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Thermally-induced deformation and effects on groundwater flow in a discontinuous granite mass

Awadalla, Awadalla Messiha, Ph.D.
The University of Arizona, 1989

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THERMALLY INDUCED DEFORMATION
AND EFFECTS ON GROUNDWATER FLOW IN A DISCONTINUOUS
GRANITE MASS

by

Awadalla Messiha Awadalla

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A Dissertation Submitted to the Faculty of the
DEPARTMENT of MINING and GEOLOGICAL ENGINEERING
In Partial Fulfillment of the Requirements
For the Degree of
DOCTOR OF PHILOSOPHY
WITH A MAJOR IN MINING ENGINEERING
In the Graduate College
THE UNIVERSITY OF ARIZONA

1989
As members of the Final Examination Committee, we certify that we have read
the dissertation prepared by Awadalla Messaia Awadalla
entitled THERMALLY INDUCED DEFORMATION AND EFFECTS ON GROUNDWATER
FLOW IN A DISCONTINUOUS GRANITE MASS

and recommend that it be accepted as fulfilling the dissertation requirement
for the Degree of Doctor of Philosophy.

Dr. Jaak Daemen  7/8/89
Dr. Charles E. Glass  9/29/89
Dr. Ian W. Farmer  8/18/89
Dr. William Davenport  8/18/89
Dr. Kenneth Keating  8/18/89

Final approval and acceptance of this dissertation is contingent upon the
candidate's submission of the final copy of the dissertation to the Graduate
College.

I hereby certify that I have read this dissertation prepared under my
direction and recommend that it be accepted as fulfilling the dissertation
requirement.

Dissertation Director Dr. Jaak Daemen  September 8, 1989
STATEMENT BY AUTHOR

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SIGNED: [Signature]
Dedicated to my parents

MARY and MESSIHA
ACKNOWLEDGEMENTS

I am indebted to Dr. Jaak Daemen, my dissertation director for his guidance and encouragements during the course of this research. I would like to express my appreciation to the members of my dissertation committee, Professors Charless E. Glass, Ian W. Farmer, William Davenport and Kenneth Keating for their advices. Their help was invaluable in completing this study. It was their interest and backing which made this dissertation possible.
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The rate of flow (Q) as a function of the thermal aperture, depicts the equivalent cubic law, (Log-log diagram).

Permeability of the joint alone as a function of joint aperture, Kj is the theoretical value assuming a flat plate model, as determined by Kranz et al. (1980).

Comparison of experimental results for radial flow through tension fracture in granite with cubic law, as determined by Witherspoon et al. (1980). [Compare with the Thermal induced stress, fig. (5.4)]

Conservation of mass as applied to a control volume

Unit element and the x-component of the forces acting on it in the x-direction.
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ABSTRACT

Existing analytical treatments of groundwater flow have mostly been founded on classical hydrodynamics, that groundwater motion is derivable from a velocity potential. This conception is in contradiction with the principle of conservation of energy, although it conforms with the principle of the conservation of mass (Hubbert, 1940, p. 785; Scheidegger, 1960, pps. 74-75; Bear, 1972, pps. 122-123).

This dissertation shows that both principles can be utilized, based on the fact that a force potential at a point is equal to the work required to transfer a unit mass from this point to another point. This potential is given the symbol $\phi - gh - gz - \frac{\rho}{\rho}$ and is incorporated in the force field $\vec{E}$. This potential is related to the flow field $(\vec{q})$ by the anisotropic hydraulic conductivity. This relation forms a solid formulation for the theory of the flow of fluids through fractured porous media. This relation is applied to develop two basic equations. One partial differential equation, representing flow in the fracture, depending on the actual geometry of the fracture and incorporating the anisotropic parameter of the hydraulic conductivity based on the thermal induced stress and the force potential. A second partial differential equation (storage equation) in two-dimensions for non-steady groundwater flow in confined and saturated aquifers. This storage equation incorporates time, hydraulic conductivity and the radial coordinates. It is solved analytically using the Bessel's functions $J_0$ and $K_0$. The two equations represent two models. Both the potential and the thermal hydraulic conductivity constitute a coupling between the two models to render the models a thermohydromechanical model.

This aspect is the essential theme underlying this work and is implemented through a matrix-fracture system based on the slow flow and the fast flow behavior.
The evaluation of the transient parameters including the aperture becomes possible and falls in line with the physics of the problem.

This comprehensive analytical model is found to satisfy the transient demands of the mathematical physics. The application of the phenomena observed in the field from different sources and from Stripa Granite, rendered the model realistic and appropriate to the fractured porous media.
CHAPTER 1
INTRODUCTION

1-1. Description of the Problem

Granite*, a hard igneous crystalline discontinuous rock mass, responds to both heat and radiation. The resultant thermo-mechanical response will induce stresses and displacements in the rock mass, altering its permeability and the flow rate of groundwater through discontinuities. In order to analyze the complex interrelations between thermally induced deformation and groundwater flow, an understanding of the structure and stress-strain behavior of discontinuous rocks is needed.

An accurate estimation of discontinuity density, orientation and characteristics is essential for this purpose. Measurement is difficult, but estimates may be made from analysis of data from experimental and theoretical studies and from modeling.

Discontinuities which are closely spaced and directional can be modeled by an equivalent anisotropic continuum approach (Goodman, 1976), where, as in the case of granite, a blocky structure may result in a rock mass that is predominately isotropic. This is the case with Stripa Granite where experiments on jointed rock (Cook and Myer, 1981) and (Heard and Page, 1982) have provided the basis for a model of a discontinuous granite mass.

Heard and Page (1982) measured the material properties of Stripa Granite at different pressures and temperatures (0-350°C) and Chan and Jeffrey (1983) have determined the in situ thermal conductivity and the in situ thermal diffusivity of

* The model can be applied to any rock, provided that the specific material properties are used. In this dissertation, granite is used as the rock of interest.
fractured granite, using standard-size core samples from Stripa Granite. The values were found to be within approximately 10% of laboratory values for intact rock.

The above-mentioned information can be used to assess the change in the fracture aperture as a result of induced thermal stresses and displacements caused by the thermal impulse from waste canisters.

Further work is, however, needed on the deformational response of granite including improved definition of mechanical properties and of time-temperature dependent relationships.

Since granite has mainly homogeneous permeability, Darcy's Law together with the continuity equation will provide the appropriate expression to manipulate the permeability as a constant by some geometrical transformation that will preserve the nature of solution. Indeed, the permeability depends on both the medium and the fluid.

As for the discontinuities in jointed granite, there is a change in the permeability across the boundary of two regions. In this case, the normal and tangential components of the charge will be manipulated to lead to an expression of the water head.

1-2. Background

Flow analysis through fractures is based on Navier-Stokes equation for steady, laminar, isothermal flow between parallel plates (Bear, 1972) separated by a distance b. The flow rate is found to be proportional to $b^3$. This cubic law is modified by Lomize (1951, after Louis 1969) by introducing a roughness factor (f) to account for local variations in aperture. Other investigators (Louis 1969, Baker 1955, Witherspoon et al. 1980) have studied the problem and established the validity of the cubic law in open fractures.
In closed fractures (fractures that have some degree of contact), some work was done by Witherspoon et al. (1980) by considering a roughness factor \( f \) (\( f \geq 1.0 \)). Iwai (1976, after Tsang et al. 1981) did experimental work on radial flow through a fracture in granite and plotted flow per unit head \( (Q/Ah) \) against normal stress for one complete cycle of loading and unloading. Gangi (1978) developed a "bed-of-nails" model of fractures which explicitly considered the geometry of the asperities and the deformation of the fracture under the load. Assuming the cubic law to hold, Gangi predicted the changes in fracture conductivity.

Walsh (1965) derived an expression for the effective modulus of elasticity \( (E_H) \) of rock containing a number of voids oriented normal to the direction of loading and found that a fracture is best represented by \( (E_H) \).

Louis (1969), based on his experimental work, made the statement that turbulence did not have an important effect on the flow in a fracture system.

The calculations by Chan, Hood and Board (1980) of the displacements and stresses that are induced in the in situ rock mass during heater experiments, demonstrate the dependence and the sensitivity of these quantities on the rock properties.

These calculations, using constant material properties, violate the standard thermomechanical equations for the dependence of displacement and stress on material properties. Recalculations, using temperature dependent material properties, demonstrate the dependence of the rock displacements on Young's modulus and the variation of the coefficient of thermal expansion with temperature. The latter parameter affects rock displacements most significantly. Variations of thermal conductivity and Poisson's ratio with temperature have little effect on the predicted rock displacement.
The predicted calculations by Chan and Cook (1979), are based on the work of Pratt et al. (1977) and Swan (1978). Two granite core samples from the Stripa site have been tested to a limited extent. Comparison of the Stripa experimental values with these predicted values, shows discrepancies. This may come from the fact that the temperature dependence of thermomechanical properties is not included in the predictive models.

Cook and Myer (1981) conducted three experiments to study the effects (Stripa program) involving sufficient field data. They produced curves relating mechanical properties to temperature at confining pressures of 55 and 2 MPa and test temperatures from room temperature to 200°C.

Although recalculation reduces the discrepancy between measured and calculated values, the improvement is not impressive since data from laboratory tests for the properties of Stripa Granite at elevated temperatures and pressures are limited.

Heard and Page (1982) have determined for, Westerly and Stripa Granite, the coefficient of thermal expansion, Young’s modulus and bulk modulus, and inferred permeability to temperatures of 350°C and pressures to 55 MPa. The experiments on Stripa Granite were in the form of a core from an extensometer drift, near the in situ heater experiment. The authors suggest that these moduli results can be used as input to the numerical models predicting the thermo-mechanical response of this granite to heating in situ.

1-3. Objectives

A two-part research program will be undertaken with two objectives:

1. To determine the rate of flow of groundwater in granite masses as a result of changes of fracture aperture due to induced thermal stresses. The material properties will be derived from published experimental work undertaken in both
Stripa mine and the laboratory. A mathematical model will result that may be solved analytically (if possible) or numerically.

2. To develop a numerical (finite difference) model for the temperature-stress-fracture dependent rate of groundwater flow as a function of the time t and of the spatial coordinate x and y, in case the analytical solution is not attainable.

1-4. Methodology

For the flow analysis, the fractures will be considered in their natural state. Normally fractures have some degree of contact, and the aperture and contact area depend on the induced thermal stress acting across the discontinuity. In this type of closed fracture, flow occurs along the paths that are not blocked by asperities bridging the fracture surfaces.

The size of the apertures is the most difficult parameter to be calculated and remains one of the most important parameters in any analysis of flow in fractured rock masses. However, the pressure head in regional aquifer systems gives rise to the rate of flow in porous media; this fact, in combination with water flow through fractured rock masses as derived from the cubic law (both expressed as a radial flow) will lead to an understanding of this parameter and to an approximate estimate of it. This should not be attempted through any modification of the cubic law as its validity is already established, but the attempt should be through spatial coordinates. Thus the rate of flow, due to the thermomechanical system, will be in some way related to the rate of flow spatially found. A recursive relation will result in an estimate of the aperture of the fracture, since both rates are in jointed granite that is thermally stressed.

In summary, there must be a relation between the flow in the porous media and the flow in the fracture system, since the groundwater flow in the immediate vicinity of a repository will also be influenced by the mechanical response of the
host rock by the subsequent thermal impulse. The two phenomena are strongly interrelated and must be considered together in predicting the performance of the waste disposal system.

In this work, the term fracture will be used for most discontinuities within a rock mass, and the term joint is used to distinguish joint rock from intact rock.

1-5 Water Percolation Through a Rock Mass

Rock percolations are controlled by the hydraulic characteristics of fracture. The extent of this control is manifested through hydraulic and stress conditions. A slight change in these conditions could have a significant effect on the pressure distribution throughout the mass of rock.

The majority of studies in this field were based on the assumption of laminar flow between parallel plates using the empirical Darcy's law. Yet, the flow and hence the hydraulic conductivity as worked out in testing and experiments showed some shortcomings to the parallel plate model. Louis (1969) carried out extensive study on laminar and turbulent flows between concrete parallel plates under a wide range of openings and roughness and established the boundary between laminar/turbulent flow and between parallel and non-parallel flow as characterized by the dashed lines $K/D_\text{h} \cdot 0.033$ and $Re_K \cdot 2300$ (Figures 1-1, 1-2), where $K$ is the absolute roughness, $D_\text{h}$ is the hydraulic diameter and $K/D_\text{h}$ is the relative roughness. It will be referred to this notable work when roughness is introduced (Sec. 2.4.2.1, eq. 2-39).

The classical pipe flow experiments by Poiseuille (1540) and Hagen (1839) were the key to parallel plate models in an attempt to formulate the empirical laws for flow through fractures. The correlation is only close when the surface roughness is small and curvature is negligible.
Fig. 1-1: Compilation of the different flow laws and their range of validity. (After Louis, 1969).
Fig. 1-2: Diagram of the proposed laws of resistance for flow in a joint. (After Louis, 1969).
However, the flow in a fracture needs careful studies as influenced by many factors. Stress is one of these factors, that affects the aperture. A small change in aperture, in turn, affects the hydraulic conductivity. Another factor is the effect of geometry on flow through fractures. These geometrical features such as tortuosity, roughness, contact area and spacing are basic and compatible terms in modeling a flow through fractures.

Parish (1963) used glass plates to which glass beads are cemented to examine wall roughness. He found friction factors higher in rough fractures and little, if any, in smooth fractures. Biaxial tests were used by Jouanna (1972) on fissured micaschist and the flow rate was measured. Repeated stress levels showed non-recoverable changes in flow rate (Figure 1-3), and as normal stress and shear stress increased, flow rate decreased. Some experimental work by Pratt et al. (1977) showed a decrease in flow rate as normal stress increased, then followed by a constant flow rate. Figures (1-4, 1-5) illustrate the results they obtained with in situ jointed granite under uniaxial and biaxial tests. Testing on Sherman granite, a massive anisotropic rock, they showed that the permeability of the fluid along a joint, is the most sensitive property to joint closure at very low stresses and that displacement behavior is highly nonlinear.

Goodman (1974) elaborated on shear stress-displacement and normal stress-displacement relations. Figure (1-6) shows that, as shear stress increases, dilation increases at constant normal stress. He observed that shearing is only possible if the asperities break and that dilatancy contributes a considerable strength increment and must be considered a joint property of first order importance.

Some authors treated fractures as rigid conductors, based on the assumptions that there is no influence of the fluid on the medium and that the medium is rigid so that permeability is independent of fluid pressure. Snow (1969) and Childs
Fig. 1-3: Repeated stress levels show non-recoverable changes in flow rate. (After Jouanna, 1972).
Fig. 1-4: Loading path dependence of flow rate of fluid along joints. (After Pratt et al., 1977).
Fig.1-5: Comparison of the effect of joint closure on displacement, sonic velocity and fluid flow. (After Pratt et al., 1977).
Fig. 1-6: Effect of test mode on shear deformation curve for dilatant joint. a) dilation, b) normal stress, c) shear stress. (A) constant normal pressure, (B) restricted normal displacement. (After Goodman, 1974).
(1957) showed the invalidity of the first assumption for soils which contains colloidal or organic matter, whose structure is influenced by clay-water chemistry. These fillings may be observed in aperture conduits in crystalline rocks. The second assumption is also invalid and violated in nature, Snow (1968).

This leads us to consider either rigidity or deformability with respect to the flow of fluid in porous medium, i.e., with respect to a volume whose boundaries are fixed in space or a volume that deforms and moves through space when the material deforms. Cooper (1966) treated both cases: 1) Equation of flow in fixed coordinates and, 2) Equation of flow in deforming coordinates. Both equations describe the non-steady flow through an elastic porous medium. The first involves the grain velocity. The second does not involve the grain velocity. Cooper supported deformability for several reasons, aside from the advantage of eliminating the grain velocity.

1. Deformability resolves a problem in the theory of ground water flow. In the fixed coordinates, two-dimensional nonsteady radial flow cannot occur in an elastic confined aquifer because the top of the aquifer constitutes a moving boundary along which vertical components of flow must occur and it is commonly assumed that two-dimensional radial flow does occur. So the contradiction disappears.

2. In laboratory experiments, the piezometers, for measuring heads, do not remain fixed in space as the material deforms. Observed gradients are likely to be in terms of deforming coordinates.

3. For a deforming medium, the equation in deforming coordinates is in a somewhat closer approximation to Darcy's law. Furthermore, the deformability of the medium is mathematically developed and involved in the equations by Cooper, Biot, Jacob Bear and others. References will
be made to these authors during the development of the fracture-storage interaction model.

1-6. Geometrical Characteristics of Fractures

Natural fractures vary widely as far as surface geometry is concerned. In sedimentary rocks, they are relatively smooth and parallel. In granite, fractures may be rough and tortuous forming an interlocking geometric pattern. This natural meandering way is perhaps due to the localized stresses and the force field \((\varepsilon - g \nabla h)\) acting upon the fluid. However, this complex pattern makes it difficult to derive the general flow laws and in such a complexity, perhaps, the flow is best interpreted by a three-dimensional model. Figure (1-7) illustrates different shapes of joint walls and Figure (1-8) illustrates a tortuous irregular fracture as visualized by Louis (1969).

In such irregular fractures, surfaces are in contact where the opening is very small. At these points, the fracture is very sensitive to the applied stresses accompanied by transient changes in the flow characteristics. In other words, transient changes take place in the surface area, in the contact area, in the flow velocity and in the aperture.

With the parallel plate model, the opening is well-defined depending on some sort of regular roughness and the flow is characterized as linear laminar depending on the width of the opening and the dominant viscous forces, figure (2-9). As the width increases, the flow starts to change from linear to non-linear, with inertial forces dominant until transition takes place to turbulent flow. These three kinds of flow are not only characterized by the width of the opening and the degree of roughness but also by the direction changes and the number of changes. The cubic law is based on parallel plate theory and was discussed by Witherspoon,
Fig. 1-20. Different shapes of the joint walls. Experiments by Lomize \(^\text{[20]}\). (After Louis, 1969).
Fig. 1-8: Flow in joints of variable gap width. (After Louis, 1969).
Wang, Iwai and Gale (1980), and others. This law will be modified for irregular fractures during the course of establishing the modified fracture equation.

The applied stress through the rock mass and hence through the fractures will affect the hydraulic conductivity and the pressure distribution. Induced thermal stress is such an applied stress that changes the groundwater pressure due to localized changes in hydraulic conductivity.

Louis and Maini (1970) showed that rock is anisotropically discontinuous because of fracturing and that the groundwater flow is determined by the joint distribution and geometry. There will be more to say about this anisotropic discontinuity when fracture-matrix system and discontinuity concept are manipulated in Sec. (3-3-1) and Sec. (3-3-3). However, the magnitude of these changes in hydraulic conductivity will depend mainly on the stress-conductivity relationship. If it is assumed that the cubic law is valid, then the groundwater pressure is significantly influenced by localized stress changes. Tsang and Witherspoon (1981) established the validity of the cubic law. When the effect of fracture roughness is taken into account, the flow follows an equivalent cubic law with \((\text{aperture})^3\) replaced by a weighted average \((\text{aperture})^3\). This equivalence will be established (eq. 2-54) in a different way from that given by Tsang and Witherspoon (1981).

In this respect, Schaffer and Daemen (1986, p. 92), in their work on rock fracture grouting, stated that the equivalent aperture of a fracture may provide a criterion governing the use of cement for fracture grouting and it can aid in the selection of a starting grout mix. As this paper will evaluate the aperture variation at different temperatures and stresses, that is, at different thermal induced stresses, this aperture variation probably will complement these requirements.
In case of parallel fracture, an assumption that considers the aperture constant, Snow (1968b) demonstrated and established the stress-permeability relation without considering the complexity of fractured blocks.

With respect to stress-permeability relations, Hsieh (1983) used single-hole tests and cross-hole tests in his studies of hydraulic testing in fractured rock. His work was mainly in the area of hydraulic characteristics of a fractured rock as a function of pressure and stress. The tests were based on the assumption that the rock behaves as a uniform anisotropic porous medium and as this assumption holds, these tests are useful. The results suggest that the rock mass responds as a classical anisotropic porous continuum.

However, the stress-conductivity relationship is a complex phenomena owing to the complex geometry of natural joints and must, therefore, be assessed in detail for designed systems.

Before proceeding to Chapter 2, it is noteworthy to observe that fractured rock is a formation characterized by distribution of fractures throughout the body of rock and can be considered a composite system of rock proper and fractures. The fracture is regarded as the equivalent of a porous medium at high permeability bounded on either side by the rock proper.

Chapter 2 will survey the development of physical models as related to fractured rock (Sec. 2-1) and the development of flow equations in fractured rock (Sec. 2-2), beside reference to the assumptions that may hold in manipulating porous media (Sec. 2-3). Once this is complete, the work will construct and solve the partial differential equation of a fracture (Sec. 2-4.1), then followed by an assessment of the irregularities or modifications to the fracture equation will be implemented by a diagram (Figure 2-16) showing the several steps in the production of the fracture equation suitable for natural media.
CHAPTER 2

2-1. DEVELOPMENT OF PHYSICAL MODELS AS RELATED TO FRACTURED ROCK.

Background

In an attempt to determine the aperture of the crack, that is, the distance between the two surfaces and hence the separation \( w \), with effective pressure \( p \), Gangi (1978) used the rod-shaped asperities for his model "bed of nails" for his analysis. His work was based on the work of Hertz as his model would be equivalent to the distribution of hemispherical, conical, wedgelike asperities of Hertz and he could obtain the identical behavior for a distribution of rods. Figure (2-1) shows the structure of the "bed of nails" model. The letters \( w \) and \( D \) are as shown on the figure and he wrote the relation equation:

\[
K_f - K_{cr} \left[ \frac{4w}{\pi D} \right] - \frac{w^3}{3\pi D} \quad \text{with reference to other authors,}
\]

where

- \( K_f \) - fracture permeability
- \( w \) - the aperture of the crack
- \( K_{cr} \) - the permeability of the planar crack - \( \frac{w}{12} \)

and assuming a power law distribution, he gave:

\[
N(x) = I_0 \left( \frac{x}{w_o} \right)^{n-1} \quad 0 \leq x \leq w_o \ ; \ 1 \leq n \leq \infty
\]

\( N(x) \) - the distribution function for the rod's 'shortness' as a power law

\( I_0 \) - the total number of rod asperities

\( w_o \) - the height of the largest rod (the spacing between the faces at zero pressure)

Using the integral theorem and assuming the rods acting as springs with a spring constant \( K_i \) - \( K \) and \( \frac{K}{E} \) - a constant, say \( bw_o \) (\( b \ll 1 \)), where \( E \) is the Young's modulus, he found
Fig. 2-1: 'Bed of nails' (After Gangi, 1978).
\[ P_x = \left[ \frac{Ebw_o^2 l_o}{nA} \right]^{x \over w_o} \]

where \( A = DL \) - area of the fracture and \( n \) - the number of rods, and \( P(x) \) - the pressure exerted on the faces of the fracture.

Elaborating on his work, one can let \( x = w - w_o \) where \( w \) is the spacing between the fracture faces and \( m = \frac{1}{n} \) and \( P_1 = \frac{Ebw_o^2}{nA} \) at zero pressure.

Then

\[ \frac{P}{P_1} = \left( \frac{Ebw_o^2}{nA} \right)^{w_o - w \over w_o} \]

or

\[ \left( \frac{P}{P_1} \right)^n = 1 - \frac{w}{w_o} = \left( \frac{P}{P_1} \right)^m \]

or

\[ \frac{w}{w_o} = 1 - \left( \frac{P}{P_1} \right)^m \]

Using

\[ K_f = \frac{w^3}{3\pi D} \cdot K \] (equation above)

- \( K_o \left[ \frac{w}{w_o} \right]^3 \) (\( K_o \) is the zero pressure permeability)

and where

\[ \frac{K}{K_o} = \left[ \frac{w}{w_o} \right]^3 \]

hence

\[ K = K_o \left[ 1 - \left( \frac{P}{P_1} \right)^m \right]^3 \] where \( m = \frac{1}{n} \)
Gangi constructed this relation from his model which gives the permeability as a function of the applied pressure, assuming cubic law holding. He obtained a good fit to flow data for a fractured sandstone with Nelson's experimental data (1975). When Tsang and Witherspoon (1981) applied Gangi's model to the flow data and the stress-strain measurements for a granite fracture, they found the discrepancy between theory and measurement too large to be ignored and hence they proceeded to seek an alternative physical model for the hydromechanical behavior for a single fracture under normal stress. Figure (2-2) shows their model. They considered the closure of a fracture as resulting from the deformation of voids between the asperities. Figures (2-3, 2-4, 2-5) are considered basic to interpret the model and to reflect the physics of the void model in conformity with the elastic property at low stresses and the gradual increase of the effective Young's modulus to approach the intrinsic value of solid rock, as Figure (2-3) shows. Their model is a collection of voids or a distribution of asperities as in Figure (2-4). They utilized the void model to describe the behavior of the fracture under normal stress as shown in Figure (2-5), the asperity model to describe the flow through a rough-walled fracture and they built their mathematical correspondence between the void model and asperity model. By taking the effect of fracture roughness into account, the flow followed an equivalent cubic law with a weighted average \((b)^3\) and thus they eliminated the incorporation of the fitting parameters to the flow data in the previous work, Witherspoon et al. (1980). They found their model in agreement with results from laboratory data on granite.

However, Tsang and Witherspoon (1981) established the validity of the cubic law through rough fractures by incorporating the numerical factor \(f(f \geq 1)\) in the
Fig. 2-2: Schematic representation of a fracture by an asperity model. (After Tsang and Witherspoon, 1981).
Fig. 2-3: Typical normal stress-strain curves for intact and jointed rock. (After Tsang and Witherspoon, 1981).
Fig. 2-4: Schematic representation of fracture by either the asperity or the void model. (After Tsang and Witherspoon, 1981).
Fig. 2-5: Deformation of voids in a sequence of increasing normal stresses. (After Tsang and Witherspoon, 1981).
fracture equation and by replacing \((\text{aperture})^3\) by a weighted average
\((\text{aperture})^3\) without altering the cubic dependence on fracture opening.

2-2. Development of Flow Equations in Fractured Rock

Basic equations leading to fracture equation in Sec (2-4-1), could be found. Consider the piezometric head \(h\) at any point in a confined aquifer and \(\rho = \rho(P)\) is the fluid density as a function of pressure at that point, neglecting the change with temperature, then the relation between \(h\) and \(\rho\) is expressed as

\[
h = z + \frac{1}{g} \int_{P_0}^{P} \frac{d\rho}{\rho}
\]

(2-1)

where

- \(P_0\) - fluid pressure intensity at reference location and time \(t = 0\).
- \(P\) - fluid pressure intensity at the point in question.
- \(g\) - gravity.
- \(z\) - the vertical coordinate.

Hubbert (1940) and other authors wrote about this relation. Differentiating eq (2-1)

\[
g\vec{v} \cdot \vec{h} = \nabla (gz \cdot \frac{P}{\rho})
\]

(2-2)

The term \(gz\) is the gravitational potential and \(\frac{P}{\rho}\) is the pressure potential.

Next consider the equation of motion (Navier - Stokes equation) derived in Appendix (B) from Newton's law.

\[
\frac{\partial \vec{q}}{\partial t} = - \rho g \cdot \nabla P + \mu \nabla^2 \vec{q}
\]

(2-3)

If the flow is steady, then \(\frac{\partial \vec{q}}{\partial t}\) = 0. If \(h\) is taken as the vertical direction, the components of acceleration due to gravity are:

\[
g_x = g \frac{\partial h}{\partial x} = 0, \quad g_y = g \frac{\partial h}{\partial y} = 0
\]

and
\( \dot{g} + g \frac{\partial h}{\partial z} \) or \( \ddot{g} = -g \nabla h \)

If \( h \) and \( z \) coincide, then \( g_z = \dot{g} \) and eq(2-3) becomes

\[ -g \nabla z \cdot \frac{1}{\rho} \nabla p - \frac{\mu}{\rho} \nabla^2 q = 0 \]

or

\[ \frac{\mu}{\rho} \nabla^2 q - g \nabla z \cdot \frac{1}{\rho} \nabla p - 0 \]  \hspace{1cm} (2-4)

g is not constant in Navier-Stokes equation. It is made constant when \( h \) coincides with \( z \) but \( h \) is not equal \( z \).

Darcy’s law could be obtained from equations (2-2) and (2-4). Comparing equations (2-2) and (2-4) we get

\[ \frac{\mu}{\rho} \nabla^2 q = \nabla \left( g \cdot \frac{P}{\rho} \right) \]

So

\[ g \nabla h \cdot \frac{\mu}{\rho} \nabla^2 q \]  \hspace{1cm} (2-5)

Differentiating both sides of equation (2-5), gives

\[ \nabla^2 h - \frac{\mu}{\rho g} \nabla^2 \left( \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} \right) - 0 \]

That is

\[ \nabla^2 h = 0 \] Laplace equation by virtue of continuity

or

\[ K \nabla^2 h = 0 \] incorporating the permeability \( K \), if the medium is homogeneous and isotropic. If the medium is anisotropic but homogeneous, then we have

\[ K_x \frac{\partial^2 h}{\partial x^2} + K_y \frac{\partial^2 h}{\partial y^2} + K_z \frac{\partial^2 h}{\partial z^2} = 0 \]  \hspace{1cm} (2-5)

However, the solution of \( K \nabla^2 h = 0 \) is given by Darcy’s law

\[ \ddot{q} = -K \nabla h \]  \hspace{1cm} (2-6)

Darcy’s law could be written

\[ q_x = -K_x \frac{\partial h}{\partial x} \]
This equation has been proved empirically by Darcy himself (1856) in terms of the pertinent physical variables involved, and mathematically by direct derivation from the Navier-Stokes equation of motion of viscous fluids as shown. This gives the relation between two fields: the force field \((\mathbf{E} - g \, \text{grad} \, h)\) per unit mass occupying the space and acting upon the fluid, and the flow field \((\bar{q})\). Both fields are linked together by the volume conductivity of the system, that is,

\[
\bar{q} = \sigma \mathbf{E}
\]  

(2-7)

This is written in the familiar relation analogous to those in electricity and heat conduction. Eq (2-7) forms a solid foundation in the theory of the flow of fluids through porous media.

Darcy's equation is a special case of Navier-Stokes equation since the inertial term \(\frac{\partial q}{\partial t}\) cannot be ignored under transient conditions and certain Reynolds numbers. Under laminar conditions, Darcy's law is simple and its applicability is required and sufficient. The inertial term is ignored in the derivation of Darcy's law from the Navier-Stokes equation. Most studies of fluid motion in porous media rely on Darcy's law and assume that steady motion is set up within a negligibly short time.

The assumption that Darcy's law could be used instead of Navier-Stokes equation, was investigated by Philip (1957) as applied to a confined saturated medium. Philip wrote the microscopic Navier-Stokes equation for an incompressible fluid (in his own terminology) as

\[
\frac{\partial \mathbf{u}}{\partial t} \cdot \mathbf{u} \cdot (\nabla \times \mathbf{u}) + \frac{1}{2} \nabla \mathbf{u}^2 - \nabla \phi - \nu \nabla^2 \mathbf{u}
\]
Subject to the approximation
\[ \ddot{u}_x(\nabla \dot{u}) = \frac{1}{2} \nabla \ddot{u}^2 \]
then
\[ \frac{\partial \ddot{u}}{\partial t} = -\nabla \phi + \nu \nabla^2 \ddot{u} \]  
(2-8)

This equation is the 'complete' (Sommerfeld 1950) Stokes-Navier equation (Philip eq 1-9). Where \( \ddot{u} \) - the microscopic velocity, \( \phi \) - the potential and \( \nu \) - the kinetic viscosity.

In most media, the velocity in eq. (2-8) depends on Reynolds number at low values for inertia term to be negligible. This lower value as suggested by Goldstein (1938), is \( 10^3 \), above which eq. (2-8) fails.

Philip's time-dependent solution of eq. (2-8), where,
\[ \nabla \cdot \ddot{u} = 0 \text{ and } \nabla^2 \phi = 0 \]
is given by
\[ \ddot{u} = \ddot{u}_\infty (1 - e^{-\beta t}) \]  
(eq. 5.9 of his paper)

where
\[ \ddot{u} \] - the microscopic flow velocity
\[ \ddot{u}_\infty \] - the velocity of the steady state
\[ \beta = \frac{\nu}{\alpha K} \] where \( \nu \) is the kinematic viscosity
\[ \alpha = \frac{\gamma P}{\rho g} \] , \( P \) is the porosity and \( \gamma \) - \( \rho g \)
\[ K \] - the medium permeability.

He showed that \( \phi \) is independent of time (t) and the distribution of \( \phi \) at all \( t \) is the same as that at the steady state.

Although this assumption depends on the physical parameters, yet, within the limits of the Reynolds number, it is proved that the steady state is generally set up within a few seconds and, in fact, within a fraction of a second. Figure (2-6)
shows the exact and approximate solutions. It is a plot of the solution in terms of 
\[ \frac{u}{u_\infty} \text{ versus } \beta t - \frac{\rho^*}{\gamma K} \frac{H}{t} \]. The transition from rest to steady motion is achieved within 0.8 seconds.

These investigations suggest that the assumption is true and that transient effects in saturated porous media will not often be large enough to invalidate the use of Darcy's law. Because of the high hydraulic resistance to porous media flow, time variations can occur only very slowly. It is therefore assumed that the solution of Navier-Stokes equation at any position and instant is very closely approximated by a steady state solution given by Darcy-type velocity relations such as represented, for incompressible flow, by eq. (2-6) unless the applied potential is periodic in which case, significant deviations may occur even for systems of quite low frequency. The following assumptions, as related to porous media, may also be expected to apply.

