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Mechanics of sampling disturbances in clay soils

Wu, Chang-Shiou, Ph.D.
The University of Arizona, 1991
MECHANICS OF SAMPLING DISTURBANCES
IN CLAY SOILS

by

CHANG-SHIOU WU

A Dissertation Submitted to the Faculty of the
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WITH A MAJOR IN CIVIL ENGINEERING
In the Graduate College
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1991
The University of Arizona
Graduate College

As members of the Final Examination Committee, we certify that we have read
the dissertation prepared by Chang-Shiou Wu
entitled Mechanics of Sampling Disturbances in Clay Soils

and recommend that it be accepted as fulfilling the dissertation requirement
for the Degree of Doctor of Philosophy

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SIGNED: Chang-Shou Wu
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ABSTRACT

This research provides an insight into the quality of a soil sample during the penetration of a soil sampler. The investigation of the mechanical disturbances in a clay soil is conducted by using an updated Lagrangian finite element formulation with the 2nd Piola-Kirchhoff stress rate (the Truesdell stress increment) to account for the large deformation behavior near the sampling tube. The penetration of the sampler is simulated by splitting a group of nodes ahead of the penetration route up to a sufficient depth and applying incremental deformation to match the geometric configuration of the sampling tube. Consolidation effect is included to account for the rate of penetration. Thin-layer elements are added into the inside wall of the sampling tube to model the soil-sampler interface. The modified Cam-clay model is used to simulate the behavior of the soils. An experimental study was conducted to study the variations of stresses and pore water pressures at the soil-sampler interface.

The numerical results show that (1) soil samples are subjected to three distinct stages of vertical strain history, compression - extension - recompression. The first stage of compression, in particular, causes irrecoverable changes in the virgin soil properties of the soil; (2) the undrained shear strength of a disturbed sample reconsolidated to the in situ stress condition is larger than the virgin soil for normally consolidated soils but it is smaller for over consolidated soils; (3) the sampling disturbances due to friction at the soil-sampler interface increase as the sampler penetrates the soil. As a result, long samples will be seriously degraded; (4) the increase of the rate of penetration can reduce the degree of disturbances; (5) the piston sampler induces much larger disturbances than the open-drive sampler.

At the soil-sampler interface, the experimental observations and the numerical results indicate that (1) the total radial stresses are subjected to stress reduction followed with stress re-increase; (2) the negative pore water pressures are developed near the tip of the sampler and extend up along the inside wall of the sampler.
during the penetration; (3) the soil-sampler interface friction is approximately a constant for the soft clay investigated.
Chapter 1 Introduction

1.1 General

Before the engineer can design a building foundation safely and economically, he/she must have a reasonable knowledge of the physical and mechanical properties of the underlying soils. This requires either taking samples for laboratory tests or field tests or both. Field tests cannot test the soil in a variety of loading configurations. Laboratory testing is, then, essential to obtain fundamental engineering properties needed for practical design and to further our understanding of soil behavior. However, ever since the emergence of soil mechanics as a science it has been recognized that the mechanical properties of naturally occurring soils may be irreversibly altered by remolding during sampling (Casagrande, 1932). Even the most sophisticated testing and sampling procedures result in some degree of disturbance and therefore, the effects of disturbance on soil properties should be considered integral parts of test data interpretation.

Of all the sampling methods, block sampling appears to disturb soils the least but this method is extremely tedious and is restricted to shallow depths. For this reason, thin-walled sampling is the usual choice for most practical purposes although soils are grossly distorted both during the sampling process itself and the subsequent laboratory sample preparation. There are, in general, two types of samplers that use thin-walled tubes for sampling, namely, open-drive samplers and piston samplers (Hvorslev, 1949). It is believed (Hvorslev, 1942) that the piston sampler can recover less disturbed samples than those by the open-drive sampler.

The standard procedure for sampling consists of excavating a hole to the desired depth, inserting the sampling tube, twisting it to break the sample from the parent soil, sealing, transporting, extruding and handling of the sample before being placed in the testing device. There are many factors which affect the quality of samples during the processes of sampling as mentioned above. These include, but are not limited to, sampler geometry, friction between sample and sample tube, rate of penetration of sampler, sampling procedures, soil type, migration of pore
water, degree of saturation, soil sensitivity, pH, stratification, handling and storage. For some soils, some of these factors may have only small effects but for others the effects may be large enough to bring the sample to a failure state.

Many investigators have attempted to understand the extent and nature of these disturbances induced by the various stages of sampling and testing. The early work of Hvorslev (1949) is the most comprehensive treatment on the subject of sampling disturbance. Research carried out since then have quantified some of the effects of sampling disturbance through laboratory investigation of the load-deformation and strength characteristics of soils. However, the detailed mechanics of sampling disturbance has received very little attention.

1.2 Hypotheses

This research is based on the following hypotheses:

1. That some, if not all, virgin fundamental properties of soil, cannot be recovered from a disturbed sample by existing laboratory methods, for example, reconsolidation back to the in situ stresses.

2. That the disturbances of the sample due to a thin-walled frictionless sampler is within tolerable limits, say, less than 10% of the strength of the sample.

3. That the soil-sampler interface friction increases the degree of sample disturbances.

4. That increasing the rate of sampling penetration can decrease the degree of sample disturbances.

5. That different in situ stresses induces different degree of sample disturbances.
1.3 Objectives

The specific objectives of this research are:

1. to provide an insight into the changes of strains, effective stresses, displacements, and porewater pressures during sampling.
2. to study the effects of (a) frictional forces at the soil-sampler interface, (b) the geometry of the sampler, (c) rate of penetration, (d) in situ stress conditions, and (e) methods of sampling on the quality of soil samples.
3. to propose a rational method to interpret test data.

1.4 Scope of Research

The scope of this research is as follows:

1. Development of an axisymmetric updated Lagrangian Finite Element algorithm incorporating constitutive relationship for soil which includes dilation, strain hardening/softening, pore water diffusion and frictional slip elements.
2. Development of a simulation procedure for sample penetration which can account for the geometry of the sampler.
3. Construction of a laboratory experimental facility to simulate field sampling procedures in a soft clay under controlled conditions and monitor the stresses and the pore water pressure changes at the interface.
4. Interpretation of the test data by means of the finite element algorithm.
Chapter 2 Sampling Disturbances

In this chapter, a discussion of the existing literature on sample quality, experimental evidences and numerical studies pertaining to sampling disturbances is presented.

2.1 Sample Quality

(a) Undisturbed Samples (Ideal Sample) - An undisturbed sample (also called ideal sample) means the sample remains intact prior to laboratory tests, that is, all the physical and chemical properties of the sample remain unchanged from the in situ condition. An undisturbed sample is practically unattainable. An undisturbed soil element subjected to in situ vertical effective stresses $v_0$ and horizontal effective stresses $\sigma_{h0} = K_0 v_0$ is shown in Fig. 2-1 where $K_0$ is the coefficient of earth pressure at rest. The undisturbed sample at depth $Z$ (Fig. 2-1) is usually consolidated one dimensionally following path OA in Fig. 2-2 where the axes $p$ and $q$ are the mean effective stresses and the deviatoric stresses respectively.

The mean effective stress is defined as

$$p = \frac{1}{3} \delta_{ij} \sigma_{ij} = \frac{1}{3} \sigma_{kk} \quad (2.1)$$

where $\delta_{ij}$ is Kronecker delta, $\sigma_{ij}$ is the effective stress tensor and the deviatoric stress $q$ is

$$q = \left( \frac{3}{2} s_{ij} s_{ij} \right)^{\frac{1}{2}} \quad (2.2)$$

where $s_{ij}$ is defined as

$$s_{ij} = \sigma_{ij} - \delta_{ij} p \quad (2.3)$$

The in situ mean effective stress and deviatoric stress at point $A$ are:

$$p_0 = \frac{(1 + 2K_0)}{3} \sigma_{v0} \quad (2.4)$$

and

$$q_0 = (1 - K_0) \sigma_{v0} \quad (2.5)$$
Fig. 2-1 Undisturbed Stress state
Fig. 2-2 Hypothetical Stress Path During Tube Sampling
(b) **Perfect Sample** - A perfect sample (point P, Fig. 2-2), defined by Ladd and Lambe (1963), denotes a sample which was subjected to a release of the in situ shear stresses from $q = q_0$ to $q = 0$ during the process of sampling. No other sources of disturbance are involved. A perfect sample then denotes the best quality sample we can practically recover from the field. Release of in situ shear stresses in this case means the stress state of the perfect sample becomes isotropic. A perfect sample would follow the stress path $AP$ (Fig. 2-2) if the soil element is unloaded directly from the in situ stress $A(p_0, q_0)$ to an isotropic stress state $P(p_{ps}, 0)$, where $p_{ps}$ denotes the effective mean stress at point P. The total stresses at this stage is zero and negative pore water pressure is developed. The effective stresses at this stage are

$$\sigma_v = \sigma_h = -u_p$$

where $-u_p$ is the negative pore water pressure expressed by Skempton and Sowa (1963) as

$$u_p = -\frac{1 + 2K_0}{3} + (A_u - \frac{1}{3})(1 - K_0)\sigma_{v0}$$

where $A_u$ is the pore pressure coefficient during undrained unloading. For elastic axisymmetric unloading, $A_u = \frac{1}{3}$. Normally $A_u$ varies between -0.1 and 0.3.

(c) **Disturbed Sample (Actual Sample)** - To recover a sample at depth $Z$, a bore hole is drilled in its close proximity (Fig. 2-3). An element of soil near to the top of the sample will then become unloaded (Fig. 2-3) The path $AB$ (Fig. 2-2) depicts the stresses which are likely to occur for this element during the excavation. The path $AP$ may not coincide with the path $AB$ since the stresses may be different.

The effect of this stress release is not only to bring that element of soil to an overconsolidated state but also to change the stress state from "compression" to "extension". Curve $BC$ depicts the possible stress path during tube sampling. Curve $CD$ denotes the effects of the extrusion from the sampling tube. Line $DE$ denotes cavitation and water redistribution and $EF$ represents the disturbances caused by trimming and mounting in test machine. Line $FG$ represents the application of
Fig. 2-3 Stress State After Excavation
cell pressure for the triaxial undrained-unconsolidated test. The point $G$ with an effective mean stress $p_r$ (called residual stress) is the stress state of an actual sample at the start of regular laboratory shear test. The residual stress is equal to residual pore water pressure $u_r$, which can be measured prior to the shear test. The ratio $\frac{p_r}{p_p}$ was proposed by Ladd and Lambe (1963) as a measure of sample disturbance.

### 2.2 Experimental Evidence

The literature review of experimental evidence is most conveniently subdivided into studies relating to disturbances which occur (i) due to stress relief, (ii) during sampling, and (iii) after sampling.

#### 2.2.1 Stress Relief

The effects of stress release (either the path A-P or A-B-C-D-E-F-G shown by the broken line in Fig. 2-2) on the undrained shear strength and Young's modulus of soils have received a good deal of attention (Alonso et. al., 1981; Atkinson and Kubba, 1981; Davis and Poulos, 1966; Hanzawa, 1977; Kimura et.al., 1982; Kirkpatrick and Khan, 1984; Okumura, 1971; Seed et. at., 1964; Skempton and Sowa, 1963). Skempton and Sowa (1963), for example, consolidated Weald clay from a slurry state under $K_0$ conditions in a conventional triaxial chamber. To simulate the undisturbed state of the sample (termed an ideal sample by Skempton and Sowa (1963)) one specimen was tested under undrained conditions immediately after consolidation. Stress relief was simulated by reducing the vertical stress of another specimen under undrained conditions until the vertical stress became equal to the cell pressure (lateral stress). This second sample was termed a perfect sample. The difference between the undrained strengths of these two samples at failure was less than 2 %. Ladd and Lambe (1963) conducted experiments similar to those of Skempton and Sowa (1963) on Kawasaki clay and Boston blue clay but also studied tube samples. Their results showed that the undrained shear strengths...
at failure of undisturbed samples were about 12 percent higher than perfect samples. But, the disturbance during tube sampling of normally consolidated clays can decrease the mean effective stress of laboratory specimens by 80 percent compared to perfect sampling, that is, $\frac{p_r}{p_{ps}} = 0.2$. The resultant low values of $p_r$ would cause the undrained strength, $c_u$, from undrained-unconsolidated triaxial tests of the disturbed sample (at point G in Fig. 2-2) to be too low compared with undisturbed sample.

Skempton and Sowa (1963) concluded that the undrained strength of samples of saturated clay will be closely equal to the strength in the ground, provided the strains during sampling are not sufficiently large to cause significant microstructural changes and the water content remains unchanged. Thus, laboratory samples can provide realistic values of $c_u$ if the water content does not change during sampling. This is perhaps the reason for the success of the $\phi = 0$ (undrained) analysis in "end of construction" problem. Ladd and Bailey (1964), however, disagreed with Skempton and Sowa's (1963) opinion as stated above, claiming that the success of the $\phi = 0$ analyses is due to compensating errors. Ladd and Bailey (1964) argued that the in situ loading condition involves a significant rotation of principal stresses such that $c_u$ in the ground may be appreciably lower than perfect sample. Ladd and Lambe (1963) also found that disturbed tube samples yield strengths appreciably lower than the strength in compression of perfect sample (due to the low value of $p_r$, stated above). These effects, compensating each other, lead to the success of the $\phi = 0$ analysis.

Ladd and Foott (1974) applied the concept that the ratio $\frac{\sigma_u}{\sigma_{vo}}$ (undrained stress ratio) is a function of over consolidation ratio (OCR), and proposed a method called stress history and normalized soil engineering properties (SHANSEP) to obtain in situ $c_u$. The procedure consists of consolidating the sample to a stress state larger than in situ (usually 2-4 times the in situ stress) to overcome sample disturbance effects, evaluating the stress history by a series of undrained tests with different value of OCR to determine the relationship of $\frac{\sigma_u}{\sigma_{vo}}$ and OCR and then use this relationship to determine the in situ undrained strength.
Noorany and Seed (1965), Seed et. al. (1964), Okamura (1971), Atkinson and Kuba (1981) also carried similar tests but using different soils than those reported by Skempton and Sowa (1963). These investigators reported that the differences between the undrained shear strengths of ideal samples and perfect samples range from 0 to 6%. But, Kirkpatrick and Rennie (1975), Alonso et al. (1981), Hanzawa (1977) and Atkinson and Kubba (1981) showed that the undrained elastic moduli of perfect samples could be as much as 50% lower than ideal samples.

Kirkpatrick and Khan (1984) prepared laboratory samples from a slurry and consolidated these samples to serve as undisturbed in situ samples. The disturbed samples were simulated by releasing the total stresses to zero and then reconsolidating them back to the in situ stress condition. Kirkpatrick and Khan (1984) found that reconsolidating the sample to the in situ anisotropic stresses and then carrying out the undrained test gave good simulation of strength and stress-strain behavior. They claimed that the method of SHANSEP (Ladd and Foott, 1974), that is, reconsolidation to a stress state larger than the in situ, is not necessary.

2.2.2 Disturbances during sampling

Soil remolding, migration of pore water, sampler geometry, friction between soil and sample tube, soil type, rate of penetration of sampler and the method of advancing the sampler have all been cited as pertinent factors in the mechanics of sampling disturbance.

Hvorslev (1949) stated that the principal causes of sampling disturbance during sampling are:

1. displacement of the soil by the sampler,
2. inside friction between the sample and the sampler,
3. pressure on top of the sample.

The effects of all of these can be reduced but not eliminated by a suitable choice of equipment, material and sampling procedures. For example, a thin-walled tube will certainly reduce soil displacement while a stationary piston sampler will reduce the pressure on the top of the sample.
Lang (1971) found that in stiff clays the average wall friction was 64% of the undrained shear strength of the soil. He also reported that the ratio of frictional forces on the outside to those inside an open-drive thin-walled tube sampler varied between thirty to fifty percent.

Radiography has been used to study qualitatively the distortions introduced by wall friction (Hvorslev, 1949; Arman and McManis, 1976, 1977). The radiographs showed that the soil below the sampler tends to compact before entering the sampler and large convex distortions of the sample occur. The distortions are most severe in the soil adjacent to the inside wall. Arman and McManis (1976) used photoelastic methods to study the stress patterns during tubing using a gelatin-water mixture as the medium. These tests reproduced the convex distortions of the sample observed in radiographic studies (Hvorslev, 1949; Arman and McManis 1977). The inside wall friction was generally observed to increase with depth of penetration of the sampler and the length of the sample. Wall friction may become so large as to prevent soil from entering the sampler. Friction developed on the outside of samplers may also contribute to sample disturbance particularly if the soil clings to this wall as the sampler is pushed downwards. Inside friction apart from causing sample disturbance imposed a definite limit on the length of "undisturbed" soil which can be obtained in a single sampling operation.

Schjetne (1971) inserted a small peizometer connected to a vibrating - wire pore water pressure transducer into the piston of a Norwegian Geotechnical Institute fixed piston sampler. Two tests - one on a plastic clay, the other on a quick clay - revealed that the excess pore water pressures developed by pushing the sampler into the clay were between 150 and 200 percent of the initial pore water pressure. When the sampler was removed, negative pore water pressures were developed which were about 20% of the initial effective overburden pressure. After a few hours, the negative pore water pressures dropped to zero. From these results, it is apparent that the center of the sample swelled and that the pore water migrated from the distorted outer zone to the relatively undisturbed central zone. The drawback of this test is that the insertion of the peizometer itself had caused disturbance to the sample. Shackel (1971) using a nuclear method found that most
of the changes in water content and density occurred when the sampler was pushed into the soil.

Hvorslev (1949) recognized that the geometry of the sampler is an important factor in soil sampling. During sampling, soil must be displaced to make way for the wall of the sampler which results in gross deformation of the sample and concomitant stress changes. Displacement of a large amount of soil may cause soil distortions below the sampler even though the entrance of excess soil can be prevented by using a stationary piston. Such distortions are not visible to the naked eye or appear as visible planes of failure in the sample. Perhaps surprisingly, no detailed study of an experimental or theoretical nature has been published on the effect of wall thickness (or wall thickness to diameter ratio) on the quality of tube samples.

Lang (1967) cut two specimens from a sample tube - one from the top, the other from the bottom - and found that the bottom specimen was stiffer, (its secant modulus was higher) and had a higher peak strength than the top specimen. He concluded that these differences could not be attributable to the variation of depth or water content, but arose as a consequence of sample disturbance.

Eden (1971) compared the undrained shear strengths and preconsolidation pressures of tube samples recovered by four different types of fixed piston samplers from a highly sensitive Canadian clay (Leda Clay) with those from block samples. He found that the undrained shear strengths of tube samples were about one half those of block samples. His investigations supported LaRochelle and LeFebvre (1971) findings on this clay that small lateral strains induced by the sampler result in extensive disturbances.

Adachi et. al. (1981) found that the strengths obtained by percussion boring with the open-drive sampler are almost one-half of those obtained by rotary borings with the fixed piston, thin-walled sampler. Thus, the open-drive and percussion boring seems to cause more disturbances than fixed piston, thin-walled sampler.
2.2.3 Disturbance after sampling

Disturbances after sampling relate to sealing, transportation, storage and extrusion. Coating of samples with hot paraffin wax is a universally accepted practice to preserve the in situ moisture content. This, however, results in increased moisture content in the outer region of the sample due to sweating of the paraffin wax (Arman and McManis, 1976). Campanella and Mitchell (1968) found that the rate of reduction of mean effective stress (assumed to be equal to the negative pore water pressure) due to temperature changes was $1.9kPa/°C$ for illite clay. Kimura and Saitoh (1982) reported losses of mean effective stresses of $1.6kPa/°C$ and $2.2kPa/°C$ for two types of Kawasaki clay preserved by a hot coating of paraffin wax.

It is believed (Bjerrum, 1973; Casagrande, 1932; Kimball, 1936; Rowe, 1971; Schjette, 1971) that in insensitive to moderate sensitive clays, the migration of pore water during storage is responsible for the observed reduction of the undrained shear strength whereas for very sensitive clays, in addition to migration of pore water, changes in chemistry such as leaching of salts (Bjerrum, 1954; Lambe, 1961; Rosenquist, 1953; Skempton and Northey, 1952) significantly affect the behavior of these sensitive soils. Bjerrum (1954) found that after three days storage the reduction in undrained shear strength for a quick clay was about 15%. Sandegren (1961) reported reductions of 8-20% for storage times between 2 to 4 weeks. Skempton and Henkel (1957) did not notice any strength reduction with time for London clay. And only after 10 days storage did Arman and McManis (1976) notice any significant reduction in undrained shear strength for a Mississippi River alluvial soil. Generally, clays with low plasticity indices are highly susceptible to substantial decreases in shear strength following migration of pore water. Marsland (1975) detected a 50% reduction of the modulus of elasticity for cohesive soils stored for a few weeks. Kirkpatrick and Khan (1984) reported that the negative pore water pressure ($u_r$) developed due to stress relief and the value of Young’s modulus drop progressively with sample age. The value of $u_r$ may drop as low as about 15% of estimated initial value. The processes of losing negative pore water pressure...
are not fully understood but they appear to be related to cavitation and diffusion effects.

Sone et al. (1971) investigated the force required to extrude a clayey silt and found that the stress induced by this force may be as much as thirteen times the unconfined compressive strength of the soil. Arman and McManis (1976) reported similar but somewhat lower values. The maximum strain measured during extrusion was approximately equal to the failure strain in unconfined compression (Arman and McManis, 1976). The largest reduction in unconfined compressive strength was found (as expected) to occur at the end of the samples subjected to the extruding force.

2.2.4 Summary of experimental findings on sampling disturbances

Clearly, the process of sampling results in significant irreversible changes in soil properties. Of all the various ways in which a soil sample is disturbed before testing, two links in the chain of disturbance - tubing and extrusion - appear to be the most significant. Indeed, Hvorslev (1949) stated that "the inside wall friction is the most important single source of disturbance of soil during sampling operation". Nevertheless, tubing and extrusion have received relatively scant attention compared with investigations on the changes in soil strength due to stress release; despite the fact that losses in undrained strength due to stress release amount to only about 10% or less. Part of the reason for this disparity of effort seems to be the difficulties of monitoring and controlling the requisite tests.

2.3 Theoretical and Numerical Studies

2.3.1 Literature Review

Alonso et. al. (1981) described a numerical analysis of sampling disturbance by means of an axisymmetric finite element algorithm. They adapted Zienkiewicz's (1977) 'flow approach' formerly used to solve a variety of extrusion, rolling and
other forming processes. In Alonso's work the soil was modelled very simply as a Drucker-Prager solid; that is a uni-phase material. Sample tube advancement was simulated by displacing three surface nodes progressively downwards into the soil which was modelled by means of fifty nine-noded elements (Fig. 2-4). The simulation of tube advancement (which, realistically must incorporate slip at the soil interface) and the discretization employed are too crude to permit quantitative interpretation of the results of this study. Although Alonso et al. (1981) suggest that the contours of mean pressure derived from their analysis may be interpreted as contours of equal excess pore water pressure; this contention is difficult to justify in real soils subjected to gross distortions. Moreover, the dissipation of pore water pressure during sampling and during the delay before laboratory testing is carried out cannot of course be predicted by this algorithm which does not differentiate between the solid and fluid phases. It should be noted, however, that the authors of this study view their algorithm as a first step to a more rigorous analysis of the problem. But, no further work to my knowledge has been reported by them in the literature.

Karim (1984) developed a finite element formulation to try to investigate the response of soils during pile and sample tube penetration. He used a finite element formulation based on the Lagrangian stress rate (1st Piola-Kirchhoff stress rate). Since the Lagrangian stress tensor is not symmetric (Fung, 1965), Karim chose a non-symmetric deformation increment gradient as the conjugate strain rate. There were drawbacks in Karim's simulation. Firstly, the geometry of sampler was not simulated properly. Secondly, he modeled the interface friction by imposing constant frictional forces equivalent to a fraction of the soil's undrained shear strength on nodes adjacent to the inside wall of the sampler. However, experimental evidences (Hvorslev, 1949; Arman and McManis, 1977) show that the inside wall friction increase with depth of penetration. Thus, constant value of friction is not a good simulation. Thirdly, only five steps of penetration with different sampler tip angle was used. This is not enough to simulate the continuous changes of stresses and strains during sampling. A typical result from Karim's analysis is shown in Fig. 2-5. This result was obtained from a sample tube of wall thickness 0.15 cm,
Fig. 2-4 Numerical Simulation of Tube Sampling (Alonso, et. al, 1981)
Friction = 0.42C_e
Large Def. Analysis
Thickness of Tube = 1.5 mm
Radius = 3.75 mm
Area ratio = 8.2 %

Fig. 2-5 Growth of Failure Zones at Various Stages of Penetration

(after Karim, 1984)
diameter 3.75 cm and a frictional force equivalent to 0.42 \( c_u \) ( \( c_u \) is the undrained shear strength). Failure was shown to originate from the adjacent soil mass inside and outside of the sampler. The width of the outside failure zone remained approximately constant, but the inside failure zone extended rapidly towards the center and below the cutting edge as penetration continued to the final penetration depth of 6.25 cm. It appears from Fig. 2-5 that if penetration were to continue beyond 6.25 cm, the whole sample entering the tube would have already failed. This overestimation of failure zone is probably due to the absence of interface slip element which will tend to prevent too much friction to be transferred to the soil when slip occurs at the soil-sampler interface.

