is found to be a wake instability of the separated steady flow, leading to a time-periodic two-dimensional flow with considerable concentration of bound vorticity in the near wake adjacent to the body. As Re is increased, this flow may in turn give rise to the shedding of fluid parcels with concentrated vorticity ("vortex shedding"), and eventually to turbulence.

For axisymmetric bodies, the transition is less well understood, despite the considerable experimental progress that has been made for axisymmetric wakes of slender (Hama & Peterson 1976, Peterson & Hama 1978, Sato & Okada 1966) and bluff (Achenbach 1974, Fuchs, Mercker & Michel 1979, Goldburg & Florsheim 1966, Jayaweera & Mason 1965, Kim & Durbin 1988, Masliyah 1972, Willmarth, Hawk & Harvey 1964) bodies of revolution. We note that for nonspherical bodies of revolution, experiments may be plagued by the need for extremely precise alignment of the body's axis with the flow direction (Hama et al. 1977).

On the theoretical side, there have been several studies of the stability of axisymmetric parallel velocity profiles as models of axisymmetric wakes (Batchelor & Gill 1962, Gold 1963, Lessen & Singh 1973, Monkewitz 1988b). Batchelor & Gill
performed an inviscid stability analysis of a family of parallel axisymmetric velocity profiles appropriate to jets and wakes, and found the critical azimuthal wavenumber to be \( m = 1 \). Gold extended this approach to compressible flows, and found the same critical azimuthal wavenumber. Lessen and Singh extended the approach of Batchelor & Gill by including the viscous terms in the disturbance equations, and also found \( m = 1 \) to be the critical azimuthal wavenumber. More recently, Monkewitz (1988b) has also considered, on a viscous basis, the linear stability of incompressible axisymmetric parallel wake profiles, including those appropriate to bluff bodies of revolution. Monkewitz has established conditions under which these parallel flows are absolutely unstable with respect to \( m = 1 \) disturbances.

The prototypical axisymmetric flow is that generated by a sphere in a uniform flow. At low Reynolds numbers, the flow is steady, axisymmetric, and remains attached. In the classical experiments of Taneda (1956), separation is first observed at \( Re \approx 24 \). (All Reynolds numbers referred to herein are based on diameter.) As Re increases, the downstream extent of the recirculating wake becomes progressively longer, and the separation circle on the sphere moves forward from the rear stagnation point. At values of Re in the range \( 120 < Re < 300 \), experiment shows that the steady axisymmetric flow loses its stability (henceforth referred to as the first transition, with corresponding Reynolds number \( Re_1 \)) and the motion becomes unsteady and nonaxisymmetric. There is considerable uncertainty in the experimental value of \( Re_1 \) (Goldburg & Florsheim 1966, Möller 1938, Nakamura 1976, Roos & Willmarth 1971, Taneda 1956, Toulcova & Podzimek 1968, Zikmundova 1970).

The experimental determination of \( Re_1 \) in conventional wind/water tunnel facilities is complicated by three factors. First, the presence of a sting or other
support is known to have a strong influence on the stability of the sphere wake (Roos 1968, Roos & Willmarth 1971). This leads to the hypothesis that $Re_1$ depends rather delicately on the details of the base flow. Second, because the onset of unsteadiness occurs as a weak, low-frequency oscillation, the instability may not be easily discernible. Third, free-stream turbulence is known to have a influence on the Reynolds number at which vortex shedding occurs (Zarin 1970), and may also effect $Re_1$. The first of these problems can be avoided by use of a magnetic suspension balance or some other noninterfering support (Zarin 1970). The third can be avoided by towing the sphere through otherwise quiescent fluid (Möller 1938, Roos & Willmarth 1971). The first problem can be avoided and the third can in large part be eliminated by using freely falling or rising spheres. Unfortunately, this introduces a possible coupling of the rigid body motion of the sphere to the nonaxisymmetric instability of the wake (through the no-slip boundary condition on the surface of the sphere). This may in turn lead to a reduction in $Re_1$ for freely falling/rising spheres. For either fixed or falling/rising spheres, the weakness of the oscillations at the onset of instability presents a formidable challenge to the experimental determination of $Re_1$.

The stability of the steady axisymmetric flow is especially important because the breakdown to nonaxisymmetric vortex shedding at higher Re appears to occur via an intermediate stage involving nonaxisymmetric wake oscillations. Although the recent work of Kim & Durbin (1988) has substantially clarified the previously muddled (Achenbach 1974, MacCready & Jex 1964, Möller 1938, Preukschat 1962, Shafrir 1965, Sheth 1970, Stringham, Simons & Guy 1969, Viets 1971, Viets & Lee 1971) nature of the vortex shedding at higher Re, there is still relatively little known about the
mechanism by which the transition from nonaxisymmetric wake oscillations to vortex shedding occurs. Obviously, a first step in unravelling the transition process is the determination of the Re at which the first transition occurs.

In the one previous theoretical investigation of the problem, Kawaguti (1955) used a Galerkin approximation to the steady axisymmetric base flow. A Galerkin calculation showed that the base flow became unstable with respect to axisymmetric disturbances at Re₁ = 51, which is at variance with both the range of experimentally determined Re₁, and with the nonaxisymmetric nature of the flow above Re₁ (Kendall 1964, Goldburg & Florsheim 1966, Roos & Willmarth 1971).

1.3 Development of the Governing Equations

In this section, we present the Navier-Stokes equations for the flow of a viscous incompressible fluid about a sphere, along with the rigid body equations of motion for the sphere.

We begin with the equations of motion for a constant property, incompressible Newtonian fluid

\[ \nabla \cdot V_f = 0 \quad (1.1a) \]

\[ \frac{\partial V_f}{\partial t} + (V_f \cdot \nabla) V_f = -\frac{1}{\rho_f} \nabla p + \nu \nabla^2 V_f - gk \quad (1.1b) \]

and a sphere that falls (or rises) freely through the fluid due to gravity (Happel & Brenner 1965, p. 30, and Likins 1973, p. 440)

\[ m_s \frac{dV_s}{dt} = -m_s gk + \int_{S_t} dS \cdot \sigma \quad (1.1c) \]
\[ I \cdot \frac{d\Omega}{dt} + \Omega \times I \cdot \Omega = \int_{S_t} \mathbf{r} \times (d\mathbf{S} \cdot \sigma) \]  

(1.1d)

\[ \mathbf{V}_f = \mathbf{V}_s + \Omega \times \mathbf{r} \quad \text{on } S_t. \]  

(1.1e)

Here, we have defined the velocities in an inertial reference frame for which the fluid is at rest far away from the sphere. Equation (1.1e) is the no-slip boundary condition at the surface of the sphere. We have used the notation:

- \( \mathbf{V}_f \) - local fluid velocity
- \( \mathbf{V}_s \) - center of mass velocity of the sphere
- \( p \) - pressure
- \( g \) - gravitational acceleration
- \( k \) - unit vector in vertical direction (positive upwards)
- \( m_s \) - mass of the sphere
- \( I \) - inertia tensor of the sphere
- \( \nu \) - kinematic viscosity of the fluid
- \( \rho_f \) - density of the fluid
- \( \sigma \) - stress tensor
- \( \Omega \) - angular velocity of the sphere
- \( \mathbf{r} \) - position vector from the center of the sphere
- \( d\mathbf{S} \) - surface element on the sphere
- \( S_t \) - instantaneous locus of points on the sphere

If we define a hydrodynamic pressure \( P = p + \rho_f g z \), we then have

\[ \nabla \cdot \mathbf{V}_f = 0 \]
\[
\frac{\partial V_f}{\partial t} + (V_f \cdot \nabla) V_f = -\frac{1}{\rho_f} \nabla P + \nu \nabla^2 V_f
\]

\[
m_s \frac{dV_s}{dt} = (\rho_f V_s - m_s)gk + \int_{S_t} dS \cdot H
\]

\[
I \cdot \frac{d\Omega}{dt} + \Omega \times I \cdot \Omega = \int_{S_t} r \times (dS \cdot H)
\]

\[
V_f = V_s + \Omega \times r \quad \text{on } S_t,
\]

where \( V_s \) is the sphere volume, \( H = \sigma - \rho_f g z E \), and \( E \) is the isotropic unit tensor.

This formulation has the decided disadvantage that the sphere's location is an unknown function of time, to be determined as a part of the solution of, say, an initial value problem. We therefore refer the fluid velocity to a nonrotating, noninertial frame moving with the center of mass of the sphere. Thus, \( V_{fm} = V_f - V_s \) is the fluid velocity in the moving frame. Hence, the location of the sphere is known, and we have

\[
\nabla \cdot V_{fm} = 0
\]

\[
\frac{dV_s}{dt} + \frac{\partial V_{fm}}{\partial t} + (V_{fm} \cdot \nabla) V_{fm} = -\frac{1}{\rho_f} \nabla P + \nu \nabla^2 V_{fm}
\]

\[
m_s \frac{dV_s}{dt} = (\rho_f V_s - m_s)gk + \int_{S_0} dS \cdot H
\]

\[
I \cdot \frac{d\Omega}{dt} + \Omega \times I \cdot \Omega = \int_{S_0} r \times (dS \cdot H)
\]
\[ V_{fm} = \Omega \times r \quad \text{on } S_o, \]

\[ V_{fm} \to -V_s \quad \text{as } |r| \to \infty, \]

where \( S_o \) is the (known) surface of the sphere.

Introducing the nondimensionalization

\[
t = \frac{a}{V_o} t' \quad \Omega = \frac{V_o}{a} \Omega' \quad V_s = V_o U_s \quad V_{fm} = V_o U_f
\]

\[
I = \frac{2}{5} m_s a^2 I' \quad r = ar' \quad P = \rho I V_o^2 \phi \quad H = \rho I V_o^2 H',
\]

where \( a \) is the radius of the sphere, and \( V_o \) is the steady downward velocity at which the sphere will fall according to the "standard drag curve", we obtain

\[
\nabla \cdot U_f = 0 \quad (1.2a)
\]

\[
\frac{dU_s}{dt} + \frac{\partial U_f}{\partial t} + (U_f \cdot \nabla) U_f = -\nabla \Phi + \frac{2}{Re} \nabla^2 U_f \quad (1.2b)
\]

\[
\frac{dU_s}{dt} = -C_D \frac{3\beta}{8} k + \frac{3\beta}{4\pi} \int_{S_o} dS \cdot H \quad (1.2c)
\]

\[
I \cdot \frac{d\Omega}{dt} + \Omega \times I \cdot \Omega = \frac{15\beta}{8\pi} \int_{S_o} r \times (dS \cdot H) \quad (1.2d)
\]

\[ U_f = \Omega \times r \quad \text{on } S_o \quad (1.2e) \]

\[ U_f \to -U_s \quad \text{as } |r| \to \infty, \quad (1.2f) \]

where \( Re \) is the Reynolds number for steady fall \((Re = 2V_o a/\nu)\), \( \beta = \rho_f/\rho_s \), \( C_D \) is the
drag normalized by the dynamic pressure associated with the steady velocity $V_o$ and the projected area of the sphere, and we have dropped the primes.

1.4 Overview of Present Work

It is the purpose of the present work to computationally investigate the first stage in the transition process for the geometrically simplest axisymmetric body, namely the sphere. In contrast to the work of Gold (1963), Lessen & Singh (1973), and Monkewitz (1988b), our work uses a numerically computed base flow which is a converged nonparallel solution of the Navier-Stokes equations. The disturbance equations are exact linearizations of the three-dimensional Navier-Stokes equations, with full treatment of the viscous and (on a linear basis) the inertial terms. To the best of our knowledge, this is the first viscous three-dimensional stability analysis of a nonparallel axisymmetric flow. The only previous attempt to compute $Re_1$ for a sphere is that of Kawaguti (1955).
CHAPTER 2

Accurate Computation of Steady Axisymmetric Flow

2.1 Motivation and Introduction

The steady axisymmetric viscous flow past a sphere in a uniform stream has been investigated analytically and numerically by many authors. In this chapter, we present a method which is ideally suited for accurate calculation of the flow near the sphere, and in particular, in the near wake. Accurate computation of the flow is a necessary prerequisite for the determination of the stability of the steady axisymmetric flow, which is the first step in the transition to turbulence for this simplest three-dimensional bluff body flow. In addition, accurate knowledge of the flow near the body is of interest in a number of situations in which heat or mass transfer occurs between a dispersed phase (particles or droplets) and a fluid medium. A problem of current interest is the scavenging of gaseous pollutants (e.g., oxides of sulfur and nitrogen) by raindrops.

The length of the separation bubble and the angle of separation are important features of the near wake, so it is natural to use the accuracy of these quantities to infer the accuracy of the computed velocity, vorticity, and streamfunction fields in the region near the sphere. The accuracy of the computed length of the separation bubble and the angle at which it separates from the sphere can be assessed by comparison to experiment, as well as by checking convergence (as the size of the domain and the resolution of the computation are increased).

In the present work, we present the results of a new calculation of the steady
axisymmetric flow. A useful technique for extrapolating the results to infinite resolution is proposed and demonstrated. The superiority to previous results is demonstrated by comparison to experiment. The results allow for an assessment of the accuracy of previous solutions.

2.2 Previous Computations

When the Reynolds number is sufficiently low, approximate theoretical calculations of the flow past a sphere can be made by the methods of Stokes and Oseen, or by the extension of Proudman & Pearson (1957), employing linearizing approximations. These solutions begin to diverge from experiment (e.g., Maxworthy 1965) for Re > 1.3.

Jenson (1959) obtained numerical solutions of a vorticity-streamfunction formulation of the Navier-Stokes equations using a finite difference method for 5 ≤ Re ≤ 40. He used a desk calculator and a relaxation method to obtain an approximate solution of the finite-difference equations. Hamielec, Hoffman & Ross (1967) and Le Clair, Hamielec & Pruppacher (1970) followed Jenson’s approach, and used a similar relaxation technique and a digital computer to solve the problem on a finer mesh. They extended the range of Reynolds number to 0.1 ≤ Re ≤ 100 and 0.01 ≤ Re ≤ 400, respectively. Numerical solutions of a time-dependent vorticity-streamfunction formulation were obtained by Rimon & Cheng (1969) using the Dufort-Frankel method for 1 ≤ Re ≤ 1000. Another finite-difference solution of the unsteady equations was given by Lin & Lee (1973) for 5 ≤ Re ≤ 300. In all of these finite-difference calculations, the far-field boundary conditions were enforced relatively close to the sphere (r∞ ≤ 20 for Re ≥ 40). As per the observation of Tiwary & Reethof (1986), it appears that this has been done in order to match numerical results.
to the experimentally observed drag on the sphere and to render the relaxation scheme stable.

Dandy & Leal (1986) transformed the unbounded physical domain to a finite computational domain and used an alternating-direction-implicit technique for $20 \leq Re \leq 180$. More recently, Fornberg (1988) obtained steady solutions for $100 \leq Re \leq 5000$, using Newton's method to solve the nonlinear equations arising from a finite-difference discretization. A rectangular computational domain was obtained by conformal transformation of a physical domain chosen to concentrate grid points in the wake. The location of the most remote grid point (in the wake) was at $r_\infty = 148$.

Dennis & Walker (1971) used a series truncation method in the polar direction, followed by finite-difference discretization of the resulting system of ordinary differential equations in the radial coordinate, to solve a vorticity-streamfunction formulation in the range $0.1 \leq Re \leq 40$. Similarly, Oliver & Chung (1987) used a combined series/finite-element method to solve a fourth-order streamfunction formulation for $0.5 \leq Re \leq 50$. Dennis & Walker (1971) and Oliver & Chung (1987) used $r_\infty = 148$ and $r_\infty = 300$, respectively, as the radius at which the far-field boundary condition was enforced.

Cliffe & Lever (1983) solved a vorticity-streamfunction formulation for $1 \leq Re \leq 100$ using finite-element methods. They used $r_\infty = 37$, except for $Re = 1$, for which $r_\infty = 67$ was used. In a later paper, Cliffe & Lever (1986) used a finite-element technique to solve both a primitive variables formulation, as well as a vorticity-streamfunction formulation in the same range of $Re$.

A spectral Galerkin method was used by Kawaguti (1955) to solve a fourth-order streamfunction formulation for $1 \leq Re \leq 80$. As noted by Flumerfelt & Slattery
(1965), Kawaguti's results are incorrect. Another spectral Galerkin method was used by Zebib (1984) to solve the fourth-order streamfunction formulation for \( 1 \leq \text{Re} \leq 10 \).

2.3 Governing Equations and Numerical Solution

Our first task is to compute the basic state, so we specialize (1.2a-f) to the case of steady axisymmetric flow, corresponding to a purely vertical motion of the sphere, for which \( U_z = -k \), \( U_r = \bar{U}_r(r, \theta) \), \( \Phi = \bar{\Phi}(r, \theta) \), and \( \Omega = 0 \). This yields

\[
\nabla \cdot \bar{U}_r = 0
\]

(2.1a)

\[
\bar{U}_r \cdot \nabla \bar{U}_r = -\nabla \Phi + \frac{2}{\text{Re}} \nabla^2 \bar{U}_r
\]

(2.1b)

\[
\bar{U}_r = 0 \quad \text{on } S_0
\]

(2.1c)

\[
\bar{U}_r \to k \quad \text{as } |r| \to \infty
\]

(2.1d)

where an overbar denotes a steady, axisymmetric quantity.