2-3. Other Assumptions Related to Porous Media

Beside the basic assumption discussed in the previous section, other assumptions may hold.

a) It is assumed that the vertical direction is a principal axis of the permeability tensor \( \bar{K} \) defined by eq. (3-4), that is, the coefficients in the Darcy's law (eq. (2-6)') must be given the values \( K_x, K_y, K_z \). These values are equal in the isotropic media. Hence, when there is a flow component normal to the bedding planes, the apparent velocity components are expected to be given by eq. (2-6)' as explained by Muskat (1949, p 270).

b) The medium is homogeneous and anisotropic, that is, \( K \) in eq. (2-6) is no longer a constant. It will be found that \( K \) turns out to be \( K_e = \sqrt{K_x K_y} \) defined by eq. (3-7) in a two-dimensional system. This parameter changes from rock to rock and also as temperature and stress change.
Fig. 2-6: Transition from rest to steady motion. Full curve gives approximate solution. Broken curve shows divergence of two exact solutions from the approximate solution. (After Philip, 1957).
c) If eq. (2-6) and its components \( q_x, q_y, q_z \) (eq. (2-6)') satisfy the boundary conditions of a particular problem, then they represent the unique solution to this problem, Muskat (1949, p 195), as could be proved by the uniqueness theorem, Kreyszig (1966, p 89, 127).

d) Equation (2-6) is restricted to the limits of Reynolds number as given by Dybbs and Edwards (1982). These limits are shown in Figures (2-10 a, -b, -c, -d, -e). Figures (2-8, 2-9) also show such limits. Louis (1969) established some limits as shown in figures (1-1, 1-2) and stated on page (26): the flow resistance laws for laminar flow are dependent on the shape of the cross section. These Reynolds numbers will be investigated in detail when the topic of the friction factor is discussed (Sec. 2-4).

e) The thermomechanical energy dominates the flow in the system, in a sense, as Louis (1969, p 19) put it, that the sum of \( z \cdot \frac{P}{\gamma} \) and \( \frac{v^2}{2g} \) is constant (according to Bernouilli) for every point of the fluid, whereas, for viscous fluids, e.g. water, it decreases during the flow. Also Bird et al. (1960, p 82) explained that in all flow systems, there is a degradation of mechanical to thermal energy and that therefore no real processes are reversible. The concept of thermomechanical energy associated with fluid flow is given in detail by Hubbert (1940, p 37-42). He stated (p 37): if such flow occurs at all, it must be accomplished by an irreversible transformation of mechanical to thermal energy through the mechanism of fluid friction. He concluded (p 42) that the value of \( \phi \) (the velocity potential) must continually decrease in the direction of flow and obtained finally

\[
\phi \cdot gz \cdot \int_{P_0}^{P} \frac{dP}{\rho} \text{ or } \frac{\phi}{g} \cdot h \cdot z \cdot \frac{1}{g} \int_{P_0}^{P} \frac{dP}{\rho}
\]

as given by eq. (2-1) of this paper.
Louis (1969, p 18-23) showed, in his treatment of flow in joints, that the total energy can be separated into two parts: the piezometric head \( \phi - z \cdot \frac{P}{\gamma} \), and the velocity head \( \frac{v^2}{2g} \) and he obtained from \( \phi - z \cdot \frac{P}{\gamma} \), the hydraulic gradient \( \overline{J} = -\nabla \phi \) and stated that the correlations \( \phi - z \cdot \frac{P}{\gamma} \) and \( \overline{J} = -\nabla \phi \) are valuable for jointed media. With reference to Figure (11) of his work, he gave Navier-Stokes equation and its simplifications and worked out the fracture equation in his own way to get his eq. (12) which is the fracture equation. The parameters \( z \) and \( \frac{P}{\gamma} \) in terms of \( g \) and \( \frac{P}{\rho} \) are defined in connection of equation (2-2).

Hence, this could be considered one of the basic assumptions if the flow is not frictionless. This piezometric head appears in different places of this dissertation.

f) The viscous fluid is Newtonian, homogeneous and incompressible.

g) Because the above assumptions are inherent in both Navier-Stokes equation and Darcy's equation and because from a practical point of view (Muskat 1949, p 192), it is not feasible to give a quantitative description of the microscopic geometry of the flow system, Darcy's law is used to the effect that the macroscopic fluid velocity is directly proportional to the pressure gradient acting on the fluid.

2-4. The Work on Fractures in Porous Media

The treatment of this subject passed through many stages, that is, the development of parallel plate flow to other stages with consideration of roughness (f), friction (f') and spacing (\( \Delta \)). All work showed the validity of the cubic law as proved by Tsang and Witherspoon (1981).

However, we will defer irregularity of closed conduits to a later treatment (Sec 2.4.2) and pick up first the theory of parallel plate which assumes constant aperture, and then modify the cubic law according to the parameters affecting it, taking into account the effect of fracture geometry on fluid flow.
2-4-1. Parallel Plate Theory

With reference to basic equations mentioned in Sec. (2.2), the Navier-Stokes equation (2-3) and the continuity equation.

\[ \nabla \cdot \mathbf{q} = 0 \]  \hspace{1cm} (2-9)

(eq. (2-9) is the conservation of mass and is derived in App (A).

Eq. (2-3) could be written

\[ \frac{\partial \mathbf{q}}{\partial t} - g \nabla z - \frac{1}{\rho} \mathbf{V} \cdot \frac{\mu}{\rho} \nabla^2 \mathbf{q} \]

where

\[ g \nabla z = -g \]

Using eq. (2-2) and set

\[ \frac{\partial \mathbf{q}}{\partial t} = 0 \]

then

\[ 0 = - \nabla \left[ gz + \frac{P}{\rho} \right] + \frac{\mu}{\rho} \nabla^2 \mathbf{q} \]

In this equation \( z = \frac{P}{\rho g} - h \), the peizometric head.

Writing the above equation of motion in component form, it becomes

\[ g \frac{\partial h}{\partial x} = \frac{\mu}{\rho} \left[ \frac{\partial^2 V_x}{\partial x^2} \cdot \frac{\partial^2 V_x}{\partial y^2} \cdot \frac{\partial^2 V_x}{\partial z^2} \right] \]

\[ g \frac{\partial h}{\partial y} = \frac{\mu}{\rho} \left[ \frac{\partial^2 V_y}{\partial x^2} \cdot \frac{\partial^2 V_y}{\partial y^2} \cdot \frac{\partial^2 V_y}{\partial z^2} \right] \]

\[ g \frac{\partial h}{\partial z} = \frac{\mu}{\rho} \left[ \frac{\partial^2 V_z}{\partial x^2} \cdot \frac{\partial^2 V_z}{\partial y^2} \cdot \frac{\partial^2 V_z}{\partial z^2} \right] \]

Provided that \( h \) and \( z \) coincide (as mentioned in sec. 2-2) and \( h \) is vertical and \( z \) is perpendicular to the plates, then \( \frac{\partial h}{\partial z} = 1 \) if \( \frac{\partial h}{\partial x} = 0 \). Also \( \frac{\partial V_x}{\partial x} = 0 \) as a result of continuity making \( \frac{\partial^2 V_x}{\partial x^2} = 0 \). For 2-dimensions \((x,z)\), \( V_y \) and its derivatives are equal to zero, leaving \( g \frac{\partial h}{\partial x} \neq 0 \) and \( \frac{\mu}{\rho} \frac{\partial^2 V_x}{\partial z^2} \neq 0 \) and the equation of motion (2-5) is reduced to:
There are two cases of treatment to this 2-dimensional problem.

**Case 1** Either the piezometric head \( h \) is constant and hence \( \frac{\partial h}{\partial x} = 0 \) and \( V \) at \( b \) is not zero and the upper plate is moving while the lower plate is stationary. This kind of flow is of no interest to this paper.

**Case 2** If \( V = 0 \), then we have a parabolic flow in which \( z = \frac{b}{2} \) at \( V_x = V_{\text{max}} \) between the fixed parallel plates and hence we have Poiseuille type flow (shown in Fig. 2-11). Fig. (2-7a) shows the velocity distribution between the fixed parallel plates.

Solving the equation of the fracture system as it stands in both cases, we have the system

\[
g \frac{\partial h}{\partial x} - \mu \frac{\partial^2 V_x}{\rho \partial z^2} = 0 \tag{2-10}
\]

with Boundary Conditions (Case 1)

\[
V_x = 0 \text{ at } z = 0 \quad (b)
\]

\[
V_x = V \text{ at } z = b \quad (c)
\]

with Boundary Conditions (Case 2)

\[
V_x = 0 \text{ at } z = 0 \quad (d)
\]

\[
V = 0 \text{ at } z = \frac{b}{2} \text{ at } V_x = V_{\text{max}}
\]

From (2-11a)

\[
\frac{\partial^2 V_x}{\partial z^2} = \frac{\rho g}{\mu} \frac{\partial h}{\partial x}
\]

Integrating twice

\[
V_x = \frac{\rho g}{\mu} \frac{\partial h}{\partial x} \frac{z^2}{2} + c_1 z + c_2
\]

Applying (2-11b)

\[
c_2 = 0
\]
Fig. 2-7a: Velocity distribution.
Applying (2-11c)

\[ V = \frac{\rho g}{\mu} \frac{dh}{dx} \left( \frac{b^2}{2} \right) c_1 b \]

or

\[ c_1 = \frac{V}{b} \frac{\rho g}{\mu} \frac{dh}{dx} \frac{b}{2} \]

and

\[ v_x = \frac{\rho g}{\mu} \frac{dh}{dx} \left( \frac{z^2}{2} \right) \left[ \frac{V}{b} \frac{\rho g}{\mu} \frac{dh}{dx} \frac{b}{2} \right] z \]

or

\[ v_x = \frac{V}{b} \frac{bz \rho g}{\mu} \frac{dh}{dx} \left[ \frac{1}{2} \right] \]

With respect to Case 2 (of interest), \( V = 0 \) and \( z = \frac{b}{2} \) at \( v_x = v_{\text{max}} \)

\[ v_{x,\text{max}} = \frac{b^2}{4} \frac{\rho g}{\mu} \frac{dh}{dx} \left[ \frac{1}{2} \right] \]

\[ = \frac{\rho g b^2}{8} \frac{dh}{dx} \]

(2-12)

The average velocity is considered as given by

\[ v = \frac{Q}{A} = \frac{2}{3} v_{\text{max}} \quad \text{(Bird et al. 1960)} \]

\[ v = \frac{2}{3} \left( \frac{\rho g b^2}{8} \frac{dh}{dx} \right) \quad \text{or} \quad v = \frac{\gamma}{12\mu} b^2 \frac{dh}{dx} \quad (2-13) \]

where \( \gamma = \rho g \), \( \mu \) - viscosity and \( b \) - aperture of fracture (it is taken a constant in parallel plate flow). The negative sign appears because the acceleration due to gravity is in the negative sense. The group \( \frac{\gamma b^2}{12\mu} \) or \( \frac{\gamma nb^2}{12\mu} \) denotes the hydraulic conductivity (or the coefficient of permeability) of the flow and \( n \) - the porosity.

The group \( \frac{nb^2}{12} \) is termed the intrinsic permeability of the medium.

The above treatment of fracture is in \( x, y, z \) coordinate system. It could be treated in \( r, \theta, z \) system (cylindrical coordinates). Writing the Navier-Stokes equation (given by eq. 2-8 in Philips terminology or given by eq. 5 in Appendix B in our terminology) in its entirety in cylindrical coordinates and using (2-9)
\[ \rho g \frac{\partial p}{\partial r} + \rho \left\{ \frac{\partial}{\partial r} \left[ \frac{1}{r} \frac{\partial}{\partial r} (ru_r) + \frac{1}{r^2} \frac{\partial^2 u_r}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} \right] \right\} \]

\[ \frac{\partial \rho r}{\partial t} + u_r \frac{\partial \rho r}{\partial r} + u_\theta \frac{\partial \rho r}{\partial \theta} + u_z \frac{\partial \rho r}{\partial z} = 0 \]

(2-14)

The bracketed term on the left of eq. (2-14) is \( \mu \nabla^2 q \) and appears in the work of Jaeger and Cook (1979, p 121); the terms \( \rho g r \) and \( \frac{\partial p}{\partial r} \) are due to gravitation and pressure head, respectively. The right bracketed terms are the convective accelerations whose \( x \)- and \( r \)-components are

\[ a_x - \frac{du}{dt} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \]

or

\[ a_r - \frac{du_r}{dt} + u_r \frac{\partial u_r}{\partial r} + u_\theta \frac{\partial u_r}{\partial \theta} + u_z \frac{\partial u_r}{\partial z} \]

and the continuity equation

\[ \nabla \cdot q - \frac{1}{r} \frac{\partial}{\partial r} (ru_r) - \frac{1}{r} \frac{\partial}{\partial \theta} (u_\theta) - \frac{\partial}{\partial z} (u_z) = 0 \]

(2-16)

These Navier-Stokes equations (2-14 to 2-16) have not yet been solved because of the nonlinearity of the partial differential equations. So approximations are necessary to any particular solution of interest. Here we are concerned with radial flow in a fracture.

**Radial Flow Equation in a Fracture System**

Setting \( u_z = 0, u_t = 0, u_\theta = 0 \) in both (2-14) and (2-16) equations

\[ \rho g r \frac{\partial p}{\partial r} + \rho \left\{ \frac{\partial}{\partial r} \left[ \frac{1}{r} \frac{\partial}{\partial r} (ru_r) + \frac{\partial^2 u_r}{\partial z^2} \right] \right\} = \rho u_r \frac{\partial u_r}{\partial r} \]

(2-17)

and

\[ \frac{1}{r} \frac{\partial}{\partial r} (ru_r) = 0 \]

(2-18)

Another approximation is necessary as the inertia term \( u_r \frac{\partial u_r}{\partial r} \) makes the analytical solution impossible. This term results from convective accelerations and may be neglected as being small. Eq. (2-17), then, becomes
\[ \rho gr - \frac{\partial p}{\partial r} \cdot u \frac{\partial^2 u_r}{\partial z^2} = 0 \]  \hspace{1cm} (2-19)

where the first term in the bracket of eq. (2-17) is set to zero by virtue of eq. (2-18). Eq. (2-19) is analogous to eq. (2-10) when the first two terms could be written as

\[ \nabla \left[ gz + \frac{P}{\rho g} \right] - g \frac{\partial h}{\partial r} \]

where

\[ g' = g \bar{v} z \]

and therefore

\[ \frac{\partial h}{\partial r} - \mu \rho g \frac{\partial^2 u_r}{\partial z^2} \]  \hspace{1cm} (2-20)

Integrating eq. (2-18)

\[ ru_r - A \text{ giving } u_r = \frac{A}{r} \]  \hspace{1cm} (2-21)

where \( A \) is a function of \( z \) in accordance with eq. (2-20). We don't know if the attempt will lead to a fracture equation, yet it is worth a trial.

Substitution eq. (2-21) in (2-20) giving

\[ \frac{\partial h}{\partial r} - \mu \rho g \frac{\partial^2 A}{\partial z^2} - \mu \rho g \frac{1}{r} \frac{\partial^2 G(z)}{\partial z^2} \]  \hspace{1cm} (2-22)

Assuming

\[ \frac{\partial^2 G(z)}{\partial z^2} = A_1 \]

then

\[ \frac{\partial G(z)}{\partial z} = A_1 z + A_2 \]

and

\[ G(z) = \frac{1}{2} A_1 z^2 + A_2 z + A_3 \]  \hspace{1cm} (2-23)

Assuming boundary conditions as shown in Fig. (2-7b)

\[ u_r = 0 \text{ at } z = \pm \frac{b}{2} \]
Fig. 2-7b: Velocity distribution in radial flow.

\[-\frac{\gamma n b^2}{12 u} \frac{\delta h}{\delta r}\]  \hspace{1cm} \text{eq 2-34}

If \( Q \) is the volumetric flow rate

\[Q = \frac{2 \pi}{\ln \frac{r_e}{r_w}} \frac{\gamma n}{12 u} b^3 (\Delta h)\]  \hspace{1cm} \text{eq 2-35}
where \( b \) - the aperture of the fracture and \( A_2, A_3 \) are constants of integration and remembering that \( u_r = \frac{A}{r} - \frac{G(z)}{r} \) (eq. 2-21) and applying the boundary conditions then

\[
G(z) = -\frac{1}{2} A_1 \left[ \frac{z^2 - b^2}{4} \right]
\]

(2-24)

Using eq. (2-22)

\[
\frac{\partial h}{\partial r} = \frac{\mu}{\rho g} \frac{1}{r} \frac{d}{dz^2} \left[ \frac{1}{2} A_1 \left[ \frac{z^2 - \frac{b^2}{4}}{4} \right] \right]
\]

or

\[
\frac{\partial h}{\partial r} = \frac{\mu}{\rho g} \frac{1}{r} A_1
\]

or

\[
\frac{\partial h}{\partial r} = \frac{\mu}{\gamma} \frac{A_1}{r} (\rho g - \gamma)
\]

(2-25)

Integrating eq. (2-25)

\[
h = \frac{\mu}{\gamma} A_1 \ln r + A_4
\]

(2-26)

As shown p (55), \( h \) is the piezometric head \( z + \frac{P}{\rho g} \). \( A_4 \) is a constant of integration and \( r \) is the varying parameter of the cylinder of flow.

Consider now the inner and outer diameters of the conduit of cylindrical shape, that is, \( h_{re} \) and \( h_{rw} \) and manipulating eq. (2-26)

\[
h_{re} = \frac{\mu}{\gamma} A_1 \ln r_e + A_4
\]

(2-27)

\[
h_{rw} = \frac{\mu}{\gamma} A_1 \ln r_w + A_4
\]

Subtracting

\[
h_{re} - h_{rw} = \frac{\mu}{\gamma} A_1 (\ln r_e - \ln r_w)
\]

Then

\[
A_1 = \frac{h_{re} - h_{rw}}{\frac{\gamma}{\mu} \ln \left( \frac{r_e}{r_w} \right)}
\]

(2-28)
substitution of (2-28) in (2-26)

\[ h_{re} = \frac{\mu \gamma}{\mu} \ln \left[ \frac{re}{rw} \right] \]

or

\[ h_{re} = \frac{h_{re} - h_{rw}}{\ln \left[ \frac{re}{rw} \right]} \]

Hence

\[ A_4 = \left[ \frac{h_{re} \ln \left[ \frac{re}{rw} \right] - (h_{re} - h_{rw})\ln r_e}{\ln \left[ \frac{re}{rw} \right]} \right] \]

\[ A_4 = \frac{h_{rw} \ln r_e - h_{re} \ln r_w}{\ln \left[ \frac{re}{rw} \right]} \]

From (2-21), (2-24) and (2-28)

\[ u_r = \frac{G_2}{r} - \frac{A_1}{2} \left( \frac{z^2 - b^2}{4} \right) \]

\[ u_{r, max} = \frac{\gamma}{2\mu} \ln \left[ \frac{re}{rw} \right] \left( \frac{b^2}{4} \right) \]

Hence eq. (2-25), using (2-28) becomes
\[
\frac{\partial h}{\partial r} = \frac{1}{\gamma r \mu} \frac{h_{re} - h_{rw}}{\ln \left( \frac{r_e}{r_w} \right)}
\]

or

\[
\frac{\partial h}{\partial r} = \frac{1}{r} \frac{h_{re} - h_{rw}}{\ln \left( \frac{r_e}{r_w} \right)}
\]

Using (2-32) in (2-31)

\[
u_{r_{\text{max}}} = -\frac{\gamma}{8\mu} b^2 \frac{\partial h}{\partial r}
\]

and

\[
\bar{u} = \frac{2}{3} \nu_{r_{\text{max}}} - \frac{Q}{A}
\]

or

\[
\bar{u} = 2 \left( \frac{\gamma b^2}{8\mu} \frac{\partial h}{\partial r} \right)
\]

or

\[
\bar{u} = -\frac{\gamma b^2}{12\mu} \frac{\partial h}{\partial r}
\]

where \((n)\) is the porosity and \(\frac{\partial h}{\partial r}\) is given by eq. (2-32)

Writing eq. (2-34) in terms of Darcy's law and recognizing

\[q = \bar{u} b\text{ (flow rate/unit area)}\]

\[q_r = \frac{\gamma b^2}{12\mu} b^3 (-\nabla h_r)\]

where the hydraulic conductivity of the fracture \(K = \frac{\gamma b^2}{12\mu}\) and the permeability of

the medium \(K' = \frac{nb^2}{12}\).

If \(Q\) is the volumetric flow rate

\[
Q = \bar{u} A - 2\pi rb \frac{\gamma b^2}{12\mu} (-\nabla h_r)
\]

\[
Q = \frac{\gamma b^3}{12\mu} (2\pi r) \frac{1}{r} \frac{h_{re} - h_{rw}}{\ln \left( \frac{r_e}{r_w} \right)}
\]
This equation appeared in the work of Tsang and Witherspoon (1981) in treating statistically the average aperture of the fracture giving the aperture the average \( \langle b \rangle^3 \) which could be better termed the statistical equivalent of the variable aperture. There will be more to say, when the tortuosity of the medium is discussed in Sec.(2-4-2), about this parameter where it will be derived in a different way.

Eq. (2-35) is the radial flow equation across the fracture in polar coordinates when the inertia term and the convective accelerations are ignored and hence, the attempt is successful. It reveals the geometry of the model and the pressure head difference.

Eq. (2-35) illustrates how the flow in fractured rock is a function of the \( \langle \text{aperture} \rangle^3 \), no matter how the geometry is, whether rectangular or cylindrical and/or spherical. Geometry differs from one system to another but still the \( \langle \text{aperture} \rangle^3 \) dominates the flow. Hence the cubic law can be derived and the permeability of the fracture is a unique function of the \( \langle \text{aperture} \rangle^2 \).

2-4-2. Irregularities and Closed Conduit (natural pattern)

The previous work shows the radial average velocity being considered with respect to smooth parallel wall fractures. This idealism is hardly found in natural media. Sharp (1970, proceedings, 2nd Congress, p 519) stated,

"Laboratory flow tests on natural fissures have shown that the commonly assumed parallel plate flow is only valid for truly planar fissures with near smooth surfaces. These are rarely encountered in practice and generally the hydraulic conductivity of a fissure will depend on:
opening of fissure, roughness or irregularity of fissure, angularity or non-planarity of fissure and fluid property."

Hence, in order to approach this problem, it seems reasonable to assume Poiseuille work as the equation of the radial flow across a conduit of irregular cross section, that is, eq. (2-43) as expressed by Bear (1972) or the same equation (2-47) expressed in hydraulic radius where \( \delta - D_h - 4R_h - 2b \) (Louis, 1969, p 24), with reference to Figure (2-11), since we have a parabolic function in cylindrical geometry. Eq. (2-47) is exactly eq. (2-33) where \( R_h = \frac{b}{2} \) and \( \delta - D_h \) - hydraulic diameter.

Hence, with Poiseuille equation as a basic unit, modification will be introduced to build a fracture model that is suitable for natural media. Generality is assumed all the way. The diagram of Figure (2-16) shows these modifications as applied to the fracture eq. (2-35) in sequence of their treatment.

2.4.2.1. Friction Factor and the Roughness of the Fracture

Rose (1945), Todd (1959) and Bear (1972) used the Darcy-Weisback formula:

\[
\frac{\Delta \phi}{L} = \left( \frac{f'}{d} \right) \frac{v^2}{2g}
\]

to establish the relation between the friction factor \( (f') \) and the Reynolds number \( (Re) \). Fanning (after Bear, 1972) used another definition for the friction factor by replacing the hydraulic radius \( R \) for \( d \) (the grain diameter). They expressed Reynolds number as

\[
Re = \frac{gd}{\nu} \text{ inertial forces}
\]
\[
- \frac{\nu}{\text{viscous forces}}
\]

and the friction factor as

\[
f' = \frac{2gd}{v^2} \frac{\Delta \phi}{L}
\]

where \( \Delta \phi \) - the piezometric head over a length \( L \) in the direction of the flow.
the apparent velocity as defined by Darcy's law, \( \nu \) - the kinematic viscosity - \( \frac{\mu}{\rho} \)

\( \Delta \phi = \phi_1 - \phi_2 \) (the difference in piezometric head, that is, the difference in energy of water per unit weight). It is also the difference in energy per unit mass expressed as \( \Delta \phi = \nabla \left( gz + \frac{P}{\rho} \right) \) and is called the driving head.

\[ \frac{\Delta \phi}{L} = \Delta h \] - the hydraulic gradient (i.e. the driving force causing the fluid to move toward the point of lower energy). For energy and hydraulic gradient, please see Sec (2-3-e). The hydraulic gradient has the components:

\[ \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \] in the x, y, z directions.

Eq. (2-36) will lead to Darcy's law by applying the relation \( Re \cdot \frac{c}{f} \) (as shown by the straight curve in Figure (2-8)), where \( c \) is some constant.

Hence

\[ q \frac{d}{\nu} = \frac{c}{2gd} \frac{\Delta \phi}{L} \]

or

\[ q = \left( \frac{2}{c} \right) \frac{d^2 g}{\nu} \frac{\Delta \phi}{L} \]

or

\[ q = K \frac{\Delta \phi}{L} \] Darcy's law (2-37)

The results of the experimental work of many investigators have been plotted in Figure (2-9). The figure shows two different segments of the curve. The straight line portion verifies Poiseuille's law (velocity in laminar flow is proportional to the first power of the hydraulic gradient). Since Darcy's law applies to laminar flow in porous media, the Reynolds number serves as a criterion to distinguish between laminar and turbulent flow.
Fig. 2-8: Relationship between Fanning's friction factor and Reynolds number for flow through granular porous media (Rose, 1945). (After Bear, 1972).
Fig. 2-9: Schematic classification of flow through porous media. (After Bear, 1972).
The second segment represents the departure from the laminar flow and the transition passage to turbulent flow, indicating an upper limit for the validity of Darcy's law. The curve also shows that no lower limit is known to exist for Darcy's law.

However, a range of values rather than a unique value must be considered because the distribution of grain sizes of natural media is limitless and because grain shapes and packings may vary widely so that deviations can only be considered where they are most frequently observed.

Muskat (1949, p 126-128), Todd (1959, p 47-48) and Todd (1980, p 67-68) restricted Reynolds number to \( < 1 \) or \( 1 - 10 \) for laminar flow. Others suggested \( \text{Re}_{100} \) (Bear, 1972) for the upper limit of the laminar flow and one should be careful not to identify this deviation with a transition from laminar flow, for which Darcy's law is valid, to turbulent flow. The \( \text{Re}_{100} \) is often referred to as the nonlinear laminar flow regime.

However, Dybbs and Edwards (1982) showed that the steady non-linear laminar flow persists to a Reynolds number of 150. In Figures (2-10-a, -b, -c) streaklines indicative of laminar flow, are well defined. In Figures (2-10-d and -e), the dye streakline boundaries are smooth and the flow is still able to conform to local geometry. At Reynolds 150, oscillations start to appear with regular periods which are measurable. This supports the assumption made at the end of Sec. (2-2) that the solution of Navier-Stokes equation is very closely approximated by a steady-state solution as given by Darcy's law as long as potential is not periodic.

Now, using (2-36) to evaluate the friction factor \( (f') \) in terms of the Reynolds number and the constant \( c \) and to give the constant \( c \) a value.

\[
f' = \frac{2gd \Delta \phi}{Q^2 L}
\]
(After Dybbs and Edwards, 1982).
Fig. 2-10e Re = 30±10, I = 0.30 cc/min.

Fig. 2-10g Re = 215±18, I = 2 x 1.1 cc/min.

Fig. 2-10f Re = 190±12, I = 2 x 1.1 cc/min.

Fig. 2-10h Re = 225±15, I = 2 x 2.2 cc/min.

"(After Dvbbis and Edwards, 1982)."
The value \( f' \) has been mentioned by Witherspoon et al. (1980). This value has not reached the turbulent flow and suggests that both roughness and Re are small. The value of \( c \) will change from model to model and there will be more about it later.

However, as Re number increases the inertia forces become more significant and turbulent flow takes place. Here all assumptions for a laminar flow break down and the laws of flow can only be established by experiments.

Eq. (2-38) is a simple relationship expressing the cubic law for a laminar flow in an open fracture. Louis (1969) elaborated on his model of parallel plates and investigated the effects of rough walls instead of smooth walls and developed the empirical relationship (Figures 1-1, 1-2).

\[
f' = \frac{96}{\text{Re}} \left[ 1 + 8.8 \left( \frac{K}{D} \right)^{1.5} \right]
\]

(2-39)

where the quantity \( 1 + 8.8 \left( \frac{K}{D} \right)^{1.5} \) replaces the implied \( (f) \) in eq. (2-38), that is \( \left[ f' = \frac{96}{\text{Re}} f \right] \) where \( f = 1 \) for smooth surfaces.

\( K \) - the absolute roughness and \( (f) \) whose value differs from rock to rock, is a factor that accounts for deviation from the ideal conditions, and \( \frac{K}{D} = 0.033 \) is the
relative roughness where \( D - D_h \) is the hydraulic diameter - \( 4R - 4R_h \) where \( R_h \) is the hydraulic radius - \( \frac{b}{2} \).

These definitions are related to eq. (2-39) as defined by Louis (1969, p 25) and are shown in Figures (1-1, 1-2). The parameter 'K' as defined must be distinguished from 'K': the coefficient of permeability (that is, the hydraulic conductivity) of the media which appears through this dissertation.

Introducing the roughness (f) into the structure of the model (equations 2-34, 2-35), we have

\[
\bar{u}_r - \frac{\gamma b^2}{12\mu} \frac{\partial h}{\partial r} - \frac{\gamma b^2}{12\mu} \Delta h
\]

and the flow rate

\[
Q = \left( \frac{1}{f} \right) (\bar{u}_r A) - \left( \frac{1}{f} \right) \frac{\gamma b^2}{12\mu} \frac{\partial h}{\partial r} 2\pi b
\]

\[
Q = \left( \frac{1}{f} \right) \frac{2\pi}{\ln \left( \frac{r_e}{r_w} \right)} \frac{\gamma b^3}{12\mu} \Delta h
\]

(2-40)

where 'b' is the distance between the assumed rough parallel walls.

Eq. (2-40) could be written as

\[
\frac{Q}{\Delta h} = c (f(b))^3
\]

which is the cubic law and \( c = \frac{2\pi}{\ln \left( \frac{r_e}{r_w} \right)} \frac{\gamma n}{12\mu} \)

and \( f \) takes the value that corresponds to the media under investigation.

At this point, it is of interest to investigate the volumetric flow rate \( (Q) \) in porous media in natural rock as illustrated by different models.

**Capillary Tube Model**

Bird et al. (1960) gave the parabolic velocity distribution for laminar flow in a tube, Figure (2-11) as
Parabolic velocity distribution $V_z(r)$

Linear momentum flux distribution $\tau_{rz}(r)$

$v_z = 0$

$v_{z,\text{max}}$

$\tau_{rz} = 0$

$(v - v_o) \frac{R}{t}$

$\tau_{rz,\text{max}} = \frac{(v - v_o) R}{2L}$

Fig. 2-11: Momentum flux and velocity distribution in flow in cylindrical tubes. (After Bird et al., 1960).
where $P - p - \rho g z$ represents the combined effect of the static pressure and the gravitational forces, and $L$ is the length of the tube, $r$ is the radial variable, $R$ is the radius of the tube and $\mu$ is the viscosity.

$$V_{\text{max}} = \frac{P_o - P_L}{4\mu L} R^2$$

and the average velocity

$$\bar{V}_z = \frac{1}{2} V_{\text{max}} \cdot \frac{(P_o - P_L) R^2}{8\mu L}$$

(2-41)

and the volume flow rate ($Q$)

$$Q = \bar{V}_z A = \frac{\pi (P_o - P_L) R^4}{8\mu L}$$

(2-42)

where $A$ - cross-section area.

This is the Hagen-Poiseuille law.

Poiseuille (1840) expression for the above result is given by

$$Q = K \frac{P D^4}{L}$$

(2-43)

and expressed by Bear (1972) as

$$Q_s = -\frac{\pi D^4}{128\mu} \frac{\rho g}{\delta s}$$

and

$$V_s = -\frac{\delta^2}{32} \frac{\rho g}{\mu} \frac{\delta \phi}{\delta s}$$

(2-43)

where

$$\delta = D - 2R \text{ and } \rho g \frac{\delta \phi}{\delta s} = \frac{P_o - P_L}{L}$$

Comparing equations (2-42) and (2-43)

$$K^* = \frac{\pi \rho g}{128\mu} \frac{D^4}{\delta^4} \text{ and } \frac{P}{L} \frac{\delta \phi}{\delta s}$$

So

$$Q_s = -\frac{\pi \rho g}{128\mu} \frac{\delta^4}{\delta s} \frac{\delta \phi}{\delta s}$$
The friction factor, in terms of velocity and pressure head is given by

\[ f' = \frac{1}{4} \left( \frac{D}{L} \right) \frac{P_o - P_L}{\frac{1}{2} \rho \overline{V}^2} \]  

(2-44)

Substitution of eq. (2-41) in eq. (2-44)

\[ f' = \frac{1}{4} \left( \frac{D}{L} \right) \left[ \frac{P_o - P_L}{\frac{1}{2} \rho \overline{V} (P_o - P_L)R^2} \right] \]

\[ = \frac{1}{2} \left( \frac{D}{L} \right) \frac{8 \mu L}{\rho R^2 \overline{V}} \]

\[ - \frac{4D^2}{D \overline{V} \mu R^2} = \frac{4(4R^2)}{Re R^2} - \frac{16}{Re} \]

(2-45)

The friction factor \( \left( \frac{16}{Re} \right) \) is then for laminar flow in a capillary tube model.

Eq. (2-45) is different from eq. (2-38) in the constant (c) in the relation \( f' = \frac{c}{Re} \).

This relation is therefore best written as

\[ f' \text{ is a function of } Re \]

and (c) depends on the type of media under investigation. This statement is made clear by the following fissure model.

Fissure Models

Irmay (1955) and Bear (1972) represented the porous media by a fissure model. A fractured rock would probably be the porous media closest to such a model.

Figure (2-12). These authors showed that the solution of Navier-Stokes equation for average velocity in a single fissure of width (b) is given by

\[ \overline{V} = b^2 \frac{\gamma}{12 \mu} \frac{\partial h}{\partial r} \]  

(2-46)

This is the same solution given by the analysis of this dissertation (eq. (2-34)) and leads to \( f' = \frac{96}{Re} \)
Fig. 2-12: A capillary fissure model. (After Bear, 1972).
Hydraulic Models

With respect to the idea of the specific surface of a porous material as defined by the total interstitial surface area of the pores per unit bulk volume of the porous medium, if the \( \delta \) (eq. 2-43) \( - 4R \) where \( R \) represents the average hydraulic radius as defined by the ratio of volume of a conduit filled with fluid to its wetted surface (Bear (1972)), this definition if combined with a visualization of the porous medium as a network of interconnected channels, leads to a relationship between \( R \) and the specific surface. Then Poiseuille equation (eq. 2-42 or 2-43) is written as

\[
\frac{Q_s}{n\delta^2} - \frac{\rho g}{32} \frac{\partial \phi}{\partial s} - \frac{\rho q}{\mu} \frac{\partial \phi}{\partial s} - K' \frac{\partial \phi}{\partial s} - \frac{R^2}{2} \frac{\rho q}{\mu} \frac{\partial \phi}{\partial s}
\]

(2-47)

and

\[
1' = \frac{2g}{V^2} \frac{\delta}{4} \frac{\partial \phi}{\partial s} - \frac{5}{4} \frac{\partial \phi}{\partial s} - \frac{R}{2g} \frac{\rho q}{\mu} \frac{\partial \phi}{\partial s} - \frac{R^2}{2} \frac{\rho q}{\mu} \frac{\partial \phi}{\partial s} \frac{V}{2g} - \frac{2 \times 2}{2} \frac{16}{Re} \frac{\delta}{V} \frac{\partial \phi}{\partial s} \frac{1}{\nu}
\]

Now, it is obvious from these several models that \( K' \) (the medium permeability) is given by (and introducing the porosity (n)).