Baligh (1985) presented a method called the 'strain path method' to analyze pile, cone and sampling tube penetration. Chin (1986), Baligh et. al. (1987) used the strain path method (SPM) to analyse the penetration of a simple sampler with a rounded cutting edge (Fig. 2-6 (a)). The semi-analytical solution is obtained by the superposition of a single ring source and a uniform vertical flow. The ring source is used to model radial movement and the uniform vertical flow (involving no distortions or strains to the single ring source) is used to model vertical movement. They found that the soil distortions are visible in the vicinity of the sampler wall and causes significant nonuniformities. Thus, the outer half of tube sample should not be used as representative specimen. However, soil distortions occurring near the sample centerline were negligible.

Although the strain path method is potentially a good method to gain an insight into problems involving penetration, there are three important deficiencies: (i) the use of a rounded sampler tip (Fig. 2-6 (a)) will cause expansions in both inner and outward directions. This would cause overestimated distortions inside the sampler. Rochelle et. al. (1987) also pointed out that such type of sampler is impractical. It is usual to use a sharp wedge shaped cutting edge (Fig. 2-6 (b)) which forces the soil in the vicinity of the sampler walls outwards only; (ii) the assumption of uniform vertical velocity field is approximate and applies only in the region where the streamlines are nearly straight. For the soil near the tip of the sampling tube the streamlines are significantly curved.
Fig. 2-6 The Geometries of the Sampling Tube

(a) Tip of the S-sampler (Baligh et. al., 1987)  
(b) Tip of a practical sampling tube
However, the problem of the strain field around the tip of the sampling tube remains unsolved; (iii) the sampler was assumed to be frictionless. This assumption may oversimplify the problem since Hvorslev (1949) claimed that the inside wall friction is the most important single source of disturbance of soil during sampling operation.

2.3.2 Summary of numerical/analytical studies

It appears from the existing state of knowledge that a realistic analytical solution to the sample tube penetration problem seems intractable due to complexity of the problems involved. Such problems include frictional forces along the wall of sampling tube and surrounding soil, unloading of in situ stresses, the difficulty in simulating the cutting edge, large deformation behavior near the sampling tube during sampling penetration, the non-linear properties of the soil, and the changing frictional force at the soil-sampler wall interface. However, an incremental finite element formulation which is suitable for large strain, large rotation and interface slip between the soil and the sampler may provide a good insight into the soil disturbances caused by sample tube penetration.
Chapter 3  Finite Deformation Analyses

It is expected that large geometric changes will occur near the wall of the sampling tube during sampling penetration. Thus, the development of an incremental finite element formulation which is suitable for finite strain analysis is necessary. The algorithm which is developed here is based on an updated Lagrangian finite element formulation. The objective of this chapter is to provide the necessary mathematical background of finite deformation for the development of an updated Lagrangian formulation. The in-depth discussion of the theory of finite deformation is presented in many excellent texts, for example, Fung (1965), Malvern (1969), Bathe (1982), Washizu (1982), Truesdell and Toupin (1965). Only the basic formulation and the field equations needed are given.

3.1 Kinematics

Consider a body point $P_0$ at time step $t_0$ in a body with volume $V^0$ and surface $A^0$ which was moved to $P_n$ at time step $t_n$ in the Cartesian coordinate system shown in Fig. 3-1. We assume that the body point arrived at $P_n$ through a sequential movement from a time step $t_0$ to a time step $t_n$. The associated volume and surface area of the body at time step $t_n$ are $V^n$ and $A^n$ respectively. The position of the body point at time step $t_0$ is defined by coordinates $^0x_1, ^0x_2, ^0x_3$. Similarly, the position of the body point at time step $t_n$ is defined by coordinates $x_1, x_2, x_3$. The body point which was at $P_n$ has moved to $P_{n+1}$ with new coordinates $X_1, X_2, X_3$ at a new time step $t_{n+1}$. Our objective is to determine the unknown static and kinematic variables when the body point is moved from $P_n$ to $P_{n+1}$ through a time interval $t_{n+1} - t_n$. The superscripts and subscripts 0, $n$ and $n + 1$ denote the body configurations corresponding to time step $t_0$, $t_n$ and $t_{n+1}$ respectively. We shall call the configuration at time step $t_0$ initial configuration, at time step $t_n$, current configuration, and at time step $t_{n+1}$, deformed configuration.
Fig 3-1 Motion of Body in Cartesian Coordinate
Quasi static deformation is assumed. Thus, at each instant of time, equilibrium is satisfied for the stresses at a particular configuration.

Assuming the point transformation from $P_0$ to $P_n$ is one-to-one and that the change of body configurations is continuous, the transformation which maps the position of $P_0$ at time step $t_0$ to the new position of $P_n$ at time step $t_n$ can be expressed as:

$$x_i(0x_i, t_n) = 0x_i + \Xi(0x_i, t_n)$$  \hspace{1cm} (3.1)

where $0x_i$ is the position of the body point corresponding to time step $t_0$, $x_i$ is the position of the body point associated with time step $t_n$ and $\Xi$ is the displacement of the body point from time step $t_0$ to time step $t_n$. In a similar way, the transformation which maps the position of $P_0$ at time step $t_0$ to the new position of $P_{n+1}$ at time step $t_{n+1}$ can be expressed as:

$$X_i(0x_i, t_{n+1}) = 0x_i + \Xi(0x_i, t_{n+1})$$  \hspace{1cm} (3.2)

In equations (3.1) and (3.2), the positions of the particle are referred to the initial configuration. The independent variable is $0x_i$. This is called Lagrangian description or material description (Malvern, 1969). The coordinates $0x_1$, $0x_2$ and $0x_3$ are called Lagrangian coordinates.

Similarly, we define the transformation which maps the position of $P_n$ to the position of $P_{n+1}$ as

$$X_i(x_i, t_{n+1}) = x_i + \Xi(x_i, t_{n+1})$$  \hspace{1cm} (3.3)

where $\Xi$ is the displacement of the body point from time step $t_n$ to time step $t_{n+1}$. In equation (3.3), the position of the particle is referred to the current configuration. The independent variable is $x_i$. This is called relative description (Malvern, 1969). The relative description applies to all the time domain. This means that time $t_{n+1}$ may be larger or smaller than time $t_n$ and the time interval $t_{n+1} - t_n$ could be very large. In the finite strain approach adopted here, the body
configuration is updated after each loading increment to serve as a new referential configuration for the calculation of the next increment. The referential coordinates (Lagrangian coordinates) are updated each time within a small time increment. We shall call this approach, a relative description with small positive time interval, the updated Lagrangian description. The coordinates $x_1, x_2$ and $x_3$ are called updated Lagrangian coordinates and $X_1, X_2$ and $X_3$ are called Eulerian coordinates.

In the updated Lagrangian description, we have

$$du_i = u_i^{n+1} - u_i^n = \int_{t_n}^{t_{n+1}} \dot{u}_i \, dt$$

(3.4)

where the superimposed dot (·) denotes the material derivative defined as (Fung 1965):

$$\dot{G} = (\frac{\partial G}{\partial t})_{x=\text{constant}} = (\frac{\partial G}{\partial t})_{X=\text{constant}} + v_i \frac{\partial G}{\partial X_i}$$

(3.5)

where $G$ is any function attributable to the moving particle and $v_i(X_i, t)$ is the velocity vector. If the time interval were small, $du_i$ becomes the incremental displacement from $t_n$ to $t_{n+1}$. Denoting the time interval $\tau (= t_{n+1} - t_n)$, we have

$$X_i(x_i, \tau) = x_i + du_i(x_i, \tau)$$

(3.6)

The above equation is used in the updated Lagrangian formulation. The time $t_{n+1}$ in equation (3.3) is replaced with $\tau$ in equation (3.6). This implies the configuration at time step $t_n$ is set to be the new reference configuration. The solution of $X_i$ depends on time increment $\tau$ and the configuration at time step $t_n$. The initial configuration has no effects on the new motion from $t_n$ to $t_{n+1}$.

According to equation (3.1), the differential $dx_i$ can be expressed as

$$dx_i = \frac{\partial x_i}{\partial x_j} d^0 x_j \bigg|_0 = F_{ij} d^0 x_j = (\delta_{ij} + \frac{\partial^0 u_i}{\partial x_j}) d^0 x_j$$

(3.7)

where

$$\delta_{ij} = \frac{\partial x_i}{\partial x_j}$$

(3.8)
is the deformation gradient with respect to initial configuration from time step $t_0$ to time step $t_n$ (Lagrangian description) and $\delta_{ij}$ is Kronecker delta. According to equation (3.2), the differential $dX_i$ can be expressed as:

$$dX_i = \frac{\partial X_i}{\partial x_j} dx_j = n^{n+1} F_{ij} dx_j = (\delta_{ij} + \frac{\partial u_i}{\partial x_j}) dx_j$$

(3.9)

where

$$n^{n+1} F_{ij} = \frac{\partial X_i}{\partial x_j}$$

(3.10)

is the deformation gradient with respect to initial configuration from time step $t_0$ to time step $t_{n+1}$ (Lagrangian description). Similarly, according to equation (3.6), we define

$$dX_i = \frac{\partial X_i}{\partial x_j} dx_j = n^n F_{ij} dx_j = (\delta_{ij} + \frac{\partial u_i}{\partial x_j}) dx_j$$

(3.11)

where

$$n^n F_{ij} = \frac{\partial X_i}{\partial x_j}$$

(3.12)

is the deformation gradient from $t_n$ to $t_{n+1}$ in updated Lagrangian description.

The Green-Lagrange strain tensor $n^n E_{ij}$ at time step $t_n$ is defined as:

$$n^n E_{ij} = \frac{1}{2} \frac{\partial x_k}{\partial x_i} \frac{\partial x_k}{\partial x_j} - \delta_{ij} = \frac{1}{2} \frac{\partial^n u_i}{\partial x_i} + \frac{\partial^n u_j}{\partial x_j} + \frac{\partial^n u_k}{\partial x_i} \frac{\partial^n u_k}{\partial x_j}$$

(3.13)

The Green-Lagrange strain tensor $n^{n+1} E_{ij}$ at time step $t_{n+1}$ is defined as

$$n^{n+1} E_{ij} = \frac{1}{2} \frac{\partial x_k}{\partial x_i} \frac{\partial x_k}{\partial x_j} - \delta_{ij} = \frac{1}{2} \frac{\partial^{n+1} u_i}{\partial x_i} + \frac{\partial^{n+1} u_j}{\partial x_j} + \frac{\partial^{n+1} u_k}{\partial x_i} \frac{\partial^{n+1} u_k}{\partial x_j}$$

(3.14)

Similarly, the incremental Green-Lagrange strain tensor $dE_{ij}$ is defined as

$$dE_{ij} = \frac{1}{2} \frac{\partial x_k}{\partial x_i} \frac{\partial x_k}{\partial x_j} - \delta_{ij} = \frac{1}{2} \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j}$$

(3.15)

The difference between $n^{n+1} E_{ij}$ and $n^n E_{ij}$ is

$$\Delta E_{ij} = \frac{1}{2} \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial^n u_k}{\partial x_i} \frac{\partial^n u_k}{\partial x_j} + \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j}$$

(3.16)
From equations (3.15) and (3.16), it can be seen that the strains $\Delta E_{ij}$ and $dE_{ij}$ refer to different configuration; $\Delta E_{ij}$ is referred to the initial configuration whereas $dE_{ij}$ is referred to the current configuration. A formulation based on $\Delta E_{ij}$ and $\varepsilon E_{ij}$ together with the 2nd Piola-Kirchhoff stress tensor (explained in the next section), referred to state $t_0$, is called the Total Lagrangian formulation. A formulation based on $dE_{ij}$ together with the 2nd Piola-Kirchhoff stress tensor referred to state $t_n$, is called the Updated Lagrangian formulation (Bathe 1982). Both formulations are applicable to finite deformation.

The Green-Lagrange strain tensor $dE_{ij}$ can be expressed in a more compact form as

$$dE_{ij} = \frac{1}{2}(du_{i,j} + du_{j,i} + du_{k,i}du_{k,j})$$  \hspace{1cm} (3.17)$$

where

$$du_{i,j} = \frac{\partial du_i}{\partial x_j}$$ \hspace{1cm} (3.18)$$

The comma (,) denotes partial differentiation. Equation (3.17) can be decomposed into two parts,

$$dE_{ij} = d\varepsilon_{ij} + d\eta_{ij}$$ \hspace{1cm} (3.19)$$

where

$$d\varepsilon_{ij} = \frac{1}{2}(du_{i,j} + du_{j,i})$$ \hspace{1cm} (3.20)$$

$$d\eta_{ij} = \frac{1}{2}du_{k,i}du_{k,j}$$ \hspace{1cm} (3.21)$$

and $d\varepsilon_{ij}$ is called the incremental Cauchy strain tensor while $d\eta_{ij}$ is called the incremental geometric non-linear strain tensor.

For small strain and rotation the incremental Green-Lagrangian strain $dE_{ij}$ is reduced to the incremental Cauchy strain tensor $d\varepsilon_{ij}$ since the derivatives of the displacement $du_{i,j}$ are small compared to unity. When $du_{i,j}$ is larger than unity, the incremental geometric non-linear strain tensor $d\eta_{ij}$ becomes larger than the incremental Cauchy strain tensor $d\varepsilon_{ij}$. This delineates the major difference.
between small deformation analysis and finite deformation analysis. If we define the incremental rotation tensor $d\omega_{ij}$ as

$$d\omega_{ij} = \frac{1}{2} \left( \frac{\partial du_j}{\partial x_i} - \frac{\partial du_i}{\partial x_j} \right)$$  \hspace{1cm} (3.22)

the incremental Green-Lagrangian strain $dE_{ij}$ can be expressed as (Fung, 1965):

$$dE_{ij} = de_{ij} + \frac{1}{2} (de_{ki} - d\omega_{ki})(de_{kj} - d\omega_{kj})$$  \hspace{1cm} (3.23)

The above equation shows that the incremental Green-Lagrangian strain involves both linear and the rotation effects. The Green-Lagrangian strain ($\varepsilon_{ij}$) has similar effects. Thus, both the total Lagrangian formulations and the updated Lagrangian formulations are applicable to solve problems of large displacement and finite rotation. However, most constitutive laws are based on small strain theory. Thus, these constitutive laws cannot be applied if the material were subjected to large strains and rotations. The Green-Lagrangian strain tensor $\varepsilon_{ij}$ in the total Lagrangian formulation is a finite value provided the displacement is large and $\Delta E_{ij}$ is difficult to be decomposed into a form similar to equation (3.23), that is, it is difficult to separate linear and rotation effects explicitly. Thus, it is difficult to develop an appropriate finite strain constitutive law to apply with it. So, the total Lagrangian formulation is most effectively employed only for large displacement, large rotation but small strain analysis (Bathe, 1982). In the updated Lagrangian formulation, a solution can be found in a quasi small strain way by means of small time steps and updating of the coordinates. The advantages of this formulation are (i) the non-linear strain is taken into account and (ii) most of the model based on the assumption of small strain can be applied without difficulty. In this research, the updated Lagrangian formulation was chosen because large strain is expected near the cutting edge of the sampling tube. So, only the incremental Green-Lagrangian strain tensor is used in the present analysis. From now on, all the formulations are in updated Lagrangian description, that is, all the stresses and strains are referred to time step $t_n$. 
The expressions above are valid for rectangular Cartesian coordinates only. By defining the Green deformation tensor $n+1C_{ij}$ as:

$$n+1C_{ij} = n+1F_{ki} n+1F_{kj} = \frac{\partial X_k}{\partial x_i} \frac{\partial X_k}{\partial x_j}$$  \hspace{1cm} (3.24)

the incremental Green-Lagrangian strain tensor can be expressed, according to equations (3.15) and (3.24), as:

$$dE_{ij} = \frac{1}{2}(n+1C_{ij} - \delta_{ij})$$  \hspace{1cm} (3.25)

The above equation is valid for any coordinate system. Generally, in cylindrical coordinate system, the location of a particle after deformation is expressed as $X_i = X_i(R, \Theta, Z, t_{n+1})$ and the location of the particle before deformation is expressed as $x_i = x_i(r, \theta, z, t)$, where $\Theta$ and $\theta$ are position angles on the plane $Z = 0$ expressed with radius which has no dimension unit. Since the components $\Theta$ and $\theta$ have different dimension units compared with $R, r, Z$ and $z$, we have to use physical components in the transformation. The corresponding physical terms related to $dX_i$ and $dx_i$ are shown below:

<table>
<thead>
<tr>
<th>Cartesian</th>
<th>Cylindrical</th>
<th>Cartesian</th>
<th>Cylindrical</th>
</tr>
</thead>
<tbody>
<tr>
<td>$dX_1$</td>
<td>$dR$</td>
<td>$dx_1$</td>
<td>$dr$</td>
</tr>
<tr>
<td>$dX_2$</td>
<td>$dz$</td>
<td>$dx_2$</td>
<td>$dz$</td>
</tr>
<tr>
<td>$dX_3$</td>
<td>$Rd\Theta$</td>
<td>$dx_3$</td>
<td>$rd\theta$</td>
</tr>
</tbody>
</table>

Thus, the deformation gradient $n+1F_{ij}$ becomes:

$$[n+1F_{ij}] = \begin{bmatrix}
\frac{\partial R}{\partial r} & \frac{\partial R}{\partial z} & \frac{\partial R}{\partial (r \Theta)} \\
\frac{\partial Z}{\partial r} & \frac{\partial Z}{\partial z} & \frac{\partial Z}{\partial (r \Theta)} \\
\frac{\partial (R \Theta)}{\partial r} & \frac{\partial (R \Theta)}{\partial z} & \frac{\partial (R \Theta)}{\partial (r \Theta)}
\end{bmatrix}$$  \hspace{1cm} (3.26)

The bracket $[\ ]$ denotes matrix. For axial symmetric deformation $\Theta$ equals $\theta$. The deformation gradient is reduced to

$$[n+1F_{ij}] = \begin{bmatrix}
\frac{\partial R}{\partial r} & \frac{\partial R}{\partial z} & 0 \\
\frac{\partial Z}{\partial r} & \frac{\partial Z}{\partial z} & 0 \\
0 & 0 & \frac{R}{r}
\end{bmatrix}$$  \hspace{1cm} (3.27)
Substituting equation (3.27) into equations (3.24) and (3.25), we obtain the explicit form of the incremental Green-Lagrangian strain in axisymmetric coordinates as

\[
\begin{bmatrix}
\frac{dE_{11}}{dE} & \frac{dE_{12}}{dE} & 0 \\
\frac{dE_{21}}{dE} & \frac{dE_{22}}{dE} & 0 \\
0 & 0 & \frac{dE_{33}}{dE}
\end{bmatrix}
\]  \hspace{1cm} (3.28)

where

\[
\frac{dE_{11}}{dE} = du_{1,1} + \frac{1}{2}((du_{1,1})^2 + (du_{2,1})^2) \hspace{1cm} (3.29)
\]

\[
\frac{dE_{22}}{dE} = du_{2,2} + \frac{1}{2}((du_{1,2})^2 + (du_{2,2})^2) \hspace{1cm} (3.30)
\]

\[
\frac{dE_{33}}{dE} = \frac{du_1}{x_1} + \frac{1}{2} \left( \frac{du_1}{x_1} \right)^2 \hspace{1cm} (3.31)
\]

\[
\frac{dE_{12}}{dE} = \frac{dE_{21}}{dE} = \frac{1}{2}(du_{1,2} + du_{2,1}) + \frac{1}{2}(du_{1,1} du_{1,2} + du_{2,1} du_{2,2}) \hspace{1cm} (3.32)
\]

and subscripts 1 and 2 denote radius (r) and vertical (z) direction respectively.

### 3.2 Definitions of Stresses

Our objective is to determine the stress state on the deformed configuration (time step \( t_{n+1} \)). However, the configuration at time step \( t_{n+1} \) is unknown at time step \( t_n \). Thus, it is necessary to have a stress measure which refers to the configuration at \( t_n \).

We define, firstly, the Eulerian stress tensor \( n^{+1}\sigma_{ij} \). If \( dTF^{n+1}_i \) denotes the traction force tensor acting on a surface \( dA^{n+1} \) of a small element at time step \( t_{n+1} \), then we can define the **Eulerian stress tensor** \( n^{+1}\sigma_{ij} \) as (Fung 1965):

\[
dTF^{n+1}_i = n^{+1}\sigma_{ij} n^{+1}_j dA^{n+1} = n^{+1}T_i dA^{n+1} \hspace{1cm} (3.33)
\]

where \( n^{+1}_j \) is a unit outer normal vector at time \( t_{n+1} \), \( n^{+1}\sigma_{ij} \) is the Eulerian stress tensor with respect to time step \( t_{n+1} \) and \( n^{+1}T_i \) is traction stress normal to the element surface at time \( t_{n+1} \). The equation (3.33) is called Cauchy formula. In small strain approach, the configuration is not updated and the Eulerian stress is calculated based on the initial configuration. The Eulerian stress (Cauchy stress),
calculated based on the initial configuration, is sometimes called the engineering stress.

The equation of equilibrium at time step $t_{n+1}$ is:

$$\frac{\partial n+1\sigma_{ij}}{\partial X_j} + n+1BF_i = 0$$  \hspace{1cm} (3.34)

where $n+1BF_i$ is body force per unit volume referred to deformed configuration.

There are two stress transformation rules to relate stresses at time $t_{n+1}$ to time $t_n$. These are the 1st Piola-Kirchhoff stresses and the 2nd Piola-Kirchhoff stresses (Malvern, 1969). These stresses are also called Lagrangian stress and Kirchhoff stress respectively (Fung, 1965). Both of them are stresses at state $t_{n+1}$ but referred to the current configuration. The 1st Piola-Kirchhoff stress is not symmetric while the 2nd Piola-Kirchhoff stress is symmetric (Fung, 1965). It would be inconvenient in the constitutive law to relate non-symmetric stresses with symmetric strains. Thus, the 2nd Piola-Kirchhoff stress is adopted in the present analysis.

If $dT^{F_i^n}$ denotes the corresponding traction force assigned to the corresponding current configuration at time step $t_n$, and if the traction forces $dT^{F_i^n}$ and $dT^{F_i^{n+1}}$ are related by the same rule as the transformation,

$$dx_i = \frac{\partial x_i}{\partial X_r}dX_r$$  \hspace{1cm} (3.35)

then we can define the 2nd Piola-Kirchhoff stress tensor $n+1S$ according to the relation

$$dT_i^n = \frac{\partial x_i}{\partial X_r}dT_r^{n+1} = n+1S_{ij} n^n_j dA^n$$  \hspace{1cm} (3.36)

where $n^n_j$ is the unit outer normal vector at time step $t_n$. Thus, the 2nd Piola-Kirchhoff stress tensor $n+1S_{ij}$ and the Eulerian stress tensor $n+1\sigma_{rs}$ can be related by the equation (Fung, 1965)

$$n+1S_{ij} = \frac{\rho^n}{\rho^{n+1}} \frac{\partial x_i}{\partial X_r} \frac{\partial x_j}{\partial X_s} n+1\sigma_{rs}$$  \hspace{1cm} (3.37)
where \( \frac{\rho^n}{\rho^{n+1}} \) is the ratio of mass densities at time \( t_n \) and time \( t_{n+1} \). Alternatively, we may write equation (3.37) as

\[
\n_{n+1}^{n+1} \sigma_{ij} = \frac{\rho^{n+1}}{\rho^n} \frac{\partial X_i}{\partial x_r} \frac{\partial X_j}{\partial x_s} \n_{r}^{n+1} \n_{s}^{n+1} S_{rs}
\]

(3.38)

It should be noted that the 2nd Piola-Kirchhoff stress tensor defined here is in the updated Lagrangian description. The left superscript \( n+1 \) denotes the stress state at time step \( t_{n+1} \). The left subscript \( n \) denotes the referenced configuration at time step \( t_n \). We call \( n_{n}^{n+1} S_{ij} \) as updated Kirchhoff stress tensor (Washizu, 1982).

The conservation of mass requires \( \rho^n dV^n = \rho^{n+1} dV^{n+1} \). This implies that:

\[
\frac{\rho^n}{\rho^{n+1}} = \frac{dV^{n+1}}{dV^n} = \text{det} \left| \frac{\partial X_i}{\partial x_j} \right| = J
\]

(3.39)

where \( J \) is called Jacobian determinant. The volumetric strain (\( \Delta \)) is defined as

\[
\Delta = \frac{dV^{n+1} - dV^n}{dV^n} = J - 1
\]

(3.40)

3.3 Constitutive Relationships

After defining the stresses and the strains with respect to the current configuration, the next step is to establish the constitutive relationships. There are three basic considerations needed to formulate the constitutive equations in finite strain analysis. Firstly, we need to consider the state of strains. Most of the available constitutive laws are based on the assumption of small strain in which the Euclidean stresses are used to formulate the constitutive equations. Secondly, soils are one type of materials which exhibit non-linear stress-strain behavior. Most of the constitutive laws which have been developed for soils are based on elastic-plastic behavior and stress history dependency. Solutions to soil deformation problems then require that the loading be applied in very small increments. This requirement suggests that an incremental approach should be adopted. Thirdly, the constitutive relationships must satisfy the principle of material frame-indifference or the
principle of material objectivity (Malvern, 1969). This requires that the constitutive equations be invariant under changes of frame of reference. This requirement is automatically satisfied in a small deformation analysis since the rotation effect is small. However, the rotation effect cannot be eliminated when the strains are finite.

In what follows, I shall develop a constitutive equation based on the modifications of the constitutive laws proposed by Hibbitt et. al. (1970), Osias and Swedlow (1974), Tesng and Lee (1985) and Washizu (1982). The derivations, in an incremental form, is given below.

