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Rigorous grid truncation for the finite element solution of electromagnetic scattering problems

Lee, Robert, Ph.D.
The University of Arizona, 1990
RIGOROUS GRID TRUNCATION
FOR THE FINITE ELEMENT SOLUTION OF
ELECTROMAGNETIC SCATTERING PROBLEMS

by

Robert Lee

A Dissertation Submitted to the Faculty of the
DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING
In Partial Fulfillment of the Requirements
For the Degree of
DOCTOR OF PHILOSOPHY
WITH A MAJOR IN ELECTRICAL ENGINEERING
In the Graduate College
THE UNIVERSITY OF ARIZONA

1990
THE UNIVERSITY OF ARIZONA
GRADUATE COLLEGE

As members of the Final Examination Committee, we certify that we have read the dissertation prepared by Robert Lee

entitled Rigorous Grid Truncation for the Finite Element Solution of Electromagnetic Scattering Problems

and recommend that it be accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy.

Final approval and acceptance of this dissertation is contingent upon the candidate's submission of the final copy of the dissertation to the Graduate College.

I hereby certify that I have read this dissertation prepared under my direction and recommend that it be accepted as fulfilling the dissertation requirement.

Dissertation Director

Date
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SIGNED: Robert Le
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ABSTRACT

The bymoment method is presented for the analysis of electromagnetic wave scattering from cylinders in an unbounded region. The method introduces a conforming surface to geometrically decouple the interior region containing the scatterer from the exterior region extending to infinity. The solution in the interior is generated from the standard finite element solution of an interior Dirichlet boundary-value problem. The interior solution is then coupled to the exterior by applying Green's theorem to the scattered field and each one of the members in a set of properly chosen testing functions. Because the finite element method is used for the solution in the interior region, the cylinder may be of arbitrary shape, and its material properties may, in general, be inhomogeneous. To demonstrate the capabilities of the method several problems involving cylindrical geometries are considered. The first is the problem of electromagnetic scattering from a single infinitely long cylinder in free space. The wave vector of the incident field is assumed to be normal to the axis of the cylinder so that the problem becomes two dimensional. The second is the case where there are a multiple number of parallel, infinitely long cylinders in free space. In this instance, an individual finite element grid is generated for each cylinder, and the coupling between the cylinders results from the application of Green's theorem. Next, the cylinder is placed in the presence of two semi-infinite half-spaces. This requires the evaluation of Sommerfeld-type integrals for the testing functions. Finally, the wave vector of the incident field is allowed
to be obliquely incident on the cylinder. Numerical results are presented and compared to eigenfunction series and integral equation solutions in order to validate the method.
CHAPTER 1

INTRODUCTION

The application of the finite element method for the analysis of electromagnetic wave scattering by complex structures has been a major research topic over the past fifteen years. This is mainly due to the fact that the numerical implementation of the finite element method results in very sparse matrices, in contrast to the full matrices generated by the more traditional integral equation formulation. Sparse matrices are computationally attractive because they allow significant reductions in computer memory as well as computation time. Therefore, the finite-element method appears to be the perfect candidate for the analysis of electromagnetic wave scattering by very large, complex structures.

The major obstacle in the application of the finite element method to problems involving unbounded regions is that differential equation-based methods are always formulated as boundary-value problems. From a computational point of view, this implies that an artificial boundary is required for the truncation of the computational domain. However, such a boundary should be transparent to the solution in the sense that an appropriate mechanism should be used to couple the solution inside the truncation boundary to the solution outside, ensuring the proper continuity of the fields along the boundary, as well as the radiation condition at infinity. This coupling can be accomplished by either local or non-local methods. In the local methods, we apply an absorbing boundary condition at the nodes on the boundary of the mesh by establishing a relationship between the value of the
unknown quantity and its derivatives at each node. Assuming that the truncation boundary is a circle, the simplest absorbing boundary condition is actually the Sommerfeld radiation condition since this condition relates the radial derivative of the field to the field itself. Thus, the absorbing boundary relationship attempts to simulate the condition in which the waves are entirely outgoing such that the truncation boundary is as transparent as possible to the waves impinging on it from the interior. Obviously, in the limit where the truncation boundary recedes to infinity, the absorbing boundary condition becomes exact.

Unfortunately, for truncation boundaries at a finite distance from the scatterer an exact relationship cannot be established without the coupling of all the nodes along the boundary. This fact destroys the optimum banded structure of the finite element matrix. Instead, approximate methods are formulated which decouple the nodes on the boundary. Examples of such methods are the absorbing boundary conditions introduced by Engquist and Majda [1977], and Bayliss, Gunzburger, and Turkel [1982]. One of the major advantages of absorbing boundary conditions is that they keep the structure of the finite element matrix intact so that the node numbering can be done in such a way that the optimum bandwidth is attained. Another advantage is that these methods tend to be easily implemented into any finite element code. The major disadvantage is that they may give inaccurate results when the boundary of the mesh is truncated close to the scatterer because of the approximate nature of the boundary condition. To assure an accurate solution, the boundary must be extended a large distance from the scatterer. Thus, the finite element solution must be generated for a larger problem domain. This large domain can dramatically increase the both the computational time and the storage requirements. The choice of where to place the boundary is also not that
clear. Currently, the only way to determine the correct placement of the boundary for complex geometries is empirical. The minimum distance between the scatterer and the grid boundary is dependent upon the geometry, the characteristics of the excitation, and the degree of accuracy desired; so, in principle, numerical tests must be run for each new geometry.

On the other hand, non-local methods are used to apply an accurate boundary condition at the nodes on the truncation boundary. These methods are computationally more intensive than the local methods, and their implementation is in some cases very complex. The first non-local methods were introduced by Silvester and Hsieh [1971], and McDonald and Wexler [1972]. In their formulation, a surface integral equation for the fields in the exterior region was imposed as a boundary constraint on the finite element solution in the interior region. This led to the development of the so-called hybrid finite element method (HFEM). Since then several modifications and/or extensions of the method have appeared such as that by MacCamy and Marin [1980]. Further work was done by Lynch, Paulsen, and Strohbehn [1986]. They then extended the work to three dimensions [Paulsen, Lynch, and Strohbehn, 1988]. The HFEM was also applied to coated cylinders by Jin and Liepa [1988a]. One of the disadvantages of the HFEM is that the optimum bandwidth of the matrix cannot be attained since the applied boundary condition couples all the nodes on the boundary. Recently, however, an alternative formulation of the final matrix equations has been proposed that allows for the optimum bandwidth, and it has improved considerably the numerical efficiency of the HFEM [Jin and Liepa, 1988b].

Another approach, introduced by Mei [1974], for coupling the interior and exterior solutions was the unimoment method. This method was later applied
extensively to electromagnetic scattering problems involving two-dimensional inhomogeneous structures [Chang and Mei, 1976], as well as inhomogeneous penetrable bodies of revolution [Morgan and Mei, 1979]. A further generalization of the method, which removed the axisymmetric restriction of the inhomogeneous media, was presented recently [Morgan, 1988]. The body of revolution problem was extended by Chang [1981] to handle the case where the exterior region has a media interface. This extension required the evaluation of generalized Sommerfeld integrals [Chang, 1980]. Cheng [1983] then modified the theory to allow the body of revolution to have an arbitrary orientation over the media interface. The unimoment method uses a separable surface for the truncation boundary to decouple the solution in the interior region where the scatterer is present, from the solution in the exterior. Then the fields in the interior are found from the finite-element solution of a standard Dirichlet boundary-value problem, while appropriate eigenfunction expansions for the specific coordinate system are used to represent the fields in the exterior region. Finally, the interior and exterior problems are coupled on a separable surface where the continuity of the tangential electric and magnetic fields are imposed.

At this point, we also mention the recently developed transfinite element method. This method uses the basic ideas of the unimoment method with a different functional that results in a computationally efficient solution [Lee and Cendes, 1987]. The use of a separable surface was thought to be an inherent limitation of the unimoment method since the finite element solution within a circular/spherical surface becomes rather inefficient for elongated scatterers that occupy only a small portion of the enclosed volume. A new method was then developed by Morgan, Chen, Hill and Barber [1984], which combined the finite element solution of the
interior region with the surface integral equation used in the extended boundary condition method to circumvent the need for a separable boundary. However, this hybrid method was found to have convergence difficulties when dealing with very elongated objects.

Recently, the field feedback formulation has been proposed in an attempt to overcome some of the disadvantages of the aforementioned methods [Morgan and Welch, 1986]. This method employs a surface that conforms to the scatterer to decouple interior and exterior regions. The solution in the interior is then found for an interior Dirichlet boundary value problem with prescribed values of the tangential electric and/or magnetic fields on the enclosing surface. These boundary conditions are given in terms of an appropriate expansion of vector basis functions whose coefficients are found from the coupling of the interior region solution to the unbounded region and the equivalence principle.

In this dissertation, we introduce a new method called the bymoment method. The bymoment method decouples the exterior and interior regions by using a surface that conforms to the scatterer. On this surface, the tangential electric and/or magnetic fields are expressed in terms of a set of appropriate basis functions whose coefficients are to be determined. The interior solution is then generated in exactly the same manner as the unimoment method and is expressed in terms of the unknown coefficients from the expansion of the tangential fields on the enclosing surface. Finally, an application of Green’s theorem over a volume enclosed by a surface just inside the finite-element mesh and the surface at infinity allows us to couple the interior solution to the exterior region and solve for the unknown coefficients.
Because of the interest in problems involving cylinders near earth, we shall apply the bymoment method to the problem of a cylinder in the presence of a media interface. This problem has applications to such areas as geophysics, radar design, microwave or high speed integrated circuits, light scattering from wafers, and target identification. There have been numerous papers written on the problem of cylinders near an interface, but none of them involved the use of the finite element method. Some of the earliest investigations were done by Hohmann [1971] and Parry and Ward [1971]. They applied integral equation techniques to obtain numerical solutions to homogeneous cylinders. Howard [1972] and Mahmoud, Ali, and Wait [1981] used a multipole expansion of the scattered field to obtain a solution. Howard and Kretzschmar [1972] developed a technique which they called the volume current method. Recently, a method of moments approach was adopted by Butler, Xu and Glisson [1985]. This work was further extended by Xu and Butler [1986,1987].

Another interesting problem is the case where the incoming wave is obliquely incident. Results for the general case of oblique incidence have been rather sparse, since the early results of Wait [1955]. The problem has important practical applications in various areas, such as diffraction grating analysis and design and scattering by finite composite cylinders with cross-sectional dimensions much smaller than their length. Indeed, results for scattering from infinite cylinders may be usefully applied to the finite case so long as the angle between the incoming wave vector and the vector normal to the cylinder axis is small. This idea has been experimentally verified by Lind and Greenberg [1966]. Wu and Chen [1986] used the variational reaction theory [1985] for a finite element solution of the problem for the general case of inhomogeneous, anisotropic cylinders with arbitrary shapes. An electric field integral equation formulation for the case of non-magnetic cylinders was given by Su
[1987]. He used the conjugate gradient method and FFT to expedite the inversion of the resulting matrix equation. More recently, Rojas [1988] presented a coupled integral equation formulation whose unknowns are the total electric and magnetic field vectors. He also accounted for the magnetic properties of the materials in the cylinder.

This dissertation is organized into six chapters in addition to the introduction and conclusion. In Chapter 2, the basic theory of the finite element method is discussed. This discussion includes the derivation of the variational expression, the formation of the finite element matrix, and the solution of the resulting matrix equation. In the next four chapters, the bymoment method is formulated and applied to various geometries in both free space and half space regions. The chapter concludes with a discussion of numerical problems associated with the bymoment method and the finite element method.

In chapter 3, we formulate the bymoment method for the single cylinder. We include comparisons to both series and integral equation solutions in order to validate the method.

Another problem of interest is the case where there are a multiple number of cylinders. The formulation for the multiple cylinder geometry is presented in Chapter 4. Many of the finite element techniques, such as the absorbing boundary condition methods and the unimoment method, require that the mesh enclose all the cylinders. This fact makes these methods inefficient, especially for problems where the cylinders are far apart. Because of the way the bymoment method is formulated, it allows the use of multiple meshes, each of which conforms to one of the cylinders. In addition, when the cylinders are identical, the finite element
solutions need only be generated for one of the cylinders. Thus, the interaction between a large number of identical cylinders can be modeled very efficiently. To demonstrate the accuracy of the resulting solutions, comparisons are made to series solutions given by Ragheb and Hamid [1985].

In Chapter 5, we consider the problem of plane wave scattering from a cylinder in the presence of a media interface. Because of the media interface, the coupling between the interior finite element solution and the exterior region is more complicated than for a free space exterior. The formulation requires the evaluation of the Green's function for the media interface geometry. This evaluation, which is in the form of an infinite integral, is numerically difficult because of the rapid variation and slow decay of the integrand. Numerical results for several geometries are presented along with comparisons to method of moments results.

In Chapter 6, we generalize the formulation in Chapter 3 so that the excitation can be either a plane wave impinging on the cylinder at an arbitrary angle or a traveling wave field with \( z \) dependence of the form \( \exp(-i\beta z) \). The \( z \) direction in this case is parallel to the axis of the cylinder. The finite element formulation used in the bymoment method is based on a variational expression derived from the method of weighted residuals which couples the \( z \) components of the electric and magnetic fields. Numerical results are generated for various homogeneous and inhomogeneous cylinders and compared to series and integral equation solutions.

In Chapter 7, various problems that can occur in the numerical implementation of both the bymoment method and the finite element method are discussed. Numerical results are presented to show these errors, and suggestions are given to remedy some them.
Finally, the dissertation is summarized in Chapter 8. Recommendations for future work are also provided.
CHAPTER 2

FINITE ELEMENT ANALYSIS

In this chapter, we review the necessary theory to solve a two-dimensional boundary value problem using the finite element method with either Dirichlet or Neumann boundary conditions. We begin by applying the method of weighted residuals to the Helmholtz equation and incorporating the correct boundary conditions in the resulting expression. The finite element mesh is generated with quadrilateral elements. The associated sub-domain basis functions are assumed to have bilinear variation within each element. These basis functions lead to the generation of a sparse banded matrix. A modified Gauss-Choleski algorithm is then used to factor and solve this matrix.

2.1 The Weak Formulation

Consider the case where the fields do not vary in the \( z \) direction. From Maxwell's equations, we can reduce the problem to that of solving the two-dimensional Helmholtz equation in \( z \) and \( y \) for either \( E_z \) or \( H_z \). The polarization where \( H_z = 0 \) is called transverse magnetic with respect to \( z \) (TMz) and conversely, the polarization where \( E_z = 0 \) is called transverse electric with respect to \( z \) (TEz). The Helmholtz equation for both cases is given by

\[
(\nabla^2 + k_z^2) \begin{pmatrix} E_z \\ H_z \end{pmatrix} = 0
\]  

(2 - 1)
where \( k_c = k_0 \sqrt{\mu_{rc}(\varepsilon_{rc} - i\sigma_c/\omega\varepsilon_0)} \) is the wave number for the problem region. The variable \( k_0 \) denotes the wave number in free space, and \( \mu_{rc} \) is the relative permeability. The variable \( \varepsilon_{rc} \) represents the relative permittivity, and \( \sigma_c \) is the the conductivity of the region. The symbol \( \nabla^2 \) represents the two-dimensional Laplacian which is given by \( \nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 \). In equation (2-1), we make the assumption that the problem region has no sources. In order to simplify the programming we restrict the material properties to be constant within each element in the finite element mesh. The \( e^{j\omega t} \) time variation has been suppressed in (2-1).

The variational expression for the finite element problem is determined from the method of weighted residuals. In order to see how the method of weighted residuals works and where the term originated, consider the differential equation

\[
L u = f \tag{2-2}
\]

where \( L \) is the differential operator, \( f \) is the forcing term, and \( u \) is the solution defined over some domain \( D \). Now choose \( \hat{u} \) to be an approximate solution for \( u \) where \( \hat{u} \) is defined over the domain \( D' \). In this case \( D' \subset D \), but \( D' \not\subset D \). Therefore, if we define the residual error term to be

\[
R = L\hat{u} - f, \tag{2-3}
\]

\( R \) can never vanish if \( u \not\in D' \). Instead, by weighting the residual term with some function \( W \) and integrating the result over the problem region, we obtain a solution for \( \hat{u} \) which satisfies

\[
\int \int RW \, dS = 0. \tag{2-4}
\]
For the finite element problem, (2-4) must be converted into a system of equations. Therefore, let us consider a set of weighting functions

\[ \{\psi\} = (\psi_j : j = 1, 2, \ldots) \quad (2 - 5) \]

The application of the method of weighted residuals to (2-1) leads to

\[ \int \int_{\Omega} \psi_j (\nabla^2 + k_z^2) \left( \frac{\hat{E}_z}{\hat{H}_z} \right) = 0 \quad (2 - 6) \]

where \( \hat{E}_z, \hat{H}_z \) are the approximate solutions for \( E_z, H_z \) and \( \Omega \) represents the surface of the finite element mesh. \( \Omega \) is replaced by a collection \( \Omega_h \) which is composed of simple surfaces \( \Omega_e \) for each element as follows:

\[ \Omega_h \simeq \Omega, \quad \Omega_h = \bigcup_{e=1}^{N_e} \Omega_e \quad (2 - 7) \]

where \( N_e \) is the number of elements in the mesh. Therefore, (2-6) can be rewritten in terms of an integration over all the elements,

\[ \sum_{1}^{N_e} \int \int_{\Omega_e} \psi_j (\nabla^2 + k_z^2) \left( \frac{\hat{E}_z}{\hat{H}_z} \right) = 0 \quad (2 - 8) \]

To obtain the optimum expression for the evaluation of (2-8), we use Green's first identity over each one of the surfaces \( \Omega_e \) and produce the following equation:

\[ \sum_{1}^{N_e} \int \int_{\Omega_e} \left[ \nabla \left( \frac{\hat{E}_z}{\hat{H}_z} \right) \cdot \nabla \psi_j - k_z^2 \left( \frac{\hat{E}_z}{\hat{H}_z} \right) \psi_j \right] dS = \sum_{1}^{N_e} \int_{C_e} \left[ \psi_j \hat{n} \cdot \nabla \left( \frac{\hat{E}_z}{\hat{H}_z} \right) \right] dl \quad (2 - 9) \]
where \( C_e \) is the curve enclosing \( \Omega_e \) and \( \hat{n} \) is the outward unit normal vector on \( C_e \) (Figure 2-1). The weighting function \( \psi_f \) is chosen such that its derivative is integrable. The derivatives of both \( E_x \) and \( H_x \) also satisfy this condition. It is interesting to note that for the Helmholtz equation with either Neumann or Dirichlet boundary conditions where both \( k_e \) and the boundary conditions are real, the traditional technique of functional minimization [Huebner, 1975] yields the same variational expression as the method of weighted residuals.

To generate the finite element matrix, (2-9) must be evaluated over each of the elements. To solve (2-9), we must either know the values of \( \hat{n} \cdot \nabla \hat{E}_x \) and \( \hat{n} \cdot \nabla \hat{H}_x \), or else we must somehow eliminate the right hand side of (2-9). To eliminate the right hand side, we use the following expressions from Maxwell’s equations,

\[
\hat{n} \cdot \nabla \hat{E}_x = -i\omega \mu_0 \mu_r \hat{H}_t \quad (2 - 10)
\]

\[
\hat{n} \cdot \nabla \hat{H}_x = i\omega \epsilon_0 \left( \epsilon_r - i \frac{\sigma_e}{\omega \epsilon_0} \right) \hat{E}_t \quad (2 - 11)
\]

where \( \hat{H}_t \) and \( \hat{E}_t \) are the magnetic and electric field components tangential to the interfaces which separate the elements. From (2-10), (2-11), and the continuity of \( \psi_j \), it is apparent that the line integral term vanishes when we introduce the modified variation expressions,

\[
\sum_{1}^{N_e} \int_{\Omega_e} \frac{1}{\mu_r} \left[ \nabla \hat{E}_x \cdot \nabla \psi_j - k_e^2 \hat{E}_x \psi_j \right] dS = 0 \quad (2 - 12)
\]

\[
\sum_{1}^{N_e} \int_{\Omega_e} \frac{1}{\epsilon_r - i \frac{\sigma_e}{\omega \epsilon_0}} \left[ \nabla \hat{H}_x \cdot \nabla \psi_j - k_e^2 \hat{H}_x \psi_j \right] dS = 0 \quad (2 - 13)
\]

Equations (2-12) and (2-13) are used to evaluate every element except for those on the boundary since the line integral in (2-9) remains along the boundaries of the
Figure 2-1  Geometry for a single element.
grid. For these elements, the given Dirichlet or Neumann boundary conditions are used to reduce the unknowns in (2-9).

2.2 Formulation of Finite Element Matrix

In this section, we describe the discretization of (2-12) and (2-13) and the actual procedure for filling the matrix. Since the easiest way of doing this is to look at a simple case, a 3-element problem is presented (Figure 2-2). The elements are four node quadrilaterals. The global node numbers are labeled next to the appropriate nodes, and the element numbers are located within each element. Only the \( TM_z \) polarization will be considered since the \( TE_z \) polarization can then be easily derived. Using a Galerkin’s formulation, we choose the basis functions to be the same as the weighting functions. Therefore, the variable \( \hat{E}_z \) in (2-12) can be represented by

\[
\hat{E}_z = \sum_{i=1}^{N_n} E_z^{(i)} \psi_i \quad (2-14)
\]

where \( N_n \) is the total number of basis functions and is equal to the number of nodes in the mesh. The basis and weighting functions that are used for the finite element formulation are the standard ones associated with the isoparametric quadrilateral elements [Becker, Carey, and Oden, 1981]. The function \( \psi_i \) takes on the value of 1 at the \( i^{th} \) node and 0 at all the other nodes with a bilinear variation between the nodes. This means that \( \psi_i \) is zero in every element except those which are connected to the \( i^{th} \) node. It also means that \( \hat{E}_z^{(i)} \) is the value of \( E_z \) at the \( i^{th} \) node.
If (2-14) is substituted into (2-12), a system of \( N_n \) equations corresponding to the \( N_n \) weighting functions is formed. The equations are given by

\[
\sum_{i=1}^{N_n} E_z^{(i)} \left\{ \sum_{l=1}^{N_n} \int_{\Omega_{rl}} \frac{1}{\mu_{rc}} \left[ \nabla \psi_i \cdot \nabla \psi_j - k_c^2 \psi_i \psi_j \right] \, dS \right\} = 0 \quad (2 - 15)
\]

The resulting \( N_n \times N_n \) matrix equation is given by

\[
\begin{pmatrix}
  K_{11} & K_{12} & \cdots & K_{1N_n} \\
  K_{21} & K_{22} & \cdots & K_{2N_n} \\
  \vdots & \vdots & \ddots & \vdots \\
  K_{N_n1} & K_{N_n2} & \cdots & K_{N_nN_n}
\end{pmatrix}
\begin{pmatrix}
  E_z^{(1)} \\
  E_z^{(2)} \\
  \vdots \\
  E_z^{(N_n)}
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (2 - 16)
\]

where the matrix term \( K_{ij} \) is

\[
K_{ij} = \sum_{l=1}^{N_n} \int_{\Omega_{rl}} \frac{1}{\mu_{rc}} \left[ \nabla \psi_i \cdot \nabla \psi_j - k_c^2 \psi_i \psi_j \right] \, dS \quad (2 - 17)
\]

It is inefficient to evaluate (2-16) directly since many of the matrix terms are zero. Instead, the method that is used here is to evaluate a small matrix for each element and then incorporate this matrix into the global matrix in (2-16). In this way, only the nonzero terms in the matrix need to be evaluated.

Let us consider element 1 of the three element grid (Figure 2-3). The nodes are numbered from 1 to 4 in a counter-clockwise manner. These numbers are the local node numbers associated with the specific element under consideration. The associated global node numbers are given in parentheses. Since only the basis functions associated with the four nodes contribute to the solution for \( E_z \) within the element, we can represent the electric field by

\[
E_z = \sum_{I=1}^{4} \psi_I \quad (2 - 18)
\]
Figure 2-2  Geometry for a three-element mesh.
where \( \psi_I \) is the basis function associated with the \( I^{th} \) local node number. Likewise, \( \psi_J \) represents the weighting function associated with the \( J^{th} \) local node number. Therefore, we can rewrite (2-12) for a single element as

\[
\sum_{I=1}^{4} E_x^{(I)} \left\{ \int_{\Omega_e} \frac{1}{\mu_e} \left[ \nabla \psi_I \cdot \nabla \psi_J - k_c^2 \psi_I \psi_J \right] \, dS \right\} = 0 \quad (2 - 19)
\]

where \( \Omega_e \) in this case represents the cross sectional surface of the element under consideration. This leads to the following matrix equation:

\[
\begin{pmatrix}
  k_{11} & k_{12} & k_{13} & k_{14} \\
  k_{21} & k_{22} & k_{23} & k_{24} \\
  k_{31} & k_{32} & k_{33} & k_{34} \\
  k_{41} & k_{42} & k_{43} & k_{44}
\end{pmatrix}
\begin{pmatrix}
  E_x^{(1)} \\
  E_x^{(2)} \\
  E_x^{(3)} \\
  E_x^{(4)}
\end{pmatrix}
= \begin{pmatrix}
  0 \\
  0 \\
  0 \\
  0
\end{pmatrix} \quad (2 - 20)
\]

where

\[
k_{IJ} = \int \int_{\Omega_e} \frac{1}{\mu_e} \left[ \nabla \psi_I \cdot \nabla \psi_J - k_c^2 \psi_I \psi_J \right] \, dS \quad (2 - 21)
\]

The specific details of evaluating (2-21) are outlined in Appendix A. To create the global matrix, let us define an \( N \times N \) matrix named \( [K_0] \) whose matrix elements are all zeroes. Then we add all the element matrices to \( [K_0] \) in such a way that the \( IJ^{th} \) term of each element matrix is added to the \( ij^{th} \) term of \( [K_0] \).