Parallel planes (fracture model, eq. 2-34)

\[
K' = \frac{nb^2}{12} \text{ leading to } 1' = \frac{96}{Re}
\]
Capillary tube model (Poiseuille flow, eq. 2-43)

\[ K' = \frac{nb^2}{32} \text{ leading to } f' = \frac{16}{Re} \]

Fissure model (parallel planes, eq. 2-46)

\[ K' = \frac{nb^2}{12} \text{ leading to } f' = \frac{96}{Re} \]

Average hydraulic radius model (Poiseuille flow, eq. 2-47)

\[ K' = \frac{nr^2}{2} \text{ leading to } f' = \frac{16}{Re} \]

These analyses lead us to believe that the flow as represented by eq. (2-47) is the type expected in a closed conduit of irregular cross section as shown in Figure (1-8) by Louis (1969) or by a modified Figure (2-13) similar to that of Louis to illustrate the concept of tortuosity which will be discussed shortly.

It is noteworthy to observe that the numerical coefficients \[ \left\{ \frac{1}{12}, \frac{1}{32}, \frac{1}{2} \right\} \] are meaningless so far as an actual porous medium is concerned. It is commonly replaced by some arbitrary coefficient that must be determined experimentally. As a more general expression for \( K' \) (the permeability of the medium) and based on the hydraulic radius, Bear (1972) gave it the expression

\[ K' = f'_1(s)f'_2(n)R^2 - cd^2 \quad (2-48) \]

where

- \( f'_1(s) \) is the dimensionless shape factor
- \( f'_2(n) \) is a function of the porosity factor
- \( c = f'_1(s)f'_2(n) \), \( R \) is the hydraulic radius
- \( d \) - the effective diameter of the grain

If the solution of parallel plate fracture is given by

\[ \bar{u} - \bar{V} = \frac{nb^2}{12} \frac{\gamma}{\mu} \frac{\partial h}{\partial r} \quad \text{eq. (2-34)} \]
The spatial average of the square of the hydraulic radius \( R \) gives the varying shape of the tortuous conduit.

\[ T = \frac{L^2}{L_x} = \frac{2}{3} \]  
(page 88)

\[ U = \frac{R}{c\mu} \cdot \frac{\Delta \phi}{L_e} \cdot \frac{L}{L_e} \]  

\[ \frac{nR}{c\mu} = \gamma (T) \nabla \phi \]  
(page 36)
and the Poiseuille solution in porous media

\[ \bar{V} = \frac{nR^2}{2} \frac{\gamma}{\mu} \frac{\partial \phi}{\partial s} (\phi - h) \]  

(2-49)

s - the length measured along the tube

Then letting \( R = \frac{n}{M} \) where \( n \) is the porosity and \( M \) is the specific surface with respect to unit volume of solid, then

\[ \bar{V} = \frac{n^3}{3M^2} \frac{\gamma}{\mu} (\Delta h) \]  

(a)

and

\[ \bar{V} = \frac{n^3}{2M^2} \frac{\gamma}{\mu} (\Delta h) \]  

(b)

Evaluating equation (b) in terms of the work of Koseny (1927) and Carman (1937), then

\[ f'_1(s) f'_2(n) = c_0 \frac{n^3}{M^2} - \frac{n^3}{2M^2} \cdot K' \]  

Koseny equation

and

\[ f'_1(s) f'_2(n) = c_0 \frac{n^2}{M^2} - \frac{n^3}{5M^2} \cdot K' \]  

Carman equation

where

\[ c_0 = \frac{1}{2} \]  

in Koseny equation and

\[ c_0 = \frac{1}{5} \]  

in Carman equation.

It was found that \( c_0 = \frac{1}{5} \) fits well with experimental data due to the fact that tortuosity is included and hence \( c_0 T = \frac{1}{5} \), and \( c_0 = \frac{1}{2} \) does not take into account velocity components normal to the tubes axes.

However, this part of discussion about the various models as investigated by the authors, reveals the most important point that improvements and modifications to the assumed parallel plate model, as Sharp (1970) suggested, are plausible as far as natural porous media is concerned, and the concept of the hydraulic radius as discussed above and the author's results based on it, dominate the irregularity of
the porous medium closed conduits as established by Poiseuille (1840). Such a hydraulic radius model is supported further by the cylindrical flow as a result of the radial flow eq. (2-20) and its solution eq. (2-33) and where (r) is the varying parameter of the cylinder of flow.

Writing both models (eq. (2-34) and eq. (2-49)) in terms of the hydraulic radius where \( R = \frac{b}{2} \) (as \( a_i \) in terms of Louis terminology)

\[
\bar{V} = \frac{nR^2}{3} \frac{\gamma}{\mu} (\Delta h) \quad \text{(Parallel plate model or parallel planes)} \tag{2-50}
\]

\[
\bar{V} = \frac{nR^2}{2} \frac{\gamma}{\mu} (\Delta h) \quad \text{(hydraulic radius model)} \tag{2-51}
\]

The substitution of \( R = \frac{b}{2} \) in eq. (2-50) will give eq. (2-34) and eq. (2-51) becomes

\[
\bar{V} = \frac{nb^2}{8} \frac{\gamma}{\mu} (\Delta h)
\]

and hence eq. (2-40) is modified to

\[
Q = \left( \frac{1}{8} \right) \frac{2\pi}{\ln \left( \frac{r_e}{r_w} \right)} \frac{nb^3}{8\mu} (\Delta h) \tag{2-52}
\]

where

\[
\Delta h = \frac{1}{r} \frac{h_{re} - h_{rw}}{\ln \left( \frac{r_e}{r_w} \right)}
\]

and (f) is the roughness of the medium.

2.4.2.2. Tortuosity of the Medium

In the above section, the term 'tortuosity' denoted by (T) is mentioned associated with Koseny and Carman equations p (82). The idea of the hydraulic radius gives too large a throughput for a given pressure gradient and the liquid traverses a tortuous path, the length of which may be double the length \( L \). Figure (2-13) shows the lengths \( L \) and \( L_e \). Tortuosity is a result of irregular closed conduit.
Scheidegger (1960, p 119) stated, "A serious drawback of the parallel type models is that all the pores are supposed to go from one face of the porous medium right through to the other. This supposition is evidently far remote from what happens in an actual porous medium." And on page (120), he defined tortuosity as "Actually $T - \frac{s}{x}$ would be an excellent measure of the tortuosity as it gives the ratio of the length of the flow channel for a fluid particle with respect to the length of the porous medium, if the model satisfactorily represents a porous medium."

Bird et al. (1960) showed that experimental measurements indicated that the theoretical formula can be improved if the $2$ in the denominator of Poiseuille equation (eq. 2-49, in terms of the hydraulic radius) is changed to $\frac{25}{6}$ (analysis of a great deal of data has lead to this value) which he accepted. However, Bird made the statement: This result is generally good for void fractions less than 0.5 and is valid only in the laminar region with $Re < 10$. Figure (2-14) illustrates this statement.

Carman (1937) believes that $\frac{Le}{L} - \sqrt{2}$ and $T - \left(\frac{Le}{L}\right)^2 - 2$ and he experimentally got this result.

Bear (1972) showed that the definition of tortuosity as $\left(\frac{Le}{L}\right)^2 > 1$ is a mistake. Since tortuosity affects both the velocity and the pressure gradient, then $T - \left(\frac{L}{Le}\right)^2 < 1$ is the tortuosity of the porous medium and by Carman measurements:

$T - \left(\frac{L}{Le}\right)^2 - \left(\frac{1}{\sqrt{2}}\right)^2 < 0.5$ and is very close to the value given by Bear et al. (1968, p 59) as 0.4 which Carman (1956) accepted as plausible for non-circular sections.
Fig. 2-14: Sketch showing the general behavior of the Ergun equation on a log-log plot. [S. Ergun, Chem. Eng. Prog., 48, 93 (1952).] (After Bird et al. 1960).
It must be noted that discussion is based on irregularity of porous media regardless of the type of soil.

By introducing Dupuit's assumption (1863) that the interstitial velocity \( u_e \) is equal to \( \frac{u}{\xi} \) where \( u \) is the mean value \( \bar{u} \) and \( \xi \) is the porosity \( (n) \), it follows

\[
u_e = \frac{\bar{u}}{n} - \bar{V} \frac{L}{L_e}
\] (A)

where \( \bar{V} \) is the magnitude of the average tangential velocity.

Extending Hagen-Poiseuille law to a flow in a noncircular tube (irregular closed conduit) by introducing the concept of the hydraulic radius \( R \) as defined by

\[
R = \frac{\text{cross section area normal to flow}}{\text{perimeter presented to fluid}}
\]

And since the varying shape of the tortuous conduit will be accounted for by a spatial average of the square of the hydraulic radius \( (R)^2 \), then (A) is written as

\[
\bar{u} = n\bar{R}^2 \frac{\gamma}{c\mu} \left( \frac{L}{L_e} \right) - \frac{\Delta \phi}{L_e} \frac{L_e}{L} \frac{L_e}{L}
\]

\[
\bar{u} = n\bar{R}^2 \frac{\gamma}{c\mu} \left( \frac{L}{L_e} \right)^2 \frac{\Delta \phi}{L_e} \frac{L_e}{L} \frac{L_e}{L}
\]

\[
\bar{u} = n\bar{R}^2 \frac{\gamma}{c\mu} \frac{\gamma L}{L_e} \frac{\Delta \phi}{L_e} \frac{L_e}{L} (B)
\]

where

\[ L \] - the x-component of the true path \( L_e \)

\[ \bar{u} \] - the mean velocity value

\[ \bar{V} \Phi \] - the absolute value of the component of the mean hydraulic gradient and equals \( \frac{\Delta \phi}{L_e} \frac{L_e}{L} \) (Carman, 1937).

\( c \) - shape factor accounting for the non-circular shape

- 2 in hydraulic radius model eq. (2.51)
- 3 in parallel plate model eq. (2.50)

\( \phi \) - h is the piezometric head
In terms of the fracture aperture where

\begin{align*}
\bar{D_h} &= 4(\bar{R_h}) - 4 \left( \frac{b}{2} \right) - 2b \\
\bar{u} &= \frac{nb^2}{8} \frac{1}{\mu} \nabla \phi
\end{align*}

(2-53)

This reasoning gives rise to what is called the statistical equivalent of the variable aperture and \( \bar{b}^2 \) will be written henceforth as \( \langle b \rangle^2 \) or in cubic law as \( \langle b \rangle^3 \). This variable aperture is a result of the spatial average of the square of the hydraulic radius \( \bar{R}^2 \).

It is noteworthy that the cubic law with fracture hydraulic radius model holds and eq. (2-52) is written as

\begin{equation}
Q = \frac{1}{i} \left[ \frac{2\pi}{\ln \left( \frac{\bar{r}_a}{\bar{r}_w} \right)} \right] \frac{n \langle b \rangle^3}{8} \frac{1}{\mu} \nabla \phi \frac{r}{(\Delta h)} \tag{2-54}
\end{equation}

Bear (1972), in his treatment of tortuosity tensor, formulated the scalar value \( \bar{T}^* \)

and gave

\[ \bar{T}^* = 2 \bar{T} - \frac{2}{3} \]

This value is the outcome of his mathematical analysis. In order to introduce the effect of divergence of streamlines in a porous media, \( \bar{T} \) must be multiplied by \( \left( \frac{d\sigma}{ds} \right) ^2 \) to help to visualize capillary tubes of irregular cross-section where \( \bar{T}^* \)

is the medium tortuosity and \( \bar{T}^* \equiv \bar{T} \) for a straight tube \( \left( \frac{d\sigma}{ds} \right) ^2 \) and where \( s \)

denotes the length measured along the axis of the channel and \( \sigma \) denotes the length measured along the streamline from the entrance to the channel.

For any point \( N \) on the cross-section and a given \( s \), there is a unique value of \( \sigma \).
For irregular cross-section, the angle \( \theta \) between a channel axis and a streamline is assumed to vary between \( \theta = 0 \) and \( \theta = 90^\circ \), such that \( \theta = 45^\circ \) can be chosen as a representative value.

He obtained 
\[
\left( \frac{d \sigma}{ds} \right)^2 - \sec^2 \theta \cdot \sec^2 45^\circ - 2 \text{ and hence } T = \frac{2}{3},
\]
a general result of mathematical analysis that could be applied to irregularity of any tube system. This result is supported by other authors and conforms to Figure (1-8) by Louis (1969) and to Figure (2-13) as visualized by this dissertation to illustrate the length \( L \) and \( L_e \).

Hence 
\[
T = \left( \frac{L}{L_e} \right)^2 \cdot \frac{2}{3}
\] in a single-phase fluid saturating the porous media.

However, it is Carman who proposed the tortuosity factor when studying the relationship among the average velocity in a tortuous capillary tube, the piezometric head difference between the ends of the tube and the straight line distance between the ends.

Now, if we switch \( T \) from numerator to denominator in eq. (2-54), then two cases arise:

(1) With Carman
\[
cT = (2.5)(2) - 5
\]
where \( c \) (shape factor) is evaluated by Carman (1937) to be (2.5). Wyllie and Spangler (1952, after Bear, 1972), trying to compensate for the noncircular shape of the conduits in an actual porous medium, suggested that the factor (2) in the denominator (as in the hydraulic model), be replaced by a factor in the range 2.5 - 3. This is so we get \( cT \cdot 5 \) in the denominator of eq. (2-54). It is noteworthy to observe that Carman was treating granular soil.

(2) If we follow Bear (1972) closely, with regard to his mathematical tensor analysis and using the fracture hydraulic radius model, we have
cT \cdot 2\left(\frac{3}{2}\right) \cdot 3

\left[\text{That is, if } T \text{ is in the denominator, then } \frac{1}{T} \cdot \frac{1}{2^{3/2}} \cdot \frac{3}{2}\right]

This surprising result calls the attention to the assumed parallel plate model as shown by eq. (2-50), (in terms of $R_h$) and carries the proof that the mathematical solution is correct.

This Result Explains the Following Facts

(a) It is interesting to note in the work of these pioneers that the value of $c$ in equation (B) does not necessarily denote a circular cross-section, nor even a shape resembling a circle. $c$ is just a shape factor and it is a numerical coefficient that appears in the solution of the particular problem for each particular porous medium. This probably explains much of the success in applying the Poiseuille law.

(b) As the actual path pursued by an element of the fluid is sinuous, this represents only the component of velocity parallel to the direction of flow. Thus the time taken for such an element of fluid to pass over a sinuous track of length $L_e$ at a velocity $\frac{u}{n} \cdot \frac{L_e}{L}$ (Dupuit's assumption, 1863), corresponds to that taken to pass over a distance $L$, at a velocity $\frac{u}{n}$. Thus the true value of the interstitial velocity $u_e$ is

\[ u \cdot \frac{L_e}{L} \text{ where } n \text{ is the porosity.}\]

(c) It is quite clear, then, that the stress-displacement effects on a fracture, produce a roughness profile in the walls of the fracture causing a reduction in flow with average velocity $\left(\frac{u}{n} \cdot \frac{L_e}{L}\right)$. This corresponds to the flow through a fracture composed of two smooth parallel walls with average velocity $\left(\frac{u}{n}\right)$. This reduction in flow, due to the asperities protruding in the flow stream,
causing variation in the aperture, is equivalent to the value \( \frac{L_e}{L} \) with average velocity \( \frac{u}{n} \cdot \frac{L_e}{L} \).

(d) It is very obvious, then, that the concept of contact area that appears in literature and which is difficult and still impossible to evaluate as a definite reliable value, is just the factor \( \frac{L_e}{L} \) which explains the physical situation as the aperture gets narrower due to the forces acting (as pressure and stress). The result is a spatial change that affects the aperture leading to its equivalent value \( \langle b \rangle \). Hydraulic conductivity is then a function of this spatial change \( \langle b \rangle \) and so is also the rate of flow \( Q \).

At this point, it is enough to state briefly that the above modifications to the Hagen-Poiseuille law are accounting for Dupuit's assumption (1863) as followed by Blake (1922), Koseny (1927), Carman (1937), and Bear (1972) and the deductive method followed in this work. The concept of the average hydraulic radius \( \bar{R} \) has lead to the tortuosity concept, revealed the contact area idea and hence to the equivalent aperture value \( \langle b \rangle \) in a way different from that given by Tsang and Witherspoon (1981).

With these results, eq. (2-54) is set to be

\[
Q = \left[ \frac{1}{f} \right] \frac{2\pi}{\ln \left( \frac{r_e}{r_w} \right)} \frac{n \langle b \rangle}{c_T} \frac{3}{2} \frac{r}{\mu} (\Delta h)
\]

\[
Q = \left[ \frac{1}{f} \right] \frac{2\pi}{\ln \left( \frac{r_e}{r_w} \right)} \left[ 8 \left( \frac{3}{2} \right)^{\frac{3}{2}} \right] \frac{n \langle b \rangle}{12} \frac{3}{2} \frac{r}{\mu} (\Delta h)
\]

(2-55)
Eq. (2-55) appears as a parallel plate model, but it is not. It is the fracture hydraulic radius model involving the tortuosity factor $T$ and the shape factor as given by Poiseuille law. It truly represents a closed conduit of irregular cross-section.

However, in both cases, it is indicative of the correct mathematical solution.

2-4-2-3. Fracture Spacing (Discontinuity Spacings)

Discontinuity spacings in rock were considered and investigated by Priest and Hudson (1976), and in crystalline granite rock by Wallis and King (1980). Any combination of evenly spaced, clustered and randomly positioned discontinuities, leads to a negative exponential form of frequency vs. spacing value curve. The work showed that the negative exponential form was confirmed by field discontinuity scanline surveys.

In their work, they used the method proposed by Deer (1964, after Priest and Hudson 1976). The Rock Quality Designation (RQD) is the proportion of scanline or borehole core that consists of intact lengths that are 0.1 m or larger. These intact lengths are summed and expressed as a percentage of the total length.

$$
RQD = 100 \sum_{i=1}^{n} \frac{x_i}{L} \quad (2-56)
$$

where

- $x_i$ - the length of the $i$th length $\geq 0.1$ m.
- $n$ - the number of intact lengths $\geq 0.1$ m.
- $L$ - the length of scanline or borehole along which RQD value is required.

Priest and Hudson, for any arrangement of randomly positioned discontinuities, defined a curve as

$$
f(x) = \lambda e^{-\lambda x} \quad (2-57)
$$
where

\[ f(x) = \text{the frequency of discontinuity spacing } x \]

\[ \lambda = \text{the average number of discontinuities per meter.} \]

This probability density distribution (f(x)) is the derivative of a negative exponential cumulative probability distribution \( P = 1 - e^{-\lambda x} \).

This pattern of Poisson distribution was also followed by Snow (1968) and other authors. Snow used packer method and parallel plate fracture model. Louis and Pernot (1972) determined the hydraulic characteristics from a statistical analysis of the fractures.

\( \lambda \) (the average number of discontinuities) and the index RQD are related by

\[ \text{RQD} = 100e^{-0.1\lambda} (0.1\lambda + 1) \quad (2-58) \]

Priest and Hudson did their experimental work on sandstone, limestone and mudstone horizons. Wallis and King experimented in a porphyritic granite of Precambrian age, intruding a complex series of Precambrian meta-volcanic and meta-elastic sedimentary rocks. They showed their results in histograms Figure (2-15).

Considering the data in Table (2-1) and eq. (2-57) that is

\[ f(x) = \lambda e^{-\lambda x} \quad (2-57) \]

Taking the logarithm of both sides

\[ \ln f(x) = \ln \lambda - \lambda x \quad (2-59) \]

From data (Table 2-1)

\[ \tilde{\lambda} = 4.86 \text{ as corresponding to the parameter } r = 0.94 \text{ suggested by the authors} \]

to be the best fit.

Now

\[ x = \frac{1}{\lambda} \cdot \frac{1}{4.86} = 0.2058 \]
Note 2.4% of data greater than 0.8m spacing

Fig. 2-15: Discontinuity spacing histogram. Sum of all scanlines and drill-core data 3. (After Wallis and King, 1980).
Table 2-1: Summary of Field Data and Negative Exponential Curve Parameters.

<table>
<thead>
<tr>
<th>Source of data</th>
<th>Total length (L (m))</th>
<th>Number of discontinuities (n)</th>
<th>Theoretical curve parameter $\left(\frac{\lambda \cdot n}{L}\right)$</th>
<th>Best fit curve parameters a</th>
<th>-b</th>
<th>r</th>
</tr>
</thead>
<tbody>
<tr>
<td>Drill-core, 1</td>
<td>61.0</td>
<td>143</td>
<td>2.34</td>
<td>1.51</td>
<td>2.30</td>
<td>0.69</td>
</tr>
<tr>
<td>All of site 1</td>
<td>167.0</td>
<td>453</td>
<td>2.71</td>
<td>2.99</td>
<td>2.90</td>
<td>0.95</td>
</tr>
<tr>
<td>Drill-core, 2</td>
<td>31.6</td>
<td>207</td>
<td>6.55</td>
<td>9.78</td>
<td>12.33</td>
<td>0.56</td>
</tr>
<tr>
<td>All of site 2</td>
<td>61.1</td>
<td>400</td>
<td>6.55</td>
<td>6.63</td>
<td>7.82</td>
<td>0.68</td>
</tr>
<tr>
<td>Drill-core, 3</td>
<td>60.5</td>
<td>389</td>
<td>6.43</td>
<td>7.75</td>
<td>8.44</td>
<td>0.91</td>
</tr>
<tr>
<td>All of site 3</td>
<td>163.5</td>
<td>795</td>
<td>4.86</td>
<td>5.15</td>
<td>5.17</td>
<td>0.94</td>
</tr>
<tr>
<td>All Data</td>
<td>391.7</td>
<td>1648</td>
<td>4.21</td>
<td>5.20</td>
<td>5.46</td>
<td>0.89</td>
</tr>
</tbody>
</table>

(After Wallis and King, 1980).
\[
\ln f(x) = \ln \bar{x} - \ln x \\
- \ln 4.86 - (4.86)(0.2058) \\
- 0.5809 \\
f(x) = e^{0.5809} - 1.7876
\]

This point is marked (x) on Figure (2-15).

If spacing is given the symbol (\(\Delta\)), then (\(\Delta\)) equals mean discontinuity spacing (x) - 0.2058. (2-60)

Introducing eq. (2-60) into eq. (2-55), we get

\[
Q = - \left[ \frac{1}{f} \left( \frac{1}{\Delta} \right)^{2n} \right] \frac{n (b_3)^{3 \gamma (\Delta h)}}{\mu} \
\ln \left[ \frac{f_e}{f_w} \right]
\]

Eq. (2-61) represents the fracture model and will be used to assess the value of the aperture (\(b\)) when the second part of the model (storage equation) is completed. As the aperture (\(b\)) is a sensitive parameter, it is worth a while to concentrate on the hydraulic conductivity (K) concept. Chapter 3 will treat this concept and Chapter 4 will treat the storage equation in porous media. There must be a relation between the flow in the porous media and the flow in the fracture system. Both form a composite system, since fractures are distributed throughout the body of rock, hence the concept of anisotropy. This statement is made clear when both fracture-matrix system and discontinuity system will be treated in Chapter 3. A common ground for the fracture equation (eq. 2-61) and the storage equation (Chapter 4) is the fact that both are proportional to the hydraulic head (\(\phi\)), that is, the pressure field is common to both. As mentioned in Sec (1-4), the rate of flow, due to the thermomechanical system, will be in some way related to the rate of flow spatially found; the two phenomena are interrelated through the common pressure field in porous media.
The work will proceed to assess the permeability (hydraulic conductivity) concept.
Fig 2-16:

Diagram showing the sequence scheme of modifications as applied to the fracture equation 2-35, giving in order the equations 2-40, 2-52, 2-54, 2-55 and 2-61 as the final modified fracture equation.

\( f \) = roughness

\( R_b \) = hydraulic radius

\( T \) = tortuosity

\( \Delta \) = fracture spacing

\( \langle b \rangle \) = equivalent aperture

\[ Q = - \left( \frac{1}{f} \right) \left( \frac{1}{\Delta} \right) \frac{2\pi}{\langle b \rangle} \cdot \frac{n\langle b \rangle}{12} \cdot \frac{T}{\nu} (\Delta h) \]  

(eq 2-61)
CHAPTER 3
PERMEABILITY OF THE ANISOTROPIC POROUS MEDIUM

3-1. Introduction

The hydraulic conductivity of the media-sometimes termed coefficient of permeability—depends on both fluid properties and matrix. The hydraulic conductivity $K$ (dim $L^2 T^{-1}$ sec) is given the expression in the isotropic media

$$K = \frac{k' \gamma}{\mu}$$

(3-1)

where

$k'$ (dim $L^2$) is the intrinsic permeability of the medium

$\gamma = \rho g$, $\rho$ is the density of the fluid

$\mu$ - the viscosity and $\frac{\gamma}{\mu}$ represents the effect of fluid properties.

This parameter, in Darcy's law, expresses an interaction between the porous medium and the flowing fluid. It has been shown (page 80, eq. 2-48) that this parameter is related to the shape factor, the porosity, the hydraulic radius and the effective diameter of the material grain. Other factors produce changes in permeability. Bear (1972, p 134-135) shows that it varies with time due to external loads that change the structure and texture of the porous matrix. An example of these external loads is the thermal induced stress due to high level waste. This work is concerned with this last change and the approach to fluid mechanics is limited to the permeability as a function of temperature and pressure.

Causes of Anisotropy

Muskat (1937, Table 8, p 103-113) did extensive work on permeability of sands and showed that permeabilities parallel to the bedding planes are greater than those normal to them and that this is strictly correct when the flow system is two-dimensional with the planes of flow parallel to the bedding planes.
Bear (1972, p 124) showed that, in geological formations, permeability along the planes of deposition is greater than across them. This is due to the orientation of the particles with their longest dimensions parallel to the plane of deposition and hence the medium becomes anisotropic. It is not only the orientation but it is the size of the grains; the larger the grain, the less the degree of grain orientation and the resulting difference in the directional hydraulic conductivity. Layers of different textures are another cause of anisotropy. Moreover, the hydraulic conductivity is not the same in all directions but depends on the direction of the flow.

It is clear then that the flow channels parallel to the flow are differently shaped from those normal to the flow and hence the medium becomes anisotropic. This gives rise to the principal directions of anisotropy, which directions coincide with the horizontal \( k_x \) and the vertical \( k_y \) directions. This leads to the fact that the anisotropic hydraulic conductivity in a saturated medium is only a function of direction but not of the coordinates since the variation of hydraulic conductivity with direction is of the same nature over the whole medium.

Consider the non-steady state storage equation

\[
\frac{d\phi}{dt} = k_x \frac{\partial^2 \phi}{\partial x^2} + k_y \frac{\partial^2 \phi}{\partial y^2}
\]

(3-2)

where \( \phi \) is the hydraulic gradient.

This non-steady state equation could be regarded a momentarily steady-state. Since the medium is saturated, below the water table, and the moisture content is everywhere constant, Childs (1957, p 75), the hydraulic conductivity no longer varies with coordinates but still depends upon the direction. Another proof will be provided when the ratio \( \frac{k_y}{k_x} \) is related to the direction of the directional hydraulic conductivities (eq. 3-25).
Many authors assigned measured values to \((K)\). Muskat (1937, p 103 Table 8) assigned a ratio \(\frac{k_x}{k_y}\) varying from 1 to 42, where \(k_x\) and \(k_y\) are the horizontal and vertical components of \((K)\). Most of his work was on sands. Pratt (1977) gave data for \(k_x\) and \(k_y\) in granite rock. Heard and Page (1982) measured the effective \(K\) in Stripa granite. Marcus (1962) plotted the ratio \(\frac{k_y}{k_x}\) versus \% difference between the two directional permeabilities at different angles. Persons (1966) gave limits to the values of the angle for anisotropy orientation. The last five authors' data will be considered in this dissertation as experimental data.

3-2. Hydraulic Conductivity in Fractured Porous Media

Some work must be done on \(K\)-relations with directions; in the direction of the flow and in the direction of the hydraulic gradient, and to show whether the fractured medium is an anisotropic medium. Finally, an application to these analyses will be performed using the experimental data referred to in the previous section.

Anisotropy has been treated by Vreedenburgh (1936), Dachler (1936, p 135), Ferrandom (1948, 1954), Stevens (1938), Irmay (1951), Scheidegger (1960, p 76), among others.

In the anisotropic media, the hydraulic conductivity is not so simple as given by eq. (3-1). It is a second rank tensor. Darcy's law gives

\[
q_x = k_{xx} \phi_1 + k_{xy} \phi_2
\]

\[
q_y = k_{yx} \phi_1 + k_{yy} \phi_2
\]

and in a matrix form

\[
\overrightarrow{Q} = \begin{bmatrix}
k_{xx} & k_{xy} \\
k_{yx} & k_{yy}
\end{bmatrix}
\begin{bmatrix}
\phi_1 \\
\phi_2
\end{bmatrix}
\]

(3-3)

where \(\phi_1\) and \(\phi_2\) are the hydraulic gradients.
Hence the permeability is a symmetrical tensor which is simply transformed to a diagonal tensor by applying the Kronecker delta (Kreyszig, 1962 p 313)

\[ \delta_{nm} = \begin{cases} 0 & (n \neq m) \\ 1 & (n = m) \end{cases} \]

in which case the determinant in eq. (3-3) must vanish and

\[ (k_{nm} - k\delta_{nm}) = 0 \]

or

\[ k_{nm} = k\delta_{nm} \quad (\delta_{nm} = 1 \text{ for } n = m) \]

hence

\[ \bar{K} = \begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix} \]

(3-4)

where \( \bar{K} \) denotes the permeability diagonal tensor; \( k_x \) and \( k_y \) are the eigenvalues or the principal values.

With the help of the eigenvalues of the diagonal tensor and the transformation into another coordinate system, the components of the symmetrical tensor are obtained. With reference to Figure (3-1) and following the work of Jaeger and Cook (1979, p 11, 13, 14) by similarity to stress tensor and writing \( K^s \) for \( \sigma^s \) and \( \tau^s \), we have

\[ k_{xx} = k_x \cos^2 \theta + k_y \sin^2 \theta \]

\[ = \frac{1}{2}(k_x + k_y) + \frac{1}{2}(k_x - k_y) \cos 2\theta \]

(3-5)

\[ k_{yy} = \frac{1}{2}(k_x + k_y) - \frac{1}{2}(k_x - k_y) \cos2\theta \]

\[ k_{xy} = k_{yx} = -\frac{1}{2}(k_x - k_y) \sin2\theta \]

Eq. (3-5) gives the components of the symmetric tensor eq. (3-3).

Shih Te Yang (1949), using Mohr's circle, was able to derive the relations given by eq. (3-5). The Mohr's circle is usually a graphical method to determine mining
Fig. 3-1. Transformation of the components of the diagonal tensor (eq. 3-4), from $x'$-$y'$ system to $x$-$y$ system making an angle $\theta$ with the $x'$-$y'$ system, is given by eq. (3-5).
forces and stresses. Castillo (1972) was able to do the same transformation from one coordinate system to another.

The four variables in a two-dimensional tensor are thus reduced to two variables in the diagonal tensor; this leads to a great result that the streamline and the hydraulic gradient (potential line) do not coincide but they do along the principal directions of anisotropy. As Castillo (1972) put it: There exist some particular orientation having this property.

Anisotropy affects the hydraulic conductivity from one point to another. The coordinates of any point in the flow suffer an expansion or a shrinkage in order to reduce the anisotropic medium to its equivalent isotropic medium. Muskat (1937, p 226) referred to these changes.

In eq. (3-2), \( x \) and \( y \) could be substituted for \( x' \) and \( y' \) giving

\[
\begin{align*}
x &= x' \sqrt{k_x} \\
y &= y' \sqrt{k_y}
\end{align*}
\]

where \( k_x \) and \( k_y \) are the principal permeabilities in the anisotropic medium. This geometrical transformation distorted the linear dimensions and render the equation isotropic in the \( x' \) and \( y' \) space. Once the solution is obtained, it is transformed back to \( x \) and \( y \) space by computing the hydraulic head \( \phi \) at \( \frac{x}{\sqrt{k_x}} \) and \( \frac{y}{\sqrt{k_y}} \).