It is postulated here that the soil behavior can be described by a constitutive law in an incremental form defined by

\[ d\sigma^*_{ij} = D_{ijrs}de_{rs} \]  \hspace{1cm} (3.41)

where \( D_{ijrs} \) is a tensor of rank 4 defining the soil behavior in Eulerian description, \( d\sigma^*_{ij} \) is a frame-indifferent tensor of rank 2 defining stresses in Eulerian description, and \( de_{rs} \) is incremental Cauchy strain. The 2nd Piola-Kirchhoff stress \( ^{n+1}S_{ij} \) at time step \( t_{n+1} \) can be decomposed into two parts:

\[ ^{n+1}S_{ij} = ^n\sigma_{ij} + dS_{ij} \]  \hspace{1cm} (3.42)

where \( dS_{ij} \) is the increment of the 2nd Piola-Kirchhoff stress tensor from time step \( t_n \) to time state \( t_{n+1} \) with respect to the current configuration and \( ^n\sigma_{ij} \) is the Eulerian stress tensor at time step \( t_n \). Similarly, we have

\[ ^{n+1}\sigma_{ij} = ^n\sigma_{ij} + d\sigma_{ij} \]  \hspace{1cm} (3.43)

where \( d\sigma_{ij} \) is the increment from time step \( t_n \) to time state \( t_{n+1} \). Substituting equations (3.11),(3.39),(3.40),(3.42) and (3.43) into equation (3.38), we obtain

\[ (1 + \Delta)(^n\sigma_{ij} + d\sigma_{ij}) = (\delta_{im} + \frac{\partial dU^i}{\partial x_m})(\delta_{jn} + \frac{\partial dU^j}{\partial x_n})(^n\sigma_{mn} + dS_{mn}) \]  \hspace{1cm} (3.44)
Neglecting higher order terms, and taking $\Delta$ as approximately equal to $du_{k,k}$ (provided the increments are small), we obtain (Washizu, 1982)

$$dS_{ij} = d\sigma_{ij} + du_{k,k} n^{\sigma_{ij}} - du_{i,m} n^{\sigma_{mj}} - du_{j,n} n^{\sigma_{in}}$$

(3.45)

d$S_{ij}$ is sometimes called the Truesdell stress increment tensor (Prager, 1961; Washizu, 1982). Prager (1961) pointed out that the Truesdell stress increment tensor is closely related to the 2nd Piola-Kirchhoff stress rate.

It was shown that $d\sigma_{ij}$ is not frame-indifferent (Malvern, 1969). Thus, we cannot apply constitutive equation using $d\sigma_{ij}$ directly. To satisfy the requirement of frame-indifference, we selected the Jaumann stress increment tensor $d\sigma_{ij}^J$, defined as (Washizu, 1982):

$$d\sigma_{ij}^J = d\sigma_{ij} - n^{\sigma_{ip}} \omega_{pj} - n^{\sigma_{jp}} \omega_{pi}$$

(3.46)

It should be noted that the original Jaumann stress rate $\dot{\sigma}_{ij}^J$ was proposed as a 'rate' type, that is,

$$\dot{\sigma}_{ij}^J = \dot{\sigma}_{ij} - n^{\sigma_{ip}} \Omega_{pj} - n^{\sigma_{jp}} \Omega_{pi}$$

(3.47)

where

$$\Omega_{ij} = \frac{1}{2} \left( \frac{\partial v_j}{\partial X_i} - \frac{\partial v_i}{\partial X_j} \right)$$

(3.48)

is called the spin tensor.

It is assumed that the Jaumann stress increment is close to the Jaumann stress rate in updated Lagrangian analysis, provided that the loading increment is small. The Jaumann stress rate is an objective stress rate tensor (Malvern 1969). So, the Jaumann stress increment tensor ($d\sigma_{ij}^J$) can be used as the frame-indifferent tensor $d\sigma_{ij}^*$ in the equation (3.41). The application of Jaumann stress increment tensor may predict oscillatory stresses in finite simple shear. This is discussed in section 5.2.

Substituting equations (3.20), (3.22) and (3.46) into equation (3.45), we obtain

$$dS_{ij} = d\sigma_{ij}^J + du_{k,k} n^{\sigma_{ij}} - d\epsilon_{ik} n^{\sigma_{kj}} - d\epsilon_{jk} n^{\sigma_{ik}}$$

(3.49)
or

\[ dS_{ij} = D_{ijrs}d\varepsilon_{rs} + du_{k,k}^\sigma \sigma_{ij} - d\varepsilon_{ik}^\sigma \sigma_{kj} - d\varepsilon_{jk}^\sigma \sigma_{ik} = D_{ijrs}^Gd\varepsilon_{rs} \quad (3.50) \]

It should be noted that the constitutive matrix \([D_{ijrs}^G]\) is non-symmetric even though \([D_{ijrs}]\) is symmetric.

The constitutive equation (3.50) was first proposed by Hibbitt, et. al. (1970). Similar formulation had been applied for a number of structural problems successfully, for example, Osias and Swedlow (1974); Tesng and Lee (1985). However, these application are concerned mainly with plain strain conditions and single phase material. This is different from (and simpler than) the application for this research where axisymmetry and a two phase material are assumed.

Finite strain studies of geotechnical problems can be found in a number of papers in the literature. For example, Banerjee and Fathallah (1979) applied the same constitutive equation proposed by Hibbitt et. al. (1970) but neglected the term \(du_{k,k}^\sigma \sigma_{ij}\) in the undrained analysis of pile driving. Desai and Phan (1980) conducted a three-dimensional finite element analysis including material and geometric non-linearities. Kiousis, et. al. (1986, 1988) postulated that the elastic material strain is a linear function of the 2nd Piola-Kirchhoff stress tensor and proposed a constitutive law directly related to the 2nd Piola-Kirchhoff stress rate with material strain rate in the undrained analysis of a footing and cone penetration. These contributions are limited to drained or undrained analyses.

Carter et. al. (1977) used Jaumann stress rate and the continuity equation to develop a theory of finite consolidation to study the rate effect of footing (a plane strain study). Karim (1984) used the 1st Piola-Kirchhoff stress rate (a non-symmetric stress rate) and rate of deformation to analyze the large deformation behavior of pile, footing and sampling disturbances.

In this research, the equation (3.50), together with Biot consolidation theory (1955), is applied to investigate the axisymmetric large deformation behavior during sampling tube penetration.
Chapter 4  Finite Element Formulation

In this chapter, the updated Lagrangian finite element formulation is derived by applying the Galerkin weighted residual method and the 2nd Piola-Kirchhoff stress tensor. Following Bathe (1976, 1982), Desai and Phan (1980), and Banerjee et. al. (1979), an incremental finite element procedure suitable for large deformation is established. The modified Cam-clay model (Roscoe and Burland, 1968) in association with the Biot's consolidation theory (Biot, 1955) is built into the formulation to account for time dependent drainage and loading. The interface element (Desai et. al., 1984) is extended for finite deformation to investigate the interaction between the soil and the wall of the sampling tube.

4.1 Governing Equation

Starting with the equations of equilibrium (3.34) and integrating with respect to volume \(V^{n+1}\) at time step \(t_{n+1}\), we obtain:

\[
\int_{V^{n+1}} \left( \frac{\partial n+1\sigma_{ij}}{\partial X_j} + n+1B F_i \right) dV^{n+1} = 0
\]

By using Galerkin weighted residual method with \(\delta du_i\) as weighting function, equation (4.1) can be expressed as:

\[
\int_{V^{n+1}} \left( \frac{\partial n+1\sigma_{ij}}{\partial X_j} + n+1B F_i \right) \delta du_i dV^{n+1} = 0
\]

where \(\delta\) means a small incremental quantity. Applying the divergence theorem to equation (4.2), it yields

\[
\int_{V^{n+1}} n+1\sigma_{ij} \frac{\partial \delta du_i}{\partial X_j} dV^{n+1} = \int_{A^{n+1}} n+1\sigma_{ij} n+1\delta du_i dA^{n+1} + \int_{V^{n+1}} n+1BF_i \delta du_i dV^{n+1}
\]

Substituting equation (3.38) into the left hand side of last equation, we have

\[
\int_{V^{n+1}} n+1\sigma_{ij} \frac{\partial \delta du_i}{\partial X_j} dV^{n+1} = \int_{V^{n+1}} \left( \frac{n+1p}{n\rho} \frac{\partial X_i}{\partial x_r} \frac{\partial X_j}{\partial x_s} \frac{\partial \delta du_i}{\partial X_j} \right) dV^{n+1}
\]
\[
\int_{V_{n+1}} (n+1 \rho \frac{\partial X_i}{\partial x_r} \frac{\partial \delta u_i}{\partial x_s} n^{+1} S_{rs}) \, dV^{n+1}
\]
(4.4)

The variation of \(X_i\) is
\[
\delta X_i = \delta(x_i + d u_i) = \delta d u_i
\]
(4.5)

and the variation of \(dE_{rs}\) is
\[
\delta dE_{rs} = \frac{1}{2} \left( \frac{\partial \delta X_i}{\partial x_r} \frac{\partial X_i}{\partial x_s} + \frac{\partial X_i}{\partial x_r} \frac{\partial \delta X_i}{\partial x_s} \right)
\]
(4.6)

Applying the above two equations and knowing that \(dE_{rs}\) and \(n+1 S_{rs}\) are symmetric we have
\[
\frac{\partial X_i}{\partial x_r} \frac{\partial \delta u_i}{\partial x_s} n^{+1} S_{rs} = \delta dE_{rs} n^{+1} S_{rs}
\]
(4.7)

Substituting equations (4.4), (4.7), (3.33) and (3.39) into (4.3) and changing the indices \(rs\) to \(ij\), we obtain
\[
\int_{V_n} n^{+1} S_{ij} \delta dE_{ij} dV^n = \int_{V_{n+1}} n^{+1} B_{Fi} \delta d u_i dV^{n+1} + \int_{A_{n+1}} n^{+1} T_{ij} \delta d u_i dA^{n+1}
\]
(4.8)

The integration at left hand side of equation (4.8) is with respect to the configuration at time step \(t_n\) while the right hand side is with respect to time step \(t_{n+1}\).

A similar formulation can be obtained by applying the principle of virtual work (Malvern 1969, Bathe 1982).

Denoting \(t_n\) the \(n^{th}\) loading step and \(t_{n+1}\) the \((n+1)^{th}\) loading step, that is,
\[
n^{+1} B_{Fi} = n B_{Fi} + dB_{Fi} = body force at t_{n+1}
\]
(4.9)
\[
n^{+1} T_{ij} = n T_{ij} + dT_{ij} = surface traction at t_{n+1}
\]
(4.10)

and substituting equations (3.42) and (3.19) into equation (4.8), we obtain
\[
\int_{V_n} n \sigma_{ij} \delta e_{ij} dV^n + \int_{V_n} n \sigma_{ij} \delta \eta_{ij} dV^n + \int_{V_n} dS_{ij} \delta dE_{ij} dV^n = \int_{V_{n+1}} n^{+1} B_{Fi} \delta d u_i dV^{n+1} + \int_{A_{n+1}} n^{+1} T_{ij} \delta d u_i dA^{n+1}
\]
(4.11)
The equations of equilibrium with incremental decompositions are:

\[ \int_{V_n} dS_{ij} \delta dE_{ij} dV^n + \int_{V_n} n \sigma_{ij} \delta d\eta_{ij} dV^n = 0 \]

\[ = \int_{V_{n+1}} (n+1)B F_i \delta d\epsilon_{ij} dV^{n+1} + \int_{A_{n+1}} n+1 T_i \delta d\epsilon_i dA^{n+1} - \int_{V_n} n \sigma_{ij} \delta d\epsilon_{ij} dV^n \]  

(4.12)

From equation (3.19), we have

\[ dS_{ij} \delta dE_{ij} = dS_{ij} \delta d\epsilon_{ij} + dS_{ij} \delta d\eta_{ij} \]  

(4.13)

Neglecting the second term on the right side of above equation which involves higher order product \((dS_{ij} \delta d\eta_{ij})\) and applying equation (3.50) into equation (4.12), we obtain the approximate equations of motion as:

\[ \int_{V_n} (D_{ijrs}^{LG} \delta \epsilon_{rs}) \delta d\epsilon_{ij} dV^n + \int_{V_n} n \sigma_{ij} \delta d\eta_{ij} dV^n = 0 \]

\[ = \int_{V_{n+1}} (n+1)B F_i \delta d\epsilon_{ij} dV^{n+1} + \int_{A_{n+1}} n+1 T_i \delta d\epsilon_i dA^{n+1} - \int_{V_n} n \sigma_{ij} \delta d\epsilon_{ij} dV^n \]  

(4.14)

The formulation above is applicable to general elastic-plastic analysis with large displacements, rotations and large strains. It is assumed that the loading is carried out in very small increment and the configurations of the volume and surface area at loading (time) step \(t_{n+1}\) are approximately equal to the loading (time) step \(t_n\). Substituting equations (4.9), (4.10) into equation (4.14), we obtain

\[ \int_{V_n} D_{ijrs}^{LG} \delta \epsilon_{rs} \delta d\epsilon_{ij} dV^n + \int_{V_n} n \sigma_{ij} \delta d\eta_{ij} dV^n = dR + C \]  

(4.15)

where

\[ dR = \int_{V_n} dB F_i \delta d\epsilon_{ij} dV^n + \int_{A_n} dT_i \delta d\epsilon_i dA^n \]  

(4.16)

and

\[ C = \int_{V_n} n B F_i \delta d\epsilon_{ij} dV^n + \int_{A_n} n T_i \delta d\epsilon_i dA^n - \int_{V_n} n \sigma_{ij} \delta d\epsilon_{ij} dV^n \]  

(4.17)

Up to this point, two approximations have been made. Firstly, \(dS_{ij} \delta d\eta_{ij}\) is neglected and secondly, the strained configuration is approximated to the unstrained
configuration in calculating incremental external work \(dR\). The second assumption is valid provided that the loading is deformation independent. In the case of deformation dependent loading, this assumption is not valid. Thus, it is likely that out-of-balance forces will develop. Equation (4.17) is the equation for the out-of-balance forces. Actually, the out-of-balance forces are simply the unbalanced forces at the previous loading state or previous iteration state. The out-of-balance equation can be corrected either by iteration or by carrying it over to the next increment. The geometric configuration, however, must be updated after each increment or iteration.

The basic equations employed are:

(a) a constitutive matrix \(D_{ijr,s}^{LC}\) from equation (3.50) to relate \(dS_{ij}\) and \(de_{r,s}\),

(b) an incremental virtual work equation (4.15) to calculate the incremental displacements \(du_i\), and

(c) the 2nd Piola-Kirchhoff stress equation (3.38) to calculate Cauchy stresses which can be directly used to describe the material behavior.

4.2 Isoparametric Finite Element Representation

A brief description of necessary isoparametric finite element representation is given below. Details of isoparametric finite element representation can be found in Zienkiewicz (1977); Owen and Hinton (1980) and Bathe (1982).

Adopting the standard isoparametric finite element formulation, the displacements are interpolated as:

\[
du_1 = \sum_{k=1}^{m} N_k d\phi_1^k
\]

\[
du_2 = \sum_{k=1}^{m} N_k d\phi_2^k
\]

where \(N_k\) is shape function, \(d\phi_1^k, d\phi_2^k\) are incremental nodal displacements and \(du_1, du_2\) are incremental displacements at any point in the element. The subscripts
1 and 2 denote radial and vertical direction respectively. The 8-noded quadrilateral element (Owen and Hinton, 1980) is used in this research.

The strain-displacement interpolation can be decomposed into linear strain-displacement interpolation matrix \([B]_L\) and nonlinear strain-displacement interpolation matrix \([B]_{NL}\). For axisymmetric analysis, \([B]_L\) is expressed as:

\[
[B]^k_L = \begin{bmatrix}
N_{k,1} & 0 \\
0 & N_{k,2} \\
N_{k,2} & N_{k,1}
\end{bmatrix}
\]  

(4.28)

where

\[
N_{k,j} = \frac{\partial N_k}{\partial x_j}
\]  

(4.29)

\[
x_1 = \sum_{k=1}^{m} N_k x^k_1
\]  

(4.30)

and \(x^k_1\) is the nodal position tensor. The linear strain is calculated by the equation

\[
[de] = \sum_{k=1}^{m} [B]^k_L \begin{bmatrix}
d\phi^k_1 \\
d\phi^k_2
\end{bmatrix}
\]  

(4.31)

The non-linear strain matrix takes a little different form. Denoting

\[
[B]^k_{NL} = \begin{bmatrix}
N_{k,1} & 0 \\
N_{k,2} & 0 \\
0 & N_{k,1} \\
0 & N_{k,2}
\end{bmatrix}
\]  

(4.32)

we have

\[
[U] = \begin{bmatrix}
\frac{\partial d\psi_1}{\partial z_1} \\
\frac{\partial d\psi_2}{\partial z_1} \\
\frac{\partial d\psi_1}{\partial z_2} \\
\frac{\partial d\psi_2}{\partial z_2} \\
\frac{\partial d\psi_1}{\partial x_1}
\end{bmatrix} = \sum_{k=1}^{m} [B]^k_{NL} \begin{bmatrix}
d\phi^k_1 \\
d\phi^k_2
\end{bmatrix}
\]  

(4.33)

and the incremental nonlinear strain is given by (Bathe, 1982):

\[
[d\gamma] = \frac{1}{2} [G] [U]
\]  

(4.34)
where
\[
[G] = \begin{bmatrix}
    \frac{\partial dU_1}{\partial x_1} & 0 & \frac{\partial dU_2}{\partial x_1} & 0 & 0 \\
    0 & \frac{\partial dU_1}{\partial x_2} & 0 & \frac{\partial dU_2}{\partial x_2} & 0 \\
    0 & 0 & 0 & 0 & \frac{dU_1}{x_1} \\
    \frac{\partial dU_1^k}{\partial x_2} & \frac{\partial dU_2^k}{\partial x_2} & \frac{\partial dU_2^k}{\partial x_1} & \frac{\partial dU_k}{\partial x_1}
\end{bmatrix}
\] (4.35)

The incremental Lagrangian strain becomes
\[
[\delta E] = [\delta e] + [\delta \eta]
\] (4.36)

It should be noted that it is \([B^k]_{NL}\) instead of \(\frac{1}{2}[G]\{U\}\) which is used in constructing the stiffness matrix described in the next section.

4.3 Stiffness Matrices

The stiffness matrices are found by invoking equation (4.15) and substituting the element coordinates and displacement interpolations, we obtain

\[
K_{ij}d\phi_j = dR_i + C_i
\] (4.37)

or

\[
[K][d\phi^k] = [dR] + [C]
\] (4.38)

where

(a) \(K_{ij}\) (or \([K]\)) is the incremental large strain stiffness tensor (matrix) which is a function of the current state of stresses and geometry.

\[
[K] = [K]_L + [K]_{NL}
\]

\[
= \int_{V_n} [B^k]_L^T[D]^{LG} [B^k]_L dV_n + \int_{V_n} [B^k]_{NL}^T [^{n}\sigma][B^k]_{NL} dV_n
\] (4.39)

or

\[
K_{ij} = \int_{V_n} D^{LG}_{k_{imn}} \frac{\partial N_{ki}}{\partial x_l} \frac{\partial N_{mi}}{\partial x_n} dV_n + \int_{V_n} \sigma_{ki} \frac{\partial N_{mi}}{\partial x_k} \frac{\partial N_{mj}}{\partial x_l} dV_n
\] (4.40)
where \([\sigma]\) takes the form shown below in axisymmetric coordinates (Bathe, 1982).

\[
[\sigma] = \begin{bmatrix}
\sigma_{11} & \sigma_{12} & 0 & 0 & 0 \\
\sigma_{21} & \sigma_{22} & 0 & 0 & 0 \\
0 & 0 & \sigma_{11} & \sigma_{12} & 0 \\
0 & 0 & \sigma_{21} & \sigma_{22} & 0 \\
0 & 0 & 0 & 0 & \sigma_{33}
\end{bmatrix}
\]

(4.41)

(b) \(dR_i\) (or \([dR]\)) is the incremental nodal force tensor (matrix)

\[
[dR] = \int_{V_n} [N][BF]dV^n + \int_{A^n} [N][T]dA^n
\]

or

\[
dR_i = \int_{V_n} BF_k N_{ki} dV^n + \int_{A^n} T_k N_{ki} dA^n
\]

(4.42)

(4.43)

(c) \(C_i\) is the out-of-balance tensor, \([C]\) is the out-of-balance matrix.

\[
[C] = \int [N][BF]dV^n + \int [N][T]dA^n - \int [B^k]\sigma dV^n
\]

or

\[
C_i = \int_{V_n} BF_k N_{ki} dV^n + \int_{A^n} T_k N_{ki} dA^n - \int_{V_n} \sigma_{ki} \partial N_{ki} dV^n
\]

(4.44)

(4.45)

4.4 Equilibrium Check

Since the equation of equilibrium (equation (3.34)) has to be satisfied throughout the loading process, an equilibrium check after each loading increment is then necessary. Assume that we have reached a certain stress state at \(t_n\). The equation to solve at state \(t_{n+1}\) is

\[
K_{ij}d\phi_j = dR_i + C_i
\]

(4.46)

where \(C_i\) is the out-of-balance load at the end of state \(t_n\) as calculated by equation (4.45). After solving equation (4.46), the new out-of-balance force \(n^{+1}C_i\) is calculated with reference to the new configuration, that is,

\[
n^{+1}C_i = \int_{V_{n+1}} n^{+1}BF_k N_{ki} dV^{n+1} + \int_{A_{n+1}} n^{+1}T_k N_{ki} dA^{n+1}
\]
This equilibrium check is done after the coordinates are updated. The first two parts in the right side of the above equation are for the calculation of the external loads (body force and tractions). The last part is to evaluate the internal forces. If the body forces are neglected and equilibrium is reached, then all the internal nodal forces not adjacent to the boundary should cancel each other. The internal nodal forces adjacent to the boundary should be equal to the external nodal forces calculated from tractions. Due to the nonlinearity of the material at large strains, a relative large out-of-balance load is possible. This means that the net nodal forces \( C_i \) may not equal to zero. The error will be accumulated if the out-of-balance load \( C_i \) is not corrected after each increment. As mentioned in section (4.1), there are two ways to correct the error; \( C_i \) can be carried over to the next increment or be corrected by iteration. If it is to be carried over to the next increment, the loading force in the next increment will be:

\[
 n^+1 K_{ij} d\phi_j^{n+1} = dR_i^{n+1} + n^+1 C_i
\]  

There are three iteration methods in wide usage - Newton-Raphson, Modified Newton-Raphson and Quasi-Newton method (see details in Bathe, 1982; Chen and Han, 1988). Although the Modified Newton-Raphson technique is considered economical with respect to computer time and storage requirement in elastic-plastic small strain problems, it was suggested that the original Newton-Raphson is convenient and economical in the analysis of finite deformation (Desai and Phan, 1980). The Newton-Raphson iteration is implemented into the finite element procedure. Assuming the system has reached equilibrium at state \( t_n \) and the incremental external loading \( dR_i \) is applied thereafter, the procedures for iteration can be described as follows:

1. calculate \( K_{ij}^0 d\phi_j^0 = dR_i + C_i \) to get nodal displacement \( d\phi_j^0 \)
2. update coordinates and calculate equation (4.45) to get \( C_i^0 \)
3. calculate \( K_{ij}^1 d\phi_j^1 = C_i^0 \) to obtain \( d\phi_j^1 \)
4. update coordinates and calculate equation (4.45) to get \( C_i^1 \)
5. calculate $K_{ij}^2 d\phi_j^2 = C_i^1$ to obtain $d\phi_j^2$

6. repeat the procedure until $C_i^n$ reach required accuracy.

7. add $C_i^n$ to next loading increment and proceed the analysis at state $t_{n+1}$

The superscripts 0, 1 and 2 denote the iteration number. The stiffness matrices are calculated at every iteration.

The first scheme (adding the out-of-balance load to the next increment) is economical. To avoid the accumulated error, the second scheme (iteration) is better than the first one. However, the iteration scheme is time consuming and is difficult to control the total computing time, especially when a large number of elements are used. Thus, sometimes, it would be better to use very small loading increment without iteration instead of using large loading increment with iteration. In this research, the first scheme (adding the out-of-balance load to the next increment) is adopted in the study of sampling disturbances because the large size of mesh used (a total of 364 elements and 1198 nodes) and the inclusion of consolidation effects (resulting in one more degree of freedom to each node). The iteration scheme is only used in the validation of algorithm (chapter 5).

4.5 Nonlinear Elastic-Plastic Constitutive Formulation

For an elasto-plastic material, we denote the constitutive tensor $D_{ijrs}$ in equation (3.50) as $D_{ijrs}^{EP}$. The derivation of constitutive tensor $D_{ijrs}^{EP}$ follows the same procedure as in small deformation theory which is derived by utilizing three requirements of plasticity (see details in many texts, for example, Bathe (1982); Desai and Siriwardane (1984); Chen and Han (1988)) as follows:

(1) The Yield Function:

Yield function defines a surface which separates the elastic behavior from the plastic behavior. The criterion for yielding is expressed mathematically as

$$f(\sigma_{ij}, H(\varepsilon_{ij}^p)) = 0$$

(4.49)

where $f$ is a yield function and $H(\varepsilon_{ij}^p)$ is hardening parameter that depends on the plastic strains $\varepsilon_{ij}^p$. The function $f$ must be equal to or less than zero at all times.
(2) Associated Flow rule:

It is assumed that the normality principle applies to the problem under investigation here. The associative flow rule which relates the plastic strain increments to the current stresses subsequent to yielding is described by:

\[ d\varepsilon_{ij}^P = d\lambda \frac{\partial f}{\partial \sigma_{ij}} \]  

(4.50)

\( d\lambda \) is a positive scalar variable which, in general, is a function of current stresses, strains and loading history.