Because the base flow is axisymmetric, we use a streamfunction formulation of the Navier-Stokes equations (2.1b), which we write as

\[
\frac{1}{r^2} \left\{ \frac{\partial \Psi}{\partial r} \frac{\partial}{\partial j} - \frac{\partial \Psi}{\partial j} \frac{\partial}{\partial r} + \frac{2k}{r^2} \frac{\partial \Psi}{\partial r} + \frac{2}{r} \frac{\partial \Psi}{\partial j} \right\} \Delta^2 \Psi - \frac{2}{\text{Re}} \Delta^4 \Psi = 0
\]

(2.2)

with boundary conditions

\[
\Psi(1, j) = 0 \quad \frac{\partial \Psi(1, j)}{\partial r} = 0
\]

(2.3a, b)

At large \( r \), we have the asymptotic boundary condition
In (2.2), we have defined $\xi = \cos \theta$, and

$$\lim_{r \to \infty} \Psi(r, \xi) = \frac{1}{2} r^2 \ (1 - \xi^2). \tag{2.4}$$

The radial coordinate is transformed to the finite domain $-1 \leq \eta \leq 1$ according to

$$r = \exp \left[ \frac{\eta(n+1)}{2} \right]. \tag{2.6}$$

Equations (2.2, 2.3, and 2.5) are straightforwardly transformed into a finite domain. If the far-field boundary condition (2.4) is satisfied at $\eta = 1$ (i.e., at finite $r$), then it is of the "hard" type discussed by Fornberg (1980). The fourth-order streamfunction formulation requires two far-field conditions on the function $\psi(\eta, \xi) = \Psi(r, \xi)$, which we take to be the "soft" conditions

$$\frac{\partial \psi(\eta, \xi)}{\partial \eta} = \frac{1}{2} (1 - \xi^2) \ a^{a(n+1)} \quad \frac{\partial^3 \psi(\eta, \xi)}{\partial \eta^3} = \frac{1}{2} (1 - \xi^2) \ a^3 \ e^{a(n+1)} \tag{2.7a,b}$$

applied at $\eta = 1$.

We follow the approach of Zebib (1984) and expand the highest $\eta$-derivative of $\psi$ in terms of the fourth radial derivative of the uniform flow and complete sets of functions in the $\eta$- and $\xi$-coordinates.
where the radial functions are in turn expanded in terms of Chebyshev polynomials

\[ \frac{d^4 H_i(\eta)}{d\eta^4} = \sum_{j=0}^{\infty} A_{ij} T_j(\eta), \quad (2.9a) \]

and the angular functions

\[ W_i(\xi) = \int_{\xi}^{1} P_i(\xi') \, d\xi' \quad (2.9b) \]

individually satisfy the symmetry conditions (2.5a, b), where \( P_i(\xi) \) is the \( i \)-th Legendre polynomial.

We next truncate (2.8) and (2.9a) to \( K \) and \( L+1 \) terms in the polar and radial directions, respectively, and integrate with respect to \( \eta \) to obtain expressions for \( \psi \) and its first three \( \eta \)-derivatives as

\[ \frac{\partial^4 \psi(\eta, \xi)}{\partial \eta^4} = \frac{1}{2} (1-\xi^2) a^4 \, e^{a(\eta+1)} + \sum_{i=1}^{K} W_i(\xi) \sum_{j=0}^{L} A_{ij} T_j(\eta) \]

\[ \frac{\partial^3 \psi(\eta, \xi)}{\partial \eta^3} = \frac{1}{2} (1-\xi^2) a^3 \, e^{a(\eta+1)} + \sum_{i=1}^{K} W_i(\xi) \sum_{j=0}^{L} A_{ij} \sum_{q=0}^{L+1} f_{0j}^{(3)} T_q(\eta) + C_1(\xi) T_0(\eta) \]

\[ \frac{\partial^2 \psi(\eta, \xi)}{\partial \eta^2} = \frac{1}{2} (1-\xi^2) a^2 \, e^{a(\eta+1)} + \sum_{i=1}^{K} W_i(\xi) \sum_{j=0}^{L} A_{ij} \sum_{q=0}^{L+2} f_{0j}^{(2)} T_q(\eta) + C_1(\xi) T_1(\eta) \]

\[ + C_{2(\xi)} T_0(\eta) \]
\[
\frac{\partial \psi(\eta, \lambda)}{\partial \eta} = \frac{1}{2} (1-\lambda^2) \alpha \psi(\eta, \lambda) + \sum_{i=1}^{K} W_i(\lambda) \sum_{j=0}^{L} A_{ij} \sum_{q=0}^{L+3} f_{qj}^{(1)} T_q(\eta) + \frac{1}{4} C_1(\lambda) T_2(\eta)
\]

\[
+ C_2(\lambda) T_1(\eta) + C_3(\lambda) T_0(\eta)
\]

\[
\psi(\eta, \lambda) = \frac{1}{2} (1-\lambda^2) \alpha \psi(\eta, \lambda) + \sum_{i=1}^{K} W_i(\lambda) \sum_{j=0}^{L} A_{ij} \sum_{q=0}^{L+4} f_{qj}^{(0)} T_q(\eta) + \frac{1}{24} C_1(\lambda) T_3(\eta)
\]

\[
+ \frac{1}{4} C_2(\lambda) T_2(\eta) + \left[ C_3(\lambda) - \frac{1}{8} C_1(\lambda) \right] T_1(\eta) + C_4(\lambda) T_0(\eta)
\]

where the coefficients \( f_{qj}^{(n)} \) are defined in the Appendix. The constants of integration are chosen so that \( \psi(\eta, \lambda) \) satisfies all of the boundary conditions (2.3a, b) and (2.7a, b), and are evaluated as

\[
C_1 = - \sum_{i=1}^{K} W_i(\lambda) \sum_{j=0}^{L} A_{ij} \sum_{q=0}^{L+1} f_{qj}^{(3)}
\]

\[
C_2 = - \frac{1}{2} \sum_{i=1}^{K} W_i(\lambda) \sum_{j=0}^{L} A_{ij} \sum_{q=0}^{L+3} \left[ 1 - (-1)^q \right] f_{qj}^{(1)} + \frac{1}{2} (1-\lambda^2) \frac{a}{2}
\]

\[
C_3 = - \frac{1}{2} \sum_{i=1}^{K} W_i(\lambda) \sum_{j=0}^{L} A_{ij} \left\{ \sum_{q=0}^{L+3} \left[ 1 + (-1)^q \right] f_{qj}^{(1)} - \frac{1}{2} \sum_{q=0}^{L+1} f_{qj}^{(3)} \right\} - \frac{1}{2} (1-\lambda^2) \frac{a}{2}
\]

\[
C_4 = - \sum_{i=1}^{K} W_i(\lambda) \sum_{j=0}^{L} A_{ij} \left\{ \sum_{q=0}^{L+4} (-1)^q f_{qj}^{(0)} + \frac{1}{8} \sum_{q=0}^{L+3} f_{qj}^{(0)} \right\} + \frac{1}{3} \sum_{q=0}^{L+3} \frac{5(-1)^q f_{qj}^{(1)}}{8} - \frac{1}{3} \sum_{q=0}^{L+1} f_{qj}^{(3)}
\]

\[
+ \frac{1}{2} (1-\lambda^2) \left[ - \frac{5}{8} a - 1 \right]
\]

The final result is
\[
\frac{d^n \psi}{d \eta^n} = \frac{1}{2} (1-\eta^2) \left\{ a^n \ e^{a(\eta+1)} + \sum_{p=0}^{2} B_p^{(n)} T_p(\eta) \right\} \\
+ \sum_{i=1}^{K} W_i(\xi) \sum_{j=0}^{L} A_{ij} \sum_{q=0}^{L+4-n} G_{qj}^{(n)} T_q(\eta) , \quad 0 \leq n \leq 3 , \tag{2.10}
\]

where the coefficients \(B_p^{(n)}\) and \(G_{qj}^{(n)}\) are defined in the Appendix.

That the boundary and symmetry conditions are automatically satisfied by any linear combination of the functions in (2.10) allows further consideration to be restricted to the partial differential equation (2.2). Rewriting this as

\[
N(\psi) - \frac{2}{Re} D^4 \psi = 0 , \tag{2.11}
\]

we formulate a weighted residuals procedure whereby the residual of the discretization of (2.11) is made orthogonal to the trial functions according to

\[
\int_{1}^{\pi} \int_{0}^{\pi} \left[ N(\psi) - \frac{2}{Re} D^4 \psi \right] V_i(\theta) S_j(r) r^k \sin \theta \ d\theta \ dr = 0
\]

for \(1 \leq i \leq K\) and \(0 \leq j \leq L\), where

\[
V_i(\theta) = W_i(\xi) , \quad S_j(r) = \sum_{q=0}^{L+4} G_{qj}^{(0)} T_q(\eta) .
\]

Thus, the problem is reduced to a system of \(K(L+1)\) nonlinear equations, which we solve by Newton-Raphson iteration.

The computational technique applied above borrows from an idea due to

\[ \Psi(r, \theta) = \sum_{n=1}^{N_\theta} r f_n(r) P_n(\cos \theta) \]

and obtained a system of nonlinear ordinary differential equations

\[ \frac{2}{Re} D_l^2(t_l) = \sum_{n=1}^{N_\theta} \sum_{m=1}^{N_n} C_{nm} \left( \frac{f_n D_m f_m}{r} \right) + C_{nm} \left( \frac{(rf_n)'}{r^2} \right), \quad 1 \leq l \leq N_\theta \]

in the radial direction, where the differential operator \( D_m \) is given by

\[ D_m = \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{m(m+1)}{r^2}. \]

Zebib then expanded the fourth derivative of the solution of (2.13) in terms of a set of Chebyshev polynomials, such that each term in (2.12) satisfies the radial boundary conditions at \( r = 1 \) and \( r_\infty \).

In our technique, we have decomposed the streamfunction into a uniform flow and a Galerkin series. In the latter, the \( i \)-th angular trial function is an integral of the \( i \)-th Legendre polynomial and satisfies the proper symmetry conditions (2.9a,b). We have also used a correct asymptotic implementation of the "soft" far-field boundary conditions of Fornberg (1980) at the outer boundary of the computational domain. Finally, our Galerkin integrations are done with weightings of the form \( r^k \sin \theta \). (We have used \( k = 1 \) herein; essentially identical results, requiring the same number of Newton iterations to solve the nonlinear algebraic equation system, are obtained for \( k = 0, 1, 2 \).)
2.4 Results

We divide this section into two parts. In the first, we establish the convergence of the present results, with respect to changes in the size of the computational domain and the number of spectral functions retained in the expansion. In the second part, we compare our results to previous work (experimental and theoretical), and assess the errors in the latter.

The results are presented in terms of the drag coefficient \( C_D \), the length of the separation bubble \( l_s \), measured from the rear stagnation point in units of sphere radii, and the separation angle \( \theta_s \), also measured relative to the rear stagnation point.

The total drag on the sphere consists of two parts.

\[
\text{Drag} = D_p + D_v
\]

where \( D_p \) is the pressure or form drag caused by normal stresses, and \( D_v \) is the viscous drag caused by shear stresses on the body surface. Recalling the nondimensionalization in §1.3, we have

\[
D_p = 2\pi \int_0^\pi (\sigma_{rr})_{r=1} \cos^2 \theta \sin \theta \, d\theta
\]

\[
D_v = -2\pi \int_0^\pi (\tau_{r\theta})_{r=1} \sin^2 \theta \, d\theta
\]

The expressions for these two components of the stress tensor are
where the continuity equation requires that $\partial u_r / \partial r$ is zero on the surface of the sphere, and pressure is evaluated from the $\theta$-component of the momentum equation (2.1b). The drag coefficient is defined as the drag force divided by the product of the free-stream dynamic pressure and the projected area of the sphere.

As the Reynolds number increases, the flow may separate from the rear part of the sphere, with a secondary flow appearing in the form of a standing eddy. If the Reynolds number is not too large, the recirculating wake remains laminar and is confined within the streamline $\psi = 0$. The dimensionless radial distance $r_s$ from the sphere center to the downstream end of the eddy is obtained by letting $u_r = 0$ at $\theta = 0$. The root of the corresponding nonlinear algebraic equation is found by Newton-Raphson iteration. We note that $l_s$, defined before, corresponds to $r_s - 1$.

When the flow becomes separated, the $\phi$-component of vorticity on the surface of the sphere changes sign on a circle on the rear of the sphere surface, indicating a change in direction of the surface streamline. This separation circle marks the surface streamline bounding the wake. The exact position of separation is determined as follows. The $\phi$-component of vorticity may be expressed in terms of the streamfunction as

$$\omega_\phi = - \frac{D^2 \psi}{r \sin \theta} .$$

By setting the right hand side of (2.15) to zero on the sphere, we get a nonlinear
algebraic equation, which is again solved by the Newton-Raphson method. We denote the location of the separation circle by $\theta_s$, measured relative to the rear stagnation point.

Figures 2-10 show the streamlines at Reynolds numbers 0.1, 5, 20, 40, 70, 100, 140, 180, and 200. The contour values are 0.0, 0.125, 0.5, 1.125, 2.0, 3.125, and -0.006, -0.02, -0.04 (in the recirculating wake).

We choose Reynolds numbers 0.1, 40 and 180 to show the convergence of our scheme for the length and angle of the separation bubble and the drag coefficient. The flows at these values of Re are representative of, respectively, flows at low Re, intermediate Re, and at the upper range of Re for which the steady axisymmetric flow is stable (Kim & Pearlstein, 1989).

For $Re = 0.1$ and twenty expansion functions in each of the radial and polar directions, Table 1 shows the variation of the drag coefficient ($CD$) as the size of the computational domain ($r_\infty = 0^a$) increases. The error falls below one percent only for $r_\infty$ in excess of 40. The drag coefficient converges monotonically. For $r_\infty$ chosen sufficiently large for finite domain effects to be unimportant (according to Table 1) at $Re = 0.1$, Table 2 shows that $CD$ converges quite rapidly as the number of spectral functions increases. We note that for small numbers of expansion functions, $CD$ varies nonmonotonically with the number of terms retained.

For $Re = 40$ and twenty expansion functions in the radial and polar directions, Table 3 shows the dependence of $l_s$, $\theta_s$, and $CD$ on the size of the computational domain. In this case, none of the computed quantities is a monotonic function of the domain size. Moreover, the magnitude of the relative errors also depends nonmonotonically on $r_\infty$. This is the source of the "false convergence" of $l_s$ at
\( r_\infty = e^{2.5} \) and \( e^3 \), for which one might deduce that \( l_s \) has converged (to values nearly identical to the fully converged result at \( r_\infty = e^5 \)), even though the results for the intervening values of \( r_\infty (e^{3.5} \) and \( e^4 \)) differ significantly. We also note that the values of \( \theta_s \) and \( C_D \) converge considerably faster than the value of \( l_s \). Table 4 shows the dependence of \( l_s, \theta_s, \) and \( C_D \) on the number of expansion functions. We note that the drag coefficient converges monotonically, whereas \( l_s \) and \( \theta_s \) do not. Table 5 shows that when \( K \) (the number of angular terms retained) is equal to twenty, \( C_D \) is essentially independent of \( L \) (the number of radial functions), even though \( l_s \) and \( \theta_s \) still depend significantly on \( L \). Table 5 also shows that for \( K = 20, C_D \) is essentially independent of \( L \), even though Table 4 shows that \( C_D \) is not fully converged. From this it is clear that for sufficiently large \( K \) and \( L \), the drag coefficient depends primarily on \( K \). The dependence of \( l_s \) and \( \theta_s \) on \( K \) and \( L \) is more complex.

For \( \text{Re} = 180, \) Table 6 shows a similar nonmonotonic dependence of \( l_s, \theta_s, \) and \( C_D \) on the size of the computational domain. As noted for \( \text{Re} = 40, \) this nonmonotonic dependence can easily give rise to false convergence. In this case, we see that \( \theta_s \) converges faster than \( l_s \) and \( C_D \) with increasing \( r_\infty \). None of the computed quantities converges monotonically as \( r_\infty \) increases. For \( \text{Re} = 180, \) Table 7 shows that as \( K \) and \( L \) are increased, \( l_s \) and \( \theta_s \) converge faster than does \( C_D \).

Finally, Table 8 shows \( l_s \) and \( \theta_s \) near the Reynolds number \((\text{Re}_s)\) at which the flow separates from the sphere at the rear stagnation point. At \( \text{Re}_s, \) we would have \( l_s = \theta_s = 0. \) We note that as the Reynolds number approaches \( \text{Re}_s, \) the absolute errors in \( l_s \) and \( \theta_s \) are nearly constant, thus rendering a precise estimate of \( \text{Re}_s \) rather difficult. However, when we plot \( \theta_s \) versus \( 1/K = 1/L, \) we find that the points nearly lie on a straight line. This plot has the obvious advantage of requiring extrapolation
to the origin rather than to infinity (Domb & Sykes 1957, Van Dyke 1975). More important, the extrapolation is often nearly linear as in the present work. Thus, the converged result can be estimated by extrapolation, rather than by using increasingly large expansions, which would require excessive computing time and storage. In Table 8, we show $\theta_{\infty}$ obtained by this extrapolation, along with the standard deviation of the computed points from the straight line obtained by a linear least squares fit. The fit to a straight line is excellent. Table 9 shows the effect of the location of the outer computational boundary for $Re = 20$ with $K = L = 28$. We see that $r_{\infty} = e^6$ is sufficiently large. On the basis of the results shown in Table 8, we estimate $Re_s = 20.49$.

For $Re = 0.1$ Table 10 shows a comparison of the values of $C_D$ computed by our spectral technique and by previous authors. In particular, we note that the pressure contribution to $C_D$ is slightly more than one-half the viscous contribution, in agreement with results at higher $Re$ (Jenson 1959, Dennis & Walker 1971, Lin & Lee 1973). (At $Re = 0$, the ratio of the pressure and viscous contributions is exactly one-half; we expect it to decrease monotonically with increasing $Re$.) We note that in the calculation of Le Clair et al. (1970), the ratio of pressure to viscous contributions was in excess of one-half, thus indicating a problem in their algorithm or code.

There have been many previous experimental and computational estimates of the Reynolds number at which the flow separates from the sphere at the rear stagnation point. Table 11 shows a fairly wide range of measured and computed values of $Re_s$ between 10 and 24. In fact, the experimental determination of $Re_s$ by flow visualization is very difficult due to the very small size of the separation bubble. Thus, it is not surprising that the most definitive estimate of $Re_s$ is by a mass transfer
measurement, in which Garner et al. (1959) measured the rate of dissolution in a fluid of a slightly soluble solid sphere. Our results definitively confirm two previous fortuitous estimates of Re$_s$. Kalra & Uhlherr (1973) estimated Re$_s = 20$, on the basis of measurements of $l_s$ for Re $> 27$ (with considerable scatter) and extrapolation to $l_s = 0$. Lin & Lee (1973) computed the axisymmetric steady flow as a long-time solution of the unsteady Navier-Stokes equations for Re $= 5, 20, 40, 100, 200, \text{ and } 300$. They found that the steady flow exhibited a small separated region for Re $= 20$.