The relation between the hydraulic conductivity \( k_e \) for the equivalent isotropic medium and the hydraulic conductivities of the anisotropic mediums \( k_x \) and \( k_y \) is given by

\[
k_e = \sqrt{k_x k_y}
\]  

Eq. (3-7) has been developed through the numerical analysis by Samsiöe (1931) in the form \( k_e = \sqrt{k_h k_v} \). Vreedenburgh (1936) proved it. The following lines show a different way, using Darcy's law:
It is simply a consideration of a volume element having \( \Delta x, \Delta y, \Delta z \) in \( x, y, z \) space and \( \nabla \tilde{x}, \nabla \tilde{y}, \nabla \tilde{z} \) in \( \tilde{x}, \tilde{y}, \tilde{z} \) space. Consider the flow in the direction of \( x \) and apply Darcy's law

\[
q - k_x \frac{\nabla \phi}{\nabla x} \nabla y \nabla z \tag{3-8}
\]

and since the transformation affects only the spatial parameters

\[
q - k_e \frac{\nabla \phi}{\nabla x} \nabla y \nabla z \tag{3-9}
\]

Using eqs. (3-6) and (3-9) along with \( z - \tilde{z} \sqrt{k_z} \)

\[
q - k_e \frac{\nabla \phi}{\nabla x} \left[ \frac{\sqrt{k_x}}{\sqrt{k_y}} \frac{\nabla y}{\sqrt{k_z}} \right]
\]

or

\[
q - k_e \frac{\nabla \phi}{\nabla x} \left[ \frac{\sqrt{k_x}}{\sqrt{k_y}} \frac{\sqrt{k_z}}{\sqrt{k_z}} \right] \nabla y \nabla z \tag{3-10}
\]

comparing (3-10) with (3-8)

\[
k_x - k_e \frac{\sqrt{k_x}}{\sqrt{k_y} \sqrt{k_z}} \quad \text{or} \quad k_e = \frac{k_x k_y k_z}{\sqrt{k_x \sqrt{k_y} \sqrt{k_z}}}
\]

in two-dimensional case

\[
k_e = \frac{\sqrt{k_x k_y}}{\sqrt{k_z}} \tag{3-7}
\]

This important relation relates the isotropic equivalent conductivity \( k_e \) (usually obtained in testing) to the principal conductivities in the anisotropic medium. The above derivation is bending the condition that the transformation affects only the spatial parameters but the flow is not affected. The effects of shrinkage or expansion could be shown by a consideration of Darcy's law. Let \( Q \) be the rate of flow and let \( m \) be the expansion or the shrinkage factor, then the hydraulic conductivity of the original medium divided by a factor \( m \) is equal to the hydraulic conductivity.
conductivity of the expanded or the shrunk medium. Using Darcy's law and let $k$ represent the hydraulic conductivity in the isotropic medium and $I$ the flow direction, then

$$Q = kA \frac{dh}{dl}$$

where $A$ is the surface area normal to the flow. Introducing the factor $m$

$$Q = \frac{k}{m} A m^2 \frac{dh}{dl}$$

where $\frac{k}{m}$ is the hydraulic conductivity of the anisotropic medium and $A m^2$ replaces $A$ in the anisotropic medium and $I$ becomes $lm$. Eliminating $m$, then

$$Q = kA \frac{dh}{dl}$$

provided the hydraulic head remains the same, it follows that the rate of flow is not affected, so is also $k$. It is the spatial parameters that are solely affected.

Henceforth, to avoid confusion, $k_e$ refers to the equivalent hydraulic conductivity in the isotropic medium (eq. 3-7) while $k$ denotes the term permeability or hydraulic conductivity of the media and $k'$ is given to intrinsic permeability (eq. 3-1).

3-3. Directional Permeability

It has been mentioned with reference to the diagonal tensor (eq. 3-4) that the hydraulic gradient and the streamline do not coincide but they do along the principal direction of anisotropy. In other words, the streamlines do not coincide with the normals to the equipotentials, but there exist a particular orientation that permits this coincidence.

3-3-1. The Development of a Permeability Expression

In developing a permeability expression, Parsons (1966) considered a fracture-matrix system in which the fracture and hence the flow along the fracture are perpendicular to the pressure gradient. Figure (3-2) shows his model. He derived the equation
Fig. 3-2: Slabs from ideal reservoirs (After Parsons, 1966).
\[ k_{fr} = k_r \cdot \cos^2 \alpha \]  

(3-11)

where

- \( k_{fr} \) - permeability of fracture-matrix system
- \( k_r \) - permeability of the matrix
- \( a \) - geometry of fracture
- \( \cos \alpha \) - the component of the pressure gradient along the fracture direction.

He expressed also the permeability in the direction of the pressure gradient in the anisotropic medium as

\[ k_p = k_x \cos^2 \theta \cdot k_y \sin^2 \theta \]  

Appendix C eq. (AC-4).  

(3-12)

To keep things straight in conformity with this dissertation terminology: his \( \alpha \) will be \( \lambda \) and his \( \theta \) will be \( \phi \) and his \( \psi \) will be \( \delta \) in our model which expands to include the other directional permeability denoted by \( (k_\delta) \) in the direction of the flow. This last permeability is the most widely used (Marcus, 1962).

Parsons related eqs. (3-11) and (3-12) in some way to prove that both equations are identical, that is, a fracture-matrix system can be replaced by an anisotropic medium. Elaborating on his work in Appendix C, some useful results may be obtained to help to apply the experimental data in evidence of his model and in compliance with other authors' conclusions.

Figure (3-3) after Parsons (1966) shows the anisotropy orientation. The work in this dissertation, based on Pratt et al. (1977) experimental data, Heard and Page (1982) experimental data and the work of Marcus (1962), will be shown (Sec 3-4) in close agreement with the work of Parsons (1966).

The above statement will be clear when experimental data is applied to our model. Moreover, the orientation of the directional permeabilities in the direction of the pressure gradient and in the direction of the flow along the fracture as shown in figure (3-4) after Marcus (1962) will be found in Figure (3-3) and
Fig. 3-3: Angle for anisotropy orientation.
(After Parsons, 1966).
Fig. 3-4: Percentage difference between $k_p$ and $k_\delta$ (After Marcus, 1962).

\[ \frac{K_y}{K_x} = 0.4 \]
table (3-1) after Parsons, based on the experimental ratio \( \frac{k_y}{k_x} \) given by Pratt et al. (1977).

It is of interest to note that, in both systems, (fracture-matrix system and anisotropic system), the flow is proportional to the pressure gradient and hence the pressure fields are identical, a situation in which the superposition of the two systems will cause the flow in each to behave as if the other were not present.

Since the pressure fields are assumed indentical, it is interesting to consider an instantaneous point between fracture and matrix at which the pressure gradient makes an angle (\( \lambda \)) with the direction of fracture and with the flow along the fracture as figure (3-2) shows. With this assumption in mind, the anisotropic transient state could be applied to a fracture-matrix system. This assumption could be shown reasonable by considering the following equations and figure (3-5).

\[
k_p = k_x \cos^2 \phi \cdot k_y \sin^2 \phi \quad (3-12)
\]

Incorporating the fracture angle (\( \lambda \)) between the fracture and the pressure gradient and the flow-fracture angle (\( \delta \)) with x-axis, where \( \phi \) is the angle between the pressure gradient and x-axis as shown in the figure.

Let \( \phi = \lambda - \delta \)  \( (3-13) \)

Substitution in eq. (3-12), we have

\[
k_p = \cos^2 \lambda (k_x \cos^2 \delta \cdot k_y \sin^2 \delta) + \sin^2 \lambda (k_x \sin^2 \delta \cdot k_y \cos^2 \delta) \]
\[
+ 2 \sin \lambda \cos \lambda [(k_x - k_y) \sin \delta \cos \delta] \quad (3-14)
\]

Eq. (3-14) is the equation of fracture-matrix system identical to eq. (3-11). Giving the bracketed terms the constants A', B' and C', hence

\[
k_{fr} = k_r \cdot \cos^2 \lambda \]
\[
- \cos^2 \lambda (A') \cdot \sin^2 \lambda (B') \cdot 2 \sin \lambda \cos \lambda (C') \quad (3-15)
\]
<table>
<thead>
<tr>
<th>If C' is</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>and A' - B' is</td>
<td></td>
<td></td>
</tr>
<tr>
<td>then $\psi$ is</td>
<td>-90° to -45°</td>
<td>-45° to 0°</td>
</tr>
<tr>
<td>in the range</td>
<td></td>
<td></td>
</tr>
<tr>
<td>and the true $\psi$ - $\psi'$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(After Parsons, 1966).
where

\[ k_x \cos^2 \delta \cdot k_y \sin^2 \delta = A' \]

\[ k_y \sin^2 \delta \cdot k_y \cos^2 \delta = B' \]

(3-16)

and

\[ (k_x - k_y) \sin \delta \cos \delta = C' \]

Eq. (3-16) tells that the fracture-rock system depends solely on the variables \( k_x \), \( k_y \) and \( \delta \). Once the specific values of these variables, with respect to a specific rock, are known, then eq. (3-15) could apply since it is made of the two identical eqs. (3-11) and (3-14), using eq. (3-16).

As a priori to the numerical application (Sec. 3-4) based on the experimental data, consider the following values and figure (3-5).

\[ \phi = 56.18^\circ \text{ (calculated), } \delta = 15^\circ \text{ (Marcus 1962)} \]

\[ \lambda = 71.18^\circ \text{ (calculated eq. (3-13)) and } \]

\[ k_x = 0.1 \text{ md, } k_y = 0.04 \text{ md (Pratt et al. 1977) and the result of Parson's model (eq. AC-3, App. C) and the calculated values from eq. (3-16)} \]

\[ A' = 0.0960 \]

\[ B' = 0.0440 \]

\[ C' = 0.0150 \]

We find from eq. (AC-3) Appendix C

\[ \frac{C'}{A' \cdot B'} = 0.2885 \]

and

\[ \frac{\sin \delta \cos \delta}{\cos^2 \delta \cdot \sin^2 \delta} = \frac{\sin 2\delta}{2 \cos 2\delta} = \frac{0.5}{1.7321} = 0.2887 \]

These values are unique with respect to the rock under consideration (granite).

Hence both systems are identical and each could be replaced by the other.
Fig. 3-5: Relationship of angles in anisotropic and fractured media.
We turn now to the other directional hydraulic conductivity in the direction of the flow denoted by \( (k_\delta) \) which when related to \( k_P \) (eq. 3-12) (the hydraulic conductivity in the direction of the pressure gradient) will lead, through directional orientations, to great results (eq. 3-25).

Consider Darcy’s law

\[
V_\phi = -k_\phi \frac{\nabla \phi}{\mu}
\]

Where \( V_\phi \) is the flow velocity, \( k_\phi \) is the permeability, \( \nabla \phi \) is the pressure gradient and \( \mu \) is the viscosity. With reference to figure (3-6)

\[
V_\phi = V \cos(\phi - \delta)
\]

Using trigonometry and the relation \( \frac{V_x}{V} = \cos \delta \) and \( \frac{V_y}{V} = \sin \delta \) and let \( \phi \) coincide with \( \delta \), and using Darcy’s law components \( V_x \) and \( V_y \), it is found

\[
k_\delta = \frac{\cos^2 \delta \cdot \sin^2 \delta}{k_x \cdot k_y}
\]

or

\[
\frac{1}{k_\delta} = \frac{\cos^2 \delta}{k_x} \cdot \frac{\sin^2 \delta}{k_y}
\]

Dachler (1936, p 135) wrote eq. (3-17) as

\[
k_\alpha = \frac{k_{\text{max}} \cdot k_{\text{min}}}{k_{\text{max}} \sin^2 \alpha + k_{\text{min}} \cos^2 \alpha}
\]

where

\[
\alpha = \delta, \text{ } k_{\text{max}} = k_x \text{ and } k_{\text{min}} = k_y
\]

A complete derivation of eq. (3-17) is in Appendix C (eq. AC-5). Now eq. (3-17) could be written

\[
1 - \frac{k_\delta \cos^2 \delta}{k_x} \cdot \frac{k_\delta \sin^2 \delta}{k_y}
\]
Figure 3-6: Flow and pressure gradient directions in anisotropic media.

\[ V_\phi = V \cos (\phi - \delta) \]

\[ K_\delta = \frac{\cos^2 \delta + \sin^2 \delta}{2} \]

\[ = \frac{\cos \delta}{Kx} + \frac{\sin \delta}{Ky} \]

\[ \frac{1}{K_\delta} = \frac{\cos \delta^2}{Kx} + \frac{\sin \delta^2}{Ky} \quad (eq \ 3-17) \]
or

\[ 1 - \frac{x^2}{k_x} - \frac{y^2}{k_y} \] 

(3-18)

where

\[ x = \sqrt{k_5} \cos \delta \quad \text{and} \quad y = \sqrt{k_5} \sin \delta \] 

(3-19)

Comparing eq. (3-18) with the equation of the ellipse

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \]

it is seen that the semi-axes \( a = \sqrt{k_x} \) and \( b = \sqrt{k_y} \) are the principal directions of anisotropy. This ellipse is shown in figure (3-7a).

At this point, it is interesting to work out the transformation that this ellipse would assume if transformed to an isotropic medium. Consider the theory of conformal mapping, Pipes and Harville (1970, p 370) and Kreyszig (1966, p 741), and using the transformation

\[ w = z + \frac{1}{z} \] 

(3-20)

Differentiating

\[ \frac{dw}{dz} = 1 - \frac{1}{z^2} \]

This mapping is conformal except at the points \( z = 1 \) and \( z = -1 \).

From eq. (3-20) \( w = 2 \) and \( w = -2 \).

This leads to \( z = \frac{w}{2} \pm \frac{1}{2} \sqrt{(w + 2)(w - 2)} \)

Hence for any value of \( w \), not equal to 2 or -2, there are two values of \( z \) mapping the \( z \)-plane onto a two-sheeted Riemann surface.

Again consider the analytic function of the complex variable

\[ z = x + iy, \quad \text{then} \quad w(z) = u + iv = z + \frac{1}{z} \] 

(3-21)

where \( u \) and \( v \) are the real and imaginary parts respectively. Eq. (3-21) could be written in the polar form, setting \( z = re^{i\theta} \).
\[ U = (\gamma + \frac{1}{\gamma}) \cos \theta \text{ and } V = (\gamma - \frac{1}{\gamma}) \sin \theta \]

and \( a + b = 2\gamma = \sqrt{kx} + \sqrt{k\gamma} \)

\[ \frac{U^2}{a^2} + \frac{V^2}{b^2} = 1 \text{ where } a = \gamma + \frac{1}{\gamma} \]
\[ b = \gamma - \frac{1}{\gamma} \]

**Fig. 3-7:** Flow in an isotropic medium. (a) The ellipse of direction. (b) The fictitious isotropic circle into which the anisotropic ellipse is transformed.
\[ w(z) = u + iv - re^{i\theta} \cdot \frac{1}{r} e^{-i\theta} \]
\[ = \left[ r + \frac{1}{r} \right] \cos \theta + i \left[ r - \frac{1}{r} \right] \sin \theta \]

Hence

\[ u = \left[ r + \frac{1}{r} \right] \cos \theta \text{ and } v = \left[ r - \frac{1}{r} \right] \sin \theta \]

or

\[
\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1 \quad \text{where } a = r + \frac{1}{r} \text{ and } b = r - \frac{1}{r} \quad (3-22)
\]

where \( a \) and \( b \) are the semi-axes of the ellipse

From eq. (3-22)

\[ a + b = 2r \]

or in terms of eq. (3-18)

\[
\sqrt{k_x} + \sqrt{k_y} = 2r - 2 \sqrt{k_0} \quad (3-23)
\]

This important result (eq. 3-23) shows that the circle \( r = \text{constant} \) is mapped onto the ellipse whose principal axes lie in the \( w \) - and \( v \) - axes and having lengths \( a \) and \( b \) respectively.

Hence the anisotropic system, represented by the ellipse of \( a = \sqrt{k_x} \) and \( b = \sqrt{k_y} \) (eq. 3-18), is transformed into a fictitious isotropic system represented by a circle of radius \( r = \sqrt{k_0} \) as shown in figure (3-7b). Adding to this important result (eq. 3-23) the equations:

\[
x = \sqrt{k_x} \cdot x' \quad y = \sqrt{k_y} \cdot y'
\]

and

\[
x = \sqrt{k_\delta} \cos \delta \quad y = \sqrt{k_\delta} \sin \delta
\]

will help to find \( x, y \) of the anisotropic medium and \( x', y' \) of the isotropic medium, once the experimental values of \( k_x, k_y \) and \( \delta \) are known, since transformation affects only the spatial parameters \( x \) and \( y \) (as shown in the evaluation of eq. (3-7)).
Now the \( r \cdot \sqrt{k_o} \) will help to scale \( k_e \cdot \sqrt{k_x k_y} \) from Heard and Page data shown in figure (3-8) and to give \( k_x \) and \( k_y \) at different temperatures and pressures through the relations.

\[
k_e \cdot \sqrt{k_x k_y} = k_x \sqrt{\frac{k_y}{k_x}} \cdot k_y \sqrt{\frac{k_x}{k_y}} \cdot k_o
\]

(3-24)

It is to be observed that the experimental data (Heard and Page, 1982) is given in terms of \( \frac{k}{k_o} \) and the ratio \( \frac{k_y}{k_x} \) or \( \frac{k_x}{k_y} \) is given by Pratt et al. (1977). Both data are shown in figures (3-8 and 3-4) and eq. (3-24) is expanded to meet the data requirements. When the numerical values are applied (Sec 3-4), it will be shown that: \( r \cdot \sqrt{k_o} \leq 0.2581 \), or, \( k_o \approx 0.0666 \text{ md.} \), which is less than \( k_x \approx 0.1 \text{ md.} \), and greater than \( k_y \approx 0.04 \text{ md.} \). This is in evidence of the conformal mapping already manipulated. The values of \( k_x \) and \( k_y \) are given by Pratt et al. (1977) at standard pressure and temperature.

3-3-2. Relation between \( k_p \) and \( k_s \)

Another important relation relating the orientations of the streamlines and the equipotentials (as stated on page 99), to the hydraulic conductivities in the principal directions of anisotropy could be formulated with reference to figure (3-9) using Darcy’s law:

where

- \( n \) - the normal to the equipotential line \( h \) (the dotted line \( h \) represents the conic section of the ellipse of the anisotropic medium).
- \( v \) - velocity of flow, making an angle \( \delta \) with \( x \)-axis.
- \( \theta \) - the angle between the potential line \( h \) and the \( x \)-axis.

Now consider \( \Delta OCD \) and \( OBC \)
Fig. 3-8: Porosity and permeability in Stripa granite. (a) Calculated porosity versus temperature for Stripa granite. (b) Normalized permeability (calculated) $k/k_0$ versus temperature for Stripa granite. (After Heard and Page, 1982).
Fig. 3-9: The ratio of the principal hydraulic conductivities \( \frac{k_y}{k_x} \) depends solely on the orientation, eq. (3-25).

\[
\tan \delta \tan \phi = \frac{k_y}{k_x} \quad \text{(eq. 3-25)}
\]
In the limit
\[ v_x = v \cos \delta - k_x \frac{\partial h}{\partial x} - k_x \frac{\partial h}{\partial n} \sin \phi \]
also
\[ v_y = v \sin \delta - k_y \frac{\partial h}{\partial y} - k_y \frac{\partial h}{\partial n} \cos \phi \] (A)

Dividing B by A
\[ \frac{v_y}{v_x} = \tan \delta - \frac{k_y}{k_x} \cot \phi \]
or
\[ \tan \delta \tan \phi = \frac{k_y}{k_x} \] (3-25)

This relation shows that the ratio of the principal hydraulic conductivities, depends solely on the orientation. Once the orientation changes, the ratio changes from rock to rock. This ratio is a basic parameter with respect to the specific rock under consideration. Eq. (3-25) says that streamlines and equipotentials are conjugate with respect to the ellipse (eq. 3-18). Once \( k_y, k_x \) and \( \delta \) are experimentally known for a specific rock, \( \phi \) could be found and hence the direction of the fracture (\( \lambda \)) (eq. 3-13).

3-3-3. A Different Look at Anisotropy

Another way of treating anisotropy is by considering any discontinuity whatsoever (fracture, fault, joint, fissure . . . ). This is the concept of stratification and anisotropy, that is, an anisotropic medium is identical to a stratified medium, provided the layers are thin and of a few millimeters in thickness (Childs, 1957, p 55) but have different hydraulic conductivities \( k_1 \) and \( k_2 \). Some relations could be established between \( k_1 \) and \( k_2 \) of the alternating layers, the principal hydraulic
conductivities $k_x$ and $k_y$, and the angles of refraction at the boundary of the two layers separated by a discontinuity.

Consider figure (3-10) where $t_1$ and $t_2$ are the thicknesses of the layers of conductivities $k_1$ and $k_2$, and the angles of refraction are as shown in the figure, and the lines AB and CD are drawn as if there were no refraction, and $k_1 > k_2$. The goal is to establish a relation between refraction and the orientation of the streamline and the equipotential $\delta$ and $\phi$.

From figure (3-10)

$$t_1 \tan \delta_1 + t_2 \tan \delta_2 = (t_1 + t_2) \cot \delta$$

let $t_1 - t_2$, then

$$t_1 \tan \delta_1 + t_1 \tan \delta_2 = 2t_1 \cot \delta$$

and

$$\tan \delta = \frac{1}{\cot \delta} = \frac{2}{\tan \delta_1 \cdot \tan \delta_2}$$

similarly

$$\tan \phi = \frac{1}{\cot \phi} = \frac{2}{\tan \phi_1 \cdot \tan \phi_2}$$

then

$$\tan \delta \tan \phi = \frac{2^2}{(\tan \delta_1 \cdot \tan \delta_2)(\tan \phi_1 \cdot \tan \phi_2)}$$

(3-27)

Childs (1957, p 55) gives $k_y$, $k_n$($i e k_y$, $k_x$) as

$$k_y = \frac{2k_1 k_2}{k_1 + k_2} \quad \text{and} \quad k_x = \frac{k_1 \cdot k_2}{2}$$

(3-28)

These are also the relations given by Terzaghi (1943, p 244) between $k_y$, $k_x$ $k_1$ and $k_2$.

The condition of continuity yields the tangent law

$$\frac{\tan \delta_1}{\cot \phi_1} = \frac{k_1}{k_2} \quad \text{and} \quad \frac{\cot \delta_2}{\cot \phi_2} = \frac{k_1}{k_2}$$

(3-29)
Fig. 3-10: Stratification and anisotropy; an anisotropic medium is identical to a stratified medium, provided the layers are thin.

\[ \tan \delta \tan \phi = k_y \cdot \frac{1}{k_x} = \frac{k_y}{k_x} \quad (\text{eq 3-25}) \]
Now
\[ \delta_1 + \phi_1 = 90^\circ - \delta_2 + \phi_2 \]
\[ \phi_1 = 90 - \delta_1 \text{ and } \phi_2 = 90 - \delta_2 \]

So
\[ \tan \phi_1 = \cot \delta_1 \text{ and } \tan \phi_2 = \cot \delta_2 \]

Using eq. (3-29) in the form
\[ \tan \delta_1 = \frac{k_1}{k_2} \tan \delta_2 \text{ and } \cot \delta_1 = \frac{k_2}{k_1} \cot \delta_2 \]

eq (3-27) becomes
\[
\tan \delta \tan \phi \cdot \frac{4}{(\tan \delta_1 + \tan \delta_2)(\cot \delta_1 + \cot \delta_2)} 
- \frac{4k_1 k_2}{(k_1 + k_2)^2} 
- \frac{2k_1 k_2}{k_1 + k_2} \cdot \frac{2}{k_1 + k_2}
\]
or
\[ \tan \delta \tan \phi = k_y \cdot \frac{1}{k_x} - \frac{k_y}{k_x} \]
\[ (3-25) \]

This is eq. (3-25) as obtained by consideration of Darcy's law and figure (3-9).

An interesting result is obtained from eq. (3-28)
\[ k_x k_y = k_1 k_2 \]
\[ (3-30) \]

and from eqs. (3-29, 3-26), it is found
\[ \tan \delta_1 = \frac{k_1}{k_2} \tan \delta_2 \]
\[ \frac{k_1}{k_2} (2 \cot \delta - \tan \delta_1) \]
or
\[
\tan \delta_1 \left(1 + \frac{k_1}{k_2}\right) = \frac{2k_1}{k_2} \cot \delta
\]
then
\[
\tan \delta_1 = \frac{2k_1}{k_1 \cdot k_2} \cot \delta
\]
and
\[
\tan \delta_2 = \frac{2k_2}{k_1 \cdot k_2} \cot \delta
\]

Since the hydraulic head \( \phi \) is continuous across the boundary, \( \frac{d\phi}{dt} \) is also continuous, hence (by virtue of Darcy's law)
\[
\frac{q^1}{k_1} = \frac{q^2}{k_2} = \frac{\partial \phi}{\partial t}
\]
where \( q^1 \) and \( q^2 \) are the rates of flow and \( k_1 \) and \( k_2 \) are conductivities of layers 1 and 2 respectively. Physically, all water leaving layer 1 must enter layer 2, that is, the rate of flow along the straight streamline (AB) must be the same as the rate of flow along the refracted AOB, if the hydraulic head across the system remains unchanged, hence
\[
\frac{1}{k_x} \cdot \frac{\cos^2 \delta}{k_1} \cdot \frac{\sin^2 \delta}{k_y}
\]
This completes the concept of the anisotropic stratified medium. It considers a discontinuity between two isotropic layers (as a fracture between two surfaces) of thickness \( t_1 \) and \( t_2 \) (\( t_1 \) assumed equal to \( t_2 \) for simplicity) with hydraulic conductivities \( k_1 \) and \( k_2 \). The relationship between \( k_x \) and \( k_y \) and the hydraulic conductivities of the layers is given by eq. (3-28). The same flow will occur in the fictitious anisotropic medium, provided that the hydraulic head at the boundaries is the same in both cases. The directions \( \delta \) and \( \phi \) of the non-refracted flow lines of
figure (3-10) in this fictitious anisotropic medium are given by eq. (3-25) and the directional hydraulic conductivity $k_5$ is given by eq. (3-17).

Hence, equations (3-25, 3-28, 3-29, 3-17) establish all relations required to solve for the unknowns, once the experimental values of $k_x$, $k_y$ and $\delta$ are known for the rock under consideration.

It is of interest to observe that a fracture-matrix system is replaced by an anisotropic system (Parsons model) and a discontinuity system is replaced by an anisotropic system (this dissertation); both cases require the pressure field (hydraulic gradient) to be the same.

There are now two definitions of the directional permeability represented by equations (3-12) and (3-17). Marcus (1962) showed that the difference is a function of the anisotropy of the material and of the directions of the flow and the gradient with respect to the principal permeabilities. He plotted the difference versus the ratio $\frac{k_y}{k_x}$ as shown in figure (3-4) using

$$\text{difference} = \frac{k_x}{k_y} \left(1 - \frac{k_y}{k_x}\right)^2 \sin^2 \delta \cos^2 \delta$$

(3-32)

(see Appendix C eq. (AC-6).

Scheidegger (1960, p 78-79) gave this difference in the form

$$\frac{k_n^-}{k_n^+} = 1 - \frac{(k_1 - k_2)^2}{k_1 k_2} \cos^2 \alpha \sin^2 \alpha$$

where

$$\frac{k_n^-}{k_n^+} = \frac{k_P}{k_5}, k_1, k_2$$

are $k_x$, $k_y$ and $\alpha - \delta$ and he showed that the maximum excess $m$ over 1 of the ratio $\frac{k_n^-}{k_n^+}$ is given by

$$m = \frac{1}{4} \frac{(k_1 - k_2)^2}{k_1 k_2}$$

(3-33)
and if this excess is less than 1, both \( k_n \) \(^{'}\) and \( k_n \) \(^{-}\) are in approximation, freely interchangeable. It is to be observed here that \( k_1 \), \( k_2 \) related to Scheidegger work, are \( k_x \), \( k_y \) and not to be confused with \( k_1 \), \( k_2 \) of the stratified system.

Taking \( \delta = 15^\circ \) (Marcus, 1962, figure 3-4) within the limits of \( k_x \cdot 0.1 \) md. and \( k_y \cdot 0.04 \) md. (Pratt et al. 1977 for granite rock), the excess over 1, that is, \( m \) is found to be 0.225 \( \ll 1 \) whereas eq. (3-32) gives 0.0563 \( \ll 0.225 \) as a maximum excess over 1. Hence these numerical data prove Scheidegger's results (eq. 3-33) and 5.63\% is shown in figure (3-4) at \( \frac{k_y}{k_x} = 0.4 \).

The analysis of the theory of the fluid flow as represented either by a fracture-matrix system or by the stratified layers has been so far carried out through the numerous equations with one sole purpose: to evaluate the hydraulic conductivity (k) as accurately as possible, since the aperture of the fracture is very sensitive to this parameter. The following section (Sec. 3-4) is solely conducted to apply the experimental data of the several authors mentioned on page (100). Some other relations will be mentioned in course of the numerical application, in case they are required.

3-4. Numerical Application of the Experimental Data

The numerical application in using the rare data available in literature, with a view to ensure that the experimental work as applied to this model, is in close proximity and in evidence of the analysis worked out in the previous pages.

Using figure (3-4) after Marcus (1962) where \( \delta = 15^\circ \) and \( \alpha - \phi = 75^\circ \) within the limits:

\[
\frac{k_y}{k_x} = 0.4, k_y = 0.04 \text{ m.d. and } k_x = 0.1 \text{ m.d.} \quad \text{(Pratt et al. 1977),}
\]

we get from eqs. AC-1, AC-3 (Appendix C) with \( \delta = 15^\circ \)

\[
0.1(\cos^2 15^\circ) + 0.04(\sin^2 15^\circ) = A' = 0.0960
\]
\[ 0.1(\sin^2 15^\circ) \cdot 0.04(\cos^2 15^\circ) - B' - 0.0440 \]

and

\[ (0.1 - 0.04)\sin 15^\circ \cos 15^\circ - C' - 0.0150 \]

and

\[ \frac{C'}{A' - B'} = \frac{0.0150}{0.0960 - 0.0440} \approx 0.2885 \]

and

\[ \frac{\sin 2\delta}{2\cos 2\delta} = \frac{\sin(2\times15^\circ)}{2\cos(2\times15^\circ)} \approx 0.2887 \]

With \( \delta = 75^\circ \)

\[ \frac{C'}{A' - B'} \approx 0.2885 \]

and

\[ \frac{\sin 2\delta}{2\cos 2\delta} \approx 0.2887 \]

Now if \( C' \) is positive and \( (A' - B') \) is positive then

\[ \delta(0^\circ - 45^\circ) = 15^\circ \]

if \( C' \) is positive and \( (A' - B') \) is negative then

\[ \delta(45^\circ - 90^\circ) = 75^\circ \]

Figure (3-3) and table (3-1) after Parsons (1966) show these results.

It is observed that the values of \( A', B' \) and \( C' \) are the components of the symmetrical tensor (eq. 3-3) whose eigenvalues are 0.1 and 0.04.

Consider figures (3-7a and -b) which show the transformation from the anisotropic ellipse to the isotropic circle and eq. (3-23)

\[ \sqrt{k_x} \cdot \sqrt{k_y} \cdot 2r \]

with

\[ k_x = 0.1 \text{ and } k_y = 0.04 \]
then
\[ r = \frac{0.3162 \cdot 0.2}{2} - 0.2581 \cdot \sqrt{k_0} \]
where \( k_0 \) is the permeability of the isotropic medium and

\[
\begin{align*}
\frac{r}{a} &= \frac{\sqrt{k_0}}{\sqrt{k_x}} - \frac{0.2581}{0.3162} - 0.8163 \\
\frac{r}{b} &= \frac{\sqrt{k_0}}{\sqrt{k_y}} - \frac{0.2581}{0.2} - 1.2905
\end{align*}
\]
where \( a \) and \( b \) are the semiaxes of the ellipse. It follows that the isotropic circle has
\[
\begin{align*}
r &= \sqrt{k_0} \cdot 0.8163 \sqrt{k_x} \\
&= 1.2905 \sqrt{k_y}
\end{align*}
\]
Hence
\[
k_0 = (0.2581)^2 - 0.0666156 - k_x - k_y \text{ in the isotropic medium. Applying the}
\]
important relation
\[
\tan \delta \tan \phi = \frac{k_y}{k_x} - \frac{k_0}{k_0} = 1
\]
If \( \delta = 15^\circ \), then \( \phi = 75^\circ \) and \( \alpha + \phi = 90^\circ \) as shown in Figure (3-7b). This result means streamlines and potential lines are orthogonal.

In the case of anisotropy (Pratt et al. 1977)
\[
\tan \delta \tan \phi = \frac{k_y}{k_x} - \frac{0.04}{0.1} = 0.4
\]
if \( \delta = 15^\circ \) then \( \phi = 56.18^\circ \) and \( \delta + \phi = 71.18^\circ \)
Hence non-orthogonality as shown in figure (3-7a).

Consider now equations (3-13, 3-12 and 3-14) and figure (3-5)
\[
\begin{align*}
\phi &= \lambda - \delta \\
\lambda &= 56.18^\circ + 15^\circ + 71.18^\circ
\end{align*}
\]
or \( \lambda = 56.18^\circ + 75^\circ = 131.18^\circ \)

\( \delta = 15^\circ \) or \( 75^\circ \) are the values given in figure (3-4).

With these values of \( \delta, \phi \) and \( \lambda \) at hand eqs. (3-12) and (3-14) would give the same numerical values (that is, \( k_p = 0.0586 \)) as could be shown by substitution whether \( \lambda = 71.18^\circ \) or \( 131.18^\circ \). This means that incorporating fracture in the fracture-matrix system is identical to an anisotropic medium.

Also the stratified system is identical to an anisotropic system as the following relations hold

\[
\frac{1}{k_\delta} = \frac{\cos^2 \delta}{k_X} \cdot \frac{\sin^2 \delta}{k_Y} \tag{3-17}
\]

\[
x = \sqrt{k_\delta \cos \delta}, \quad y = \sqrt{k_\delta \sin \delta} \tag{3-19}
\]

\[
\tan \delta \tan \phi = \frac{k_Y}{k_X} \tag{3-25}
\]

\[
k_y = \frac{2k_1k_2}{k_1 + k_2}, \quad k_x = \frac{k_1 + k_2}{2} \tag{3-28}
\]

\[
k_yk_x = k_1k_2 \tag{3-30}
\]

\[
\tan \delta_1 = \frac{2k_1}{k_1 + k_2} \cot \delta \tag{3-31}
\]

\[
\tan \delta_2 = \frac{2k_2}{k_1 + k_2} \cot \delta
\]

and

\[
\frac{\tan \delta_1}{\tan \delta_2} = \frac{k_1}{k_2} \tag{3-29}
\]

In this respect, with reference to the values of \( k_X \) and \( k_Y \) (Pratt et al. 1977)

\[
k_e = \sqrt{k_Xk_Y} = 0.0632 \text{ m.d.}
\]

where

\[
k_X = 10^{-4} \text{ darcy} \cdot 9.613 \times 10^{-8} \text{ cm sec} \cdot 9.613 \times 10^{-10} \text{ m sec}
\] and

\[
k_Y = 0.40 \times 10^{-4} \text{ darcy} \cdot 3.8452 \times 10^{-8} \text{ cm sec} \cdot 3.8452 \times 10^{-10} \text{ m sec}
\]
Neuzil (1986) in his review of groundwater flow in low-permeability environments, referred on page 1165 to the work of Olsen (1966) and Olsen et al. (1985) on permeability giving a hydraulic conductivity (k) in the range $10^{-7}$ to $10^{-10}$ m/sec. He also referred to the work of Sangal et al. (1971, 1972) on page (1166) on Precambrian chert, giving conductivities as small as $10^{-10}$ m/sec. These values are in close agreement with the experimental values of Pratt et al. (1977).