(3) Hardening Rule:

The hardening parameter \( H \), during plastic flow, specifies how the yield surface grows during loading and is assumed to be a function of the current plastic strain \( d\varepsilon_{ij}^P \) only.

The first step in incremental plasticity is to decompose the incremental strains \( d\varepsilon_{ij} \) into incremental elastic strains \( d\varepsilon_{ij}^{E} \) and incremental plastic strains \( d\varepsilon_{ij}^{P} \) as

\[ d\varepsilon_{ij} = d\varepsilon_{ij}^{E} + d\varepsilon_{ij}^{P} \]  

(4.51)

where superscript \( E \) denotes elastic and superscript \( P \) denotes plastic. The above decomposition is valid in small deformation theory. However, the suitability of additive decomposition of incremental strain at finite deformation had been criticized. For example, Lee (1969) proposed a multiplicative decomposition (explained later) and concluded that the additive decomposition is in error. But, Nemat-Nasser (1979, 1982) proved that "infinitesimal incremental deformations from a given finitely deformed configuration of an elasto-plastic solids must also satisfy the additivity, provided that the same measure of strain increment with respect to the same configuration is used to define the total, the elastic and plastic strain increments." Following the interpretation of Guo (1981), this latter point can be explained as follows. Referring to the Fig. 4-1, a body point was moved from
Fig. 4.1 Decomposition of Displacement at Finite Strain
position $A(x_i)$ to a new position $D(X_i)$. We assume that the total displacement $u_i$ can be decomposed into elastic part $u_i^E$ and plastic part $u_i^P$, that is,

$$u_i = u_i^E + u_i^P \quad (4.52)$$

Now, we can express the position tensor $X_i$ as

$$X_i = x_i^P + u_i^E = x_i^E + x_i^P - x_i \quad (4.53)$$

where $x_i^E$ and $x_i^P$ are position tensors with respect to points $C$ and $B$ respectively.

The deformation gradient is

$$[F] = \frac{\partial X_i}{\partial x_j} = \left[ \frac{\partial x_i^E}{\partial x_j} \right] + \left[ \frac{\partial x_i^P}{\partial x_j} \right] = \left[ F^E \right] + \left[ F^P \right] - [I] \quad (4.54)$$

such that

$$[\dot{F}] = [\dot{F}^E] + [\dot{F}^P] \quad (4.55)$$

The spatial gradient of the velocity is

$$[\dot{L}] = [\dot{F}][F]^{-1} = \frac{\partial \dot{X}_i}{\partial x_j} = \left[ \frac{\partial x_i^E}{\partial x_j} \right] [F]^{-1} + \left[ \frac{\partial x_i^P}{\partial x_j} \right] [F]^{-1} = [L^E] + [L^P] \quad (4.56)$$

such that the rate of deformation

$$[D] = \frac{1}{2}([L] + [L]^T) = \frac{1}{2}([L^E] + [L^E]^T) + \frac{1}{2}([L^P] + [L^P]^T) = [D^E] + [D^P] \quad (4.57)$$

So, if the strain measure is referred to position $A(x_i)$, the decomposition of the rate of deformation $[D]$ is additive. The incremental strain is evaluated by integrating the last equation with respect to time. This proves the suitability of the additive decomposition (equation (4.51)) of incremental strain for finite strain. It is of interest that the multiplicative decomposition (Lee, 1969)

$$[F] = [\tilde{F}^E][\tilde{F}^P] = \frac{\partial X_i}{\partial x_k} \frac{\partial x_i^P}{\partial x_j} \quad (4.58)$$

is conceptually correct, but mathematically difficult to implement into the finite element procedure. Besides, the intermediate deformation gradient $[\tilde{F}^E]$ is dependent on the position tensor $x_i^P$ (point $B$), not position tensor $x_i$ (point $A$). The
deformation gradients \([\tilde{F}^E]\) and \([F^P]\) are referred to different configurations and therefore the additive rule cannot be applied in Lee’s (1969) formulation.

The elastic-plastic constitutive tensor \(D_{ijrs}^{EP}\) is given (see Bathe, 1982; Desai and Siriwardane, 1983; Chen and Han, 1988) as

\[
D_{ijrs}^{EP} = D_{ijrs}^{E} - \frac{D_{ijkl}^{E} \frac{\partial f}{\partial \sigma_{kl}} \frac{\partial f}{\partial \sigma_{mn}} D_{nmrs}^{E} - \frac{\partial f}{\partial \sigma_{mn}} \frac{\partial f}{\partial \sigma_{mn}}}{D_{ijkl}^{E} \frac{\partial f}{\partial \sigma_{kl}} \frac{\partial f}{\partial \sigma_{mn}} D_{nmrs}^{E}} \tag{4.59}
\]

where \(D_{ijrs}^{E}\) is elastic constitutive tensor.

In matrix notation, we define

\[
[Q]^T = \{ \frac{\partial f}{\partial \sigma_{ij}} \}^T = \{ \frac{\partial f}{\partial \sigma_{11}}, \frac{\partial f}{\partial \sigma_{22}}, \frac{\partial f}{\partial \sigma_{33}}, \frac{\partial f}{\partial \sigma_{12}} \} \tag{4.60}
\]

\[
[H]^T = \{ \frac{\partial f}{\partial \epsilon_{ij}^P} \}^T = \{ \frac{\partial f}{\partial \epsilon_{11}^P}, \frac{\partial f}{\partial \epsilon_{22}^P}, \frac{\partial f}{\partial \epsilon_{33}^P}, \frac{\partial f}{\partial \epsilon_{12}^P} \} \tag{4.61}
\]

Then, we obtain

\[
[D^{EP}] = [D^E] - \frac{[D^E]^T [Q]^T [Q] [D^E]}{[Q]^T [D^E] [Q] - [H]^T [Q]} \tag{4.62}
\]

Since, the constitutive relationship must be in a form that is independent of the current rate of rigid body rotation, we employ Jaumann stress increment \(d\sigma_{ij}^J\) (see section 3.4) and the corresponding deformation increment \(d\epsilon_{ij}\). In doing so, the elastic-plastic constitutive relationship can be formulated as

\[
d\sigma_{ij}^J = D_{ijrs}^{EP}(d\epsilon_{rs} - d\epsilon_{rs}^P) = D_{ijrs}^{EP} d\epsilon_{rs} \tag{4.63}
\]

where \(D_{ijrs}^{EP}\) is a function of current stresses and is related to Jaumann stress rate instead of the 2nd Piola-Kirchhoff stress rate.

By substituting equation (4.63) into equation (3.50), we obtain the Truesdell stress increment tensor (the 2nd Piola-Kirchhoff stress rate tensor) \(dS_{ij}\) as:

\[
dS_{ij} = D_{ijrs}^{LG} d\epsilon_{rs} = D_{ijrs}^{EP} d\epsilon_{rs} + du_{k,k} n^i \sigma_{ij} - d\epsilon_{ik} n^i \sigma_{kj} - d\epsilon_{jk} n^i \sigma_{ik} \tag{4.64}
\]
Updating $s_{ij}^{n+1}$ by $s_{ij}^{n+1} = s_{ij}^n + dS_{ij}^{n+1}$ and substituting into equation (3.38), we have

$$
n_{i}^{+1} = \frac{\rho^{n+1} \partial X_r}{\rho^n} \frac{\partial X_r}{\partial x_i} \frac{\partial X_r}{\partial x_j} s_{ij}^{n+1}
$$

(4.65)

4.6 Consolidation

Hvorslev (1949) suggested that fast pushing of the sampler produces longer and less disturbed samples than either slow pushing (hammering) or slow jacking. This motivates the author to study the rate dependent analysis of sampling disturbances. In this investigation, the rate dependent analysis is emphasised on the effect of permeability (consolidation). The effect of viscoplasticity is ignored. Formulations and stiffness matrices listed in sections 3.2, 3.3 and 4.1 are valid for solids only. Soils are made up of a soil skeleton and pore volume. When a saturated soil is loaded, excess porewater pressures are generated. It is then necessary to extend the finite strain formulation to incorporate the coupled effects of soil and water. This is accomplished by using Terzaghi's principle of effective stress. The principle of effective stress at time step $t_n$ is:

$$
\sigma_{ij}^n = \sigma_{ij} + \delta_{ij}^n u_w
$$

(4.66)

where $\sigma_{ij}$ is the total stress, $\sigma_{ij}$ is the effective stress and $u_w$ is porewater pressure at time step $t_n$; $\delta_{ij}$ is Kronecker delta. The pore water pressure is hydrostatic in nature and is, therefore, unaffected by rigid body motion. The above equation can be rewritten in matrix form as

$$
[n\sigma] = [n\sigma] + [m]^T u_w
$$

(4.67)

where

$$
[m]^T = \{ 1 1 1 0 \}
$$

(4.68)
in axisymmetric case. At the state \( t_{n+1} \), the principle of effective stress is expressed in a similar way, that is,

\[
\dot{\sigma}_{ij} = \sigma_{ij} + \delta_{ij}u_{w}^{n+1}
\]

where \( u_{w}^{n+1} \) is pore water at state \( t_{n+1} \), that is,

\[
u^{n+1} = u_{w} + du_{w}
\]

Following an approach described by Carter et al. (1979), a finite element formulation capable of solving the problem of the time dependent consolidation of an elasto-plastic soil subjected to finite deformations is developed.

In drained or undrained analysis, the nodal displacements are the only unknowns to be specified. In the consolidation analysis, however, the nodal pore water pressure is added to account for the time marching behavior of excess pore water pressures. We used the same interpolation function described in section 4-2 to describe the pore water pressure, that is,

\[
u = [N]^{T}[u_{w}^{k}] = \sum_{k=1}^{m} N_{k} u_{w}^{k}
\]

where \( N_{k} \) is the shape function for pore water pressure and \( u_{w}^{k} \) is nodal pore water pressure. The incremental excess pore water pressure is interpolated in the same way, that is,

\[
 du_{w} = [N]^{T}[du_{w}^{k}] = \sum_{k=1}^{m} N_{k} du_{w}^{k}
\]

Carter et al. (1983) had showed that the 8-noded quadrilateral element used here (three degrees of freedom associated with nodal displacements and one degree of freedom associated with pore pressure) can provide accurate predictions of consolidation response. The coupled effects of soil and water must consider equilibrium and continuity simultaneously.
(1) Equilibrium Equation:

The equation derived in section 4.1 must now include the pore water pressures and equation (4.14) becomes

\[ \int_{V_n}^{n+1} \left( D_{ijr}^{LG} d \epsilon_{rs} + \delta_{ij} d u_w \right) d \epsilon_{ij} dV + \int_{V_n}^{n} \sigma_{ij}^{1} d \eta_{ij} dV = 0 \]

or, in matrix form,

\[
[K][d \phi] + [L][d u_w] = [C] + [d R]
\]

where

\[
[L] = \int_{V_n}^{n} [B^k]^T [m][N]^T dV
\]

and \( D_{ijr}^{LG} \) use the equation (4.64).

(2) Continuity equation:

The flow of water through the soil is assumed to obey Darcy's Law, that is,

\[
v_i = - \sum_{j=1}^{3} k_{ij} \frac{\partial u_w}{\partial x_j}
\]

where \( v_i \) is superficial velocity, \( k_{ij} \) is coefficient of permeability. Assuming the off-diagonal terms of \( k_{ij} \) are negligible, the equation of continuity for the soil under axisymmetric condition becomes

\[
\frac{k_r}{\gamma_w} \frac{\partial^2 u_w}{\partial r^2} + \frac{k_r}{\gamma_w} \frac{1}{r} \frac{\partial u_w}{\partial r} + \frac{k_z}{\gamma_w} \frac{\partial^2 u_w}{\partial z^2} + \frac{\partial \epsilon_v}{\partial t} = 0
\]

where \( k_z \) is permeability in \( z \) direction, \( k_r \) is the permeability in radial direction and

\[
\epsilon_v = \epsilon_{11} + \epsilon_{22} + \epsilon_{33} = [m]^T [\epsilon]
\]

By using the Galerkin method with virtual pore pressure \( \delta u_w \) as weighting function and applying divergence theorem, equation (4.77) becomes

\[
\int_{A_n}^{n} \left( \frac{k_r}{\gamma_w} \frac{\partial u_w}{\partial r} n_r + \frac{k_z}{\gamma_w} \frac{\partial u_w}{\partial z} n_z \right) dA = 0
\]
\[ = \int_{V^n} \left( \frac{k_r}{\gamma_w} \frac{\partial \delta u}{\partial r} \frac{\partial u}{\partial r} + \frac{k_z}{\gamma_w} \frac{\partial \delta u}{\partial z} \frac{\partial u}{\partial z} \right) dV^n - \int_{V^n} \delta u_w \frac{\partial V_{vol}}{\partial t} dV^n \quad (4.79) \]

where \( n_r \) and \( n_z \) are direction cosines of the unit outward normal vector. The pressure gradients are given by

\[ \frac{\partial u_w}{\partial r} = [B_r]^T[u_w^k] = \sum_{k=1}^{m} \frac{\partial N_k}{\partial r} u_w^k \quad (4.80) \]

\[ \frac{\partial u_w}{\partial z} = [B_z]^T[u_w^k] = \sum_{k=1}^{m} \frac{\partial N_k}{\partial z} u_w^k \quad (4.81) \]

By writing

\[ [\Gamma] = \int_{A^n} \left( \frac{k_r}{\gamma_w} [N][B_r]^T n_r + \frac{k_z}{\gamma_w} [N][B_z]^T n_z \right) dA^n \quad (4.82) \]

\[ [\Phi] = \int_{V^n} \left( \frac{k_r}{\gamma_w} [B_r][B_r]^T + \frac{k_z}{\gamma_w} [B_z][B_z]^T \right) dV^n \quad (4.83) \]

then equation (4.79) can be expressed as

\[ [\Gamma][u_w^k] = [\Phi][u_w^k] - [L] \frac{d[\phi^k]}{dt} \quad (4.84) \]

where

\[ [\Gamma][u_w^k] = - \int [N]^T[v_n] dA^n \quad (4.85) \]

and \( v_n \) is normal velocity along boundary. Equation (4.85) denotes the flux across the boundary. If there is no flux across the boundary, equation (4.85) is neglected. Equation (4.84) is a first order differential equation with respect to time. Integrating from \( t_n \) to \( t_{n+1} \) gives

\[ \int_{t_n}^{t_{n+1}} [\Phi][u_w^k] dt - \int_{t_n}^{t_{n+1}} [L] \frac{d[\phi^k]}{dt} dt = \int_{t_n}^{t_{n+1}} [\Gamma][u_w^k] dt \quad (4.86) \]

In order to solve this time marching problem, we make the following approximation

\[ \int_{t_n}^{t_{n+1}} u_w^k dt = \{(1 - \alpha)^n u_w + \alpha^{n+1} u_w\} \Delta t \quad (4.87) \]

where \( \alpha \) is a constant with a magnitude chosen to yield optimum stability. It was shown by Booker and Small (1975) that equation (4.87) is unconditionally stable provided \( \alpha \geq 0.5 \). In the analysis, \( \alpha \) is chosen as 1.
We can rewrite equation (4.86) as

\[
[\Phi][n^{+1} u_w^k] \Delta t - [L][\phi^k]_{t_n}^{t_{n+1}} = [\Gamma][n^{+1} u_w^k] \Delta t
\]  

(4.88)

Since,

\[
[du_w] = [n^{+1} u_w] - [u_w]
\]  

(4.89)

\[
[d\phi^k] = [n^{+1} \phi^k] - [\phi^k]
\]  

(4.90)

so,

\[
[\Phi]\{ [du_w^k] + [n u_w^k] \} \Delta t - [L]^T \{ [n^{+1} \phi^k] - [\phi^k] \} = [\Gamma]\{ [n u_w^k] + [du_w^k] \} \Delta t
\]  

(4.91)

and

\[
[L]^T [d\phi^k] + \{ [\Gamma] - [\Phi] \}[du_w^k] \Delta t = \{ [\Phi] - [\Gamma] \}[n u_w^k] \Delta t
\]  

(4.92)

The right hand side of equation (4.92) is only a function of the current time. Combining equations (4.74) and equation (4.92) we obtain the coupled equation as

\[
\begin{bmatrix} [K] & [L] \\ [L]^T & [M] \end{bmatrix} \Delta t \begin{bmatrix} [d\phi^k] \\ [du_w^k] \end{bmatrix} = \begin{bmatrix} [dR] + [C] \\ [O] \Delta t \end{bmatrix}
\]  

(4.93)

where

\[
[O] = \{ [\Phi] - [\Gamma] \}[n u_w^k]
\]  

(4.94)

\[
[M] = \{ [\Gamma] - [\Phi] \}
\]  

(4.95)

The combined equations (4.93), satisfying both the equilibrium equation and the continuity equation, were used to solve the time dependent problem in this analysis.

It should be noted that the continuity equation derived here is basically a small deformation approach of Biot theory (1955). However, the constitutive equation, that is, the permeability $k_{ij}$ should be modified when the deformation is large. The permeability changes with respect to direction and void ratio. To
account for the effects of rotation, it was suggested by Carter, et. al. (1979) that the permeability $k_{ij}$ at $t_n$ can be taken as

$$k_{ij} = R_{ri}k_{rs}^0R_{sj}$$ (4.96)

where $R_{ij}$ is the rotation tensor of the coordinate axes from $t_0$ to $t_n$ and $k_{ij}^0$ is the permeability at time step $t_0$. We assume the permeability is isotropic $k_r = k_z$ such that, after rotation $k_{ij} = k_{ij}^0$. It is also assumed that the permeability remains constant at large strains. The differences between small deformation and finite deformation for one dimensional consolidation is presented in section 5.3.

4.7 Modified Cam-Clay Model

In choosing a particular model to represent soil behavior one should be aware of the test data which has been used to generate the model, the ease with which these data can be obtained and the type of soil for which the model has demonstrated success. The model chosen for this research is the modified Cam-clay model that was developed by Roscoe and Burland (1968). The choice of this model is based on the following reasons: (1) the parameters require can be obtained in one single triaxial test; (2) the soil used in the experiment is an insensitive ideal clay which would be in a soft condition when penetrated by the sampling tube and during subsequent laboratory tests; the modified Cam-clay model has successfully predicted the mechanical behavior of soft soils (Worth, 1977; Wood, 1980; Parry and Wroth, 1981) and is expected to be quite good in making predictions of the changes in effective stresses and pore water pressures and the stress-strain behavior of kaolin used in the test; (3) it is relatively easy to incorporate into a finite element algorithm as done by Britto and Gunn (1987).

The modified Cam-clay model makes use of an elliptical yield surface defined by (Fig 4-2 and Fig. 4-3)

$$f = p^2 - pp_m + \frac{q^2}{M^2} = 0$$ (4.97)
where

1. \( p \) is effective mean normal stress defined as:

\[
p = \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3}
\]  
(4.98)

2. \( q \) is deviatoric stress defined as:

\[
q = \frac{1}{\sqrt{2}} \sqrt{(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 + 6\sigma_{12}^2 + 6\sigma_{23}^2 + 6\sigma_{31}^2}
\]  
(4.99)

3. \( p_m \) works as a hardening parameter which is determined by the intersection of the past maximum ellipsoidal yield locus and the \( p \) axis. The overconsolidation ratio \( R_p \) used in this investigation is defined as

\[
R_p = \frac{p_m}{(p_m)_o}, \quad R_p \geq 1
\]  
(100)

where

\[
(p_m)_o = \frac{(p_0^2 + \frac{q_0^2}{M^2})}{p_0}
\]  
(101)

and \( p_0 \) is in situ mean effective stress and \( q_0 \) is in situ deviatoric stress.

The parameters required are \( \lambda \) (the slope of the virgin consolidation line in \( e - \ln(p) \) space), \( \kappa \) (the mean slope of the swelling and recompression line in \( e - \ln(p) \) space), \( e_{cs} \) (the value of void ratio \( e \) at unit \( p \) on the critical state line in \( e - \ln(p) \) space), \( \mu \) (the elastic shear modulus) and \( M \) (the value of the stress ratio \( \frac{p}{q} \) at the critical state line). The failure criterion in terms of stress invariants can be written as

\[
q_f = M p_f
\]  
(4.102)

where subscript \( f \) indicates the critical state (failure) condition; the stress state at which no volume change occurs during further shearing. In addition to the above parameters, a full description of the material state requires specification of the in situ stresses at each point in the soil, unit weight of water \( \gamma_w \) and the soil's permeability.
\[ p^2 - pp_m + \frac{q^2}{M^2} = 0 \]

elliptical yield surface

\[ q = M \times p \]

critical state line

Ko Consolidation

Fig. 4-2 Modified Cam-Clay Model
Fig. 4-3 Idealized Unloading and Reloading Response
The behavior of soil inside the yield surface is assumed elastic. The stresses and deformations are determined by the elastic bulk modulus $K_b$ of soil defined as

$$K_b = \frac{(1+\epsilon)}{\kappa}p$$

(4.103)

and the elastic shear modulus $\mu$. The above equation shows that the elastic bulk modulus $K_b$ is stress dependent, that is, $K_b$ is non-linearly elastic. On the contrary, $\mu$ is held constant at all time. For the stress state on the yield surface, associative flow rule is used to determine plastic flow. The explicit forms for $\frac{\partial f}{\partial \sigma_{ij}}$ (equation (4.60)) for the modified Cam-clay model are:

$$\frac{\partial f}{\partial \sigma_{11}} = \frac{1}{3}(2p - p_m) + \frac{3}{M^2}s_{11}$$

(4.104)

$$\frac{\partial f}{\partial \sigma_{22}} = \frac{1}{3}(2p - p_m) + \frac{3}{M^2}s_{22}$$

(4.105)

$$\frac{\partial f}{\partial \sigma_{33}} = \frac{1}{3}(2p - p_m) + \frac{3}{M^2}s_{33}$$

(4.106)

$$\frac{\partial f}{\partial \sigma_{12}} = \frac{6}{M^2}s_{12}$$

(4.107)

where

$$s_{ij} = \sigma_{ij} - \delta_{ij}p$$

(4.108)

is stress deviation tensor. In the modified Cam-clay, the hardening parameter $H$ depends on the volumetric plastic strain $\epsilon_v^P$ only, that is,

$$H = H(\epsilon_v^P)$$

(4.109)

Now, $\epsilon_v^P$ is a function of $p_m$ given as

$$d\epsilon_v^P = (\frac{\lambda - \kappa}{1 + \epsilon}) \frac{dp_m}{p_m}$$

(4.110)

So, the elastic-plastic constitutive tensor in equation (4.59) becomes

$$D_{ijrs}^{EP} = D_{ijrs}^{E} - \frac{D_{ijkl}^{E} \frac{\partial f}{\partial \sigma_{kl}} \frac{\partial f}{\partial \sigma_{mn}} D_{mnsr}^{E}}{\frac{\partial f}{\partial \sigma_{rs}} D_{e}^{E} \frac{\partial f}{\partial \sigma_{mn}} - \frac{\partial f}{\partial \epsilon} \frac{\partial f}{\partial \sigma_{mn}}}$$

(4.111)
where
\[ \frac{\partial f}{\partial \varepsilon_v} = (-p) \frac{P_m(1 + e)}{\lambda - \kappa} \]  
(4.112)
and \( e \) is the void ratio at the current stress state.

The equation (4.102) is used to define the critical state. In general, we have

\[ M_e \leq M \leq M_c \]  
(4.113)
where \( M_c \), the value of \( M \) obtained in conventional triaxial compression test, defined as

\[ M_c = \frac{6 \sin \phi}{3 - \sin \phi} \]  
(4.114)
and \( M_e \), the value of \( M \) obtained in triaxial extension, is equal to

\[ M_e = \frac{6 \sin \phi}{3 + \sin \phi} \]  
(4.115)
\( \phi \) is the angle of effective internal friction.

4.8 Thin-Layer Element

During sampling penetration, friction forces are developed inside and outside of the sampler. The soil-sampler friction has been classified as the major source of sampling disturbances (Hvorslev, 1949). A proper simulation of friction force during sampling tube penetration appears then to be the key to obtain the magnitude of sampling disturbances. In previous works (Alonso, et. al., 1981; Karim, 1984; Baligh, et. al., 1987), the soil-sampler interface was modelled simply as either frictionless (Alonso, et. al., 1981; Baligh, et. al., 1987) or frictional with an estimated shearing stress applied directly on the inside wall of the sampling tube (Karim, 1984). A frictionless sampler is, of course, not a good simulation. Imposing a constant shear stress may account for the effect of interface friction. However, experimental evidences (Hvorslev, 1949; Arman and McManis, 1977) revealed that the wall friction varies with respect to the depth of penetration. Constant shear force, therefore, may not be a good simulation. What's important here is that the
soil at the moving contact surface (soil-pile or soil-sampler interface) may be seriously altered such that the original soil properties cannot represent the interface properties. Using the parent soil properties along the contact surface (for example, Karim, 1984) may misrepresent the true behavior of the interface. Thus, in this research, the interface elements are added to the inside wall of the sampling tube to investigate sampling disturbances caused by a frictional sampler.