Table 12 provides a detailed comparison of our computed values of $l_s$ and $\theta_s$ with previous calculations for the entire range of Re over which the steady axisymmetric flow is stable. (The corresponding computational domains and truncation levels are shown in Table 13.) The present results are in closer agreement with the classical experiments of Taneda (1956) than are any previous calculations. Table 14 shows the detailed convergence of $l_s$ and $\theta_s$ at the Reynolds numbers used in Table 12 and 13, as well as the extrapolated result.

2.5 Comparison to Experiment and Previous Computations

The present work provides very accurate results over the entire range of Reynolds numbers in which the steady axisymmetric flow is stable and hence physically realizable, and gives closer agreement with the classical experiments of Taneda (1956) than any previous calculations, as shown in Table 12. Our extrapolation technique improves the agreement even more, and confirms two previous fortuitous estimates of Re$_s$. 
CHAPTER 3

Stability of Flow Past a Fixed Sphere

We first consider the incompressible flow of a constant property Newtonian fluid past a fixed rigid sphere, which is governed by a special case of equations (1.2a-f) for the fluid/sphere density ratio \( \beta = 0 \). The governing equations are

\[
\nabla \cdot \mathbf{U}_f = 0
\]

\[
\frac{\partial \mathbf{U}_f}{\partial t} + (\mathbf{U}_f \cdot \nabla) \mathbf{U}_f = -\nabla \Phi + \frac{2}{Re} \nabla^2 \mathbf{U}_f
\]

\[
\mathbf{U}_f = 0 \quad \text{on } S_0
\]

\[
\mathbf{U}_f \to k \quad \text{as } |r| \to \infty.
\]

3.1 Linear Disturbance Equations

We consider infinitesimally small disturbances to the steady axisymmetric flow and decompose the dimensionless velocity and pressure into

\[
\mathbf{U}_f(r, \theta, \phi, t) = \bar{\mathbf{U}}_f(r, \theta) + u_f(r, \theta, \phi, t)
\]

\[
\Phi(r, \theta, \phi, t) = \bar{\Phi}(r, \theta) + \Pi(r, \theta, \phi, t).
\]

In what follows, we shall only retain terms to first order in the disturbance quantities and obtain the following linear disturbance equations.

\[
\nabla \cdot \mathbf{u}_f = 0 \quad \text{(3.1a)}
\]
\[
\frac{\partial \bar{u}_f}{\partial t} + (\bar{u}_f \cdot \nabla) \bar{U}_f + (\bar{U}_f \cdot \nabla) u_f = -\nabla \Pi + \frac{2}{\text{Re}} \nabla^2 u_f
\]  
(3.1b)

\[u_f = 0 \quad \text{on } S_0\]  
(3.1c)

\[u_f \to 0 \quad \text{as } r \to \infty .\]  
(3.1d)

The disturbances can be Fourier decomposed into azimuthal components according to

\[
u_f(r, \theta, \phi, t) = \sum_{m=-\infty}^{\infty} u_m(r, \theta, t) e^{im\phi}\]  
(3.2a)

and

\[\Pi(r, \theta, \phi, t) = \sum_{m=-\infty}^{\infty} p_m(r, \theta, t) e^{im\phi} .\]  
(3.2b)

The remainder of this chapter is devoted to discussions of the fates of axisymmetric \((m = 0)\) and nonaxisymmetric \((m \neq 0)\) disturbances to the steady axisymmetric base flow. For each azimuthal wavenumber, we will then seek to determine whether, at fixed \(\text{Re}\), any disturbances grow in time. The critical Reynolds number for the axisymmetric base flow, \(\text{Re}_1\), is the \(\text{Re}\) for which no disturbances grow, and one or more disturbances are neutral.

### 3.2 Axisymmetric \((m = 0)\) disturbances

We begin by considering axisymmetric disturbances with arbitrary time dependence. Because the base flow and disturbance are axisymmetric, we can use a streamfunction formulation of the Navier-Stokes equations, which we write as
with boundary conditions

\[ \Psi'(1,\xi,t) = 0 \quad \frac{\partial \Psi'(1,\xi,t)}{\partial r} = 0 , \]  

(3.4a,b)

where the subscript s denotes a base flow quantity and the prime denotes a disturbance quantity. At large \( r \), we have the asymptotic boundary condition

\[ \lim_{r \to \infty} \Psi(r,\xi,t) = 0 . \]  

(3.5)

In (3.3), we have defined \( \xi = \cos \theta \), and

\[ D^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1-r^2}{r^2} \frac{\partial^2}{\partial \xi^2} . \]

Since the solution must be symmetric about \( \theta = 0 \) and \( \pi \), we also impose the conditions

\[ \Psi'(r,1,t) = 0 \quad \Psi'(r,-1,t) = 0 . \]  

(3.6a,b)

The determination of \( \text{Re}_1 \) is thus equivalent to computing the largest value of \( \text{Re} \) for which no solution of (3.3-3.6) grows in time. As (3.3-3.6) are linear and time-invariant, we investigate modal solutions of the form

\[ \Psi'(r,\xi,t) = \psi'(r,\xi) e^{\sigma t} \]

where \( \sigma \) is the temporal eigenvalue. Thus, we consider the eigenvalue problem
subject to the boundary conditions

\[ \psi'(1, \xi) = 0 \quad \frac{\partial \psi'(1, \xi)}{\partial \xi} = 0 , \quad (3.8a, b) \]

\[ \lim_{r \to \infty} \psi(r, \xi) \to 0 , \quad (3.9) \]

and the symmetry conditions

\[ \psi'(r, 1) = 0 \quad \psi'(r, -1) = 0 . \quad (3.10a, b) \]

We have solved the linear eigenvalue problem for axisymmetric disturbances by a numerical technique akin to that used in the base flow calculation, in which the radial coordinate is transformed to the finite domain \(-1 \leq \eta \leq 1\) according to

\[ r = \exp \left[ \frac{a(\eta+1)}{2} \right] . \quad (3.11) \]

Equations (3.7, 3.9, and 3.10) are straightforwardly transformed into the finite domain. If the far-field boundary condition (3.9) is satisfied at \(\eta = 1\) (i.e., at finite \(r\)), then it is of the "hard" type discussed by Fornberg (1980). The fourth-order streamfunction formulation for the axisymmetric disturbances requires two far-field conditions on the function \(\tilde{\psi}(\eta, \xi) = \psi'(r, \theta)\), which we take to be the "soft" conditions
applied at $\eta = 1$.

As in the calculation of the base flow, we have followed the approach of Zebib (1987) and expanded the highest $\eta$-derivative of $\tilde{\psi}$ in terms of complete sets of functions in the $\eta$- and $\xi$-coordinates

$$\frac{\partial^4 \tilde{\psi}(\eta, \xi)}{\partial \eta^4} = \sum_{i=1}^{\infty} W_i(\xi) \frac{d^4 H_i(\eta)}{d\eta^4},$$  \hfill (3.13)

where the radial functions are expanded in terms of Chebyshev polynomials

$$\frac{d^4 H_i(\eta)}{d\eta^4} = \sum_{j=0}^{\infty} A_{ij} T_j(\eta),$$ \hfill (3.14a)

and the angular functions

$$W_i(\xi) = \int_{\xi}^{1} P_i(\xi') d\xi'$$ \hfill (3.14b)

individually satisfy the symmetry conditions, where $P_i(\xi)$ is the $i$-th Legendre polynomial.

We next truncate (3.13) and (3.14a) to $I$ and $J+1$ terms in the polar and radial directions, respectively, and integrate with respect to $\eta$ to obtain expressions for $\psi$ and its first three $\eta$-derivatives. The constants of integration are chosen so that each radial function $H_i(\eta)$ satisfies all of the boundary conditions. The result is
\[
\frac{d^n H_{ij}}{d\eta^n} = \sum_{j=0}^{J} A_{ij} \sum_{q=0}^{J+4-n} G_{qj}^{(n)} T_q(\eta), \quad 0 \leq n \leq 3 ,
\]

where the coefficients \( G_{qj}^{(n)} \) are shown in the Appendix.

That the boundary and symmetry conditions are automatically satisfied by any linear combination of the functions in (3.13) allows further consideration to be restricted to the partial differential equation (3.7). Rewriting this as

\[
\sigma D^2 \tilde{\psi} + L(\tilde{\psi}) - \frac{2}{Re} D^4 \tilde{\psi} = 0 ,
\]

we formulate a weighted residuals procedure whereby the residual of the discretized (3.16) is made orthogonal to the trial functions according to

\[
\int_1^e \int_0^\pi \left[ \sigma D^2 \tilde{\psi} + L(\tilde{\psi}) - \frac{2}{Re} D^4 \tilde{\psi} \right] V_i(\theta) S_j(r) r^k \sin \theta \, d\theta \, dr = 0
\]

for \( 1 \leq i \leq I \) and \( 0 \leq j \leq J \), where

\[
V_i(\theta) = W_i(\xi) , \quad S_j(r) = \sum_{q=0}^{J+4} G_{qj}^{(0)} T_q(\eta) .
\]

Thus, the problem is reduced to a \( I(J+1) \times I(J+1) \) matrix eigenvalue problem

\[
A \times = \sigma \, C \times ,
\]

which is solved using standard numerical software (Garbow et al. 1977). That (3.17) is nonsingular and has no spurious eigenvalues (Gottlieb & Orszag 1977) is guaranteed by satisfying the boundary and symmetry conditions (which do not involve \( \sigma \)) on a term-by-term basis in (3.14a,b) rather than by inclusion in (3.17) (Zebib 1984).
3.3 Nonaxisymmetric ($m \neq 0$) disturbances

The linear analysis of the stability of nonaxisymmetric disturbances (i.e., $m \neq 0$) proceeds from the Fourier decompositions (3.2a, b). Because of the axisymmetry of the base flow and the linearity of the disturbance equations, the individual Fourier modes are uncoupled and can be investigated separately. Because the disturbance variables $u_t$ and $\Pi$ must be real, we require that $u_{-m} = u_m^*$ and $p_{-m} = p_m^*$, where $^*$ denotes a complex conjugate. As the case $m = 0$ has been treated in §3.2, attention will be restricted to $m \geq 1$.

We have found it convenient to use the primitive variable form of (3.1a, b) in this part of the analysis. The $m$-th Fourier component of the disturbance is then governed by

\[ \nabla \cdot u'_m = 0 \]  \hspace{1cm} (3.18)

\[ \frac{\partial u'_m}{\partial t} + (u'_m \cdot \nabla) \vec{U} + (\vec{U} \cdot \nabla) u'_m = - \nabla p'_m + \frac{2}{Re} \nabla^2 u'_m \]  \hspace{1cm} (3.19)

with the no-slip boundary condition

\[ u'_m(1, \theta, \phi, t) = 0 , \]  \hspace{1cm} (3.20)

where $u'_m(r, \theta, \phi, t) = u_m(r, \theta, t) e^{i \phi}$ and $p'_m(r, \theta, \phi, t) = p_m(r, \theta, t) e^{i \phi}$.

We then make the change of variables $\xi = \cos \theta$ and eliminate the pressure and azimuthal velocity component as follows. We first note that $\partial u'_m / \partial \phi = i u_m^*$ and use (3.18) to eliminate $u_{m\phi}$ from (3.18, 3.19). In the same manner, we can solve the azimuthal momentum equation for $p'_m$, and hence eliminate $p_m$. 
As with axisymmetric disturbances, we consider modal solutions of the form

\[ u_{m}(r, t) = e^{at} \tilde{u}_{m}(r, t) \]  
\[ u_{n}(r, t) = e^{at} \tilde{u}_{n}(r, t) . \]  

leading to two coupled equations in \( r \) and \( \xi \) which constitute an eigenvalue problem for \( \sigma \).

The final result is a pair of partial differential equations for the \( r \)- and \( \xi \)-components of the velocity of the form

\[
\sigma \left[ m^2 \tilde{u}_{m} + \sin^2 \theta \frac{\partial}{\partial r} \left( r h_{m} \right) \right] \\
+ m^2 \left[ \frac{\partial}{\partial r} \left( \tilde{U}_{r} \tilde{u}_{m} \right) - \frac{\sin \theta}{r} \left( \frac{\partial}{\partial \xi} \tilde{u}_{m} + \tilde{U}_{\xi} \frac{\partial \tilde{u}_{m}}{\partial \xi} \right) - \frac{2}{r} \tilde{U}_{r} \tilde{u}_{m} \right] \\
+ \sin^2 \theta \frac{\partial}{\partial r} \left[ r \tilde{U}_{r} \frac{\partial h_{m}}{\partial r} - \tilde{U}_{\xi} \frac{\partial}{\partial \xi} (h_{m} \sin \theta) + \tilde{U}_{r} h_{m} + h_{m} \tilde{U}_{\xi} \cot \theta \right] \\
+ \frac{2}{Re} \left\{ \frac{m^2}{r^2} \left[ - \frac{\partial}{\partial r} \left( r^2 \frac{\partial \tilde{u}_{m}}{\partial r} \right) - \frac{\partial}{\partial \xi} \left( \sin^2 \theta \frac{\partial \tilde{u}_{m}}{\partial \xi} \right) + 2 \tilde{u}_{m} + \frac{m^2}{\sin^2 \theta} \tilde{u}_{m} \right] \right. \\
- \left. \sin \theta \frac{\partial}{\partial r} \left[ \frac{\sin \theta}{r} \frac{\partial}{\partial r} \left( r^2 \frac{\partial h_{m}}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial \xi} \left( \sin^2 \theta \frac{\partial}{\partial \xi} (h_{m} \sin \theta) \right) \right] \right. \\
+ \left. (1+m^2) \frac{\partial}{\partial r} \left( \frac{h_{m}}{r} \right) + 2m^2 \left[ \cot \theta \frac{\partial}{\partial r} \left( \frac{\tilde{u}_{m}}{r} \right) - \frac{3}{r^2} \tilde{u}_{m} \right] \right\} = 0 \]  
(3.22a)
\[
\begin{align*}
\sigma & \left[ m^2 \bar{u}_{m\beta} - \sin \theta \frac{\partial}{\partial \beta} (h_m \sin^2 \theta) \right] \\
& + m^2 \left[ \bar{u}_{mr} \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) \bar{U}_r + \bar{u}_r \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) \bar{u}_{m\beta} - \frac{\sin \theta}{r} \frac{\partial}{\partial \beta} (\bar{U}_r \bar{u}_{m\beta}) \right] \\
& - \sin \theta \frac{\partial}{\partial \beta} \left[ \sin^2 \theta \left( \bar{u}_r \frac{\partial h_m}{\partial r} - \frac{\bar{U}_r}{r} \frac{\partial}{\partial \beta} (h_m \sin \theta) + \frac{\bar{U}_r}{r} h_m + \frac{\bar{U}_r}{r} h_m \cot \theta \right) \right] \\
& + \frac{2}{\mathcal{Re}} \left\{ \frac{m^2}{r^2} \left[ -\frac{\partial}{\partial r} \left( r^2 \frac{\partial \bar{u}_{m\beta}}{\partial r} \right) - \frac{\partial}{\partial \beta} \left( \sin^2 \theta \frac{\partial \bar{u}_{m\beta}}{\partial \beta} \right) + \frac{1+m^2}{\sin^2 \theta} \bar{u}_{m\beta} - 2 \sin \theta \frac{\partial}{\partial \beta} (\bar{u}_{m\beta} \cot \theta) \right] \\
& + \frac{\sin \theta}{r^2} \frac{\partial}{\partial \beta} \left[ \sin^2 \theta \frac{\partial}{\partial \beta} \left( r^2 \frac{\partial h_m}{\partial r} \right) + \sin \theta \frac{\partial}{\partial \beta} \left( \sin^2 \theta \frac{\partial}{\partial \beta} (h_m \sin \theta) \right) \right] \\
& - \frac{1+m^2}{r^2} \sin \theta \frac{\partial h_m}{\partial \beta} + \frac{2m^2}{r^2} h_m \cot \theta \right\} = 0 ,
\end{align*}
\]

where
\[
h_m = \frac{\partial}{\partial \beta} (\bar{u}_{m\beta} \sin \theta) - \frac{1}{r} \frac{\partial}{\partial r} (r^2 \bar{u}_{mr}) .
\]

Here, the velocity components must satisfy the boundary conditions
\[
\bar{u}_{mr}(1, \beta) = 0 \quad \bar{u}_{m\beta}(1, \beta) = 0 \quad (3.23a, b)
\]
on the sphere. Following Batchelor & Gill (1962), the disturbance must also satisfy the
kinematic conditions
\[
\bar{u}_{mr}(r, \pm 1) = 0, \quad m \geq 1 \quad (3.23c)
\]
\[
\bar{u}_{t\beta}(r, \pm 1) \text{ bounded} \quad (3.23d)
\]
\[ \hat{u}_{mT}(r, \pm 1) = 0, \quad m > 1 \]  \hspace{1cm} (3.23e)

on the coordinate axes \( \xi = \pm 1 \).

As in §3.2, the numerical treatment begins with the transformation (3.11) of the radial variable. The transformed counterparts of (3.22a,b) contain fourth derivatives of \( \hat{u}_{m\eta}(\eta, \xi) = \hat{u}_{m\xi}(r, \xi) \) and third derivatives of \( \hat{u}_{mT}(\eta, \xi) = \hat{u}_{mT}(r, \xi) \) with respect to \( \eta \). The boundary conditions satisfied by \( \hat{u}_{m\eta} \) and \( \hat{u}_{mT}(r, \xi) \) are then

\[ \hat{u}_{m\eta}(-1, \xi) = 0 \quad \frac{\partial \hat{u}_{m\eta}(-1, \xi)}{\partial \eta} = 0 \]  \hspace{1cm} (3.24a,b)

and

\[ \hat{u}_{mT}(-1, \xi) = 0 \]  \hspace{1cm} (3.24c)

at the surface of the sphere. Here (3.24a,c) follow directly from (3.23a,b), whereas (3.24b) is a direct consequence of the continuity equation.

The \( \eta \)- and \( \xi \)-components of the velocity are then required to satisfy two far-field boundary conditions at \( \eta = 1 \). As in the calculation of the base flow and the investigation of axisymmetric disturbances, there is a degree of choice available in the selection of these far-field conditions. We have chosen to require that each of the disturbance velocity components satisfy a "hard" and a "soft" boundary condition at \( \eta = 1 \). These are

\[ \hat{u}_{m\eta}(1, \xi) = 0 \quad \hat{u}_{mT}(1, \xi) = 0 \]  \hspace{1cm} (3.24d,e)

and

\[ \frac{\partial \hat{u}_{m\eta}(1, \xi)}{\partial \eta} = 0 \quad \frac{\partial \hat{u}_{mT}(1, \xi)}{\partial \eta} = 0 , \]  \hspace{1cm} (3.24f,g)

where the hard conditions (3.24d,e) ensure that the disturbance inflow is axisymmetric.
on the outer boundary of the computational domain.