The work of Brace (1980) gave for crystalline rocks, in situ, k in the range 1 μd to 100 m.d. (that is, 0.001 m.d. to 100 m.d.). So the Pratt et al. values, in terms of $k_e$ - $\sqrt{k_x k_y}$ - 0.0632, lie between these limits.

Using now the experimental values of Pratt et al. (1977), that is, $k_x$ - 0.1 m.d. and $k_y$ - 0.04 m.d. and $\delta$ - 15° (Marcus, 1962) and the ratio $\frac{k_y}{k_x}$ - 0.4 with respect to granite rock (every rock has its specific ratio), all parameters in the above equations could be calculated and when they are compared to Heard and Page (1982) experimental work on Stripa granite, it is found they are in close proximity showing that the stratified system is identical to an anisotropic system.

Eq. (3-25) with $\delta$ - 15° and $\frac{k_y}{k_x}$ - 0.4 gives

$\phi$ - 56.18°

Solution of eq. (3-28) with $k_x$ - 0.1, $k_y$ - 0.04 and using the quadratic equation, gives

$k_1$ - 0.02254 or 0.17746

$k_2$ - 0.17746 or 0.02254

Since $k_1 > k_2$ (figure 3-10), the values $k_1$ - 0.17746 and $k_2$ - 0.02254 are chosen and eq. (3-30) holds. With $\delta$ - 15° and the values of $k_1$ and $k_2$, eq. (3-31) gives
\[ \delta_1 = 81.4156^\circ, \delta_2 = 40.0206^\circ \text{ satisfying the tangent law eq. (3-29)}. \]

Again with \( \delta = 15^\circ \), eq. (3-17) and eq. (3-19) give

\[ k_5 = 0.0909 \]

and

\[ x = \sqrt{k_5 \cos \delta - 0.29112} \rightarrow x^2 = 0.0848 \]
\[ y = \sqrt{k_5 \sin \delta - 0.0780} \rightarrow y^2 = 0.0061 \]

and eq. (3-18, the ellipse) is satisfied.

We turn now to relate this system to Heard and Page (1982) experimental work on Stripa granite as given in figure (3-8), and to show that this work falls in line with other authors' data.

Starting with the basic data, specific for granite rock: \( k_x = 0.1 \text{ m.d.} \) and \( k_y = 0.04 \text{ m.d.} \) (Pratt et al. 1977), it is found that the ratio \( \frac{k_y}{k_x} = 0.4 \) as shown in figure (3-4) and eq. (3-7) gives

\[ k_e = \sqrt{k_x k_y} = \sqrt{k_1 k_2} = \sqrt{0.1} \times 0.04 = \sqrt{0.17746} \times 0.02254 = 0.0632456 \]

where \( k_e \) is the hydraulic conductivity of the equivalent isotropic medium, \( k_x \) and \( k_y \) are the principal hydraulic conductivities of the anisotropic medium, and \( k_1, k_2 \) are the hydraulic conductivities of the stratified layers separated by a discontinuity.

It has been proved that the ratio \( \frac{k_y}{k_x} \) is a function of direction only as shown by eq. (3-25): \( \tan \delta \tan \phi = \frac{k_y}{k_x} \) and eq. (3-7): \( k_e = \sqrt{k_x k_y} \) could be written

\[ k_e = \sqrt{k_x k_y} = k_x \sqrt{\frac{k_y}{k_x}} = k_x \sqrt{0.4} \]
\[ = k_y \sqrt{\frac{k_x}{k_y}} = k_y \sqrt{2.5} \]

(3-34)
These relations show that as $k_e$ varies due to temperature and pressure, $k_x$ and $k_y$ vary but the ratio remains constant. The calculations will show that this is correct (Table 3-2).

Heard and Page gave their data in terms of $\frac{k}{k_o}$ where $k - k_e$ in this dissertation and $k_o$ is found on page (119) to be $(0.2581)^2$ at STP as a result of applying the concept of conformal mapping.

Scaling from figure (3-8) the values of $\frac{k}{k_o}$ and $k_o - (0.2581)^2$ and using

$$k_e = \frac{k_e}{k_o} k_o$$

at the assigned temperatures and pressures, it is found as a sample of calculations:

Getting the values of $k_x$, $k_y$ to 10 decimal places for better accuracy,

$k_e$ (at $T = 19^\circ C$ and 5.9 MPa)

$$k_e = \frac{k_e}{k_o} k_o - (0.95) (0.2581)^2$$

and

$$\frac{k_e}{\sqrt{0.4}} - k_x - \frac{(0.95)(0.2581)^2}{\sqrt{0.4}} - 0.1000621013 \quad \text{(eq. 3.34)}$$

and

$$\frac{k_e}{\sqrt{2.5}} - k_y - \frac{(0.95)(0.2581)^2}{\sqrt{2.5}} - 0.0400248405 \quad \text{(eq. 3.34)}$$

and the ratio $\left(\frac{k_y}{k_x} - 0.4000000011\right)$ remains constant.

Next

$k_e$ (at $T = 100^\circ C$ and $P = 13.8$ MPa)

$$k_e = (0.5)(0.2581)^2$$

$$\frac{k_e}{\sqrt{0.4}} - k_x - \frac{(0.5)(0.2581)^2}{\sqrt{0.4}} - 0.0526642638$$
\[
\frac{k_e}{\sqrt{2.5}} = k_y \cdot \frac{(0.5)(0.2581)^2}{\sqrt{2.5}} - 0.0210657055
\]

and

\[
k_y \cdot k_x = 0.4000000063
\]

Table (3-2) shows the result of these calculations together with the long sought values of \(k_x\) and \(k_y\) at different temperatures and pressures as a result of the induced thermal stress.

There are two sources of error in this table:

1. Pratt et al. (1977) gave the values of \(k_x\) and \(k_y\) as

\[k_x = 100 \pm 10 \mu d\] and \[k_y = 40 \pm 4 \mu d\]

2. Reading from figure (3-8) is a second source of error. This error, as table (3-2) shows the values of \(\frac{k_y}{k_x}\), is limited to the 9th and 10th decimal places; a result of reading and could be neglected.

Hence the ratio \(\frac{k_y}{k_x}\) is a function of orientation only in confirmation of eq. (3-25). For a specific rock there is a specific \(\delta\) and \(\phi\) with a specific constant ratio. The calculations in table (3-2) show this ratio a constant.

As a result of the change in the values of \(k_x\) and \(k_y\) due to temperature and pressure changes, \(k_\delta\) (the directional permeability in the direction of the streamline) and the spatial parameters \(x\) and \(y\) change but \(\bar{x}\) and \(\bar{y}\) suffer no change.

As a sample of calculation, consider the equations

\[x - \bar{x} \sqrt{k_x} \text{ and } y - \bar{y} \sqrt{k_y}\] (3-6)

and

\[\frac{1}{k_\delta} = \cos^2 \delta \cdot \frac{\sin^2 \delta}{k_x} \cdot \frac{\sin^2 \delta}{k_y}\] (3-17)
Table 3-2: The Principal Hydraulic Conductivities $k_x$, $k_y$ from the Experimental Values of $k_e$.

<table>
<thead>
<tr>
<th>$T^\circ C$</th>
<th>$P$ MPa</th>
<th>$k_e$ $^\dagger$</th>
<th>$\left[\frac{k_e}{k_o}\right] k_o$</th>
<th>$k_x$ $-\frac{k_e}{\sqrt{0.4}}$</th>
<th>$k_y$ $-\frac{k_e}{\sqrt{2.5}}$</th>
<th>Ratio $\frac{k_y}{k_x}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>19</td>
<td>5.9</td>
<td>(0.95)(0.2581)$^2$</td>
<td>0.1000621013</td>
<td>0.0400248405</td>
<td>0.4000000011</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>13.8</td>
<td>(0.5)(0.2581)$^2$</td>
<td>0.0526642638</td>
<td>0.0210657055</td>
<td>0.4000000063</td>
<td></td>
</tr>
<tr>
<td>200</td>
<td>27.6</td>
<td>(0.3)(0.2581)$^2$</td>
<td>0.0315985583</td>
<td>0.0126394233</td>
<td>0.4000000038</td>
<td></td>
</tr>
<tr>
<td>300</td>
<td>41.4</td>
<td>(0.26)(0.2581)$^2$</td>
<td>0.0273854172</td>
<td>0.010954167</td>
<td>0.4000000073</td>
<td></td>
</tr>
<tr>
<td>350</td>
<td>55.2</td>
<td>(0.25)(0.2581)$^2$</td>
<td>0.0263321319</td>
<td>0.0105328528</td>
<td>0.3999999848</td>
<td></td>
</tr>
</tbody>
</table>

$^\dagger$ It is observed from Figure (3-8) that $k_e$ increases with temperature at constant pressure. The figure and the calculations show that $k_e$ decreases as a function of both temperature and pressure. This is physically sound as temperature enhances the fracture but pressure narrows it. Each curve gives a point at the assigned temperature and pressure.

$k - \frac{k_e}{k_o}$ is obtained from Figure (3-8) after Heard and Page (1982).
and

\[ x = J_x \cos \delta; \quad y = J_x \sin \delta \]  \hspace{1cm} (3-19)

From eq. (3-17) and table (3-2), using \( k_x \) and \( k_y \) at \( T = 190 \, ^\circ \text{C} \) and \( P = 5.9 \, \text{MPa} \)

\[
\frac{1}{k_\delta} = \frac{\cos^2 15^\circ}{(0.95)(0.2581)^2} \cdot \frac{\sin^2 15^\circ}{(0.95)(0.2581)^2}
\]

or

\[ k_\delta = 0.0909257916 \]

\[ x = \sqrt{k_\delta} \cos \delta = 0.2912643446 \]

\[ y = \sqrt{k_\delta} \sin \delta = 0.0780440457 \]

and from eq. (3-6)

\[ \bar{x} = \frac{x}{\sqrt{k_x}} \quad \text{and} \quad \bar{y} = \frac{y}{\sqrt{k_y}} \]

\[ \bar{x} = 0.9207728699; \quad \bar{y} = 0.3900991214 \]

where \( x \) and \( y \) are the spatial coordinates in the anisotropic medium and \( \bar{x} \) and \( \bar{y} \) are the spatial coordinates in the isotropic medium, as stated on page (118).

A summary of these calculations are listed in table (3-3). It will be useful when the solution of the storage equation in \( x \) and \( y \), is obtained in the fictitious isotropic medium and transformation is required to get the solution in the true anisotropic medium.

The calculations, shown in table (3-3), show the following expected facts as it must be in agreement with the theory and the physics of the problem.

(1) As shown on page (103), the anisotropy affects the hydraulic conductivity and the coordinates suffer a distortion in order to reduce the anisotropic medium to its equivalent isotropic medium. The parameters \( x \) and \( y \), in table (3-3) show this distortion.
Table 3.3. Changes in the directional permeability $k_6$ and in the special coordinates $x$, $y$, $\bar{x}$, $\bar{y}$ due to changes in $T$ and $P$ at an angle $\delta = 15^\circ$.

<table>
<thead>
<tr>
<th>$T$/°C</th>
<th>$P$/MPa</th>
<th>$k_x$</th>
<th>$k_y$</th>
<th>$\bar{x}$</th>
<th>$\bar{y}$</th>
<th>$k_x$</th>
<th>$k_y$</th>
<th>$x - v$</th>
<th>$y - v$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>1.9</td>
<td>0.62022517615</td>
<td>0.7912545446</td>
<td>0.0260454657</td>
<td>0.1000321013</td>
<td>0.6400454657</td>
<td>0.2207723659</td>
<td>0.2200051214</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>13.3</td>
<td>0.4765055790</td>
<td>0.2112053628</td>
<td>0.0556191013</td>
<td>0.0525642538</td>
<td>0.2165570559</td>
<td>0.2207723648</td>
<td>0.2209031197</td>
<td></td>
</tr>
<tr>
<td>200</td>
<td>27.6</td>
<td>0.02571242976</td>
<td>0.1657664317</td>
<td>0.0435564576</td>
<td>0.0315965583</td>
<td>0.2165570559</td>
<td>0.2207723673</td>
<td>0.2209031197</td>
<td></td>
</tr>
<tr>
<td>300</td>
<td>41.4</td>
<td>0.025521485403</td>
<td>0.1623066703</td>
<td>0.0434999573</td>
<td>0.0273941722</td>
<td>0.2165570559</td>
<td>0.2207723769</td>
<td>0.2209031199</td>
<td></td>
</tr>
<tr>
<td>500</td>
<td>55.2</td>
<td>0.023278399</td>
<td>0.1494154562</td>
<td>0.0400057308</td>
<td>0.0263221319</td>
<td>0.2165570559</td>
<td>0.2207723649</td>
<td>0.2209031194</td>
<td></td>
</tr>
</tbody>
</table>
(2) The special parameters $x$ and $y$ of the equivalent isotropic medium remain virtually constant.

(3) The directional permeability ($k_\theta$) shows a decrease as a result of the thermal induced stress, that is, it varies inversely with the increase in temperature and pressure. Hence, the theory is physically verified.

The work will proceed to Chapter 4 to assess the storage equation in an anisotropic medium associated with the rock deformability, making use of the fact that the flow is proportional to the pressure gradient since the pressure fields are identical, whether there is a stratified medium system or a fracture-matrix system. Both have been proved to be identical to an anisotropic medium as a result of the analysis of these concepts.
CHAPTER 4

A DIFFERENT LOOK AT THE FRACTURE FLOW PROBLEM

4-1 Introduction

In Chapter 2, the mathematical solution of the fractured media, limited to the parallel plate model (sec. 2-4-1, figures 2-7a, 2-7b), has been modified, through various variables, to approach the natural media, that is, the conventional model approach under the influence of the porous media theory (Darcy 1856) is evaluated in terms of the behavior of the flow as related to the exact function of the fractured medium whatever be the kind of porosity present. This is a meaningful approach, since the behavior of the flow is described by Boulton (1963) as a transition curve connecting the early time-drawdown and the late time-drawdown curves. The early time-drawdown is described by Jenkins and Prentice (1982) as linear in the vicinity of the test well whereas the late time-drawdown curve depicts the curvilinear curve as worked out by Theis (1935). Hence there is a basic two flow relationship: low-permeability or slow flow in the matrix and high-permeability or fast flow in fracture. This two flow relationship is illustrated in figure (4-15). This figure shows the intersections of the Boulton curves and the Theis curves. These intersections are depicted by Parsons (1966) as instantaneous points.

This different look at the fracture flow problem, under the influence of the induced thermal stress and the control of the pressure gradient, will render the model more meaningful provided that the anisotropy of the medium is accounted for and the field experimental observations fit properly in the model. The instantaneous point could be defined as a transition point from slow flow to fast flow at which the pressure gradient is same at the specific temperature and the specific pressure.
4.2 A Geological Look at the Model

Natural phenomena affect the saturated crystalline rocks in some way or another. Tectonic activities are a major factor. Thermal and chemical reactions lead to the dissolution of the rock. As a result of these factors, the water-bearing capacity of crystalline rocks depends upon the degree of interconnections of the fractures, depth, size and the degree of weathering affecting the rocks.

The above statements can be supported by the work of many authors. Marsily (1935) in his tests on granite in Massif Central in Central France, introduced two terms: regionalized variables and percolation threshold. He described fracture properties as regionalized variables which can be estimated from local measurements, that is, "fracturation" is a regionalized phenomenon as shown by its imbedded structures characteristic of different classes of features (faults, fractures, fissures, joints, cracks, microcracks, etc.). Stochastic analysis of these properties is meaningful as ensemble averages and not as space averages. Hence the question of the representative elementary volume (REV) or of the scale is variable especially when experimental work is made in a poorly connected network. So an increase of the size of the REV doesn't represent the medium any better. Connectivity is then the right answer to replace the general REV by a restricted REV to the location of interest using aerial photographs and geophysics. Then the connectivity of fractures is defined and their properties are obtained by hydraulic tests.

The term percolation threshold defines a number (1.5-3.0) above which flow is possible and below which fractures are not connected. Above the percolation threshold, large scale in situ hydraulic tests can be used to characterize the global behavior (Marsily 1985). In this respect, it is observed, as a result of the above, that tests are approached in different ways: In situ measurements, pumping tests, packer tests, tracer tests, slug tests and in situ inferred permeability from various
large scale phenomena. The latter approach includes Heard and Page (1982) and Neretnieks (1985). Both works are in Stripa Granite at a depth ≈ 350m and both give the same permeability measurements.

The same reasoning, parallel to Marsily (1985), is given by Long et al. (1985). They showed that interconnection of a fracture network is proportional to a factor Z. As Z increases, the size and number of fracture clusters increase and that on the infinite scale of observation, systems with $Z > Z^*$ are always connected. Here $Z^*$ corresponds to the percolation threshold of Marsily (1985). They also pointed out that the magnitude of the average permeability will only vary by about one order of magnitude due to interconnection effects. Hence connectivity concept dominates what is traditionally called REV or scale effect.

In the light of these findings, a common feature of inconsistency where permeability varies by one or more orders of magnitude at a particular site could be explained. Witherspoon et al. (1983) showed on a 33-m length of drift in Stripa mine that the average permeability is of the order of $10^{-11} \text{m sec}^{-1}$, an order of magnitude less than that of Heard and Page $\left(10^{-10} \text{m sec}^{-1}\right)$.

Landström and Stille (1978) measured permeability in a large scale test and gave the value of $0.4 \times 10^{-10} \text{m sec}^{-1}$ at a temperature $10^0 \text{C}$ and a value of $0.2 \times 10^{-10} \text{m sec}^{-1}$ at $36^0 \text{C}$.

Brace (1980, 1984) compiled permeability from various sources where permeability varies by over four orders of magnitude at a particular site. Figure (4-1) shows permeability values as a function of scale where variation is clear to observe.

So connectivity of the fracture network, that is the degree of intersections of the fractures, is the only phenomenon that can cast light on this discrepancy.
Fig. 4-1: Summary of permeability values as a function of scale. The two left-hand boxes refer to drill hole measurements, and the two right-hand boxes refer to crustal values. Shale values are shown dashed. Nearly all reported values fall within the bounds shown. (After Brace, 1984).
Neretnieks (1985) observed in the Stripa mine that water flow is unevenly distributed over the 100m-long tunnel sections (a monitored area covers more than 800 m²) and a third of the water flow takes place in about 2% of the covered area. In about 70% of the area, no water flow has been found to take place. Figure (4-2) shows the flow rate into a 3-D drift at Stripa before drilling the injection holes. This figure reveals the connectivity concept as suggested by Marsily (1985) in his tests on granite, and explains why permeability measurements are different. The vertical lines are proportional to the flow that depends on the degree of interconnection of fractures, otherwise there is no flow where connectivity is poor. Neretnieks observed fracture frequencies varying between 1 and 4 fractures per meter. Only a fraction of the fractures are conducting water. Hence the scale or REV depends on the individual fractures and the fracture clusters. Neretnieks used aerial photographs and geophysical measurements to locate fracture zones over areas on the scale of tens of kilometers, and rock blocks between these fracture zones of several cubic kilometers could be found in which a repository could be sited. Figure (4-3) shows such sites.

In support of Marsily (1985) and Neretnieks (1985), two more proofs could be furnished to give a meaningful definition to REV. Pratt et al. (1972), working on Diorite rock, plotted stress vs. length, (figure (4-4)) and showed, using different volumes of rock, that maximum stress decreases by a factor of 10 as specimen size increases and asymptotically approaches a constant value for specimens 3 ft. in length and greater. This length could then be considered a REV for this rock at the specified location or better to say, this is the value of the regionalized variable REV at this location.
Fig. 4-2: Water flow rate into 3-D drift at Stripa before drilling the injection holes. (After Neretnieks, 1985).
Fig. 4-3: Location of fracture zones at 600 m depth in Gidea. Available area for possible repository at 500 and 600 m depth is indicated. (After Neretnieks, 1985).
EFFECT OF SPECIMEN SIZE ON UNJOINTED DIORITE

Fig. 4-4: Maximum stress vs. specimen length—Cedar City quartz diorite.
(After Pratt et al., 1972).

- In situ—this study
- Laboratory—this study
\( \Delta \) Brown and Swanson (9)
\( + \) Walsh et al. (10)
A still better definition is given by Long et al. (1982) and figure (4-5). They defined REV as the volume at which the parameter of interest first ceases to vary. With respect to permeability, the REV of a medium can be sought by measuring the average permeability of increasing volumes of rock until the value does not change significantly.

Among other authors, Heuze (1980) illustrated the same trend with other properties of rock (as strength). The trend suggests that the size effect nearly disappears when the volume reaches 1m³. Figure (4-6) illustrates this concept.

In summary, the REV with respect to permeability as well with other properties (as stress and strength) is defined as a regionalized variable. It is a specific parameter of a specific value at the location of interest.

Would the concept of connectivity and REV as a regionalized variable apply to the data used by this dissertation (Data from Heard and Page 1982)? The answer is yes. The temperature from HLW induces stress. This stress affects the fracture (or the clusters of fractures) and hence the permeability. Plotting pressure vs. permeability produces figure (4-7). Three observations could be made from this figure:

1. The permeability decreases as pressure increases.
2. At the higher pressure (55.2 MPa), the permeability starts to stabilize as the fracture closes.
3. The increases are by a factor of \( \frac{55.2}{9.36} \approx 10 \) as permeability decreases until the latter asymptotically approaches a constant value for a pressure \( \geq 55.2 \) MPa and greater. Both pressure and permeability stabilize, as the fracture closes and becomes hydraulically indistinguishable from the rock mass. This case also has been illustrated by Kranz et al. (1979) as shown in figure (4-8) while working on jointed Barre granite. figure (4-9) shows the work of many authors.
Fig. 4-5: Statistical definition of a representative elementary volume (REV). (After Long et al., 1982).
Fig. 4-6: Effect of specimen size on measured uniaxial compressive strength of coal and iron ore. (After Heuze, 1979).
Fig. 4-7: Permeability vs. pressure (data scaled from fig. (3-8), Heard and Page, 1982).
Fig. 4-8: Permeability at various pressures of jointed Barre granite with joint surface prepared with 0.3 μ aluminia polish. (After Kranz et al., 1979).

\[ P_c - P_f \] (bars)

PERMEABILITY (microdarcies)

- 0.3μ polish
- \[ P_c \] (bars)
  - 372
  - 690
  - 1381

\( P_c = \) confining pressure
\( P_f = \) internal fluid pressure
Fig. 4-9: Variation of hydraulic conductivity in a fracture with increasing stress for three different-size rock samples. Results for the 0.15-m and 0.95-m specimens are with radial divergent flow. Results for in situ specimen are with linear flow. (After LBL 8571, 1979).
(Witherspoon et al. 1979). Each curve approaches a zero slope at some stress (for different rocks). This suggests a limiting aperture in each case and hence a stabilized permeability at a definite stress.

If we consider permeability parallel size figure (4-4), then we have a parallel situation: the factor \( \frac{55.2}{5.9} \approx 10 \) (figure 4-7) parallel stress (figure 4-4) with the factor \( \frac{10000\text{Psi}}{1000\text{Psi}} = 10 \); both size (figure 4-4) and permeability (figure 4-7) stabilize when they approach a constant value, some important conclusions could be inferred from this situation:

1. At a specific location in a specific rock, there is a specific REV. There is no general specific scale that could be applied and connectivity concept dominates as stated previously.

2. Within the limits of the specific REV, fracture permeability increases as the size of fracture increases.

3. As the fracture closes, the flow transforms from fast flow to slow flow (an alternative approach, sec. (4-3)).

Neretnieks (1985) plotted his data (figure (4-10)) correlating hydraulic conductivity and depth for the rock mass. The hydraulic conductivity of the fracture zone was about an order of magnitude higher than that of the rock mass. With reference to this figure and at a depth between 300m and 400m, and considering the fracture permeability an order higher than that of the rock mass, we find the data of Heard and Page (1982) fits properly, that is, between \( 10^{-10} \) and \( 10^{-9} \text{ m sec}^{-1} \) regardless of the different techniques used by both authors. Both data are at Stripa mine with 3-year difference in time between their experimental works.

In Sec (4-5), published data is applied to the model. This will lead to deduce the anisotropic parameters as shown in table (3-3) in confirmation of the
Fig. 4-10: Correlation between hydraulic conductivity and depth for the rock mass at Kamlunge. (After Neretnieks, 1985).
mathematical analysis in Chapter (3). Thus all analysis is a unified work whatever the source of data is.

Geology of Stripa Granite

LBL 7052 (Landström and Stille, 1978) describes Stripa granite as a rock made of mineral components: grey-white quartz, red-grey potassium felspar and mica in the form of biotite and chlorite. Fractures in granite form different families (same orientation in space). The chlorite coating gives granite a degree of anisotropy. These fracture families are nearly horizontal with varying fracture widths and filling materials. This probably means that the water-paths through the rock are irregular and complicated. This description supports the modifications already applied to fractures in chapter 2.

4·3 Fast Flow and Slow Flow

As implied in the introduction (sec. 4·1), there must be a transition between the fast flow in the fracture region and the slow flow in the rock mass in the vicinity of the fracture. This transition takes place at an instantaneous point where the two force fields (chapter 2) acting on the flow are equal.

In this respect, we can enunciate what Parsons (1966, p. 133) stated: Assuming the pressure equalization between fracture and matrix at a given point is instantaneous, then some unsteady state results from anisotropic media can be applied directly to a fracture-matrix system. The usual pressure drawdown equations are found to apply.

This author is referred to in chapter 3 (sec. 3·3) and his work goes side by side with the experimental data and agrees accurately with the porous fractured media approach by this dissertation (this chapter).

This approach is different from the conventional methods which assume isotropic homogeneous media. Hsieh and Kranz, among others, treated permeability
as a function of pressure in anisotropic media. Natural media does not satisfy these idealized conditions and hence this dissertation treats the porous fractured media in an alternative way compatible to the flow relationships in a natural media.

Pumping tests support this approach in a porous fractured media. Summers et al. (1978), measuring the mass flow rate of water on the basis of the input pumping rate, using Westerly granite demonstrated that the initial high permeability at elevated temperature was due to thermal stress cracking. The high permeability did not persist with time and decreased significantly as shown by the dramatic decrease of flow rate with time (figure 4-11). Their results show that the permeability is time and temperature dependent. The point of interest here is to show that the permeability is a transient parameter and it responds to thermal effects similar to the changes in rock properties as temperature changes. Cook and Myer (1981) did this work and termed it 'Themomechanical' studies in granite at Stripa, Sweden. They wrote "Another possible contribution to the discrepancies between predicted and measured values may come from the fact that the temperature dependence of the thermomechanical properties was not included in the predictive model."

This high-low permeability relationships that is fast-slow flow relationships, show that the fractured system responds similar to an equivalent porous medium after stress is applied for a long time as the fracture closes.

Raven and Gale (1985) gave a relation between the flow rate and the stress (figure 4-12). Figures (4-11, 4-12), if superimposed will show the flow rate as a function of both time and stress. The physical situation as shown by these curves, clearly imply the fast flow in the fractured zone and the slow flow as the fracture closes since the rate of flow and hence the permeability is a function of stress and time. Moreover, if temperature is applied, the temperature enhances the fracture
Fig. 4-11: Flow rate at 300°C plotted as a function of time. (After Summers et al., 1978).

$P_c =$ confining pressure

$P_p =$ inlet pore pressure

$\sigma_D =$ axial differential stress
Fig. 4-12: Fracture flow rate per unit head as a function of normal stress. (After Raven and Gale, 1985).
and stress narrows it. Consequently the permeability also is a function of both temperature and pressure. Figure (3-8) and table (3-2) illustrate this fact. Hence the permeability is a temperature-pressure dependent variable and it is therefore a thermohydro-mechanical parameter by analogy to the Cook and Myer (1981) terminology.

It is noteworthy to distinguish the curves (figures 4-11, 4-12) from the drawdown curves. These latter curves as shown by figure (4-15) describing the linear flow and the curvilinear flow intersections at instantaneous points as depicted by Passons (1966). But, figure (4-11) is limited to the flow rate as a function of time under the control of temperature and figure (4-12) illustrates the flow rate as a function of stress. They have the same pattern to that shown in figure (4-7) and this is the reason why these curves are included. Hence the porous fractured medium should be treated in terms of the behavior of the flow (linear and curvilinear) as shown by figure (4-15) and not in terms of the other models which are hard to verify in the field.

Based on the fracture network and hence on the normalized mean square error of the fracture network (nomenclature and definitions are on figure 4-13), the behavior of the flow in fracture networks departs from that of a porous medium because of lack of connection spread in the aperture distribution and the scale of observation (Long et al., 1985). The normalized mean square error approaches zero as the behavior of the flow approaches that of an equivalent porous medium. Long et al. did experimental work and plotted the normalized mean square errors (NMSE) and the average permeability vs. the length of the fracture. Figures (4-13, 4-14) show their work. In figure (4-13), the system becomes connected for length > 8 cm, and in figure (4-14), the percolation threshold ≥ 10 cm. The percolation
Fig. 4-13: Permeability and NMSE versus fracture length. (After Long et al., 1985).

- $K_1$: hydraulic conductivity for the inner region
- $K_2$: hydraulic conductivity for the region
- NMSE: The mean square error normalized by dividing by the product of the principal permeabilities
- $\lambda_A$: number of fractures per unit area
- $\lambda_L$: number of fractures per unit length
- $L$: the dimension of the flow region
Fig. 4-14: Summary of all permeability results plotted as a function of scale of measurements. (After Long et al., 1985).

$K_1$: hydraulic conductivity for the inner region
$K_2$: hydraulic conductivity for the region
RMSE: (see figure 4-13)
threshold is defined on page (141). Hence the scale effect due to connectivity concept.

The Behavior of Flow in Porous Media

The double porosity approach refers to the primary intergranular porosity as controlled by deposition, geometry and size distribution of the grains, and to the secondary porosity as controlled by fracturing, jointing and refilling as a result of precipitations (Warren et al. 1963). Warren et al. gave an answer to this statement (p. 252). They investigated the primary intergranular porosity and the secondary (fissure /or vugular) porosity model on the assumption that primary-porosity region contributes significantly to the pore volume but contributes negligibly to the flow capacity. This means two flows occur: a double porosity flow in which a pseudo-steady flow occurs from blocks to fissures and a fissure flow in which flow occurs under transient conditions. Hence the two responses are present in a single fracture, violating the assumptions governing the flow in the aquifer.

While Warren et al. (1963) superimpose an independent system of secondary porosity on the primary porosity, Shapiro and Anderson (1985) couple a discrete medium with a continuum medium. Their model suffers as the scale associated with each medium is significantly different. Physically, it is an inaccurate description of a system based on a steady state fluid movement (Shapiro and Anderson, 1985, p. 268).

The discrete model is treated since the analogy between the Hele-Shaw flow and the Darcian flow makes it easy to deduce the permeability of a fracture. It is inappropriate as a modeling approach when applied to highly fractured orientations (that is, fractures of non-well defined orientations), since the location of all fractures can never be explicitly specified and since the geometry of the fractures are often difficult to ascertain. The discrete model is therefore difficult to use in the analysis
of highly fractured media. However, the flow pattern is difficult to predict, since most fracture systems have irregular geometrics (an attempt was made in chapter 2 of this paper in the light of the information given by Louis (1969) and Sharp (1970).

Also the continuum flow (Shapiro and Anderson, 1985) neglects any geologic information regarding the location of a fracture and it is difficult to verify with field data. A deficiency which does exist is its inability to deal with a nonuniform hydraulic conductivity field. Also, research, in this kind of modeling is mainly theoretical and field and laboratory data are scanty making verification difficult. Moreover, the author (Shapiro) fell short of verifying it with field data as he did with discrete-continuum model.

As mentioned in the introduction of this chapter and sec. (4-3), the fluid experimental observations are very important for a model verification. Pumping tests provide practical confirmation of the two flow model. Hurr (1966) plotted drawdown vs. time as shown in figure (4-15). The straight line shows the time dependent linear flow and the curve illustrates the flow in fractured zones. Assuming there is no skin effects due to mineral deposition at the interfaces between fractures and blocks, the transition is then instantaneous without any period of transition (as explained in sec. (4-1)). The curves by Theis (1935) and by Boulton (1963) have been superimposed by Hurr (p. 660) based on one of the original assumptions considered by Theis (1935) that the yield from storage is instantaneous. He used the apparent specific storage to account for the instantaneous yield as suggested by Boulton (1963). The intersections of the two curves: linear and curvilinear (that is, between the slow flow and the fast flow or the earlier-time and the late-time segments of the time drawdown curve) as shown in figure (4-15), are used to derive figure (4-16). Here a question arises: Would the storage equation associated with appropriate boundary conditions, as a
**EXPLANATION**

--- Boulton curve for long-term specific yield of 17.5% and r/B of 0.01
--- Theis curves for indicated specific yield

Time and apparent specific yield values used to derive figure 4

--- Figure 4-15: Drawdown curves for determining apparent specific yield-time relation by superimposing Theis (1935) curves on Boulton (1963) curve.
(After Hurr, 1966).

--- Figure 4-16: Relation of apparent specific yield to time. (After Hurr, 1966).
representative of these intersections in porous fractured rock, under the control of the pressure gradient, give rise to the curve shown in figure (4-16). The following work will answer this question.

Before proceeding to build the storage equation, it is appropriate to refer to the work of Lewis and Burgy (1964). Working on hydraulic characteristics of fractured and jointed rock, they obtained the data necessary for the analysis of pumping tests using a nonequilibrium equation for radial flow into wells. Finding that the pumping test data did not match the type curve: $uW(u)$ vs. $u$ as suggested by Theis (1935), where $u[W(u)]$ is a mathematical relation for evaluating drawdown during nonequilibrium conditions, they stated (p. 7 and 9): "There is an apparent similarity in the curvature of the semi-log plots of all the well data; thus the possibility of a site characteristic causing the deviation from a straight line can be ruled out "and" the flow of water to wells in fractured and jointed rock would be invalid."

Jenkins and Prentice (1982) showed that test data plot curvilinear on semi-log plots and straight lines on log-log plots suggesting that traditional methods of aquifer test analysis are not applicable. They stated (p. 16): "Rather than resulting in a single 'Theis curve', the data plot as two parallel straight lines" as figure (4-17) shows, and (p. 12) that "linear flow in the vicinity of the test well may be a common phenomenon that has been overlooked in fractured rock aquifers."