The algorithm of interface element chosen here is based on the theory of thin-layer element (Desai et al., 1984), in which the interface is represented by an equivalent solid or continuum element with a (small) finite thickness \( t \) and with an appropriate soil properties to model the interface. The validation and implementation of thin-layer element are presented in papers, for example, Sharma and Desai, 1990; Desai and Nagaraj, 1988; Desai et al. 1984, and will not be repeated here. In this research, the concept of thin-layer element is incorporated into the updated Lagrangian formulation (a finite strain formulation) to deal with two phase (soil and water) interface material. This is done as follows:

(a) interface properties:

As stated before, some limited experimental evidence shows that inside wall friction varies with depth. Thus, the use of material model such as Von-Mises and Tresca are not appropriate. The Mohr-Coulomb model (Fig. 4-4) is, however, widely used to represent the contact surface. In this research, it is postulated that the thin-layer element follows the generalized Mohr-Coulomb criterion expressed as (Owen and Hinton, 1981)

\[
f = -\frac{1}{3} J_1 \sin \delta + (J_2)^\frac{3}{2} (\cos \theta + \frac{1}{\sqrt{3}} \sin \theta \sin \delta) - c_a \cos \delta = 0 \tag{4.116}
\]

where \( J_1 (= \sigma_{ii}) \) is first stress invariant, \( J_2 (= \frac{1}{2} \sigma_{ij} \sigma_{ij}) \) is second stress invariant, \( \theta \) is Lode's angle, \( c_a \) is the adhesion, \( \delta \) is the angle of internal friction at the interface of the soil and the sampling tube.
Fig. 4-4 Mohr-Coulomb Model
The compressive stresses are taken as positive in above equation. The drift correction for yielding function follows the algorithm suggested by Owen and Hinton (1981) and will not be repeated here. It is assumed that if the material reaches failure point A (Fig. 4-4), the post-failure behavior will follow the path AC (along the failure line).

(b) interface for two phase material:

It is postulated here that the principle of effective stress \( \sigma^\prime_{ij} = \sigma_{ij} + \delta_{ij} u_w \) applies to the thin-layer elements such that they can be handled as a two phase material similar to the parent soils. The Biot consolidation theory described in section 4.6 is also extended to apply to these thin-layer elements.

(c) behavior modes:

In general, there are four interface modes, that is, stick, slip, debonding and rebonding (Desai, et. al., 1984). In this research, debonding and rebonding are not observed. The stick mode and the slip mode are handled as follows: (1) stick - the stress state of a thin-layer element is inside the yield surface (within the elastic range); (2) slip - the stress state is on the yield surface.

(d) element thickness:

It was found (Desai et. al., 1984) that the distribution of stresses in thin-layer element is not affected significantly by the thickness of the element provided that

\[
0.01 \leq \frac{t}{B} \leq 0.1
\]  

(4.117)

where \( t \) is the thickness of the thin-layer element and \( B \) is the element width adjacent to thin-layer element. The thickness ratio \( \frac{t}{B} = 0.1 \) is used in this research.

(e) accommodation of large strain:

It is expected that the soil-sampler interface may be subjected to large strains. Thus, the theory of thin-layer element is extended and incorporated into
the updated Lagrangian finite element formulation described previously to deal with the large strain interface problems.

In summary, the interface element is modelled as a thin-layer element characterized as a two phase material. The thickness ratio is 0.1. This thin-layer element is treated as the regular element in the updated Lagrangian finite element formulation. The four behavior modes are dependent on the predicted stress state as explained above.

4.9 Incremental finite element Procedure

The procedure described below requires the specification of a small incremental displacement or load.

1. For each increment, calculate the matrices \([K], [O], [C], [dR], [L], [M]\) in equation (4.103) where
   a. \([K]\) is new stiffness matrices based on the previous stress variable. (equation (4.40))
   b. \([dR]\) represents the incremental external force in the boundary and body force. (equation (4.42))
   c. \([O]\) is calculated from equation (4.94).
   d. \([M]\) is calculated from equation (4.95).
   e. \([L]\) is calculated from equation (4.75).

2. Solve the equation (4.93) to obtain the nodal displacement \([d\phi^k]\) and excess pore pressure \([du_{w}^k]\) for the current increment.

3. Calculate \(\Delta E_{ijrs}^{EP}\) from equation (4.59)

4. Knowing that \([du] = [N][d\phi^k], [\epsilon] = [B][d\phi^k]\), calculate the incremental 2nd Piola-Kirchhoff stress from equation (4.64).

5. Calculate the 2nd Piola-Kirchhoff Stress \(\sigma^{n+1}_{ij}\) by equation (3.42) and transform back to Cauchy stress using equation (3.38). We, then, obtain the stress \(\sigma^{n+1}_{ij}\).

6. Determine the new yield surface according to the new stresses.
(7) Update the variables, i.e., \( \varepsilon_{ij}^{n+1} = \varepsilon_{ij}^n + d\varepsilon_{ij} \); \( n^{+1}\phi = n\phi + d\phi \); \( n^{+1}u_w = n_uw + du_w \).

(8) Calculate the new out-of-balance load \([C]\) from equation (4.45) according to the updated stresses and new geometry.

(9) Add the \([C]\) to the next increment and repeat the incremental procedure from step (1) to step (8).
Chapter 5 Validation of the Algorithm

The implementation of any finite element algorithm needs to be checked against closed form solutions for problems as close as possible to the intended problem. The purpose of this chapter is to validate the present algorithm (the updated Lagrangian finite element formulation) by comparing the numerical results with closed form solutions in four cases - uniaxial deformation, simple shear strain, one-dimensional consolidation and cylindrical cavity expansion. These four types of deformation are commonly used in geotechnical engineering to determine the fundamental properties of soils. The differences between the predictions of finite strain solutions and small deformation solutions are also given.

To obtain the finite strain closed form solution, rate-type constitutive equation is desirable. In this research, hypoelasticity in which the current stress rate is assumed to be a function of the current values of Cauchy stress and rate of deformation (Truesdell, 1955) was chosen. That is, \( \dot{\sigma}_{ij} = \lambda d_{kk} \delta_{ij} + 2\mu d_{ij} \), where \( \dot{\sigma}_{ij} \) is the Jaumann stress rate (an objective stress rate), \( d_{ij} \) is rate of deformation, and \( \lambda \) and \( \mu \) are Lame’s constants. The inclusion of Jaumann stress rate is to fulfill the requirement of frame-indifference in formulating the constitutive equation at finite strain (see details in chapter 3).

In the literature, most of the available finite strain closed form solutions (with the above assumption of constitutive equation) were solved for plane strain conditions using a rectangular coordinate system. The finite strain closed formed solutions with axisymmetrical loading, which is the main deformation pattern in this research, remain unsolved. I present finite strain closed form solutions for the above cases using cylindrical coordinates. These solutions are shown in Appendices A to D and will be compared with the updated Lagrangian finite element results.

It is aware that different assumption of constitutive law will lead to different closed form solution (Herrmann, 1991). Other closed form solutions with different assumption of constitutive equations can be found in the literature, for example, Desai and Siriwardane (1984), Truesdell (1955, 1956).
5.1 Uniaxial Deformation

Consider a bar subjected to one dimensional deformation as shown in Fig. 5-1, where $z_0$ and $r_0$ are the initial length and radius of the bar, respectively, while $Z_0$ and $R_0$ denote the length and the radius after deformation. Points $A'$, $B'$, $C'$ and $D'$ are the deformed coordinates corresponding to initial points $A$, $B$, $C$ and $D$ respectively. For convenience, it is assumed that the deformation at the top of the bar takes place at a constant rate $k_0$. It is also assumed that the deformation is homogeneous such that the rotational effect is eliminated.

For small deformations, the solution for uniaxial deformation is simply $\sigma_{zz} = E_y \varepsilon_{zz}$; if logarithmic strain is adopted, the solution is $\sigma_{zz} = E_y \ln \lambda_z$, where $\lambda_z$ is the stretch (defined in appendix A) and $E_y$ is Young's modulus. The finite strain closed form solution for a hypo-elastic material (see Appendix A for details) is exactly similar to the logarithmic strain solution. The updated Lagrangian finite element results were obtained by using an 8-noded isoparametric element.

The predictions of vertical stresses developed during finite uniaxial extension from zero vertical displacement to twice its initial length are shown in Fig. 5-2. In Fig. 5-2, the horizontal axis denotes the vertical strain $\frac{u_z}{l_0}$, where $u_z$ is the vertical displacement and $l_0$ is the initial length; the vertical axis denotes the stresses normalized with respect to Young's modulus. The finite strain closed form solutions is plotted as solid line; the small deformation solution is plotted with dash line and the updated Lagrangian finite element results are shown as asterisks. The results of the updated Lagrangian finite element formulation, which were obtained with 20 increments ($\varepsilon_{zz} = 5\%$ per step), are in good agreement with the finite strain closed form solution. The comparison between the finite strain closed form solution and the small deformation solution shows that the small deformation analysis gives higher vertical stress. This implies that, for a given vertical stress at failure, employing the theory of finite strain will predict the failure at larger strain than small strain theory. Also, finite strain solution indicates that the tangent modulus $\frac{d\sigma_{zz}}{d\varepsilon_{zz}}$ is decreasing, that is, the material is getting weaker in comparison to the small strain solution.
Fig. 5-1 Uniaxial Extension
Fig. 5-2 Finite Uniaxial Extension

(a) Updated Lagrangian
(b) Hypoelasticity

- Hypoelasticity
- Small deformation
- Updated Lagrangian
The predictions of finite uniaxial compression from zero vertical displacement to one half the initial length of the bar are shown in Fig. 5-3. The updated Lagrangian finite element results, which were obtained with 20 increments (εzz = 2.5% per step), also show good agreement with finite strain closed form solution for the case of finite uniaxial compression. However, the finite strain closed form solution predicts larger vertical stresses σzz than the small deformation solution and an increasing tangent modulus \( \frac{d\sigma_{zz}}{d\varepsilon_{zz}} \). This implies that, in comparison to uniaxial extension, the material is getting stronger.

In summary, the accuracy of the updated Lagrangian finite element algorithm is satisfactory. The results for the extension and the compression of a bar indicate that material is strengthened or softened at large strain depending on the loading direction and, of course, the assumed constitutive law.

5.2 Simple Shear Strain

The simple shear strain is defined as a strain state with homogeneous lateral displacement and zero volumetric change. The simple shear strain is a plane strain condition in which, say, a cubical element is deformed into a parallelepiped (Budhu, 1979). The major difference between uniaxial deformation and simple shear strain is that the principal stress axes are rotated throughout the process of shearing. The deformation mode of simple shear strain in a rectangular coordinate system is shown in Fig. 5-4. The deformation mode of simple shear strain in a cylindrical coordinate system, for a pile as an example, is shown in Fig. 5-5. In these two figures, capital letters (A, B, C, D, E) denote initial positions and primed letters (A', B', C', D', E') denote deformed positions. The major difference between these two modes is that, in rectangular coordinates (Fig. 5-4), the simple shear strain can take in any given direction while, in axisymmetric coordinates, simple shear strain occurs through deformation along the Z axis (Fig. 5-5). A practical example where the simple shear occurs is along the pile shaft or the wall of a sampling tube is given in Budhu (1979) and Randolph & Wroth (1981).
Fig. 5-3 Finite Uniaxial Compression

---

hypo-elasticity
small deformation
updated Lagrangian

Vertical Strain $\frac{u_z}{l_o}$

Normalized Vertical Stress $rac{\sigma_{zz}}{E_y}$
Fig. 5-4 Simple Shear Strain
Fig. 5-5 Simple Shear Strain in Axial Symmetry
In small deformation analysis, for an elastic material obeying Hooke’s law, only shearing stresses are predicted in simple shear strain, that is, $\sigma_{rz} = \mu \gamma$ and $\sigma_{rr} = \sigma_{zz} = 0$, where $\mu$ is the shear modulus and $\gamma$ is the rotational angle defined in Fig. 5-5; $\gamma$ is called the engineering shear strain if the deformation is small. Following a procedure described in the paper of Dienes (1979) for rectangular plane strain condition, the solution of finite simple shear strain in axial symmetry is (see Appendix B for details):

$$\sigma_{rz} = \mu \sin \phi \quad (5.1)$$
$$\sigma_{rr} = +\mu(1 - \cos \phi) \quad (5.2)$$
$$\sigma_{zz} = -\mu(1 - \cos \phi) \quad (5.3)$$

where $\phi = \tan \gamma$. Equations (5.1)-(5.3) reveal that finite simple shear strain predicts (1) oscillatory shearing stresses; (2) two non-zero normal stresses which are equal in magnitude but opposite in sign, that is, one is tensile ($\sigma_{zz}$) while the other ($\sigma_{rr}$) is compressive. The prediction of two non-zero normal stresses in finite simple shear strain differs from the small strain analysis, which predicts no normal stresses. But, the finite simple shear strain solution converges to the small strain solution if the shear strain is small, that is, $\gamma \approx \tan \gamma \approx \phi \approx \sin \phi$ and $\cos \phi \approx 1$ such that $\sigma_{rz} = \mu \gamma$ and $\sigma_{rr} = \sigma_{zz} = 0$. The updated Lagrangian finite element results were obtained by an 8-noded element with the boundary conditions as shown in Fig. 5-6. A rotational angle up to $\frac{\pi}{4}$ ($45^\circ$) was specified and a total of 100 increments was used to obtain the numerical results. The final shear strain $k_{rz} / l_r$ (small deformation definition) was 100%.

The numerical results together with analytical solutions are plotted in Fig 5-6. In Fig. 5-6, the closed form solutions are plotted with solid line ($\sigma_{rz}$) and dotted line ($\sigma_{zz} = -\sigma_{rr}$); the numerical results are shown with symbols *, o, and +; the dash line denotes the small deformation solution of $\sigma_{rz}$. Again, the updated Lagrangian algorithm agree extremely well with the closed form solutions. All the stresses predicted were within an accuracy of 1% in the range of $0 \leq \gamma \leq 45^\circ$. 
Fig. 5-6 Finite Elastic Simple Shear Strain
Dienes (1979) pointed out that the use of Jaumann stress rate cannot provide correct predictions at shear strains larger than 40%. However, for shear strains less than 0.4, the above closed form solution does provide good agreement compared with the corotational rates solutions solved by Dienes (1979). Thus, I limit the application of the current algorithm within 40% of shear strain. This limitation does not apply for the normal strains.

5.3 One-Dimensional Elastic Consolidation

A finite element mesh with six 8-noded elements, shown in Fig. 5-7, was used to simulate an elastic porous material under one-dimensional consolidation. The material properties are as shown in Fig. 5-7 (a). The loading, $\Delta \sigma_{zz}$, is applied instantaneously at the top surface at time $t = 0$ and an initial excess pore water pressure $\Delta u_w$ ($= \Delta \sigma_{zz}$) is generated throughout the whole region. Thereafter, drainage is allowed at the top surface and the excess pore water pressure is dissipated with time. For small deformation, Terzaghi's consolidation theory is often used. However, when the deformation is large, Terzaghi's theory fails. The variations of excess pore water pressure under vertical traction $\Delta \sigma_z = 1.0 \times E' \times 1.0$ are presented in Fig. 5-7(b). The dissipation of excess pore water pressures predicted by the updated Lagrangian algorithm is much faster than the small strain Terzaghi's theory.

The degree of consolidation ($U$) is defined as $\frac{u_z}{(u_z)_f} \times 100\%$, where $(u_z)_f$ is the total displacement at the top surface at the end of consolidation. The finite strain closed form solution of the total displacement $(u_z)_f$ under one-dimensional constrained compression is (see Appendix C):

\[
(u_z)_f = (1 - \lambda_z) \times H_0 = (1 - e^{\frac{\beta(1+v)(1-2v)}{(1-v)}}) \times H_0 \tag{5.4}
\]

where $\beta = \frac{\Delta \sigma_{zz}}{E_y}$ and $\beta$ is taken as negative if $\sigma_{zz}$ is compressive stress. The small strain solution of $(u_z)_f$ is

\[
(u_z)_f = \beta \times \frac{(1 + v)(1-2v)}{(1 - v)} \times H_0 \tag{5.5}
\]
Elasticity

\[ E_y = 7428.57 \text{ mN/(mm}^2 \text{)} \]

\[ v=0.3 \]

\[ K_r=K_z=1.0 \times 10^{-6} \text{ mm/sec} \]

\[ C_v=1.0 \text{ mm}^2/\text{sec} \]

---

**Fig. 5-7 Variation of Excess Pore Water Pressure vs. Time Factor**
The degree of consolidation vs. time factor $T_v \left(= \frac{C_v t}{H_0^2} \right)$ is plotted in Fig. 5-8. It shows that 90 percent of consolidation can be obtained at $T_v = 0.36$ in finite consolidation prediction. It is about three times faster than small deformation prediction.

The main reason of faster rate of consolidation is due to the reduced drainage path resulting from the large vertical displacement. Reducing the path of drainage is, of course, accompanied with faster rate of consolidation.

The variation of total settlement with respect to time factor $T_v$ is shown in Fig. 5-9. It can be seen that the updated Lagrangian finite element algorithm predicts larger settlement at the beginning but end up with less settlement than small deformation. This prediction is in conflict with what Karim (1984) reported, i.e. finite consolidation predicts larger settlement during the process of consolidation but has good agreement with what Carter et. al.(1979) reported. The total settlement $(u_z)_f$ predicted by the current algorithm also agrees with the finite strain closed form solution solved by equation (5.4) $(= 0.5242 H_0)$ as shown in Fig. 5-9.

It should be noted that the permeability was assumed constant in this research. The permeability decreases as the void ratio decreases. However, error induced due to the assumption of constant permeability is neglected in the current research (sampling disturbances) since the penetration of the sampling tube is so fast that nearly no void ratio is likely to occur. A modification of permeability according to void ratio and effective stresses into the finite element procedure is recommended for the future research.

5.4 Cylindrical Cavity Expansion

The deformation mode of a cylindrical cavity expanding from an initial radius $r_0$ to its final radius $R_0$ in an incompressible material is shown in Fig. 5-10. It is postulated that the rate of expansion is constant such that

$$R_0 = r_0 + u_{r_0} = r_0 + k_0 t \quad (5.6)$$
Fig. 5-8 Degree of Consolidation vs. Time Factor
Fig. 5-9 Vertical Displacement vs. Time Factor
where \( u_{r0} \) is the radial displacement of the cavity and \( k_0 \) is the constant rate of expansion. The finite strain closed form solutions for (a) an ideal hypo-elastic material and (b) an ideal hypo elastic-perfect plastic material are derived in the Appendix D.

The comparison of the predictions between the closed form finite strain hypo-elastic solution and the updated Lagrangian finite element analyses is shown in Fig. 5-11. The finite element results were obtained with a mesh consisting of fifty 8-noded finite element. The match between the two techniques is almost perfect. It should be noted that the excess pore water pressure is negative at finite elastic cylindrical cavity expansion. The pattern of deformation is extension in the tangential direction and compression in the radial direction.

The predictions of the closed form finite strain solution of a hypo elastic-perfect plastic material under cylindrical cavity expansion are presented in Fig. 5-12. Again, the updated Lagrangian finite element results are very close to the closed form solution particularly at far field. However, there is some differences (1% to 10%) between these predictions near the cavity wall.

For the excess pore water pressures inside the plastic region, the finite strain closed form solution solved here is in good agreement with the prediction of Randolph and Worth (1978). However, Randolph and Worth (1978) assumed zero excess pore pressures in the elastic domain while the finite strain closed form solution predicts small negative pore pressures.
Fig. 5-10 Deformation Mode of Cylindrical Cavity Expansion
material properties:
- Elasticity
  \( E = 1.0 \times 10^4 \text{kPa} \)
  \( v = 0.3 \)
- permeability \( K_r = K_s = 1.0 \times 10^{-9} \text{m/sec} \)
- loading time = 0.1 sec

--- : analytical results
* : FEM results

Fig. 5-11 Cavity Expansion of An Elastic Material
material properties:
Mohr-Coulomb
$E_p = 1.0 \times 10^4 \text{kPa}$
$\nu = 0.3$
cohesion ($c$) = 50.0 kPa
angle of internal friction = 0.0
$K_r = K_s = 1.0 \times 10^{-9} \text{m/sec}$
loading time = 0.1 sec

Fig. 5-12 Cavity Expansion of A Mohr-Coulomb Material
Chapter 6 Experimental Study

The experimental study in this research was mainly directed to study the variations of stresses and pore water pressure at the soil-sampler interface. The study of stresses and pore water pressure at the center of sample was excluded because that the insertion of pore pressure measuring devices, for example, a hypodermic needle peizometer as used by Schjetne (1971), will cause disturbances to the core of the sample. In this chapter, a description of experimental setup for the investigation of the normal forces, shearing forces (friction forces) and pore water pressures at three different locations along the length of a thin walled sampling tube during sampling is presented. The test data obtained will be compared with the numerical results in the next chapter.

6.1 Instrumented Consolidation Tank

A consolidation tank (600 mm in diameter and 1400 mm high) shown in Fig. 6-1 was used to consolidate kaolin clay slurry under one dimensional \( K_0 \) consolidation. The inside perimeter of the tank was lined with a smooth porous plastic filter (1.6 mm thick) to allow radial drainage of pore water to occur. A 100 mm thick layer of gravel was placed at the bottom of the tank and covered with a 3.2 mm thick disc of porous plastic. The porous plastic disc was used to separate the soil and the gravel. The gravel was divided into two layers. The top layer was 40 mm thick with the gravel having an average grain size of 5 mm. The bottom layer was 60 mm thick and the average grain size of the gravel was 20 mm in diameter. Two drainage valves (V1 and V2) provided drainage of water from the tank to the laboratory drainage system.

The tank consisted of two sections. The bottom section (480 mm height) was split into two vertical sections and clamped together by angle side plates with a rubber gasket between the seams. The top section was 920 mm high and was bolted to the bottom section via flanges on each section.
Fig. 6-1 Instrumented Consolidation Tank
A circular steel plate (12.7 mm in thickness) with a diameter of 600 mm (equivalent to the internal diameter of the consolidation tank) was placed on top of the sample. Two holes were drilled on the plate to serve as either drainage ports during consolidation or for plate removal using threaded bolts when consolidation was completed. A porous plastic disc (3.2 mm) was placed between the top surface of sample and the steel plate.

The vertical pressure was applied via dead load. During consolidation, the water level was kept 100 mm above the steel plate to prevent the evaporation of water from the sample.

A rigid frame with a movable platform was connected to the top section of the consolidation tank via four 19 mm diameter columns. A motor was installed on the platform and was connected to a gear box. The gear box was used to transfer the rotary movement of the motor to vertical movement. The rate of vertical movement resulting from the motor/gear box mechanism was from 0.2 mm/sec to 20 mm/sec. A brass adapter with a diameter of 100 mm was fixed to the bottom of the gear box to connect the instrumented sampling tube (see section 6.3).

6.2 Instrumented Thin-Walled Sampling Tube

A thin-walled sampling tube of length of 460 mm was instrumented with three load transducers and three miniature pore water pressure transducers placed at three different locations along the length of the tubes (Figs 6-2 and 6-3). These transducers provided data on the distribution of shear stresses (friction), normal stresses and pore water pressures along the length of the inner wall of the tube. The transducers were housed in two opposing aerofoiled shaped wings on the sample tube and located at sufficient distance away from the cutting edge. These wings were greased to minimize external disturbance. We assumed that the wings had no significant influence on the quality of the soil sample recovered.
Fig. 6-2 Instrumented Sampling Tube
Fig. 6-3 Instrumentation of Sampling Tube (section DD)
The inner diameter \((B_i)\) of the sampling tube was 101 mm and its outer diameter \((B_o)\) was 105 mm. The thickness of the tube wall was 2 mm. The area ratio \(A_r\) was equal to 8\% where \(A_r\) is defined as
\[
A_r = \frac{B_o^2 - B_i^2}{B_i^2} \times 100\%
\] (6.1)

The cutting edge had a tip angle of 26.56 degree (Fig. 6-2).

### 6.3 Load Transducer

#### 6.3.1 Introduction

Load transducers capable of measuring normal and shear stresses simultaneously were developed to measure the friction and normal stresses at the sampling tube - soil interface. A typical load transducer is shown in Fig. 6-4. The load transducer was made from aluminium alloy. The overall dimensions of the load transducer were 25.4 mm (length) \(\times\) 5.1 mm (width) \(\times\) 15.5 mm (height) as shown in Fig. 6-4 (b). Two thin columns \((v_1\) and \(v_2\) in Fig. 6-4 (a)\) 5.1 mm in height and 0.25 mm in thickness were located at opposite side along the length of the load transducer. Two thin beams \((h_1\) and \(h_2\) in Fig. 6-4 (a)\) 5.1 mm in length and 0.25 mm in thickness were located at the top of the load transducer. Two bonded wire strain gages were glued to each column and each beam. These wire strain gages were linked together to form two Wheatstone Bridge circuits - one, a shear bridge circuit and the other, a normal bridge circuit. Under a constant input voltage \((5V)\), the changes in the output voltages resulting from the changes in resistances of each circuit were monitored. The normal and shear stresses were obtained from these output voltages based on the calibration described in section 6.3.2.

Aluminium caps (same material as the sampling tube) were glued to each load transducer as shown in Fig. 6-4 (a) and Fig. 6-6 (a). The cap which served as the contacting interface had a concave surface to match the inner surface of the sampling tube. A gap \((0.05\ mm)\) was kept between each load transducer and the wall of the sampling tube to ensure that beams and columns can compress freely.
Fig. 6-4 A typical Load Cell
6.3.2 Calibration of Load Transducer

The calibration arrangement used is shown in Fig. 6-5. A calibration block (87 mm x 87 mm x 9.5 mm) was fabricated with two channels (87 mm x 3 mm x 2 mm) in the center to route the transducer cables. The load transducer was mounted on the calibration block and fixed by two screws. The calibration block was then bolted to a steel frame. The load transducer cables (one for the normal circuit, the other for the shear circuit) were connected to a data acquisition system. A steel calibration cap with positioning holes at the center line and a hook at one end as shown in Fig. 6-6 (b) was placed on the top of the load transducer. The normal load was applied by means of dead weights on a hanger. A sharp tool steel spike on the under side of the horizontal bar of the hanger was used to center the point of the application of the load on the positioning hole on the calibration cap. The shear load was applied by attaching dead weights to a plate connected to a pulley which was mounted on the steel frame. A nylon string was used to link the plate to a hook on the calibration cap.