We now consider solution of an eigenvalue problem consisting of the partial differential equations (3.22a, b) and the boundary and kinematic conditions (3.23, 3.24). This time, we expand the disturbance quantities directly in terms of a complete set of functions. (A similar procedure had earlier been attempted for the base flow calculation, but was abandoned due to slow convergence of the Newton iteration required to solve the nonlinear algebraic equation system.) The expansion for \( m = 1 \)

\[
\tilde{u}_{1\eta}(\eta, \xi) = \sum_{i=1}^{I} W_{i}(\xi) \sum_{j=0}^{J} B_{ij} e^{-a(\eta+1)/2} (\eta^2 - 1)^2 \eta^j
\]  
(3.25a)

\[
\tilde{u}_{1\xi}(\eta, \xi) = \sum_{i=0}^{I} P_{i}(\xi) \sum_{j=0}^{J} Q_{ij} e^{-a(\eta+1)/2} (\eta^2 - 1) (\eta - 1) \eta^j.
\]  
(3.25b)

differs from the general case on account of the kinematic condition (3.23e), and will be treated first. We denote by \( R_1(\eta, \xi) \) and \( \Theta_1(\eta, \xi) \) the residuals that result when the expansions (3.25a, b) are substituted into the differential equations (3.22a) and (3.22b), respectively. We then make these residuals orthogonal to the trial functions according to

\[
\int_{-1}^{1} \int_{-1}^{1} R_{1}(\eta, \xi) W_{i}(\xi) (\eta^2 - 1)^2 \eta^j \, d\xi \, d\eta = 0, \quad 1 \leq i \leq I, \quad 0 \leq j \leq J \quad (3.26a)
\]

\[
\int_{-1}^{1} \int_{-1}^{1} \Theta_{1}(\eta, \xi) P_{i}(\xi) (\eta^2 - 1) (\eta - 1) \eta^j \, d\xi \, d\eta = 0, \quad 0 \leq i \leq I, \quad 0 \leq j \leq J. \quad (3.26b)
\]

These linear equations constitute a matrix eigenvalue problem of the form (3.17),
where \( C \) is again nonsingular. In the nonaxisymmetric case, the order of the eigenvalue problem is \((J+1)(2I+1)\). Again, the eigenproblem is solved using standard EISPACK software. No spurious eigenvalues were found.

For \( m > 1 \), the linear disturbance equations are still (3.22a,b) and (3.23c, 3.24), with (3.23d) replaced with (3.23e). We employ the expansions

\[
\tilde{u}_{mq}(\eta, \xi) = \sum_{i=1}^{I} W_i(\xi) \sum_{j=0}^{J} F_{ij} e^{-a(\eta+1)} (\eta^2 - 1)^2 \eta^j
\]

\[
\tilde{u}_{mj}(\eta, \xi) = \sum_{i=1}^{I} W_i(\xi) \sum_{j=0}^{J} X_{j} e^{-a(\eta+1)} (\eta^2 - 1) (\eta - 1) \eta^j.
\]

As before, these expressions are substituted into (3.26a,b), and the residuals are made orthogonal to the trial functions. The resulting set of \( 2I(J+1) \) homogeneous linear algebraic equations again constitutes an eigenvalue problem of the form (3.24), which is solved as before.

### 3.4 Results

Table 15 shows, for \( m = 0 \), the least stable temporal eigenvalue of (3.17) for \((I, J) = (10, 10), (14, 14), \) and \((18, 18)\). These values were computed with \( a = 5 \), corresponding to a computational domain consisting of a spherical annulus with radii 1 and \( r_{\infty} \equiv e^a = 148 \). Additional calculations show that when the base flow and disturbance equations are solved on a smaller domain \((r_{\infty} = 90.0)\), the computed temporal eigenvalues (with \( I = J = 14 \)) differ by 4.5% and 3.3% at \( Re = 130 \) and 190, respectively. The results clearly indicate that for \( Re \leq 190 \), all of the eigenvalues \( \sigma \) of the discretization are in the left half-plane. We therefore conclude that the flow is
stable with respect to axisymmetric disturbances for $\text{Re} \leq 190$. For $m = 0$, we also plot the real part of the least stable temporal eigenvalue as a function of Reynolds number in figure 11.

The stability of the flow with respect to axisymmetric disturbances at higher $\text{Re}$ was not investigated because of the discovery of an instability with respect to nonaxisymmetric disturbances with azimuthal wavenumber $m = 1$ at $\text{Re} = 175.1$.

Table 16 shows, for $m = 1$ and $160 \leq \text{Re} \leq 190$, the temporal eigenvalue $\sigma$ having the largest real part, for $r_\infty = 148$ and $(I, J) = (12, 12), (14, 14), (16, 16), \text{and } (14, 16)$. (We have considered $I = J$ and $I \neq J$ in order to examine the possible dependence of $\text{Re}_1$ and the disturbance eigenfunction on the truncation scheme employed (Steen & Aidun, 1988).) As in the $m = 0$ case, calculations using $r_\infty = 90.0$ gave least stable (most unstable) temporal eigenvalues differing only modestly from the results for $r_\infty = 148$ at $\text{Re} = 130$ and 190 (6.5% and 9.5%, respectively). For $m = 1$, we also plot the real part of the least stable temporal eigenvalue as a function of Reynolds number in figure 12.

The results indicate that disturbances with azimuthal wavenumber $m = 1$ become unstable at $\text{Re} \approx 175.1$ with an imaginary part of approximately $\omega = 0.300$. This corresponds to a Hopf bifurcation, and indicates that the onset of instability occurs via oscillatory disturbances. If the frequency is nondimensionalized in the same way as a Strouhal number (based on sphere diameter), we obtain $\text{St}_1 = \omega/\pi = 0.0955$. The computed values of $\text{Re}_1$ and $\text{St}_1$ are both in excellent agreement with experiment, as will be discussed in §3.5.

Figures 14(a-b) show the distribution of the temporal eigenvalues at $\text{Re} = 175$ with $I = J = 14$ and $r_\infty = 148$. From figure 14(a), we note that the eigenvalues cluster
along a curve which passes near to the origin of the \( \sigma \)-plane. Figure 14(b) shows that although a pair of eigenvalues with \( \text{Im}(\sigma) = 0.300 \) is the least stable at \( \text{Re} = 175 \), there are many other eigenvalues having real parts which are only slightly negative.

The number of functions retained in the expansion of the base flow in the computation of the above results for \( m = 1 \) was 28. Table 17 shows that this number is sufficient.

The results shown in Table 18 clearly indicate that the flow is stable for disturbances with \( m = 2 \) for \( \text{Re} < \text{Re}_1 \). For \( m = 2 \), we also plot the real part of the least stable temporal eigenvalue as a function of Reynolds number in figure 13. Table 19 suggests that the flow becomes increasingly stable with respect to disturbances having larger azimuthal wavenumbers, at least near \( \text{Re}_1 \).

Here, we make a few remarks regarding the interpretation of our analysis. We note that for either axisymmetric or nonaxisymmetric disturbances, the problem reduces to one of deciding whether the spectra of certain linear partial differential operators lie entirely in the left half-plane. We have approximated the linear operators by finite-dimensional matrices, the eigenvalues of which are taken to approximate those of the corresponding differential operators. This approach leaves in some doubt the status of the continuous spectrum of the original operator. Although we cannot prove that the continuous spectra (if in fact they exist) lie entirely in the left half-plane for all of the combinations of \( \text{Re} \) and \( m \) investigated, there is every reason (based on the reasonable agreement with experiment and the distribution of eigenvalues shown in figures 14(a-b)) to believe that the present results are in fact not corrupted by the discretization.

In order to better understand the three-dimensional character of the
nonaxisymmetric disturbance that becomes unstable at \( \text{Re}_1 \), we have examined an approximation to the oscillating wake. In particular, we have considered a composite flow consisting of the axisymmetric base flow to which is added some multiple of the eigenfunction of the linear stability problem, i.e.,

\[
u_c(r, \theta, \phi, t) = \bar{U}_1(r, \theta) + u_1(r, \theta, \phi, t),
\]

with the amount of the linear eigenfunction added to the base flow chosen so that the maximum speed of the disturbance flow \(|u_1|\) is 0.0174, where the characteristic velocity of the base flow is unity, corresponding to the uniform flow. Equation (3.27) consists of a system of ordinary differential equations as follows:

\[
\begin{align*}
\frac{dr}{dt} &= u_{cr} = \bar{U}_r(r, \theta) + u_{1r}(r, \theta, \phi, t) \\
r \frac{d\theta}{dt} &= u_{c\theta} = \bar{U}_{\theta}(r, \theta) + u_{1\theta}(r, \theta, \phi, t) \\
r \sin \theta \frac{d\phi}{dt} &= u_{c\phi} = u_{1\phi}(r, \theta, \phi, t),
\end{align*}
\]

where the \( r \)-component of the perturbation velocity is expressed as

\[
u_{1r}(r, \theta, \phi, t) = [ \bar{U}_{1r}(r, \theta) \cos(\phi + \sigma_1 t) - \bar{U}_{1\theta}(r, \theta) \sin(\phi + \sigma_1 t) ] \exp(\sigma_1 t),
\]

and the subscripts \( R \) and \( I \) denote the real and imaginary parts of the complex eigenfunction and eigenvalue. The \( \theta \)-component of the perturbation velocity has a similar expression and the \( \phi \)-component is obtained from the continuity equation.

Although (3.27) is no longer a numerical solution of the full Navier-Stokes equations, it should be a good approximation to the actual flow just above \( \text{Re}_1 \) if instability sets in via oscillations of small amplitude (infinitesimal amplitude at the
For this composite flow, we have selected $Re = 175$, and computed the pathline for a fluid particle having an initial position $r = 1.2$ and $\theta = 30^\circ$. This point lies within the recirculating flow region for the base flow with $Re = 175$. Figure 15 shows two perspective views of the pathlines at a sequence of dimensionless times. We observe that the pathline resembles a trajectory on a torus having a noncircular, $\phi$-dependent cross-section. In fact, the perimeter of the cross-section of the torus closely approximates a pathline (in a constant $\phi$-plane) of the steady axisymmetric base flow. The helical nature of the disturbance is clearly evident. At larger $Re$ (corresponding to a larger azimuthal velocity component), the pathlines will be less tightly wound. We can easily show that the particle trajectories will not be temporally periodic unless $Re$ is chosen so that the number of times the pathline winds around the toroidal cross-section during a $2\pi$ change in $\phi$ is an integer.

The $m = 1$ eigenfunction can also be characterized in terms of the disturbance vorticity field. Figures 16(a-c) show the $r$- and $\theta$-dependence of the azimuthal, polar, and radial components of the disturbance vorticity, with the real and imaginary parts in the upper and lower halves, respectively, of each figure. We note that the disturbance is strongest near the equator of the sphere, upstream of the separation point, which is located at $\theta = 63^\circ$ for $Re = 175$. In each of the six cases (real and imaginary part of the three vorticity components), inspection of the entire field out to $r = 148$ revealed no other vorticity contours with level equal to or exceeding the contours shown. It is thus clear that although the disturbance manifests itself as a helical motion in the wake, its origin is in the instability of the boundary layer flow on the sphere.
3.5 Comparison to Experiment

In this section, our computational results are compared to a number of previous experiments. As discussed below, the experimental determination of Re_1 is complicated by the fact that the wake oscillations have a very low frequency. In light of this problem, we believe that the comparisons presented here are very good.

Möller (1938) vertically towed spheres equipped with dye ports through water in tanks of square cross-section. He found that the flow was steady below Re = 170, and that at Re = 200, the wake oscillated with a very low frequency. The ratio of tank width to sphere diameter was eight at Re = 170 and twenty at Re = 200. In light of the rather large blockage factor, we consider the agreement between Möller's experiment and our calculation to be very good.

Taneda (1956) reported that when a "Reynolds number of about 130 is reached, however, the faint periodic pulsatile motion with a very long period occurs at the rear of the vortex-ring". A possible explanation for the observation of unsteadiness at an Re below our computed Re_1 is given by Taneda, who noted "that a slight asymmetry of the flow pattern was caused by the presence of the support; namely the separation point just behind the support was moved slightly forward." The fact that the spheres were suspended by a single vertical support (with the base flow being primarily in the horizontal direction) means that, per force, the disturbance to the base flow had a substantial m = 1 component, which is the azimuthal mode with respect to which the flow is most unstable. In addition, Taneda's use of a thin piano wire (with diameter 0.3 mm or 0.5 mm and aspect ratios of 50 and 30, respectively) to support a much larger sphere (diameters between 9.52 mm and 19.82 mm) may allow considerable coupling between disturbances in the flow and the rigid body motion of
the sphere. It seems likely that $Re_1$ for a fixed (i.e., rigidly supported) sphere will be
greater than or equal to the $Re_1$ for a sphere which is allowed to respond to flow
disturbances, because the disturbances to which the latter sphere are subject are less
constrained than those to which the former are subject. Thus, the lack of rigidity of
the support may also contribute to Taneda's observation of instability at a lower $Re$
than predicted by us.

Goldburg & Florsheim (1966) observed that the wake behind a falling sphere
lost its axisymmetry at an $Re$ of about 210, although they did not detect the loss of
steadiness until $Re = 270$. We consider our results to be in good agreement with
these, given the low frequency nature of the instability, which renders more difficult
the discernment of unsteadiness.

Toulcova & Podzimek (1968) dropped aluminum alloy spheres through various
solutions of glycerol and water, and reported that the transition to unsteadiness
occurred at $Re = 130$. In a later study with the same spheres, Zikmundova (1970)
reported a transition to unsteadiness in the range $130 < Re < 150$. Since $Re_1$ for
falling spheres approaches the fixed-sphere value when the sphere/fluid density ratio
($\beta = \rho_f/\rho_s$) in the freely falling case approaches zero, we might expect $Re_1$ to be
significantly lower for these experiments, in which $\beta = 0.46$. (Although neither the
composition nor the density of the solutions was given, subsequent inquiry (private
communication, J. Goodman (formerly J. Toulcova and J. Zikmundova) 1984) revealed
that in all cases, the glycerol mass fraction was in excess of 0.99.)

Roos & Willmarth (1971) found that for a sphere towed through water and
supported by a sting in the rear, a side force first became apparent at $Re \approx 290$,
corresponding to a nonaxisymmetric force exerted on the sphere by the flow. They
also found that if a weighted sphere was supported by two fine threads, the onset of the side force occurred at \( \text{Re} \approx 215 \). Roos (1968) concluded that the presence of a sting in the near wake had the effect of "tying down the tail of the wake bubble through the no-slip condition at the sting surface", and that this prevented oscillation of the wake under conditions where it otherwise would have been unstable (i.e., for \( 215 \leq \text{Re} \leq 290 \)). We think that our computed value of \( \text{Re}_1 = 175.1 \) is in good agreement with the value (\( \text{Re} \approx 215 \)) of Roos & Willmarth (1971).

Nakamura (1976) observed the trajectories of falling spheres to be vertical for \( \text{Re} < 190 \), with meandering at higher \( \text{Re} \). Since it can be shown that the trajectory of a freely falling sphere subject to disturbances containing an \( m = 1 \) azimuthal component cannot be vertical, the onset of meandering corresponds to the onset of an \( m = 1 \) instability. Thus, we consider our results to be in excellent agreement with Nakamura's flow visualization experiments.

As for the frequency of the wake oscillations near the onset of instability, there is relatively little quantitative data with which our results can be compared. Probably the best data at low \( \text{Re} \) (below the onset of vortex shedding) is that of Roos (1968), who towed a 3 in (7.62 cm) diameter sphere supported by a side-sting in a 380 in (9.6 m) long tank having a 24 inch x 26 inch (60.96 cm x 66.04 cm) cross section, and found \( \text{St}_1 \approx 0.087 \). (This is an average of measurements at the six lowest \( \text{Re} \), in the range \( 250 < \text{Re} < 450 \), with all values lying between 0.084 and 0.095.) We consider this to be in excellent agreement with our computed value \( \text{St}_1 = 0.0955 \).

In connection with the observations of Möller (1938) and Taneda (1956) regarding the low frequency nature of the wake oscillations, it is of interest to note that the dimensional frequency
corresponding to \( \omega = 0.300 \) is very low in most of the previous experimental work in which \( \text{Re}_1 \) was estimated. Table 20 shows the sphere radii, fluids, and dimensional frequencies (calculated according to (3.28) with \( \text{Re} = 175.1 \)) for previous experiments. The low frequencies undoubtedly contribute to the difficulty in observing the wake oscillations, especially at or near \( \text{Re}_1 \). (The experimental work of Stringham et al. (1969) is not included in Table 20 because their measured viscosity data for aqueous glycerol solutions are in serious disagreement with accepted sources (Segur & Oberstar 1951). We cannot correct for this by interpolating from the data of Segur & Oberstar because it is not known whether the discrepancy is due to poor viscometry or the absorption of atmospheric water by the extremely hygroscopic glycerol. We have also not used the data of Toulcova & Podzimek (1968) because of uncertainty about the degree to which the velocity of the falling sphere had approached its terminal value.)

### 3.6 Discussion

The present work provides the first accurate calculation of the critical Reynolds number at which the nonparallel flow past any body of revolution becomes unstable. Our results for a sphere are in good agreement with previous experimental work, for both \( \text{Re}_1 \) as well as the frequency of the wake oscillations. The predicted instability mode \((m = 1)\) is the same as that observed experimentally at higher \( \text{Re} \).

Our results are also in good agreement with the theoretical prediction by Monkewitz (1988b) of local absolute instability for a class of parallel axisymmetric wake profiles. His results are for the \( m = 1 \) azimuthal mode and are parameterized
by a centerline reversed flow parameter ($\Lambda$) and a shape parameter. In light of the fact that both the degree of reversed flow on the centerline, as well as the wake shape depend on downstream distance (due to the nonparallel nature of the flow), we have not attempted a direct comparison of our results for the sphere to Monkewitz's theory. However, we do note that our computed value $Re_1 = 175.1$, lies in the range $(10^2 < Re < 10^3)$ in which Monkewitz has predicted the occurrence of local absolute instability for two values of $\Lambda$.