The resulting global finite element matrix is both sparse and banded. A sparse matrix is one which contains very few nonzero terms. This is true for the finite element matrix since the integral in (2-21) is nonzero only for those elements \( \Omega_e \) over which both the basis and weighting functions have support. Of course, in standard finite element procedures the \( \psi_I \)'s are chosen such that any two of them overlap only over a small number of elements. A matrix is banded if the
Figure 2-3  Element one of the three-element mesh
nonzero terms exist in only a banded region around the diagonal of the matrix as shown in Figure 2-4. The bandwidth of a specific row in a matrix is the number of terms counting from the leftmost nonzero term in the row to the rightmost nonzero term. The bandwidth of the matrix is defined to be the maximum bandwidth of all the rows in the matrix. Its value determines both the computational efficiency of the matrix solution and the memory requirements of the matrix itself. For the Helmholtz equation, the resulting matrix is also symmetric. This further reduces the storage and computational requirements. For a symmetric matrix, the important parameter is the half-bandwidth of a matrix. Counting from the main diagonal to the rightmost nonzero term, we define the half-bandwidth to be the maximum number of terms for all rows. The bandwidth is related to the half-bandwidth by

\[ \text{half-bandwidth} = \frac{\text{bandwidth} + 1}{2}. \]

The half-bandwidth of a matrix is determined by the numbering of the nodes in the finite element grid. There are four global node numbers associated with each element, one for each node. From these four numbers there is a maximum and minimum global node number. The difference between the maximum and minimum global node number yields the half-bandwidth for that specific element. The matrix half-bandwidth is the maximum of the half-bandwidths for all the elements. The most popular algorithm for ordering the global node numbers and thereby minimizing the matrix bandwidth is the one developed by Cuthill and McKee [Carey and Oden, 1984]. Readers are referred to this paper the details of this algorithm.
Figure 2-4  The structure of a banded matrix.
2.3 Matrix Solution

There have been many methods developed for efficiently solving a banded matrix. In this section three of the more commonly used methods are discussed. This is followed by a detailed description of the technique which is actually implemented into the computer program.

Consider the matrix equation

\[ [A] \vec{x} = \vec{b} \]  

(2-22)

where \([A]\) is the \(N \times N\) matrix, \(\vec{x}\) is the vector of length \(N\) to be determined, and \(\vec{b}\) is the \(N\) length forcing vector. The conjugate gradient method [Axelsson and Barker, 1984] is an iterative technique which is very efficient at solving a large, sparse matrix. For a very large matrix, this method is probably the most efficient one to use. One major disadvantage of the conjugate gradient method is that the entire solution must be redone for each \(\vec{b}\). In the bymoment method the matrix equation must be solved for several different forcing vectors \(\vec{b}\), but for the same matrix \([A]\). Therefore, the conjugate gradient method is inefficient when used in the bymoment method and will not be considered further.

An interesting technique that was developed in the late 1960's is the frontal method [Carey and Oden, 1984]. Its main advantage is its ability to factor a matrix without the need to store the entire matrix. This is due to the fact that the calculation of the matrix elements and the factorization is done simultaneously. Once all the element matrices associated with global node \(j\) has been formed, the \(j^{th}\) row in the global matrix can be factored and then eliminated from memory. Therefore, by optimizing the order in which the evaluation of the element matrices is done, we can
reduce the storage requirements to just a fraction of that required for other methods. There is a tradeoff since extra computation must be performed to keep track of the rows being swapped into and out of memory. If memory becomes the limitation in solving a problem, this technique should be utilized. The frontal method can be efficiently implemented into the finite element section of the bymoment method since, for the case where there are different forcing vectors $\vec{b}$, the factorization need only be done once for the matrix $[A]$, and only the computationally less intensive backsolve must be done for each forcing vector.

Because the problems solved in this dissertation do not have large memory requirements, a simpler method is implemented into our computer program. This method is called the modified Gauss-Choleski algorithm [Akin, 1982]. Like the frontal method, the factorization only needs to be done once for any number of forcing vectors, but in this case the entire matrix must be stored. The matrix is stored in a banded format so that the extraneous zero terms outside the band do not have to be stored. To understand the Gauss-Choleski algorithm, consider (2-22). The matrix $[A]$ can be factored as follows:

$$[A] = [L][d][L]^T$$ (2 - 23)

where $[L]$ is a lower triangular matrix with its diagonal terms being 1, $[d]$ is a diagonal matrix, and $[L]^T$ is the transpose of $[L]$. The diagonal matrix can be determined from

$$d_{ii} = A_{ii}$$

$$i = 1$$

$$= A_{ii} - \sum_{k=1}^{i-1} d_{kk} L_{ik}^2$$

$$i > 1$$ (2 - 24)
The matrix $[L]$ is given by

$$L_{ij} = \begin{cases} 1 & i = j \\ \frac{1}{d_{jj}} \left[ A_{ij} \sum_{k=1}^{j-1} d_{kk} L_{ik} L_{jk} \right] & i > j \\ 0 & i < j \end{cases} \quad (2-25)$$

In order to determine $\bar{x}$, we use the following procedure. First, define

$$\bar{g} = d [L]^T \bar{x} \quad (2-26)$$

Then

$$[L] \bar{g} = \bar{b} \quad (2-27)$$

Since $[L]$ is a lower triangular matrix, $\bar{g}$ can be obtained by forward substitution, i.e.,

$$g_i = b_i - \sum_{k=1}^{i-1} g_k L_{ik} \quad (2-28)$$

From (2-26), $\bar{x}$ can be obtained using back substitution since $[L]^T$ is an upper triangular matrix. In equation form,

$$x_i = \frac{g_i}{d_{ii}} - \sum_{k=1}^{N-1} x_{i+k} L_{i(i+k)} \quad (2-29)$$

The number of floating point operations (flops) required to factor a full matrix is on the order of $N^3/6$. For a symmetric banded matrix, the number decreases to $N w^2/2$ where $w$ is the half-bandwidth of the matrix. The backsolve, i.e., the forward and back substitution procedure shown in (2-28) and (2-29), requires approximately $N^2$
flops for a full matrix and $2Nw$ flops for a banded matrix. Therefore, the time required to factor a matrix is significantly larger than that required to perform the backsolve.
CHAPTER 3

SCATTERING FROM A SINGLE CYLINDER IN FREE SPACE

The first (and simplest) case considered is that of an infinitely long cylinder in free space where the wave vector of the incident field is normal to the axis of the cylinder. The geometry is shown in Figure 3-1 with the cylinder parallel to the $z$-axis. The excitation is assumed to be independent in $z$. The material properties and shape of the cylinder are also invariant in $z$, but may vary arbitrarily in $x$ and $y$. Only isotropic cylinders are considered. By placing these restrictions on the problem, we see that the fields are invariant in the $z$ direction; therefore, the Helmholtz equation can be used to evaluate either $E_z$ for the $TM_z$ polarization or $H_z$ for the $TE_z$ polarization. The other field components can then be determined from Maxwell's equations. For the $TM_z$ case, the only field components which exist are $E_z, H_z, H_y$. Once $E_z$ is determined from the Helmholtz equation, the magnetic field can be found from

$$\vec{H} = -\frac{1}{i\omega \mu} \nabla \times \hat{z} E_z$$  \hspace{1cm} (3 - 1)

For the $TE_z$ case only $H_z, E_z, E_y$ survive. Likewise, the electric fields are given by

$$\vec{E} = \frac{1}{\sigma + i\omega \epsilon} \nabla \times \hat{z} H_z$$  \hspace{1cm} (3 - 2)

In the following formulation of the bymoment method only the $TM_z$ polarization is considered since the $TE_z$ formulation is virtually identical.
Figure 3-1  Cross sectional geometry of a single cylinder
3.1 Formulation of the Bymoment Method

In this section a step by step description of the bymoment method is given for the case of scattering from a single cylinder. In Figure 3-1, a single cylinder is shown with several labeled lines. The actual cylinder is denoted by the shaded region and is totally enclosed by the mesh. The line $\partial S$ defines the boundary of the mesh. We assume that there is at least one layer of elements between the cylinder boundary and the boundary of the enclosing mesh. The line $\partial S'$ is wholly enclosed by $\partial S$ and passes through the interior of all the elements on the boundary of the mesh. The region inside $\partial S$ is denoted by $S_c$. The region exterior to the mesh is denoted by $S_0$. To solve the finite element problem, boundary conditions must be specified on the nodes located along $\partial S$. However, these values are not known apriori and can be determined only after the interior and exterior region solutions have been coupled via the enforcement of the proper boundary conditions. The bymoment method accomplishes this in the following way. $E_z$ on $\partial S$ is expanded in terms of a sum of known basis functions multiplied by unknown coefficients. Finite element solutions are then generated using these basis functions as Dirichlet boundary conditions. To determine the coefficients, Green's theorem is applied to the region exterior to the cylinder for the scattered electric field $E_z^s$ and for each one of the members in a set of testing functions chosen to satisfy the Helmholtz equation and the Sommerfeld radiation condition at infinity. The solution of the resulting matrix equation completes the solution to the scattering problem.
3.1.1 Interior Solution

To determine the set of boundary conditions on \( \partial S \), we must first develop the notation for representing a function on the mesh boundary. Let the variable \( t \) vary from 0 to \( d \) where \( d \) is the length of \( \partial S \). We may then represent \( E_z \) along \( \partial S \) by a function of the single variable \( t \) with each value of \( t \) corresponding to a point on \( \partial S \). Let \( e^s(t) \) represent \( E_z \) on the boundary. Since the points \( t = 0 \) and \( t = d \) correspond to the same point on \( \partial S \), \( e^s(t) \) must be chosen such that

\[
e^s(0) = e^s(d)
\]  

(3 - 3)

Because the electric field has been reduced to a function of one variable on \( \partial S \), it can be written in terms of a set of linearly independent functions \( \Psi_n(t) \) and unknown coefficients \( a_n \) as follows:

\[
e^s(t) = \sum_{n=1}^{\infty} a_n \Psi_n(t) \quad t \in [0, d]
\]  

(3 - 4)

Since \( e^s \) is continuous, the above equation is true for any \( e^s \) so long as the set \( \Psi_n(t) \) is complete on the space of continuous function. Actually, since our problem must be evaluated numerically, the summation in (3-4) must be truncated to a finite number of terms \( N \). Therefore, \( e^s(t) \) can almost never be represented exactly, but by choosing a large enough \( N \), we can approximate \( e^s(t) \) to any degree of precision desired. The truncation results in an approximate representation for (3-4) which is given by

\[
e^s(t) \approx \sum_{n=1}^{N} a_n \Psi_n(t) \quad t \in [0, d]
\]  

(3 - 5)
The value of $N$ depends upon the convergence requirements and the choice of basis functions $\Psi_n$.

From the uniqueness theorem we know that, with the presence of any physical loss mechanism, the fields inside a given volume are determined uniquely by either the tangential electric or tangential magnetic fields on the surface enclosing the volume [Harrington, 1961]. Therefore, $\epsilon^*$ uniquely determines the solution inside $S_e$. If $N$ finite element solutions are generated where $\Psi_n(t)$ are the Dirichlet boundary condition for the $n^{th}$ finite element problem, then from the uniqueness theorem and the properties of linearity, (3-5) leads to

$$E_x(x, y) = \sum_{n=1}^{N} a_n \Lambda_n(x, y) \quad (x, y) \in S_e$$

where $\Lambda_n(x, y)$ is the solution to the $n^{th}$ finite element problem. The interior solution is now complete except for the determination of the coefficients $a_n$.

3.1.2 Coupling of the Interior Solution to the Exterior Region

To evaluate the coefficients, an additional region must be defined in the geometry. Let $S'_0$ be the region exterior to the line $\partial S'$. Note that $S'_0$ is homogeneous since $\partial S'$ lies outside the cylinder. The electric field in $S'_0$ can be defined as the superposition of an incident field which is defined to be the field in the absence of the cylinder and a scattered field, which is the field due to the cylinder. The electric field is

$$E_x(x, y) = E_x^{inc}(x, y) + E_x^{sc}(x, y) \quad (x, y) \in S'_0$$

(3-7)
It is assumed that $E_{x}^{inc}$ is a quantity that can be determined either analytically or numerically. The scattered field satisfies the Helmholtz equation,

$$(\nabla^2 + k_0^2)E_x^{sc}(x,y) = 0 \quad (x,y) \in S_0'$$ \hspace{1cm} (3-8)

and the Sommerfeld radiation condition at infinity. The variable $k_0$ is the wave number in $S_0'$. For the case presented here, $k_0$ is the free space wave number.

Now consider a set of linearly independent testing functions $\{\Phi\} = (\Phi_j: j = 1, 2, \ldots)$ which are chosen to satisfy the Helmholtz equation in $S_0'$, i.e.,

$$(\nabla^2 + k_0^2)\Phi_j(x,y) = 0 \quad (x,y) \in S_0'$$ \hspace{1cm} (3-9)

and the Sommerfeld radiation condition at infinity. Applying Green's theorem for $E_x^{sc}$ and $\Phi_j$ over $S_0'$, we obtain the following expression:

$$\int_{S_0'} \left[ \Phi_j(\nabla^2 + k_0^2)E_x^{sc} - \Phi_j(\nabla^2 + k_0^2)\Phi_j \right] dS$$

$$= \int_{\partial S_0'} \left[ \Phi_j \frac{\partial E_x^{sc}}{\partial n'} - E_x^{sc} \frac{\partial \Phi_j}{\partial n'} \right] dl + \int_{\partial S_\infty} \left[ \Phi_j \frac{\partial E_x^{sc}}{\partial n_\infty} - E_x^{sc} \frac{\partial \Phi_j}{\partial n_\infty} \right] dl$$ \hspace{1cm} (3-10)

where $\partial S_\infty$ is the line at infinity, $n'$ is the outward normal to $\partial S_0'$ and $n_\infty$ is the outward normal along the line at infinity. Substituting (3-8) and (3-9) into (3-10), we see that the surface integral vanishes since both $E_x$ and $\Phi_j$ satisfy the Helmholtz equation in $S_0'$. Also, the integral along $\partial S_\infty$ vanishes since $E_x$ and $\Phi_j$ satisfy the Sommerfeld radiation condition. Therefore (3-10) reduces to

$$\int_{\partial S_0'} \left[ \Phi_j \frac{\partial E_x^{sc}}{\partial n'} - E_x^{sc} \frac{\partial \Phi_j}{\partial n'} \right] dl = 0$$ \hspace{1cm} (3-11)
Substituting (3-7) into (3-11) and then using (3-6) for $E_z$, we can rewrite (3-11) to be

$$\sum_{n=1}^{N} a_n \int_{\partial S'} \left[ \Phi_j \frac{\partial \Lambda_n}{\partial n'} - \Lambda_n \frac{\partial \Phi_j}{\partial n'} \right] \,dl = \int_{\partial S'} \left[ \Phi_j \frac{\partial E_{z}^{inc}}{\partial n'} - E_{z}^{inc} \frac{\partial \Phi_j}{\partial n'} \right] \,dl \quad (3 - 12)$$

Both $E_{z}^{inc}$ and $\Phi_j$ can be determined either analytically or numerically. In the case where analytical expressions are available, their normal derivatives also have analytical expressions. For the case where numerical evaluation is necessary, their normal derivatives can be determined from a finite difference approximation. Therefore, $\Phi_j$, $\partial \Phi_j / \partial n'$, $E_{z}^{inc}$, and $\partial E_{z}^{inc} / \partial n'$ can be determined in (3-12). Also, the numerical values for both $\Lambda_n$ and $\partial \Lambda_n / \partial n'$ along $\partial S'$ can be obtained from the finite element solutions. At this point, it is important to explain why Green's theorem is applied along $\partial S'$ rather than $\partial S$. The reason is that $\partial \Lambda_n / \partial n'$ is determined from the evaluation of the derivative on (2-11) in the finite element solution. Because the basis functions vary bilinearly within the element, the normal derivative along the sides of an element cannot be evaluated accurately. Thus, we choose a line $\partial S'$, which passes through the interior of the boundary elements, to evaluate the normal derivatives of $\Lambda_n$. We find that the best results are obtained when this line passes through the centroids of the elements.

The only unknowns left in (3-12) are the coefficients $a_n$. By using the first $N$ testing functions in the set $\{\Phi\}$, we can assemble an $N \times N$ matrix equation to solve for the coefficients. We write the matrix equation as follows:

$$
\begin{pmatrix}
S_{11} & S_{12} & \cdots & S_{1N} \\
S_{21} & S_{22} & \cdots & S_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
S_{N1} & S_{N2} & \cdots & S_{NN}
\end{pmatrix}
\begin{pmatrix}
a_1 \\
a_2 \\
\vdots \\
a_N
\end{pmatrix}
=
\begin{pmatrix}
T_1 \\
T_2 \\
\vdots \\
T_N
\end{pmatrix}
\quad (3 - 13)
$$
where \( S_{jn} \) and \( T_j \) are given by,
\[
S_{jn} = \int_{\partial S'} \left[ \Phi_j \frac{\partial \Lambda_n}{\partial n^i} - \Lambda_n \frac{\partial \Phi_j}{\partial n^i} \right] \, dl \tag{3 - 14}
\]
\[
T_j = \int_{\partial S'} \left[ \Phi_j \frac{\partial \phi^{inc}}{\partial n^i} - \phi^{inc} \frac{\partial \Phi_j}{\partial n^i} \right] \, dl \tag{3 - 15}
\]

Once the coefficients are determined, \( e'(t) \) can be evaluated directly from (3-5). Then by using \( e'(t) \) as the Dirichlet boundary condition, we can obtain the finite element solution.

3.2 Basis and Testing Functions

Up to this point, the descriptions of the basis and testing functions have been rather abstract. The choice of basis functions \( \Psi_n \) and testing functions \( \Phi_j \) has a major impact on the efficiency and accuracy of the bymoment method. In this section several concrete examples of both are provided. Also, a discussion detailing the advantages and disadvantages of various basis and testing functions is given.

Let us begin by looking at the basis functions. There are two types of basis functions that will be considered. The first are the entire-domain functions, i.e., functions that have support over the entire domain of \( t \) from 0 to \( d \). The second are the sub-domain functions, i.e., functions which have support over only a part of the domain. Because of the large number of possible sub-domain basis functions, we will consider the specific case where the basis functions are the triangle functions (Figure 3–2). Each of these functions has support over two of the boundary elements with a maximum value of one at the boundary node common to the two elements and a minimum value of zero at the adjacent boundary nodes. The variation from
the common node to the two adjacent nodes is assumed to be linear. Therefore, the number of triangle functions is fixed depends on the number of nodes on \( \partial S \). The major advantage of using entire-domain basis functions over the triangle function is that, in most case, a smaller number of basis functions is required to obtain accurate results. Another advantage is the flexibility in choosing the number of basis functions. Because the exact solution requires an infinite number of basis functions, the number \( N \) is determined by a convergence criteria that is specified by the user. In cases where the accuracy is not as important as speed, a smaller value of \( N \) can give a good approximate solution. One disadvantage of using entire-domain basis functions results directly from the flexibility in choosing \( N \). The value of \( N \) for a given convergence criteria is not easily determined since it is highly dependent on the geometry of the mesh and the object inside the mesh. One method to circumvent this problem is to iterate the solution using different values of \( N \). This is computationally efficient for small values of \( N \) since the finite element solutions from the previous iteration can be used in the current iteration. In this way, the effect of additional terms on the solution can be seen directly. For large values of \( N \), the additional computation time for each iteration may become large since the coefficient matrix must be evaluated for each iteration. If the iterations become too expensive, an approximation, based on an empirical relationship between the geometry characteristics and the value of \( N \) can be found. The major consideration in this relationship is the electrical size of the scatterer and the enclosing mesh. An attempt is made, later in this chapter, to determine this relationship. Note that this method is very approximate and does not account for many geometrical anomalies such as sharp corners on a perfectly conducting cylinder.
Figure 3-2 Triangular basis functions on $\partial S$. 
Since in the finite element method we assume the continuity of $E_z$ throughout the mesh, the entire-domain basis functions must be continuous on $\partial S$ to obtain an accurate finite element solution. Also, $\Psi_n$ must be a function whose endpoints have the same value, as was mentioned earlier. Therefore, $\Psi_n$ must be a continuous, periodic function. The obvious choice are the sinusoidal functions (Figure 3–3),

$$
\Psi_n(t) = \begin{cases} 
\cos \left( \frac{n-1}{2} \theta(t) \right) & n \text{ odd} \\
\sin \frac{n}{2} \theta(t) & n \text{ even}
\end{cases}
$$

(3–16)

where $t \in [0, d]$ maps to $\theta(t) \in [0, 2\pi]$. Because of the oscillatory behavior of the sinusoidal functions, the size of $N$ is limited by how fine the mesh is on the boundary. Let the number of nodes on the boundary be given by $N_b$. If we assume that four nodes are needed to approximate each period of the sine wave, then the number of nodes must be at least twice the number of basis functions, i.e., $N \leq N_b/2$. This limitation should never be a problem since it is assumed that the mesh is discretized fine enough to accurately evaluate a finite element problem that has basis functions with bilinear variation. We expect that $N$ should be significantly smaller than $N_b$.

Let us now consider the triangle basis functions. The major advantage of this basis function is that the implementation into the finite element code is trivial. Since the behavior of the triangle function along the boundary is the same as the basis functions $\psi_i$ used in the finite element problem, we can obtain the corresponding finite element basis function $A_n$ by setting the value on the $n^{th}$ node on the boundary to one and setting the other nodes on the boundary to zero. In order to obtain an accurate solution using this triangle function for $\Psi_n$, we must have a basis function for each node on the boundary, so $N = N_b$. For meshes with a large number of nodes on the boundary the solution may be computationally intensive in both time
Figure 3-3 Sinusoidal basis functions on $\partial S$. 
and storage requirements. Because of this, we found ourselves using the entire-domain functions much more often than the subdomain ones. On the positive side, the solution on the boundary is guaranteed to be as accurate as the finite element solution since the basis function are discretized in the same manner as the finite element problem. One way to alleviate some of the numerical calculations is to modify the triangle function so that it has support over four elements rather than two. This has the effect of halving the number of basis functions. The resulting loss in accuracy is dependent on the electrical spacing of the nodes on the boundary. As will be shown in the next section, the loss of accuracy is insignificant in most cases. Note that the number of nodes on the boundary must be even to use this basis function. Otherwise, the requirement in (3-3) will be violated.

We now consider the testing functions. Up to this point, the only description of the testing functions is that they satisfy the Helmholtz equation in $S_0$ and the Sommerfeld radiation condition at infinity. The choice can be further narrowed by requiring the testing functions to have closed form expressions. This still leaves several possibilities. One choice is the set of cylindrical harmonic functions. These are

$$\Phi_j(x, y) = H^{(2)}_m(\rho, \phi) \begin{cases} \cos m \phi & j = 2m + 1 \\ \sin m \phi & j = 2m \end{cases}$$

(3 − 17)

where $j = 1, 2, \ldots, N$. The variables, $\rho$ and $\phi$, represent the usual transformation from cartesian to cylindrical coordinates. For $j$ small the evaluation of the Hankel function is fairly straightforward, but as $j$ becomes large, problems arise in the numerical evaluation of the Hankel function. The large-order approximation for
$H_\nu^{(2)}(x)$ is given by [Abramowitz and Stegun, 1972]

$$H_\nu^{(2)}(x) \sim \frac{1}{\sqrt{2\pi \nu}} \left( \frac{ex}{2\nu} \right)^\nu + i\sqrt{\frac{2}{\pi \nu}} \left( \frac{ex}{2\nu} \right)^{-\nu}. \quad (3-18)$$

As $\nu \to \infty$, the real part of the Hankel function goes to zero while the imaginary part of the Hankel function goes to infinity. Because both the real and imaginary parts of the Hankel function are important in the evaluation of (3-12), their values must be retained. Unfortunately, even using double precision, there is a limit to the value of $j$ for which an accurate result in (3-12) can be obtained. Therefore, we must be careful in using this testing function when $N$ is large.

Another suitable testing function is the free space scalar Green's function with the line source located in the region $S'_c$, where $S'_c$ is the region containing the cylinder and enclosed by the line $\partial S'$. The two-dimensional Green's function is given by

$$g(\vec{r}, \vec{r}') = H_0^{(2)}(k_0|\vec{r} - \vec{r}'|) \quad (3-19)$$

where $|\vec{r} - \vec{r}'| = \sqrt{(x-x')^2 + (y-y')^2}$ and the $1/4i$ factor is excluded. Let $\rho_j$, $j = 1, 2, ..., N$ be a set of points in $S'_c$. By substituting $\vec{r}_j$ for $\vec{r}'$ in (3-19), we can form a set of linearly independent functions which can be used for $\Phi_j$. The mathematical expression for $\Phi_j$ is

$$\Phi_j(x, y) = H_0^{(2)}(k_0|\vec{r} - \vec{r}_j|) \quad (3-20)$$

where $(x_j, y_j) \in S'_c$ are the locations of the line sources. Therefore, the different testing functions represent solutions of (3-9) for different positions of the line
source in \( S'_c \). This is similar to the basis functions used in the generalized multiplypole technique [Hafner, 1989]. Note that this testing function does not have a size limitation on \( N \), but the proper placement of the line source is vital for obtaining an accurate solution. A random choice for \( \rho_j \) may cause the matrix in (3-13) to be ill-conditioned. This results in gross inaccuracies in the solution. This is analogous to the situation in the method of moments. Consider the case where we apply collocation to a method of moments problem. Then the basis functions are pulse functions given by

\[
P_n(x) = \begin{cases} 
1, & x_{n-1} < x < x_n \\
0, & \text{otherwise} 
\end{cases}
\]

(3-21)

where \( x_{n-1} \) and \( x_n \) denote the boundary of the \( n^{th} \) \((n = 1, \ldots, N)\) pulse. The weighting functions are delta functions located at \( x = y_i, i = 1, \ldots, N \). The location of the delta functions cannot be arbitrary. In fact, for the case where \( y_i \notin (x_{n-1}, x_n) \) for all values of \( i \), the resulting matrix is ill-conditioned. Thus, we must place \( y_i \) such that \( y_i \in (x_{n-1}, x_n) \) for \( i = n \). From this example, we see that although ideally the location of the line sources can be chosen anywhere in \( S'_c \), realistically some care must be taken in choosing their location. The choice for the placement of the testing functions in (3-20) is not quite as obvious as in the method of moments example. We found that the best results are achieved by placing the line sources along contours which are concentric and conforming with \( \partial S \). This is also the conclusion reached by people who work with the generalized multipole technique. The method that they use to place their basis functions is based on the \textit{sphere of influence} of each basis function [Leuchtmann, 1988]. Actually their restrictions are tighter than ours because the \textit{sphere of influence} dictates the number of basis functions which must be used to obtain a good solution. A typical placement of the
line sources for a circular cylinder and a dielectric half shell are shown in Figures 3-4 and 3-5, respectively. The reason for the staggered line source placement is explained in Chapter 7.