Assuming that the material of the load transducer behaves linearly elastic and the changes of the output voltages from each Wheatstone bridge circuit are proportional to each of the applied loads, the relation between the difference of output voltages and applied forces can be expressed as

\[
\begin{bmatrix}
\Delta V_1 \\
\Delta V_2
\end{bmatrix} =
\begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix}
\begin{bmatrix}
N \\
S
\end{bmatrix}
\]  

(6.2)

where \( \Delta V_1 \) is the difference in the output voltages of the normal bridge circuit, \( \Delta V_2 \) is the difference in the output voltages of the shear bridge circuit, \( N \) is the normal force, \( S \) is the shear force and \( a_{11}, a_{12}, a_{21}, a_{22} \) are calibration constants.

The calibration constants were obtained from the procedures below.

1. The transducer was loaded (or unloaded) by a vertical load applied incrementally at the center of the calibration cap. During each loading step or unloading step, the output voltages from each Wheatstone bridge were recorded.
Fig. 6-5 The Calibration System Arrangement
(a) load transducer cap

(b) calibration cap

Fig. 6-6 Transducer and Calibration cap
By plotting the output voltages from each Wheatstone bridge against the vertical load the constants $a_{11}$, $a_{21}$ were determined from the slopes of the curves. A typical normal force calibration curve of load transducer B is shown in Fig. 6-7. The constant $a_{11}$ is determined from the average slope of the calibration curve. The input voltage used in the calibration was 5 volt.

2. With a constant vertical load, the shear force was applied incrementally. By plotting the shear force vs. the output voltages from each Wheatstone bridge the constants $a_{12}$, $a_{22}$ were obtained from the slopes of the curves. A typical shear force calibration curve of load transducer B is shown in Fig. 6-8. The constant $a_{22}$ is determined from the average slope of the calibration curve.

A summary of the calibration constants for the three load transducers used in the instrumented sampling tube are shown in the Table 6-1. It can be seen that the diagonal constants ($a_{11}$ and $a_{22}$) are 5 to 20 times those of the off-diagonal constants ($a_{12}$ and $a_{21}$). This indicates that the stress measured, say normal stress, is almost independent of the change of the other stress, say shear stress.

**Table 6.1 Calibration Constants of Load Cells**

(unit=mV/V/kPa)

<table>
<thead>
<tr>
<th>Load Cell</th>
<th>$a_{11}$ x10^-3</th>
<th>$a_{12}$ x10^-3</th>
<th>$a_{21}$ x10^-3</th>
<th>$a_{22}$ x10^-3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Load Cell A</td>
<td>1.86</td>
<td>0.36</td>
<td>0.13</td>
<td>3.26</td>
</tr>
<tr>
<td>Load Cell B</td>
<td>2.82</td>
<td>0.46</td>
<td>0.21</td>
<td>2.85</td>
</tr>
<tr>
<td>Load Cell C</td>
<td>2.73</td>
<td>0.16</td>
<td>0.38</td>
<td>2.55</td>
</tr>
</tbody>
</table>
Calibration of Load Cell B
Normal Bridge

Fig. 6-7 Calibration of Load Cell B
Calibration of Load Cell B
Shear Bridge

Fig. 6-8 Calibration of Load Cell B
6.4 Pore Water Transducer

Three diaphragm pore water pressure transducers a, b and c (6.4 mm in diameter) were installed at three different locations opposite to the load cells along the length of the instrumented thin-walled sampling tube as shown in Fig. 6-2. I expected that the porewater pressure at a (Fig. 6-2) would be large and therefore used a porewater pressure transducer with an allowable pressure of 689 kPa. At locations b and c the porewater pressure transducers were of lower range pressure (35 kPa).

6.5 Data Acquisition System

A total of 10 channels - 3 for shear stresses, 3 for normal stresses, 3 for porewater pressures and 1 for the power supply to the load transducers - of data were read continuously during sampling tube penetration and extraction. A HP3497A data acquisition controller coupled to a HP9816S computer served as the main data acquisition system.

6.6 Sample Preparation

Kaolin (liquid limit 63% and plastic limit 42%) was mixed with water up to twice its liquid limit, stirred thoroughly and then poured into the consolidation tank. In the first stage, the slurry was poured to a height of 1300 mm and then allowed to consolidate under a dead weight of 91 kg for two days. After then, the steel plate was removed and the tank was replenished with a fresh quantity of Kaolin slurry to a height of 1400 mm and again allowed to consolidate under a vertical load of 91 kg. Additional weights were added approximately every three days to bring the final consolidation load to 546 kg (19 kPa) within two weeks. The loading was kept on the sample for three months.
6.7 Penetration

The final loading (546 kg) was removed in 15 minutes. The water, originally kept on the top of the steel plate, was allowed to drain out. Two hooks were fixed to the holes on the steel cover plate. The steel plate was pulled upward by connecting a rope through the two hooks to the brass adapter on the movable platform and reversing the motion of the motor. The porous plastic disc on the soil was carefully taken off. The final height of the soil was 1108 mm.

The instrumented sampling tube was connected to the brass adapter as shown in Fig. 6-1. The cutting edge of the sampling tube was placed flush with the top of the soil and pushed down at rate of penetration of 2 mm/sec to a total penetration depth of 400 mm. During the penetration, the porewater pressure and the stresses along the wall of the sampling tube were continuously recorded by the data acquisition system. The total penetration time spent was 204 seconds. The test data recorded during penetration are presented in the next chapter.
Chapter 7 Numerical Study

In this chapter, an algorithm for the numerical simulation of sampling tube penetration is presented. Based on this algorithm, the details of the mechanical disturbances due to a frictional sampler penetrating a soft clay are presented. Subsequently, the effects of excavation, soil-sampler friction, wall thickness of sampling tube, rate of penetration, in situ condition and methods of sampling on the sampling disturbances are investigated. The numerical results of excess pore water pressures and the shear forces along the interface of the soil-sampler are compared with the experimental results described in chapter 6.

7.1 Finite Element Discretization

A typical finite element mesh used in the investigation is shown in Fig. 7-1. In the figure, Z and R are the vertical axis and the radial axis respectively. The diameter of the sampling tube is B (100 mm). The size of the mesh is 3B (300 mm) \times 3.5B (350 mm). For a typical sampling tube with wall thickness \( t = 2 \text{mm} \), the size of mesh is equal to 150\( t \times 175t \). The boundary HI is fixed in the R direction to simulate the wall of the casing which is usually used to keep the borehole open in soft clays. The boundary GH is assumed fixed in the Z direction due to the effect of the overburden pressure. It is assumed that the boundaries EF and FG are far away from the sampling tube so that the excess pore water pressures generated during sampling penetration are equal to zero at these boundaries. Drainage of water is allowed along boundary IJ except during the simulation of the deep penetration.

For simplicity, only the results inside the region a-b-c-d (marked by dashed line in Fig. 7-1) are presented. To make the interpretation of results easier, the region a-b-c-d is sub-divided into eight regions as shown in the Fig. 7-2. Regions 1, 2 and 3 are portions inside the sampling tube; regions 5 and 6 represent regions outside the sampling tube; region 4 denotes the portion in front of the sampling tube; regions 7 and 8 represent regions away from the sampling tube.
Fig. 7-1 Finite Element Mesh
Fig. 7-2 Subregions of Mesh
7.2 Simulation of Sampling Tube Penetration

The simulation of the sampling tube penetration involves a proper representation of the geometry of the sampling tube, the soil-sampler interface conditions and the maintenances of equilibrium at new boundaries and of penetration trajectory. The details of the simulation used in this investigation are presented in the following:

(a) Geometry

The geometry of a typical sampling tube are shown in the Fig. 7-3. The thickness of the sampling tube is 2 mm. The cutting edge is 4 mm high with a tip angle of 26.5°. The numerical simulation of tube penetration was conducted by a nodal separation process as follows. Suppose we wish to penetrate the first soil layer of thickness 4 mm with the tip of the sampling tube cutting edge at node A as shown in Fig. 7-3 (a). We approximate this penetration process by splitting node A such that a new node A' is created. This new node A' is then displaced incrementally to follow the profile of the sampling tube cutting edge as shown in Fig. 7-3(a). The old node A is fixed in the radial direction and is allowed to slide along the internal wall of the sampling tube. The penetration of the next layer is simulated in a similar way by splitting node B (midside node of element 1) creating a new node B' which is then displaced incrementally to match the geometric configuration of the sampler as shown in Fig. 7-3 (b). Once nodes such as A' and B' have displaced an amount equal to the thickness of the sampling tube, no further displacement is imposed and these nodes are forced to slide along the external wall (boundary A'B') of the sampling tube.

(b) Equilibrium at the New Boundary

Along the new boundaries (for example, AB and BA'), either tractions or displacement constraints must be specified to maintain the system in an equilibrium condition. Out-of-balance forces from equilibrium checks at the current step of penetration are imposed at nodes such as A and A' to act as external forces to equilibrate the system.
Fig. 7-3 Simulation of Tube Penetration
(c) Trajectory

During the penetration process, the nodal points which were initially along the penetration route from A to E (Figs. 7-3 (a) and (b)), could displace away from this route before the tip of sampling tube reaches them. The trajectory of the penetration is maintained by moving these wayward nodes back to the penetration path after each increment or iteration. In doing so, a small amount of unbalanced forces is created which is then redistributed by iteration.

(d) Interface

A group of thin-layer elements (Desai et. al., 1984) with the same material property as the parent soil was specified at the inside of the sampler along the intended penetration path. The thin-layer elements were re-characterized with the interface material property each time a layer was penetrated. Thin-layer interface elements at elements below the sampling tip retained the properties of the parent soil. Slip occurred when the shear strength exceeded the failure shear strength as determined from the Mohr-Coulomb failure criterion. Stick and slip were handled in the algorithm following the suggestions of Desai et. al. (1984). The effect of friction along the outside wall of sampling tube was assumed to be a minor factor in the disturbances to the sample inside the tube and neither thin-layer elements nor friction was imposed at outside soil-sampler interface.

7.3 Soil Properties

The modified Cam-clay (Roscoe and Burland, 1968) was adopted as the soil model for this study. The necessary parameters and selected variables required for the analysis are given below. Among these data, only $\lambda$ and $\kappa$ are taken from the triaxial test results using the same soil (Kaolin) as were used in the experimental study described in chapter 6. Others are fictitious.

$\lambda$ (the slope of the one dimensional consolidation line) = 0.20
$\kappa$ (the slope of the unloading line) = 0.04
$\lambda M$ (ratio of mean effective stress and deviatoric stress at failure) = 1.0
$e_0$ (in situ void ratio) = 1.35
\[ p_0 \text{ (in situ mean effective stress)} = 141.3 \text{ kPa} \]
\[ q_0 \text{ (in situ deviatoric stress)} = 88 \text{ kPa} \]
\[ p_m = \frac{p^2 + q^2}{p} = 196.1 \text{ kPa} \]
\[ (p_m)_o \text{ (initial past maximum mean stress)} = 196.1 \text{ kPa} \]
\[ \sigma_{zz} = 200 \text{ kPa} \]
\[ \sigma_{rr} = \sigma_{\theta \theta} \text{ (horizontal stresses)} = 112 \text{ kPa} \]
\[ K_0 \text{ (coefficient of lateral earth pressure at rest)} = 0.56 \]
\[ R_p \text{ (over consolidation ratio)} = \frac{(p_m)_o}{p_m} = 1 \]
\[ \mu \text{ (Poisson’s ratio)} = 0.3 \]
\[ k_r = k_z \text{ (permeability)} = 1.0 \times 10^{-6} \text{ mm/sec} \]

A typical prediction of the undrained triaxial compression behavior of the soil is shown in Figs 7-4 and 7-5. From Fig. 7-4, it is seen that the soil fails at \( p_f = q_f = 105.7 \text{ kPa} \) at a vertical strain of nearly 10%. It is also seen that the mean effective stress decreases \( (p > p_0) \) while the deviatoric stress increases \( (q < q_0) \) during the process of undrained loading. The hardening parameter \((p_m)_f\) at failure equals 211.3 kPa. At a strain of 2.8\%, the ratio of \( \frac{q}{p} \) reaches 0.95. The soil is classified as "approaching critical state" if the ratio \( \frac{q}{p} \) is larger than 0.95. From a practical point of view, soils having \( \frac{q}{p} \) larger than 0.95 should be considered as "failed". The undrained shear strength \( (c_u) \) of the soil is 52.85 kPa \( (= \frac{1}{2} q_f) \). However, \( c_u \) is assumed to be 50 kPa which is equal to 95% of the determined value (52.85 kPa). The results in Fig. 7-5 show that the total changes in stresses at failure are \( \Delta(\sigma_{zz})_f = -23.9 \text{ kPa} \), \( \Delta(\sigma_{rr})_f = \Delta(\sigma_{\theta \theta})_f = -41.5 \text{ kPa} \) and \( (\Delta u_w)_f = 41.5 \text{ kPa} \).

The interface element properties are assumed as follows:
\[ c_a \text{ (adhesion)} = 25 \text{ kPa} \]
\[ \text{angle of internal friction} = 30^\circ \]
Fig. 7-4 Variation of Stresses (p and q) of A Clay Soil
Fig. 7-5 Variation of Stresses, Variations of A Clay Soil Under Undrained Compression

- Excess pore water pressure: \((\Delta u_w)_f = 41.5\)
- Change of horizontal stresses
- Change of vertical stress
- \(\Delta \sigma_{rr} = \Delta \sigma_{\theta\theta}\)

\[
\frac{\Delta l}{l_0} \times 100\%
\]
7.4 Disturbances due To Excavation

In order to obtain a soil sample at a particular depth, it is often necessary to drill a bore hole down to the necessary level. For a soft soil, a casing is used to keep the bore hole from collapsing. This is simulated by fixing the radial displacement of the boundary HI (Fig. 7-1). A fictitious overburden pressure was applied on the boundary IJ initially. The process of excavation is simulated by removing this overburden pressure in 100 seconds.

For a soil with the modified Cam-clay properties described in section 7.3, the contour of normalized excess pore water pressures generated after excavation is plotted in Fig. 7-6. Except for the region near the right side of casing, the soil sample acquires significant negative pore water pressures as a result of the excavation. The maximum negative excess pore water pressure generated ( \( \Delta u_w > 0.8 p_0 \) ) is located in region 1 and decreases radially towards the far field (region 8).

Figure 7-7 shows the contours of deviatoric stress \( q \) normalized with respect to the initial mean effective stress. In this figure, the in situ stress ratio \( \frac{p_s}{p_0} \) is 0.62. The top of the soil sample reaches a zero effective stress condition (quick condition) and a small region (inside region 5) of the soil under the casing reaches critical state. The maximum heave predicted from this numerical simulation is 1.49 mm.

The deviatoric stresses of the soil mass inside the sampling region (regions 1, 2 and 3) decreases, as shown in Fig. 7-7, indicating unloading, as expected, during the excavation process. However, the deviatoric stresses at the right upper corner (region 5) increases ( \( \frac{q}{p_0} \) is larger than 0.62). Thus, soil elements near to the casing have approached a failure state and the elements just under the casing bottom edge have failed. In parts of regions 1 and 2, the horizontal stresses became larger than the vertical stresses after excavation. The area at the left of the broken curve XY in Fig. 7-7 denotes where the horizontal stresses (radial stress and tangential stress) are greater than the vertical stress. It is concluded that, due to the relief of vertical stress, the direction of maximum principal stresses changes from the vertical to the horizontal plane and the sample becomes overconsolidated.
Fig. 7-6 Contours of Normalized Excess Pore Water Pressure After Excavation
Fig. 7-7 Contours of Normalized Deviatoric Stress After Excavation
The soil elements along the line XY have equal vertical and horizontal stresses but these elements are not in an isotropic stress state due to the presence of shearing stresses. Ladd and Lambe (1963) have shown that the unloading stress path intersects the mean stress axis at point P (Fig. 2-2) after perfect sampling. From the numerical results obtained here, it appears that the observation made by Ladd and Lambe (1963) is correct only for laboratory prepared samples and not for actual tube sampling in the field.

7.5 Disturbances Due to Tube Penetration

In order to investigate the disturbances caused only by tube penetration, the effects of excavation are neglected. This is carried out by imposing an overburden pressure equal to the in situ vertical stress on the surface IJ (Fig. 7-1). As a result, the tube penetration starts from an undisturbed state. In the field, this means that an external pressure (water or air) is applied to maintain the in situ vertical pressure at the bottom of the bore hole during excavation. Such a tube penetration is termed ideal penetration in this research.

A total of 12 types of numerical analysis was conducted. Details of the differences amongst these analyses are given in Table 7-1. The results presented were obtained with 25 steps of penetration with each step penetrating 4 mm. In these analyses, the total depth of penetration was 1.0 B. The type A analysis is used as the bases of comparison with the other types.

At the outset, a hypothetical pattern of deformation is proposed. In conjunction with this hypothesis, three factors that affect the deformation of the soils during the sampling penetration are introduced. The mechanical disturbances of a sampling tube penetrating a clay are then described in terms of strain fields, stresses fields, excess pore water pressures, displacement fields and failure zones. In this section, all the numerical results are based on the type A analysis (Table 7.1).
## Table 7.1 Summary of Numerical Simulation

<table>
<thead>
<tr>
<th>Type</th>
<th>Analysis</th>
<th>Interface Stress</th>
<th>Penetration Rate</th>
<th>( R_p ) ((O.C.R))</th>
<th>Tube Thickness</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>ideal penetration</td>
<td>( \tau = 1.0 , c_u )</td>
<td>0.4 mm/sec</td>
<td>1.0</td>
<td>2.0 mm</td>
</tr>
<tr>
<td>B</td>
<td>ideal penetration</td>
<td>( \tau = 0.0 )</td>
<td>0.4 mm/sec</td>
<td>1.0</td>
<td>2.0 mm</td>
</tr>
<tr>
<td>C</td>
<td>ideal penetration</td>
<td>( \tau = 1.0 , c_u )</td>
<td>0.4 mm/sec</td>
<td>1.0</td>
<td>2.5 mm</td>
</tr>
<tr>
<td>D</td>
<td>ideal penetration</td>
<td>( \tau = 1.0 , c_u )</td>
<td>0.4 mm/sec</td>
<td>1.0</td>
<td>3.0 mm</td>
</tr>
<tr>
<td>E</td>
<td>ideal penetration</td>
<td>( \tau = 1.0 , c_u )</td>
<td>4.0 mm/sec</td>
<td>1.0</td>
<td>2.0 mm</td>
</tr>
<tr>
<td>F</td>
<td>ideal penetration</td>
<td>( \tau = 1.0 , c_u )</td>
<td>40 mm/sec</td>
<td>1.0</td>
<td>2.0 mm</td>
</tr>
<tr>
<td>G</td>
<td>ideal penetration (isotropic ( K_0 = 1 ))</td>
<td>( \tau = 1.0 , c_u )</td>
<td>0.4 mm/sec</td>
<td>1.0</td>
<td>2.0 mm</td>
</tr>
<tr>
<td>H</td>
<td>ideal penetration (( K_0 = 2 ))</td>
<td>( \tau = 1.0 , c_u )</td>
<td>0.4 mm/sec</td>
<td>1.0</td>
<td>2.0 mm</td>
</tr>
<tr>
<td>I</td>
<td>ideal penetration (piston sampler)</td>
<td>( \tau = 1.0 , c_u )</td>
<td>0.4 mm/sec</td>
<td>1.0</td>
<td>2.0 mm</td>
</tr>
<tr>
<td>J</td>
<td>ideal penetration (piston sampler)</td>
<td>( \tau = 1.0 , c_u )</td>
<td>4 mm/sec</td>
<td>1.0</td>
<td>2.0 mm</td>
</tr>
<tr>
<td>K</td>
<td>exca + pene. (open-drive sampler)</td>
<td>( \tau = 1.0 , c_u )</td>
<td>4 mm/sec</td>
<td>1.0</td>
<td>2.0 mm</td>
</tr>
<tr>
<td>L</td>
<td>exca + pene. (piston sampler)</td>
<td>( \tau = 1.0 , c_u )</td>
<td>4 mm/sec</td>
<td>1.0</td>
<td>2.0 mm</td>
</tr>
</tbody>
</table>
7.5.1 Deformation Pattern

Consider a sampling tube in the soil undergoes an increment of deformation as shown in Fig. 7-8 (a). The penetration increment of the sampling tube will displace the soil and cause the soil to "flow" out radially. The movement of the soil will be termed deformation flow. The term "deformation flow" is used to distinguish the incremental finite element approach adopted here from the fluid approach used by Baligh (1985). In the incremental finite element approach, the sampler penetrates into the ground to deform the soil; the soil is displaced, not flowed. However, in the fluids approach, only the relative motion between the sampler and the soil is considered; the soil is considered to be in a state of "flowing". If the sampling tube is penetrating very fast, the soil, together with the water, will behave virtually as an incompressible material. With the assumption of incompressibility, Baligh (1985) solved the approximate solutions of a rounded-end sampler by using a ring source to model the deformation flow. By moving this ring source downwards, the total strains can be obtained by integrating the incremental strains caused by the displaced ring source (Baligh, 1985).

The rounded-end sampler assumed by Baligh (1985) is impractical. The geometry of a practical sampler is shown in Fig. 2-6. The realistic flow lines of practical samplers are unclear. However, the hypothetical flow lines of a practical sampler are shown in Fig. 7-8 (a) with dash lines. The arrows ahead of the flow lines indicate the direction of deformation flow. The flow lines inside the sampler near the tip initially radiate inwards and then bend towards the vertical direction. No flow line crosses the centerline due to axisymmetry. If the incremental deformation of the tip of the sampler is considered as a source of deformation flow and the deformed soils "flow" out radially as the hypothetical flow lines, it is then expected that: (1) large increment of compressive strains are induced near the tip of the sampling tube due to the displacement of the tip of the sampler; (2) these compressive strains spread out and cause the soil to flow out according to the hypothetical flow lines; (3) at the region along the inside wall of the sampler, the radial stresses is reduced before the flow lines bend (element E in Fig. 7-8 (a)) and is increased thereafter (element D in Fig. 7-8 (a)); (4) the elements
Fig. 7-8 The Effects of Deformation Flow and Cohesion
Fig. 7-9 The Effect of Inside Wall Friction
along the centerline (elements A and C) is compressed vertically and the element B is extended vertically.

The conditions of axisymmetry and incompressibility demand that the strain state of the element along the centerline is $\varepsilon_{rr} = \varepsilon_{\theta\theta} = -\frac{1}{2} \varepsilon_{zz}$ and $\varepsilon_{rr} = 0$. This is proved in the Appendix E in two cases - vertical compression and radial compression by the theory of finite strain. Thus, if the rate of penetration is fast, it is expected that the strains in elements A and C will be compressive vertically and extensive radially. The element B would be extended vertically and compressed radially. However, in both cases, the vertical strains are always twice the radial strains.

The hypothetical pattern of deformation described above is expected to be predominant during sampling penetration, especially at a fast rate of penetration. There are other three factors that affect the deformation of soil during the penetration. These are:

1. cohesion: The effect of deformation flow described above is to push the soil away from the tip of the sampler. On the other hand, the effect of the cohesion of the soil is to keep the soil particles together. Thus, the flows caused by the cohesion (shown in Fig. 7-8 (b)) oppose the direction of the deformation flow. This effect is to drag the soil inside the sampler down and to drag the soil in region 4 up. The induced increments of strains will be in extension for elements A and C and, will be compressive for element B (Fig. 7-8 (b)). The effect of cohesion flow is expected to dominate when the rate of penetration is low.

2. wall friction: The effect of inside wall friction is shown schematically in Fig. 7-9. The friction is acting downwards along the sample perimeter up to the tip of the sampler. The friction forces are transferred to the inside region of the sample and results in vertical compression and distortion as shown in the Fig. 7-9 (shown with small squares and arrows). The distorted soil at the top of sample is shown by dashed lines. The curvature of distorted layer is convex. However, for the soil ahead of the sampler, there is no friction acting along the perimeter area but with a force $F$ acting downwards which is equal to the total wall friction developed along the wall of the sampler. As
a result, the soil is deformed into a concave curvature as shown in Fig. 7-9. Thus, the effect of inside wall friction creates vertical compressive strains but induce different distortion shapes on the top and bottom of the sample. It should be noted that, if the sampling tube is penetrating very fast, the soil will behave virtually like a fluid. In this case, the effect of inside wall friction becomes a minor factor.

3. boundary effect: The boundary at the top of the sample (boundary IJ in Fig. 7-1) affects the results in two ways, that is, the relaxation of the stresses and the direction of flow lines. If the boundary IJ is free, it will release the excess stresses at the boundary. However, if it is fixed, the excess stresses will not be released. The flow lines inside the sampler are directed upwards if the boundary IJ but are directed downwards if the boundary is fixed. The downward deformation flow will increase the compression in region 4. The boundary effect is expected to be a major factor between the degree of disturbances caused by the open-drive sampler and the piston sampler.

7.5.2 Strain Fields

The results of the total vertical strain $\varepsilon_{zz}$ for type A analysis are presented in Fig. 7-10. The strains shown are calculated as $\varepsilon_{ij} = \sum d\varepsilon_{ij}$, where $\sum$ denotes summation of increments. It should be noted that the strain measure used here is close to natural strain. It may not have significant physical meaning if large rotation is involved, especially in the region near the cutting edge of the sampler. However, for the regions inside the sampler, the strains are small and have no significant rotation effect. Thus, the strain measures we adopted still carry good physical meaning for the interpretation of sampling disturbances, especially at the regions along the centerline. There is no intention to use these strain measures to interprete the physical meaning at the regions near the outside wall of the sampler.