Since our analysis is a linear one that finds the smallest $Re$ at which infinitesimally small disturbances will grow, the possibility remains that disturbances of finite amplitude might lead to the onset of subcritical instability at smaller $Re$. That the onset of instability in the steady axisymmetric base flow probably occurs by a linear mechanism is suggested by the agreement with experiment. First, the computed value of $Re_1$ and the associated Strouhal number are in good agreement with previous experimental values. (In those cases where the agreement between computed and experimental values of $Re_1$ leaves something to be desired, differences between experiment and the ideal situation considered in our analysis are such as to justify the magnitudes and signs of the discrepancies.) Moreover, the predicted azimuthal wavenumber ($m = 1$) is the same as that observed in experiments at higher $Re$.

Since in the two-dimensional case transition from steady flow to vortex shedding seems to be the result of a Hopf bifurcation, it would be interesting to determine whether the transition to vortex shedding in the three-dimensional case also proceeds via a linear mechanism. This question could be investigated by introducing disturbances into a three-dimensional time-dependent numerical simulation.
CHAPTER 4

Stability of the Motion of a Falling Sphere

4.1 Motivation

When a sphere falls (or rises) due to gravity, dimensional analysis and existing experimental data show that an additional parameter, the ratio of fluid density to sphere density, is involved if the flow is unsteady. This is because the sphere may be subjected to lateral forces due to nonaxisymmetric pressure and velocity distributions; the resulting acceleration and rigid body motion of an unconstrained sphere will depend on the sphere’s mass (or in dimensionless terms, on the ratio of the fluid density to the sphere density).

In §1.3, we presented the Navier-Stokes equations for the flow of a viscous incompressible fluid about a sphere, along with the rigid body equations of motion for the sphere. We now specialize them to a form useful in the linear stability analysis for general small disturbances. In so doing, we combine the Navier-Stokes equations and the no-slip boundary condition in such a way that the linear stability problem does not explicitly involve the dynamic variables associated with the sphere’s rigid body motion (i.e., $U_s$ and $\Omega$).

4.2 Linear Disturbance Equations

The linear stability of the steady axisymmetric flow can be investigated by expanding the pressure and velocity about their nominally steady axisymmetric values and retaining only the terms that are linear in the disturbance quantities. The zeroth order terms satisfy the steady equations (2.1a–d). Thus,
\[
U_s = -k + u_s \quad \Omega = 0 + \omega_z \quad \phi = \tilde{\phi} + \Pi
\]
\[
U_f = \tilde{U}_f + u_f \quad H = \tilde{H} + h
\]

and we obtain the linear perturbation equations

\[
\nabla \cdot u_f = 0 \quad (4.1a)
\]
\[
\frac{du_s}{dt} + \frac{\partial u_f}{\partial t} + (u_f \cdot \nabla) \tilde{U}_f + (\tilde{U}_f \cdot \nabla) u_f = -\nabla \Pi + \frac{2}{Re} \nabla^2 u_f \quad (4.1b)
\]
\[
\frac{du_s}{dt} = \frac{3\beta}{4\pi} \int_{S_0} dS \cdot h \quad (4.1c)
\]
\[
\mathbf{I} \cdot \frac{d\omega_z}{dt} = \frac{15\beta}{8\pi} \int_{S_0} \mathbf{r} \times (dS \cdot h) \quad (4.1d)
\]
\[
u = \omega_z \times \mathbf{r} \quad \text{on } S_0 \quad (4.1e)
\]
\[
u = -u_s \quad \text{as } r \to \infty , \quad (4.1f)
\]

Clearly, (4.1b,c) can be combined to eliminate \( u_s \) and (4.1d,e) can be combined to eliminate \( \omega_z \), leaving a boundary value problem involving only the disturbance pressure \( \Pi \) and velocity \( u_f \) as independent variables.

The resulting integro-differential equations are

\[
\nabla \cdot u_f = 0 \quad (4.2a)
\]
\[
\frac{3\beta}{4\pi} \int_{S_0} dS \cdot h + \frac{\partial u_f}{\partial t} + (u_f \cdot \nabla) \tilde{U}_f + (\tilde{U}_f \cdot \nabla) u_f = -\nabla \Pi + \frac{2}{Re} \nabla^2 u_f \quad (4.2b)
\]
As in Chapter 3, the disturbances can be Fourier decomposed into azimuthal components according to

\[ u_f(r, \theta, \phi, t) = \sum_{m=-\infty}^{\infty} u_m(r, \theta, t) \, e^{im\phi} \quad (4.3a) \]

and

\[ \Pi(r, \theta, \phi, t) = \sum_{m=-\infty}^{\infty} p_m(r, \theta, t) \, e^{im\phi}. \quad (4.3b) \]

The Fourier decompositions are substituted into the equations (4.2a-d). Because of the axisymmetry of the base flow and the linearity of the disturbance equations, the individual Fourier modes are uncoupled and can be investigated separately.

4.2.1 Boundary Conditions

We now evaluate the force and torque on the sphere due to perturbation stresses, which appear in the boundary conditions (4.2c, d) and the first term in the momentum equation (4.2b). The forces due to \( \sigma_{rr}, \tau_{r\theta}, \) and \( \tau_{r\phi} \) are calculated separately. By using the Fourier decompositions (4.3a, b) and Cartesian unit base vectors converted from the unit base vectors in a spherical coordinate system, we can perform the \( \phi \)-integrations in the surface integrals and get force components in the \( x-, \)
... and z-directions. After summing the contributions from $\sigma_{rr}$, $\tau_{r\theta}$, and $\tau_{r\phi}$ in the x-, y-, and z-directions, we obtain

$$\int_{S_0} dS \cdot \mathbf{h} = F_x(t) \mathbf{e}_x + F_y(t) \mathbf{e}_y + F_z(t) \mathbf{e}_z ,$$

which can be rewritten as

$$\int_{S_0} dS \cdot \mathbf{h} = [ F_x(t) \sin\theta \cos\phi + F_y(t) \sin\theta \sin\phi + F_z(t) \cos\theta ] \mathbf{e}_r$$

$$+ [ F_x(t) \cos\theta \cos\phi + F_y(t) \cos\theta \sin\phi - F_z(t) \sin\theta ] \mathbf{e}_\theta$$

$$+ [ -F_x(t) \sin\phi + F_y(t) \cos\phi ] \mathbf{e}_\phi ,$$

(4.4)

in which $F_x$, $F_y$, and $F_z$ are the components of the perturbation force acting on the sphere and are given by

$$F_x(t) = \pi \int_0^\pi \sin\theta \left\{ f_1(\theta, t) + f_{-1}(\theta, t) \right\} \sin\theta + \left[ g_1(\theta, t) + g_{-1}(\theta, t) \right] \cos\theta$$

$$- i \left[ b_1(\theta, t) - b_{-1}(\theta, t) \right] d\theta$$

$$F_y(t) = i\pi \int_0^\pi \sin\theta \left\{ f_1(\theta, t) - f_{-1}(\theta, t) \right\} \sin\theta + \left[ g_1(\theta, t) - g_{-1}(\theta, t) \right] \cos\theta$$

$$- i \left[ b_1(\theta, t) + b_{-1}(\theta, t) \right] d\theta$$
\[ F_z(t) = 2\pi \int_0^\pi \sin\theta \left[ f_o(\theta,t) \cos\theta - g_o(\theta,t) \sin\theta \right] d\theta , \]

where we have used the Fourier decompositions, and defined

\[ f_m(\theta,t) = - p_m(1,\theta,t) + \frac{4}{\text{Re}} \frac{\partial u_m(1,\theta,t)}{\partial r} \] (4.5a)

\[ g_m(\theta,t) = \frac{2}{\text{Re}} \left[ \frac{\partial u_m(1,\theta,t)}{\partial r} - u_{m\theta}(1,\theta,t) + \frac{\partial u_m(1,\theta,t)}{\partial \theta} \right] \] (4.5b)

\[ b_m(\theta,t) = \frac{2}{\text{Re}} \left[ \frac{\partial u_m(1,\theta,t)}{\partial r} - u_{m\theta}(1,\theta,t) \right] . \] (4.5c)

Next, we evaluate the torque acting on the sphere due to perturbation stresses, according to the boundary condition (4.2c). The torques due to \( \tau_{r\theta} \) and \( \tau_{r\phi} \) are calculated, separately. Again, by using the Fourier decompositions (4.3a,b) and Cartesian unit base vectors converted from the unit base vectors in a spherical coordinate system, we can perform the \( \phi \)-integrations in the components of (4.2c) to obtain the torque components in the \( x \)-, \( y \)-, and \( z \)-directions. Summing the contributions from \( \tau_{r\theta} \) and \( \tau_{r\phi} \) in the \( x \)-, \( y \)-, and \( z \)-directions, we get

\[ \int_{S_0} r \times (dS \cdot \mathbf{h}) = M_x(t) \mathbf{e}_x + M_y(t) \mathbf{e}_y + M_z(t) \mathbf{e}_z , \] (4.6)

where \( M_x \), \( M_y \), and \( M_z \) are the components of the perturbation torque acting on the sphere. These are evaluated as
\[ M_x(t) = -\pi \int_0^\pi \sin \theta \left\{ i [ g_1(\theta, t) - g_{-1}(\theta, t) ] + [ b_1(\theta, t) + b_{-1}(\theta, t) ] \cos \theta \right\} d\theta \]

\[ M_y(t) = \pi \int_0^\pi \sin \theta \left\{ [ g_1(\theta, t) + g_{-1}(\theta, t) ] - i [ b_1(\theta, t) - b_{-1}(\theta, t) ] \cos \theta \right\} d\theta \]

\[ M_z(t) = 2\pi \int_0^\pi b_0(\theta, t) \sin^2 \theta \, d\theta , \]

where we have used the Fourier decompositions, and \( g_m \) and \( b_m \) are given in (4.5b) and (4.5c), respectively.

Using (4.6), we can evaluate the boundary condition (4.2c). The right hand side of (4.2c) is evaluated on the sphere as

\[ \int_{S_0} r \times (dS \cdot \mathbf{h}) \times r \cdot [ -M_x(t) \sin \phi + M_y(t) \cos \phi ] \, e_\theta \]

\[ = \{ -[ M_x(t) \cos \phi + M_y(t) \sin \phi ] \cos \theta + M_z(t) \sin \theta \} \, e_\phi . \]  

(4.7a)

The left hand side of (4.2c) is evaluated as

\[ \mathbf{I} \cdot \frac{\partial \mathbf{u}}{\partial t} = \sum_{m=-\infty}^{\infty} e^{im\phi} \left\{ \frac{\partial u_{mr}}{\partial t} \, e_r + \frac{\partial u_{m\theta}}{\partial t} \, e_\theta + \frac{\partial u_{m\phi}}{\partial t} \, e_\phi \right\} . \]  

(4.7b)

By equating the right hand sides of (4.7a) and (4.7b), we evaluate the boundary condition represented by (4.2c) on the surface of the sphere. The result is decoupled about the azimuthal wavenumber \( m \) and is expressed as
\[
\frac{\partial u_{m\theta}(1, \theta, t)}{\partial t} = 0 \quad \text{for all } m \tag{4.8a}
\]

\[
\frac{\partial u_{m\phi}(1, \theta, t)}{\partial t} = 0 \quad \text{for } m \neq 1 \tag{4.8b}
\]

\[
\frac{\partial u_{1\theta}(1, \theta, t)}{\partial t} = \frac{15\beta}{16\pi} [ M_y(t) + i M_x(t) ] \tag{4.8c}
\]

\[
\frac{\partial u_{1\phi}(1, \theta, t)}{\partial t} = \frac{15\beta}{8\pi} M_z(t) \sin \theta \tag{4.8d}
\]

\[
\frac{\partial u_{1\delta}(1, \theta, t)}{\partial t} = i \frac{15\beta}{16\pi} \cos \theta [ M_y(t) + i M_x(t) ] \tag{4.8e}
\]

\[
\frac{\partial u_{m\delta}(1, \theta, t)}{\partial t} = 0 \quad \text{for } m \geq 2 \tag{4.8f}
\]

\[
\frac{\partial^2 u_{m\phi}(1, \theta, t)}{\partial t \partial r} = 0 \quad \text{for all } m \tag{4.8g}
\]

where (4.8g) is obtained by taking the time derivative of (4.2a), and

\[
M_y(t) + i M_x(t) = \frac{4\pi}{\Re} \int_0^\pi \sin \theta \left[ (1+\cos^2 \theta) \left( \frac{\partial u_{1\theta}}{\partial r} - u_{1\theta} \right) \right] \sin \theta \cos \theta \left[ \frac{\partial^2 u_{1\theta}}{\partial r \partial \theta} + \frac{\partial^2 u_{1\phi}}{\partial r^2} \right] d\theta. \tag{4.8h}
\]

By using (4.4), we evaluate the boundary condition at infinity (4.2d). The result is again decoupled about the azimuthal wavenumber m, and is written as

\[
\frac{\partial u_{\theta r}(r, \theta, t)}{\partial t} \rightarrow -\frac{3\beta}{4\pi} F_z(t) \cos \theta \tag{4.9a}
\]
\[ \frac{\partial u_{1r}(r, \theta, t)}{\partial t} \rightarrow -\frac{3\beta}{\partial r} \sin\theta \left[ F_x(t) - i F_y(t) \right] \] \tag{4.9b}

\[ \frac{\partial u_{m\theta}(r, \theta, t)}{\partial t} \rightarrow 0 \quad \text{for } m \geq 2 \] \tag{4.9c}

\[ \frac{\partial u_{0\theta}(r, \theta, t)}{\partial t} \rightarrow \frac{3\beta}{4\pi} F_x(t) \sin\theta \] \tag{4.9d}

\[ \frac{\partial u_{1\phi}(r, \theta, t)}{\partial t} \rightarrow -\frac{3\beta}{\partial r} \cos\theta \left[ F_x(t) - i F_y(t) \right] \] \tag{4.9e}

\[ \frac{\partial u_{m\phi}(r, \theta, t)}{\partial t} \rightarrow 0 \quad \text{for } m \geq 2 \] \tag{4.9f}

\[ \frac{\partial u_{1\theta}(r, \theta, t)}{\partial t} \rightarrow -\frac{3\beta}{\partial r} i \left[ F_x(t) - i F_y(t) \right] \] \tag{4.9g}

\[ \frac{\partial u_{m\phi}(r, \theta, t)}{\partial t} \rightarrow 0 \quad \text{for } m \neq 1 \] \tag{4.9h}

as \( r \to \infty \), where

\[ F_x(t) - i F_y(t) = \frac{4\pi}{(1-\beta) \Re} \int_0^\theta \left[ -i \sin^3\theta A_1(1, \theta, t) + B_1(1, \theta, t) \right] d\theta \] \tag{4.10a}

\[ F_x(t) = \frac{4\pi}{(1-\beta) \Re} \int_0^\theta \left[ \sin\theta \cos\theta D_1(1, \theta, t) - \sin^2\theta \frac{\partial u_{0\theta}}{\partial r}(1, \theta, t) \right] \right] d\theta \] \tag{4.10b}

\[ A_1(1, \theta, t) = -i \left[ \sin\theta \left( 6 \frac{\partial^2 u_{1r}}{\partial r^2} + 3 \frac{\partial^3 u_{1r}}{\partial r^3} \right)_{r=1} + \cos\theta \left( 2 \frac{\partial u_{1\theta}}{\partial r} + \frac{\partial^2 u_{1\theta}}{\partial r^2} \right)_{r=1} ight. \]

\[ + \sin\theta \left( 2 \frac{\partial^2 u_{1\phi}}{\partial r \partial \theta} + \frac{\partial^3 u_{1\phi}}{\partial r^2 \partial \theta} \right)_{r=1} - 2 \cos^2\theta (u_{1\theta})_{r=1} \left. \right] \]

\[ B_1(1, \theta, t) = 2 \sin\theta \cos\theta \left( \frac{\partial u_{1r}}{\partial r} - u_{1r} \right)_{r=1} + \sin^2\theta \left( \frac{\partial^2 u_{1r}}{\partial r^2} + \frac{\partial^2 u_{1\theta}}{\partial r \partial \theta} \right)_{r=1} \]
Finally, we notice that the boundary conditions used for the stability of flow past a fixed sphere in Chapter 3 are recovered when the density ratio $\beta$ approaches zero in (4.8) and (4.9).

### 4.2.2 Navier-Stokes Equations

In this section, we rewrite the linearization of the momentum equation (4.2b). First, we use the Fourier decomposition (4.3a,b), and resolve (4.2b) into three scalar components, using (4.4) to represent the rigid body coupling term. Because of the axisymmetry of the base flow and the linearity of the disturbance equations, the individual Fourier modes are uncoupled and can be investigated separately.

For axisymmetric ($m = 0$) disturbances, we can eliminate the pressure and rigid body coupling term by subtracting the $r$-derivative of the $\theta$-component of the momentum equation multiplied by $r$, from the $\theta$-derivative of the $r$-component of the momentum equation. The combined equation does not involve $u_{0\theta}$, and we define a streamfunction-like variable $\psi_{ps}(r,\theta,t)$ such that

$$u_{0r} = \frac{1}{r^2 \sin \theta} \frac{\partial \psi_{ps}}{\partial \theta}$$,

$$u_{0\theta} = -\frac{1}{r \sin \theta} \frac{\partial \psi_{ps}}{\partial r}$$

satisfies the axisymmetric continuity equation. The final expression is
where the subscript $s$ denotes a base flow quantity. In (4.11a), we have defined $\zeta = \cos \theta$, and

$$D^2 = \frac{\partial^2}{\partial r^2} + \frac{1-\xi^2}{r^2} \frac{\partial^2}{\partial \xi^2}.$$ 

The $\zeta$-component of the momentum equation is expressed as

$$\frac{\partial u_{0\zeta}}{\partial t} + \bar{U}_r \frac{\partial u_{0\zeta}}{\partial r} + \frac{\bar{U}_\theta}{r} \frac{\partial u_{0\zeta}}{\partial \theta} + \frac{\bar{U}_r}{r} u_{0\zeta} + \frac{\bar{U}_\theta}{r} u_{0\zeta} \cot \theta = \frac{2}{Re} \left[ \nabla^2 u_{0\phi} - \frac{u_{0\phi}}{r^2 \sin^2 \theta} \right],$$

which is decoupled from the $r$- and $\theta$-components of the momentum equation, and can be solved separately. This equation is further simplified to

$$\frac{D u_{0\zeta}}{Dt} = \frac{2}{Re} \nabla^2 u_{0\phi} - \frac{2u_{0\phi}}{r^2 Re \sin^2 \theta} \left[ 1 + \frac{Re}{2} \frac{r \sin \theta (\bar{U} \cdot i_\omega)}{2} \right], \quad (4.11b)$$

where $i_\omega$ is the unit vector normal to the symmetry axis.