However, a closer look at figure (4-17) may suggest a curve rather than a straight line or a curve as extension to a straight line, yet the work extended by Jenkins and Prentice as shown in figures (4-18, 4-19) suggest the idea of a curve on a semi-log plot and a straight line on log-log plot.
Fig. 4-17: Log-log plot of drawdown (s) versus $r^2/t$. (From Jenkins and Prentice, 1982).
Fig. 4-18: Semi-log plot of drawdown(s) versus time(t) and residual drawdown(s') versus t/t' in observation wells EGH-1 and EGH-2. Silver City, New Mexico. (From Jenkin and Prentice, 1982).

Fig. 4-19: Log-log plot of drawdown(s) versus t/t^2 in observation wells EGH-1 and EGH-2. Silver City, New Mexico. (From Jenkins and Prentice, 1982).
Cooper and Jacob (1946) analysed the radial flow and plotted the drawdown vs. distance, time and \( \frac{t^2}{1} \) and got straight lines in all cases, figures (4-20, 4-21, 4-22).

It seems then from this dissertation standpoint that the complete curve as made of the intersections of both linear curve and curvilinear curve to produce the transition curve as termed by Boulton (1963) and as explained in the previous page, is oversimplified and that data is somewhat misinterpreted to the extent that linear flow is ignored.

In summary, the models referred to (double porosity, discrete and continuum, ps. 163-164) may theoretically be consistent but are not compatible with the physical systems of the fractured crystalline rock observed in the field, the theoretical solutions must go side by side with the field observations in order to model accurately the flow in porous fractured media.

This dissertation is interested in the time drawdown curve, that is, the transition curve (as termed by Boulton, 1963) that connects the early-time drawdown curve (linear) and the late-time drawdown curve (curvilinear). Both curves are referred to on page (164) and figure (4-15) with instantaneous transition, to give rise to the transition curve, from the linear inflow to the curvilinear flow in fracture proper.

This dissertation will design a storage equation with appropriate initial and boundary conditions to represent the flow in porous fractured media without a need for superposition. Both curves referred to above are parts of the time drawdown curve. Their intersection at any step of pressure and temperature (as given by Heard and Page, 1982, in Stripa Granite) will describe a curve that is a solution of the porous fractured medium.
Fig. 4-20: Distance-drawdown graph based on drawdowns in three wells after 18 days of continuous discharge from an unconfined sand, $Q=2.23$ cfs. (After Cooper and Jacob, 1946).

Fig. 4-21: Time-drawdown graph for a well 1200 feet from another well discharging from a confined sand, $Q=3.00$ cfs. (After Cooper and Jacob, 1946).
Fig. 4-22: Composite drawdown graph based on drawdowns observed in a discharging well and two neighboring wells in a confined sand. (After Cooper and Jacob, 1946).
4-4 Storage Equation, Deformability of Rock and Compressability of Water

In order to build the storage equation to involve all the parameters related to the fractured rock in combination with hydraulics involved, the conservation of mass in porous media is introduced.

Rate of fluid mass flow - time rate of fluid mass storage

that is,

$$\iint_{S} \rho \bar{q} \cdot d\bar{s} \cdot \frac{d}{dt} \iiint_{V} nsp \rho dv$$  \hspace{1cm} (4-1)

where

- \( \bar{q} \) - apparent fluid velocity vector, \( \rho \) - density
- \( d\bar{s} \) - the outward normal to the control surface
- \( v \) - the control volume
- \( nsp\Delta v \) - mass of fluid, \( n \) - porosity, \( s \) - saturation.

Applying the divergence theorem to the left side of equation (4-1) (details are in Appendix A) and integrating the right side, we get

$$- \nabla \cdot \rho \bar{q} \Delta v = nsp \frac{\partial v}{\partial t} \cdot \frac{\partial}{\partial t} nsp \Delta v$$

or

$$- \nabla \cdot \rho \bar{q} \Delta v = nsp \frac{\partial s}{\partial t} \Delta v + nsp \frac{\partial \bar{v}}{\partial t} \Delta v + nsp \frac{\partial v}{\partial t}$$  \hspace{1cm} (4-2)

By definition, the compressibility of water is given by

$$\beta = \frac{\Delta v_w}{v_w \Delta P} \text{ or } \beta \Delta P = \frac{\Delta v_w}{v_w} \cdot \frac{\partial \rho}{\partial \rho}$$

or

$$\beta \rho \frac{\partial \rho}{\partial t} = \frac{\partial}{\partial t} (2nd \text{ right term in eq 4-2})$$  \hspace{1cm} (4-3)

where \( P \) - pore water pressure, \( \rho \) - density, \( v_w \) - volume of water (a relation between volume strain \( \varepsilon \) and pore water pressure will shortly be shown).

Terzaghi (1943, p. 267) shows that the flow of water in confined layers, takes place only along vertical lines and is therefore an example of linear flow. So the
volume element deforms vertically in response to a change in pore pressure and hence

Vertical compressibility $\beta' \cdot \frac{\Delta V_v z / V_v z}{\Delta \sigma_z}$

or

$$\beta' \frac{\partial \sigma_z}{\partial t} = \frac{1}{V} \frac{\partial V_v}{\partial t}$$

where

$V_v$ - void volume

So $\Delta \sigma_z = - \Delta p$ [Terzaghi, 1943, p. 270: eq 3, $\frac{\partial u}{\partial t}$ - $\frac{\partial p}{\partial t}$ where (u) is the excess hydrostatic pressure and is related to the void n by $\frac{\partial V}{\partial t} - m_c \frac{\partial u}{\partial t}$ where $m_c$ is the coefficient of volume decrease]. This means $\Delta \sigma_z = - \Delta p$, that is, a change in static pore pressure produces an equal but opposite change in the intergranular stress throughout the medium.

Now, porosity $n = \frac{\text{Volume of void}}{\text{Volume of element}} - \frac{V_v}{V}$

differentiating:

$$\frac{\partial n}{\partial t} \cdot \frac{V_v}{V} (1 - \frac{V_v}{V}) = -\beta' \frac{\partial \sigma_z}{\partial t} \frac{\Delta V}{V}$$

and let

$$V \frac{\partial n}{\partial t} + \frac{\partial V_v}{\partial t}$$

then

$$ns \frac{\partial \sigma_z}{\partial t} = - sp \beta' \frac{\partial \sigma_z}{\partial t} \Delta V \text{ (3rd term in eq (4.2)}$$

(4.4)

Substitution of eqs. 4-3, 4-4 in eq 4.2, with $\Delta \sigma_z = - \Delta p$ and $\frac{\partial s}{\partial t} = 0$ (s = 1, saturated medium) gives

$$- \frac{\partial q}{\partial t} = \mu \beta' \frac{\partial \sigma_z}{\partial t} - \nu \frac{\partial \rho}{\partial t}$$

or

$$- \frac{\partial q}{\partial t} = (\beta' \cdot \nu) \frac{\partial \rho}{\partial t}$$

(4.5)
Equation (4-5) is the equation of conservation of mass and represents the flow in elastic media. With reference to eq. (2-1), chapter 2, that is, \( h - z \cdot \frac{P}{\gamma} \) in simple form, then:

\[
\frac{\partial h}{\partial t} = \frac{1}{\gamma} \frac{\partial P}{\partial t}
\]

Substituting in eq. (4-5)

\[
\cdot \vec{V} = q + \gamma(\beta' \cdot \eta) \frac{\partial h}{\partial t}
\]  \hspace{1cm} (4-6)

\( \beta' \) (deformability of Rock) could be easily shown as related to volumetric strain \( \varepsilon \).

Consider the components of strain as related to displacement (linear functions)

\[
\varepsilon_r = \frac{\partial u_r}{\partial r}, \quad \varepsilon_\theta = \frac{1}{r} \frac{\partial u_\theta}{\partial \theta}, \quad \varepsilon_z = \frac{\partial u_z}{\partial z}
\]  \hspace{1cm} (Jaeger and Cook eq. 12 p. 52, 1979) and

Hook's law (Jaeger and Cook eq. 26, p. 112, 1979)

\[
\sigma_r = 2G\varepsilon_r + \lambda \Delta
\]

\[
\sigma_\theta = 2G\varepsilon_\theta + \lambda \Delta
\]

\[
\sigma_z = 2G\varepsilon_z + \lambda \Delta
\]

where

\[
\Delta = \varepsilon_r + \varepsilon_\theta + \varepsilon_z
\]

and

\[
G \cdot \text{Bulk modulus}, \quad \lambda \quad \text{and} \quad G \quad \text{are Lame's constants.}
\]

With axial symmetry, components of \( \theta \) vanish and using effective stress as another form of Hook's law

\[
\sigma_{r_{\text{eff}}} = 2G \frac{\partial u_r}{\partial r} + \lambda \varepsilon_r
\]

\[
\sigma_{\theta_{\text{eff}}} = 2G \frac{u_\theta}{r} + \lambda \varepsilon_\theta \quad \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} = 0
\]

\[
\sigma_{z_{\text{eff}}} = 2G \frac{\partial u_z}{\partial z} + \lambda \varepsilon_z
\]

Now, assume that the vertical stress does not change during the process of consolidation and that the effective stress is acting and is equal to the water pressure \( p \). This is based on the concept of effective stress for saturated soils by
Terzaghi (1923, after Jaeger and Cook, 1979, Sec. 8.8, p. 219) and that deformation is controlled by the effective stress. Also assuming that the radial displacement vanishes (an important boundary condition in the solution of the storage equation), then
\[ \sigma_z = 0 \text{ and } \sigma_{\text{eff}} = p \] (where \( \sigma_z \) - \( \sigma_{\text{eff}} \) - \( p \))
then
\[ p = (2G+\lambda)\epsilon \] (where \( \epsilon = \frac{\partial u}{\partial z} \) - incremental volumetric strain)
or
\[ \frac{1}{2G+\lambda} \frac{\partial P}{\partial t} - \frac{\partial \epsilon}{\partial t} - \beta \frac{\partial P}{\partial t} - \gamma \frac{\partial h}{\partial t} \}
\[ \text{the first term in eq. (4-6)} \]
Now, introducing Darcy’s law into eq. (4-6)
\[ K \nabla^2 h + \gamma (\beta' + n\beta) \frac{\partial h}{\partial t} = S_s \frac{\partial h}{\partial t} \] (4-8)
Equation (4-8 is the storage equation in its condensed form. If the medium is anisotropic and the aquifer has a thickness b, then
\[ S_s \frac{\partial h}{\partial t} = b (Kx \frac{\partial^2 h}{\partial x^2} + Ky \frac{\partial^2 h}{\partial y^2}) \]
where \( S_s \) - specific storage - \( b \gamma (\beta' + n\beta) \).
Using (c) for piezometric head, then
\[ b (Kx \frac{\partial^2 \phi}{\partial x^2} + Ky \frac{\partial^2 \phi}{\partial y^2}) = S_s \frac{\partial \phi}{\partial t} \] (4-9)
It is observed in eq. (4-7) that stress-strain relations (in isotropic form) are tapered down to Lame’s constants leading to \( B' \) (compressibility of rock). This, in turn, is incorporated in the specific storage (\( S_s \)). \( S_s \) is defined by Jaeger and Cook (1979, p. 219), in hydrological terms, as the quantity of water released pr unit volume for unit fall in pore pressure without reference to stresses in the medium.
Moreover, Jaeger and Cook continued to discuss a linear relation suggested by Terzaghi (1923) and verified by Skempton (1944) and Parasnis (1960), between the void ratio and the applied pressure. Skempton and Parasnis (after Jaeger and
Cook) extended experiments to pressure of 1000 bars (100MPa) without finding important departure from this linear relation. However, the upper limit of pressure used by Heard and Page is 55.2 MPa and any displacement is already incorporated in the deformation of rock and this in turn is transformed to the fracture affecting the hydraulic conductivity.

Equation (4-9) is in cartesian coordinates. Normally the transformation from cartesian coordinates to polar coordinates, is done through the relations:

\[ x = r \cos \theta, \ y = r \sin \theta, \ r = \sqrt{x^2 + y^2} \] and \[ \theta = \tan^{-1} \frac{y}{x}, \]

which give

\[ \frac{\partial r}{\partial x} = \cos \theta, \ \frac{\partial r}{\partial y} = \sin \theta \]
\[ \frac{\partial \theta}{\partial x} = -\frac{\sin \theta}{r} \text{ and } \frac{\partial \theta}{\partial y} = \frac{\cos \theta}{r} \]

Now evaluating \( \frac{\partial^2 \phi}{\partial x^2} \) and \( \frac{\partial^2 \phi}{\partial y^2} \) (where \( \phi = \phi(r, \theta) \)) and adding to get:

\[ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \]

But we have the anisotropic equation (4-9) which requires some modifications to meet the requirements of anisotropy, that is, the effect of anisotropy on the permeability can be replaced by an equivalent shrinking or expansion of the coordinates. So to find the piezometric head at \( (x, y, eq. 4.9) \), the following transformations are necessary:

let \( k_x = \eta^2, x = x_1 - \rho_1, y = x_2, \) then \( \rho_2 = \frac{x_2}{\eta} \)

\[ T = \text{transmissibility} \cdot k_x \] b and \( \bar{r} = \rho_1^2 + \rho_2^2 \) \hspace{1cm} (4-10)

Using these transformations, equation (4-9) is rendered isotropic, that is:

\[ \frac{bk_x}{k_x} \left[ \frac{\partial^2 \phi}{\partial x_1^2} \cdot \eta^2 + \frac{\partial^2 \phi}{\partial x_2^2} \right] = T \left[ \frac{\partial^2 \phi}{\partial x_1^2} \cdot \frac{x_1^2}{\rho_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} \right]
\]

\[ - T \left[ \frac{\partial^2 \phi}{\partial \rho_1^2} + \frac{\partial^2 \phi}{\partial \rho_2^2} \right] - T \left[ \frac{\partial^2 \phi}{\bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial \phi}{\partial \bar{r}} \right] \cdot S_{xy} \frac{\partial \phi}{\partial t} \]
or

\[
\frac{\partial \phi}{\partial t} - \frac{T}{S_s} \left[ \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} \right]
\]

(4-11)

It is observed in the above work that all derivatives with respect to \( \theta \), vanish because of symmetry about an axis. The parameter \( \bar{r} \) is the isotropic parameter. It is the distance from the center of the well.

Of course, another inverse transformation is required when the solution for \( \phi \) is obtained in terms of the isotropic parameter \( \bar{r} \). This is done by replacing \( \bar{r} \) by

\[
\frac{r}{\sqrt{k_r}} = \frac{\text{isotropy}}{\text{anisotropy}} \rightarrow \bar{r} - \frac{r}{\sqrt{k_r}}
\]

where \( k_r = k_6 \) (chapter 3) according to the transformation from the isotropic circle to the anisotropic ellipse (fig. 3-7). \( K_t \) is the directional permeability (eq. 3-17).

Equation (4-11) is a mathematical statement of the storage equation in radial geometry. A fracture is assumed to represent a well in an aquifer of infinite extent. In eq. (4-11), \( S_s \) is the storage coefficient equal to the volume of water released per unit area to the well per unit decrease in piezometric head. The water is released by consolidation and compression effects since the aquifer (fracture and pores) is confined.

\( \phi - h_0 - h \), at any radial distance \( \bar{r} \) and time \( t \) of pumping. In the literature, \( \phi \) and \( h \) are interchanged to represent the drawdown. Drawdown is usually given the symbol \( s \) but this dissertation preserves \( s \) for Laplace Transform.

Then, we have the system

\[
\frac{\partial \phi}{\partial t} - \frac{T}{S_s} \left[ \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} \right]
\]

(4-11)

Initial condition

\[ \phi = 0 \text{ at } t = 0 \]

(4-11a)

Boundary conditions
This system constitutes a boundary value problem. The substitution
of \( \phi = e^{-mt} u(r) \) will reduce eq. (4-11) to the differential equation:

\[
\frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} + k^2 u = 0
\]

(4-12)

This is the Bessel differential equation

\[
x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2) y = 0
\]

where \( n \neq 0 \)

Its solution is expressible in terms of Bessel's functions (Pipes and Harvill, 1970, p. 784 and Watson, 1944 p. 70).

Then we have the general solution

\[
u = A J_0(kr) + B Y_0(kr)
\]

where \( A \) and \( B \) are constants.

Now at \( r \to 0 \), \( Y_0(kr) \) goes to infinity

hence

\( B = 0 \)

and

\[
u = A J_0(kr)
\]

and a particular solution of eq. (4-11) is given by:

\[
\phi = A \exp \left( -\frac{T}{S_s} k^2 t \right) J_0(kr)
\]

(4-13)

where

\[
m = \left[ \frac{T}{S_s} k^2 \right]
\]

Let \( \lambda_L \) replace \( k \) which has another meaning in this work (\( k \) - hydraulic conductivity)
then

\[ m = \frac{T}{S} \lambda_L^2 \text{ and } \lambda \leq \lambda_0, \lambda_1, \lambda_2 \ldots \text{ are the only permissible positive roots that satisfy } J_0(\lambda_L \tilde{r}) = 0 \]

and

\[ \phi = A_{\lambda_L} \exp \left[ -\frac{T}{S} \lambda_L^2 t \right] J_0(\lambda_L \tilde{r}) \]

and the general solution is the sum over all possible values of \( L \), hence

\[ \phi = \int_0^\infty A_{\lambda_L} \exp \left[ -\frac{T}{S} \lambda_L^2 t \right] J_0(\lambda_L \tilde{r}) \, d\lambda_L \quad (4.14) \]

Equation (4.11) has another solution given by the modified Bessel function of the second kind (\( K_n(r) \)). This solution is obtained by applying the Laplace transform to the system (4.11). Without it, the system (4.11) would not give a correct solution to the flow in fractured rock, since the linear inflow to the fracture would be overlooked. This leaves the traditional solution (eq. 4-13) in a great error.

Use of differential equation given as equation (9.3) in Pipes and Harvill (1970, p. 793) and letting the constant coefficients \( a = 0, b = 1 \) and \( c = 1 \), then

\[ \frac{d^2 y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + (-1 - \frac{n^2}{x^2}) y = 0 \]

This is a Bessel of the same form as eq. (4.12) and has a general solution:

\[ \phi = A I_n (\tilde{r}) + B K_n (\tilde{r}) \quad (4.15) \]

\( K_n (\tilde{r}) \) is also shown as a solution by Watson (1944, p. 78) and by Creyszig (1962, p. 204).

It follows then that

\[ K_0(\tilde{r}) \text{ and } \int_0^\infty A_{\lambda_L} \exp \left[ -\frac{T}{S} \lambda_L^2 t \right] J_0(\lambda_L \tilde{r}) \, d\lambda \text{ are solutions of the system (4.11).} \]
The linearity and homogeneity of eq. (4-11) allows combination of these solutions.

Hence

\[ \phi = C_1 \left[ \int_0^\infty A_{\lambda L} \exp \left( -\frac{T}{S_1} \lambda_L^2 t \right) J_0(\lambda_L \vec{r}) \, d\lambda_L \cdot K_0(\vec{r}) \right] \quad (4-16) \]

where \( C_1 \) is a constant, \( A_{\lambda L} \) is another constant depending on \( \lambda_L \) only and independent of \( r \) and \( t \). In equation (4-16), \( J_0 \) is the Bessel function of the first kind of zero order and \( K_0 \) is the modified Bessel function of the second kind of zero order. These constants (\( C_1 \) and \( A_{\lambda L} \)) are to be determined.

Applying the initial condition (4-11a) to equation (4-16) gives

\[ \phi = \int_0^\infty A_{\lambda L} J_0(\lambda_L \vec{r}) \, d\lambda_L \cdot k_0(\vec{r}) \quad (4-17) \]

Condition (4-11c) gives

\[ \frac{Q}{2\pi T} \cdot C_1 \quad (4-18) \]

Condition (4-11b) gives \( \phi = 0 \) as both

\[ J_0(\infty) = K_0(\infty) = 0 \quad (\text{Tables: Watson, 1944, and Abramowitz and Stegun, 1972}) \]

In eq. (4-17), there must be some relation between the right hand terms that will express \( K_0(\vec{r}) \) in terms of an integral of \( J_0(\lambda_L \vec{r}) \). Watson (1944, p. 425) shows that

\[ \int_0^\infty \frac{x J_0(ax)}{x^2 + k^2} \, dx \cdot K_0(ak) \]

and in terms of \( \lambda_L \) (where \( x = \lambda_L \) and \( K = (-1) \))

\[ K_0(\vec{r}) \cdot \int_0^\infty \frac{\lambda_L}{\lambda_L^2 + 1} J_0(\lambda_L \vec{r}) \, d\lambda_L \quad (4-19) \]

Then eq. (4-17 gives
or

\[ o - \int_{0}^{\infty} A_{0 L} J_{0}(\lambda_{L}, r) \, d\lambda_{L} \cdot \int_{0}^{\infty} \frac{\lambda_{L}}{\lambda_{L}^2 + 1} J_{0}(\lambda_{L}, r) \, d\lambda_{L} \]

The bracket gives

\[ A_{0 L} = \frac{\lambda_{L}}{\lambda_{L}^2 + 1} \quad (4.20) \]

Substituting 4-18, 4-20 in 4-16

\[ \phi\left(\frac{Q}{2\pi T}\right) = K_{0}(r) - \int_{0}^{\infty} \frac{\lambda_{L}}{\lambda_{L}^2 + 1} \exp \left[-\frac{T}{S_{s}} \lambda_{L}^2 t\right] J_{0}(\lambda_{L}, r) \, d\lambda_{L} \quad (4.21) \]

For integration with respect to \( \lambda_{L} \), eq. (4-21) must be modified to agree with the results obtained by Watson (1944, p. 394, eq. 4). It is a matter of replacement, substitution and change of the order of integration. Integral tables give

\[ \int_{0}^{\infty} e^{-ax} \, dx = \frac{1}{a} \quad a > 0 \]

So

\[ \frac{1}{\lambda_{L}^2 + 1} \cdot \int_{0}^{\infty} e^{-\left(\lambda_{L}^2 + 1\right)x} \, dx \]

Also let

\[ \frac{T}{S_{s}} \lambda_{L}^2 t = p(\lambda_{L}^2 + 1) \quad \text{and} \quad \lambda_{L}^2 = p \]

Substitution in eq. (4-21) gives

\[ \phi\left(\frac{Q}{2\pi T}\right) = K_{0}(r) - \int_{0}^{\infty} e^{-p x + \lambda_{L}^2 x} \int_{0}^{\infty} e^{-p(\lambda_{L}^2 + 1)} J_{0}(\lambda_{L}, r) \, d\lambda_{L} \quad (4.22) \]

Using \( p - x\lambda_{L}^2 \), the last integral in eq. (4-22) could be written in the form

\[ \int_{0}^{\infty} e^{-x\lambda_{L}^2(\lambda_{L}^2 + 1)} J_{0}(\lambda_{L}, r) \lambda_{L} \, d\lambda_{L} \]
In equation 4 of Watson (1944, p. 394), that is:

\[ \int_0^\infty J_\nu (at) \exp (-p^2 t^2) \cdot t^{\nu+1} dt - \frac{a^\nu}{(2p^2)^{\nu+1}} \exp (-\frac{a^2}{4p^2}) \]

let \( \nu = 0, p^2 = p + x, a = r \) and \( t = \lambda_L \)

then

\[ \int_0^\infty J_\nu (\lambda_L r) \exp [-(P + x) \lambda_L^2] \lambda_L d\lambda_L \]

\[ - \frac{1}{2(p + x)} \exp \left[ -\frac{r^2}{4(p + x)} \right] \]

and hence, the second term of eq. (4-22) is reduced to

\[ \int_0^\infty e^{-(P + x)} \frac{1}{2(p + x)} \exp [-\frac{r^2}{4(p + x)}] dx \]

or

\[ \frac{1}{2} \int_0^\infty \frac{1}{p + x} \exp \left[ -(p + x) \cdot \frac{r^2}{4(p + x)} \right] dx \]  \hspace{1cm} (4-23)

Using eq. 15 (Watson, 1944, p. 183) that is:

\[ K_\nu(z) = \frac{1}{2} \left[ \frac{1}{2} z \right]^\nu \int_0^\infty \exp (-\tau \cdot \frac{z^2}{4\tau}) \frac{d\tau}{\tau^{\nu+1}} \]

and putting \( \nu = 0, \tau = y, \ dr = dy, \ p + x = \tau = y \) and \( \frac{r^2}{4(p + x)} \) being small and substituting in eq. (4-21) we get
\[
\phi / \left( \frac{Q}{2 \pi T} \right) - K_o(r) - \frac{1}{2} K_o(r) - \frac{1}{2} K_o(r)
\]

where

\[
K_o(r) = \int_0^\infty \frac{1}{y} \exp (-y) \, dy
\]

then finally

\[
\phi / \left( \frac{Q}{4 \pi T} \right) - \int_0^\infty \frac{1}{y} \exp (-y) \, dy
\]

This kind of differential equations of radial symmetry can best be treated by a Laplace transform. Again we have

\[
\frac{\partial \phi}{\partial t} - \frac{T}{S} \left( \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} \right)
\]

(4-25)

\[
\phi = \phi_0 \text{ at } t = 0 \quad r \geq 0
\]

(4-25a)

\[
\phi = \phi_0 \text{ at } r = \infty \quad t \geq 0
\]

(4-25b)

\[
r \frac{\partial \phi}{\partial r} = \frac{Q}{2 \pi T} \quad \text{as } \rightarrow 0
\]

(4-25c)

Now Laplace of \( \frac{\partial \phi}{\partial t} \) is given by

\[
s \phi = \int_0^\infty \exp (-st) \frac{\partial \phi}{\partial t} \, dt
\]

where \( s \) is the Laplace symbol. Hence the radial flow (eq. 4-25) can be written:

\[
s \phi = \phi(r,0) - \frac{T}{S} \left( \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} \right) \text{ where } \quad r = 0
\]

and

\[
\phi (0,s) = \mathcal{L} (\phi (0,t)) - \frac{\phi_0}{s}
\]

where \( \mathcal{L} \) is Laplace of \( \phi \). Then the general solution is given in terms of Bessel functions as
which is eq. (4-15) after Laplace transformation.

Since $I_0 (\lambda L \bar{r})$ is unbounded when $\bar{r} \to \infty$, we must have $C_1 = 0$ and

$$\phi (\bar{r},s) = \frac{\phi_0}{s}$$

or

$$\phi (\bar{r},s) = \frac{\phi_0}{s} + C_2 K_0 (\lambda L \bar{r})$$

or

$$\phi (\bar{r},s) = \frac{\phi_0}{s} + C_2 K_0 (\lambda L \bar{r})$$

or

The constant $C_2$ is found to be $C_2 = \frac{Q}{2\pi T}$ (eq. 4-18). However, by applying the transform of the boundary condition 4-25c:

$$\bar{r} \frac{\partial \phi}{\partial \bar{r}} = \frac{Q}{2\pi T} \quad \bar{r} \to 0$$

as $\bar{r} \to r_0$ then $\frac{\partial \phi}{\partial \bar{r}} = -\frac{Q}{2\pi T r_0^3}$ \hspace{1cm} (A)

and differentiating (4-26)

$$\frac{\partial \phi}{\partial \bar{r}} = C_2 \lambda L K_1 (\lambda L \bar{r})$$

where $K_1$ is the modified Bessel function of order 1 and derivative of $K_0$. Equating (A) and (B)

$$\frac{\lambda}{2\pi T r_0^3} = C_2 \lambda L K_1 (\lambda L \bar{r})$$

or

$$C_2 = \frac{Q}{2\pi T s} \frac{1}{r_0 \lambda L K_1 (\lambda L \bar{r})}$$

The factor $r_0 \lambda L K_1 (\lambda L \bar{r})$ could be taken as 1 as $\lambda L \bar{r} \to 0$

Hence

$$C_2 = \frac{Q}{2\pi T s} \quad \text{which is eq. (4-18)}.$$

Substitution in eq. (4-26) gives

$$\phi (\bar{r},s) = \frac{\phi_0}{s} + \frac{Q}{2\pi T s} \frac{1}{s} K_0 (\lambda L \bar{r})$$

Equation (4-27) is to be transformed back to the head $\phi$.

It is found, using the standard mathematical tables (Samuel, 1971, p. 499)
\[ F(t) = \frac{1}{2t} \exp \left( -\frac{K^2}{4t} \right) \]

and

\[ F(s) = K_0(K\sqrt{s}) \quad (K > 0) \]

replacing \( k \) by \( a \) (as \( k \) is reserved for permeability)

\[ F(s) = K_0(a\sqrt{s}) \quad F(t) = \frac{1}{2t} \exp \left( -\frac{a^2}{4t} \right) \]

Using Laplace property: the integration of the original function corresponds to multiplication by \( \frac{1}{s} \) then

\[
\frac{K_0(a\sqrt{s})}{s} = \int_0^\infty e^{-st} \left( \frac{1}{2t} \right) \exp \left( \frac{a^2}{4t} \right) dt
\]

\[
= \int_0^\infty e^{-st} \left( \frac{1}{(2\pi)^{-1}} \exp \left( \frac{a^2}{4\pi} \right) \right) \exp \left( -\frac{a^2}{4t} \right) dt
\]

\[ (4-28) \]

From eq. (4-27)

\[ \lambda_L \tilde{r} = a\sqrt{s} \tilde{r} = \left( \frac{\rho g (\beta' + n\beta)}{kb} \right)^{1/2} \sqrt{s} \tilde{r} \]

As eqs. (4-8, 4-9) give: \( \rho g (\beta' + n\beta) \) b - \( S_b \) - storativity

and \( K_b \) - T - transmissibility

So \( \lambda_L \tilde{r} = a\sqrt{s} \tilde{r} = \left( \frac{S_b}{T} \right)^{1/2} \sqrt{s} \tilde{r} = \left( \frac{r^2 S_b}{T} \right)^{1/2} \sqrt{s} \]

\[
\left\{ \text{Since eq. (4-13) gives } m = \frac{T}{S_b} \lambda_L^2 \rightarrow \lambda_L = (mS_b/T)^{1/2} \right\}
\]

where

\[ \sqrt{m} = \sqrt{s} \text{ and } a = \left( \frac{S_b}{T} \right)^{1/2} \]

Hence, eq. (4-27), using eq. (4-28), becomes

\[ \phi - \phi_0 = \frac{Q}{2\pi T} \int_0^\infty \frac{1}{2\pi} \exp \left( -\frac{r^2 S_b}{4\pi T} \right) dr \]
or

\[ \phi - \phi_0 = \frac{Q}{4\pi T} \int_0^\infty \frac{1}{y} \exp(-y) \, dy \]  

(4-29)

where

\[ y = \frac{r^2 S_k}{4T} \] and \( \phi - \phi_0 \) is the drawdown.

Equation (4-29) could be written as

\[ \phi_0 - \phi = -\frac{Q}{4\pi T} \int_0^\infty \frac{1}{y} \exp(-y) \, dy \]

This form explains the negative sign attached to the drawdown as appeared in the computer results in the work of Gale (1981).

Equation (4-29) is exactly eq. (4-24) obtained by a different method as a confirmation of the solution.

The exponential function in eq. (4-29), is termed by Abramowitz and Stegun (1972, eq. 5.1.11, p. 228) as

\[ E_1(x) = \int_0^\infty \frac{t^{t-1}}{t} dt \quad (l \arg z \leq \pi) \]

and expanded in eq. 5.1.11, p. 229 (same authors) as

\[ E_1(z) = -\gamma - \ln z - \sum_{n=1}^{\infty} \frac{(-1)^n z^n}{n} \quad (l \arg z \leq \pi) \]  

(A)

where

\[ \gamma = 0.5772156649 \] is the Euler's constant.

If \( Z = \frac{r^2 S_k}{4T} \) is less than 1.

Then eq. (4-29) with eq. (A) and with the appropriate approximation can be written as

\[ \phi - \phi_0 = \frac{Q}{4\pi T} \left[ -0.577215 - \ln \frac{r^2 S_k}{4T} \right] \]
and the other terms of the series are ignored (very small for small values of \( z \) as \( r \) is small and \( t \) is large.

Then

\[
\phi - \phi_0 = \frac{Q}{4\pi T} \left[ -\ln \frac{r^2 S_s}{4Tt} - 0.5772 \right]
\]

Still more simplifications could be made.

Putting

\[
\phi - \phi_0 = 0 \text{ gives } \ln \frac{1}{Z_0} - 0.5772
\]

or

\[
Z_0 = 0.561
\]

then

\[
\phi - \phi_0 = \frac{Q}{4\pi T} \ln \frac{r^2 S_s}{(0.561)^2 Tt}
\]

or

\[
\phi - \phi_0 = \frac{Q}{4\pi T} \ln \frac{r^2 S_s}{2.244 Tt}
\]

Usually \((\phi_0 - \phi)\) is given by \((-\phi)\)

or

\[
\phi - \phi_0 = \frac{Q}{4\pi T} \ln \frac{r^2 S_s}{2.44 Tt}
\]

(4.30)

Bearing in mind that the potential and flux boundary conditions are unaffected by the transformation from anisotropy to isotropy, since the medium everywhere is saturated and the hydraulic conductivity depends solely upon direction (proved in chapter 3) the transformation to polar coordinates shows that

\[
T \text{ (transmissibility) } = k_x \text{ (thickness of aquifer)}
\]

- \( k_x b \) (page 176)

Hence eq. (4.11) is the distorted form to render the problem isotropic and eq. (4.30) is the isotropic solution of eq. (4.11). This isotropic solution is transformed
back by the inverse distortion by replacing \( \tilde{r} \) by \( \frac{r}{\sqrt{k_r}} \) (as mentioned on page 177),

where \( k_r = k_6 \) as given by eq. (3-17). \( K_6 \) is a function of \( k_x, k_y \) (the principal permeabilities) and the directional angle \( \delta \) (fig. 3-7). Equation (4-30) may, then, be written in the form,

\[
\phi = - \frac{Q}{4\pi T} \ln \left( \frac{\frac{r^2}{2.244} S_A}{T} \right)
\]

(4-31)
after fitting in the anisotropic parameter. It is a relation between the drawdown \( \phi \) and the integral of the anisotropic coordinate \( x \):

\[
x = \frac{r^2}{2.244} \frac{S_A}{T} \quad \text{(where} \ r \ \text{is the anisotropic parameter)}
\]

Although the exponential function appears in the solution, yet it is not the same equation that appeared in Theis (1935) by analogy to heat conduction. This latter equation is applicable to an isotropic medium where transmissibility is constant and to unconfined aquifers. Also it is based on the assumption that the yield from storage is instantaneous, that is, there is no storage at all. As Hurr (1966, p. 660) put it: "For water table conditions this is not true. As pumping begins and as pressure in the aquifer is reduced, water is derived from internal storage by decompression of the water and by aquifer compaction and specific yield values are small. After several seconds to several minutes, aquifer dewatering becomes significant and specific yield begins to increase. Because drainage is not instantaneous, it may take from several hours to several days for specific yield to approach its maximum."