Examining Fig. 7-10 from region 4 (bottom) to region 1 (top) along the centerline, it is observed that the soil is subjected to different strain states during
penetration. The soil is in a compressive state \((\varepsilon_{zz} > 0.0)\) in front of the sampling tube (region 4) but changes to a tensile strain state \((\varepsilon_{zz} < 0.0)\) after it enters the sampling tube (regions 1, 2 and 3). The vertical strains approach zero at the region near the top. The path followed by the vertical strain is compression-extension-recompression. These three distinct stages are consistent with the prediction of deformation flow described in section 7.5.1 which indicates compression at element C, tension at element B and recompression at element A. The boundary IJ (Fig. 7-1) is unrestricted in the vertical direction, allowing relaxation of the excess stresses in region 1. As a result, the net effect is an increase in the rate of recompression near the top.

The three distinct stages of vertical strain observed along the centerline were first reported by Baligh (1985) and his co-workers (Chin, 1986; Baligh et. al. 1987). Herein, Baligh's solution shall be termed SPM (Strain Path Method) (Fig. 7-11). The identification of the three distinct strain stages is an important contribution to the study of the disturbances of tubed samples. However, the SPM considered only the first pattern of deformation described in the last section with a non-practical (rounded-end) sampling tip. The other three factors were neglected.

A comparison between type A analysis and the SPM solution along the center line is shown in Fig. 7-12. The curve a-b-c-d represents the results of type A analysis while the curve a'-b'-c'-d' represents the SPM solution. Strain path a-b (or a' - b') is the vertical compression state; path b-c (or b' - c') is the extension state and path c-d (or c' - d') is the recompression state. The SPM solution gives a symmetric strain path with respect to the axis \(Z = 0\), where \(Z\) and \(B\) are defined in Fig. 7-12, while type A analysis gives a non-symmetric strain path. The maximum vertical strain predicted by the SPM solution (Chin, 1986) is 0.77 % for a sampling tube with \(\frac{t}{B} = \frac{1}{50}\), where \(t\) is the wall thickness. Type A analysis predicts a maximum vertical compressive strain of 0.95 % in region 4 and a maximum vertical tensile strain of -0.3 % in region 2.

In the SPM solution a ring source (Chin, 1986) is used in which the streamlines radiate evenly in all directions resulting in symmetric streamlines. Thus, the
strain history is anti-symmetric. The geometry of the sampler used here is a practical one and has a non-symmetric geometry (like a "half cone" as shown in Fig. 2-6 and Fig. 7-3). The purpose of this particular shape is to minimize the flow effect towards the regions inside the sampler. As a result, the flow lines are directed mainly towards region 8. Thus, the non-symmetric shape of strain path a-b-c-d (type A) in Fig. 7-12 is expected.

Other minor factors that contribute to the non-symmetric strain history include the inside wall friction and the boundary effect. The inside wall friction, which is acting downwards, tends to increase the vertical compressive strain. This is shown schematically in Fig. 7-9. Due to this friction, the vertical strains are developed throughout the regions 1, 2, 3, and 4. As a result, the compressive strains in region 4 will increase and the tensile strains in regions 2 and 3 will decrease.

It should be noted that point d (type A analysis) in Fig. 7-12 is nearly zero but d' is a finite value. The SPM solution assumes deep penetration such that the boundary effect is negligible and the point d' converges to zero at far field boundary as shown in Fig. 7-11. But, due to the boundary effect, the rate of recompression in type A analysis is increased at shallow penetration.

The vertical strain state changes not just vertically but also horizontally. Consider the region 2 as an example. The vertical strains change horizontally from -0.20 % to 0.0 % and finally to a positive value near the sampling tube (Fig. 7-10). The positive value of vertical strain is due to inside wall friction effect. If the sampler were to be frictionless, the predicted vertical strain would be negative (tension) as can be seen in Fig. 7-11 (b) (SPM solution). The compressive vertical strains near the sampling tube at regions 1, 2, and 3 reveal the importance of including the wall friction in simulating sampling tube penetration. Detailed comparisons of a frictionless sampler and a frictional sampler are given in section 7.6.

The results of total radial strains ($\epsilon_{rr}$) for type A analysis are shown in the Fig. 7-13. As seen from Fig. 7-13, the radial strains decrease from the center of the sampler to the sampler wall at region 1 but increases at region 3. This implies
that region 1 is expanding radially while region 3 is compressing radially. These also match the pattern of deformation flow described in section 7-5 (a).

It is observed, although it is not strictly applied, that the radial strains (Fig. 7-13) inside the sampler are approximately one half the vertical strains (Fig. 7-10) with an opposite sign. This is especially true along the centerline. On the contrary, at regions outside the sampler (region7, for example), the radial strains are approximately twice the vertical strains but with an opposite sign. Thus, inside the sampler, the vertical strains dominate while outside the sampler the radial strains predominate. The strains are larger than 10% near the tip of the sampler which suggest that a finite strain approach as used in this research is appropriate.

The soil is modeled as an elasto-plastic material in this investigation. The plastic deformation is irrecoverable. As mentioned before, the soil sample is subjected to compression-extension-recompression during the sampling penetration. The first loading stage (compression) in region 4 would have caused plastic deformation and would have changed some of the virgin soil properties before the soil enters the second loading stage (extension). The second and the third loading stages are elastic unloading and reloading. According to the modified Cam-clay model, the elastic unloading or reloading changes the stresses and strains but does not change the irrecoverable material properties (for example, plastic strain). However, the plastic deformation does induce irrecoverable changes in the material properties. These imply that the major irrecoverable disturbances are developed before the soil enters the sampler, not after. The degree of sampling disturbances then depends largely on the amount of vertical loading ahead of the sampler. Besides, the soil would fail before it enters the sampler if the predicted maximum vertical strain of the soil in region 4 is larger than the failure strain of the soil. These suggest that the magnitude of the vertical strain in region 4 can serve as an index of degree of sampling disturbances.
Fig. 7-10 Contours of Vertical Strain (type A)
(a) — Straining History at Centerline of Simple Samplers

(b) — Deviatoric Strain Contours during Undrained Simple Sampler Penetration in Saturated Claye (B/1 = 40, ICR = 1%): (a) Radial Strain, ; (b) Tangential Strain, ; (c) Meridional Shear Strain, ; and (d) Vertical Strain, 

Fig. 7-11 Strain Path Method (After Baligh et. al. (1987) )
Fig. 7-12 Straining History at Centerline of the Sampler
Fig. 7-13 Contours of Radial Strain (type A)

*Note: The image contains a diagram illustrating contours of radial strain for a specific type of casing and sampling tube configuration.*
The results of shear strains ($\epsilon_{rz}$) for type A analysis are plotted in Fig. 7-14. Clockwise shear strains are defined as positive. The parallelograms with arrows in Fig. 7-14 indicate the direction of distortion. The shear strains change from being negative in regions 3 and 4 to being positive in regions 1 and 2. This indicates that the curvatures of the soil layer in regions 1 and 2 are convex but they are concave in regions 3 and 4.

### 7.5.3 Normalized Stresses

The results of mean effective stresses and deviatoric stresses for type A analysis are plotted in Figs 7-15 and 7-16 respectively. These stresses are normalized with respect to the initial mean effective stress $p_0$. From Figs 7-15 and 7-16, it is observed that the mean effective stresses decrease about 13 % and the deviatoric stresses increase about 8 % at region 4. This means that the sample is subjected to undrained compression before entering the sampler. Inside the sampling tube (regions 1, 2 and 3), the deviatoric stresses decrease and yield a minimum deviatoric stress ratio ($\frac{\sigma}{p_0}$) of 0.4 at region 3. After then, the deviatoric stresses increase again and reach a value close to the in situ value ($\frac{\sigma}{p_0}$=0.62). The increase of deviatoric stresses denotes that the soil is subjected to compressive loading. On the other hand, the decrease of deviatoric stresses denotes extension. Thus, after entering the sampling tube, the soil is subjected to undrained extension and followed by undrained recompression. These numerical results are consistent with the three distinct vertical strain stages described in section 7.5.2.

An examination of Figs 7-15 and 7-16, it shows that the mean effective stresses and deviatoric stresses increase horizontally from the centerline to the inside wall of the sampler. High values of deviatoric stresses near the sampler are as expected. The soil-sampler friction is believed to be the main source of this phenomenon.

The undrained compression effect on the soil in region 4 will lead to an expanded the yield surface and an increase of the hardening parameter $p_m$. According to the modified Cam-clay model, the change of the hardening parameter
Fig. 7-14 Contours of Shear Strain (type A)
$p_m$ is not recoverable. This also means that the virgin properties of a undisturbed sample cannot be recovered from a disturbed tube sample. Thus, the difference between the initial $(p_m)_0$ and the current $p_m$ should serve as an index of sampling disturbances.

Schematic stress paths of the disturbed samples are given in Fig. 7-17. In Fig. 7-17, the point A denotes the in situ state with associative hardening parameter $(p_m)_0$. The in situ stress path to undrained failure is A-B-E and the in situ deviatoric stress at failure is $(q_f)_E$. The point B denotes the intermediate stress state of the soil which has been subjected to undrained loading in region 4 before it enters the sampling tube and $(p_m)_B$ is the associative hardening parameter. After point B, the soil enters the second loading stage, that is, elastic unloading. The point C denotes the final state of the soil sample prior to laboratory testing. If the sample is reconsolidated to the in situ stress state (from point C to point A) and then sheared, it will follow the stress path C-A-D-F and will fail at a deviatoric stress $(q_f)_F$. The value of $(q_f)_F$ is larger than $(q_f)_E$. The reason for this is that the reconsolidated soil has a hardening parameter $(p_m)_B$ which is larger than the in situ value $(p_m)_0$.

Similarly, for an overconsolidated sample at point G, the virgin undrained compression stress path is G-H-J while the reconsolidated sample follows the stress path G-H-I-K-G-L-M. This results in a deviatoric stress at failure $(q_f)_J$ (in situ value) which is larger than $(q_f)_M$ (reconsolidated value). Thus, if a disturbed sample was reconsolidated back to the in situ stress condition and is sheared subsequently, the shear strength of the disturbed sample should reach a higher value than the in situ shear strength for normally consolidated or lightly consolidated clay but reach at a lower value for heavily overconsolidated clay.

The first observation, $(q_f)_F > (q_f)_E$ for normally and lightly consolidated clay, had been proved experimentally. For example, Baligh et. al. (1987) conducted tests in which the samples prepared from slurry were subjected to a strain history as shown in Fig. 7-11 and followed by a release of the confining pressure and then reconsolidation back to the in situ $K_0$ condition. This was called ISP (Ideal Sampling Approach) by Baligh et. al. (1987). Baligh et. al. (1987) found
that, for a specific sample, the undrained shear strength of the reconsolidated sample exhibited a slightly higher peak strength than the "undisturbed" normally consolidated soil (prepared from slurry), but a much higher strain at peak. Similar results were also reported by Atkinson and Kubba (1981) by comparing tubed samples with "undisturbed" samples. They found that the tubed samples had larger undrained shear strengths than undisturbed samples.

Karim (1984) predicted that the deviatoric stresses decrease from region 3 gradually to region 1. No re-increase of the deviatoric stresses was found. This implies no recompression state in regions 1 and 2. Karim (1984) did not provide the solution for the strain fields so that I cannot comment in detail on his work. However, based on the solutions of strain fields (type A and SPM analysis), the deviatoric stresses should increase again to match the recompression state. Kirkpatric and Khan (1984) claimed that the reconsolidation of a sample back to its in situ $K_0$ condition and then shearing the sample will result in $(q_f)_F = (q_f)_E$. However, their results were based on the comparisons of laboratory prepared "undisturbed samples" and disturbed samples. The disturbed sample was simulated by unloading the confining stresses and then reconsolidating the sample. This process involved no strain hardening, that is, no mechanical disturbances caused by the sample tube penetration. If the modified Cam-clay model applies, then unloading is elastic and the material should return to the in situ condition after reconsolidation. This is why Kirkpatric and Khan (1984) obtained the approximate in situ undrained shear strength from a disturbed sample. This is true only for the laboratory prepared samples that had never been subjected to any vertical loading (due to sampling disturbances) prior to laboratory test. It cannot apply to field sampling.

The method of SHANSEP (Ladd and Foote, 1974) method, whereby a sample is reconsolidated to about twice the in situ stress to overcome the sampling disturbances is considered to be a rational method to obtain the in situ undrained shear strength. However, the reconsolidation pressure must be sufficient large to bring the sample to pass the new created yield surface (yield surface with $(p_m)_B$) due to sample tube penetration. The value of $(p_m)_B$ can serve as a criteria to determine the minimum recompression pressure in the SHANSEP method.
Fig. 7-15 Contours of Normalized Mean Effective Stress (type A)
Fig. 7-16 Contours of Normalized Deviatoric Stress (type A)
Fig. 7-17 Stress Paths of Disturbed Samples
7.5.4 Normalized Excess Pore Pressures

The results of normalized excess porewater pressures at a penetration depth equal to the diameter of sampling tube B (= 50t) are plotted in Fig. 7-18. In general, positive porewater pressures are generated during the penetration of the sampling tube. These positive excess porewater pressures are generated near the tip of the sampling tube (region 6) and tend to dissipate from regions outside of the sampling tube to the top surface of the sample inside the tube. Thus, the excess porewater pressures decrease gradually from the bottom inside the sampling tube to the top of the sample (boundary IJ).

7.5.5 Displacement Fields

Displacement fields in regions 1, 2, 3 and 4 from type A analysis are shown in Fig. 7-19. The arrows represent the total displacements from the beginning to the end of the sampling penetration. From Fig. 7-19, it is evident that the total displacement inside the sampler are downwards indicating that the effects of the cohesion of the soil and the inside wall friction predominate at low rate penetration (type A). For a fast rate of penetration (types E and F analyses), the movements are unlike type A analysis as explained later in the section 7.6.

The downward displacements (Fig. 7-19) along the centerline increase from region 4 to region 3 and reach a maximum vertical displacement 0.49 mm at region 3. After then, the displacement starts to decrease from region 3 to region 1. The deflection of soil layers is convex in regions 1 and 2 while the deflection is concave in regions 3 and 4. The convex curves in regions 1 and 2 are expected since the wall friction drags the soil near the sampler. The concave curves in regions 3 and 4 are due to the combined effects of deformation flow and the wall friction (discussed in section 7.5.1). These results are also consistent with the results of the shear strains (Fig. 7.14) described in section 7.5.2. The change from a concave deflection to a convex deflection was first observed by Hvorslev (1949) experimentally. The numerical predictions of this phenomenon is provided here, in this study, for the first time based on a literature review by the author.
7.5.6 Failure Zones

The growth of failure zones for type A analysis at penetration depths 20t (=0.4B) and 50t (=B) is shown in Fig. 7-20. The failure zone here means that the soil is approaching the critical state line with a stress ratio \( \frac{f}{M_p} \) larger than 0.95. A relatively large failure zone is observed at the outside of the sampling tube (regions 5 and 6). However, only a small failure region near the inside wall of the sampling tube was predicted by the updated Lagrangian finite strain algorithm.

7.6 Parametric Studies

The disturbances caused by the penetration of a frictional sampling tube was described in the last section. In this section, parametric studies are provided regarding five factors: (a) inside wall friction; (b) wall thickness; (c) rate of penetration; (d) in situ stress condition and (e) methods of sampling. These factors are considered to play an important role in sampling disturbances. Since, from the practical point of view, only the center part of a tube sample is taken as the representative sample, emphases will be placed on the condition along the center line.

Ladd and Lambe (1963) proposed the ratio \( \frac{\sigma_1}{\sigma_3} \) (defined in chapter 2) as a measure of sampling disturbances. The ratio of two mean effective stresses is too simple to define the degree of disturbances because a zero change of stress may not necessary means the soil was not disturbed before. Herein, the degree of sampling disturbances will be investigated according to: (1) the maximum compressive strain in region 4; (2) the maximum tensile strains at region 3 (or 2); (3) the ratio of the change of deviatoric stresses in region 4, defined as \( \frac{\sigma_4 - \sigma_0}{\sigma_{1f} - \sigma_0} \); (4) the ratio of the change of mean effective stresses in region 4, defined as \( \frac{\sigma_m - \sigma_m^0}{\sigma_{1f} - \sigma_{1f}^0} \); (5) the ratio of the change of the hardening parameter in region 4, defined as \( \frac{(p_m)_f - (p_m)^0}{(p_m)_0} \), where \((p_m)_f\) is the hardening parameter at failure; and (6) the displacement at the top surface. The subscript \( f \) denotes failure under undrained triaxial compression. Item (5) denotes irreversible sampling disturbances.
Fig. 7-18 Contours of Normalized Excess Porewater Pressure (type A)
Fig. 7-19 Displacement Fields (type A)
Fig. 7-20 Failure Zones (type A)
7.6.1 Inside Wall Friction

The effect of the inside wall friction is mostly easily examined by comparing a frictional sampler (type A analysis) with a frictionless sampler (type B analysis). The numerical results of vertical strains for type B analysis are plotted in Fig. 7-21. The same pattern of three distinct phases of vertical strain (compression-extension-recompression) along the center line, as was shown in type A analysis, is observed. By comparing Fig. 7-10 and 7-21, one observes that a frictionless sampler yields smaller vertical strains in region 4 but has larger vertical strains at region 3 than a frictional sampler. In addition, no positive shear strain is observed near the inside wall of the frictionless sampler (Fig. 7-21). The results of vertical strains along the centerline for type B analysis are plotted in Fig. 7-12 (line $a'' - b'' - c'' - d''$). The frictionless sampler yields a non-symmetric vertical strain path similar to the frictional sampler.

The results of normalized deviatoric stresses of a frictionless sampler is shown in Fig. 7-22. The changes of deviatoric stresses due to a frictional sampler are less than a frictional sampler (type A), especially in the regions 1 and 4 (compare Figs 7-16 and 7-22). The excess pore water pressures induced by the penetration of a frictionless sampler (Fig. 7-23) are much less than a frictional sampler (Fig. 7-18).

The quantitative comparisons of a frictional sampler with a frictionless sampler are given in Table 7-2. It is evident, by comparing the values of columns 1 and 2 in table 7-2, that a frictional sampler causes more sampling disturbances than a frictionless sampler.

The sampling disturbances along the center line after the depth of penetration equal to 0.72 $B$ for a frictional sampler and a frictionless sampler are shown in Columns 3 and 4 in Table 7-2 respectively. By comparing column 2 with column 4 in table 7-2, one observes no apparent increase of degree of disturbances for the frictionless sampler from depth=0.72$B$ to depth=1.0$B$. This indicates that the frictionless sampler has approached a steady state. However, for a frictional sampler, the degree of disturbances increases as the depth of penetration increases.
Fig. 7-21 Contours of Vertical Strain (type B)
Fig. 7-22 Contours of Normalized Deviatoric Stress (type B)
Fig. 7-23 Contours of Normalized Excess Porewater Pressure (type B)
### Table 7.2 Comparisons Between the Frictional Sampler and the Frictionless Sampler

<table>
<thead>
<tr>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
</tr>
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<tbody>
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<td>Type A</td>
<td>Type B</td>
<td>Type A</td>
<td>Type B</td>
</tr>
<tr>
<td>Wall Friction</td>
<td>1.0 ( c_u )</td>
<td>0.0 ( c_u )</td>
<td>1.0 ( c_u )</td>
</tr>
<tr>
<td>Penetration Depth</td>
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<td>1.0 B</td>
<td>0.72 B</td>
</tr>
<tr>
<td>Max. ( (\epsilon_{zz})_{comp} )</td>
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<td>0.61%</td>
<td>0.81%</td>
</tr>
<tr>
<td>Max. ( (\epsilon_{zz})_{tension} )</td>
<td>-0.27%</td>
<td>-0.45%</td>
<td>-0.32%</td>
</tr>
<tr>
<td>Max. ( \frac{x=x_0}{y=x_0} )</td>
<td>70.2%</td>
<td>57.9%</td>
<td>66%</td>
</tr>
<tr>
<td>Max. ( \frac{P-A}{P_0-A} )</td>
<td>54.2%</td>
<td>39.3%</td>
<td>45.6%</td>
</tr>
<tr>
<td>Max. ( \frac{\Delta P_n}{(P_m)_n-(P_m)_o} )</td>
<td>61.4%</td>
<td>47.4%</td>
<td>55.9%</td>
</tr>
<tr>
<td>Disp. at Top</td>
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<td>+0.123 mm</td>
<td>-0.185 mm</td>
</tr>
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</table>
The increasing disturbances is due to the increase of total frictional forces resulting from the increase of the depth of penetration. If the depth of penetration is too long, eventually, the induced compression will exceed the bearing capacity of soil under the sampler and cause failure. This phenomenon had been explained experimentally by Hvorslev (1949). The current findings, that is, a frictional sampler has larger compressive strains at region 4 than a frictionless sampler and the increasing compressive strains with respect to the depth of penetration, are in agreement with Hvorslev (1949) and provide the numerical evidences of this potential failure pattern.

In summary, the sampling disturbances is induced even with a frictionless sampler and the inside wall friction increases it. The additional disturbances caused by the inside wall friction may be as large as to cause the soil to fail. Thus, the lubrication of the wall of the sampling tube is always recommended. The increasing frictional force with respect to the depth of penetration also indicates that a steady state of penetration is not attainable. The assumption of steady state penetration as used in the SPM solution and type B analysis is only valid in the frictionless samplers. For a frictional sampler, the solutions will be different if the depth of penetration is changed. An investigation with deeper penetration is recommended.

### 7.6.2 Wall Thickness

Three wall thickness were investigated - 2 mm (type A), 2.5 mm (type C) and 3 mm (type D). The area ratio $A_r = \frac{(B+2t)^2 - B^2}{B^2} \times 100\%$ are 8.2 %, 10.3 % and 12.4 % respectively. The tip angles ($\alpha$) are 26.6°, 32.0° and 36.9° respectively.

The comparisons of these three samplers, regarding strains, stresses, $p_m$ and displacements along centerline, are listed in Table 7-3. It is observed, by comparing the values in Table 7-3, the magnitude of disturbances increases with respect to the wall thickness.

The results of type C and type D analyses follow the same pattern as the results of type A except for the excess pore water pressure predictions for the type
D analysis (Fig. 7-24) where a region of negative excess pore water pressure is located near the tip of the sampler spreading from the outside of the tip into the inner region were obtained. The maximum negative excess pore pressure predicted is 0.74 \( p_0 \). The negative pore water pressure region is, perhaps, a result of the separation (or the tendency for separation) of the soil and the sampler near the tip forming a void as was reported by Kiousis et. al. (1988) for a cone tip. However, the manner in which the separated zone spreads is a new observation. For a separation zone to occur the tip angle of a cone has to be large enough with an appropriate rate of penetration.

7.6.3 Rate of Penetration

Slow rate of penetration allows plastic deformations and volume changes to take place. It promotes entrance of excess soil and the development of wall friction and adhesion with consequent increase of penetration resistance and distortion of the soil layers. Thus, fast penetration is recommended for general use in obtaining undisturbed sample (Hvorslev, 1949). To investigate the effect of the rate of penetration on the degree of sampling disturbances, three penetration speeds were analysed, 0.4 mm/sec (type A), 4.0 mm/sec (type E) and 40 mm/sec (type F).

The numerical results of vertical strains, deviatoric stresses and excess pore water pressures for type E analysis are shown in Figs 7-25, 7-26 and 7-27 respectively. Quantitative comparisons of rate dependent disturbances along the centerline are given in Table 7-4. Indeed, a fast rate of penetration (type E, Fig. 7-25), induces less vertical compressive strains in region 4 and less tensile strains in region 2 than a slow rate of penetration (type A, Fig. 7-10). The maximum vertical strain (=0.12%) at region 4 in type E analysis is only about one eighth of type A (=0.95%). The changes of deviatoric stresses (\( \Delta q = q - q_0 \)) of type E (Fig. 7-26) inside the sampler are also less than type A (Fig. 7-16). These imply that the degree of sampling disturbances can be reduced by increasing the rate of penetration. The low values of \( \frac{\Delta q}{p_0} \) (row 4), \( \frac{\Delta p}{p_0} \) (row 5) and \( \frac{\Delta \tilde{p}_m}{(P_m) - (P_m)_0} \) (row 6) in
Table 7-4 of type E analysis provide the numerical evidences that the fast rate of penetration (type E) induces less disturbances than the lower rate (type A).

A relatively large region of negative excess pore pressures is observed in Fig. 7-27. This region spreads down from the tip of the sampler to a depth of about 0.2 B and a width of about 0.15 B and extend along the inside wall of the sampler from the tip of the sampler up to a length of about 0.7 B. As mentioned before, negative excess pore pressures indicate separation or tendency for separation of the soil and the sampler. This means a fast penetration rate creates the tendency for separation of soil-sampler interface. However, according to the results of type E analysis, the separation doesn’t actually happen because the effective radial stresses along the inside wall of the sampler are still compressive.

The displacement fields inside the sampler of type E analysis is shown in Fig. 7-28. A dramatic change from type A analysis is observed; the predicted displacements are upward for a fast rate of penetration (type E analysis) while the predicted displacements are downward for a low rate of penetration (type A analysis, Fig. 7-19). This observation provides the numerical evidence that, when the rate of penetration is fast, the deformation flow is predominant such that the soil is displaced up instead of being dragged down.