We observe that (4.11a,b) do not involve the density ratio $\beta$. The differential equation (4.11a) is identical to (3.3), but is subject to boundary conditions at infinity derived from (4.9a,d). On the other hand, the differential equation (4.11b) is a convective diffusion equation. Thus, its solution will decay if the quantity inside the square brackets in (4.11b) is nonnegative.

Because the disturbance variables $u_l$ and $\Pi$ must be real, we require that $u_{-m} = u^{*}_{m}$ and $p_{-m} = p^{*}_{m}$, where $^*$ denotes a complex conjugate. As the case $m = 0$
has been treated above, attention will be restricted to \( m \geq 1 \).

For \( m \geq 1 \), we can use the continuity equation (4.2a) to eliminate \( u_m \), and the \( \phi \)-component of the momentum equation (4.2b) to eliminate \( p_m \), where we note that
\[
\frac{\partial u_m}{\partial \phi} = im u_m \quad \text{and} \quad \frac{\partial p_m}{\partial \phi} = im p_m.
\]
When the expression for \( p_m \) (obtained from the \( \phi \)-component of the momentum equation) is substituted into the \( r \)- and \( \theta \)-components of the momentum equation, the rigid body coupling term cancels in each equation. The final result is a pair of partial differential equations for the \( r \)- and \( \theta \)-components of the velocity

\[
\frac{\partial}{\partial t} \left[ m^2 u_{mr} + \sin^2 \theta \frac{\partial}{\partial r} (r h_m) \right]
\]

\[
+ m^2 \left[ \frac{\partial}{\partial r} (\tilde{U}_r u_{mr}) - \sin \theta \frac{\partial}{\partial r} \left( \frac{\partial U_r}{\partial \theta} u_{mr} + U_r \frac{\partial u_{mr}}{\partial \theta} \right) - \frac{2}{r} \tilde{U}_r u_{mr} \right]
\]

\[
+ \sin^2 \theta \frac{\partial}{\partial r} \left[ r \tilde{U}_r \frac{\partial h_m}{\partial r} - \tilde{U}_r \frac{\partial}{\partial \theta} (h_m \sin \theta) + \tilde{U}_r h_m + h_m \tilde{U}_r \cot \theta \right]
\]

\[
+ \frac{2}{Re} \left\{ \frac{m^2}{r^2} \left[ -\frac{\partial}{\partial r} \left( r^2 \frac{\partial u_{mr}}{\partial r} \right) - \frac{\partial}{\partial \theta} \left( \sin^2 \theta \frac{\partial u_{mr}}{\partial \theta} \right) + 2 u_{mr} + \frac{m^2}{\sin^2 \theta} u_{mr} \right] \right. 
\]

\[
- \sin \theta \frac{\partial}{\partial r} \left[ \frac{\sin \theta}{r} \frac{\partial}{\partial r} \left( r^2 \frac{\partial h_m}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left( \sin^2 \theta \frac{\partial}{\partial \theta} (h_m \sin \theta) \right) \right]
\]

\[
+ (1+m^2) \frac{\partial}{\partial r} \left( \frac{h_m}{r} \right) + 2m^2 \left[ \cot \theta \frac{\partial}{\partial r} \left( \frac{u_{mr}}{r} \right) - \frac{3}{r^2} u_{mr} \right] \right\} = 0 \quad (4.12a)
\]

and
\[
\frac{\partial}{\partial t}[m^2 u_{m}\sin \theta - \sin \theta \frac{\partial}{\partial \xi} (h_m \sin^2 \theta)] \\
+ m^2 \left[ u_{mr} \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) \tilde{U}_r + \tilde{U}_r \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) u_{m}\sin \frac{\sin^2 \theta \frac{\partial}{\partial \xi} (\tilde{U}_r u_{m})}{r}ight] \\
- \sin \theta \frac{\partial}{\partial \xi} \left[ \sin^2 \theta \left( \tilde{U}_r \frac{\partial h_m}{\partial r} - \frac{\tilde{U}_r}{r} \frac{\partial}{\partial \xi} (h_m \sin \theta) + \frac{\tilde{U}_r}{r} h_m + \frac{\tilde{U}_r}{r} h_m \cot \theta \right) \right] \\
+ \frac{2}{Re} \left\{ \frac{m^2}{r^2} \left[ -\frac{\partial}{\partial r} \left( r^2 \frac{\partial u_{m\theta}}{\partial r} \right) - \frac{\partial}{\partial \xi} \left( \sin^2 \theta \frac{\partial u_{m\theta}}{\partial \xi} \right) + \frac{1+m^2}{\sin^2 \theta} u_{m\theta} - 2 \sin \theta \frac{\partial}{\partial \xi} (u_{m\theta} \cot \theta) \right] \\
+ \frac{\sin \theta}{r^2} \frac{\partial}{\partial \xi} \left[ \sin^2 \theta \left( \frac{\partial}{\partial r} \left( r^2 \frac{\partial h_m}{\partial r} \right) \right) + \sin \theta \frac{\partial}{\partial \xi} \left( \sin^2 \theta \frac{\partial}{\partial \xi} (h_m \sin \theta) \right) \right] \\
- \frac{1+m^2}{r^2} \sin \theta \frac{\partial h_m}{\partial \xi} + \frac{2m^2}{r^2} h_m \cot \theta \right\} = 0, \quad (4.12b)
\]

where

\[ h_m = \frac{\partial}{\partial \xi} (u_{m\theta} \sin \theta) - \frac{1}{r} \frac{\partial}{\partial r} (r^2 u_{mr}) \].

Finally, we notice that, as in the case \( m = 0 \), (4.12a,b) do not involve the density ratio \( \beta \). Thus, for any \( m \), the linear disturbance forms of the momentum equation are identical for spheres which are fixed or falling under gravity. However, they are subject to different boundary conditions, as will be shown in the next section.

### 4.3 Solution Procedure

For \( m = 0 \), we use a streamfunction-like variable for the \( r \)- and \( \theta \)-component of the linear disturbance equations, given as (4.11a) in the previous section. We again investigate modal solutions of the form
where $\sigma$ is the temporal eigenvalue. As mentioned in the previous section, the differential equation (4.11a) after (4.13) is substituted into it is identical to (3.7), but is subject to the boundary conditions

$$
\psi'(1, \xi) = 0 \quad \frac{\partial \psi'(1, \xi)}{\partial r} = 0 \ , \quad (4.14a, b)
$$

$$
\lim_{r \to \infty} \sigma \psi'(r, \xi) = -\frac{3\beta}{4\pi} \hat{F}_z \frac{r^2}{2} \sin^2 \theta , \quad (4.15)
$$

where (4.14a,b) are obtained from (4.8a,b), (4.15) is obtained from (4.9a,d), and $\hat{F}_z$ is found from (4.10b) after substituting the modal solution (4.13).

For $m \geq 1$, we use a modified primitive variables formulation (4.12a,b). As with axisymmetric disturbances, we consider modal solutions of the form

$$
U_{mr}(r, \xi, t) = e^{st} \hat{U}_{mr}(r, \xi) \quad (4.16a)
$$

$$
U_{m\xi}(r, \xi, t) = e^{st} \hat{U}_{m\xi}(r, \xi) , \quad (4.16b)
$$

leading to two coupled equations in $r$ and $\xi$ which constitute an eigenvalue problem for $\sigma$. The final result is a pair of partial differential equations for the $r$- and $\xi$-components of the velocity, exactly as in (3.22a,b) for the stability of flow past a fixed sphere, but subject to different boundary conditions.

For $m = 1$, the boundary conditions to be satisfied are

$$
\hat{U}_{1r}(1, \theta) = 0 \quad \frac{\partial \hat{U}_{1r}(1, \theta)}{\partial r} = 0 \ , \quad (4.17a, b)
$$
on the sphere, where (4.17a,b) are obtained from (4.8a,g), (4.17c) is obtained from (4.8c), and \( [ \hat{M}_y + i \hat{M}_x ] \) is found by substituting the modal solution (4.16a,b) into (4.8h). The far-field boundary conditions to be satisfied are

\[
\sigma \hat{u}_{1\theta}(r,\theta) \rightarrow -\frac{3\theta}{8\pi} \sin \theta \left[ \hat{F}_x - i \hat{F}_y \right]
\]

(4.18a)

\[
\sigma \hat{u}_{1\phi}(r,\theta) \rightarrow -\frac{3\theta}{8\pi} \cos \theta \left[ \hat{F}_x - i \hat{F}_y \right]
\]

(4.18b)

as \( r \rightarrow \infty \), where (4.18a,b) are obtained from (4.9b) and (4.9e), and \( [ \hat{F}_x - i \hat{F}_y ] \) is found by substituting the modal solution (4.16a,b) into (4.10a).

For \( m \geq 2 \), the boundary conditions to be satisfied are

\[
\hat{u}_{m\theta}(1,\theta) = 0 \quad \frac{\partial \hat{u}_{m\theta}}{\partial r}(1,\theta) = 0
\]

(4.19a,b)

\[
\hat{u}_{m\phi}(1,\theta) = 0
\]

(4.19c)

on the sphere and

\[
\hat{u}_{m\theta}(r,\theta) \rightarrow 0
\]

(4.20a)

\[
\hat{u}_{m\phi}(r,\theta) \rightarrow 0
\]

(4.20b)

as \( r \rightarrow \infty \). Here (4.19a,b,c) and (4.20a,b) are consequences of (4.8a,b,g) and (4.9c,f), respectively.

We note that for \( m \geq 2 \), the boundary conditions as well as the differential equations are exactly the same as in the case of the stability of flow past a fixed sphere. Thus, the stability of the motion of a falling/rising sphere with respect to
disturbances having azimuthal wavenumber $m \geq 2$ does not need to be considered further.
CHAPTER 5

Conclusions

We have presented a new spectral Galerkin technique for the steady axisymmetric flow past a sphere, which is well-suited for accurate calculation of the flow near the sphere, and in particular, in the near wake for the entire range of Reynolds numbers for which the steady axisymmetric flow is stable and hence physically realizable. The new technique also gives closer agreement with the classical experiments of Taneda (1956) than does any previous calculation.

We have performed a linear stability analysis of the steady axisymmetric base flow with respect to axisymmetric and nonaxisymmetric disturbances. The analysis shows that, for a fixed sphere, the axisymmetric base flow undergoes a Hopf bifurcation at $Re = 175.1$ with the critical disturbance having azimuthal wavenumber $m = 1$ and Strouhal number $St = 0.0955$. The computed value of $Re_1$ and the associated Strouhal number are in good agreement with previous experimental values. Moreover, the predicted azimuthal wavenumber ($m = 1$) is the same as that observed in experiments at higher Reynolds numbers.

We have also derived the differential equations and boundary conditions for a sphere falling under gravity through an otherwise quiescent fluid and obtained the equations governing the behavior of small disturbances, in a form which does not involve the dynamical variables associated with the sphere's rigid body motion. The linear disturbance equations differ from those for the fixed sphere case only in the appearance of the fluid/sphere density ratio ($\beta = \rho_f/\rho_s$) in the boundary conditions.
The problem of the stability of flow past a fixed sphere is exactly recovered when $\beta = 0$.

The methods employed in this investigation of the stability of flow past a sphere can be extended, via the use of a body-fitted coordinate system, to other finite axisymmetric bluff bodies such as disks, ellipsoids of revolution, and cones. The onset of instability for each of these has been experimentally investigated (Willmarth, Hawk & Harvey 1964, Goldburg & Florsheim 1966, Stringham, Simons & Guy 1969). Another extension would be to the prediction of the observed shape and trajectory (Magarvey & Bishop 1961a,b, Magarvey & MacLatchy 1965, Winnikow & Chao 1966, and Yeheskel & Kehat 1971) of small droplets and bubbles under the influence of gravity, with an accounting in the nominally steady axisymmetric basic flow for both the internal circulation and the drop's departure from sphericity.
Table 1. Effect of the size of the computational domain for $Re = 0.1$ with $K = L = 20$.

<table>
<thead>
<tr>
<th>$a$</th>
<th>$r_{\infty}$</th>
<th>$C_D$</th>
<th>rel error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>4.48</td>
<td>304.174</td>
<td>$2.46 \times 10^{-1}$</td>
</tr>
<tr>
<td>2.0</td>
<td>7.39</td>
<td>275.089</td>
<td>$1.27 \times 10^{-1}$</td>
</tr>
<tr>
<td>2.5</td>
<td>12.2</td>
<td>260.238</td>
<td>$6.58 \times 10^{-2}$</td>
</tr>
<tr>
<td>3.0</td>
<td>20.1</td>
<td>252.165</td>
<td>$3.27 \times 10^{-2}$</td>
</tr>
<tr>
<td>3.5</td>
<td>33.1</td>
<td>247.753</td>
<td>$1.46 \times 10^{-2}$</td>
</tr>
<tr>
<td>4.0</td>
<td>54.6</td>
<td>245.498</td>
<td>$5.41 \times 10^{-3}$</td>
</tr>
<tr>
<td>4.5</td>
<td>90.0</td>
<td>244.549</td>
<td>$1.53 \times 10^{-3}$</td>
</tr>
<tr>
<td>5.0</td>
<td>148</td>
<td>244.267</td>
<td>$3.7 \times 10^{-4}$</td>
</tr>
<tr>
<td>5.5</td>
<td>245</td>
<td>244.199</td>
<td>$9.4 \times 10^{-5}$</td>
</tr>
<tr>
<td>6.0</td>
<td>403</td>
<td>244.181</td>
<td>$2 \times 10^{-5}$</td>
</tr>
<tr>
<td>6.5</td>
<td>665</td>
<td>244.176</td>
<td>0</td>
</tr>
<tr>
<td>7.0</td>
<td>1097</td>
<td>244.176</td>
<td>0</td>
</tr>
</tbody>
</table>
Table 2. Effect of the number of the spectral coefficients for $\text{Re} = 0.1$ with $a = 6.5$ ($r_\infty = 665$).

<table>
<thead>
<tr>
<th>$K = L$</th>
<th>$C_D$</th>
<th>rel error</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>246.209</td>
<td>$8.33 \times 10^{-3}$</td>
</tr>
<tr>
<td>4</td>
<td>243.340</td>
<td>$-3.42 \times 10^{-3}$</td>
</tr>
<tr>
<td>6</td>
<td>244.402</td>
<td>$9.26 \times 10^{-4}$</td>
</tr>
<tr>
<td>8</td>
<td>244.297</td>
<td>$4.96 \times 10^{-4}$</td>
</tr>
<tr>
<td>10</td>
<td>244.225</td>
<td>$2.0 \times 10^{-4}$</td>
</tr>
<tr>
<td>12</td>
<td>244.190</td>
<td>$5.7 \times 10^{-5}$</td>
</tr>
<tr>
<td>14</td>
<td>244.179</td>
<td>$1 \times 10^{-5}$</td>
</tr>
<tr>
<td>16</td>
<td>244.177</td>
<td>$4 \times 10^{-6}$</td>
</tr>
<tr>
<td>18</td>
<td>244.176</td>
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</tr>
<tr>
<td>20</td>
<td>244.176</td>
<td>0</td>
</tr>
</tbody>
</table>
Table 3. Effect of the size of the computational domain for \( \text{Re} = 40 \) with \( K = L = 20 \).

<table>
<thead>
<tr>
<th>( a )</th>
<th>( r_\infty )</th>
<th>( C_D )</th>
<th>( l_s )</th>
<th>( \theta_s )</th>
<th>relative error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>4.48</td>
<td>1.767</td>
<td>0.538</td>
<td>35.281</td>
<td>(-1.2 \times 10^{-1})</td>
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<tr>
<td>2.0</td>
<td>7.39</td>
<td>1.774</td>
<td>0.561</td>
<td>35.684</td>
<td>(-1.1 \times 10^{-1})</td>
</tr>
<tr>
<td>2.5</td>
<td>12.2</td>
<td>1.958</td>
<td>0.634</td>
<td>37.252</td>
<td>(-2.0 \times 10^{-2})</td>
</tr>
<tr>
<td>3.0</td>
<td>20.1</td>
<td>2.041</td>
<td>0.633</td>
<td>37.870</td>
<td>(-2.2 \times 10^{-2})</td>
</tr>
<tr>
<td>3.5</td>
<td>33.1</td>
<td>2.009</td>
<td>0.675</td>
<td>37.591</td>
<td>(5.5 \times 10^{-3})</td>
</tr>
<tr>
<td>4.0</td>
<td>54.6</td>
<td>1.999</td>
<td>0.650</td>
<td>37.507</td>
<td>(5 \times 10^{-4})</td>
</tr>
<tr>
<td>4.5</td>
<td>90.0</td>
<td>1.998</td>
<td>0.634</td>
<td>37.500</td>
<td>0</td>
</tr>
<tr>
<td>5.0</td>
<td>148</td>
<td>1.998</td>
<td>0.634</td>
<td>37.498</td>
<td>0</td>
</tr>
</tbody>
</table>
Table 4. Effect of the number of the spectral coefficients for Re = 40 with a = 5 (r_∞ = 148).