Hence with a closer look to eq. (4-29), \( \phi \) (the drawdown) is considered the apparent specific yield to take care of the delayed yield (Boulton, 1963) but it is the drawdown that is produced by the apparent specific yield \( (S_A) \) in Theis work by analogy to heat conduction.
Thus equation (4-29) actually represents the curves of Theis (1935) and of Boulton (1963) as shown by fig (4-16) and as explained on page (164).
Equation (4-31) (a form of eq. 4-29) includes also the modifications introduced by the anisotropic medium in which the transient permeability is not a constant and it is the directional permeability that is in effect.

Theis derived his formula by analogy to Carslaw equation (1921, p. 152) for the temperature at any point, that is:

\[ v = \frac{Q}{4\pi k t} e^{-\frac{(x^2 + y^2)}{4kt}} \]

as he mentioned in his paper (page 520) involving two dimensional flow of heat. Whether this analogy is accurate with respect to water flow or not, or the solution in this work through the manipulations of Bessel's functions or through Laplace transform, both solutions show the exponential integral as it must be with the proper boundary conditions, otherwise the solution could be far from the truth and would appear in terms of the so-called error function or in terms of Fourier series.

The difference is mainly attributed to delayed yield that changes drastically the curve given by Theis (1935), or as Jenkins and Prentice (1982, pp 12, 16) put it: "Linear flow in the vicinity of the test well may be a common phenomenon that has been overlooked in fractured rock aquifer."

Also Singh and Gupta (1986, p. 223) wrote: "The idealistic concept of a line well, having a negligible storage as enunciated by previous workers for the interpretation of the well response data, is not really valid in many situations. The finite diameter of the well and the effect of the well storage should be taken into account while analysing the data."

Other authors support the idea that previous works are oversimplified: Atobrah (1983, p. 228), Rushton and Holt (1981, p. 505), Neuzil (1986, p. 1170), Hurr (1966, p. 660), Singh and Gupta (1986, p. 223), Rushton and Singh (1984, p. 670), Boulton and Streltsova (1977, p. 269) and Cooper and Jacob (1946, fig. 4-20). This latter figure shows clearly the relation: drawdown vs. distance as compared to Theis (1935, fig.
1, p. 52) of his paper. Both figures are in unconfined aquifers with one major difference: the linear flow in the fracture was not considered and this lead to a loss of accuracy in the parameters of several orders (Atobrah, 1983, p. 228).

4-5 Application of Experimental Data to the Analysis of Sec. (4-4)

It is time now to look at the shape function (curve) of eq. (4-31). This will be done in a series of examples, using data from different fields for different rocks (the first, in Decan jointed basalts in India, the 2nd in granitic terrain in India, and the 3rd in Stripa granite in Sweden).

As a first example:

Consider table (4-1) with data given by Deolankar and Kulkarni (1985) in Decan jointed basalts with constant discharge and using eq. (4-31) in the form:

\[ \frac{4\pi T \phi}{Q} \cdot \ln \frac{r^2 S_s}{2.244 T t} \]

Storativity \((S_s)\) is calculated and the same equation is used to evaluate the points required for the plot. The four points are plotted in figures (4-23, -24, -25) giving straight lines in arithmetic plot and log-log plot and the inflow fractured curve in semi-log plot. Hence, drawdown vs. distance or time reflects the work of Jenkins and Prentice (1982, fig. 4-17, -18, -19) and the work of Cooper and Jacob (1946, fig. 4-20, -21, -22).

As a second example, consider table (4-2) as scaled from fig. (4-26) by Gupta and Singh (1985). Data was obtained from pumping tests in granitic terrain with constant discharge. Again using equation (4-31), the five points are plotted in figures (4-27, -28, -29). We get the same pattern as in Example (1).

As a third example, and it is of great interest to this dissertation, consider LBL - 13101 by Gale (1981) and Hole R5 (p. 236) and borehole (1) page (242) and writing a data extract as shown in table (4-3). The points (1,2,3,5,6), and in this
Table 4-1. Data in jointed basalt aquifer.*
Examples involving calculation of Aquifer Response.

<table>
<thead>
<tr>
<th>Time $t$ in minutes</th>
<th>Drawdown $s$ in meters</th>
<th>Unit drawdown $\Delta s$ in meters</th>
<th>$Q_a$ in m$^3$/min.</th>
<th>$s/Q_a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>0.70</td>
<td>0.70</td>
<td>0.0060</td>
<td>116.66</td>
</tr>
<tr>
<td>60</td>
<td>1.30</td>
<td>0.60</td>
<td>0.0980</td>
<td>13.26</td>
</tr>
<tr>
<td>90</td>
<td>1.78</td>
<td>0.48</td>
<td>0.2145</td>
<td>8.29</td>
</tr>
<tr>
<td>120</td>
<td>2.20</td>
<td>0.42</td>
<td>0.2729</td>
<td>8.06</td>
</tr>
</tbody>
</table>

(A) Well No. *B2W2*
Aquifer: Jointed basalt
Pump discharge $Q_p = 0.682$ m$^3$/min.
Area of cross section = 29.22 m$^2$

Fig. (4-23): Arithmetic curve of storage equation in jointed basalt (Data from Deolankar and Kulkarni, 1985).
Fig. (4-24): Log-log curve of storage equation in jointed basalt (Data from Deolankar and Kulkarni, 1985).
Fig.(4-25): Semi-log curve of storage equation in jointed basalt (Data from Deolankar and Kulkarni, 1985).
Fig. 4-26: Time drawdown plot (After Gupta and Singh, 1985).
Table 4-2. Data from Fig. 4-26*

<table>
<thead>
<tr>
<th>time t in sec.</th>
<th>180</th>
<th>300</th>
<th>600</th>
<th>1200</th>
<th>1800</th>
<th>2400</th>
<th>3000</th>
<th>3600</th>
<th>4200</th>
</tr>
</thead>
<tbody>
<tr>
<td>Drawdown s = ϕ</td>
<td>0.019</td>
<td>0.023</td>
<td>0.046</td>
<td>0.085</td>
<td>0.14</td>
<td>0.18</td>
<td>0.23</td>
<td>0.26</td>
<td>0.30</td>
</tr>
</tbody>
</table>

Test: Pump test in granitic terrain.
Constant discharge (Q) = 0.0718 m³/min. = 0.0011966 m³/sec.***
Radius of well (r) = 2.2750 m.**

** Gupta and Singh, 1985, p. 142.
Fig.(4-27): Arithmetic curve of storage equation in granite (Data from Gupta and Singh, 1985).
Fig. (4-28): Log-log curve of storage equation in granite (Data from Gupta and Singh, 1985).
Fig.(4-29): Semi-log curve of storage equation in granite (Data from Gupta and Singh, 1985).
Table 4-3. An extract from Gale's data (1981) in Stripa granite in Sweden. (Using borehole technique in a time drawdown test)

<table>
<thead>
<tr>
<th>No.</th>
<th>Time (t) sec.</th>
<th>Drawdown ( \phi ) (m)</th>
<th>Rate of flow ( Q ) (m/sec)</th>
<th>Transmissibility ( T ) (m(^2)/sec)</th>
<th>Conductivity ( K ) (m/sec)</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>126.6</td>
<td>34.143</td>
<td>( 5.0 \times 10^{-8} )</td>
<td>( 1.3 \times 10^{-9} )</td>
<td>( 6.2 \times 10^{-10} )</td>
<td>Borehole R(_5) (p. 236)</td>
</tr>
<tr>
<td>2</td>
<td>126.6</td>
<td>31.747</td>
<td>( 2.2834 \times 10^{-4} )</td>
<td>( 6.4 \times 10^{-10} )</td>
<td>( 3.0 \times 10^{-10} )</td>
<td>Borehole R(_5) (p. 236)</td>
</tr>
<tr>
<td>3</td>
<td>126.6</td>
<td>18.325</td>
<td>( 0.8334 \times 10^{-4} )</td>
<td>( 4.0 \times 10^{-10} )</td>
<td>( 1.9 \times 10^{-10} )</td>
<td>Borehole R(_5) (p. 236)</td>
</tr>
<tr>
<td>4</td>
<td>108.0</td>
<td>30.046</td>
<td>( 1.0567 \times 10^{-4} )</td>
<td>( 3.1 \times 10^{-10} )</td>
<td>( 1.7 \times 10^{-10} )</td>
<td>Borehole 1 (p. 242)</td>
</tr>
<tr>
<td>5</td>
<td>126.6</td>
<td>32.545</td>
<td>( 1.2167 \times 10^{-4} )</td>
<td>( 3.3 \times 10^{-10} )</td>
<td>( 1.6 \times 10^{-10} )</td>
<td>Borehole R(_5) (p. 236)</td>
</tr>
<tr>
<td>6</td>
<td>126.6</td>
<td>23.083</td>
<td>( 0.95 \times 10^{-4} )</td>
<td>( 3.7 \times 10^{-10} )</td>
<td>( 1.7 \times 10^{-10} )</td>
<td>Borehole R(_5) (p. 236)</td>
</tr>
</tbody>
</table>

Notes:  
1. Discharge (Q) is not constant.  
2. No correlation is made for the orientation of the borehole.  
3. Radial distance to the pressure boundary is taken to be 10 m. No correction for erosion is made.  
4. Temperature is taken to be 12° C.  
5. Points 1, 2, 3, 5, 6 are used to plot the curves shown in Figs. 4-30, 4-31, 4-32. Points 3, 4, 5 are chosen as they give the nearest hydraulic conductivities to those of Heard and Page (1982) as if they were measured at the nearest temperatures and pressures given by Heard and Page.
order with respect to the permeability, to compensate, to some extent, for the
discharge that is not constant, are used to plot figures (4-30, -31, -32). The pattern
is the same as in examples 1 and 2.

These nine experimental curves (4-23 to 4-25, 4-27 to 4-29, 4-30 to 4-32) are
meant to show the inflow fractured curve as made of the linear inflow (Jenkins and
Prentice, 1982) and the curve-linear curve (Theis, 1935).

As Stripa granite is the focus of this dissertation, and as there is no thermally
induced stress data beside that of Heard and Page (1982), consider the last 3
points (3, 4, 5) in table (4-3). These points are in this order with respect to the
permeability and are numerically in the vicinity of those of Heard and Page (1982)
but they are not thermally induced stress dependent. The purpose of this plan is
to use this data to evaluate the anisotropic coordinates x and y and to compare
them to those obtained from the thermally induced permeability by Heard and Page,
as shown in table (3-3). Moreover, these points are numerically chosen as if they
were measured at the temperatures and pressures of Heard and Page. They are
also not equal but are still in Stripa Granite. If this attempt gives closer values of
x and y as shown in table (3-3), and if this happens, then the physics of this
mathematical model combined with the experimental work would be a unified piece
of work.

It is to be observed that this inflow fractured curve is a result of experimental
work with or without temperature (in the sense of increasing temperature or at room
temperature). The effect of temperature is to adjust the curve literally to itself
whereas the pattern remains the same. For example, if temperature is applied in an
increasing fashion associated with pressure, the same curve will result parallel to
itself and in this case, the results are termed thermomechanical results. A sounding
example to these statements, is found in literature. This dissertation, in this
Fig. (4-30): Arithmetic curve of storage equation in Stripa Granite, Sweden (Data from Gale, 1981).
Fig. (4.31): Log-log curve of storage equation in Stripa Granite (Data from Gale, 1981).
Fig. (4-32): Semi-log curve of storage equation in Stripa Granite (Data from Gale, 1981).
respect, is following suit as to the analogy of this example in assigning the terminology. Cook and Myer (1979) in the well-known thermomechanical experiments with electrical heaters simulating the thermal output of waste canisters, tested the change in rock properties due to temperature. The higher the temperature they used, the more discrepancy is reduced between the measured values and the predicted values. They termed these temperature dependence properties as thermomechanical properties. As for the pattern, I referred to above with or without temperature, it can be understood by reference to this paper titled "Thermomechanical Studies in Granite at Stripa, Sweden," and to fig. (6) of that paper. Moreover, Heard and Page (1982) wrote "The changes in crack porosity for both Granites (Westerly and Stripa) observed at all P and T conditions have interesting implications to any in situ operation or processing scheme involving a thermal perturbation in similar rock, since connected porosity affects permeability and, hence, fluid transport," and "the temperature dependence of permeability changes may be qualitatively inferred from the data presented in figure 8b (fig. (3-8) in this dissertation."

The conclusion that can be drawn from this analogy is very important. The parameters so far treated and the parameter still to be treated, are thermohydromechanical parameters since temperature and hydrology are incorporated.

Using the following equation:

\[ x (\text{anisotropic}) = \sqrt{\delta} \cos \delta \]  \hspace{1cm} (3-19)

(a mathematical result of the anisotropic ellipse fig. (3-7)) and

\[ \phi = \frac{Q}{4T \ln \frac{r^2 S_s}{2.244T}} \]  \hspace{1cm} (4-21)
where

\[ x_{\text{anisotropic}} = \frac{r^2 s}{2.244 T} \]

where

- \( x \) is the anisotropic coordinate
- \( K_\delta \) is the directional hydraulic conductivity
- \( \delta \) is the inclination of \( K_\delta \) with respect to the major axis of the anisotropic ellipse (Fig. 3-7)
- \( r \) is radial distance to the pressure boundary - \( r_b = 10 \) m (Gale, 1981)

It is to be observed that the measured values in the field are anisotropic values dictated by the flow itself \((r = \sqrt{K_s}, \text{ where } r \text{ is the isotropic value})\). Also \( \delta \) is taken as \((15^\circ) \) assigned to granite rock (Marcus, 1962, Fig. 3-4).

Using table (4-3). Point (3) gives

\[ x = 1.5951 \times 10^{-5} - \sqrt{K_s} \cos 15^\circ \]

or

\[ K_\delta = 2.837 \times 10^{-5} \text{ m sec}^{-1} \cdot 0.02837 \text{ md} \]

and the anisotropic \( x \) and \( y \) of the ellipse (Fig. 3-7):

- \( x = 0.1627 \)
- \( y = 0.0436 \)

Point (4) gives

- \( K_\delta = 0.027 \text{ md} \)
- \( x = 0.1587 \)
- \( y = 0.0425 \)

Point (5) gives

- \( K_\delta = 0.026149 \text{ md} \)
Table (4-4) compares these results to those already calculated in table (3-3) in the last 3 lines.

These examples and their results are meant to give a fair idea about the shape function of eq. (4-31) in granite (that is the curves that this equation represents) and other rocks and to make sure that the hydraulic conductivity values in chapter (3) are the most correct values as far as the literature can provide as a result of this detailed analysis and the experimental work of Heard and Page (1982, fig. (3-8), Neretnieks (1985, fig. 4-10) and Gale's detailed work (1981).

The slight differences are due to the type of work used in the techniques practiced in this field. While one technique used borehole measurements, the other used \((\text{porosity})^3\) to infer hydraulic conductivity which is more appropriate with respect to REV, and a result of the thermal induced stress.

By definition, a REV is the lower limit of a volume of a porous or a fractured medium for which the hydrogeologic parameters, processes or conceptualizations are valid (Grisak et al., 1985, p. 60). This in turn affects the measurement of the hydraulic conductivity by the choice of the measurement scale which has a significant effect on the analytical techniques required for hydraulic tests. It has been shown by Grisak et al. (p. 63) that the borehole diameter affects the measurements of the hydraulic conductivity on the order of a factor of 3. In addition, the rate of flow \((Q)\) varies, while the measurements are considered accurate if \((Q)\) is taken constant (Deolankaz and Kulkarin (1985, p. 628).

However, if we consider fig. (3-8) given by Heard and Page (1982) which is confirmed by Neretnieks (1985, fig. (4-10)); both in Stripa Site, and the exact calculations by conformal mapping to evaluate the anisotropic coordinates \(x\) and \(y\).
Table 4-4. Anisotropic ellipse parameters based on temperature-independent permeability (selected values) measured in a time drawdown test as a function of flow pressure (Gale, 1981), as compared to those based on inferred permeability as a function of temperature and pressure (Heard and Page, 1982). Both Works are in Stripa Granite, using:

\[ x = \sqrt{k_x \cos \theta}, \quad y = \sqrt{k_y \sin \theta} \quad \text{and} \quad x = \left(\frac{r^2 S_y}{2.244 T} \right) \]

<table>
<thead>
<tr>
<th>Points No.</th>
<th>Directional Permeability</th>
<th>Anisotropic Parameters</th>
<th>Permeability (drawdown test Gale, 1981)</th>
<th>Directional Permeability</th>
<th>Anisotropic Parameters</th>
<th>Inferred Permeability (Temperature-dependent)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.02837</td>
<td>0.1627 0.0436</td>
<td>1.9 \times 10^{-10}</td>
<td>0.02871</td>
<td>0.1636 0.0438</td>
<td>2.0 \times 10^{-10}</td>
</tr>
<tr>
<td>4</td>
<td>0.027</td>
<td>0.1587 0.0425</td>
<td>1.7 \times 10^{-10}</td>
<td>0.028</td>
<td>0.1623 0.0434</td>
<td>1.729 \times 10^{-10}</td>
</tr>
<tr>
<td>5</td>
<td>0.0261</td>
<td>0.1562 0.0418</td>
<td>1.6 \times 10^{-10}</td>
<td>0.0239</td>
<td>0.1494 0.0400</td>
<td>1.669 \times 10^{-10}</td>
</tr>
</tbody>
</table>

Notes: 1. The only common ground between both works is Stripa Granite.
2. The differences in the decimal fractions are as explained on page 207-209 and are mainly due to the temperature effect and to the different techniques in the evaluation of hydraulic conductivity.
(table (3-3), then the source of errors are limited to the experimental work and to the effect of temperature in the work of Heard and Page (1982), while temperature is absent in the work of Gale (1981).

In summary, then, the solution of the partial equation gave the same results that the model at hand predicted in chapter (3), while working through parameter transformation. While in chapter (3), pure mathematics, using Pratt (1977) experimental work and Heard and Page (1982) experimental work, we found the partial equation in Chapter (4), giving approximate results using selected numerical values of permeability given by Gale (1981) in the vicinity of those given by Heard and Page. These values are neither temperature-dependent nor equal to those of Heard and Page, but since they are in Stripa Granite, it is expected to give closer values of the anisotropic parameters x and y (table (4-4)).

Both works (in chapter 3 and in chapter 4) are completely unrelated: in the analytical analysis and in, as well, the source of experimental data. Even the data is obtained by different techniques. One common ground to both is the Stripa granite. A significant result is obtained which is one of the creative contribution in this paper: The ability to quantify the anisotropic parameters once the ratio of the principal permeabilities \(\frac{K_y}{k_x}\) for any rock is known. It has been explained in chapter (3) that this ratio only is a function of direction.

Then we have a model at hand that adds many contributions to the field under discussion. It modifies in detail some simplified previous works giving rise to a shape function (curve) of fractured rock, that is, fracture that is not isolated from the matrix block in the sense that the inflow from matrix is considered.

And again we have a model that could be applied to any kind of rock including the craters on the moon (so to speak), had we this kind of rock on earth, beside the quantification of the anisotropic ellipse parameters as mentioned above.
As mentioned in sec (1-3), regarding the objectives of this dissertation, the work will proceed to determine the rate of flow of ground water in granite masses as a result of changes of fracture aperture due to induced thermal stresses.

Having used the most accurate data at hand, this dissertation ensures a matching technique based on analytical and experimental solutions. The other parameters and the different values of aperture can be quantified (chapter 5) followed by a conclusion to this work (chapter 6).

This paper asks: Is there any fracture problem? The answer is 'No', when the fracture is treated on its own merits, that is, in terms of the behavior of the flow that is related to the exact function of the fractured medium, regardless of the kind of porosity present, but not to follow the abstract theories that are difficult to identify or verify in the field.

Since the data from different sources fits properly in the model, then the model is theoretically consistent and compatible with the physical systems of the fractured crystalline rock observed in the field.

There is a comment to mention before ending this chapter: a side-result as relevant to the topic. It is instructive to use the parameters already treated in this dissertation and the consolidation theory as treated by Suklje (1969).

The partial equation (4-9) or (4-11), if considered in the nondimensional form, implies dimensionless time. This time is given by

\[
t^* = \frac{\kappa t}{l^2} \quad \text{(Suklje, 1969, p. 140 and equation 9.72)}
\]

The author indicates in his work on consolidation theory that transient flow behavior persists until \( t^* \) is of the order (1).

where \( \kappa \) - diffusivity \( \left( \frac{m^2}{sec} \right) \) and \( l \) - length of the region and \( \kappa = \frac{K}{S_s} \) where \( K \) is the permeability and \( S_s \) is the storativity (storage coefficient).
So

\[ t - t^* = \frac{L^2}{K} - \frac{(1)}{K} \]  

Now considering the work of Gale (1981) or the work of Heard and Page (1982). Where Gale gives \( K = 1.9 \times 10^{-10} \frac{m}{sec} \) (at \( 12^0 C \)) and Heard and Page give \( K = 2 \times 10^{-10} \frac{m}{sec} \) (scaled from fig (3-8) as a function of temperature) and \( S_s \) [as calculated from equation (4-31) with the value of transmissibility given by Gale (T)] \[ S_s = 18.1263 \times 10^{-15} \text{ (dimensionless)} \]

then

\[ \kappa = \frac{1.9 \times 10^{-17} \text{ m}^2/\text{sec}}{18.1263 \times 10^{-15}} = 0.1048 \times 10^{-2} \frac{\text{m}^2}{\text{sec}} \]

with Heard and Page (1982)

\[ K = 2 \times 10^{-17} \frac{\text{m}^2}{\text{sec}} \text{ and } \kappa = 0.1103 \times 10^{-2} \frac{\text{m}^2}{\text{sec}} \]

Figure (4-3) by Neretnieks (1985) shows Gidea Site. He mentioned on page (302) of his paper that rock blocks between fracture zones of the order of several cubic kilometers, could be found in which a repository could be sited.

So \( L \) could be taken 1000m, yet let us consider \( L = 100 \text{m} \), a very-modest and a minimum value, and calculate the time taken by radionuclides to reach the biosphere.

\[ t = (1) \frac{(100)^2}{0.1048 \times 10^{-2}} = 9 \times 10^6 \text{ years} \]

If \( L = 1000 \text{m} \)

\[ t = 9 \times 10^8 \text{ years} \]

These calculations tell us that the response time is comparable to geologic time scales.
With reference to Neretnieks Site at Gidea and considering the radionuclides of the radioactive waste, there are 3 species of actinides: technetium, ruthenium and nodymium, at the end of the Oklo reactors criticality (DOE Report, 1981, p. 255) by Curtis et al. (1981). There are fission products and part of these species was lost and contained within a few tens of meters of their source. This supports the idea of the geologic burial.

The authors at Los Alamos National Laboratory, in this respect (p. 278) by analogy to Oklo Reactors mines in Gabon (equatorial Africa), showed that the hydrologic transport was not the primary control of elements, but it was likely that the amount removed was controlled by the rate of release by diffusion of each element from the host phase. They observed also (p. 279) that the rates of transport were on the order of $10^{-5} \frac{m}{\text{year}}$ and that sorption ratio was great. Also sorption differs from rock to rock, yet the oxidation-reduction process controls the movement.

Using the given rate of transport ($10^{-5} \frac{m}{\text{year}}$), then the distance covered by the nuclear species to reach the biosphere (assuming $l = 100m$) = (rate/\text{year}) (number of years) = ($10^{-5} \frac{m}{\text{year}}$) (9 x $10^6$ years) = 90m, less than any distance at Gidea on all sides of the proposed burial.

Within the limits of this time (9 x $10^6$ years) or greater in large regions, $^{99}$Tc is a radionuclide of particular concern. Its half life ($t_{1/2}$) = 2.13 x $10^5$ years. It decays to the stable $^{99}$Ru. However, chemical and physical processes fractionate technetium from ruthenium at some time less than a million years after the reactor ceases criticality (Report, p. 270). This figure is within the limits of $t = 9 \times 10^6$ years or greater.

So the Oklo reactors represent an excellent natural analog to the geologic repositories. Hence, Neretnieks' site fig. (4-3) or similar sites to host the waste
could be the best choice provided that man activities (as mining and underground structures) and earthquakes are not operative in such sites.
CHAPTER 5
RESULTS OF THE CALCULATIONS

5-1 Quantification of Flow Parameters

The system of equations in chapter (2) leads to the modified fracture equation (2-61) considered a realistic approximation to the natural fracture. It is time now to apply the experimental data so far considered to quantify the transient parameters and to illustrate the effect of temperature in a series of graphs. It is instructive as well, to compare these graphs to their counterpart without temperature in this underground fracture model.

Equation (4-11) has already been confirmed by the application of the published data and Stripa Granite data and leads to two basic results considered major contributions.

1. Its capability to trace the flow path in the matrix-fracture system.

2. It helps to quantify the anisotropic parameters as an essential proof to what mathematical physics revealed by applying the theory of conformal mapping in chapter (3). Hence the theory and the data observed in the field, played a twin role in this respect.

Looking now at equation (2-61) to see if the experimental data would confirm the modifications introduced in this equation in chapter (2) figure (2-16)

\[
Q = \frac{1}{f} \frac{1}{\Delta} \frac{2\pi}{\ln \frac{r_e}{r_w}} \frac{n(b)^3}{12} \frac{\gamma}{\mu} (\Delta h) \tag{2-61}
\]

where the symbols are defined during the development of the equation.

\[ Q = \text{rate of flow} \quad \frac{cc}{min} \]

\[ f = (1 - 8.8 \left( \frac{k}{D} \right)^{1.5}) \]

where

\[ \frac{k}{D} = 0.033 \quad \text{(relative roughness - Louis (1969) - page 57, eq. (2-39))}. \]

\[ \Delta = \text{spacing} = 0.2058 \quad \text{(figure 2-15)}. \]

\[ n = \text{porosity}. \]

\[ r_e = \text{radial distance to pressure boundary} = 10 \, m \quad \text{(Gale, 1981)}. \]

\[ r_w = \text{radius of borehole} = 0.038 \, m \quad \text{(Gale, 1981)}. \]

\[ \gamma = \text{specific weight of water} = 980 \, \frac{gm}{cm^2 \, sec^2} \]

\[ \mu = \text{dynamic viscosity} \quad \text{(Table 2m-1, p. 2-191, sec. 2, Mechanics, American Institute of Physics Handbook, 1973)}. \]

\[ \Delta h = \text{difference in hydraulic head} \quad \text{(values from Gale, 1981)}. \]

\[ Q, b, \mu \text{ and } \Delta h \text{ are transient variables; } f, \Delta, r_e, r_w \text{ and } \gamma \text{ are constants.} \]

\[ Q = \Delta h \cdot 34.143, 31.747, 18.325, 23.083, 32.545 \quad \frac{cc}{min} \quad \text{(Gale 1981, p. 236)}. \]

From

\[ ke = \frac{1}{f} \frac{1}{\Delta} (b)^2 \gamma \frac{n}{12\mu} \]

(where \( ke \) (eq. 3-24) is quantified in table (3-2)) and using table (5-1) for \( ke \) (-k) values at the specific temperature and pressure and the above stated values, we get:

\[ <b> = 12971 \, \mu m \text{ and so also the other values of } <b> \text{ listed in table (5-1).} \]

The graph of permeability vs. aperture is shown in fig. (5-1).

And from eq. (2-61) we get the rate of flow \( (Q) \) using table (5-2). This parameter is listed in the last column of table (5-2). Figure (5-2) reflects the relation between the rate of flow and the aperture \( (b) \).
Table (5-3) lists the values of pressure and aperture and fig. (5-3) reflects the relation between these parameters.

Table (5-4) lists the rate of flow ($Q$) and the aperture ($b$) in log-log form and fig. (5-4) reflects this relation.

Comments and Observations on the temperature-dependent quantified values and their graphs.

At higher ($k$) and larger ($b$) (fig. (5-1) and table (5-1)), the curve is advancing under the control of pressure and temperature. Then the curve starts to change as the fracture is closing. Figure (5-1) suggests the transition zone from fast flow to slow flow. Then the curve passes through the point 1 $\mu$m (at $K = 1 \times 10^{-8}$ cm$^3$/sec) which is the value Kranz (1979, p. 230) termed it as "uncertainty in ($d$) is about 1 $\mu$m."

Ultimately the aperture closes to merge with the matrix porosity at 0.01 $\mu$m ($1 \times 10^{-6}$ cm) assumed minimum in the calculation of ($b$) and becomes a part of ($b$) while the fracture is existing.

Kranz et al. (1980, fig. 5-5) worked on Barre Granite at room temperature, using the parallel plate model and they commented on page 230 (1979): "The flat plate model predicts permeability and orders of magnitude higher than determined."

Regardless of the different techniques used to collect the data in somewhat different granites and using the parallel plate or the natural fracture, the curve of fig. (5-1) bears the shape pattern to that of Kranz et al. (fig. 5-5).

Considering the above comment and had Kranz considered the temperature parameter coupled with the pressure he used, both curves would show close similarity. However, this supports the fact that the permeability ($k$) is a unique function of the aperture ($b$). Figure (5-1) shows clearly the temperature-dependent ($k$-$b$) relation as compared to that of Kranz without temperature. Moreover, as for pressure alone (Kranz, fig. (5-5)) or pressure associated with temperature (this work,
Table 5-1. Calculation of the Aperture 'b' from the experimental temperature-dependent inferred Permeability ($K_e$).

\[ K_e = \frac{1}{f} \frac{1}{\Delta} (b)^{n+1} \frac{\gamma}{2\mu} \]  
(Fig. 3-8, Table 3-2, Eq. 2-61)

<table>
<thead>
<tr>
<th>Data compiled from several sources</th>
<th>Heard and Page data (1982)</th>
<th>Temp.-dep. aperture</th>
</tr>
</thead>
<tbody>
<tr>
<td>No.</td>
<td>$f$</td>
<td>$\Delta$</td>
</tr>
<tr>
<td>-----</td>
<td>-----</td>
<td>---------</td>
</tr>
<tr>
<td>1</td>
<td>1.0528</td>
<td>0.2058</td>
</tr>
<tr>
<td>2</td>
<td>1.0528</td>
<td>0.2058</td>
</tr>
<tr>
<td>3</td>
<td>1.0528</td>
<td>0.2058</td>
</tr>
<tr>
<td>4</td>
<td>1.0528</td>
<td>0.2058</td>
</tr>
<tr>
<td>5</td>
<td>1.0528</td>
<td>0.2058</td>
</tr>
</tbody>
</table>

$^{+\dagger}$Symbols are defined on page (215).
$^{+\dagger}$\( \mu \) = Dynamic viscosity. After 100\(^\circ C\), \( \mu \) is extrapolated from Table 2M-1, p. 2-191, American Institute of Physics Handbook (1973).
Fig.(5-1): Hydraulic conductivity is a unique function of the transient aperture under the influence of the induced thermal stress.
Table 5-2. Calculation of the Rate of Flow (Q) using Eq. (2-61).

\[ Q = \frac{1}{1} \frac{1}{\Delta} \left[ \frac{2\pi}{\ln r_0} \frac{r_e}{r_w} \right] n^+ \left( \frac{b}{\mu} \right)^3 \frac{T}{\mu} \Delta h \]

<table>
<thead>
<tr>
<th>No.</th>
<th>( f )</th>
<th>( \Delta )</th>
<th>( \gamma )</th>
<th>( \mu ) poise*</th>
<th>(b)*</th>
<th>( \ln \frac{r_e}{r_w} = \ln \frac{10}{0.038} ) Ratio+++</th>
<th>( \Delta h )+++</th>
<th>Q cm(^3) min</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0528</td>
<td>0.2058</td>
<td>980</td>
<td>0.01002</td>
<td>12.971</td>
<td>5.5728</td>
<td>3414.3</td>
<td>18.9516</td>
</tr>
<tr>
<td>2</td>
<td>1.0528</td>
<td>0.2058</td>
<td>980</td>
<td>0.002820</td>
<td>4.9906</td>
<td>5.5728</td>
<td>3174.7</td>
<td>3.5662</td>
</tr>
<tr>
<td>3</td>
<td>1.0528</td>
<td>0.2058</td>
<td>980</td>
<td>0.00008</td>
<td>2.0989</td>
<td>5.5728</td>
<td>1832.5</td>
<td>0.5662</td>
</tr>
<tr>
<td>4</td>
<td>1.0528</td>
<td>0.2058</td>
<td>980</td>
<td>0.000715</td>
<td>1.8110</td>
<td>5.5728</td>
<td>2308.3</td>
<td>0.5155</td>
</tr>
<tr>
<td>5</td>
<td>1.0528</td>
<td>0.2058</td>
<td>980</td>
<td>0.000065</td>
<td>1.6052</td>
<td>5.5728</td>
<td>3254.5</td>
<td>0.5044</td>
</tr>
</tbody>
</table>

*\( \mu \) and 'b' as in Table 5-1.

+\( \mu \) = porosity (not used). It is already used in Table 5-1, assumed a minimum value of \( 1 \times 10^{-4} \), negligible with respect to aperture and becomes a part of the fracture aperture.

+++Symbols are defined on page 2.15.

+++\( r_e, r_w, \Delta h \) are taken from Gale (1981) to compare any rate of flow in the vicinity of the measured points in his work.
fig. (5-1)), both curves close in similar fashion as the effects are minimal in both cases as the fracture closes.

The rate of flow \( (Q) \) as a function of the aperture \((b)\) is shown in fig. (5-2) and table (5-2). Point \((x)\) on the graph shows \( b \approx 0.4153 \) μm and \( Q \cdot 0.0091 \approx 0.01 \text{ cm}^3/\text{min}. \) These values are obtained by calculation and extrapolation of the curve. Both merge with the matrix value beyond this point. \( Q \) gets smaller and smaller as \( b \) decreases ultimately to 0.01 μm. In fact the value of \( b \approx 0.4153 \) μm (shown by \((x)\) in figure 5-2) is in the lower zone of uncertainty (Kranz considered uncertainty about 1 μm).

Figure (5-3) and table (5-3) show the aperture \((b)\) as a function of the pressure measured at the appropriate temperature (Heard and Page, 1982). The curve reflects the relation shown in figures (4-7, 4-8). So both aperture and conductivity decrease as pressure increases. The aperture merges with the porosity of the matrix as permeability stabilizes and fracture closes.