A further simplification occurs in (3-12) when (3-20) is used for testing. Assume that a finite source \( J_z \) located in \( S_0' \) excites an incident field so that

\[
(\nabla^2 + k_0^2)E_z^{inc}(x,y) = i\omega\mu_0 J_z \quad (3-22)
\]

Also assume that the cylinder is not present. Then Green's theorem can be applied to all space for \( E_z^{inc} \) and \( \Phi_j \) as follows,

\[
\int \int \left[ \Phi_j (\nabla^2 + k_0^2)E_z^{inc} - E_z^{inc} (\nabla^2 + k_0^2)\Phi_j \right] dS = \int_{\partial S_{\infty}} \left[ \Phi_j \frac{\partial E_z^{inc}}{\partial n_{\infty}} - E_z^{inc} \frac{\partial \Phi_j}{\partial n_{\infty}} \right] dl \quad (3-23)
\]

Since \( \Phi_j \) is the Green's function with a line source located at \((x_j, y_j)\), it satisfies the differential equation

\[
(\nabla^2 + k_0^2)\Phi_j(x, y) = -4i\delta(x-x_j)\delta(y-y_j) \quad (x_j, y_j) \notin S_0' \quad (3-24)
\]

Substituting (3-22) and (3-24) into (3-23) and recognizing the fact that the fields due to a finite source satisfy the Sommerfeld radiation condition, we get

\[
4iE_z^{inc}(x_j, y_j) = -i\omega\mu_0 \int \int \Phi_j J_z dS \quad (3-25)
\]

Now, let us consider the case where the cylinder is present. Applying Green's theorem over the region \( S_0' \) for \( E_z^{inc} \) and \( \Phi_j \), we can write

\[
\int \int_{S_0'} \left[ \Phi_j (\nabla^2 + k_0^2)E_z^{inc} - E_z^{inc} (\nabla^2 + k_0^2)\Phi_j \right] dS
\]

\[
= \int_{\partial S'} \left[ \Phi_j \frac{\partial E_z^{inc}}{\partial n'} - E_z^{inc} \frac{\partial \Phi_j}{\partial n'} \right] dl + \int_{\partial S_{\infty}} \left[ \Phi_j \frac{\partial E_z^{inc}}{\partial n_{\infty}} - E_z^{inc} \frac{\partial \Phi_j}{\partial n_{\infty}} \right] dl \quad (3-26)
\]
Figure 3-4  Testing function placement for a circular cylinder.
Figure 3-5  Testing function placement for a dielectric half shell.
We now substitute (3-22) and (3-24) into (3-26) and again recognize the fact that $E^\text{inc}_z$ satisfies the Sommerfeld radiation condition. Since the integration is over $S'_0$ rather than all space, the line source in (3-24) does not come into the equation and (3-26) reduces to

$$i\omega \mu_0 \int \int_{S'_0} J_z \, dS = \int_{S'1} \left[ \Phi_j \frac{\partial E^\text{inc}_z}{\partial n'} - E^\text{inc}_z \frac{\partial \Phi_j}{\partial n'} \right] \, dl \quad (3 - 27)$$

The final step is the substitution of (3-25) into (3-27),

$$-4i E^\text{inc}_z(x_j, y_j) = \int_{S'1} \left[ \Phi_j \frac{\partial E^\text{inc}_z}{\partial n'} - E^\text{inc}_z \frac{\partial \Phi_j}{\partial n'} \right] \, dl \quad (3 - 28)$$

Therefore, the right hand side of (3-12) becomes

$$T_j = -4i E^\text{inc}_z(x_j, y_j) \quad (3 - 29)$$

A very similar derivation for the case where the source is not finite can be done to show that (3-29) holds true in this case. The testing functions in (3-20) are the ones that are used throughout this dissertation to compute numerical results.

The final set of testing functions that we present is a hybrid of the first two. These are given by

$$\Phi_j(x, y) = H^{(2)}_{in}(k_0|\vec{\rho} - \vec{\rho}_n|) \begin{cases} \cos m\phi_n & j = 2m_{\text{max}}(n - 1) + 2m + 1 \\ \sin m\phi_n & j = 2m_{\text{max}}(n - 1) + 2m \end{cases} \quad (3 - 30)$$

where we assume that there are as many sine basis functions as cosine ones used in (3-30) and the index $m$ runs from 0 to $m_{\text{max}}$. The variable $m_{\text{max}}$ represents the maximum $m$ used for each value of $n$. The quantity $|\vec{\rho} - \vec{\rho}_n| = \sqrt{(x - x_n)^2 + (y - y_n)^2}$,
and $\phi_n = \arctan[(y-y_n)/(x-x_n)]$. Basically, (3-30) represents cylindrical harmonic functions for each one of the cylindrical coordinate systems with origin at $\rho = \rho_n$ and the $z$-axis parallel to the axis of the cylinder. This allows the testing functions to have two degrees of freedom, namely, the freedom in choosing the number of harmonics used and the freedom in choosing the location of the points where the the expansion occurs. Note that the first two testing functions are special cases of the third. (3-17) is the case where $n = 1$, and (3-20) is the case where $m = 0$. Although (3-30) is not used in the work for this dissertation, it has a great deal of potential for future development since there should be a greater freedom in choosing $\rho_n$ as opposed to $\rho_j$ in (3-20). The one major disadvantage of this method is that the choice for varying the indices $n$ and $m$ is not as clear cut as for the testing function in (3-20). If an algorithm can be developed for this, this set of testing functions may be the best one to use.

3.3 Numerical Results

In this section, numerical results are obtained for various geometries. In order to numerically validate the bymoment method, comparisons are made with known series solutions for the perfectly conducting and dielectric circular cylinders [Harrington, 1961] and with method of moments solutions for the case of a dielectric shell and a dielectric half shell. Finally, results are presented for more complicated geometries. All of the results shown in this dissertation were obtained from calculations done on either a VAX 8550 or a VAX 2000 workstation. Several of the subroutines in the computer program were taken from Akin's finite element program [1982]. The preprocessor FASTQ [Blacker, 1988] was used to mesh the geometry and to optimize the node ordering in the mesh.
Although the bymoment method allows the incident field to be generated using any two dimensional source, the numerical results are presented only for the case of plane wave incidence. The plane wave is described by

\[
\begin{pmatrix}
E_z^{inc} \\
H_z^{inc}
\end{pmatrix} = e^{i k_0 (x \sin \phi_i + y \cos \phi_i)}
\]  

(3 - 31)

where \( E_z^{inc} \) is used for the \( TM_z \) case and \( H_z^{inc} \) is used for the \( TE_z \) case. The angle \( \phi_i \) is defined from the y axis as shown in Figure 3–1. The three basis functions discussed in the previous section will be used to obtain the numerical results. To distinguish between the two subdomain triangle functions, the notation \( T1 \) is used for the triangle function with support over two elements and the notation \( T2 \) is used for the triangle function with support over four elements. There are several parameters that are used to present the results in this section. The first is the surface current density on a perfectly conducting cylinder. The surface current density \( j_s \) is defined by,

\[
\vec{j}_s = \hat{n} \times \vec{H}
\]  

(3 - 32)

where \( \hat{n} \) is the unit vector normal to the surface of the cylinder. Another parameter of interest is the echo width \( L_e \). For the \( TM_z \) polarization, the echo width is defined to be

\[
L_e(\phi) = \lim_{\rho \to \infty} \left[ 2 \pi \rho \frac{|E_z^{inc}|^2}{|E_z^{inc}|^2} \right].
\]  

(3 - 33)

where \( \rho \) is the radial variable in the cylindrical coordinate system and \( \phi \) is the angular variable defined such that \( \phi = 0 \) corresponds to the positive x-axis. Similarly, for the \( TE_z \) case,

\[
L_e(\phi) = \lim_{\rho \to \infty} \left[ 2 \pi \rho \frac{|H_z^{inc}|^2}{|H_z^{inc}|^2} \right].
\]  

(3 - 34)
Once the finite element solution is generated, \( E_z^{se} \) can be calculated from Green's theorem. The expression for \( E_z^{se} \) is

\[
E_z^{se}(x, y) = \int_{S_1} \left[ g(x, y|x', y') \frac{\partial E_z(x', y')}{\partial n'} - E_z(x', y') \frac{\partial g(x, y|x', y')}{\partial n'} \right] \, dt \tag{3-35}
\]

where \( g(x, y|x', y') \) is the free space Green's function. The expression for the \( TE_z \) case can be obtained once \( E_z^{se} \) in (3-35) is replace by \( H_z^{se} \). In a couple of the cases, results are given in terms of a quantity defined by

\[
|E_n(\phi)| = \frac{\sqrt{k_0 L_z}}{2} \tag{3-36}
\]

for the \( TM_z \) polarization. For the \( TE_z \) case, replace \( |E_n(\phi)| \) by \( |H_n(\phi)| \).

The first geometry considered is a perfectly conducting circular cylinder of radius \( 0.3\lambda \) in free space. The plane wave is incident at an angle of \( \phi_i = -90^\circ \). The finite element mesh (Figure 3-6) consists of a single layer of elements around the cylinder. The mesh contains 80 nodes with a half-bandwidth of 6. In Figures 3-7 and 3-8, we show the magnitude and phase, respectively, of the surface current density \( j_s \) on the cylinder under \( TE_z \) excitation. The results using both the \( T1 \) \( (N = 40) \) and \( T2 \) \( (N = 20) \) basis functions are compared against the series solution. We see that the finite element solutions are very accurate in both cases. Actually, the two finite element solutions are practically indistinguishable, which implies that the use of the computationally less intensive \( T2 \) function is preferable for this case. In fact, if the geometry discretization is such that it corresponds to 20 nodes/wavelength, the use of the \( T2 \) function is expected to give sufficiently accurate results for any geometry.
Figure 3-6 Finite element mesh for a perfectly conducting circular cylinder with radius $0.3\lambda$
Figure 3-7  Magnitude of $j_z$ on a perfectly conducting circular cylinder ($r = 0.3\lambda$, $TE_2$ case, $\phi^i = -90^\circ$).
Figure 3-8  Phase of $j_s$ on a perfectly conducting circular cylinder ($r = 0.3\lambda$, $TE_z$ case, $\phi^i = -90^\circ$).
We next consider a dielectric circular cylinder \( r = 0.3\lambda, \phi = -90^\circ \) under TM\(_z\) excitation. The relative permeability of the cylinder is \( \mu_{rd} = 1 \), and the conductivity and relative dielectric constant are given by \( \sigma_d = 0.0 \) and \( \epsilon_{rd} = 4 \), respectively, where the subscripts \( d \) and \( rd \) are used to indicate that the parameters are associated with the cylinder itself. This notation will be used throughout this dissertation to indicate the material properties of a penetrable cylinder. Also, from now on the conductivity will only be specified if it is nonzero, and the relative permeability is assumed to be one unless otherwise indicated. The mesh for the dielectric cylinder is shown in Figure 3-9. There are 1321 nodes in this mesh, and the cylinder is located one element from the boundary. The half-bandwidth of the resulting matrix is 47. In Figures 3-10 and 3-11, we compare the series results to the finite element results which are generated using the sinusoidal functions in (3-16) with different values of \( N \). This gives us an indication of the convergence of the solution for a cylinder. The electric field is plotted as a function of \( x \) for \( y = 0 \).

We have very good agreement between the series solution and the finite element solution for \( N = 12 \). Even for \( N = 8 \), the error in the finite element solution is relatively small. There are no curves for \( N > 12 \) since we find no significant improvement in accuracy for larger values of \( N \).

For the sinusoidal basis functions, we find that we can quantify the number of functions that are required for a given geometry size. If we specify the length of the perimeter of the scatterer to be \( P_{sc} \) and the length of the perimeter of the mesh to be \( P_{mesh} \), then the number of basis functions that are required to obtain an accurate solution is given by

\[
N \geq \max \left\{ \frac{2.5P_{sc}}{\lambda_{sc}}, \frac{2.5P_{mesh}}{\lambda_0} \right\}
\]  

(3 - 37)
Figure 3-9  Finite element mesh for a dielectric circular cylinder with a radius of 0.3λ.
Figure 3-10  Magnitude of $E_x$ for a homogeneous circular dielectric cylinder ($r = 0.3\lambda$, $\epsilon_{rd} = 4$, $TM_z$ case). The field is computed along $x$ for $y = 0$. Comparison of entire function solutions to series solution.
Figure 3-11  Phase of $E_z$ for a homogeneous circular dielectric cylinder ($r = 0.3\lambda, \varepsilon_{rd} = 4, TM_z$ case). The field is computed along $x$ for $y = 0$. Comparison of entire function solutions to series solution.
where, for a homogeneous cylinder, $\lambda_{sc}$ is the wavelength inside the scatterer and $\lambda_0$ is the wavelength in free space. For scatterers which are inhomogeneous, $\lambda_{sc}$ represents the smallest wavelength in the scatterer. For our dielectric cylinder we find that $N \geq 10$, which agrees with the numerical results. Equation (3-37) is only an approximate rule and should be used with caution. In Figures 3–12 and 3–13, we consider the same geometry as before except now the series solution is compared to the finite element solution generated using the $T1$ basis functions. In addition to the results along the line $y = 0$, we present the results as a function of $y$ for $x = 0$. Again, there is excellent agreement between the series and finite element solutions.

Richmond [1965,1966] provides echo-width results for several cylindrical geometries under both $TE_1$ and $TM_1$ excitations. The method of moments is used to generate these solutions. Two of the geometries analyzed by Richmond are presented here. The first is a dielectric circular shell with an inner radius of $0.25\lambda$ and an outer radius of $0.3\lambda$. The mesh is very similar to that of the dielectric cylinder. The relative dielectric constant of the shell is 4, and the angle $\phi'$ is $-90^\circ$. In Figure 3–14, we compare Richmond's solution to the finite element solution generated using the $T2$ basis function. The two methods produce almost identical results for both the $TE_1$ and $TM_1$ cases. The second geometry is a dielectric circular half shell ($r_1 = 0.25\lambda, r_2 = 0.3\lambda, \epsilon_{rd} = 4, \phi' = -90^\circ$). This geometry is different from the previous ones in that it does not have a circular boundary. We can therefore use this geometry to demonstrate the ability of the method to solve problems involving grids that conform to the surface of the scatterer. The finite element mesh used for this problem is shown in Figure 3–15. The number of nodes is 302, and the half-bandwidth is 24. The three center rows of elements represent the dielectric
Figure 3-12 Magnitude of $E_z$ for a circular dielectric cylinder ($r = 0.3\lambda$, $\epsilon_{rd} = 4$, $TM_z$ case). The field is computed along $x$ for $y = 0$ and along $y$ for $x = 0$. Comparison of subdomain function solutions to series solution.
Figure 3-13  Phase of $E_z$ for a circular dielectric cylinder ($r = 0.3\lambda$, $\varepsilon_{rd} = 4$, $TM_z$ case). The field is computed along $x$ for $y = 0$ and along $y$ for $x = 0$. Comparison of subdomain function solutions to series solution.
shell. The resulting echo width solutions are shown in Figure 3-16, where they are compared to Richmond's solutions. The sinusoidal functions ($N = 16$) are used for this example. Once again, the agreement is excellent.

Now that the method has been validated from known solutions for some simple geometries, let us consider more complicated geometries. The first geometry is a circular layered dielectric cylinder (Figure 3-17). The radii of the layers are $r_1 = 0.15\lambda$, $r_2 = 0.2\lambda$, $r_3 = 0.25\lambda$, and $r_4 = 0.3\lambda$, while the relative dielectric constants are $\varepsilon_{rd1} = 8$, $\varepsilon_{rd2} = 6$, $\varepsilon_{rd3} = 4$, and $\varepsilon_{rd4} = 2$. The mesh is shown in Figure 3-18 and is composed of 697 nodes (50 half-bandwidth). Fourteen sinusoidal basis functions are used. The incident plane wave is oriented so that $\hat{\phi}^i = -90^\circ$. In Figure 3-19, we show the echo width results for the above geometry with $\sigma_d = 0$ for both $TM_e$ and $TE_e$ excitation. We now increase the conductivity to $\sigma_d = 0.25$. The results are given in Figure 3-20. Comparing Figure 3-19 to 3-20, we see that the effect of the conductivity is to reduce the size of the echo width and also to damp out the oscillatory behavior of the echo width as a function of $\phi$. The next geometry consists of four perfectly conducting cylinders in free space. The cylinders are arranged in the pattern of a square. The mesh (Figure 3-21) contains 875 nodes with a half-bandwidth of 58. The number of basis function is 24. In Figure 3-22, we plot $|E_n(\phi)|$ ($TM_e$) for $r = 0.08\lambda$, $\hat{\phi}^i = -135^\circ$, and $d = 1.0\lambda$. This results for this geometry will be used as a benchmark for the numerical results in the next chapter. The final geometry is that of two dielectric circular cylinders in free space ($r = .3\lambda$, $\varepsilon_{rd} = 4$, $d = 1.0\lambda$, $N = 24$). The mesh for this geometry is shown in Figure 3-23 with 2317 nodes and a half-bandwidth of 83. Results are presented for both polarizations. In Figure 3-24, we show $|E_n(\phi)|, |H_n(\phi)|$ for the two cylinders.
Figure 3-14  Echo width for a homogeneous dielectric circular shell \( r_1 = 0.25\lambda, r_2 = 0.3\lambda, \epsilon_{rd} = 4 \). Comparison of \( T_2 \) basis function solution to series solution.
Figure 3-15  Finite element mesh for a dielectric circular half shell.
Figure 3-16  Echo width for a dielectric circular half shell ($r_1 = 0.25\lambda$, $r_2 = 0.3\lambda$, $\varepsilon_{rd} = 4$). Comparison of sinusoidal basis function solution to series solution.
where the plane wave is incident at an angle of $\hat{\phi} = -100^\circ$. For the case where the angle of incidence is $\hat{\phi} = 0^\circ$, the results are shown in Figure 3–25. As we can see, the interaction between the two cylinders creates far-field patterns with many lobes. Figures 3–24 and 3–25 are used for comparisons in Chapters 2 and 3, respectively.
Figure 3-17  Geometry for a four layer dielectric circular cylinder ($r_1 = 0.15\lambda$, $r_2 = 0.2\lambda$, $r_3 = 0.25\lambda$, $r_4 = 0.3\lambda$)
Figure 3-18  Finite element mesh for the four layer dielectric circular cylinder.
Figure 3-19 Echo width for the four layer dielectric circular cylinder with $\varepsilon_{rd1} = 8$, $\varepsilon_{rd2} = 6$, $\varepsilon_{rd3} = 4$, $\varepsilon_{rd4} = 2$, $\sigma = 0$, and $\phi^i = -90^\circ$. 
Figure 3-20  Echo width for the four layer dielectric circular cylinder with $\varepsilon_{rd1} = 8$, $\varepsilon_{rd2} = 6$, $\varepsilon_{rd3} = 4$, $\varepsilon_{rd4} = 2$, $\sigma = 0.25$, and $\phi^i = -90^\circ$. 
Figure 3-21  Finite element mesh of four perfectly conducting circular cylinders.
Figure 3-22 Plot of $|E_n(\phi)|$ for the case of four perfectly conducting circular cylinders ($r = 0.08\lambda$, $d = 1.0\lambda$, $\phi^i = -135^\circ$) under $TM_\varepsilon$ excitation.
Figure 3-23  Finite element mesh of two homogeneous dielectric circular cylinders.
Figure 3-24  Plot of \(|E_n(\phi), H_n(\phi)|\) for the case of two homogeneous dielectric circular cylinders \((r = 0.3\lambda, d = 1.0\lambda, \epsilon_{rd} = 4, \phi^i = -100^\circ)\).
Figure 3-25  Plot of $|E_n(\phi), H_n(\phi)|$ for the case of two homogeneous dielectric circular cylinders ($r = 0.3\lambda$, $d = 1.0\lambda$, $\epsilon_{rd} = 4$, $\phi^i = 0^\circ$).
CHAPTER 4

SCATTERING FROM MULTIPLE CYLINDERS IN FREE SPACE

In the previous chapter, three of the examples (Figures 3-22, 3-24, and 3-25) involve geometries where there is more than one cylinder. In order to solve these problems, the solution technique in Chapter 3 requires that the cylinders be treated as one inhomogeneous cylinder. This means that a mesh which encloses all the cylinders must be generated. For cylinders which are far apart, the computational and/or storage requirements may be prohibitive. In addition, we must also consider the labor required to generate the finite element grid. Even with an automatic mesh generator, the creation of the meshes in Figures 3-21 and 3-23 is still labor intensive. By gridding only around the cylinders, the mesh generation process is greatly simplified. In this chapter a modified formulation is presented that allows each cylinder to be meshed individually. Many of the steps are the same as in the one cylinder case so the formulation is presented in an abbreviated form. Only the steps relevant to the multiple cylinder case are presented in detail. This formulation is especially efficient for the case where the cylinders are identical because the finite element solutions are the same for all the cylinders. Therefore, only the finite element solution for one cylinder needs to be considered. After the formulation, numerical results are presented. Comparisons are made to known series solutions. Also, some of the multiple cylinder cases in the previous chapter are rerun with the formulation in this chapter to show its efficiency and accuracy.
4.1 Formulation of the Bymoment Method

Let us examine $M$ infinitely long cylinders in free space, each of which has an arbitrary cross section. Since the finite element formulation assumes that the material properties are constant within each element of the corresponding mesh, we consider only cylinders which have piece-wise constant material properties or ones that can be approximated in this manner. The coordinate system is chosen such that the $z$-axis is parallel to the axis of the cylinders. As in the previous chapter, we consider only the $TM_z$ case. The $TE_z$ formulation is similar and will not be given here.

The geometry is shown in Figure 4-1. Each cylinder is wholly contained within its own specific mesh. The lines, $\partial S_1, \ldots, \partial S_M$, represent the mesh boundaries around each of the cylinders. We assume that there is at least one layer of elements between the cylinder boundary and the boundary of the enclosing mesh. Each of the artificial lines, denoted by $\partial S'_1, \ldots, \partial S'_M$, is wholly enclosed by the boundary of the corresponding mesh. We choose the paths of these lines such that they pass through the interior of all the elements which border the boundary of the enclosing mesh. We designate the surface within $\partial S_m$ ($m = 1, 2, \ldots, M$) to be $S_{cm}$ and the surface within $\partial S'_m$ to be $S'_{cm}$. Note that $S'_{cm} \subset S_m$.

Now that there is more than one cylinder the notation used in the previous derivation must be modified somewhat. Let us introduce a variable $t_m$ which varies from 0 to $d_m$ where $d_m$ is the length of $\partial S_m$. Then the electric field on $\partial S_m$ can be represented by the function $e^*_m(t_m)$ where $e^*_m(0) = e^*_m(t_m)$. Equation (3-5) for the
Figure 4-1  Geometry: $M$ arbitrary cylinders in free space.
multiple cylinder case is given by

\[ e_m(t_m) = \sum_{n=1}^{N_m} a_n^{(m)} \Psi_n(t_m) \quad t_m \in [0, d_m] \]  

(4 - 1)

where \( a_n^{(m)} \) are the unknown coefficients for the \( m \)th cylinder and \( N_m \) is the number of basis functions which is required to represent \( e_m(t_m) \). We assume that the same type of basis functions are used for all the cylinders. Using \( \Psi_n(t_m) \) \((n = 1, 2, \ldots)\) as the set of boundary conditions on \( \partial S_m \), we form the corresponding set of finite element solutions which is denoted by \( \Lambda_n^{(m)}(x, y) \). Therefore, the electric field inside \( S_{cm} \) is given by

\[ E_e(x, y) = \sum_{n=1}^{N_m} a_n^{(m)} \Lambda_n^{(m)}(x, y) \quad (x, y) \in S_{cm} \]  

(4 - 2)

To determine the coefficients let us define the region exclusive of \( S_{c1}', \ldots, S_{cM}' \) to be \( S_0' \). Green's theorem is then applied to \( S_0' \) as in the one cylinder case:

\[
\int \int_{S_0'} \left[ \Phi_j (\nabla^2 + k_0^2) E_{zc} - E_{zc} (\nabla^2 + k_0^2) \Phi_j \right] dS = \int \int_{S_{oo}} \left[ \Phi_j \frac{\partial E_{zc}}{\partial n_{oo}} - E_{zc} \frac{\partial \Phi_j}{\partial n_{oo}} \right] dS + \sum_{m=1}^{M} \int \int_{S_m'} \left[ \Phi_j \frac{\partial E_{zc}}{\partial n'} - E_{zc} \frac{\partial \Phi_j}{\partial n'} \right] dS
\]  

(4 - 3)

Substituting (3-6) through (3-9) in (4-3), we get

\[
\sum_{m=1}^{M} \sum_{n=1}^{N_m} a_n^{(m)} \int_{S_m'} \left[ \Phi_j \frac{\partial \Lambda_n^{(m)}}{\partial n'} - \Lambda_n^{(m)} \frac{\partial \Phi_j}{\partial n'} \right] dS = \sum_{m=1}^{M} \int_{S_m'} \left[ \Phi_j \frac{\partial E_{zc}^{inc}}{\partial n'} - E_{zc}^{inc} \frac{\partial \Phi_j}{\partial n'} \right] dS
\]  

(4 - 4)
Note that (4-4) reduces to the one cylinder expression (3-12) when \( M = 1 \).

The only unknowns left in (4-4) are the coefficients \( a_n^{(m)} \). To form a matrix equation for solving for the coefficients, we require \( L \) testing functions where

\[
L = \sum_{m=1}^{M} N_m. \quad (4-5)
\]

We can then form a matrix equation to solve for \( a_n^{(m)} \):

\[
\left( [A^{(1)}] \quad \cdots \quad [A^{(m)}] \quad \cdots \quad [A^{(M)}] \right) \left( \begin{array}{c} \{ b^{(1)} \} \\ \vdots \\ \{ b^{(m)} \} \\ \vdots \\ \{ b^{(M)} \} \end{array} \right) = \left( \begin{array}{c} T_1 \\ \vdots \\ T_j \\ \vdots \\ T_L \end{array} \right) \quad (4-6)
\]

where \([A^{(m)}]\) is a \( L \times N_m \) matrix which is given by

\[
[A^{(m)}] = \begin{pmatrix} Q_{11}^{(m)} & Q_{12}^{(m)} & \cdots & Q_{1N_m}^{(m)} \\ Q_{21}^{(m)} & Q_{22}^{(m)} & \cdots & Q_{2N_m}^{(m)} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{L1}^{(m)} & Q_{L2}^{(m)} & \cdots & Q_{LN_m}^{(m)} \end{pmatrix} \quad (4-7)
\]

and \( \{ b^{(m)} \} \) is a column matrix of dimension \( N_m \) given by

\[
\{ b^{(m)} \} = \begin{pmatrix} a_1^{(m)} \\ a_2^{(m)} \\ \vdots \\ a_{N_m}^{(m)} \end{pmatrix}^T \quad (4-8)
\]

The terms \( Q_{jn}^{(m)} \) and \( T_j \) are written as,

\[
Q_{jn}^{(m)} = \int_{\partial S_m} \left[ \Phi_j \frac{\partial \Delta_n^{(m)}}{\partial n^t} - \Delta_n^{(m)} \frac{\partial \Phi_j}{\partial n^t} \right] dl \quad (4-9)
\]

\[
T_j = \sum_{m=1}^{M} \int_{\partial S_m} \left[ \Phi_j \frac{\partial E_s^{inc}}{\partial n} - E_s^{inc} \frac{\partial \Phi_j}{\partial n} \right] dl \quad (4-10)
\]
As in the one cylinder case, \( T_j \) can be reduced into a simple form when the testing function is given by (3-20). Using a derivation similar to the one in Chapter 3, we obtain the following relationship:

\[
-4iE_z^{inc}(x_j, y_j) = \sum_{m=1}^{M} \int_{\partial S_m} \left[ \Phi_j \frac{\partial E_z^{inc}}{\partial n'} - E_z^{inc} \frac{\partial \Phi_j}{\partial n'} \right] dl \tag{4-11}
\]

where \((x_j, y_j) \in S'_{cm}\) is the location of the line source associated with the \(j^{th}\) testing function.