<table>
<thead>
<tr>
<th>K = L</th>
<th>C_D</th>
<th>l_s</th>
<th>θ_s</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2.593</td>
<td>0.355</td>
<td>52.206</td>
</tr>
<tr>
<td>4</td>
<td>2.631</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>6</td>
<td>2.507</td>
<td>0.000</td>
<td>43.156</td>
</tr>
<tr>
<td>8</td>
<td>2.289</td>
<td>1.003</td>
<td>22.471</td>
</tr>
<tr>
<td>10</td>
<td>2.209</td>
<td>0.583</td>
<td>38.393</td>
</tr>
<tr>
<td>12</td>
<td>2.147</td>
<td>0.706</td>
<td>37.842</td>
</tr>
<tr>
<td>14</td>
<td>2.096</td>
<td>0.646</td>
<td>37.612</td>
</tr>
<tr>
<td>16</td>
<td>2.056</td>
<td>0.652</td>
<td>37.844</td>
</tr>
<tr>
<td>18</td>
<td>2.024</td>
<td>0.657</td>
<td>37.616</td>
</tr>
<tr>
<td>20</td>
<td>1.998</td>
<td>0.634</td>
<td>37.498</td>
</tr>
<tr>
<td>22</td>
<td>1.975</td>
<td>0.629</td>
<td>37.351</td>
</tr>
<tr>
<td>24</td>
<td>1.956</td>
<td>0.628</td>
<td>37.222</td>
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<td>26</td>
<td>1.940</td>
<td>0.621</td>
<td>37.102</td>
</tr>
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<td>28</td>
<td>1.926</td>
<td>0.613</td>
<td>36.995</td>
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<tr>
<td>30</td>
<td>1.914</td>
<td>0.608</td>
<td>36.899</td>
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Table 5. Dependence of $C_D$ on the angular expansion for $Re = 40$ with $K = 20$ and $a = 5$.

<table>
<thead>
<tr>
<th>L</th>
<th>$C_{DP}$</th>
<th>$C_{DV}$</th>
<th>$C_D$</th>
<th>$l_s$</th>
<th>$\theta_s$</th>
</tr>
</thead>
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<tr>
<td>12</td>
<td>0.820</td>
<td>1.179</td>
<td>1.999</td>
<td>0.683</td>
<td>37.056</td>
</tr>
<tr>
<td>14</td>
<td>0.820</td>
<td>1.178</td>
<td>1.998</td>
<td>0.702</td>
<td>37.154</td>
</tr>
<tr>
<td>16</td>
<td>0.820</td>
<td>1.178</td>
<td>1.998</td>
<td>0.619</td>
<td>37.535</td>
</tr>
<tr>
<td>18</td>
<td>0.820</td>
<td>1.178</td>
<td>1.998</td>
<td>0.647</td>
<td>37.447</td>
</tr>
<tr>
<td>20</td>
<td>0.820</td>
<td>1.178</td>
<td>1.998</td>
<td>0.634</td>
<td>37.498</td>
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</table>
Table 6. Effect of the size of the computational domain for Re = 180 with K = L = 28.

<table>
<thead>
<tr>
<th>a</th>
<th>$r_\infty$</th>
<th>$C_D$</th>
<th>$l_s$</th>
<th>$\theta_s$</th>
<th>relative error</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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<td></td>
<td></td>
<td>$C_D$</td>
</tr>
<tr>
<td>1.5</td>
<td>4.48</td>
<td>0.701</td>
<td>2.836</td>
<td>62.972</td>
<td>-2.0 x 10^{-1}</td>
</tr>
<tr>
<td>2.0</td>
<td>7.39</td>
<td>0.861</td>
<td>2.722</td>
<td>63.035</td>
<td>-1.3 x 10^{-1}</td>
</tr>
<tr>
<td>2.5</td>
<td>12.2</td>
<td>1.061</td>
<td>2.657</td>
<td>64.883</td>
<td>7.1 x 10^{-2}</td>
</tr>
<tr>
<td>3.0</td>
<td>20.1</td>
<td>1.007</td>
<td>2.677</td>
<td>64.290</td>
<td>1.6 x 10^{-2}</td>
</tr>
<tr>
<td>3.5</td>
<td>33.1</td>
<td>0.993</td>
<td>2.637</td>
<td>64.147</td>
<td>2 x 10^{-3}</td>
</tr>
<tr>
<td>4.0</td>
<td>54.6</td>
<td>0.991</td>
<td>2.648</td>
<td>64.114</td>
<td>0</td>
</tr>
<tr>
<td>4.5</td>
<td>90.0</td>
<td>0.991</td>
<td>2.645</td>
<td>64.112</td>
<td>0</td>
</tr>
<tr>
<td>5.0</td>
<td>148</td>
<td>0.992</td>
<td>2.645</td>
<td>64.153</td>
<td>1 x 10^{-5}</td>
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</table>
Table 7. Effect of the number of the spectral coefficients for Re = 180 with a = 5 (r_\infty = 148).

<table>
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<tr>
<th>K, L</th>
<th>C_D</th>
<th>l_s</th>
<th>\theta_s</th>
</tr>
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<tr>
<td>2, 2</td>
<td>1.273</td>
<td>0.478</td>
<td>69.106</td>
</tr>
<tr>
<td>4, 4</td>
<td>1.159</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>6, 6</td>
<td>1.198</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>8, 8</td>
<td>1.108</td>
<td>2.645</td>
<td>66.516</td>
</tr>
<tr>
<td>10, 10</td>
<td>1.041</td>
<td>2.749</td>
<td>61.591</td>
</tr>
<tr>
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<td>1.048</td>
<td>3.012</td>
<td>63.952</td>
</tr>
<tr>
<td>14, 14</td>
<td>1.031</td>
<td>3.009</td>
<td>64.802</td>
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<td>2.554</td>
<td>64.240</td>
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<td>18, 18</td>
<td>1.008</td>
<td>2.496</td>
<td>64.130</td>
</tr>
<tr>
<td>20, 20</td>
<td>1.000</td>
<td>2.571</td>
<td>64.231</td>
</tr>
<tr>
<td>22, 22</td>
<td>0.992</td>
<td>2.645</td>
<td>64.153</td>
</tr>
<tr>
<td>24, 24</td>
<td>0.983</td>
<td>2.698</td>
<td>64.055</td>
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<td>26, 26</td>
<td>0.976</td>
<td>2.719</td>
<td>64.022</td>
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<td>0.969</td>
<td>2.719</td>
<td>63.973</td>
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<td>30, 30</td>
<td>0.934</td>
<td>2.714</td>
<td>63.657</td>
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<tr>
<td>32, 32</td>
<td>0.929</td>
<td>2.714</td>
<td>63.614</td>
</tr>
<tr>
<td>34, 34</td>
<td>0.925</td>
<td>2.712</td>
<td>63.573</td>
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Table 8. $l_s$ and $\theta_s$ near the Reynolds number for separation with $a = 6$ ($r_\infty = 403$).

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<th>Re</th>
<th>K = L</th>
<th>$l_s$</th>
<th>$\theta_s$</th>
<th>$\theta_{s\infty}$</th>
<th>SD</th>
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<td>20</td>
<td>26</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
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</tr>
<tr>
<td></td>
<td>28</td>
<td>0.000</td>
<td>0.000</td>
<td></td>
<td></td>
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<tr>
<td></td>
<td>30</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td></td>
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<tr>
<td>20.5</td>
<td>26</td>
<td>0.017</td>
<td>7.194</td>
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<td></td>
<td>28</td>
<td>0.015</td>
<td>6.772</td>
<td></td>
<td></td>
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<td></td>
<td>30</td>
<td>0.013</td>
<td>6.390</td>
<td></td>
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<td></td>
<td>32</td>
<td>0.011</td>
<td>6.035</td>
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<td></td>
<td>38</td>
<td>0.007</td>
<td>5.116</td>
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<td></td>
<td>40</td>
<td>0.006</td>
<td>4.849</td>
<td>0.510</td>
<td>0.013</td>
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<tr>
<td>21</td>
<td>26</td>
<td>0.033</td>
<td>10.172</td>
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<td>28</td>
<td>0.031</td>
<td>9.865</td>
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<td>9.593</td>
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<td>9.351</td>
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<td>8.762</td>
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<td>0.015</td>
<td>8.604</td>
<td>5.690</td>
<td>0.008</td>
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<td>14.243</td>
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<td>28</td>
<td>0.064</td>
<td>14.026</td>
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<tr>
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<td>13.173</td>
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<td>0.0012</td>
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<td>32</td>
<td>0.131</td>
<td>19.197</td>
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<td>34</td>
<td>0.129</td>
<td>19.085</td>
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<td>0.128</td>
<td>18.985</td>
<td>17.307</td>
<td>0.0014</td>
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Table 9. Effect of the outer boundary for Re = 20 with K = L = 28.

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<tr>
<th>a</th>
<th>r_∞</th>
<th>C_D</th>
<th>l_s</th>
<th>θ_s</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.0</td>
<td>148</td>
<td>2.817</td>
<td>0.004</td>
<td>3.388</td>
</tr>
<tr>
<td>5.5</td>
<td>245</td>
<td>2.816</td>
<td>0.000</td>
<td>0.000</td>
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<tr>
<td>6.0</td>
<td>403</td>
<td>2.816</td>
<td>0.000</td>
<td>0.000</td>
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</table>
Table 10. Drag coefficient at Re = 0.1.

<table>
<thead>
<tr>
<th>Authors</th>
<th>$C_{DP}$</th>
<th>$C_{DV}$</th>
<th>$C_D$</th>
<th>Remark</th>
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<tbody>
<tr>
<td>Hamielec et al.</td>
<td></td>
<td></td>
<td>262</td>
<td></td>
</tr>
<tr>
<td>LeClair et al.</td>
<td>81.056</td>
<td>163.445</td>
<td>244.501</td>
<td>$C_{DP} &lt; C_D/3$</td>
</tr>
<tr>
<td>Dennis &amp; Walker</td>
<td>81.4</td>
<td>162.8</td>
<td>244.2</td>
<td>$C_{DP} = C_D/3$</td>
</tr>
<tr>
<td>Present result</td>
<td>81.395</td>
<td>162.781</td>
<td>244.176</td>
<td>$C_{DP} &gt; C_D/3$</td>
</tr>
</tbody>
</table>
Table 11. Reynolds number for separation obtained by various authors.

<table>
<thead>
<tr>
<th>Authors</th>
<th>$Re_s$</th>
<th>Technique</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Taneda (1956)</td>
<td>24</td>
<td>Experiment</td>
<td>Extrapolation of $Re$ data for $Re &gt; 25$. Data contains very little scatter.</td>
</tr>
<tr>
<td>Kalra &amp; Uhlherr (1973)</td>
<td>20</td>
<td></td>
<td>Extrapolation of $Re$ data for $Re &gt; 27$. Data contains considerable scatter</td>
</tr>
<tr>
<td>Kawaguti (1948)</td>
<td>39</td>
<td>Spectral Galerkin</td>
<td>Cited by Hamielec et al. (1967)</td>
</tr>
<tr>
<td>Proudman &amp; Pearson (1957)</td>
<td>16</td>
<td>2 term Stokes expansion</td>
<td></td>
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<tr>
<td>Jenson (1959)</td>
<td>17</td>
<td>Finite difference</td>
<td>Value of $Re_s$ given in abstract. Method of determination unclear.</td>
</tr>
<tr>
<td>Hamielec et al. (1967)</td>
<td>22</td>
<td></td>
<td>Extrapolation of $Re_s$ data at $Re = 40$ and 100.</td>
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<tr>
<td>Rimon &amp; Cheng (1969)</td>
<td>10</td>
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<tr>
<td>Pruppacher et al. (1970)</td>
<td>20</td>
<td></td>
<td>Extrapolation of $Re_s$ data at $Re = 22, 24, and 30.$</td>
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<tr>
<td>Dennis &amp; Walker (1971)</td>
<td>20.5</td>
<td>Series truncation</td>
<td>Linear interpolation between the values at $Re = 20$ and 40.</td>
</tr>
<tr>
<td>Lin &amp; Lee (1973)</td>
<td>20</td>
<td>Finite difference</td>
<td>Their figure 2 shows that the flow is separated at $Re &lt; 20.$</td>
</tr>
<tr>
<td>Present work</td>
<td>20.5</td>
<td>Spectral Galerkin</td>
<td>Method of estimation discussed in §2.4</td>
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Table 12. Comparison with previous calculations

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<td>$l_s$</td>
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<td>0.61</td>
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<td>57.4</td>
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<tr>
<td>Kalra &amp; Uhlherr (1973)</td>
<td>0.80</td>
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<td>2.06</td>
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<td>2.04</td>
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<td>Rimon &amp; Cheng (1969)</td>
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<td>Pruppacher et al. (1970)</td>
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<tr>
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<td>$\infty$</td>
<td>0.62*</td>
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<td>-</td>
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<td>2.14*</td>
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<td>Oliver &amp; Chung (1987)</td>
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<td>Forkberg (1988)</td>
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<td>Proudman &amp; Pearson (1957)</td>
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<tr>
<td>(K,L) in Table 13</td>
<td>0.613</td>
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<td>2.089</td>
<td>57.16</td>
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* numerical values reported by author(s). All other values read from plots.
Table 12. continued

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<td>-</td>
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<tr>
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</tr>
<tr>
<td>Jenson</td>
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</tr>
<tr>
<td>Hamielec</td>
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<tr>
<td>Rimon &amp; Cheng</td>
<td>-</td>
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<tr>
<td>Pruppacher et al.</td>
<td>-</td>
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<tr>
<td>Dennis &amp; Walker</td>
<td>-</td>
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<td>Series truncation</td>
</tr>
<tr>
<td>Lin &amp; Lee</td>
<td>-</td>
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<td>-</td>
<td>2.45</td>
<td>60.8</td>
<td>Finite difference</td>
</tr>
<tr>
<td>Cliffe &amp; Lever</td>
<td>-</td>
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<td>-</td>
<td>-</td>
<td>-</td>
<td>Finite element</td>
</tr>
<tr>
<td>Dandy &amp; Leal</td>
<td>2.7*</td>
<td>59.5*</td>
<td>2.9*</td>
<td>62.0*</td>
<td>-</td>
<td>-</td>
<td>Finite difference</td>
</tr>
<tr>
<td>Oliver &amp; Chung</td>
<td>-</td>
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<td>-</td>
<td>Series truncation</td>
</tr>
<tr>
<td>Fornberg</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>2.871*</td>
<td>62.5</td>
<td>Finite difference</td>
</tr>
<tr>
<td>Proudman &amp; Pearson</td>
<td>2.51</td>
<td>39.04</td>
<td>2.87</td>
<td>39.45</td>
<td>3.09</td>
<td>39.65</td>
<td>2 term</td>
</tr>
<tr>
<td>Present results</td>
<td>(K,L)</td>
<td>$\theta_s$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Stokes expansion</td>
</tr>
<tr>
<td>(K,L) in Table 13</td>
<td>2.392</td>
<td>60.72</td>
<td>2.712</td>
<td>63.57</td>
<td>2.875</td>
<td>65.20</td>
<td>Spectral Galerkin</td>
</tr>
<tr>
<td>Extrapolation of $\theta_s$</td>
<td>60.36</td>
<td>63.12</td>
<td>64.90</td>
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<td></td>
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<td></td>
</tr>
</tbody>
</table>
Table 13. Size of computational domain and truncation level used in table 12.

<table>
<thead>
<tr>
<th>Re</th>
<th>$a$</th>
<th>$r_{\infty}$</th>
<th>K, L</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>5</td>
<td>148</td>
<td>28, 28</td>
</tr>
<tr>
<td>100</td>
<td>5</td>
<td>148</td>
<td>50, 28</td>
</tr>
<tr>
<td>120</td>
<td>5</td>
<td>148</td>
<td>50, 28</td>
</tr>
<tr>
<td>150</td>
<td>5</td>
<td>148</td>
<td>50, 28</td>
</tr>
<tr>
<td>180</td>
<td>5</td>
<td>148</td>
<td>50, 34</td>
</tr>
<tr>
<td>200</td>
<td>4.5</td>
<td>90</td>
<td>50, 38</td>
</tr>
</tbody>
</table>
Table 14. Detailed convergence of $l_s$ and $\theta_s$ for Reynolds numbers used in table 12 and 13.

<table>
<thead>
<tr>
<th>Re</th>
<th>K = L</th>
<th>$l_s$</th>
<th>$\theta_s$</th>
<th>$\theta_{\infty}$</th>
<th>SD</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>24</td>
<td>0.628</td>
<td>37.222</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>26</td>
<td>0.621</td>
<td>37.102</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>28</td>
<td>0.613</td>
<td>36.995</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>30</td>
<td>0.608</td>
<td>36.899</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>32</td>
<td>0.610</td>
<td>36.814</td>
<td>35.594</td>
<td>0.0027</td>
</tr>
<tr>
<td>100</td>
<td>24</td>
<td>1.829</td>
<td>54.976</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>26</td>
<td>1.818</td>
<td>54.913</td>
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</tr>
<tr>
<td></td>
<td>28</td>
<td>1.813</td>
<td>54.840</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>30</td>
<td>1.815</td>
<td>54.757</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>32</td>
<td>1.813</td>
<td>54.682</td>
<td>53.807</td>
<td>0.0144</td>
</tr>
<tr>
<td>120</td>
<td>24</td>
<td>2.102</td>
<td>57.972</td>
<td></td>
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</tr>
<tr>
<td></td>
<td>26</td>
<td>2.111</td>
<td>57.865</td>
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<tr>
<td></td>
<td>28</td>
<td>2.095</td>
<td>57.813</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>30</td>
<td>2.085</td>
<td>57.741</td>
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<tr>
<td></td>
<td>32</td>
<td>2.084</td>
<td>57.665</td>
<td>56.787</td>
<td>0.0098</td>
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<tr>
<td>150</td>
<td>24</td>
<td>2.340</td>
<td>61.490</td>
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<tr>
<td></td>
<td>26</td>
<td>2.402</td>
<td>61.427</td>
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<tr>
<td></td>
<td>28</td>
<td>2.441</td>
<td>61.318</td>
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<td></td>
<td>30</td>
<td>2.449</td>
<td>61.272</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>32</td>
<td>2.440</td>
<td>61.214</td>
<td>60.362</td>
<td>0.0112</td>
</tr>
</tbody>
</table>
Table 14. continued

<table>
<thead>
<tr>
<th>Re</th>
<th>K = L</th>
<th>$l_s$</th>
<th>$\theta_s$</th>
<th>$\theta_{s\infty}$</th>
<th>SD</th>
</tr>
</thead>
<tbody>
<tr>
<td>180</td>
<td>26</td>
<td>2.571</td>
<td>64.231</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>28</td>
<td>2.645</td>
<td>64.153</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>30</td>
<td>2.698</td>
<td>64.055</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>32</td>
<td>2.719</td>
<td>64.022</td>
<td>63.116</td>
<td>0.0116</td>
</tr>
<tr>
<td></td>
<td>34</td>
<td>2.719</td>
<td>63.973</td>
<td></td>
<td></td>
</tr>
<tr>
<td>200</td>
<td>28</td>
<td>2.733</td>
<td>65.708</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>30</td>
<td>2.801</td>
<td>65.644</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>32</td>
<td>2.847</td>
<td>65.611</td>
<td></td>
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</tr>
<tr>
<td></td>
<td>34</td>
<td>2.875</td>
<td>65.568</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>36</td>
<td>2.875</td>
<td>65.523</td>
<td>64.901</td>
<td>0.0051</td>
</tr>
</tbody>
</table>
Table 15. Least stable temporal eigenvalue for $m = 0$ with $r_{\infty} = 148$.