These results undoubtedly show the validity of the model at hand. The mathematical model, no matter how elegant it is, is measured by the degree of correspondence of the actual response observed in the field.

Figure (5-4) is a log-log plot relating \( Q \) to \( b^3 \) (table (5-4)). The appearance of a straight line depicts the equivalent cubic law. This is a result of the transient aperture continually decreasing and finally closing under the influence of the induced thermal stress. Figure (5-6) after Witherspoon et al. (1980) is a result of the parallel plate model under normal conditions and roughness \((f) \cdot 1.0\). The differences are obvious and the slope is different.

5-2 Comparison Between Parallel Plate Model and the Natural Model

The points determined by Gale (1981), are selected injection and/or withdrawal test data for each 2-m or 4-m interval in each borehole (page 95 of his report) to
Fig. (5-2): The rate of flow \( Q \) as a function of the aperture \( b \).
Table 5-3. The aperture as a function of the induced Thermal pressure.**

<table>
<thead>
<tr>
<th>Thermal Pressure† (MPa)</th>
<th>Aperture (b)* (μm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.9</td>
<td>12.971</td>
</tr>
<tr>
<td>13.8</td>
<td>4.9906</td>
</tr>
<tr>
<td>27.6</td>
<td>2.0989</td>
</tr>
<tr>
<td>41.4</td>
<td>1.8110</td>
</tr>
<tr>
<td>55.2</td>
<td>1.6052</td>
</tr>
</tbody>
</table>

† = Pressure (Fig. 3-8, Table 3-2)
* = Aperture (Table 5-1)

** (Heard and Page, 1982)
Fig. (5-3): The aperture (b) as a function of pressure (P). (b) and (K) (one to one correspondence) indicate the degree of correspondence of the actual observed response in the field. (Compare with fig(4-7).)
Table 5-4. Rate of Flow (Q) as a function of the "Thermal aperture" depicts the equivalent Cubic Law.

<table>
<thead>
<tr>
<th>Q cc/hr</th>
<th>(b)</th>
<th>(log Q)\times10</th>
<th>(log b)\times10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1137.0960</td>
<td>12.971</td>
<td>30.558</td>
<td>11.129</td>
</tr>
<tr>
<td>213.9720</td>
<td>4.9906</td>
<td>23.304</td>
<td>6.982</td>
</tr>
<tr>
<td>33.9720</td>
<td>2.0989</td>
<td>15.311</td>
<td>3.220</td>
</tr>
<tr>
<td>30.930</td>
<td>1.8110</td>
<td>14.904</td>
<td>2.579</td>
</tr>
<tr>
<td>30.2640</td>
<td>1.6052</td>
<td>14.809</td>
<td>2.055</td>
</tr>
</tbody>
</table>
Fig. (5-4): The rate of flow \((Q)\) as a function of the thermal aperture, depicts the equivalent cubic law, (Log-log diagram).
Fig. (5-5): Permeability of the joint alone as a function of joint aperture, $K_j$ is the theoretical value assuming a flat plate model. (After Kränz et al. 1980).
Fig. (5-6): Comparison of experimental results for radial flow through tension fracture in granite with cubic law. (After Witherspoon et al. 1980).
help to evaluate the hydraulic parameters. These points have different apertures with same viscosity at 12°C. In his work (page 76), he assumed that flow is taking place through a single fracture oriented normal to the borehole and computed the apertures for the measured flow pressures at the borehole. So the thermal induced stress in an increasing fashion to affect the closure of fracture is not there, but the apertures as they exist, fluctuating between 12.53,........15.32, 3.71 μm (page 236) are measured.

However, the experiment was not intended for this purpose but simply, in addition to the pumping borehole test, the aperture was evaluated as a function of the flow pressure.

What concerns this work from Gale's data is the following:

1. As a pumping test, it is used to illustrate the fractured flow curve as made of the inflow and the curvilinear flow. This is already done in chapter 4 (example 3 and figures 4-30, -31, -32) with other examples from published data.

2. As permeability is a function of aperture and as there is no published temperature-dependent data beside that of Heard and Page (1982) to the best of my knowledge, it is planned to use Gale's permeability values limited to the values in the vicinity of Heard and Page data to evaluate the anisotropic parameters x and y. Although they are neither temperature-dependent nor equal to those of Heard and Page, yet it is expected as they are in Stripa Granite, to give values near those calculated values using Heard and Page permeability. The difference would be due to the absence of temperature and the difference between the values of the permeabilities in both works as a result of the different techniques and the different acting forces. This suggested plan, as a fair approximation, was successful and the results are shown in table (4-4) and the calculations are on pages (206) and (207).
It is to be carefully observed that all curves are plotted and based on Heard and Page (1982) temperature-dependent-anisotropic permeability \( k_e = k - \sqrt{k_xk_y} \) and all the evaluated parameters are therefore thermohydromechanical parameters as discussed on page (205) by analogy to Cook and Myer (1979) in thermomechanical experiments.

Point (1) in table (5-2) is the only point that allows correlation between this work and Gale (1981). It is a starting point. Both viscosity and aperture are in close proximity.

Gale's report \((\mu = 0.0124 \text{ poise}, b = 12.53 \times 10^{-4} \text{ cm at } 12^\circ \text{C})\).
This work \((\mu = 0.01002 \text{ poise}, b = 12.971 \times 10^{-4} \text{ cm at } 19^\circ \text{C}, \text{table (5-2)})\).

Calculations will show the following:

Gale (Parallel plate and 2 \( b \equiv b \)) using page (236) of his report (1981)

\[
Q = \frac{2\pi}{2\mu} (2b)^3 \frac{1}{\ln \frac{r_e}{r_w}} (H_b - H_w) \\
\quad - \frac{2\pi(980)}{12(1.24 \times 10^{-2})} (12.53 \times 10^{-4})^3 \frac{1}{\ln \frac{10}{0.038}} \quad (3414.3)
\]

\[
- 0.05 \frac{\text{cc}}{\text{sec}} = 2.9914 \frac{\text{cc}}{\text{min}} \approx 3.0 \frac{\text{cc}}{\text{min}}
\]

(A)

With respect to temperature-dependent natural fracture (this dissertation).

\[
Q = \frac{1}{1 + \frac{1}{\Delta}} (b)^3 \frac{2\pi}{12\mu} \frac{\Delta h}{\ln \frac{r_e}{r_w}} \\
\quad - \frac{1}{1.0528} \frac{1}{0.2058} (12.971 \times 10^{-4})^3 \frac{2\pi(980)}{12(0.01002)} \frac{3414.3}{\ln \frac{10}{0.038}}
\]

\[
- 0.31586 \frac{\text{cc}}{\text{sec}} = 18.9516 \frac{\text{cc}}{\text{min}}
\]

(B)

Reversing the calculations to get (A) from (B), gives

\[
Q = 3.0 \frac{\text{cc}}{\text{min}}
\]

This shows that the setting of the parameters in eq. (2-61) is reasonable and indeed reflects the natural fracture. Also the transient aperture \( b \) is closer to the
right value based on the transient acting forces, that is, the thermal induced stress forces.

Apart from the induced thermal stress, all points will undergo the above mentioned differences. This means that the thermal induced stress passes to the fracture and affects the aperture and hence the permeability and the rate of flow. Whereas (B) is a result of the thermal induced stress, (A) is a result of the flow pressure in the fracture (no external forces acting as mentioned on page (228)). Hence with the induced thermal stress, another story starts with higher temperatures and pressures resulting in graphs (5-1, 5-2, 5-3, 5-4) which would reflect the work of Kranz et al. (fig. 5-5), had they considered the temperature parameter.

Had Prof. Gale, in his work shown \( K \cdot 6.329 \times 10^{-10} \frac{m}{sec} \) (Heard and Page, 1982) instead of \( K \cdot 6.2 \times 10^{-10} \frac{m}{sec} \) (p. 236 of his report) and used the variable viscosity (0.01002 at 19° C) instead of the constant viscosity (0.0124 at 12° C) which is correct with respect to his work (no thermal effect), \( Q \) would become (0.0553 \( \frac{cc}{sec} \approx 3.318 \frac{cc}{min} \)) as applied to the parallel plate. But now with natural fracture approximated by eq. (2-61) involving all modifications (\( f, \Delta, \mu, \) Tortuosity and Heard experimental permeability) under the control of the induced thermal stress:

\[
Q = 0.31586 \frac{cc}{sec} - 18.9516 \frac{cc}{min}
\]

For the purpose of clarification, the difference is due to the use of:

\( \mu = 0.01002 \) at 19° C instead of (0.0124 at 12° C)

\( f = 1.0528 \) instead of 1 (parallel plate)

\( \Delta = 0.2058 \) not used

\( k = 6.329 \times 10^{-10} \frac{m}{sec} \) instead of \( 6.2 \times 10^{-10} \frac{m}{sec} \)
or

\[
Q = 0.0553 \times \frac{0.0124}{0.01002} \times \frac{1}{1.0528} \times \frac{1}{0.2058} \times \frac{6.2 \times 10^{-10}}{6.329 \times 10^{-10}} \\
= 0.3094 \approx 0.31 \text{ cc/sec} - 18.9 \text{ cc/min}
\]

as compared to Gale (1981) in absence of the induced thermal stress

\[
Q = 0.05 \frac{\text{cc}}{\text{sec}} - 3.0 \frac{\text{cc}}{\text{min}} \quad \text{(Gale, 1981)}
\]

It is implicitly obvious that the values 0.05 and 0.0553 \(\frac{\text{cc}}{\text{sec}}\) reflect the use of the parallel plate in addition to the borehole errors discussed in chapter 4 and table (4-3), whereas the value 0.31586 \(\frac{\text{cc}}{\text{sec}}\) reflects the natural fracture eq. (2-61) under the influence of temperature, i.e., thermal induced stress.

Many authors cite the inadequacy of the parallel plate model. For example, Kranz et al. (1979, page 230), Tsang and Witherspoon (1985, p. 690) among others.

5.3 Applications

There are many diversified applications to the work at hand. It is an area of research among scores of researches that concentrate on the study of the geological formations as a result of the recent interests in the study of nuclear waste disposal. It is a key issue to estimate and to quantify the several parameters needed as a part of the performance of geological nuclear waste disposal. Heat as released from the decay of H.L.W., is a dominant factor. It causes mechanical deformations and movements of the rocks and the thermohydraulic flow. It is necessary then to enhance our understanding of the underground processes and the problems involved.

This work enhanced the two-fracture hydromechanical aspects to three aspects by considering the temperature aspect. This development is a progressive step down the road to a complete treatment of the fracture problem.
At the end of chapter 4, we referred to a relevant side-result concerning the transport, time and the distance covered by the nuclear species to reach the biosphere. Hence this study of fractures combined with fluids flowing in rock-fractured system under the temperature effects, constitutes a basic goal of interest beneficial to the biosphere through a knowledge of hydraulic conductivity in rocks.

Large stretches of arid zones such as Negev, Sinai, Sahara deserts and volcanic islands suffer from lack of water. These extensive areas of low permeability rocks, several kilometers underground that store large amounts of water could be explored by rock mechanics and hydrology techniques for aquifers and the evaluation of their water potential for industrial consumption and irrigation.

When researches show arid rocks of no water, these zones could be used as sites for toxic waste disposal.

Certain anomalies require careful studies, using pumping to provide the water samples and the data necessary to implement the theoretical models for the measurement of hydraulic heads and chemical properties in horizontal and vertical directions. Although technology is difficult to apply, yet it is worth to understand the flow system in such environments and to mine the hidden treasures (fluids, oil, minerals) that are vital to biosphere. However, technology and economics should combine for such achievements.

Although man normally disregards these anomalies, yet these are the valuable tools at hand and more effort is required to offset partly the world famine and partly to meet the requirements of the industrial countries.

However, fracturing is the only way for such developments in order to locate the rich water fracture zones and the arid zones and to use them accordingly.

Understanding of the flow system can help to solve the contamination problem which is a dilemma to some areas. In addition, the dilemma about to be created
by the HLW., will pose detrimental results if the flow system in fractured rock is not well understood.

Beside improving both understanding and technology at hand, this dissertation discussed, to some extent, various aspects that could be applied successfully. For instance, the hydraulic head (eq. 2-61) is essential to estimate the vertical gradient. Eq. (4-31) is an essential tool as applied to pumping tests already discussed in Sec. (4-5). Anisotropy is a very important aspect, when considered, that gives true estimate of flow parameters and a reliable evaluation of the aperture and hence the hydraulic conductivity. Anisotropy is usually determined in laboratory since field evaluation is difficult and expensive, yet this dissertation designed a mathematical method to estimate it (chapter 3) and used it successfully in chapters 3 and 4 to obtain the flow parameters as shown in tables (3-3, 4-4). The results fall in line with the experimental data obtained in the field. This mathematical method could save the effort and the money, once the principal permeabilities and the direction are known (eq. 3-25).

'High permeability' and 'low permeability' are two terms that require any investigator to look for as basis for any work suggested in this field. For instance, this is the basic idea under consideration when one thinks of the burial of the HLW.

And last but not least, the study of the underground water through rock fractured system, is a fertile project that enriches our knowledge about the flow and differentiates between various techniques used for the sake of economy and technology.
CHAPTER 6

SUMMARY AND CONCLUSIONS

Many points of interest are revealed by the model in this dissertation. It is an attempt to approximate the flow behavior of a natural fracture and to quantify the parameters involved under the transient forces. In order to reach this goal, a model representing flow in the fracture, is designed and published data is applied (chapter 2 and 3). Again, for confirmation, a non-study groundwater flow model is developed in confined and saturated aquifers (chapter 4), using a partial differential equation which is solved analytically and a different source of data is applied. Both models have a common ground (Stripa Granite). Their results are in close proximity. Hence the physics of the models is shown to account for phenomena observed in the field.

We have seen in chapter (4) that the discrete model is very difficult to use in the analysis of fractured media. Also the double porosity model shows that two responses in a single fracture are present violating the assumptions governing the flow in the aquifer.

Again, coupling, the discrete model with a continuum model proved to be inadequate as the author fell short of verifying it with field data.

This leads us to consider the high-low permeability relationships. This concept shows that the fractured system responds similar to an equivalent porous medium.

The consideration of the high-low permeability relationships, that is, the fast flow in fracture and the slow flow in the matrix, is implemented by using the published data in several fields. The fact that the data satisfied the model is a
proof that the model is realistic, otherwise, it will be inadequate as observed in other models.

The application of data in chapters (4) and (5) clearly verifies the above mentioned statements.

The parameters related to fracture are quantified for several temperatures, that is, under the thermal induced stress that changes for temperatures from 0°C-350°C. This allows quantitative predictions and measured values to approach each other without any discrepancy. This proved to be realistic as it was done by Cook and Myer (1979) in the well known thermomechanical experiments in Stripa. The parameter temperature was vital to remove the discrepancy between predicted values and the measured values of rock properties.

The conclusions that can be drawn from this model and the application of the published data to it, add several contributions to the field under discussion. It modifies in detail some simplified previous works, giving rise to a shape function of fractured rock, that is, a fracture that is not isolated from the matrix block in the sense that the inflow from matrix to fracture and from fracture to matrix is considered (chapter 4).

As another significant conclusion, the model could be applied to any kind of rock as seen in the several graphs (4-23 to 4-32) showing the behavior of the flow. Generality is observed throughout all the work.

As a third conclusion, the theory (chapter 3) is able to quantify the anisotropic parameters, using conformal mapping. The data observed in the field confirms the results and implements the model. Hence the theory and the data played a twin role in this respect. This is shown in chapter (5) implying significant results.
These results reflect the difference between the natural media model (this work) and some other models found in literature. The introduction of various variables into the fracture equation resulting in eq. (2-61) proved to be realistic when data is applied to it. For instance, the size of the aperture remains one of the most important parameters in any analysis of flow in fractured rock masses and is difficult to quantify through all models. It is a transient parameter and a result of transient forces under the control of temperature, pressure, and anisotropy. For its evaluation, measured values must correspond to predictive values as shown above. This natural media model approximates it pretty well. Moreover, these results as represented by several graphs in chapter (5), show clearly the difference between temperature-dependent parameters (this work) and those without temperature, indicating discrepancy if temperature is not used as shown in Cook and Myer experiments.

Based on these results, the model could be described as a coupled model: a thermomechanical model as represented by fracture deformation due to thermally induced stress, a thermohydrological model as represented by flow changes and a hydromechanical model as represented by pore pressure changes and variation of fracture aperture. Hence the model could be termed "Thermohydromechanical model of fractured rock" in the sense; that the stress is thermally induced leading to the change of the state of stress and to deformations and these in turn affect the opening and closing of fractures and hence the hydraulic flow.

As a summary of the conclusions and results, this work reformulated the flow equations in fractured rock to meet the requirements of the slow flow and the fast flow (low-high permeability flow, chapter 4). Basic modifications are introduced into the fracture equation (chapter 2) to achieve the transient state demands. Published data from several rock types and in particular from Stripa Granite, is
found to satisfy the physics of the problem. Moreover, the quantified parameters, to a greater extent, could be considered an essential part in the choice of the site for the burial of the high level waste. The aperture values may be needed for a successful work in related experiments (as in the work on rock fracture grouting).

In the hydromechanical couple, the heat parameter was ignored and hence the hydraulic parameters were not truly evaluated. In this thermohydromechanical model, the temperature is involved and the parameters are approximated. Chemical aspects are absent. However, this is a further step down the road.

The chemical discipline can be coupled with two or three of the 4 disciplines (chemical-, hydro-, mechanical- and thermo-disciplines). The four challenges are left to other colleagues having multidisciplinary fields including chemistry and physical chemistry majors.

I hope, finally, that these analyses will stimulate other colleagues and encourage further efforts in this multidisciplinary field.
APPENDICES
APPENDIX A

CONVERSION OF MASS AS APPLIED TO A CONTROL VOLUME

Consider a control volume at time $t$ and then at time $t + dt$.

Let control volume at time $t$ equal control volume at time $t + dt$ and mass $m$ is conserved, then

$$ (m_A)_t + (m_B)_t = (m_B)_{t+dt} + (m_C)_{t+dt} $$

or

$$ \frac{(m_B)_{t+dt} + (m_B)_t}{dt} = \frac{(m_A)_t - (m_C)_{t+dt}}{dt} $$

(A.1)

The left side of Eq. (A.1), in the limit gives

$$ \frac{\partial (m_B)}{\partial t} = \frac{\partial}{\partial t} \int_v \rho dv \cdot \text{time rate of change of mass in the original control volume} $$

and the right side of Eq. (A.1) is the flux of matter through the control surface attributable to the velocity of the flow.

Then flux-in at 1 and -out at 2 (Fig. A.1) is given by

$$ \frac{\partial}{\partial t} \int_v \rho dv = \int_s \rho \vec{q} \cdot dA_1 - \int_s \rho \vec{q} \cdot dA_2 $$

$$ = \int_s \rho \vec{q} \cdot dA $$

(A.2)

where

$\vec{q}$ is the velocity vector normal to the surface of the control volume

$dA$ is the vector differential area in the inward direction. Equation (A.2) is
\( q_1 dA_1 \) is positive

\( q_2 dA_2 \) is negative

\( q(x,y,z,t) \): Velocity field

Fig. (A-1): Conservation of mass as applied to a control volume.
written as
\[ \int_{V} \frac{\partial \rho}{\partial t} \, dv = \int_{S} \rho \mathbf{q} \cdot dA \]  \hspace{1cm} (A.3)

Since the fluid occupies the entire control volume, the differentiation is on the integrand. If the mass remains constant,
\[ \int_{S} \rho \mathbf{q} \cdot dA = 0 \rightarrow \text{applied to steady and compressible flow} \]

and
\[ \int_{S} \mathbf{q} \cdot dA = 0 \rightarrow \text{applied to steady and unsteady incompressible flow, since} \]
\[ \frac{\partial \rho}{\partial t} = 0 \]  \hspace{1cm} (A.4)

Equation (A.4), using the divergence theorem, is written as
\[ \int_{S} \rho \mathbf{q} \cdot dA = -\int_{V} \nabla \cdot \rho \mathbf{q} \, dv \]

Hence, Eq. (A.3) becomes
\[ \int_{V} \frac{\partial \rho}{\partial t} \, dv = -\int_{V} \nabla \cdot \rho \mathbf{q} \, dv \]

or
\[ \frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{q} = 0 \]  \hspace{1cm} (A.5)

Equation (A.5) gives the flux per unit volume without regard to porosity \( n \) and saturation \( s \). The terms \( n \) and \( s \) appear in the storage equation (Chapter 4).
\[ \frac{\partial \rho}{\partial t} = 0 \text{ for incompressible fluid} \]

then
\[ \nabla \cdot \mathbf{q} = 0 \]

also
Equation (A.6) is the continuity equation for steady and unsteady flow of incompressible fluid and is given by Eq. (2.9) in the text.
APPENDIX B

EQUATION OF THE MOTION
AND ITS DERIVATION FROM NEWTON'S SECOND LAW

This equation, referred to in the text as the Navier-Stokes equation (2.3) could be derived directly from Newton's second law (F = ma).

Consider Fig. (B.1) and the x components of Newton's law

\[ m a_x = F_x \]  \hspace{1cm} (B.1)

To evaluate \( F_x \), that is, the force acting on the element, the element is to be sufficiently small such that the normal and shearing stresses can be considered uniformly applied, in addition to a body force through the centroid of the element (that is, \( \rho g_x \)) where \( \rho \) is the density and \( g_x \) is the x component of gravity.

From Fig. (B.1)

\[
F_x = -\sigma_x dy \, dz + \left( \sigma_x + \frac{\partial \sigma_x}{\partial x} dx \right) dy \, dz \\
- \tau_{yx} \, dx \, dz + \left( \tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy \right) dx \, dz \\
- \tau_{zx} \, dx \, dy + \left( \tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} dz \right) dx \, dy \\
+ \rho g_x \, dx \, dy \, dz .
\]

Dividing by \( dx \, dy \, dz \), we get

\[
F_x = \rho g_x + \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} .
\]

Also \( m_a_x = \frac{m}{V} a_x = \rho a_x \) is the acceleration acting in the x direction where \( m \) and \( V \) are the mass and volume of the element.
Fig. B-1: Unit element and the x-component of the forces acting on it in the x-direction.
Hence, Eq. (B.1) could be written as

\[ \rho a_x = \rho g_x + \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \]  \hspace{1cm} (B.2)

From the elasticity theory

\begin{align*}
(1) \quad & \epsilon_x = \frac{1}{E} \left[ (\sigma_x - \nu(\sigma_y + \sigma_z)) \right] \quad \text{Hooke's law} \\
(2) \quad & G = \frac{E}{2(1+\nu)} \quad \text{shear modulus} \\
(3) \quad & \epsilon = \epsilon_x + \epsilon_y + \epsilon_z = \Delta \\
& = \frac{1-2\nu}{E} (\sigma_x + \sigma_y + \sigma_z) \quad \text{volume dilation}
\end{align*}

and let \( \theta = \sigma_x + \sigma_y + \sigma_z \)

\[ \theta = \frac{1}{3} (\sigma_x + \sigma_y + \sigma_z) = -p . \]

where \( \epsilon \) denotes the strains
\( \sigma \) denotes the stresses
\( \theta \) is the arithmetic mean of the normal stresses
\( -p \) = hydrostatic compressive pressure applied to incompressible fluids.

\( G, E, \nu \) are shear modulus, Young's modulus, and Poisson's ratio.

Equations (1), (2), and (3), give

\[ \sigma_x = 2G \left[ \epsilon_x + \frac{\nu \epsilon}{1-2\nu} \right] . \]  \hspace{1cm} (5)

Adding \( \theta - \theta \) to both sides of Eq. 5 and simplifying using Eqs. (2) and (3), then

\[ \sigma_x - \theta = 2G \epsilon_x - 2G \frac{\epsilon}{3} = 2G \left[ \epsilon_x - \frac{\epsilon}{3} \right] . \]  \hspace{1cm} (6)

For a Newtonian substance, shear modulus \( G \) is replaced by viscosity \( \mu \) and the stress is related to the rate of strain (Jaeger and Cook 1979, p. 314). Hence Eq. (6) is rewritten as

\[ \sigma_x - \theta = 2\mu \frac{\partial}{\partial t} \left( \epsilon_x - \frac{\epsilon}{3} \right) \]

or, using Eq. (4),
\[ \sigma_x = -p + 2\mu \frac{\partial e_x}{\partial t} - \frac{2}{3} \mu \frac{\partial \epsilon}{\partial t}. \]

Since \( \frac{\partial \epsilon}{\partial t} = \frac{\partial A}{\partial t} = \nabla \cdot \dot{q} = 0 \), where \( \epsilon \) is given by Eq. (3), that is, the rate of volume dilation equals zero for incompressible fluid.

Hence,

\[ \sigma_x = -p + 2\mu \frac{\partial e_x}{\partial t} - \frac{2}{3} \mu (\nabla \cdot \dot{q}) \quad (B.3) \]

Now, using the relation between shear stress, shear strain, and the displacements of a point under deformation,

\[ \tau_{yx} = G \gamma_{yx} = \mu \gamma_{yx} = \mu \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \]

and

\[ \tau_{zx} = G \gamma_{zx} = \mu \gamma_{zx} = \mu \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \quad (B.4) \]

where \( u, v, w \) are the displacements and \( \mu \) is the viscosity.

Substituting Eqs. (B.3) and (B.4) into (B.2) and manipulating the differentiation, we get

\[ \rho \dot{e}_x - \frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x \partial y} = \rho a_x. \]

or, in vector form,

\[ \rho \ddot{q} - \nabla p + \mu \nabla^2 q + \frac{\mu}{3} \nabla (\nabla \cdot \dot{q}) = \rho \frac{\partial \dot{\epsilon}}{\partial t} + \rho (\dot{q} \cdot \nabla) \dot{q}. \quad (B.5) \]

where \( \rho a_x = \rho \frac{\partial \dot{q}}{\partial t} + \rho (\dot{q} \cdot \nabla) \dot{q} \).

Now, if \( \nabla \cdot \dot{q} = 0 \) for incompressible fluid, that is,

the time rate dilation = 0 \( \left( \frac{\partial \epsilon}{\partial t} = \nabla \cdot q = 0 \right) \),

and if \( \frac{\partial \dot{q}}{\partial t} = 0 \) steady flow,

and if \( (\dot{q} \cdot \nabla) \dot{q} \) = convective accelerations, being small and neglected.

Eq. (B.5) becomes:

\[ \rho \ddot{q} - \nabla p + \mu \nabla^2 q = 0. \]
If \( h \) is the vertical direction, and \( h \) and the elevation \( z \) coincide, and \( g_x = 0 \), \( g_y = 0 \), and dividing by \( \rho \), then

\[
-g \nabla z - \frac{1}{\rho} \nabla p + \frac{\mu}{\rho} \nabla^2 q = 0
\]

This is Eq. (2.4) in the text, where \( g_z = -g \frac{\partial h}{\partial z} \) or \( \tilde{g} = -g \nabla h = -g \nabla z \)
APPENDIX C
DERIVATION OF SOME EQUATIONS AND PARAMETERS

Parsons (1966) proved that a fracture-matrix system is identical to an anisotropic system. It follows from his work (Eqs. (A.7), (A.8)) that

\[ K_x \cos^2 \delta + K_y \sin^2 \delta = A' \]  \hspace{1cm} (1)
\[ K_x \sin^2 \delta + K_y \cos^2 \delta = B' \]  \hspace{1cm} (AC-1)
\[ (K_x - K_y) \sin \delta \cos \delta = C' \]  \hspace{1cm} (3)

Now, adding Eqs. (1) and (2) and then subtracting (2) from (1),
\[ K_x + K_y = A' + B' \]  \hspace{1cm} (4)
\[ K_x - K_y = \frac{A' - B'}{\cos 2\delta} \]  \hspace{1cm} (5)

Again adding Eqs. (4) and (5):
\[ K_x = \frac{1}{2} \left[ A' + B' + \sqrt{(K_x - K_y)^2} \right] \]  \hspace{1cm} (6)

From (3):
\[ 4C'^2 = (K_x - K_y)^2 \sin^2 2\delta \]  \hspace{1cm} (7)

From (5):
\[ (A' - B')^2 = (K_x - K_y)^2 \cos^2 2\delta \]  \hspace{1cm} (8)

From Eqs. (6), (7), and (8):
\[ K_x = \frac{1}{2} \left[ A' + B' + \sqrt{4C'^2 + (A' - B')^2} \right] . \]  \hspace{1cm} (AC-2)

Similarly
\[ K_y = \frac{1}{2} \left[ A' + B' - \sqrt{4C'^2 + (A' - B')^2} \right] \]

Again from Eqs. (3) and (5)
\[ \cos 2\delta \cdot \frac{C'}{\sin \delta \cos \delta} = A' - B' \]
or
\[
\frac{c'}{A' - B'} = \frac{\sin \delta \cos \delta}{\cos 2\delta} = \frac{\sin 2\delta}{2 \cos 2\delta} \tag{AC.3}
\]

Equation (3.12) could be drawn on the following lines with reference to Fig. (3.6):

\[\nabla \phi = V \cos(\phi - \delta) \text{ in the } \nabla \phi \text{ direction}\]

\[K_p = -\mu V \cos (\phi - \delta)/\nabla \phi \text{ by definition (Darcy's law)}\]

\[K_p = -\mu \left[ \frac{V \cos \phi \cos \delta + V \sin \phi \sin \delta}{\nabla \phi} \right] \tag{A}\]

where:

\[\nabla \phi = \text{ the hydraulic head}\]

\[\phi = \text{ angle } \nabla \phi \text{ makes with the x-axis} \]

\[\delta = \text{ angle } V \text{ makes with the x-axis} \]

\[\mu = \text{ viscosity of fluid} \]

\[V = \text{ velocity of fluid} \]

\[K_p = \text{ directional permeability in the direction of the hydraulic gradient.} \]

The components of \(V\) (by Darcy’s law) are

\[V_x = -\frac{k_x}{\mu} \frac{\partial \phi}{\partial x}; \quad V_y = -\frac{k_y}{\mu} \frac{\partial \phi}{\partial y}\]

and from the figure,

\[\frac{\partial \phi}{\partial y} = \nabla \phi \cos \phi\]

and

\[\frac{\partial \phi}{\partial y} = \nabla \phi \sin \phi .\]

\[V_x = V \cos \delta\]

and

\[V_y = V \sin \delta .\]

Hence \(V_x = -\frac{k_x}{\mu} \nabla \phi \cos \phi; \quad V_y = -\frac{k_y}{\mu} \nabla \phi \sin \phi\). Substitution in (A) gives
Again, with reference to Fig. (3.6), the permeability in the direction of the flow, Eq. (3.17) could be found.

Consider Darcy's law

\[ \nabla \phi = - \frac{\mu}{V_x \cos \phi \cos \delta + V_y \sin \phi \sin \delta} \]

or

\[ K_p = K_x \cos^2 \phi + K_y \sin^2 \phi \quad (AC.4) \]

From Fig. (3.6),

\[ \nabla \phi = V \cos(\phi - \delta), \quad V_x = V \cos \delta, \quad V_y = V \sin \delta \]

Hence, \( K_\phi = -\mu \left[ \frac{V \cos \phi \cos \delta + V \sin \phi \sin \delta}{\nabla \phi_x + \nabla \phi_y} \right] \]

\[ = -\mu \frac{V \phi}{\nabla \phi} = -\mu \left[ \frac{V \cos \phi \cos \delta + V \sin \phi \sin \delta}{-\mu \frac{V_x}{K_x} \cos \phi - \frac{\mu V_y}{K_y} \sin \phi} \right] \]

Since Darcy's law gives

\[ V_x = -\frac{K_x}{\mu} \frac{\partial \phi}{\partial x}, \quad V_y = -\frac{K_y}{\mu} \frac{\partial \phi}{\partial y} \]

and
\[ \frac{\partial \phi}{\partial x} = \nabla \phi \cos \phi, \quad \frac{\partial \phi}{\partial y} = \nabla \phi \sin \phi \]

Hence

\[ \nabla \phi_x = -\mu \frac{V_x}{K_x} \cos \phi \text{ and } \nabla \phi_y = -\mu \frac{V_y}{K_y} \sin \phi \]

So

\[ K_\phi = \frac{\cos \phi \cos \delta + \sin \phi \sin \delta}{\frac{1}{K_x} \cos \phi \cdot \frac{V_x}{V} + \frac{1}{K_y} \frac{V_y}{V} \sin \phi} \]

Letting \( \phi \) coincide with \( \delta \),

\[ K_\delta = \frac{\cos^2 \delta + \sin^2 \delta}{\frac{1}{K_x} + \frac{1}{K_y}} \]

or

\[ \frac{1}{K_\delta} = \frac{\cos^2 \delta + \sin^2 \delta}{K_x} \frac{K_x}{K_y} \]

The difference between \( K_p \) and \( K_\delta \) (the permeabilities in the directions of the pressure gradient and the streamline respectively).

Consider Eqs. (AC.4) and (AC.5):

From (AC.4):

\[ \frac{K_p}{K_x} = \frac{K_y}{K_x} \sin^2 \phi \quad (A) \]

From (AC.5):

\[ \frac{K_x}{K_\delta} = \frac{K_x}{K_y} \sin^2 \delta \quad (B) \]

From (A) and (B) and assuming \( \delta = \phi \):

So

\[ \frac{K_p}{K_x} \cdot \frac{K_x}{K_\delta} = \frac{K_p}{K_\delta} = \left[ \frac{\cos^2 \delta + \frac{K_y}{K_x} \sin^2 \delta}{\frac{K_x}{K_y} \sin^2 \delta} \right] \left[ \cos^2 \delta + \frac{K_x}{K_y} \sin^2 \delta \right] \]

\[ = \cos^4 \delta + \sin^4 \delta + \left[ \frac{K_x}{K_y} - \frac{K_y}{K_x} \right] \sin^2 \delta \cos^2 \delta \]

\[ = 1 + \left[ \frac{K_x}{K_y} + \frac{K_x}{K_y} \left[ \frac{K_y}{K_x} - \frac{K_y}{K_x} \right]^2 \right] \sin^2 \delta \cos^2 \delta \]
or

\[ \frac{K_p}{K_\delta} = 1 + \frac{K_x}{K_y} \left( 1 - \frac{K_y}{K_x} \right)^2 \sin^2 \delta \cos^2 \delta \]

Hence the difference = \( \frac{K_x}{K_y} \left( 1 - \frac{K_y}{K_x} \right)^2 \sin^2 \delta \cos^2 \delta \) \hspace{1cm} (AC.6)

This is Eq. (3.32) in the text.
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