### 4.2 Identical Cylinder Case

For the special case where the cylinders have the same geometries and material properties, the finite element solutions for all the separate meshes are identical if we choose the basis functions \( \Psi_n(t_m) \) to be the same on \( \partial S_1, \ldots, \partial S_M \). Therefore, \( N_m \) and \( \Lambda_n^{(m)} \) are the same for all values of \( m \). Since they are independent of \( m \), we may redefine \( N_m \) to be \( N \) and \( \Lambda_n^{(m)} \) to be \( \Lambda_n \). Thus, only the finite element solution of a single cylinder is necessary to determine the solution of a multiple cylinder problem. This is very advantageous since, for large penetrable scatterers, the majority of the computational time is devoted to the finite-element calculations. The testing functions in (3-20) are especially suitable for the identical cylinder problem because the matrix in (4-6) can be separated into submatrices where each submatrix describes a physical interaction between the cylinders. Let us consider the case where we place an equal number of line sources in each surface \( S'_{cm} \). To formulate the submatrices, let us re-index the testing functions to explicitly show the specific
surface to which the line source of a testing function belongs. The testing functions are given by

\[ \Phi_i^\ell = H_0^{(2)}(k_0|\vec{r} - \vec{r}_i^\ell|) \]

where \( \vec{r}_i^\ell \) is the location of the source for the \( i \)'th testing function in \( S_i^\ell \). Note that \( i \) goes from 1 to \( N \). Equation (4-6) can then be written as follows:

\[
\begin{pmatrix}
[Z_{11}] & [Z_{12}] & \cdots & [Z_{1M}] \\
[Z_{21}] & [Z_{22}] & \cdots & [Z_{2M}] \\
\vdots & \vdots & \ddots & \vdots \\
[Z_{M1}] & [Z_{M2}] & \cdots & [Z_{MM}]
\end{pmatrix}
\begin{pmatrix}
\{b_1\} \\
\{b_2\} \\
\vdots \\
\{b_M\}
\end{pmatrix}
= 
\begin{pmatrix}
T_1 \\
T_2 \\
\vdots \\
T_L
\end{pmatrix}
\]

(4-13)

where the \( N \times N \) submatrix \([Z_{tm}]\) is given by

\[
[Z_{tm}] = 
\begin{pmatrix}
Q_{11} & Q_{12} & \cdots & Q_{1N} \\
Q_{21} & Q_{22} & \cdots & Q_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
Q_{N1} & Q_{N2} & \cdots & Q_{NN}
\end{pmatrix}
\]

(4-14)

The matrix element \( Q_{in} \) is described by

\[
Q_{in} = \int_{S_i^m} \left[ \Phi_i^\ell \frac{\partial \Lambda_n}{\partial n'} - \Lambda_n \frac{\partial \Phi_i^\ell}{\partial n'} \right] dl
\]

(4-15)

Although it is not shown explicitly in (4-14), we can see from (4-15) that \( Q_{in} \) depends on the \( \ell \) and \( m \) indices. These indices also determine the interaction to which a given submatrix corresponds. Each of the submatrices \([Z_{tm}]\) describes a separate interaction between the different cylinders. The diagonal submatrix \([Z_{mm}]\) describes the scattering from the \( m^{th} \) cylinder with the other cylinders absent. The submatrices \([Z_{tm}]\) and \([Z_{ml}]\) describe the scattering interaction between the \( m^{th} \) and
\( \ell^{th} \) cylinder. These submatrices are analogous to the self impedance and mutual impedance terms in an impedance matrix.

For cases where there are a large number of cylinders, the amount of computational time required to fill the matrix in (4-13) becomes a significant part, if not the majority, of the overall time. By choosing the locations of the line sources wisely, we can reduce the calculation time. If we specify the pattern of the source locations to be the same within each mesh, then \([Z_{mm}]\) would be identical for all \(m\). We would therefore need to evaluate \([Z_{mm}]\) for only one value of \(m\). In general, unlike an impedance matrix, the matrix in (4-13) is not reciprocal, i.e.,

\[
[Z_{\ell m}] \neq [Z_{m\ell}] \quad \ell \neq m. \tag{4 - 16}
\]

In order to form a reciprocal matrix, there are several requirements which must be met. As an example, consider the case of two identical circular cylinders in free space (Figure 4-2) where four finite element basis functions are used to determine the solution over each of the cylinders. The source locations of the testing functions are represented in the figure by the filled circles. The artificial plane \(P\) is located half-way between the cylinders. For the matrix to be reciprocal, the geometry of the scatterer, the position of testing functions, and the position of the basis functions on \(\partial S_1\) and \(\partial S_2\) must be mirror symmetric with respect to the plane \(P\). For our example, the geometry of the scatterer is such that it is mirror symmetric with respect to the plane \(P\). Furthermore, if we let the numbers in Figure 4-2 denote the index \(i\) of the testing function, the testing functions will also be mirror symmetric. Finally, let us choose the basis such that they satisfy the symmetry conditions. We now satisfy the conditions for a reciprocal matrix. Note that when these conditions are satisfied, the diagonal submatrix terms are still identical.
Figure 4-2  Testing function placement for two identical circular cylinders.
As the number of cylinders and the complexity of the cylinders' geometry increase, it becomes more difficult to choose the basis and testing functions such that the reciprocal relationship of the matrix holds. Therefore, we suggest that this method of reducing the fill time of the matrix only be used for a simple configuration that consists of a large number of cylinders such as a large array of cylinders along one dimension or a two dimensional array where the cylinders are placed in a rectangular pattern. Also, note that the geometry must have mirror symmetry.

4.3 Numerical Results

The bymoment method is first applied to the cases of scattering from two, three, four, and nine perfectly conducting circular cylinders in free space under a $TM_\perp$ excitation. These cases were chosen so that we could compare our results to the series solutions given by Ragheb and Hamid [1985]. The parameter of interest is the far-field pattern given in terms of $|E_n(\phi)|$ for $TM_\perp$ excitation and $|H_n(\phi)|$ for $TE_\perp$ excitation. These parameters are defined in (3-36). The incident wave is the plane wave defined in (3-31). In all the results presented in this section sinusoidal functions are used for the basis functions, and the free space Green’s functions are used for the testing functions. The meshes for all the perfectly conducting circular cylinders have one layer of elements on each cylinder and are the same in all cases except for a scaling factor to account for different radii. The mesh is composed of 48 nodes with a half-bandwidth of 6.

In the comparison with Ragheb and Hamid, the series solution is denoted by the solid line, whereas the bymoment solution is denoted by the dashed line overlaid by the triangle symbols. The first case that we consider is the scattering from two
identical perfectly conducting circular cylinders. The radius of the cylinders is given by \( r = 0.16\lambda \), and \( N = 6 \) for each cylinder. The separation distance of the cylinders is \( d = 0.477\lambda \), and the angle of incidence is \( \hat{\phi} = -100^\circ \). In Figure 4–3, we show \(|E_n(\phi)|\) for this case. There is excellent agreement between the series and bymoment solution. In Figure 4–4, we present the results for the three cylinder case \((N = 6)\). The parameters are given by \( r = 0.12\lambda \), \( d = 1.0\lambda \), and \( \hat{\phi} = 180^\circ \). The differences between the bymoment and series solutions are mainly due to the coarse sampling used to graphically reproduce the series solution results.

In Figures 4–5 and 4–6, numerical results are presented for the four and nine cylinder cases, respectively. The parameters used for these two cases are \( r = 0.08\lambda \), \( k_0d = 1.0\lambda \), \( \hat{\phi} = -135^\circ \), and \( N = 4 \). We see that there are major differences between the solutions in both cases. In order to determine which solution is the correct one, a comparison to results from the previous chapter is made for the four cylinder case (Figure 4–7). The solution using a single mesh is given by the solid line, whereas the solution using four meshes is given by the dashed line overlaid with triangles. We see that there is excellent agreement. Therefore, we believe that there is an error in the series solution, possibly due to a premature truncation of the series.

To show the advantages of using multiple meshes rather than a single mesh, we compare the computational time needed to obtain the results in Figure 4–7. The CPU time required to obtain the single mesh solution was approximately nine times greater than that for the multiple mesh solution.

We next consider scattering from non-perfectly conducting cylinders. The first case involves scattering from two homogeneous dielectric cylinders where \( \epsilon_{rd} = \ldots \)
Figure 4-3  Plot of $|E_n(\phi)|$ due to a plane wave for the case of two perfectly conducting circular cylinders in free space; $r = 0.16\lambda$, $d = 0.477\lambda$, $\delta^i = -100^\circ$, $TM_z$ case.
Figure 4-4  Plot of $|E_n(\phi)|$ due to a plane wave for the case of three perfectly conducting circular cylinders in free space; $r = 0.12\lambda$, $d = 1.0\lambda$, $\phi^i = 180^\circ$, TM\textsubscript{z} case.
Figure 4-5  Plot of $|E_n(\phi)|$ due to a plane wave for the case of four perfectly conducting circular cylinders in free space; $r = 0.08\lambda$, $d = 1.0\lambda$, $\phi^i = -135^\circ$, $TM_z$ case.
Figure 4-6  Plot of $|E_n(\phi)|$ due to a plane wave for the case of nine perfectly conducting circular cylinders in free space; $r = 0.08\lambda$, $d = 1.0\lambda$, $\hat{\phi}^i = -135^\circ$, TM$_z$ case.
Figure 4-7 Plot of $|E_n(\phi)|$ for the case in Figure 4-5: Comparison of results from a single mesh to that of multiple meshes.
4, r = 0.3λ, d = 1λ, and N = 12. The mesh is composed of 556 nodes with a half-bandwidth of 43. Two different angles of incidence are considered (\( \phi^i = -90^\circ \) and \( \phi^i = -100^\circ \)). The TM\(_2\) and TE\(_2\) cases are shown in Figures 4–8 and 4–9, respectively. The comparison to Figure 3–24 shows excellent agreement between the multiple mesh and single mesh cases. For \( \phi^i = -90^\circ \), we see that both \(|E_n(\phi)|\) and \(|H_n(\phi)|\) are symmetrical around \( \phi = 0^\circ \). This is expected since both the incident field and the geometry is symmetrical around the x-axis. It is interesting to note that even for a small deviation of the incident field from the x-axis, such as in the case of \( \phi^i = -100^\circ \), \(|E_n(\phi)|\) and \(|H_n(\phi)|\) become strongly asymmetric.

Because of the lack of numerical results in the literature for cases involving inhomogeneous multiple scatterers, we consider the geometry of multiple inhomogeneous semicircular dielectric cylinders under TM\(_2\) excitation. The numerical results generated for this geometry can be used to validate future work on multiple scatterers. The mesh used for the semicircular cylinders is shown in Figure 4–10. The mesh has 248 nodes, and the resulting finite element matrix has a half-bandwidth of 20. Fourteen basis functions are used. Results are shown for the case of two semicircular cylinders in Figure 4–11. We specify \( r = 0.3\lambda, d = 0.6\lambda, \) and \( \phi^i = -90^\circ \). Three different material properties are considered. The solid curve represents the homogeneous case with the relative dielectric constant given by \( \epsilon_{rd} = 4 \). The dashed curve, labeled by “inhom1”, represents an inhomogeneous dielectric with \( \epsilon_{rd} = 8 - 6\rho/r \) where \( \rho \) is the radial coordinate centered at the origin of each of the semicircular dielectric cylinders. The dotted curve, denoted by “inhom2”, represents the results for another inhomogeneous cylinder with \( \epsilon_{rd} = 4/(1 + (\rho/r)^2)^2 \). Finally, we consider the case where four semicircular cylinders are present. The parameters are the same
Figure 4-8 Plot of $|E_n(\phi)|$ due to a plane wave for the case of two homogeneous dielectric ($\varepsilon_r = 4$) circular cylinders in free space; $r = 0.3\lambda$, $d = 1.0\lambda$, $TM_z$ case.
Figure 4-9  Plot of $|H_n(\phi)|$ due to a plane wave for the case of two homogeneous dielectric ($\varepsilon_{rd} = 4$) circular cylinders in free space; $r = 0.3\lambda$, $d = 1.0\lambda$, $TE_x$ case.
as those in the two semicircular cylinder case. The results are shown in Figure 4–12. From Figures 4–11 and 4–12, we see that the choice of the material properties has a significant effect on the far-field pattern. These figures also indicate that the bymoment method can be used as a tool to design a scattering structure to form a specific far-field pattern.

In formulating the finite-element problem, we made the assumption that the material properties did not vary within each element. We therefore use a piece-wise constant approximation of the equation for $\varepsilon_{rd}$ within each element. The constant approximation for $\varepsilon_{rd}$ in each element is chosen to be the exact value at the centroid of that element.
Figure 4-10  Finite element mesh for the semicircular dielectric cylinder with arc radius of 0.3\(\lambda\).
Figure 4-11 Plot of $|E_n(\phi)|$ due to a plane wave for the case of two dielectric semicircular cylinders in free space; $r = 0.3\lambda$, $d = 0.6\lambda$, $\phi^i = -90^\circ$, $TM_2$ case.
Figure 4-12  Plot of $|E_n(\phi)|$ due to a plane wave for the case of four dielectric semicircular cylinders in free space; $r = 0.3\lambda$, $d = 0.6\lambda$, $\phi^i = -90^\circ$, $TM_z$ case.
CHAPTER 5

CYLINDERS NEAR A PLANAR MEDIA INTERFACE

In Chapter 3 the problem of scattering from a single cylinder in free space is considered. In this chapter the formulation is extended to handle the case where the exterior region is composed of two semi-infinite half-spaces. Since the formulation for the interior solution and the solution on the boundary of the mesh are the same as in Chapter 3, it will not be considered here. The major differences are in the evaluation of the coefficients in (3-6) and the expression for the testing functions. These testing functions must satisfy the correct boundary conditions along the interface joining the two half-spaces and at infinity. These two issues are addressed in the first two sections of this chapter. In the third and final section, we present numerical results to validate our theory. Comparisons are made to method of moments solutions available in the literature.

5.1 Coupling of the Interior Solution to the Media Interface

Let us consider the case of an infinitely long cylinder in the presence of a media interface which divides the region into two half-spaces (Figure 5-1). Although this figure shows the cylinder to be above the interface, we will also consider the case where the cylinder is wholly contained or partially buried in the lower half-space. The material properties of the upper half-space, exclusive of the cylinder, are that of free space. The material properties of the lower half-space are homogeneous and in general, may be either lossy or a pure dielectric. The permeability of the lower
half-space is assumed to be that of free space \((\mu_2 = \mu_1 = \mu_0)\). The coordinate system is cartesian and is oriented such that the \(z\)-axis is parallel to the axis of the cylinder and the \(y = 0\) line is lying on the interface itself. Since we are applying the finite element method only over the region occupied by the cylinder, we define a mesh which totally encloses the cylinder. The mesh is constructed such that the elements on the boundary of the mesh are exterior to the cylinder. The boundary of the mesh is given by \(\partial S\). The line \(\partial S'\) follows a closed path which passes through the interior of all the elements on the boundary of the mesh. We denote the region inside of \(\partial S\) by \(S_c\). The upper half-space, exclusive of \(S_c\), is defined to be \(S_1\), and the lower half-space, exclusive of \(S_c\), is called \(S_2\).

Unlike the previous two chapters, there are major differences between the formulations of the \(TM\) and \(TE\) polarizations. Because the permeability \(\mu\) is constant throughout the exterior region, the \(TM\) formulation does not explicitly incorporate the permeability in the general derivation. Since the permittivity is not uniform in the exterior region, the \(TE\) case is a better model for presenting the general formulation. Therefore, the formulation will be done in terms of the \(TE\) rather than the \(TM\) polarization. On occasions where the two cases differ, we will show the steps for both. For the derivation, equation (3-5) and (3-6) are rewritten for the \(TE\) case as follows:

\[
h^e(t) \approx \sum_{n=1}^{N} a_n \Psi_n(t) \quad (5 - 1)
\]

\[
H_z(x, y) \approx \sum_{n=1}^{N} a_n \Lambda_n(x, y) \quad (5 - 2)
\]

where \(h^e(t)\) is the magnetic field on the boundary of the mesh.
Figure 5-1  Geometry for an arbitrary cylinder in the presence of an interface.
As in Chapter 3, let the surface enclosed by $\partial S'$ be denoted by $S'_0$ and the surface $S'_0$ be the region exterior to $\partial S'$. In this case $S'_0$ contains either all or part of both $S_1$ and $S_2$. The field in $S'_0$ can be divided into the incident field and the scattered field. This is written as,

$$H_z(x, y) = H_z^{inc}(x, y) + H_z^{sc}(x, y) \quad (x, y) \in S'_0. \quad (5-3)$$

where the scattered field satisfies the Helmholtz equation,

$$(\nabla^2 + k_i^2)H_z^{sc}(x, y) = 0 \quad (x, y) \in S'_0 \quad (5-4)$$

and the Sommerfeld radiation condition at infinity. The variable $k_i$ represents the wave number of the region where the Helmholtz equation is evaluated, i.e., $k_1$ for $S_1$, and $k_2$ for $S_2$. Similar subscripts are used to denote the conductivity and relative dielectric constant. The incident field $H_z^{inc}$ is defined to be the field everywhere in the absence of the cylinder. In other words, it is the field in the presence of the interface. This definition of the incident field implies that the boundary conditions for the continuity of the tangential electric and magnetic field at the interface are satisfied by $H_z^{inc}$.

Let $\Phi_j (j = 1, 2, \ldots, N)$ be a set of $N$ linearly independent testing functions which are chosen to satisfy the Helmholtz equation over the surface $S'_0$, i.e.,

$$(\nabla^2 + k_i^2)\Phi_j(x, y) = 0 \quad (x, y) \in S'_0 \quad (5-5)$$

and the radiation condition at infinity. Applying Green's theorem for $H_z^{sc}$ and $\Phi_j(\epsilon_{ri} - i(\sigma_i/\omega \epsilon_0))^{-1}$ over the region denoted by the intersection of $S'_0$ and the
upper half-space and the region denoted by the intersection of \( S'_0 \) and the lower half-space, we obtain the following expression from (5-4) and (5-5):

\[
\int_{\partial S'} \frac{1}{\varepsilon_{ri} - i\frac{\sigma}{\omega_0}} \left[ \Phi_j \frac{\partial H^{sc}_x}{\partial n'} - H^{sc}_x \frac{\partial \Phi_j}{\partial n'} \right] \, dl \\
+ \int_{\partial S_{int}} \frac{1}{\varepsilon_{r1} - i\frac{\sigma}{\omega_0}} \left[ \Phi_j^{(+)} \frac{\partial H^{sc(+)}_x}{\partial y} - H^{sc(+)}_x \frac{\partial \Phi_j^{(+)}}{\partial y} \right] \, dl \\
- \int_{\partial S_{int}} \frac{1}{\varepsilon_{r2} - i\frac{\sigma}{\omega_0}} \left[ \Phi_j^{(-)} \frac{\partial H^{sc(-)}_x}{\partial y} - H^{sc(-)}_x \frac{\partial \Phi_j^{(-)}}{\partial y} \right] \, dl = 0
\]

(5 - 6)

where \( \partial S_{int} \) denotes the plane interface and the superscripts \((+)\) and \((-)\) denote the values of the field quantities just above and just below the interface, respectively. Note that the contributions from the integral at infinity vanishes since both \( H^{sc}_x \) and \( \Phi_j \) satisfy Sommerfeld's radiation condition. Since \( H^{sc}_x \) satisfies the interface boundary conditions which are given by

\[
H^{sc(+)}_x = H^{sc(-)}_x
\]

(5 - 7a)

\[
\frac{1}{\varepsilon_{r1} - i\frac{\sigma}{\omega_0}} \frac{\partial H^{sc(+)}_x}{\partial y} = \frac{1}{\varepsilon_{r2} - i\frac{\sigma}{\omega_0}} \frac{\partial H^{sc(-)}_x}{\partial y}
\]

(5 - 7b)

we observe that the contribution from the line integral along the interface will vanish if we choose \( \Phi_j \) such that they satisfy (5-7a) and (5-7b) also. This leaves only the line integral term,

\[
\int_{\partial S'} \frac{1}{\varepsilon_{ri} - i\frac{\sigma}{\omega_0}} \left[ \Phi_j \frac{\partial H^{sc}_x}{\partial n'} - H^{nt}_x \frac{\partial \Phi_j}{\partial n'} \right] \, dl = 0.
\]

(5 - 8)
By substituting (5-3) into (5-8) and then using (5-2), we may write the following equation:

\[
\sum_{n=1}^{N} a_n \int_{\partial S'} \frac{1}{\varepsilon_{ri} - i \frac{\sigma_i}{\omega_0}} \left[ \Phi_j \frac{\partial A_n}{\partial n'} - A_n \frac{\partial \Phi_j}{\partial n'} \right] \, dl = \int_{\partial S'} \frac{1}{\varepsilon_{ri} - i \frac{\sigma_i}{\omega_0}} \left[ \Phi_j \frac{\partial H_{z}^{inc}}{\partial n'} - H_{z}^{inc} \frac{\partial \Phi_j}{\partial n'} \right] \, dl \tag{5-9}
\]

For the TM$_x$ polarization the boundary conditions are different from the TE$_x$ case. The boundary conditions for E$_x^{sc}$ is

\[
E_{x}^{sc(+)} = E_{x}^{sc(-)} \tag{5-10a}
\]

\[
\frac{1}{\mu_1} \frac{\partial E_{x}^{sc(+)}}{\partial y} = \frac{1}{\mu_2} \frac{\partial E_{x}^{sc(-)}}{\partial y} \tag{5-10b}
\]

Since we assume that $\mu_1 = \mu_2$, (5-10b) just enforces the continuity of $\partial E_{x}^{sc}/\partial y$. In order to eliminate the integral along the interface when Green’s theorem is applied, the testing functions must satisfy the same boundary conditions as $E_{x}^{sc}$ in (5-10a) and (5-10b). Therefore, the testing function is not the same for the two polarizations. To obtain the TE$_x$ expression for (5-9) just replace $H_{z}^{inc}$ by $E_{x}^{inc}$ and remove the $1/(\varepsilon_{ri} - i \sigma_i/\omega_0)$ term. The testing function and finite element solutions are those for the TM$_x$ case. In the next section a description of the testing functions is given.
5.2 Testing Functions

The testing function that we will consider is the Green's function that incorporates the correct boundary conditions at the interface. Unlike the free space case, the Green's function for the media interface problem cannot be evaluated analytically but rather it must be evaluated numerically. It is well known for an electric line source ($TM_z$ case) and is given by Felsen and Marcuvitz [1973]. Similarly, the solution for a magnetic line source ($TE_z$ case) can be easily obtained. In order to differentiate between the two solutions, let $g^I$ be the Green's function due to an electric line source, and let $g^M$ be the Green's function due to a magnetic line source. Note that for the Green's function geometry, $S_1$ encompasses the entire upper half-space, and $S_2$ encompasses the entire lower half-space.

Let us consider the problem of an electric line source of unity strength located in $S_1$. We must therefore solve the following differential equation:

$$-(\nabla^2 + k_i^2)g^I_{i1}(x,y|x',y') = \delta(x-x')\delta(y-y') \quad i = 1, 2 \tag{5 - 11}$$

where the primed coordinates indicate the location of the source and the unprimed coordinates indicate the observer location. The subscript $i1$ of the Green's function indicates that the observation point is in $S_i$ and the source location is in $S_1$. Since we are considering an electric line source, $g^I_{i1}$ must satisfy the boundary conditions of $E_x$ and $\partial E_x/\partial y$ along the interface. The resulting solution for the observation point in $S_1$ is given by

$$g^I_{i1}(x,y|x',y') = \frac{1}{4i}H_0^{(2)}(k_1|r-r'|) + \frac{1}{\pi} \int_0^\infty \frac{u_1 - u_2}{2u_1(u_1 + u_2)} e^{-u_1(y+y')} \cos k_x(x-x') \, dk_x \tag{5 - 12}$$
where $|\vec{r} - \vec{r}'| = \sqrt{(x - x')^2 + (y - y')^2}$ and $u_i = \sqrt{k_z^2 - k_i^2}; \text{Re}(u_i) > 0, i = 1, 2.$

The form of (5-12) is not the same as that given by Felsen and Marcuvitz but is more suitable for numerical evaluation [Baertlein, 1988]. Also, we have corrected for differences in coordinate system and time variation definitions. In studying the expression in (5-12), we note that the first term represents the Green's function in the absence of the interface, and the second term represents the reflection from the interface. The solution in $S_2$ is given by

$$g_{21}^f(x, y|x', y') = \frac{1}{\pi} \int_0^\infty \frac{e^{-u_1 y'} e^{u_2 y}}{u_1 + u_2} \cos k_z(x - x') dk_z. \quad (5 - 13)$$

To obtain the Green's functions for the line source in $S_2$, we simply have to switch $u_1$ and $u_2$ and replace $(y, y', k_1)$ by $(-y, -y', k_2)$. Then (5-12) becomes the solution for the observation point in $S_2$, and (5-13) becomes the solution for the observation point in $S_1$.

To determine the values of the Green's functions we must numerically evaluate the integrals in (5-12) and (5-13). This is done by repeated application of Gauss quadrature integration on finite sub-intervals along the real axis until a specific convergence is achieved. Because of the highly oscillatory behavior and slow decay of the integrand as a function of $k_z$, the evaluation of the integral is both tedious and computationally expensive. For most problems, the majority of the computational time is spent evaluating the testing functions rather than the finite-element solutions. Thus, it is very important to minimize the testing function calculations. Several techniques are utilized to improve the convergence of the integral [Johnson and Dudley, 1983], including asymptotic extraction and variable transformations.
In Appendix B, a detailed description of the integral evaluation as well as the derivation of the Green's functions is given.