<table>
<thead>
<tr>
<th>Re</th>
<th>$I = J = 10$</th>
<th>$I = J = 14$</th>
<th>$I = J = 18$</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>$-9.638 \times 10^{-4} \pm 9.688 \times 10^{-3} i$</td>
<td>$-8.661 \times 10^{-4} \pm 9.749 \times 10^{-3} i$</td>
<td>$-9.062 \times 10^{-4} \pm 9.740 \times 10^{-3} i$</td>
</tr>
<tr>
<td>130</td>
<td>$-6.781 \times 10^{-4} \pm 9.525 \times 10^{-3} i$</td>
<td>$-6.002 \times 10^{-4} \pm 9.670 \times 10^{-3} i$</td>
<td>$-6.301 \times 10^{-4} \pm 9.689 \times 10^{-3} i$</td>
</tr>
<tr>
<td>160</td>
<td>$-5.355 \times 10^{-4} \pm 9.524 \times 10^{-3} i$</td>
<td>$-5.127 \times 10^{-4} \pm 9.671 \times 10^{-3} i$</td>
<td>$-5.311 \times 10^{-4} \pm 9.691 \times 10^{-3} i$</td>
</tr>
<tr>
<td>170</td>
<td>$-4.804 \times 10^{-4} \pm 9.533 \times 10^{-3} i$</td>
<td>$-4.817 \times 10^{-4} \pm 9.675 \times 10^{-3} i$</td>
<td>$-4.921 \times 10^{-4} \pm 9.695 \times 10^{-3} i$</td>
</tr>
<tr>
<td>180</td>
<td>$-4.139 \times 10^{-4} \pm 9.564 \times 10^{-3} i$</td>
<td>$-4.510 \times 10^{-4} \pm 9.677 \times 10^{-3} i$</td>
<td>$-4.612 \times 10^{-4} \pm 9.698 \times 10^{-3} i$</td>
</tr>
<tr>
<td>190</td>
<td>$-3.585 \times 10^{-4} \pm 9.615 \times 10^{-3} i$</td>
<td>$-4.175 \times 10^{-4} \pm 9.681 \times 10^{-3} i$</td>
<td>$-4.308 \times 10^{-4} \pm 9.701 \times 10^{-3} i$</td>
</tr>
</tbody>
</table>
Table 16. Least stable temporal eigenvalue for $m = 1$ with $r_\infty = 148$.

<table>
<thead>
<tr>
<th>Re</th>
<th>$I = J = 12$</th>
<th>$I = J = 14$</th>
<th>$I = 14, J = 16$</th>
<th>$I = J = 16$</th>
</tr>
</thead>
<tbody>
<tr>
<td>160</td>
<td>$-2.153 \times 10^{-4} \pm 0.0986 \text{i}$</td>
<td>$-1.985 \times 10^{-4} \pm 0.0996 \text{i}$</td>
<td>$-2.075 \times 10^{-4} \pm 0.0995 \text{i}$</td>
<td>$-2.095 \times 10^{-4} \pm 0.0996 \text{i}$</td>
</tr>
<tr>
<td>170</td>
<td>$-6.428 \times 10^{-5} \pm 0.3111 \text{i}$</td>
<td>$-5.316 \times 10^{-5} \pm 0.3001 \text{i}$</td>
<td>$-5.773 \times 10^{-5} \pm 0.3012 \text{i}$</td>
<td>$-5.937 \times 10^{-5} \pm 0.3004 \text{i}$</td>
</tr>
<tr>
<td>175</td>
<td>$-3.041 \times 10^{-6} \pm 0.3113 \text{i}$</td>
<td>$-2.261 \times 10^{-7} \pm 0.3001 \text{i}$</td>
<td>$-1.002 \times 10^{-6} \pm 0.3013 \text{i}$</td>
<td>$-1.052 \times 10^{-6} \pm 0.3004 \text{i}$</td>
</tr>
<tr>
<td>175.2</td>
<td>$+1.425 \times 10^{-6} \pm 0.3115 \text{i}$</td>
<td>$+4.458 \times 10^{-7} \pm 0.3000 \text{i}$</td>
<td>$+1.486 \times 10^{-6} \pm 0.3010 \text{i}$</td>
<td>$+1.512 \times 10^{-6} \pm 0.3002 \text{i}$</td>
</tr>
<tr>
<td>180</td>
<td>$+8.349 \times 10^{-5} \pm 0.3115 \text{i}$</td>
<td>$+9.453 \times 10^{-5} \pm 0.3000 \text{i}$</td>
<td>$+9.035 \times 10^{-5} \pm 0.3010 \text{i}$</td>
<td>$+8.912 \times 10^{-5} \pm 0.3002 \text{i}$</td>
</tr>
<tr>
<td>190</td>
<td>$+2.134 \times 10^{-4} \pm 0.3115 \text{i}$</td>
<td>$+2.012 \times 10^{-4} \pm 0.3000 \text{i}$</td>
<td>$+2.102 \times 10^{-4} \pm 0.3010 \text{i}$</td>
<td>$+2.105 \times 10^{-4} \pm 0.3002 \text{i}$</td>
</tr>
</tbody>
</table>
Table 17. Least stable eigenvalue for \( m = 1, \text{Re} = 175 \) with \( r_\infty = 148 \).

<table>
<thead>
<tr>
<th>I = J</th>
<th>( K = L = 28 )</th>
<th>( K = 28, L = 34 )</th>
<th>( K = 40, L = 28 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>(-3.041 \times 10^{-6} \pm 0.3113 \ i)</td>
<td>(-3.050 \times 10^{-6} \pm 0.3110 \ i)</td>
<td>(-3.048 \times 10^{-6} \pm 0.3114 \ i)</td>
</tr>
<tr>
<td>14</td>
<td>(-2.261 \times 10^{-7} \pm 0.3001 \ i)</td>
<td>(-2.972 \times 10^{-7} \pm 0.3005 \ i)</td>
<td>(-2.756 \times 10^{-7} \pm 0.3002 \ i)</td>
</tr>
<tr>
<td>16</td>
<td>(-1.052 \times 10^{-6} \pm 0.3004 \ i)</td>
<td>(-1.143 \times 10^{-6} \pm 0.3000 \ i)</td>
<td>(-1.095 \times 10^{-6} \pm 0.3003 \ i)</td>
</tr>
</tbody>
</table>
Table 18. Least stable temporal eigenvalue for \( m = 2 \) with \( r_\infty = 148, \, I = J = 14 \).

<table>
<thead>
<tr>
<th>Re</th>
<th>(-8.116 \times 10^{-4} \pm 0.0) i</th>
</tr>
</thead>
<tbody>
<tr>
<td>160</td>
<td>-8.007 \times 10^{-4} \pm 0.0 i</td>
</tr>
<tr>
<td>170</td>
<td>-7.687 \times 10^{-4} \pm 0.0 i</td>
</tr>
<tr>
<td>180</td>
<td>-7.439 \times 10^{-4} \pm 0.0 i</td>
</tr>
</tbody>
</table>
Table 19. Least stable temporal eigenvalue for $Re = 175$, with $r_\infty = 148$, $I = J = 14$.

<table>
<thead>
<tr>
<th>m</th>
<th>Least stable eigenvalue</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$- 7.775 \times 10^{-4} \pm 0.0 \text{ i}$</td>
</tr>
<tr>
<td>3</td>
<td>$- 1.193 \times 10^{-3} \pm 2.348 \times 10^{-2} \text{ i}$</td>
</tr>
<tr>
<td>4</td>
<td>$- 1.285 \times 10^{-3} \pm 2.377 \times 10^{-2} \text{ i}$</td>
</tr>
<tr>
<td>5</td>
<td>$- 1.302 \times 10^{-3} \pm 2.375 \times 10^{-2} \text{ i}$</td>
</tr>
</tbody>
</table>
Table 20. Wake oscillation frequencies for previous experimental work, computed using equation (4.3), for \( Re = 175.1 \).

<table>
<thead>
<tr>
<th>Authors</th>
<th>Liquid</th>
<th>Distance Traveled (cm)</th>
<th>( U_\infty ) (cm/sec)</th>
<th>Sphere Radius (cm)</th>
<th>Remarks</th>
<th>Computed ( f ) (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Möller</td>
<td>Water</td>
<td>120</td>
<td>0.286</td>
<td>2.975</td>
<td>Towed</td>
<td>0.00472</td>
</tr>
<tr>
<td></td>
<td>Water</td>
<td>120</td>
<td>0.278</td>
<td>3.050</td>
<td>Towed</td>
<td>0.00449</td>
</tr>
<tr>
<td>Taneda</td>
<td>Water</td>
<td>100</td>
<td>0.86</td>
<td>0.991</td>
<td>Towed</td>
<td>0.0425</td>
</tr>
<tr>
<td></td>
<td>Water</td>
<td>100</td>
<td>1.13</td>
<td>0.754</td>
<td>Towed</td>
<td>0.0735</td>
</tr>
<tr>
<td>Nakamura</td>
<td>Water</td>
<td>100</td>
<td>1.373</td>
<td>0.619</td>
<td>Falling</td>
<td>0.109</td>
</tr>
<tr>
<td>Goldburg &amp;</td>
<td>30% glycerol</td>
<td>120</td>
<td>10.0</td>
<td>0.286</td>
<td>Falling</td>
<td>1.29</td>
</tr>
<tr>
<td>Florsheim</td>
<td>70% water</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Figure 1. Coordinate system
Figure 2. Steady axisymmetric flow past a sphere at Re = 0.1
Figure 3. Steady axisymmetric flow past a sphere at Re = 5.
Figure 4. Steady axisymmetric flow past a sphere at $Re = 20$. 
Figure 5. Steady axisymmetric flow past a sphere at $Re = 40$. 
Figure 6. Steady axisymmetric flow past a sphere at Re = 70.
Figure 7. Steady axisymmetric flow past a sphere at Re = 100.
Figure 8. Steady axisymmetric flow past a sphere at Re = 140.
Figure 9. Steady axisymmetric flow past a sphere at Re = 180.
Figure 10. Steady axisymmetric flow past a sphere at Re = 200.
Figure 11. Growth rate versus Re for $m = 0$
Figure 12. Growth rate versus $Re$ for $m = 1$
Figure 13. Growth rate versus Re for m = 2
Figure 14.a Distribution of the temporal eigenvalues for $m = 1$ and $Re = 175$. All of the temporal eigenvalues.
Figure 14.b Distribution of the temporal eigenvalues for $m=1$ and $Re=175$.
Detail showing the least stable eigenvalues.
Figure 15. Views of the pathline of a particle beginning at $t = 0$, $r = 1.2$, $\theta = 30^\circ$, $\phi = 0$. The views on the right are from a point 100 sphere radii above from the plane $\phi = 0$. The views on the left are from a point 100 sphere radii upstream of the sphere center. The pathline is shown at $t = 30$, 60, 120, 240, 480, 960, 1920, and 3840. (See next page for continuation.)
Figure 15. continued
Figure 16.a  $r-\theta$ dependence of the real and imaginary parts of the $\phi$-component of vorticity at $Re = 175$. The upper and lower halves depict the real and imaginary parts of $\omega_\phi$. 
Figure 16.b  $r-\theta$ dependence of the real and imaginary parts of the $\theta$-component of vorticity at $Re = 175$. The upper and lower halves depict the real and imaginary parts of $\omega_\theta$. 
Figure 16.c  
$r$-$\theta$ dependence of the real and imaginary parts of the $r$-component of vorticity at $Re = 175$. The upper and lower halves depict the real and imaginary parts of $\omega_r$. 
Appendix

Zebib (1984) used the recurrence formulae

\[
2 T_n(z) = \frac{c_n}{n+1} T'_{n+1}(z) - \frac{d_{n-2}}{n-1} T'_{n-1}(z)
\]

\[
2z T_n(z) = c_n T_{n+1}(z) + d_{n-1} T_{n-1}(z)
\]

where \( c_n = d_n = 0 \) if \( n < 0 \) and \( c_0 = 2, d_0 = 1, c_n = d_n = 1 \) if \( n > 0 \), to derive the expressions

\[
T^{(3)}_{q_l} = \delta_{q,l+1}\beta^{(3)} + \delta_{q,l-1}\gamma^{(3)}
\]

\[
T^{(2)}_{q_l} = \delta_{q,l+2}\beta^{(2)} + \delta_{q,l}\gamma^{(2)} + \delta_{q,l-2}\delta^{(2)}
\]

\[
T^{(1)}_{q_l} = \delta_{q,l+3}\beta^{(1)} + \delta_{q,l+1}\gamma^{(1)} + \delta_{q,l-1}\delta^{(1)} + \delta_{q,l-3}\epsilon^{(1)}
\]

\[
T^{(0)}_{q_l} = \delta_{q,l+4}\beta^{(0)} + \delta_{q,l+2}\gamma^{(0)} + \delta_{q,l}\delta^{(0)} + \delta_{q,l-2}\epsilon^{(0)} + \delta_{q,l-4}\eta^{(0)}
\]

for the coefficients \( T^{(n)}_{q_l} \), where \( \delta_{q,l} \) is the Kronecker delta and the nonzero values of \( \beta_j, \gamma_j, \delta_j, \epsilon_j, \) and \( \eta_j \) (for \( j \geq 0 \) only) are

\[
\beta^{(3)}_j = \frac{c_1}{2(j+1)}, \quad \beta^{(n-1)}_j = \frac{\beta^{(n)}_j}{2(j+5-n)}, \quad n = 1,2,3, j \geq 0
\]

\[
\gamma^{(3)}_j = \frac{-1}{2(j-1)}, \quad j \geq 2, \quad \gamma^{(n-1)}_j = \frac{-\beta^{(n)}_j + \gamma^{(n)}_j}{2(j+3-n)}, \quad n = 1,2,3, j \geq n-2
\]

\[
\delta^{(2)}_j = \frac{-\gamma^{(3)}_j}{2(j-2)}, \quad j \geq 3, \quad \delta^{(n-1)}_j = \frac{-\gamma^{(n)}_j + \delta^{(n)}_j}{2(j+1-n)}, \quad n = 1,2 j \geq n
\]

\[
\epsilon^{(1)}_j = \frac{-\delta^{(2)}_j}{2(j-3)}, \quad j \geq 4, \quad \epsilon^{(0)}_j = \frac{-\delta^{(1)}_j + \epsilon^{(1)}_j}{2(j-2)}, \quad j \geq 3
\]
The $G_{qj}^{(n)}$ consist of $f_{qj}^{(n)}$ and terms such that $\psi$ satisfies the boundary conditions.

These coefficients satisfy the homogeneous version of (3.2a, b) and (3.6a, b) as

$$
\sum_{q=0}^{L+1} G_{qj}^{(3)} = 0 \quad \sum_{q=0}^{L+3} G_{qj}^{(1)} = 0
$$

$$
\sum_{q=0}^{L+3} (-1)^q G_{qj}^{(1)} = 0 \quad \sum_{q=0}^{L+4} (-1)^q G_{qj}^{(0)} = 0
$$

for $0 \leq j \leq L$. Thus,

$$
G_{qj}^{(3)} = f_{qj}^{(3)} - \kappa_j^{(3)} \delta_{q,0}
$$

$$
G_{qj}^{(2)} = f_{qj}^{(2)} - \kappa_j^{(3)} \delta_{q,1} - \frac{1}{2} [\kappa_j^{(1)} - \nu_j^{(1)}] \delta_{q,0}
$$

$$
G_{qj}^{(1)} = f_{qj}^{(1)} - \frac{1}{4} \kappa_j^{(3)} \delta_{q,2} - \frac{1}{2} [\kappa_j^{(1)} - \nu_j^{(1)}] \delta_{q,1} - \frac{1}{2} \left[ \kappa_j^{(1)} + \nu_j^{(1)} - \frac{1}{2} \kappa_j^{(3)} \right] \delta_{q,0}
$$

$$
G_{qj}^{(0)} = f_{qj}^{(0)} - \frac{1}{24} \kappa_j^{(3)} \delta_{q,3} - \frac{1}{8} [\kappa_j^{(1)} - \nu_j^{(1)}] \delta_{q,2} - \frac{1}{2} \left[ \kappa_j^{(1)} + \nu_j^{(1)} - \frac{3}{4} \kappa_j^{(3)} \right] \delta_{q,1} - \left[ \nu_j^{(0)} + \frac{3}{8} \kappa_j^{(1)} + \frac{5}{8} \nu_j^{(1)} - \frac{1}{3} \kappa_j^{(3)} \right] \delta_{q,0},
$$

where

$$
\kappa_j^{(n)} = \sum_{q=0}^{L+4-n} f_{qj}^{(n)} \quad \nu_j^{(n)} = \sum_{q=0}^{L+4-n} (-1)^q f_{qj}^{(n)}.
$$
The coefficients $B_p^{(n)}$ are determined by the inhomogeneous boundary conditions (3.6a,b) and depend only on the transformation parameter $a$ in (3.5). They are expressed as

\[
\begin{align*}
B_0^{(3)} &= 0 & B_1^{(3)} &= 0 & B_2^{(3)} &= 0 \\
B_0^{(2)} &= \frac{1}{2} a & B_1^{(2)} &= 0 & B_2^{(2)} &= 0 \\
B_0^{(1)} &= -\frac{1}{2} a & B_1^{(1)} &= \frac{1}{2} a & B_2^{(1)} &= 0 \\
B_0^{(0)} &= -\frac{5}{8} a - 1 & B_1^{(0)} &= -\frac{1}{2} a & B_2^{(0)} &= \frac{1}{8} a
\end{align*}
\]
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