For a magnetic line source of unit strength in $S_1$, we have the following differential equation:

$$-(\nabla^2 + k_1^2)g_{i1}^M(x, y|x', y') = \delta(x - x')\delta(y - y') \quad i = 1, 2 (5 - 14)$$

The definitions are the same as that in the $TM$ case except that $g_{i1}^M$ must satisfy the boundary conditions for $H_z$ and $\partial H_z/\partial y$ at the interface. The resulting solution for the observation point in $S_1$ is given by

$$g_{i1}^M(x, y|x', y') = \frac{1}{4i} H_0^{(2)}(k_1|\vec{r} - \vec{r}'|) + \frac{1}{\pi} \int_0^\infty \frac{\epsilon_1^* u_1 - u_2}{2\epsilon_1^* u_1 + u_2} e^{-u_1(y+y')} \cos k_s(x-x') dk_s (5 - 15)$$

where we define $\epsilon_1^* = (\epsilon_{r2} - i\sigma_2/\omega\epsilon_0)/(\epsilon_{r1} - i\sigma_1/\omega\epsilon_0)$. Since $S_1$ is free space, we see that $\epsilon_1^* = \epsilon_{r2} - i\sigma_2/\omega\epsilon_0$. The solution for the observation point in $S_2$ is given by

$$g_{i1}^M(x, y|x', y') = \frac{1}{\pi} \int_0^\infty \frac{\epsilon_2^* e^{-u_1 y'} e^{u_2 y}}{\epsilon_2^* u_1 + u_2} \cos k_s(x-x') dk_s (5 - 16)$$

For a line source in $S_2$, we make the same modifications as in the $TM$ case. In addition, $\epsilon_1^*$ is replaced by $1/\epsilon_2^*$. To numerically evaluate the $TE_z$ Green's functions, we use the same numerical techniques as those used to evaluate the $TM_z$ Green's functions. Again, the details are given in Appendix B.

We may now relate the Green's functions to the testing functions $\Phi_j$. Since the Green's function must satisfy the Helmholtz equation in $S_0'$, we locate the line source in $S_0'$. Let $\{x_j, y_j\}, j = 1, 2, \ldots, N$ be points in $S_0'$. Then for the $TE_z$ case,

$$\Phi_j = g_{jm}^M(x, y|x_j, y_j) \quad i, m = 1, 2 (5 - 17)$$
where the subscript \( im \) indicates whether the observation and source locations are above or below the interface. This results in \( N \) linearly independent testing functions for any choice of \( (x_j, y_j) \in S_c^j \) as long as the locations of the line sources are different for each \( j \).

By substituting the Green's function expressions into (5-9), we obtain

\[
\sum_{n=1}^{N} a_n \int_{\partial S'} \frac{1}{\varepsilon_r - i \frac{\sigma}{\sqrt{2} \omega_0}} \left[ \frac{M_n}{g_{im}} \frac{\partial A_n}{\partial n'} - \frac{\Delta_n}{\partial n'} \right] \, dl
= \int_{\partial S'} \frac{1}{\varepsilon_r - i \frac{\sigma}{\sqrt{2} \omega_0}} \left[ \frac{M_n}{g_{im}} \frac{\partial H_z^{inc}}{\partial n'} - H_z^{inc} \frac{\partial g_{im}}{\partial n'} \right] \, dl \tag{5 - 18}
\]

The normal derivative of the Green's function \( \partial g_{im}^M / \partial n \) is determined from evaluation the Green's function at the nodes of the elements which contain \( \partial S' \). We then numerically compute the derivative along \( \partial S' \). Because we choose the Green's function to be the testing function, the right hand side of (5-18) can be simplified since

\[
\frac{1}{\varepsilon_r - i \frac{\sigma}{\sqrt{2} \omega_0}} H_z^{inc}(x_j, y_j) = - \int_{\partial S'} \frac{1}{\varepsilon_r - i \frac{\sigma}{\sqrt{2} \omega_0}} \left[ \frac{M_n}{g_{im}} \frac{\partial H_z^{inc}}{\partial n'} - H_z^{inc} \frac{\partial g_{im}}{\partial n'} \right] \, dl \tag{5 - 19}
\]

5.3 Numerical Results

For the cases considered in this section the incident field is a plane wave traveling from the upper half-space into the lower half-space. The solution to this problem is well known [Wait, 1985]. For the \( TM_s \) case,

\[
E_s^{inc} = \begin{cases} 
    e^{ik_1 y \cos \phi'} + \Gamma \, e^{-ik_1 y \cos \phi'} e^{ik_1 z \sin \phi'} , & y > 0 \\
    [1 + \Gamma] e^{ik_1 (z \sin \phi' + y \cos \phi')} , & y < 0 
\end{cases} \tag{5 - 20}
\]
where \( \Gamma_\perp = (\cos \phi^i - \cos \phi^t)/(\cos \phi^i + \cos \phi^t) \) and \( \cos \phi^t = \sqrt{\varepsilon_r - \sin^2 \phi^i} \). For the \( \text{TE}_2 \) case,

\[
H^\text{inc}_x = \begin{cases} 
[\epsilon_r k_1 y \cos \phi^i + \Gamma_\parallel e^{-ik_1 y \cos \phi^i}] e^{ik_1 z \sin \phi^i}, & y > 0; \\
[1 + \Gamma_\parallel] e^{ik_1 (z \sin \phi^i + y \cos \phi^i)}, & y < 0
\end{cases}
\tag{5 - 21}
\]

where \( \Gamma_\parallel = (\epsilon_r \cos \phi^i - \cos \phi^t)/(\epsilon_r \cos \phi^i + \cos \phi^t) \).

Sinusoidal basis functions are used for \( \Psi_n(t) \) to calculate all of our results. The geometries are either perfectly conducting circular cylinders or dielectric circular cylinders. Therefore, Figure 5–2 can be used to describe all the geometries for this section. The mesh for the perfectly conducting circular cylinder consists of one layer of elements around the cylinder such as the one in Figure 3–6.

The numerical results are given in terms of a normalized surface current \( j^\text{norm} \) for the perfectly conducting circular cylinder and a normalized far-field expression for the dielectric circular cylinder. The surface currents are normalized to the incident magnetic field at a specific point on the cylinder. For the \( \text{TE}_2 \) case, the normalized surface current is given by

\[
j^\text{norm}_\phi(\phi) = \frac{H_x(\lambda a, \phi)}{H_2^\text{inc}(\lambda a, \phi_a)} \tag{5 - 22}
\]

where the \( \rho, \phi \) cylindrical coordinate system is centered on the cylinder as shown in Figure 5–2. The radius of the cylinder is specified to be \( \rho/\lambda = a \) where \( \lambda \) is the free space wavelength. The cylinder is centered at the point \( x/\lambda = 0, y/\lambda = h \). For cylinders which are totally or partially in the upper half-space, we define \( \phi_a = 90^\circ \). For cylinders which are totally in the lower half-space, we define \( \phi_a = -90^\circ \).
Figure 5-2  Geometry of a circular cylinder in the presence of an interface.
the \( TM_\varepsilon \) case, where the incident magnetic field is given by \( H_{t\,\text{inc}} \), we obtain the following expression for the normalized surface current:

\[
j_{x\,\text{norm}}(\phi) = \frac{H_t(\lambda a, \phi)}{H_{t\,\text{inc}}(\lambda a, \phi_a)}
\]

(5 - 23)

The unit vector \( \hat{t} \) is the vector tangential to the cylinder in the \( x-y \) plane. \( H_t \) and \( H_{t\,\text{inc}} \) can be obtained from the derivatives of \( E_x \) and \( E_{x\,\text{inc}} \), respectively. The derivatives of \( E_x \) are evaluated numerically from the finite element solution. Note that since \( H_t \) involves normal (to the surface of the cylinder) derivatives of \( E_x \), we calculate \( H_t \) at the centroids of the elements bordering the cylinder rather than on the element boundary at the surface of the cylinder. Therefore, the results for the \( TM_\varepsilon \) case have some inaccuracies. To minimize these inaccuracies, we keep the elements which border the cylinder small. For the \( TE_\varepsilon \) case, this inaccuracy does not exist since \( H_x \) is calculated on the nodes along the cylinder. Another problem with the mesh in Figure 3-6 is that it cannot be used for a circular cylinder which touches the interface since the material properties of each element are assumed to be constant. Because the cylinder contacts the interface only at one point, we could not obtain a suitable mesh for this geometry with quadrilateral elements. This problem can be overcome with triangular elements, but unfortunately, our mesh generator can only use quadrilateral elements. When comparisons need to be made to results for a cylinder sitting on the interface, finite element results are generated for a cylinder which is lifted slightly off the interface.

For the dielectric cylinder (\( TM_\varepsilon \)), the normalized far-field expression is

\[
|E_x^{\text{ff}}(\phi)| = \lim_{\rho \to \infty} \frac{|E_x^{\text{c}}(\rho, \phi)|}{|E_{\text{max}}(\rho, \phi)|}
\]

(5 - 24)
where \(|E_{\text{max}}(\rho, \phi)|\) is defined to be the maximum value of \(|E_z^{\text{sc}}(\rho, \phi)|\) as \(\rho \to \infty\) for all values of \(\phi\) so that \(0 \leq |E_z^{\text{sf}}| \leq 1\). The scattered field can be calculated from Green's theorem, which results in the expression

\[
E_z^{\text{sc}}(x, y) = \int_{\partial S'} \left[ g_{im}^J(x, y|x', y') \frac{\partial E_z(x', y')}{\partial n'} - E_z(x', y') \frac{\partial g_{im}^J(x, y|x', y')}{\partial n'} \right] \, dx'
\]

(5 - 25)

where \(g_{im}^J\) is the Green's function from (5-12) and (5-13). In general, the above form is rather complicated since the Green's function must be evaluated numerically, but for the case where \(\rho \to \infty\), this can be simplified when a far-field approximation for the Green's function is used. Applying the method of steepest descent to our Green's functions, we obtain the following far-field expressions:

\[
g_{11}^J(\rho, \phi) \sim \sqrt{\frac{1}{8\pi}} e^{i\pi/4} e^{ik_1z' \sin \tilde{\phi}} \left[ e^{ik_1y' \cos \tilde{\phi}} + R_1 e^{-ik_1y' \cos \tilde{\phi}} \right] \frac{e^{-ik_1\rho}}{\sqrt{k_1\rho}} \]  

(5 - 26)

\[
g_{21}^J(\rho, \phi) \sim \sqrt{\frac{1}{8\pi}} e^{i\pi/4} \left[ 1 + R_2 \right] e^{-ik_2z' \sin \tilde{\phi}} e^{-ik_1y' \sqrt{1 - \epsilon_{22} \sin^2 \tilde{\phi}}} \frac{e^{-ik_2\rho}}{\sqrt{k_2\rho}} \]  

(5 - 27)

\[
g_{12}^J(\rho, \phi) \sim \sqrt{\frac{1}{8\pi}} e^{i\pi/4} \left[ 1 + R_1 \right] e^{ik_1z' \sin \tilde{\phi}} e^{-ik_2y' \sqrt{1 - \sin^2 \tilde{\phi} / \epsilon_{22}}} \frac{e^{-ik_1\rho}}{\sqrt{k_1\rho}} \]  

(5 - 28)

\[
g_{22}^J(\rho, \phi) \sim \sqrt{\frac{1}{8\pi}} e^{i\pi/4} e^{-ik_2z' \sin \tilde{\phi}} \left[ e^{-ik_2y' \cos \tilde{\phi}} + R_2 e^{ik_2y' \cos \tilde{\phi}} \right] \frac{e^{-ik_2\rho}}{\sqrt{k_2\rho}} \]  

(5 - 29)

where we have used the definitions \(z = \rho \sin \tilde{\phi}, y = \rho \cos \tilde{\phi}\) and \(\tilde{\phi} = \tilde{\phi} - \pi\). Thus, the angle \(\tilde{\phi} = 0\) is defined in the same manner as the incident field. The reflection
terms are defined by

$$R_1 = \frac{\cos \phi - \sqrt{\epsilon_{r2} - \sin^2 \phi}}{\cos \phi + \sqrt{\epsilon_{r2} - \sin^2 \phi}}$$

$$R_2 = \frac{\cos \phi - \sqrt{1/\epsilon_{r2} - \sin^2 \phi}}{\cos \phi + \sqrt{1/\epsilon_{r2} - \sin^2 \phi}}$$

(5-30)  

(5-31)

The derivation of the above expressions are given in Appendix C. In performing the steepest descent integral, we ignore the contributions due to the branch cut. If we choose $\rho$ sufficiently large, the branch cut contribution is only significant at the grazing angles which are at $\phi = \pi/2$ or $\phi = -\pi/2$. Therefore, by substituting (5-26) through (5-29) into (5-25), we can obtain an accurate far-field expression for $E_x$ as long as $\phi$ is not too close to grazing.

Method of Moments results for a perfectly conducting circular cylinder were obtained by Butler, Xu, and Glisson [1985] and Xu and Butler [1986,1987] for both the $TM_x$ and $TE_x$ cases. We will compare our results to theirs. In the numerical results which follow, the bymoment solutions are shown by the solid and dashed lines while the method of moment results are denoted by the triangular and asterisk symbols for the perfectly conducting cylinder cases. For the first dielectric cylinder case, the results from the method of moments is given by the solid line, and the bymoment result is given by the dashed line.

We begin by considering a perfectly conducting cylinder with $\alpha = 0.175$. The lower half-space is a pure dielectric with $\epsilon_{r2} = 4$. A plane wave is assumed to be normally incident on the interface ($\phi^i = 0^\circ$). Results were obtained from the method of moments for $h = -0.175$ and $h = 0.175$. Because the cylinder resides on
the interface, the bymoment solution uses \( h = -0.185 \) and \( h = 0.185 \) and assumes that the effects of this perturbation are small. The mesh consists of 80 nodes with a half-bandwidth of 6. The number of basis functions used is \( N = 14 \). For the \( TM_x \) case, a comparison between the two methods in terms of the magnitude and phase of \( j_x^{\text{norm}} \) as a function of \( \phi \) are shown in Figures 5–3 and 5–4. We observe that the agreement is very good for both the magnitude and phase. For the \( TE_x \) case, the magnitude and phase of \( j_x^{\text{norm}} \) are given in Figures 5–5 and 5–6. Again, we see that there is good agreement between the method of moments and bymoment results. In addition, the agreement between the results indicate that the slight shift in the value of \( h \) did not adversely affect the results.

Let us now consider a geometry where we vary the material properties of the lower half-space rather than the cylinder position. For the parameters \( h = 0.425, \quad a = 0.375, \quad \phi^i = 0^\circ \), and \( N = 18 \), we consider cases where \( \epsilon_{r2} = 4 \) and \( \epsilon_{r2} = 16 \). The mesh consists of 80 nodes with a half-bandwidth of 6. In Figures 5–7 and 5–8, we show the magnitude and phase of \( j_x^{\text{norm}} \) for the \( TM_x \) case, and similarly in Figures 5–9 and 5–10, we present the magnitude and phase of \( j_x^{\text{norm}} \) for the \( TE_x \) case. The agreement between the two methods is excellent except for the phase plot of the \( TM_x \) case with \( \epsilon_r = 4 \). In the region near \( \phi = -90^\circ \), there is a discrepancy between the two solutions. Note that in this region, the magnitude of the \( j_x^{\text{norm}} \) is very close to zero. Therefore, we believe the phase error is due to numerical computation errors rather than errors in the method. To show the variation with angle of incidence, solutions are obtained for \( \phi^i = 0^\circ \) and \( \phi^i = 60^\circ \). For these solutions we only have method of moment results for the \( TM_x \) case [Xu and Butler, 1987]. Therefore, let us only consider this polarization. The perfectly conducting circular cylinder under
Figure 5-3 Magnitude of the normalized surface current density on a perfectly conducting circular cylinder for $h = -0.175$ and $h = 0.175$. Parameters are $a = 0.175$, $\phi^i = 0^\circ$, $\varepsilon_r = 4$, $TM_z$ case.
Figure 5-4: Phase of the normalized surface current density on a perfectly conducting circular cylinder for $h = -0.175$ and $h = 0.175$. Parameters are $a = 0.175$, $\phi = 0^\circ$, $\varepsilon_r = 4$, $TM_x$ case.
Figure 5-5  Magnitude of the normalized surface current density on a perfectly conducting circular cylinder for $h = -0.175$ and $h = 0.175$. Parameters are $a = 0.175$, $\phi_i = 0^\circ$, $\epsilon_r = 4$, $TE_z$ case.
Figure 5-6  Phase of the normalized surface current density on a perfectly conducting circular cylinder for $h = -0.175$ and $h = 0.175$. Parameters are $a = 0.175$, $\phi^i = 0^\circ$, $\epsilon_r = 4$, $TE_2$ case.
consideration has a radius given by \( a = 0.125 \). We specify \( \varepsilon_r = 4 \), \( h = -0.065 \), and \( N = 14 \). The mesh contains 60 nodes with a half-bandwidth of 6. The results for the magnitude and phase of \( J_x^n \) are shown in Figures 5-11 and 5-12, respectively.

For the case of the homogeneous dielectric cylinder under \( TM_x \) excitation, we have method of moments results [Butler and Xu, 1989] for \( |E_z^f(\phi)| \) where \( a = 0.175 \), \( h = -0.175 \), \( \varepsilon_r = 4 \), and \( \phi^i = 0^\circ \). The cylinder has a relative dielectric constant given by \( \varepsilon_{rd} = 8 \). Again, because the cylinder is only touching the interface at one point, we solve the problem using the bymoment method with \( h = -0.185 \). A comparison of the two methods is shown in Figure 5-13. The next case involves a larger dielectric cylinder \( (a = 0.6, h = -0.65, \varepsilon_r = 4, \varepsilon_{rd} = 8, \text{ and } \phi^i = 0^\circ) \). In order to obtain accurate results for this size cylinder, 8161 nodes are used to mesh the cylinder. The half-bandwidth is 172, and the number of basis functions is \( N = 24 \).

The field pattern \( |E_z^f(\phi)| \) is shown in Figure 5-14. To demonstrate the capabilities of the bymoment method in handling arbitrary shapes and inhomogeneities, we consider the case of two dielectrics cylinders. Each cylinder has a radius of \( a = 0.3 \), and their centers are separated by a distance of one free space wavelength. For the first case, the cylinders are located in the upper half-space with \( h = 0.35 \), \( \varepsilon_{rd} = 4 \), and \( \phi^i = 0^\circ \). The mesh is the same as that used previously in Figure 3-23. Curves are shown in Figure 5-15 for the field pattern when \( \varepsilon_r = 1 \) and \( \varepsilon_r = 4 \). The free space result \( (\varepsilon_r = 1) \) agrees with the result shown in Figure 3-25 once that data is normalized. It is interesting to note that the forward scatter is focused by the presence of the interface and the backscatter is suppressed. Let us next consider the case where the two cylinders are buried in the lower half-space. We choose \( h = -0.35 \), \( \varepsilon_{rd} = 8 \), and \( \varepsilon_r = 4 \). Two angles of incidence are used
Figure 5-7 Magnitude of the normalized surface current density on a perfectly conducting circular cylinder for \( \epsilon_r = 4 \) and \( \epsilon_r = 16 \). Parameters are \( a = 0.375 \), \( \phi^i = 0^\circ \), \( h = 0.425 \), \( TM_z \) case.
Figure 5-8  Phase of the normalized surface current density on a perfectly conducting circular cylinder for $\epsilon_{r2} = 4$ and $\epsilon_{r2} = 16$. Parameters are $a = 0.375$, $\phi^i = 0^\circ$, $h = 0.425$, $TM_z$ case.
Figure 5-9 Magnitude of the normalized surface current density on a perfectly conducting circular cylinder for $\varepsilon_r = 4$ and $\varepsilon_r = 16$. Parameters are $a = 0.375$, $\phi_i = 0^\circ$, $h = 0.425$, $TE_2$ case.
Figure 5-10  Phase of the normalized surface current density on a perfectly conducting circular cylinder for \( \epsilon_r = 4 \) and \( \epsilon_r = 16 \). Parameters are \( a = 0.375 \), \( \phi^i = 0^\circ \), \( h = 0.425 \), \( TE_2 \) case.
Figure 5-11  Magnitude of the normalized surface current density on a perfectly conducting circular cylinder for $\phi^i = 0^\circ$ and $\phi^i = 60^\circ$. Parameters are $a = 0.125$, $\varepsilon_{r2} = 4$, $h = -0.065$, $TM_z$ case.
Figure 5-12  Phase of the normalized surface current density on a perfectly conducting circular cylinder for $\phi^i = 0^\circ$ and $\phi^i = 60^\circ$. Parameters are $a = 0.125$, $\varepsilon_r = 4$, $h = -0.065$, $TM_z$ case.
(\phi^i = 0^\circ, \phi^i = 45^\circ). Because the relative dielectric constant is greater than in the previous case, a finer mesh must be used. This mesh consists of 3025 elements with a half-bandwidth of 95. The far-field results are shown in Figure 5–16.
Figure 5-13  Plot of $|E_{z}^{ff}|$ for a homogeneous dielectric circular cylinder with $a = 0.175$, $\phi^i = 0^\circ$, $\epsilon_{r2} = 4$, $h = -0.175$, $\epsilon_{rd} = 8$, and $TM_z$ case.
Figure 5-14  Plot of $|E_z'|$ for a homogeneous dielectric circular cylinder with $a = 0.6$, $\phi^i = 0^\circ$, $\varepsilon_r = 4$, $h = -0.65$, $\varepsilon_{rd} = 8$, and $TM_z$ case.
Figure 5-15  Plot of $|E_Z|$ for two homogeneous dielectric circular cylinder with $a = 0.3$, $\phi^i = 0^\circ$, $h = 0.35$, $\epsilon_{rd} = 4$, and $TM_z$ case.
Figure 5-16 Plot of $|E_z^{ff}|$ for two homogeneous dielectric circular cylinder with \( a = 0.3, \varphi^i = 0^\circ, \epsilon_r = 4, h = -0.35, \epsilon_{rd} = 8, \) and $TM_z$ case.
In this chapter the restriction from the previous chapters on the variation of the source in the $z$ direction is removed. Thus, assuming that the cylinder axis is parallel to the $z$ axis of a cartesian coordinate system, the excitation can be either a plane wave of arbitrary polarization incident at an angle other than perpendicular to the $z$ axis or any traveling-wave field with $z$ dependence of the form $\exp(-i\beta z)$, where $\beta$ is a constant. Because of this excitation, the expressions in (2-9) and (2-10) do not hold for material cylinders due to the coupling of the $TM_z$ and $TE_z$ polarizations. To account for the coupling, new expressions are derived using the method of weighted residuals. These expressions directly incorporate the correct boundary conditions at material interfaces and show the coupling between the two polarizations. Next, the by moment formulation is modified to handle the case of oblique incidence. Finally, numerical results are presented from the application of the method to scattering from homogeneous and inhomogeneous material cylinders of various shapes, along with comparisons to results obtained using exact eigenfunction expansion and integral equation methods.

6.1 Finite Element Formulation

Consider a single cylinder in free space with the incident field of the form

$$
\left( \vec{E}^{inc}, \vec{H}^{inc} \right) = \left( \vec{e}^{inc}(x,y), \vec{h}^{inc}(x,y) \right) e^{-i\beta z}
$$

(6 - 1)
where $\beta$ is a constant. The incident field is defined to be the field everywhere in the absence of the cylinder, and its general representation in (6-1) includes various cases, such as a plane wave propagating at an angle $\theta$ with respect to the $z$-axis, in which case $\beta = \omega \sqrt{\mu_0 \varepsilon_0} \cos \theta$, or the field due to an infinitely thin straight wire of infinite length running parallel to the $z$ axis and carrying a current $I(z) = I_0 \exp(-i\beta z)$. The geometry of the problem is shown in Figure 6-1 where $\theta$ is then angle of incidence of the incoming wave. As in the previous cases, the material properties can vary in $x$ and $y$ but not in $z$.

The presence of the cylinder causes a scattered field $(\vec{E}^{sc}, \vec{H}^{sc})$, which has the same space harmonic variation in $z$ as the incident field due to phase matching

$$\left( \vec{E}^{sc}, \vec{H}^{sc} \right) = \left( \vec{e}^{sc}(x,y), \vec{h}^{sc}(x,y) \right) e^{-i\beta z} \quad (6 - 2)$$

The total field is the superposition of the incident and scattered fields

$$\left( \vec{E}, \vec{H} \right) = \left( \vec{e}(x,y), \vec{h}(x,y) \right) e^{-i\beta z} = \left( (\vec{e}^{inc} + \vec{e}^{sc}), (\vec{h}^{inc} + \vec{h}^{sc}) \right) e^{-i\beta z} \quad (6 - 3)$$

It can be shown that, for the assumed $z$ dependence of the fields and for a homogeneous medium, the transverse electric and magnetic fields $\vec{e}_t = \hat{z} e_x + \hat{y} e_y$ and $\vec{h}_t = \hat{z} h_x + \hat{y} h_y$ can be expressed in terms of the transverse derivatives of the axial components $e_x, h_x$ [Wait, 1955]

$$\vec{e}_t = -\frac{1}{k^2 - \beta^2} (i\beta \nabla_t e_x - i\omega \mu \hat{z} \times \nabla_t h_x), \quad (6 - 4)$$

$$\vec{h}_t = -\frac{1}{k^2 - \beta^2} (i\beta \nabla_t h_x + (\sigma + i\omega \varepsilon) \hat{z} \times \nabla_t e_x) \quad (6 - 5)$$
Figure 6-1  Geometry for the oblique incidence case.
where $\nabla_t = \frac{\partial}{\partial x} + i\beta \frac{\partial}{\partial y}$ and the common $z$ dependence $\exp(-i\beta z)$ has been suppressed for simplicity. Of course, the axial components satisfy the following Helmholtz equation

$$\nabla_t^2 \left\{ \begin{array}{c} \epsilon_x \\ h_x \end{array} \right\} + (k^2 - \beta^2) \left\{ \begin{array}{c} \epsilon_x \\ h_x \end{array} \right\} = 0 \quad (6-6)$$

As in the previous chapters, it is assumed that the material properties $\epsilon, \mu, \sigma$ are constant in each element of the finite element grid. Therefore, (6-6) is satisfied within each element of the grid. Finally, from (6-4) and (6-5) it is apparent that the continuity of the tangential components of the transverse electric and magnetic fields at material interfaces couples $e_x$ and $h_x$ for $\beta \neq 0$ as first observed by Wait [1955].

The equation in (2-8) can be considered for a single element and modified to account for the axial variation so that

$$\int \int_{\Omega_e} (v^{-2}i\omega \mu \nabla_t^2 h_x + i\omega \mu h_x) \psi_j \, dS = 0 \quad (6-7)$$

$$\int \int_{\Omega_e} [v^{-2}(\sigma + i\omega e) \nabla_t^2 e_x + (\sigma + i\omega e) e_x] \psi_j \, dS = 0 \quad (6-8)$$

where $v^2 = k_e^2 - \beta^2$ and $\Omega_e$ is the surface in the $x-y$ plane of the element under consideration. Equations (6-7) and (6-8) may be integrated by parts and the above weak form becomes

$$\int \int_{\Omega_e} (v^{-2}i\omega \mu \nabla_t h_x) \cdot \nabla_t \psi_j - i\omega \mu h_x \psi_j \, dS = \oint_{C_e} (v^{-2}i\omega \mu \hat{n} \cdot \nabla_t h_x) \psi_j \, dl, \quad (6-9)$$

$$\int \int_{\Omega_e} [v^{-2}(\sigma + i\omega e) \nabla_t e_x] \cdot \nabla_t \psi_j - (\sigma + i\omega e) e_x \psi_j \, dS = \oint_{C_e} [v^{-2}(\sigma + i\omega e) \hat{n} \cdot \nabla_t e_x] \psi_j \, dl, \quad (6-10)$$
where $C_e$ is the curve enclosing $\Omega_e$ and $\mathbf{\hat{n}}$ is the outward unit normal vector on $C_e$.

It should be noted that the boundary integral terms in these equations involve the normal derivatives of $h_x$ and $e_x$ at the boundary. These terms are used to applying boundary conditions at material interfaces and to introduce the forcing terms at boundaries where $h_x$ and $e_x$ are not specified. In addition, they may be used to compute the normal derivatives of $h_x$ and $e_x$ on boundaries where $h_x$ and $e_x$ are specified.

An alternative form of (6-9) and (6-10) may be derived for both $\mathbf{\hat{n}} \cdot \nabla_{t} h_x$ and $\mathbf{\hat{n}} \cdot \nabla_{t} e_x$ in terms of the tangential derivatives of $e_x$ and $h_x$ along the boundary $C_e$ and the components of $\mathbf{\vec{e}}_t$ and $\mathbf{\vec{h}}_t$ tangent to $C_e$. We use (6-4) and (6-5) in conjunction with a local coordinate system $(\mathbf{\hat{n}}, \mathbf{\hat{\tau}}, \mathbf{\hat{z}})$ along $C_e$ as shown in Figure 6-2, where $\mathbf{\hat{\tau}}$ is the unit vector tangent to $C_e$. Dot multiplying (6-4) and (6-5) by $\mathbf{\hat{\tau}}$, we get

$$v^{-2}i\omega \mu \mathbf{\hat{n}} \cdot \nabla_{t} h_x = \mathbf{\hat{\tau}} \cdot \mathbf{\vec{e}}_t - i\beta v^{-2} \mathbf{\hat{\tau}} \cdot \nabla_{t} e_x$$

(6-11)

$$v^{-2}(\sigma + i\omega \epsilon) \mathbf{\hat{n}} \cdot \nabla_{t} e_x = i\beta v^{-2} \mathbf{\hat{\tau}} \cdot \nabla_{t} h_x - \mathbf{\hat{\tau}} \cdot \mathbf{\vec{h}}_t$$

(6-12)

In view of (6-11) and (6-12), (6-9) and (6-10) take the form

$$\int \int_{\Omega_e} (v^{-2}i\omega \mu \nabla_{t} h_x) \cdot \nabla_{t} \psi_j - i\omega \mu h_x \psi_j \, dS = \oint_{C_e} (\mathbf{\hat{\tau}} \cdot \mathbf{\vec{e}}_t - i\beta v^{-2} \mathbf{\hat{\tau}} \cdot \nabla_{t} e_x) \psi_j \, dl,$$

(6-13)

$$\int \int_{\Omega_e} [v^{-2}(\sigma + i\omega \epsilon) \nabla_{t} e_x] \cdot \nabla_{t} \psi_j - (\sigma + i\omega \epsilon) e_x \psi_j \, dS = \oint_{C_e} (i\beta v^{-2} \mathbf{\hat{\tau}} \cdot \nabla_{t} h_x - \mathbf{\hat{\tau}} \cdot \mathbf{\vec{h}}_t) \psi_j \, dl.$$  

(6-14)

This alternative form shows explicitly the coupling of $e_x$ and $h_x$, and makes possible the direct enforcement of the continuity of the tangential components of the electric and magnetic field at material interfaces.
Figure 6-2  The localized coordinate system for an element.
Unlike the previous formulations, the line integral around each element does not cancel when the global matrix is formed. The terms survive along boundaries where the material properties are discontinuous. As mentioned at the beginning of the section, material properties are assumed constant within each element of the finite element grid; hence, any boundaries where material properties change abruptly coincide with interelement boundaries. Since the finite element interpolation functions \( \psi_j \) used to approximate the \( z \) components of the field are continuous at interelement boundaries, the numerical approximations of \( \varepsilon_x \) and \( h_x \) are continuous by construction. Thus it only remains to enforce the continuity of the other two tangential components \( \hat{r} \cdot \hat{e}_t \) and \( \hat{r} \cdot \hat{n}_t \) at the interelement boundaries. However, in performing the usual element-by-element assembly of (6-13) and (6-14), we see that the line integral contributions for a typical interelement boundary \( AB \) are (Figure 6-3)

\[
\int_{AB} \left[ (\hat{r} \cdot \hat{e}_{t(1)} - \hat{r} \cdot \hat{e}_{t(2)}) - (i\beta v_1^{-2} \hat{r} \cdot \nabla_t \varepsilon^{(1)}_z - i\beta v_2^{-2} \hat{r} \cdot \nabla_t \varepsilon^{(2)}_z) \right] \psi_j \, dl, \quad (6 - 15)
\]

\[
\int_{AB} \left[ (i\beta v_1^{-2} \hat{r} \cdot \nabla_t h^{(1)}_z - i\beta v_2^{-2} \hat{r} \cdot \nabla_t h^{(2)}_z) - (\hat{r} \cdot \hat{n}^{(1)}_t - \hat{r} \cdot \hat{n}^{(2)}_t) \right] \psi_j \, dl \quad (6 - 16)
\]

where the indices 1 and 2 refer to elements (1) and (2), respectively. It is then apparent that the enforcement of the continuity of \( \hat{r} \cdot \hat{e}_t \) and \( \hat{r} \cdot \hat{n}_t \) along the interface \( AB \) nullifies the first term in (6-15) and the second term in (6-16). Furthermore, it suggests the following simplified variational statement

\[
\int \int_{\Omega} (v^{-2} i\omega \mu \nabla_t h_z) \cdot \nabla_t \psi_j - i\omega \mu h_z \psi_j \, dS = \oint_{C_s} \left( -i\beta v^{-2} \frac{\partial \varepsilon_z}{\partial r} \right) \psi_j \, dl, \quad (6 - 17)
\]
\[ \int \int_{\Omega_e} [v^{-2}(\sigma + i\omega) \nabla_t e_x] \cdot \nabla_t \psi_j - (\sigma + i\omega) e_x \psi_j \, dS = \int_{C_e} (i\beta v^{-2} \frac{\partial h_x}{\partial r}) \psi_j \, dl. \]

Note that the coupling terms are proportional to \( \beta \) and thus become zero when the fields are \( z \) invariant (\( \beta = 0 \)). \( e_x \) and \( h_x \) are then decoupled, and (6-17) and (6-18) reduce to the standard \( TE_x \) and \( TM_x \) scattering problems, respectively. This modified variational statement has built-in all the continuity conditions at material interfaces. Thus, it can be used directly for the usual element-by-element assembly of the finite element equations over all the computational domain \( \Omega \) excepting only those elements with boundaries coinciding with the boundaries of the mesh. These boundaries are handled in the next section using the bymoment method.

Because both \( e_x \) and \( h_x \) are coupled, there are two unknowns for each node. Therefore, the resulting finite element matrix is \( 2N_n \times 2N_n \) where \( N_n \) was previously defined to be the number of nodes in the mesh. Also, the half-bandwidth of the matrix is doubled.

### 6.2 Formulation of the Bymoment Method

In the previous chapters, the incident field has always been invariant in \( z \). In what follows, the method is extended to the case of scattering from inhomogeneous cylinders at oblique incidence, where there are two unknown functions, \((e_x(x, y), h_x(x, y))\). These two field components are determined from the numerical solution of two coupled scalar wave equations where the variational statement in (6-17) and (6-18) are utilized. The cross sectional geometry is the same as that in Figure 3-1. The boundary curve \( \partial S \) divides the unbounded region where the solution is sought into an interior region \( \mathcal{S}_c \) containing the inhomogeneous scatterer.
Figure 6-3  The localized coordinate system along the interelement boundary AB.
and an exterior homogeneous region $S_0$ which extends to infinity as in Chapter 3. For the case of oblique incidence both $e_x$ and $h_z$ must be specified on $\partial S$ for the solution to be unique. Because both field components exist, the formulation of the bymoment method is significantly different than for the normal incidence case.

Let $e^*(t), h^*(t)$ be the unknown functions that describe the restrictions of $e_x$ and $h_z$ on $\partial S$, respectively. The variable $t$ denotes the position on $\partial S$ and varies from 0 to $d$, where $d$ is the length of $\partial S$. The functions $e^*(t)$ and $h^*(t)$ are expanded in terms of the set of basis functions $\Psi_n(t)$ as follows:

$$\begin{align*}
\{ e^*(t) \} &= \sum_{n=1}^{N} \left\{ a_n \right\} \psi_n(t) \quad t \in [0, d]. \\
\{ h^*(t) \} &= \sum_{n=1}^{N} \left\{ b_n \right\} \psi_n(t) \quad t \in [0, d].
\end{align*} \tag{6 - 19}$$

where $a_n$ and $b_n$ are unknown coefficients and $N$ is the number of expansion functions required to represent $e^*(t)$ and $h^*(t)$. Next, by specifying each one of the following $2N$ Dirichlet boundary conditions on $\partial S$

$$\begin{align*}
\{ e_x(t) = \psi_n(t), h_z(t) = 0 \}, & \quad n = 1, 2, \ldots, N \quad \tag{6 - 20} \\
\{ e_x(t) = 0, h_z(t) = \psi_n(t) \}, & \quad n = 1, 2, \ldots, N \quad \tag{6 - 21}
\end{align*}$$

we generate a set of $2N$ finite element solutions inside $S_e$. Let $\left( \Lambda_n^{(e)}, \Xi_n^{(e)} \right)$ be the numerical solution for $(e_x, h_z)$ inside $S_e$ corresponding to the $n^{th}$ boundary condition in (6-20), and $\left( \Lambda_m^{(h)}, \Xi_m^{(h)} \right)$ be the numerical solution for $(e_x, h_z)$ corresponding to the $n^{th}$ boundary condition in (6-21). Then, from linearity and the uniqueness theorem, $e_x$ and $h_z$ inside $S_e$ can be written as a superposition of these finite element solutions multiplied by the appropriate coefficients $a_n$ and $b_n$ as follows:

$$e_x(x, y) = \sum_{n=1}^{N} \left[ a_n \Lambda_n^{(e)} + b_n \Lambda_n^{(h)} \right] \quad \tag{6 - 22}$$
From (6-22) and (6-23) it becomes apparent that in order to complete the solution the coefficients $a_n$ and $b_n$ need to be determined. To find these coefficients, we couple the interior solution to the exterior homogeneous region as explained next.

Considering the geometry of Figure 3-1, let $\partial S'$ be a curve that passes through the interior of all the elements on the boundary of the grid. This curve lies entirely in the homogeneous region exterior to the scatterer and is totally enclosed by $\partial S$. Let $S_0'$ be the surface which extends from $\partial S'$ all the way to infinity. In $S_0'$ the scattered field components $e^{sc}_x$, $h^{sc}_z$ satisfy the source-free Helmholtz equation and the Sommerfeld radiation condition at infinity. Let $\Phi_j$, $j = 1, 2, \ldots$, be a set of linearly independent testing functions which are also chosen to satisfy the source-free Helmholtz equation inside $S_0'$ and the radiation condition at infinity. Application of Green's theorem for $e^{sc}_x$ and any of the $\Phi_j$ over $S_0'$ gives

$$
\int_{\partial S'} \left[ \Phi_j \frac{\partial e^{sc}_x}{\partial n'} - e^{sc}_x \frac{\partial \Phi_j}{\partial n'} \right] dl = 0,
$$

(6 - 24)

where $\hat{n}'$ is the outward pointing unit normal vector on $\partial S'$. Expressing the scattered field as the difference of the total and incident fields and using (6-22), we see that (6-24) becomes

$$
\sum_{n=1}^{N} a_n \int_{\partial S'} \left[ \Phi_j \left( \frac{\partial \Delta^{(e)}_n}{\partial n'} - \Delta^{(e)}_n \frac{\partial \Phi_j}{\partial n'} \right) \right] dl + \sum_{n=1}^{N} b_n \int_{\partial S'} \left[ \Phi_j \left( \frac{\partial \Delta^{(h)}_n}{\partial n'} - \Delta^{(h)}_n \frac{\partial \Phi_j}{\partial n'} \right) \right] dl
$$

$$
= \int_{\partial S'} \left[ \Phi_j \frac{\partial e^{inc}}{\partial n'} - e^{inc}_x \frac{\partial \Phi_j}{\partial n'} \right] dl
$$

(6 - 25)
A similar procedure for $h_x^{ce}$ yields
\[
\sum_{n=1}^{N} a_n \int_{\partial S'} \left[ \Phi_j \frac{\partial E_n^{(e)}}{\partial n'} - E_n^{(e)} \frac{\partial \Phi_j}{\partial n'} \right] dl + \sum_{n=1}^{N} b_n \int_{\partial S'} \left[ \Phi_j \frac{\partial \Phi_n^{(h)}}{\partial n'} - E_n^{(h)} \frac{\partial \Phi_j}{\partial n'} \right] dl \\
= \int_{\partial S'} \left[ \Phi_j \frac{\partial h_x^{inc}}{\partial n'} - h_x^{inc} \frac{\partial \Phi_j}{\partial n'} \right] dl \tag{6-26}
\]

Equations (6-25) and (6-26) constitute the testing statements for the calculation of the unknown coefficients $(a_n, b_n)$, $n = 1, 2, \ldots, N$. Indeed, applying (6-25) and (6-26) for $N$ of the independent test functions $\Phi_j$, we can construct a system of $2N$ independent equations to solve for the $2N$ coefficients. The resulting matrix equation is
\[
\begin{bmatrix}
[S^{ce}] & [S^{ch}] \\
[S^{he}] & [S^{hh}]
\end{bmatrix}
\begin{bmatrix}
\{a\} \\
\{b\}
\end{bmatrix}
= \begin{bmatrix}
\{T^{ce}\} \\
\{T^{ch}\}
\end{bmatrix}, \tag{6-27}
\]

where $\{a\}$ is an $N$ dimensional vector containing the coefficients $a_n$, $\{b\}$ is an $N$ dimensional vector containing the coefficients $b_n$, and $[S^{ce}]$, $[S^{ch}]$, $[S^{he}]$, and $[S^{hh}]$ are $N \times N$ matrices with elements
\[
S_j^{ce} = \int_{\partial S'} \left[ \Phi_j \frac{\partial \Phi_n^{(e)}}{\partial n'} - \Phi_n^{(e)} \frac{\partial \Phi_j}{\partial n'} \right] dl, \tag{6-28}
\]
\[
S_j^{ch} = \int_{\partial S'} \left[ \Phi_j \frac{\partial \Phi_n^{(h)}}{\partial n'} - \Phi_n^{(h)} \frac{\partial \Phi_j}{\partial n'} \right] dl, \tag{6-29}
\]
\[
S_j^{he} = \int_{\partial S'} \left[ \Phi_j \frac{\partial E_n^{(e)}}{\partial n'} - E_n^{(e)} \frac{\partial \Phi_j}{\partial n'} \right] dl, \tag{6-30}
\]
\[
S_j^{hh} = \int_{\partial S'} \left[ \Phi_j \frac{\partial E_n^{(h)}}{\partial n'} - E_n^{(h)} \frac{\partial \Phi_j}{\partial n'} \right] dl. \tag{6-31}
\]
Finally, \{T^e \} and \{T^h \} are vectors of dimension \( N \) with elements

\[
T^e_j = \int_{\partial S'} \left[ \Phi_j \frac{\partial e^\text{inc}_z}{\partial n'} - e^\text{inc}_z \frac{\partial \Phi_j}{\partial n'} \right] \, dt, \tag{6-32}
\]

\[
T^h_j = \int_{\partial S'} \left[ \Phi_j \frac{\partial h^\text{inc}_z}{\partial n'} - h^\text{inc}_z \frac{\partial \Phi_j}{\partial n'} \right] \, dt. \tag{6-33}
\]

Since the incident field is specified everywhere, \( e^\text{inc}_z \), \( h^\text{inc}_z \), and their normal derivatives on \( \partial S' \) are known. Also, the numerical values for \( \Lambda_n^{(e)} \), \( \Lambda_n^{(h)} \), \( \Sigma_n^{(e)} \), \( \Sigma_n^{(h)} \), and their normal derivatives are obtained from the finite element solutions of the \( 2N \) interior Dirichlet problems.

6.3 Numerical Results

Numerical results are given for the case of plane wave scattering from material cylinders. The incident plane wave can be either \( TM_z \) or \( TE_z \) and is defined by

\[
\left( \begin{array}{c}
e^i_z \\
h^i_z
\end{array} \right) = \sin \theta \, e^{-ik_0 \sin \theta (x \cos \phi_i + y \sin \phi_i)} \tag{6-34}
\]

where \( k_0 \) is the free space wave number, \( \theta \) is the angle of incidence along the \( z \)-axis, and \( \phi_i \) is the angle of incidence in the \( x \)-\( y \) plane as shown in Figure 3-1. For all the geometries considered, entire-domain sinusoidal functions are used for the basis functions \( \Psi_m \), and the testing functions \( \Phi_j \) are the normalized Green’s function solutions to the scalar wave equation in free space due to a traveling wave line source located at points \( (x_j, y_j) \), \( j = 1, 2, \ldots, M \), in the region interior to \( \partial S' \). They are given by

\[
\Phi_j = H_0^{(2)}(k_0 \sin \theta \sqrt{(x - x_j)^2 + (y - y_j)^2}) \tag{6-35}
\]
where the $z$-dependence is suppressed. For this testing function, (6-32) and (6-33) can be reduced to a much simpler form since

$$\int_{\partial S'} \left[ \Phi_j \frac{\partial e_{z}^{\text{inc}}}{\partial n'} - e_{z}^{\text{inc}} \frac{\partial \Phi_j}{\partial n'} \right]\, dl = -4ie_{z}^{\text{inc}}(x_j, y_j), \quad (6 - 36)$$

$$\int_{\partial S'} \left[ \Phi_j \frac{\partial h_{z}^{\text{inc}}}{\partial n'} - h_{z}^{\text{inc}} \frac{\partial \Phi_j}{\partial n'} \right]\, dl = -4ih_{z}^{\text{inc}}(x_j, y_j). \quad (6 - 37)$$

The first geometry considered is that of a homogeneous dielectric circular cylinder in free space. An eigenfunction solution for a $TM_z$ incident wave is computed using the formulation from Wait [1955] and then compared to the bymoment solution. The cylinder has a radius of $r/\lambda = 0.3$ and a relative permittivity $\varepsilon_r = 4.0$. The incident wave is impinging on the cylinder at an angle of $\theta = 45^\circ$ and $\phi^i = -90^\circ$. The mesh contains 1321 nodes (2642 unknowns) resulting in a stiffness matrix with a half-bandwidth of 94. $N$ is set to 14 and thus the full matrix in (6-27) has dimensions of $28 \times 28$. Results for the magnitude and phase of $e_z$ and $\eta h_z$ along the line labeled $ab$ are given in Figures 6-4 and 6-5 where $\eta$ is the free space wave impedance. The line $ab$ is oriented at a $45^\circ$ angle from the horizontal. Wait's series solution is denoted by SER, and the bymoment solution is denoted by FEM. The agreement between the two methods is excellent.

Next the case of a $2\lambda \times 0.1\lambda$ rectangular cylinder is considered with the longer length in the horizontal direction. Because of the large perimeter length, $N = 28$. The grid (Figure 6-6) contains 394 nodes (788 unknowns), and the half-bandwidth of the finite element matrix is 30. Echo-width results are obtained for
Figure 6-4  Magnitude of the z-component of the fields along the interior line \(ab\) for a homogeneous dielectric circular with \(r = 0.3\lambda, \varepsilon_{rd} = 4,\) and \(\phi^i = -90^\circ.\)
Figure 6-5  Phase of the z-component of the fields along the interior line $ab$ for a homogeneous dielectric circular with $r = 0.3\lambda$, $\epsilon_{rd} = 4$, and $\phi = -90^\circ$. 
this geometry and compared to the method of moments (MOM) results given by Rojas [1988]. The expression for the echo width is

\[
W = \lim_{\rho \to \infty} k_0 \rho \frac{|E_{Z}^{sc}|^2 + |\eta H_{Z}^{sc}|^2}{|E_{Z}^{inc}|^2 + |H_{Z}^{inc}|^2}
\]

(6 – 38)

The direction of propagation of the incident \(TE_z\) plane wave is defined by \(\theta = 45^\circ\) and \(\phi_i = 0^\circ\). The results in Figure 6–7 are for two separate cases. The first is the case where the cylinder is homogeneous with \(\varepsilon_{rd} = 3 - j0.3\) and \(\mu_{rd} = 2 - j0.1\). The corresponding curves are labeled by MOM(H) and FEM(H). The results show good agreement. The second is the case where the cylinder is inhomogeneous. The curve denoted by MOM(I) has material properties of \(\varepsilon_{rd} = 3 - j0.3 + 2\cos(\pi x/2\lambda)\) and \(\mu_{rd} = (2 - j0.1)|x|/2\lambda\) where the origin for \(x\) is taken to be the center of the cylinder. The material properties used for the curve marked FEM(I) are an approximation to that used in MOM(I). Instead of using a continuously varying function for \(\varepsilon_r\) and \(\mu_r\), we divide the cylinder into 5 homogenous regions of equal area with the material properties of each region taken to be the value of the continuously varying case evaluated at the center of each region. The material properties used for the five regions are \(\varepsilon_{rd1} = \varepsilon_{rd5} = 3.62 - j0.3, \varepsilon_{rd2} = \varepsilon_{rd4} = 4.62 - j0.3, \varepsilon_{rd3} = 5 - j0.3, \mu_{rd1} = \mu_{rd5} = 1.8 - j0.04, \mu_{rd2} = \mu_{rd4} = 1.4 - j0.02,\) and \(\mu_{rd3} = 1.0.\) This division was done in order to see whether one can approximate a continuously inhomogeneous cylinder with one that has piecewise homogeneous properties. In comparing the two curves, it can be seen that there is very little difference between the two solutions. The next set of cases (Figure 6–8) uses the same geometry and parameters as those used in Figure 6–7 except that \(\phi_i\) is now set to \(-90^\circ\). For this angle of incidence,
the differences between MOM(I) and FEM(I) are more evident than in the previous case. This is especially true for the forward scattering data.

The final case considered is that of two homogeneous dielectric cylinders separated by a distance of $1\lambda$. The radius of each cylinder is $0.3\lambda$, and the relative permittivity is given by $\varepsilon_{rd} = 4$. In Figure 6-9, echo width results are shown for both the $TM_z$ and $TE_z$ polarized incident plane waves for $\phi^i = 0^\circ$, $\theta = 45^\circ$. The solution is generated using a grid of 2317 nodes (4634 unknowns) resulting in a finite element matrix with a half-bandwidth of 166. $N$ is set to 24.
Figure 6-6  Finite element mesh for the $2\lambda \times 0.1\lambda$ rectangular dielectric cylinder.
Figure 6-7  Echo width of a $2\lambda \times 0.1\lambda$ rectangular cylinder for $TE_2$ polarized incident plane wave with $\theta = 45^\circ$ and $\phi^i = 0^\circ$. 
Figure 6-8 Echo width of a $2\lambda \times 0.1\lambda$ rectangular cylinder for $TE_z$ polarized incident plane wave with $\theta = 45^\circ$ and $\phi^i = -90^\circ$. 
Figure 6-9  Echo width of two homogeneous dielectric circular cylinders of radius \( r = 0.3\lambda \). The parameters are \( \varepsilon_r = 4, \ d = 1.0\lambda, \ \theta = 45^\circ, \) and \( \phi' = 0^\circ \).
CHAPTER 7

NUMERICAL CONSIDERATIONS

Much of the work presented thus far deals with the principles of the bymoment method. There are various difficulties which may be encountered in the actual numerical implementation. Some of them are specific to the bymoment method, and some are common to all finite element methods. In this chapter, several of the major difficulties are discussed.

The first important issue concerns the accuracy of the bymoment method as a function of the truncation distance from the cylinder. Ideally, the method allows the truncation distance to be only one element from the cylinder. In practice, this may not always be true. For smooth cylinders, we can generally place the boundary one element from the cylinder. An example of this is shown in Figure 7-1 where the numerical results for the magnitude of the surface current on a perfectly conducting circular cylinder with one, three, and five layers of elements are compared to the series solution. The cylinder has a radius of $0.3\lambda$, and the excitation is a $TE_z$ polarized plane wave incident at an angle of $\phi^i = -90^\circ$. The element size is chosen so that the thickness of each layer is $0.05\lambda$. The results indicate that only one layer of elements is necessary for this case.

For geometries which have sharp edges, more than one layer of elements are sometimes necessary because of the rapid variations of the fields near the edges or corners. To test this, we consider the geometry of a $0.25\lambda \times 0.1\lambda$ perfectly conducting rectangular cylinder under $TM_x$ plane wave excitation with $\phi^i = 0^\circ$. 
Figure 7-1  Comparison of the magnitude of the surface current on a perfectly conducting circular cylinder for various truncation distances from the cylinder ($r = 0.3\lambda$, $TE_z$ case, $\phi^i = -90^\circ$).
From quasi-static theory [Meixner, 1972], even though $E_z$ is zero at the perfectly conducting edges, its normal derivative has a $\rho^{-1/3}$ singularity there. The mesh that is used to compute the finite element solution is shown in Figure 7–2 for the case where there are three layers of elements. In order to accurately model the singular behavior, non-uniform spacing of the elements is used. Also, the thickness of each layer of elements is small ($\sim 0.005\lambda$) since the surface current is determined from the derivative of $E_z$ and the derivative is computed from a central difference approximation of $E_z$ at the nodes. In Figures 7–3 and 7–4, we present the magnitude and phase of the surface current on the upper surface of the cylinder for one and three layers of elements and compare them to the result obtained by the method of moments [Butler, Xu, and Glisson, 1985]. The surface current is normalized by the incident magnetic field at the upper surface of the cylinder. The results are obtained using 24 sinusoidal basis functions along the boundary. The large number of basis functions is required to accurately represent the rapidly varying fields at the edges. The two figures demonstrate the need for more than one layer of elements in this geometry. It should be mentioned that increasing the number of sinusoidal basis functions beyond 24 did not improve the accuracy of the one layer case.

Since, in theory, one layer should be enough to obtain an accurate solution, the error must be due to the numerical implementation of the method. The most probable source of error is in the evaluation of the integral in (3-14). Because two-point Gauss quadrature is used for the numerical integration along $\partial S'$, it may not be accurate enough near singularities. There are two methods to counter this inaccuracy. The first is to increase the order of the quadrature along the parts of $\partial S'$ which are near the edges. The second is to use a quadrature scheme that
Figure 7-2  Finite element grid for a $0.25\lambda \times 0.1\lambda$ perfectly conducting rectangular cylinder.
Figure 7-3 Magnitude of the normalized surface current on the upper surface of a $0.25\lambda \times 0.1\lambda$ perfectly conducting rectangular cylinder ($TM_2$ case, $\phi = 0^\circ$) for two different truncation distances. Comparison to MOM.
Figure 7-4  Phase of the normalized surface current on the upper surface of a 0.25λ × 0.1λ perfectly conducting rectangular cylinder (TMx case, ϕ = 0°) for two different truncation distances. Comparison to MOM.
accounts for the singularity exactly. Neither of these schemes have been attempted in this dissertation. Although it is usually desirable to grid as closely as possible to the cylinder, it may not always be the best choice for cylinders with sharp edges. As the truncation boundary is moved closer to the edges, the number of basis functions on the boundary must be increased accordingly due to the rapid variation of the fields at the edges. Therefore, one must consider both parameters when attempting to optimize for efficiency. As the number of basis functions on the boundary increases, the number of finite element solution that need to be generated increases. In addition, the computation time required to fill and solve the matrix in (3-13) may grow significantly. Therefore, for most of the geometries with sharp edges, it is not efficient to truncate the grid one element from the cylinder.

As we mentioned in the previous chapters, the placement of the line sources for the testing functions in (3-30) is critical to obtaining an accurate solution. The general rule is to place the line sources in a pattern which is concentric and conforming to the boundary of the mesh. For most geometries, there is a great deal of leeway on this rule, and many different configurations can be used to produce accurate solutions as long as they do not stray too far from this rule. There are some geometries where the restrictions on the placement of the line sources are much tighter. These are geometries where the placement of the line sources results in symmetries in the line sources with respect to the integration path and to the basis function on the boundary. To show what we mean by symmetries, let us consider the simple case of a circular geometry with a circular mesh. The line $\partial S'$ is also assumed to be in a circular pattern. The basis functions $\Psi_n(t)$ are sinusoidal, and we specify the number of these basis functions to be $N = 4$. Because the geometry is circular, we can choose $t = \phi$ so that $t$ goes from 0 to $2\pi$ where $t = 0$ is along the
positive $x$ axis. Now let us place the testing functions in a circular pattern at angles of $\pm 90^\circ$ where $j = 0, \ldots, 3$ as shown in Figure 7–5. We observe from this figure that the first and third row in the Green's theorem matrix in (3-13) are identical as well as the second and fourth row. Therefore, this matrix becomes ill-conditioned. Two of the worst geometries for these symmetries are the circular cylinder and the square cylinder. In order to eliminate this problem, it is important to place the line sources in a non-uniform manner as shown in Figures 3–4 and 3–5.

The mesh in Figure 7–2 for the rectangular cylinder does not have a rectangular shape. The reason for this is that a small but significant error occurs in a localized region around corners in a grid. The source of error seems to involve the placement of the testing functions. It is similar to the problem which occurs in the generalized multipole technique where the sphere of influence can never cover a corner as shown in Figure 7–6. The sphere of influence is the region for which the Hankel function in (3-20) contributes to the integral in (3-12). The sphere (or circle for two-dimensions) is centered at the line source, and its radius is defined to be the shortest distance from the line source to a point on $\partial S'$. For more details on the sphere of influence, the reader is directed to Leuchtmann [1988]. In order to show the errors at the corner, let us grid a $-0.325\lambda \times 0.325\lambda$ two-dimensional region of free space as shown in Figure 7–7 and propagate a $TM_0$ plane wave incident at an angle of $\phi' = -135^\circ$ through it. We then generate a solution using the bymoment method which we call $E_{z}^{\text{num}}$. The percentage error in the magnitude is calculated by

$$\text{error(Mag.)} = 100 \times \frac{|E_{z}^{\text{num}}| - |E_{z}^{\text{ana}}|}{|E_{z}^{\text{ana}}|}$$  \hspace{1cm} (7 - 1)
Figure 7-5  Testing function placement to induce symmetries.
where \( E_{z}^{\text{ana}} \) is the analytical solution. Figures 7-8 and 7-9 are surface plots of the percentage error in the magnitude of \( E_z \) and error in the phase of \( E_z \) in degrees, respectively. Sixteen sinusoidal basis functions are used to compute the solution. The maximum percentage error in the magnitude is only 2.2%. The major error occurs in the phase where there is a much as a 11° phase shift.

Looking at Figure 7-9, we see that most of the phase error is very localized and is concentrated in two of the corner regions. Since the error is localized, we feel that a mesh containing corner elements can be used as long as the solution is not sought in the corner regions. To demonstrate this, we use a rectangular mesh for the cylinder in Figure 7-2. The grid is shown in Figure 7-10 where three layers of elements surround the cylinder. Because of the field singularities near the corners of the perfectly conducting cylinder, we expect this geometry to be very sensitive to any inaccuracies in the corner element of the grid. Therefore, we are interested in determining whether these errors will have any effect on the the surface current which is calculated three elements from the grid corner. In Figures 7-11 and 7-12, we compare the magnitude and phase of the surface current density generated using the rectangular grid to that of the three layer case from Figures 7-3 and 7-4. We see that the difference is minimal.

Another potential source of error occurs at frequencies which coincide with the eigenfrequencies of the cavity formed by the application of a Dirichlet boundary condition on the geometry under consideration. The geometry in this case means both the cylinder and the homogeneous region between the cylinder and the truncation boundary. If \( -\beta^2 \) (\( \beta^2 \) real) is the eigenvalue, then the scalar Helmholtz
Figure 7-6  Diagram showing the effective coverage of the sphere of influence at a corner.
Figure 7-7  Finite element mesh used to test the effect of the corner element on the accuracy of the finite element solution in a free space region.
Figure 7-8  Surface plot to show the distribution of the percentage magnitude error in the finite element solution for the grid in Figure 7-6 (TMz case, $\phi = -135^\circ$).
Figure 7-9  Surface plot to show the distribution of the phase error in the finite element solution for the grid in Figure 7-6 ($TM_x$ case, $\hat{\phi} = -135^\circ$).
Figure 7-10  Finite element grid for the geometry in Figure 7-2 except that corner elements are used in this mesh.
Figure 7-11 Magnitude of the normalized surface current on the upper surface of a $0.25\lambda \times 0.1\lambda$ perfectly conducting rectangular cylinder ($TM_z$ case, $\phi = 0^\circ$) for two different finite element grids.
Figure 7-12 Phase of the normalized surface current on the upper surface of a $0.25\lambda \times 0.1\lambda$ perfectly conducting rectangular cylinder ($TM_x$ case, $\phi_i = 0^\circ$) for two different finite element grids.
equation in (2-1) can be written as

$$(k_c^2 - \beta^2)E_x = 0$$

From (7-2) it is evident that $E_x$ is non-unique when $k_c^2 = \beta^2$. This non-uniqueness results in a singular finite element matrix. In most geometries, the eigenvalue is an irrational number; therefore, $k_c^2$ can be very close to $\beta^2$, but it can never be equal to it. In addition, the numerical wave number $\hat{k}_c$, not $k_c$, is the actual wave number of the finite element solution (as explained later). Since $\hat{k}_c$ is dependent on both the element type and its position in the grid, it adds to the fact that the value of $\beta^2$ is very hard to approach exactly throughout the grid. Thus, when $k_c^2$ is close to $k_c$, the matrix is not singular, but it could be very ill-conditioned. To check how close the frequency of interest must be to an eigenfrequency in order to produce errors in our solution, we consider the simple case of a circular free space region similar to the mesh shown in Figure 3-9. By using the condition number as a gauge and by trial and error, we are able to approach the eigenvalue to within 8 digit accuracy (calculations in single precision). Even for frequencies this close to the eigenvalue of the geometry, there is no noticeable increase in the error of the solution. Although it is theoretically possible to pick up errors for specific geometries at specific frequencies, we were not able to detect anything in our numerical experiments.

Now let us turn our attention to discretization errors in the finite element solution. Note that these errors are not specific to the bymoment method, but rather occur in all finite element methods. The choice of nodal density in a finite element grid is one of the most important factors in determining both the accuracy of finite element solution and computational intensity needed to obtain that
solution. Because the Helmholtz equation in (2-1) is evaluated from finite element basis functions which can only approximate the exact solution, the effective wave number of the numerical solution is different from that of the analytical solution. Let us call this numerical wave number \( \hat{k} \). The dependence of \( \hat{k} \) on grid size, type of element, and type of finite element interpolation function has been analyzed by various groups such as Mullen and Belytschko [1982], Platzman [1981], and Lynch, Paulsen, and Strohbehn [1985]. One of the key findings from these studies is that the ratio \( \hat{k}/k \) is dependent on the electrical size of the elements. They show that as the electrical size of the elements approaches zero, \( \hat{k}/k \) approaches one. In order to show the effects of the numerical wave number on a finite element solution, we consider an \( a \times a \) square region of free space gridded with four different nodal densities (10, 15, 20, and 25 nodes/\( \lambda \)) and for three different wavelengths (\( a = 0.4\lambda \), \( a = 1.0\lambda \), \( a = 4.0\lambda \)). The grid is composed of square elements with sides of length \( h \). The variation over each element is chosen to be bilinear. An example of such a grid is shown in Figure 7-13. We assume that the excitation is a \( TM_x \) plane wave propagating at angles of \( \phi^x = -90^\circ \) and \( \phi^y = -135^\circ \). Since the solution is known on the mesh boundary, Dirichlet boundary conditions are implemented. In Figure 7-14, we show the error in the solutions as a function of wavelength, nodal density, and angle of incidence. The magnitude error is calculated from (7-1) and the phase error is given in degrees. Both the maximum error and the average error over all the nodes are shown. The data presented in this chart indicates some interesting things. The first thing to note is that the error is much smaller for an incident plane wave approaching along the diagonal of the square. Lynch, Paulsen, and Strohbehn [1985] proved this mathematically and showed that this was due to the fact that
the variation along the diagonal is bilinear, whereas the variation along the $x$ and $y$ direction is linear. Because of this, the worst error occurs at $\phi^i = -90^\circ \pm n90^\circ$ and the smallest error occurs at $\phi^i = -135^\circ \pm n90^\circ$ where $n$ is an integer; therefore the error for any other angle should be in between the errors for these two angles.

Another very interesting characteristic of the chart is that the error is dependent on the size of the mesh. As the size of the mesh increases, the nodal density must increase to maintain the same accuracy. In previous literature, there has been some disagreement over the nodal density necessary for an accurate solution. Most felt that the correct number of nodes was somewhere between 10 and 20 nodes per wavelength. The data presented here indicate that for large scatterers this may not be enough. In fact for the case of a $4\lambda \times 4\lambda$ cylinder, we show that even with 25 nodes per wavelength, the error is large when the incident plane wave propagates along either the $x$ or $y$ directions. This discretization error brings up another disadvantage of the finite element solutions in which absorbing boundary conditions are used for the truncation of the grid. Not only must the mesh extend some distance beyond the scatterer, but the nodal density of the entire mesh may have to be significantly increased due to the added mesh region. For large scatterers, this may make the use of the absorbing boundary condition prohibitively expensive in both storage and computational costs. It should be stated here that this study assumes bilinear variation within the elements so that our study applies only to linear and bilinear elements.

Although the chart in 7–13 shows the maximum and average errors, it does not indicate the distribution of the error over the mesh. Therefore, surface plots are
Figure 7-13  Finite element mesh for a free space square region to determine the discretization error.
<table>
<thead>
<tr>
<th>a</th>
<th>Nodes per λ</th>
<th>Incident Angle (°)</th>
<th>Max. Mag. Error (%)</th>
<th>Avg. Mag. Error (%)</th>
<th>Max. Phase Error (°)</th>
<th>Avg. Phase Error (°)</th>
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Figure 7-14  Magnitude and phase data of the discretization error for various geometry sizes, element sizes, and angles of incidence.
Figure 7-15  Surface plot of the distribution of the percentage magnitude error for a $TM_x$ plane wave propagating at an angle of $\phi^i = -90^\circ (a = 1.0\lambda$, 10 nodes/$\lambda$).
Figure 7-16  Surface plot of the distribution of the phase error in degrees for a $TM_z$ plane wave propagating at an angle of $\phi^i = -90^\circ$ ($a = 1.0\lambda$, 10 nodes/\lambda$).
Figure 7-17  Surface plot of the distribution of the percentage magnitude error for a $TM_2$ plane wave propagating at an angle of $\phi = -135^\circ$ ($a = 1.0\lambda$, 10 nodes/$\lambda$).
Figure 7-18  Surface plot of the distribution of the phase error in degrees for a TM$_z$ plane wave propagating at an angle of $\phi^i = -135^\circ$ ($a = 1.0\lambda$, 10 nodes/$\lambda$).
Figure 7-19 Surface plot of the distribution of the percentage magnitude error for a $TM_z$ plane wave propagating at an angle of $\phi^i = -90^\circ$ ($\alpha = 1.0\lambda$, 20 nodes/$\lambda$).
Figure 7-20  Surface plot of the distribution of the phase error in degrees for a $TM_z$ plane wave propagating at an angle of $\psi^i = -90^\circ$ ($a = 1.0\lambda$, 20 nodes/$\lambda$).
Figure 7-21  Surface plot of the distribution of the percentage magnitude error for a $TM_z$ plane wave propagating at an angle of $\phi^i = -135^\circ$ ($a = 1.0\lambda$, 20 nodes/$\lambda$).
Figure 7-22  Surface plot of the distribution of the phase error in degrees for a $TM_z$ plane wave propagating at an angle of $\phi^i = -135^\circ$ ($a = 1.0\lambda$, 20 nodes/\lambda$).
provided for the case where \( a = 1.0\lambda \) and the nodal density is 10 and 20 nodes per wavelength. Both angles of incidence are considered. In Figures 7-15 and 7-16, we plot the magnitude and phase error, respectively, over the square region for the case of 10 nodes per wavelength and \( \hat{\phi}^i = -90^\circ \). The plots for \( \hat{\phi}^i = -135^\circ \) are shown in Figures 7-17 and 7-18. In Figures 7-19 through 7-22, we consider the case where the nodal density is 20 nodes per wavelength. Otherwise, the parameters are the same as those in Figures 7-15 through 7-18.

In observing the surface plots in Figures 7-15 through 7-22, it is evident that the error occurs in a rippled pattern. To discern the cause of this ripple let us formulate the analogous one-dimensional problem which has an analytical solution. Since the Dirichlet boundary condition for our finite element problem simulates a propagating wave, the equivalent analytical problem is the one-dimensional Helmholtz equation,

\[
\left( \frac{d^2}{dx^2} + k^2 \right) E_z = 0 \quad (7-3)
\]

with boundary conditions,

\[
E_z(0) = 1, \quad \left( \frac{d}{dx} + ik \right) E_z(0) = 0 \quad (7-4)
\]

The solution to this differential equation is the analytical solution for the case of a \( TM_z \) polarized plane wave incident at an angle of \( \hat{\phi}^i = -90^\circ \) in free space. The application of the boundary conditions in (7-4) should give a good indication of the source of the ripples in the magnitude error. Because the one-dimensional boundary condition in (7-4) cannot accurately model the two-dimensional Dirichlet boundary condition in the finite element solution, the phase error cannot be recovered from
the analytical one-dimensional solution. Also, this difference between the analytical model and the numerical model does not allow us to predict the size of the error in the finite element solution. The analytical solution for \( E_x \) is

\[
E_x = e^{-i k z} \quad (7-5)
\]

Let us now consider the Helmholtz equation using the numerical wave number,

\[
\left( \frac{d^2}{dx^2} + \hat{k}^2 \right) \hat{E}_x = 0 \quad (7-6)
\]

with the boundary conditions in (7-4). Then \( \hat{E}_x \) is given by

\[
\hat{E}_x = \frac{1}{1-q} e^{-i k z} - \frac{q}{1-q} e^{i k z} \quad (7-7)
\]

where \( q \) is

\[
q = \frac{k - \hat{k}}{k + \hat{k}} \quad (7-8)
\]

Equation (7-7) can be rewritten so that

\[
\hat{E}_x = \cos \hat{k} x - i \frac{k}{\hat{k}} \sin \hat{k} x \quad (7-9)
\]

If we let \( k = \hat{k} + \delta \) and consider only the first order terms involving \( \delta \), then

\[
|\hat{E}_x| \sim \sqrt{1 - 2 \delta \hat{k} \sin^2 \hat{k} x} \quad (7-10)
\]

where \( \delta \) is assumed small. We can approximate (7-10) as follows:

\[
|\hat{E}_x| \approx 1 - \delta \hat{k} \sin^2 \hat{k} x \quad (7-11)
\]
If we compare the ripple to the second term on the right hand side of (7-11), we see that they have the same period, assuming that \( \hat{k} \) is close to \( k \). Thus, our analysis predicts the ripples correctly.

To improve the accuracy of the solution, we can of course increase the nodal density, but this can greatly increase the storage and computation time. Another possibility is to increase the order of the finite element basis functions. In the above examples, the solutions are generated using bilinear basis functions with quadratic behavior along the diagonal and linear behavior along \( x \) and \( y \). For a given nodal density, the bilinear variation greatly improves the accuracy of the solution over that generated from the linear variation. This suggests that the bilinear element is a much better element to use compared to the triangular element, which only has linear variation. Extending this premise, it follows that the solution for a given nodal density should improve by using more complex basis functions. This possibility should be explored in the future. A third idea is not to implement the desired wave number in the finite element solution, but rather to use a wave number that produces a numerical wave number equal to the desired one. In order to do this, we must calculate the numerical wave number that propagates through the grid. For waves propagating along the \( x \) or \( y \) axis, the numerical wave number for a square four node element is given by Lynch, Paulsen, and Stroehn to be

\[
\frac{\hat{k}}{k} = \frac{1}{kh} \cos^{-1} \left[ \frac{1 - (kh)^2/3}{1 + (kh)^2/6} \right]
\]  

(7-12)

where \( h \) is the length of the element sides. To eliminate the errors in Figure 7-14, let \( \hat{k} \) be the desired wave number. Then \( k \) can be determined from (7-12) and substituted for \( k_e \) in (2-17). For the case where the nodal density is 10 nodes per
wavelength, \( \hat{k}/k \) is 0.98425011; therefore, we must choose a value for \( k \) which is 1.016001919 times the desired value. Note that we must maintain as many digits of accuracy as possible in order to minimize the error, which grows as the size of the finite element region grows. Let us now compare the resulting errors to those in Figure 7-14 for the case where \( \hat{\phi} = -90^\circ \) and the nodal density is 10 nodes per wavelength. The maximum and average magnitude error for \( a = 1.0 \lambda \) is 0.56\% and 0.17\%, respectively. The maximum and average phase error is 0.17\° and 0.05\°, respectively. When \( a \) is increased to 4\( \lambda \), the maximum and average errors in both the magnitude and phase are given by 8.7\%, 3.0\%, 2.8\°, and 1.1\°, respectively. Comparing these results to those in Figure 7-14, we see that the error has been greatly reduced.

In most finite element problems, the elements are not simple squares. They are usually quadrilaterals with arbitrary orientation. Therefore, we must determine \( \hat{k} \) for any arbitrary four node quadrilateral due to a plane wave propagating in any arbitrary direction. This is not an easy task; but if it can be done, our numerical experiment implies that the discretization error can be greatly reduced for a finite element problem under plane wave excitation. The procedure is as follows. First, we must calculate the numerical wave number for each element in the mesh. We then determine a value for \( k \) in each element such that \( \hat{k} \) is the desired wave number. Once this is done, \( k \) is used in (2-15) to calculate the finite element solutions while \( \hat{k} \) is used to determine the testing function in (3-20) and the incident field.
CHAPTER 8

CONCLUSIONS AND RECOMMENDATIONS

This treatise concerned the application of the bymoment method to the problem of electromagnetic scattering from geometries which consisted of infinitely long cylinders with arbitrary cross sections and arbitrary but isotropic material properties residing in an unbounded region. The bymoment method is a technique for accurately truncating the finite element grid a small distance away from a scatterer. A surface conforming to the shape of the scatterer was used to divide the computational domain into an unbounded region and an interior region where the scatterer exists. By using either the tangential electric or magnetic field on this surface as the key unknown in the problem, we completely decoupled the numerical solution in the interior from the solution in the exterior. This unknown tangential field was represented as a series expansion in terms of known basis functions with unknown coefficients. The solution in the interior was then expressed in terms of these unknowns coefficients by superposing the finite element solutions generated for each one of the known expansions functions. Finally, a linear system of equations for the unknown coefficients was formed when Green’s theorem is applied to the exterior unbounded region for the scattered field and a set of linearly independent functions which satisfies the Helmholtz equation and the Sommerfeld radiation condition in the exterior homogeneous region.

In the bymoment method, we borrowed ideas from the unimoment method and the field feedback formulation and tried to combine them in an attempt to
increase their numerical efficiency and versatility. Indeed, by using an enclosing surface that conforms to the shape of the scatterer, we bypassed the requirement of the unimoment method for a separable surface and thus were able minimize the volume that needed to be discretized. Furthermore, by choosing the testing functions to be the fields produced by line sources located at different points in the interior region, we introduced a straightforward and computationally simple way of coupling the interior numerical solution to the exterior region.

The first case considered was that of plane wave scattering from a single cylinder in free space where the plane wave was normally incident with respect to the axis of the cylinder. Several numerical examples were presented for $TE_z$ and $TM_z$ scattering by conducting and dielectric cylinders. The excellent agreement of our numerical solutions to those generated by series and integral equation solutions demonstrated the validity of the bymoment method. The next application was to the case where there were a multiple number of cylinders. One of the major advantages of the bymoment method was that it allowed each cylinder to be enclosed by its own individual mesh so that the solution in the region between the cylinders need not be computed. This technique was especially suitable for the case where the cylinders were identical since the finite element solution with respect to a given basis function $\Psi_n(t)$ is identical over each cylinder. In addition, by choosing the set of testing functions properly we demonstrated a decomposition of the matrix formed from Green's theorem into a set of submatrices where each of the submatrices represented a specific interaction between the scatterers. We showed that under certain circumstances some of the submatrices were identical. Numerical results were obtained and compared to series solutions to show the accuracy of the method.
Next, we complicated the exterior region by placing a media interface there. The finite element solutions were generated in the same manner as the free space case; but, in order to satisfy the boundary conditions at the interface, the evaluation of the testing functions have become much more difficult. Instead of evaluating a Hankel function, we must now evaluate a Sommerfeld-type integral. Thus, unlike the free space problem, the majority of the computation time was usually spent in the evaluation of the testing functions rather than the finite element solutions. A comparison was made to method-of-moment solutions to demonstrate the validity of our technique. The final geometry studied in this dissertation was the problem of electromagnetic scattering from a single cylinder in free space where the behavior of the fields along the axis of the cylinder was given by \( \exp(-i \beta z) \) with the \( z \) direction parallel to axis of the cylinder. A major complication in the formulation occurred due to the coupling of the \( TE_z \) and \( TM_z \) polarizations. A new variational expression was derived which incorporates the correct boundary conditions along material interfaces. It resulted in the expected coupling of the \( z \) components of the electric and magnetic fields. Additional modifications to the bymoment formulation were necessary in order to handle this coupling. The modified formulation decoupled \( E_z \) and \( H_z \) along the boundary of the mesh, so that its implementation was both simple and efficient. Numerical results were presented for the special case where the incident field is an obliquely incident plane wave. Again, the results were compared to series and integral equation solutions. The last section in this work discussed some of the relevant numerical issues of the bymoment method. We showed that the discretization error may require the use of a very high nodal density.

The work presented in this dissertation opens up a large number of potential applications of the bymoment method. Some immediate extensions that
we recommend are as follows: 1) Scattering from a cylinder in the presence of a media interface due to an obliquely incident plane wave, 2) Scattering from a cylinder in the presence of layered media, 3) Scattering from a periodic array of cylinders in free space or near a media interface, 4) Scattering from cylinders composed of anisotropic material properties, and 5) Scattering from cylinders due to a dipole source. Up to now all of the geometries considered have been infinitely long cylinders, but the method is not limited to these cases. Another very important two-dimensional geometry is the body of revolution. Although the present formulation requires some modification to handle this geometry, it can certainly be done. This includes incident fields which are obliquely incident on the body of revolution. Of course, ultimately we would like to solve a three-dimensional problem. The bymoment method seems very suitable. One of the major obstacles is finding a good three-dimensional basis function for the solution on the truncation boundary. Once this is done, the rest of the formulation should be fairly straightforward.

There are several other issues that are not directly related to the choice of geometry. One is the problem of a large scatterer. Because of precision problems and numerical dispersion, the finite element solution from a large scatterer may be inaccurate. In this case, iterative techniques such as the diakoptic method [Butler, 1990] should be applied to the finite element method. Another very important problem is the eigenvalue problem. Although the eigenvalue problem has been solved for closed regions with the finite element method, it has yet to be done for open region geometries. The bymoment method seems like a possible candidate to do this although, at this point, the formulation for this type of problem has not been done.
In principle, the applications of the bymoment method are almost limitless. Consider the choice of the exterior region. In one case the choice that we made was free space; in another, it was the two semi-infinite half-spaces. The choice is not limited to only a small number of geometries. Note that the exterior region is coupled into the solution by using a testing function which is really the Green’s function of the exterior region. Therefore, any exterior region can be used as long as an expression for the Green’s function of the exterior region is found. This implies that the bymoment method can be coupled into other methods, such as the method of moments, to take advantage of the strengths of both techniques. In this instance, the method of moments can be used to generate a numerical Green’s function for exterior regions for which the Green’s function has no closed-form expression.

Lastly, we recommend that some experimental work be done. Much of the work here was verified by comparisons to series and integral equation solutions, but the strength of the finite element method is in solving complex geometries. For these cases, the best form of validation is by comparison to experimental results.
APPENDIX A

EVALUATION OF FINITE ELEMENT MATRIX

Because the function $\psi_I$ in (2-20) is dependent upon the shape and location of the elements on which it is defined, the mathematical description for $\psi_I$ is different for each value of $I$. In order to write an efficient computer program, a systematic procedure for evaluating $\psi_I$ in any element must be formulated. The procedure described herein is the standard one used in most finite element programs. The main foundation of the formulation is the master element $\Omega_m$. For a mesh composed of four node quadrilaterals, the master element is a square defined on a $\xi-\eta$ coordinate system (Figure A-1) with corner coordinates of (-1,-1), (1,-1), (1,1), and (-1,1). The local node numbers are written next to the corresponding node.

By defining basis functions on $\Omega_m$, we can obtain the basis functions for any element from a coordinate transformation. We previously defined the function $\psi_I$ to have value one at the $I^{th}$ node and zero at all the other nodes with bilinear variation between the nodes. Let $\hat{\psi_I}$ be $\psi_I$ for the master element. Then

$$\hat{\psi_1}(\xi,\eta) = \frac{1}{4}(1 - \xi)(1 - \eta) \quad (A - 1)$$

$$\hat{\psi_2}(\xi,\eta) = \frac{1}{4}(1 + \xi)(1 - \eta) \quad (A - 2)$$

$$\hat{\psi_3}(\xi,\eta) = \frac{1}{4}(1 + \xi)(1 + \eta) \quad (A - 3)$$

$$\hat{\psi_4}(\xi,\eta) = \frac{1}{4}(1 - \xi)(1 + \eta) \quad (A - 4)$$

Now consider the following expression

$$\int \int_{\Omega_m} \left[ \frac{\partial \psi_I}{\partial x} \frac{\partial \psi_J}{\partial x} + \frac{\partial \psi_I}{\partial y} \frac{\partial \psi_J}{\partial y} - k_c^2 \psi_I \psi_J \right] dx dy \quad (A - 5)$$
Figure A-1  Geometry for the master element.
This is basically equation (2-20) with the $1/\mu_{re}$ term removed. To write (A-5) in terms of the master element, we must establish a relationship between $\psi_I, \partial \psi_I / \partial x, \partial \psi_I / \partial y$ and $\hat{\psi}_I, \partial \hat{\psi}_I / \partial \xi, \partial \hat{\psi}_I / \partial \eta$. A simple mapping between the two coordinate systems is given by

$$
x = \sum_{I=1}^{4} x_I \hat{\psi}_I(\xi, \eta) \tag{A-6}
$$

$$
y = \sum_{I=1}^{4} y_I \hat{\psi}_I(\xi, \eta) \tag{A-7}
$$

where $(x_I, y_I)$ is the coordinate of the $I^{th}$ node in $\Omega_e$. Therefore,

$$
\psi_I(x, y) = \hat{\psi}(\xi(x, y), \eta(x, y)) \tag{A-8}
$$

The relationship between the derivatives is more complicated. Let us first relate $dx, dy$ to $d\xi, d\eta$ as follows:

$$
\begin{bmatrix}
    dx \\
    dy
\end{bmatrix} = 
\begin{bmatrix}
    \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\
    \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta}
\end{bmatrix}
\begin{bmatrix}
    d\xi \\
    d\eta
\end{bmatrix} \tag{A-9}
$$

where the $2 \times 2$ matrix is the Jacobian matrix $[J]$. After further manipulation of (A-9), it can be shown that

$$
\begin{bmatrix}
    d\xi \\
    d\eta
\end{bmatrix} = [J]^{-1}
\begin{bmatrix}
    dx \\
    dy
\end{bmatrix} = \frac{1}{|J|}
\begin{bmatrix}
    \frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi} \\
    -\frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \xi}
\end{bmatrix}
\begin{bmatrix}
    dx \\
    dy
\end{bmatrix} \tag{A-10}
$$

where $|J|$ is the determinant of the Jacobian matrix and is usually just referred to as the Jacobian. Similar to (A-9), we see that

$$
\begin{bmatrix}
    d\xi \\
    d\eta
\end{bmatrix} = 
\begin{bmatrix}
    \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\
    \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y}
\end{bmatrix}
\begin{bmatrix}
    dx \\
    dy
\end{bmatrix} \tag{A-11}
$$
Equating the matrix terms in (A-10) and (A-11) and substituting (A-7) and (A-8) for \( x \) and \( y \), we get

\[
\frac{\partial \xi}{\partial x} = \frac{1}{|J|} \sum_{i=1}^{4} y_i \frac{\partial \psi_i}{\partial \eta}
\]

(A - 12)

\[
\frac{\partial \xi}{\partial y} = -\frac{1}{|J|} \sum_{i=1}^{4} x_i \frac{\partial \psi_i}{\partial \eta}
\]

(A - 13)

\[
\frac{\partial \eta}{\partial x} = -\frac{1}{|J|} \sum_{i=1}^{4} y_i \frac{\partial \psi_i}{\partial \xi}
\]

(A - 14)

\[
\frac{\partial \eta}{\partial y} = \frac{1}{|J|} \sum_{i=1}^{4} x_i \frac{\partial \psi_i}{\partial \xi}
\]

(A - 15)

Equations (A-12) through (A-15) are used to calculate the derivatives of \( \psi_I \) in (A-5) since these derivatives can be written as

\[
\frac{\partial \psi_I}{\partial x} = \frac{\partial \psi_I}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial \psi_I}{\partial \eta} \frac{\partial \eta}{\partial x}
\]

(A - 16)

\[
\frac{\partial \psi_I}{\partial y} = \frac{\partial \psi_I}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial \psi_I}{\partial \eta} \frac{\partial \eta}{\partial y}
\]

(A - 17)

The substitution of (A-12) through (A-15) into (A-16) and (A-17) gives

\[
\frac{\partial \psi_I}{\partial x} = \frac{1}{|J|} \left\{ \frac{\partial \psi_I}{\partial \xi} \sum_{k=1}^{4} y_k \frac{\partial \psi_k}{\partial \eta} - \frac{\partial \psi_I}{\partial \eta} \sum_{k=1}^{4} y_k \frac{\partial \psi_k}{\partial \xi} \right\}
\]

(A - 18)

\[
\frac{\partial \psi_I}{\partial y} = \frac{1}{|J|} \left\{ -\frac{\partial \psi_I}{\partial \xi} \sum_{k=1}^{4} x_k \frac{\partial \psi_k}{\partial \eta} + \frac{\partial \psi_I}{\partial \eta} \sum_{k=1}^{4} x_k \frac{\partial \psi_k}{\partial \xi} \right\}
\]

(A - 19)

where the Jacobian can be written in terms of \( \xi \) and \( \eta \) as follows:

\[
|J| = \left[ \sum_{k=1}^{4} x_k \frac{\partial \psi_k}{\partial \xi} \right] \left[ \sum_{k=1}^{4} y_k \frac{\partial \psi_k}{\partial \eta} \right] - \left[ \sum_{k=1}^{4} x_k \frac{\partial \psi_k}{\partial \eta} \right] \left[ \sum_{k=1}^{4} y_k \frac{\partial \psi_k}{\partial \xi} \right]
\]

(A - 20)
The application of (A-8), (A-18), and (A-19) converts everything in (A-5) to the $\xi-\eta$ coordinate system except the actual variables of integration. However, the change in the variables of integration can be easily accomplished since

$$dx\,dy = |J| d\xi d\eta$$

(A - 21)

where the limits of integration for both $\xi$ and $\eta$ are from -1 to 1. The final expression is then evaluated using Gauss quadrature.
APPENDIX B

GREEN’S FUNCTION FOR THE PLANAR MEDIA INTERFACE

In this appendix the derivation of the Green’s function for a line source in the upper half-space is presented for both the TM and TE polarizations. Once we determine these solutions, the extension to a line source in the lower half-space is trivial. In addition, the numerical evaluation of the resulting infinite integrals in the Green’s functions are described for one of the integrals.

We begin by considering the Green’s function for the TM case. The Green’s function geometry is shown in Figure B-1 where the line source is located at a point \((x', y')\) above the interface. The upper half-space is denoted by Region 1, and the lower half-space is denoted by Region 2. The associated differential equation is

\[- \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_i^2 \right) g_{i1}^T(x, y|x', y') = \delta(x - x')\delta(y - y') \quad (B - 1)\]

where the subscript \(i\) denotes the region that the observation point \((x, y)\) is located and the subscript “1” denotes the region that the line source is located.

Because the interface is parallel to the \(x\)-axis, the Fourier transform pair

\[G_{i1}^T(k_x, y) = \int_{-\infty}^{\infty} g_{i1}^T(x, y)e^{ik_xz} \, dx \quad (B - 2)\]

\[g_{i1}^T(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_{i1}^T(k_x, y)e^{-ik_xz} \, dk_x \quad (B - 3)\]

can be used to solve (B-1). In the transform domain, (B-1) becomes

\[- \left( \frac{\partial^2}{\partial y^2} + k_i^2 - k_x^2 \right) \tilde{G}_{i1}^T(k_x, y) = \delta(y - y') \quad (B - 4)\]
Figure B-1  The Green’s function geometry for the media interface case.
where $\tilde{G}^I_{i1} = G^I_{i1} \exp(-ikzx')$. The boundary conditions on $G^I_{i1}$ are as follows: 1) $G^I_{i1}$ must vanish at $y = \pm\infty$, 2) $G^I_{i1}$ is continuous at $y = y'$, 3) the $y$ derivative $G^I_{i1}$ has a negative unit jump discontinuity at $y = y'$, 4) $G^I_{i1}$ must be continuous at $y = 0$, and 5) $\mu^{-1}dG^I_{i1}/dy$ is continuous at $y = 0$. Note that $\mu$ is the same in both regions so that the $y$ derivative of $G^I_{i1}$ is continuous. Application of the first three boundary conditions yields

$$\tilde{G}^I_{11} = Ae^{-u_1y} + \frac{1}{2u_1}\exp(-u_1(y'-y)) \quad 0 \leq y < y' \quad (B-5)$$

$$= \left(A + \frac{1}{2u_1}\exp(u_1y)\right) \exp(-u_1y) \quad y > y' \quad (B-6)$$

$$\tilde{G}^I_{21} = Be^{u_2y} \quad y < 0 \quad (B-7)$$

where $A$ and $B$ are the undetermined coefficients and

$$u_1 = \sqrt{k^2 - k_1^2} \quad \text{Re}(u_1) \geq 0 \quad (B-8)$$

$$u_2 = \sqrt{k_2^2 - k_1^2} \quad \text{Re}(u_2) \geq 0 \quad (B-9)$$

The last two boundary conditions are then applied at the media interface. Solving in terms of $G^I_{i1}$, we get

$$G^I_{11} = \frac{1}{2u_1} \left[ e^{-u_1|y-y'|} + \frac{u_1-u_2}{u_1+u_2}e^{-u_1(y+y')} \right] e^{ikzx'} \quad (B-10)$$

$$G^I_{21} = \frac{1}{u_1+u_2} e^{-u_1y'+u_2y} e^{ikzx'} \quad (B-11)$$

Finally, an inverse Fourier transform converts (B-10) and (B-11) into our desired solution,

$$g^J_{11} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{2u_1} \left[ e^{-u_1|y-y'|} + \frac{u_1-u_2}{u_1+u_2}e^{-u_1(y+y')} \right] e^{-ikz(x-x')} \, dk_x \quad (B-12)$$
\[ g_{21}^{J} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{u_1 + u_2} e^{-u_1 y' + u_2 y} e^{-ik_z(z - z')} dk_z \] (B-13)

Some additional forms of (B-12) can be obtained since

\[ H_0^{(2)}(k_1 \sqrt{x^2 + y^2}) = \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{1}{u_1} e^{ik_zz} e^{-u_1 |y|} dk_z \] (B-14)

Also (B-12) is an even function of \( k_z \). Two alternative forms are

\[ g_{11}^{J} = \frac{1}{4\pi} H_0^{(2)}(k_1 |r' - r'|) \]

\[ + \frac{1}{\pi} \int_{0}^{\infty} \frac{u_1 - u_2}{2u_1(u_1 + u_2)} e^{-u_1 (y+y')} \cos k_z (x - x') \, dk_z \] (B-15)

\[ = \frac{1}{4\pi} H_0^{(2)}(k_1 |r' - r'|) - \frac{1}{4\pi} H_0^{(2)}(k_1 \sqrt{(x - x')^2 + (y + y')^2}) \]

\[ + \frac{1}{\pi} \int_{0}^{\infty} \frac{1}{u_1 + u_2} e^{-u_1 (y+y')} \cos k_z (x - x') \, dk_z \] (B-16)

where \( |r' - r'| = \sqrt{(x - x')^2 + (y - y')^2} \).

For the \( TE_z \) polarization the formulation is exactly the same as the \( TM_M \) case up to equation (B-9). The difference comes in the fifth boundary condition where we now must satisfy the continuity of \((1/\epsilon^*) d\tilde{G}_1^M/dy\). The variable \( \epsilon^* \) implicitly incorporates the conductivity and is given by \( \epsilon^* = \epsilon - i\sigma/\omega\epsilon_0 \). Therefore,

\[ G_{11}^M = \frac{1}{2u_1} \left[ e^{-u_1 |y-y'|} + \frac{\epsilon_2 u_1 - \epsilon_1 u_2}{\epsilon_2^2 u_1 + \epsilon_1^2 u_2} e^{-u_1 (y+y')} \right] e^{ik_zz'} \] (B-17)

\[ G_{21}^M = \frac{\epsilon_2}{\epsilon_2^2 u_1 + \epsilon_1^2 u_2} e^{-u_1 y' + u_2 y} e^{ik_zz'} \] (B-18)
The final forms are

\[ g_{11}^M = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{2u_1} \left[ e^{-u_1|\nu-\nu'|} + \frac{\epsilon_2^2 u_1 - \epsilon_1^2 u_2}{\epsilon_2^2 u_1 + \epsilon_1^2 u_2} e^{-u_1(y+y')} \right] e^{-ik_z(x-x')} \, dk_z \quad (B-19) \]

\[ g_{21}^M = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\epsilon_2^*}{\epsilon_2^2 u_1 + \epsilon_1^2 u_2} e^{-u_1\nu'+u_2\nu} e^{-ik_z(z-z')} \, dk_z \quad (B-20) \]

Two alternative forms of (B-19) are

\[ g_{11}^M = \frac{1}{4i} H_0^{(3)}(k_1|r-r'|) \]

\[ + \frac{1}{\pi} \int_0^\infty \frac{\epsilon_2^* u_1 - \epsilon_1^2 u_2}{2\epsilon_2^2 u_1(u_1 + u_2)} e^{-u_1(y+y')} \cos k_z(x-x') \, dk_z \quad (B-21) \]

\[ = \frac{1}{4i} H_0^{(3)}(k_1|r-r'|) - \frac{1}{4i} H_0^{(2)}(k_1 \sqrt{(x-x')^2 + (y+y')^2}) \]

\[ + \frac{1}{\pi} \int_0^\infty \frac{\epsilon_2^*}{\epsilon_2^2 u_1 + \epsilon_1^2 u_2} e^{-u_1(y+y')} \cos k_z(x-x') \, dk_z \quad (B-22) \]

To obtain the results for the line source in the lower half-space, we simply switch \((u_1, \epsilon_1^2, k_1)\) and \((u_2, \epsilon_2^2, k_2)\) and replace \((\nu, \nu')\) by \((-\nu, -\nu')\). Then (B-12) \(\rightarrow g_{22}^F\), (B-13) \(\rightarrow g_{12}^F\), (B-19) \(\rightarrow g_{22}^M\), and (B-20) \(\rightarrow g_{21}^M\).

Because of the oscillatory behavior of the integrands in (B-12), (B-13), (B-19), and (B-20), the evaluation of these integrals is numerically intensive. Since the behavior of the integrand is basically the same as that of the Sommerfeld integral, the integrals in the Green’s functions are called Sommerfeld-type integrals. There have been numerous papers describing the efficient evaluation of Sommerfeld integrals. Depending upon the location of the source and observation points, we see that some methods are more effective than others. Since the purpose of this
dissertation is to show the applications of the bymoment method rather than finding the most efficient ways of computing the Sommerfeld-type integrals for a given source and observation points, only one method is chosen for implementation into the computer program without any studies done on its efficiency relative to other methods. The method that we use is the real axis integration technique [Johnson and Dudley, 1983]. We shall use equations (B-15) to demonstrate this technique.

Let us begin by making a change of variable to \( \gamma \) where \( k_2 = k_1 \gamma \) so that the integral in (B-14) can then be written as

\[
\int_0^\infty \frac{\sqrt{\gamma^2 - 1} - \sqrt{\gamma^2 - \epsilon_r}}{2\sqrt{\gamma^2 - 1}(\sqrt{\gamma^2 - 1} + \sqrt{\gamma^2 - \epsilon_r})} e^{-k_1(y+y')\sqrt{\gamma^2-1}} \cos [k_1(x-x')\gamma] \, d\gamma
\]

(B-23)

where \( \epsilon_r = \epsilon_e^2/\epsilon_i^2 \). If we plot the integrand from 0 to 2, we see that the integrand is highly oscillatory around the point \( \gamma = 1 \), especially for large values of \( k_1(y+y') \). Also for larger values of \( \gamma \), the integrand decays slowly for small values of \( k_1(y+y') \) and is highly oscillatory for large values of \( k_1(x-x') \). To deal with all these cases we divide the integral into three sections:

\[
\int_0^\infty F(\gamma) \, d\gamma = I_1 + I_2 + I_3 = \int_0^1 F(\gamma) \, d\gamma + \int_1^2 F(\gamma) \, d\gamma + \int_2^\infty F(\gamma) \, d\gamma \quad (B-24)
\]

where \( F(\gamma) \) is the integrand of (B-23). To reduce the oscillatory behavior around \( \gamma = 1 \) for \( I_1 \), a change a variable is made (\( \gamma = \sin \nu \)). This results in a change in the limits of integration from \([0,1]\) to \([0,\pi/2]\). Also, \( d\gamma = \cos \nu \, d\nu \) so that

\[
I_1 = \int_0^{\pi/2} \frac{i \cos \nu - \sqrt{\sin^2 \nu - \epsilon_r}}{2i(i \cos \nu + \sqrt{\sin^2 \nu - \epsilon_r})} e^{-ik_1(y+y') \cos \nu} \cos [k_1(x-x') \sin \nu] \, d\nu
\]

(B-25)
For $I_2$, we make the change of variable $\gamma = \sec \nu \ (d\gamma = \tan \nu \sec \nu \, d\nu)$. Hence, $I_2$ is

$$I_2 = \int_0^\delta \frac{\tan \nu - \sqrt{\sec^2 \nu - \epsilon_r}}{2(\tan \nu + \sqrt{\sec^2 \nu - \epsilon_r})} \cos \left[k_1(x - x') \sec \nu \right] \sec \nu \, d\nu \quad (B - 26)$$

To evaluate the third integral $I_3$, we divide the integration path from 2 to $\infty$ into a sum of integrals with finite limits as follows:

$$I_3 = \sum_{j=1}^{\infty} \int_{\gamma_j}^{\gamma_{j+1}} \frac{e^{-k_1(y' + y')} \cos \left[k_1(x - x') \gamma \right]}{2\sqrt{\gamma^2 - 1} + \sqrt{\gamma^2 - 1 + \sqrt{\gamma^2 - \epsilon_r}}} \, d\gamma \quad (B - 27)$$

where $\gamma_1 = 2$ and $\gamma_i, i = 2, 3, \ldots$ are the zero crossings of the integrand $F(\gamma)$. The sum is truncated at the index $j = J_{\text{max}}$ when the $J_{\text{max}}$ term of the sum satisfies some convergence requirement which the programmer must specify. To speed up the convergence of this sum, we consider the integrand for large values of $\gamma$:

$$F(\gamma) \sim \tilde{F}(\gamma) = \frac{\epsilon_r - 1}{8\gamma^3} \left[1 + \frac{k_1(y + y')}{2\gamma} \right] e^{-k_1(y' + y') \gamma} \cos \left[k_1(x - x') \gamma \right] \quad (B - 28)$$

It should be noted that the integrand decays asymptotically as $\exp(-k_1(y + y')/\gamma^3)$. Let us call the integral of this asymptotic approximation $I_4$. It can be evaluated in terms of the exponential integral $E_n(x)$ and is given by

$$I_4 = \int_2^\infty \tilde{F}(\gamma) \, d\gamma$$

$$= \frac{\epsilon_r - 1}{32} \text{Re} \left\{ E_0 \left[2k_1(y + y' + i(x - x'))\right] + \frac{k_1(y + y')}{4} E_1 \left[2k_1(y + y' + i(x - x'))\right] \right\} \quad (B - 29)$$
Since $\tilde{F}$ is very close to $F$ for large values of $\gamma$, the difference of the two should be very small. Therefore, the convergence of the sum in (B-27) will be greatly improved with the following modifications,

$$I_3 = \sum_{n=1}^{\infty} \int_{n}^{n+1} \left[ F(\gamma) - \tilde{F}(\gamma) \right] d\gamma + I_4 \quad (B-30)$$

As noted earlier, the asymptotic approximation $\tilde{F}(\gamma)$ converges as $\exp(-k_1(y+y')/\gamma^3$. If the representation in (B-16) had been used instead, the asymptotic decay would only be $\exp(-k_1(y+y')/\gamma$. Thus the representation in (B-15) is a much better one to use for fast convergence of the integral. Unfortunately, this is not true for the $TE_\varepsilon$ case. The greater rate of decay is achieved in (B-28) because the asymptotic approximation for $\sqrt{\gamma^2 - 1 - \sqrt{\gamma^2 - \epsilon_r}}$ is

$$\sqrt{\gamma^2 - 1 - \sqrt{\gamma^2 - \epsilon_r}} \sim \gamma - \frac{1}{2\gamma} - \gamma + \frac{\epsilon_r}{2\gamma}$$

$$= \frac{\epsilon_r - 1}{2\gamma} \quad (B-31)$$

For the $TE_\varepsilon$ polarization, the equivalent term is $\epsilon_r \sqrt{\gamma^2 - 1 - \sqrt{\gamma^2 - \epsilon_r}}$. Its asymptotic approximation is given by

$$\epsilon_r \sqrt{\gamma^2 - 1 - \sqrt{\gamma^2 - \epsilon_r}} \sim \epsilon_r \gamma - \frac{\epsilon_r}{2\gamma} - \gamma + \frac{\epsilon_r}{2\gamma}$$

$$= (\epsilon_r - 1) \gamma \quad (B-32)$$

Therefore, for the $TE_\varepsilon$ polarization the choice between using either (B-21) or (B-22) does not seem to be that important for convergence purposes. The other integrals used in the Green's functions are evaluated in a similar manner.
APPENDIX C

FAR-FIELD EVALUATION

The far-field approximations of the Green's functions given in Chapter 5 are obtained via the method of steepest descent [Felsen and Marcuvitz, 1973]. To demonstrate this method, a step by step derivation of the approximation in (5-25) is presented here. Because the derivation for the other Green's functions are very similar, only the derivation for (5-25) is shown. The best form of $g_{11}^I$ to use is the one in (B-12). With the change of variable $k_z = k_1 \gamma$, (B-12) becomes

$$g_{11}^I = \frac{1}{4\pi i} \int_{-\infty}^{\infty} \frac{1}{1 - \gamma^2} \left[ e^{-ik_1(y-y')\sqrt{1-\gamma^2}} + R(\gamma)e^{-ik_1(y+y')\sqrt{1-\gamma^2}} \right] e^{-ik_1(x-x')\gamma} d\gamma$$

(C - 1)

where

$$R(\gamma) = \frac{\sqrt{1 - \gamma^2} - \sqrt{\varepsilon_r - \gamma^2}}{\sqrt{1 - \gamma^2} + \sqrt{\varepsilon_r - \gamma^2}}$$

(C - 2)

We assume that $y > y'$ since we are considering the approximation for $(x, y)$ large. A further change in the variable of integration ($\gamma = \sin \alpha$) results in the path of integration $C_\alpha$ being that shown in Figure C-1. The equation then becomes

$$g_{11}^I = \frac{1}{4\pi i} \int_{-\pi}^{\pi} \left[ e^{-ik_1(y-y')\cos \alpha} + R(\sin \alpha)e^{-ik_1(y+y')\cos \alpha} \right] e^{-ik_1(x-x')\sin \alpha} d\alpha$$

(C - 3)

Note that the integrand does not have any poles, but it does have branch cuts. The branch cuts are also shown in Figure C-1.
Figure C-1 The complex $\alpha$-plane for evaluating the steepest descent path.
Let us now make a coordinate transformation to the \((\rho, \phi)\) coordinate system. The appropriate relationships between the two coordinate systems are \(x = \rho \sin \phi\) and \(y = \rho \cos \phi\). Substituting this into (C-3), we get

\[
g_{11}^J = \frac{1}{4\pi i} \int_{-\frac{\pi}{2} + i\infty}^{\frac{\pi}{2} + i\infty} \left[ e^{-ik_1 y' \cos \alpha} + R(\sin \alpha) e^{-ik_1 y' \cos \alpha} \right] e^{ik_1 z' \sin \alpha} e^{-ik_1 \rho \cos(\alpha - \phi)} d\alpha
\]  

\[ (C-4) \]

Using a standard trigonometric identity, (C-4) reduces to

\[
g_{11}^J = \frac{1}{4\pi i} \int_{-\frac{\pi}{2} - i\infty}^{\frac{\pi}{2} - i\infty} \left[ e^{-ik_1 y' \cos \alpha} + R(\sin \alpha) e^{-ik_1 y' \cos \alpha} \right] e^{ik_1 z' \sin \alpha} e^{-ik_1 \rho \cos(\alpha - \phi)} d\alpha
\]

\[ (C-5) \]

The contour \(C_\alpha\) can be deformed by setting \(ik_1 + s^2 = ik_1 \cos(\alpha - \phi)\). The resulting contour \(C_s\) has a saddle-point located at \(s = 0\) (\(\alpha = \phi\)) (Figure C-1). The integral in (C-5) can be written in terms of \(s\),

\[
g_{11}^J = \frac{1}{4\pi i} \int_{-\infty}^{\infty} \left[ e^{-ik_1 y' \cos \alpha(s)} + R(\sin \alpha(s)) e^{-ik_1 y' \cos \alpha(s)} \right] e^{ik_1 z' \sin \alpha(s)} e^{-ik_1 \rho \cos(\alpha - \phi)} \sqrt{\frac{2}{k_1}} e^{i\frac{3\pi}{4}} ds
\]

\[ (C-6) \]

where we use the fact that \(d\alpha = \sqrt{2/k} \exp(i3\pi/4) \, ds\). Since the saddle-point is located at \(s = 0\), the integrand except for the \(\exp(-\rho s^2)\) term is evaluated at \(s = 0\). Note that \(\alpha(0) = \phi\). Therefore,

\[
g_{11}^J \sim \frac{1}{4\pi i} \left[ e^{-ik_1 y' \cos \phi} + R(\sin \phi) e^{-ik_1 y' \cos \phi} \right] e^{ik_1 z' \sin \phi} e^{-ik_1 \rho \sqrt{\frac{2}{k_1}} e^{i\frac{3\pi}{4}}} \int_{-\infty}^{\infty} \exp(-\rho s^2) ds
\]

\[ (C-7) \]
The integral in (C-7) has an analytical solution, and its evaluation gives our final answer

\[ g_{11}^{\tau}(\rho, \phi) \sim \sqrt{\frac{1}{8\pi}} e^{i\pi/4} e^{i k_1 x'} \sin \phi \left[ e^{i k_1 y' \cos \phi} + R(\sin \phi) e^{-i k_1 y' \cos \phi} \right] \frac{e^{-i k_1 \rho}}{\sqrt{k_1 \rho}} \quad (C - 8) \]

The above solution assumes that the majority of the contribution to the integral comes from the saddlepoint for \( \rho \) large. This is not true when the path of integration intersects the branch cut. For \( \rho \to \infty \) the branch cut contribution is only important at grazing angles, i.e., \( \phi \) near \( \pm \pi/2 \).
REFERENCES