It is evident, as stated by Hvorslev (1949) and was proved above, that the increase of rate of penetration will reduce the sample disturbances. However, this is not always true. The failure zone of type F analysis (Fig. 7-29) which simulates a penetration rate of 40 mm/sec, reveals a larger failure zone than the failure zone of type A analysis (Fig. 7-20). Besides, this failure area extends towards the inside of the sample. It seems that too fast a penetration rate will cause additional disturbances along the perimeter of the sample and the current standard penetration rate 20 mm/sec (Hvorslev, 1949) seems within the speed range of the least disturbances.
Fig. 7-24 Contours of Normalized Excess Porewater Pressure (type D)
Table 7.3 The Effects of Wall Thickness of Sampler

<table>
<thead>
<tr>
<th></th>
<th>Type A</th>
<th>Type C</th>
<th>Type D</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) Wall Thickness</td>
<td>2 mm</td>
<td>2.5 mm</td>
<td>3.0 mm</td>
</tr>
<tr>
<td>(2) Area Ratio $A_r$</td>
<td>8.2%</td>
<td>10.3%</td>
<td>12.4%</td>
</tr>
<tr>
<td>(3) Tip Angle $\alpha$</td>
<td>25.6°</td>
<td>32.00°</td>
<td>36.9°</td>
</tr>
<tr>
<td>(4) Max. $(\varepsilon_{zz})_{comp}$</td>
<td>0.95%</td>
<td>1.07%</td>
<td>1.32%</td>
</tr>
<tr>
<td>(5) Max. $(\varepsilon_{zz})_{tension}$</td>
<td>-0.27%</td>
<td>-0.34%</td>
<td>-0.32%</td>
</tr>
<tr>
<td>(6) Max. $\frac{\pi-\pi_0}{\pi_f-\pi_0}$</td>
<td>70.2%</td>
<td>75.0%</td>
<td>78.2%</td>
</tr>
<tr>
<td>(7) Max. $\frac{E_{eff}}{P_0-P_f}$</td>
<td>54.2%</td>
<td>57.2%</td>
<td>54.8%</td>
</tr>
<tr>
<td>(8) Max. $\frac{\Delta P_m}{(P_m)_{eff}-(P_m)_0}$</td>
<td>61.4%</td>
<td>65.4%</td>
<td>72.7%</td>
</tr>
<tr>
<td>(9) Disp. at Top</td>
<td>-0.33 mm</td>
<td>-0.34 mm</td>
<td>-0.19 mm</td>
</tr>
</tbody>
</table>
Fig. 7-25 Contours of Vertical Strain (type E)
Fig. 7-26 Contours of Normalized Deviatoric Stress (type E).
Fig. 7-27 Contours of Normalized Excess Porewater Pressure (type E)
Fig. 7-28 Displacement Fields (type E)
Fig. 7-29 Failure Zone

- Sampling tube
- Casing
- Type F
- Failure Zone
- $t = 2\text{mm}$
### Table 7.4 Comparisons of Penetration Rates

<table>
<thead>
<tr>
<th>(1) Penetration Rate</th>
<th>Type A</th>
<th>Type E</th>
<th>Type F</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.4 mm/sec</td>
<td>4.0 mm/sec</td>
<td>40 mm/sec</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(2) Max. ((\varepsilon_{xx})_{comp})</th>
<th>0.95%</th>
<th>0.12%</th>
<th>none</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>(3) Max. ((\varepsilon_{xx})_{tension})</th>
<th>-0.27%</th>
<th>-0.26%</th>
<th>-0.45%</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>(4) Max. (\frac{q-q_0}{q_f-q_0})</th>
<th>70.2%</th>
<th>23.2%</th>
<th>-</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>(5) Max. (\frac{P_n-P_f}{P_0-P_f})</th>
<th>54.2%</th>
<th>8.7%</th>
<th>6.7%</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>(6) Max. (\frac{\Delta P_m}{(P_m)_f-(P_m)_0})</th>
<th>61.4%</th>
<th>36.7%</th>
<th>35.5%</th>
</tr>
</thead>
</table>

| (7) Disp. at Top | -0.33 mm | +0.443 mm | +0.92 mm |
7.6.4 In Situ Stress Condition:

The in situ stress condition also affects the degree of disturbances. Consider elements of soil A, G and H (Fig 7.30, (a), (b) and (c)) with in situ conditions $K_0 = 0.56$, $K_0 = 1$ and $K_0 = 2.0$ respectively. All of these elements represent a normally consolidated soil ($R_p = 1$) with the same in situ vertical stress. The corresponding in situ stress states on the $p - q$ space are shown in Fig. 7-30 (d). The lines with arrow in Fig. 7-30 (d) A-A', G-G' and H-H'-H'' represent the stress paths of elements A, G and H subjected to undrained vertical compression respectively. Both elements G and H have the in situ undrained strength larger than element A. A check is made by comparing three tests, namely, type A ($K_0=0.56$), type G ($K_0=1$) and type H ($K_0=2$). Quantative values of strains and degree of disturbances along the centerline are given in Table 7-5. As seen from Table 7-5, the sampling disturbances of type G and type H analyses are less than type A analysis.

The contours of the vertical strains for type H analysis are plotted in Fig. 7-31. The pattern of three distinct strain stages described in section 7.5.2 still applies to this analysis but the maximum center vertical compressive strain in region 4 of type H is less than type A (Fig. 7-10). Compressive strains are also predicted in regions 1 and 2 for type H analysis (Fig. 7-31) while none is predicted in type A analysis (Fig. 7-10). The reason of this change is due to the combined effects of the deformation flow and the cohesion. Because of the effect of the deformation flow, the vertical strain history for type H analysis is expected to follow a similar trend as type A analysis, that is, compressive strain in front of sampler and tensile strain inside the sampler. However, the element H has a larger undrained strength and a lower rate of penetration is imposed. The net effect is that the cohesion is predominant and drags the soil inside the sampler downwards. This action reduces the tensile strain in region 3 and increase the recompression effects in region 1. In summary, the effect of cohesion reduces the disturbances. This is especially true when the cohesion is large and the penetration rate is low.
Fig. 7-30 Stress Paths r.s.t. Different In Situ Stress Conditions
Fig. 7-31 Contour of Vertical Strain (type H)
### Table 7.5 The Effects of In Situ Stress Condition

<table>
<thead>
<tr>
<th></th>
<th>Type A</th>
<th>Type G</th>
<th>Type H</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) $K_0$</td>
<td>0.56</td>
<td>1.0</td>
<td>2.0</td>
</tr>
<tr>
<td>(2) $R_p$</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>(3) Max. $(\epsilon_{zz})_{comp}$</td>
<td>0.95%</td>
<td>0.45%</td>
<td>0.51%</td>
</tr>
<tr>
<td>(4) Max. $(\epsilon_{zz})_{tension}$</td>
<td>-0.27%</td>
<td>-0.24%</td>
<td>-0.09%</td>
</tr>
<tr>
<td>(5) Max. $\frac{e-e_0}{f_f-e_0}$</td>
<td>70.2%</td>
<td>46.2%</td>
<td>15.9%</td>
</tr>
<tr>
<td>(6) Max. $\frac{P_a-P}{P_0-P_f}$</td>
<td>54.2%</td>
<td>17.2%</td>
<td>23.9%</td>
</tr>
<tr>
<td>(7) Max. $\frac{\Delta P}{(P_m)_f-(P_m)_0}$</td>
<td>61.4%</td>
<td>8.4%</td>
<td>1.2%</td>
</tr>
<tr>
<td>(8) Disp. at Top</td>
<td>-0.33 mm</td>
<td>-0.37 mm</td>
<td>-0.92 mm</td>
</tr>
</tbody>
</table>
7.6.5 Methods of Sampling

There are two types of sampler that use thin-walled tubes for sampling. These are open-drive samplers and piston samplers (Hvorslev, 1949). In the piston sampler a piston is used at the bottom of borehole to prevent the soil from flowing into the sampling tube through the bottom of the sampling tube. The piston sampler is especially useful when the soil is soft and Hvorslev (1949) suggested that it will not disturb the sample as much as the open-drive sampler.

To investigate the differences of between two methods on the quality of sample, six numerical tests were conducted. These are type A (low rate, open-drive), type E (high rate, open-drive), type I (low rate, piston), type J (high rate, piston), type K (high rate, open-drive) and type L (high rate, piston) (Table 7.6). The first four tests are to simulate ideal penetration (no excavation). The last two cases are to simulate the field situation when a bore hole is excavated and the sampler is then pushed into the ground. The piston sampler is simulated by fixing the vertical movement at the top boundary IJ (Fig. 7-1) while the open-drive sampler is kept free.

The piston sampler does not appear to improve the quality of the sample very much for low rate penetration as indicated in rows (6) to (8) for type A and I analyses. The maximum tensile strain ($\varepsilon_{zz} = -0.56\%$) for the piston sampler (type I) is larger than the open-drive sampler ($\varepsilon_{zz} = -0.27\%$, type A). The reason for this is that the potential movement of the top surface is downwards for type A analysis. However, in the piston sampler (type I analysis), this downward movement is prevented such that suction and negative pore water pressures are generated at region 1. This suction effect creates tension in regions 1, 2 and 3.

The results of a fast rate of penetration, namely, type E analysis (open-drive) and type J analysis (piston) show that the piston sampler produces larger degree of disturbances (see rows (6), (7) and (8) in Table 7-6) than the open-drive sampler. The top boundary IJ is fixed in the piston sampler, the upward flow inside the sampler at regions 1, 2 and 3 is, then, directed downwards. As a result, the maximum vertical strain in region 4 increases from 0.12\% for the open-drive
sampler (type E analysis) to 0.74% for the piston sampler (type J analysis). The increase of the vertical strains in region 4 also indicates increase in the degree of disturbances for a piston sampler.

The above results are contrary to the suggestion made by Hvorslev (1949), and generally accepted in practice, that the piston sampler is better than the open-drive sampler. The four numerical tests above are cases of ideal sample tube penetrations. In practice, an excavation is made without adding any external pressure to balance the pressure at the bottom of the borehole. Thus, the sample is unloaded before the penetration starts. This is an extension effect. Now, consider the standard penetration of a open-drive sampler with a rate of penetration 20 mm/sec. This is a fast penetration. From the results of type E (rate=4 mm/sec, Fig. 7-25) and type F (rate=40 mm/sec, Table 7-4), we can expect that most of the induced vertical strains of the standard penetration rate would be tensile. The excavation of the bore hole already causes extension on the sampling area. The combination of these two effects turns out to be "double-extension". Soils are generally weak in tension. Thus, an open-drive sampler with the standard penetration speed can cause extension failure in clays, especially soft clays, resulting in excess soil flowing into the sampling tube. On the contrary, the piston sampler provides compression strain and the extension effect caused by excavation is offset. Thus, the piston sampler with the standard penetration rate is better than the open-drive sampler.

The choice as to whether to use open-drive sampler or piston sampler should be determined by the loading history and the soil type. If the soil is strong enough to sustain the induced tensile strains without causing any hardening effect, then, the open-drive sampler, in stead of the piston sampler, should be used because of its simplicity of operation. For example, the results of a field open-drive sampler (type K) has a maximum tensile strain -2.1% (table 7-6) along the centerline. This tensile strain is too small to induce any hardening effect. On the contrary, the piston sampler (type L) creates compressive strains ahead of the sampler and results in a 30% change of hardening parameter $p_m$. 
Table 7.6 Comparisons Between Methods of Sampling

<table>
<thead>
<tr>
<th></th>
<th>Type A</th>
<th>Type E</th>
<th>Type I</th>
<th>Type J</th>
<th>Type K</th>
<th>Type L</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) Sampler</td>
<td>open</td>
<td>open</td>
<td>piston</td>
<td>piston</td>
<td>open</td>
<td>piston</td>
</tr>
<tr>
<td>(3) Rate (mm/sec)</td>
<td>0.4</td>
<td>4.0</td>
<td>0.4</td>
<td>4.0</td>
<td>4.0</td>
<td>4.0</td>
</tr>
<tr>
<td>(4) Max. $+\varepsilon_{zz}$</td>
<td>0.95%</td>
<td>0.12%</td>
<td>0.88%</td>
<td>0.74%</td>
<td>none</td>
<td>0.13%</td>
</tr>
<tr>
<td>(5) Max. $-\varepsilon_{zz}$</td>
<td>-0.27%</td>
<td>-0.26%</td>
<td>-0.56%</td>
<td>-0.41%</td>
<td>-2.1%</td>
<td>-1.47%</td>
</tr>
<tr>
<td>(6) Max. $\frac{q-q_0}{q_f-q_0}$</td>
<td>70.2%</td>
<td>23.2%</td>
<td>68.6%</td>
<td>62.6%</td>
<td>no hardening</td>
<td>10.2%</td>
</tr>
<tr>
<td>(7) Max. $\frac{p_{m}-p_{f}}{p_{m}}$</td>
<td>54.2%</td>
<td>8.7%</td>
<td>51.3%</td>
<td>43.5%</td>
<td>no hardening</td>
<td>10.1%</td>
</tr>
<tr>
<td>(8) Max. $\frac{\Delta p_m}{(p_m)<em>{f}-(p_m)</em>{c}}$</td>
<td>61.4%</td>
<td>36.7%</td>
<td>55.6%</td>
<td>53.9%</td>
<td>no hardening</td>
<td>30.0%</td>
</tr>
<tr>
<td>(9) Disp.(mm)</td>
<td>-0.33</td>
<td>+0.443</td>
<td>0.0</td>
<td>0.0</td>
<td>+1.94</td>
<td>+1.49</td>
</tr>
</tbody>
</table>
7.7 Soil-Sampler Interface effects

The experimental results of porewater pressures, normal forces and shear forces observed during the sampling penetration at three different locations along the length of the instrumented thin-walled sampling tube are shown in Figs 7-32 to 7-34. In these figures, the horizontal axis represents elapsed time and the vertical axis represents porewater pressures or stresses. The total time spent for a full penetration depth of 400 mm was 204 seconds. The average penetration speed was about 2 mm/sec. It should be noted that the experimental results only shows the values at the three fixed points (points a, b and c in Fig. 6-2) on the instrumented thin-wall tube during the process of the penetration. The raw experimental data showed a large drop in all the measured stresses at t=180 sec which was probably due to a short stoppage due to the mechanical problem during the experiment. The results after then are suspicious. Thus, only the experimental results before t=180 sec are used to interpretate the behavior of the soil-sampler interface.

The numerical results of type E analysis along the inside soil-sampler interface at two different depths of penetration (0.72 B and 1.0 B) are presented in Figs 7-35 and 7-36. Only the results at the center of the thin-layer elements are presented. In Fig. 7-35 and 7-36, the horizontal axis denotes the distance above the tip of the sampler. At the elevation of the tip of the sampler, Z is zero. The vertical axis denotes the normalized stresses with respect to $p_0$ ($=141.3$ kPa). Emphasis is made on the description of soil-sampler behavior. A discussion between the numerical results and the experimental results are given below.

7.7.1 Porewater Pressures

The variations of pore water pressures ($u_w$) measured by the porewater transducers located at three positions (as shown in Fig. 6-2) on the sampling tube with respect to time are shown in Fig 7-32. Due to the difficulty of estimating the initial negative excess pore water pressure caused by removing the steel plate (consolidation pressure), these data are interpreted by observing the differences (called $\Delta u_w$) between the estimated static porewater pressures and the total
porewater pressures measured by the porewater transducers. The referential static pore water pressures are plotted by dash lines in Fig. 7-32.

The estimated times for contact between the sample and the three pore water pressure transducers "a", "b" and "c" are 31.25 sec, 81.25 sec and 131.25 sec respectively. For the pore water transducer "a", the measured $\Delta u_w$ was negative at the beginning of contact due to the effect of stress relief; at t=54 sec, $\Delta u_w$ changed to positive and after t=100 sec, it changed to negative again. These changes indicated that the pore water pressure transducer "a" was gaining $\Delta u_w$ before t=100 sec (penetration depth = 200 mm) and was decreasing after then. For pore water pressure transducers "b" and "c", $\Delta u_w$ were negative at the beginning (due to stress relief) of contact and then remained positive as penetration proceeded.

For the numerical analysis in type E the predicted excess porewater pressures extended up along the inside wall of the sampler (Figs 7-27). It is shown in Fig. 7-35 that negative excess pore pressure extends up to a length equals 0.47B for a depth of penetration equals 0.72B. It is also shown in Fig. 7-36 that negative excess pore pressure extends up to a length equal to 0.73B for a depth of penetration equals 1.0B. These indicate the total length of negative excess pore pressure along the inside wall of the sampler is increasing with respect to the depth of penetration. Porewater pressure transducer "a" (Fig. 6-2) is located 0.625 B (62.5 mm) above the tip of the sampler. Thus, it gained positive excess pore pressure at the beginning of penetration. After 100 sec, the region of negative excess pore pressure extended up to pore pressure transducer "a" such that the pore water pressure at transducer "a" started to drop down. Porewater pressure transducer "b" and "c" are located at 1.625 B (162.5 mm) and 2.625 B (262.5 mm) above the tip of sampler respectively. The negative pressure did not rise up to these positions during the penetration such that they gained positive pore pressure continuously.

**7.7.2 Lateral Stresses**

The monitored radial stresses ($\sigma_{rr}$) normal to the internal wall of the sampling tube measured by the three load cells (load cells A, B, and C in Fig.
6-2) during the penetration of the sampling tube are shown in the Fig 7-33. The estimated static radial stresses (calculated by $K_0 \times \sigma_{zz} + u_w$) is shown in Fig. 7-33 by dash lines. The differences between the experimental and the estimated static radial stresses ($\Delta \sigma_{rr}$) give an indication of the changes of the total radial stresses due to stress relief and the penetration of the sample. For load cell A, the change in total radial stress ($\Delta \sigma_{rr}$) was positive at the very beginning of the penetration and then became negative as the penetration proceeded. For load cell B and C, $\Delta \sigma_{rr}$ was also positive at the beginning of the penetration of the sampler and became negative thereafter. The initial positive values of $\Delta \sigma_{rr}$ recorded by these load cells indicated that the soil was compressed radially at the beginning of contact. After then, the negative $\Delta \sigma_{rr}$ indicates that the soil relaxed radially. These experimental evidences are consistent with the pattern of deformation flow described in section 7.5.1. Specifically, at the beginning of contact, the load cells were located at the region where the flow had bent (element D in Fig. 7-8(a)), the flow was against them such that they gained positive $\Delta \sigma_{rr}$. As the penetration proceeded, they moved to the position E (Fig. 7-8(a)), the flow was heading away such that the load cells had negative $\Delta \sigma_{rr}$.

The initial radial stress of type E is $0.79p_0$. The numerical results (type E analysis) of total radial stresses (Figs 7-35 and 7-36) show that, except at the region near the top surface ($\frac{z}{H} = 1.0$), the change of total radial stresses is positive ($\sigma_{rr} > (\sigma_{rr})_0 = 0.79p_0$) after the penetration. The soil then was compressed before passing the tip of the sampler. After the soil passes the tip of the sampler, the soil is subjected to radial extension, $\Delta \sigma_{rr}$ decreases as shown in Figs 7-35 and 7-36.

These results confirm the pattern of deformation flow described in section 7.5.1.
7.7.3 Shear Stresses

The monitored shear stresses ($\sigma_{rz}$) along the internal wall of the sampling tube measured by the three load cells during penetration of the sampling tube are shown in the Fig. 7-34. The magnitude of shear stresses measured at load cell A increased with the depth of penetration and reached a peak value at $t = 75$ sec. This peak value of shear stress remained constant for about 20 seconds and then decreased to a smaller value. For load cell B, a small drop at $t = 125$ sec was observed. This was consistent with the dropping of radial stress (negative $\Delta\sigma_{rr}$) at load cell B because the magnitude of shear stress is a function of normal stress. The final shear stress at $t = 175$ sec of load cell B was a little lower than the peak shear stress of load cell A but was a little larger than the load cell A at the end of experiment. But, in general, they were reaching a constant value. The results from load cell C were not consistent with the results from load cells A and B (Fig. 7-34). No rational explanation can be given. The constant shear stresses in load cell A and B indicated that the soil was in a state of "plastic flow". The numerical results shown in Fig. 7-35 and 7-36 also indicate that the shear stress along the inside wall of the sampler is approximatively constant. Thus, qualitatively, the experimental results provided good agreement with the numerical results of type E analysis.
Fig. 7-32 Variations of Pore Water Pressures
Along the Wall of the Sampling Tube
Variations of Total Radial Stress Along the Wall of Sampling Tube

Fig. 7-33 Variations of Total Radial Stresses Along the Wall of The Sampling Tube
Fig. 7-34 Variations of Shear Stresses
Along the Wall of The Sampling Tube
Stresses at Soil-Sampler Interface
Depth of Penetration = 0.72 B

Fig. 7-35 Stresses at Soil-Sampler Interface (Depth = 0.72 B)
Stresses at Soil-Sampler Interface
Depth of Penetration = 1.0 B

Fig. 7-36 Stresses at Soil-Sampler Interface (Depth = 1.0 B)
Chapter 8 Conclusions and Recommendations

In this research, the quality of a soil sample during penetration of a thin-walled sampling tube was examined by using (1) an updated Lagrangian finite element formulation suitable for axisymmetric finite strain consolidation analysis; (2) an experiment, with an instrumented sampling tube, to investigate the variations of stresses and pore water pressures at the soil-sampler interface. Based on the results of this research, an insight into the changes in strains, stresses, displacement and excess pore water pressure during the penetration of a sampler were provided. Friction at the soil sampler interface, rates of penetration, loading history, sampling methods (piston and open-drive) were all shown to have varying degrees of disturbances to the soil sample.

Based on the analyses, using the modified Cam-clay, the following conclusions are drawn:

1. Soil samples are subjected to three distinct stages of vertical strain history: compression-extension-recompression. The first stage of compression causes irrecoverable changes of the virgin soil properties.
2. The undrained shear strength of a disturbed sample reconsolidated to the in situ stress condition is larger than the virgin soil for normally consolidated soils but it is smaller for over consolidated soils.
3. The sampling disturbances due to the soil-sampler friction increases as the sampler penetrates. As a result, the longer the sample is, the greater the disturbances.
4. An increase in the rate of penetration can reduce the degree of disturbances.
5. The degree of sampling disturbances decreases as the in situ strength (cohesion) increases.
6. The piston sampler induces larger disturbances than the open-drive sampler. For stiff soils, the open-drive sampler is likely to produce a better quality sample than the piston sampler.
The above conclusions are consistent with the hypothesis of this work except that the disturbances, even for a frictionless sampler, is larger than 10%, especially near the wall of the sampler. However, the undrained shear strength of a disturbed sample can be obtained by reconsolidating the sample as suggested by the method SHANSEP. Only one soil model was used and it is expected that other types of soil models will give different predictions.

For future research, it is recommended that the investigations on sampling disturbances should be conducted with an anisotropic soil model and a deeper depth of penetration. To investigate the optimum rate of penetration, a visco-plastic soil model is also recommended.
Appendix A : Uniaxial Compression ( Extension )

Loading a cylindrical sample one-dimensionally (Fig. 5-1) is one of the most common laboratory tests used in Geotechnical engineering. Loading with zero confining pressure is called unconfined compression (or extension) test. Loading with a constant confining pressure is called triaxial compression (or extension) test. In this appendix, two finite strain closed form solutions for a material subjected to uniaxial deformation are provided. These two types of solution are obtained with different constitutive laws as explained later.

Assuming the deformation is homogeneous, the displacement fields of uniaxial extension (or compression) in a cylindrical bar (as shown in Fig 5.1) can be expressed as:

\[ R = r + u_r \]
\[ Z = z + u_z \]

where \( u_r \) and \( u_z \) are displacements in the radial and the vertical directions respectively; \( R \) and \( Z \) are coordinates after deformation while \( r \) and \( z \) are initial coordinates. We can specify the deformation at the top of the bar by the equation

\[ Z_0 = z_0 + k_0 t \]

where \( k_0 \) is a constant rate of deformation.

The stretches, by definition, are:

\[ \lambda_r = \frac{R}{r} \]
\[ \lambda_z = \frac{Z}{z} \]

The assumption of homogeneous deformation also demands

\[ \frac{R_0}{r_0} = \frac{R}{r} \]
\[ \frac{Z_0}{z_0} = \frac{Z}{z} \]

such that the vertical displacement field becomes

\[ Z = z + k_0 \frac{z}{z_0} t \]

or

\[ z = Z - k_0 \frac{Z_z}{Z_0} t \]

If the superimposed dot denotes material derivative, we have,

\[ [L] = \left[ \frac{\partial \dot{X}_i}{\partial X_j} \right] = \begin{bmatrix} \frac{\partial \dot{R}}{\partial R} & 0 & 0 \\ 0 & \frac{\partial \dot{z}}{\partial z} & 0 \\ 0 & 0 & \frac{\partial \dot{R}}{\partial R} \end{bmatrix} \]

and the rate of deformation \( d_{ij} \)

\[ [D] = [d_{ij}] = \frac{1}{2}([L]^T + [L]) = \begin{bmatrix} \frac{\partial \dot{R}}{\partial R} & 0 & 0 \\ 0 & \frac{\partial \dot{z}}{\partial z} & 0 \\ 0 & 0 & \frac{\partial \dot{R}}{\partial R} \end{bmatrix} \]
The material derivative of spatial coordinate Z (equation (A-5)) is

\[ \dot{Z} = \frac{k_0}{z_0} Z \]  

(A - 12)

such that

\[ \frac{\partial \dot{Z}}{\partial Z} = \frac{\partial \dot{Z}}{\partial Z} \frac{\partial Z}{\partial z} = \frac{k_0}{z_0 + k_0 t} \]  

(A - 13)

Thus, the total vertical strain is obtained as \( \int \frac{\partial \dot{Z}}{\partial Z} dt = \ln \lambda_z \). The radial strain and tangential strain are determined by the assumed constitutive law.

Assume that the material obeys the constitutive law:

\[ \dot{\sigma}_{ij} = \lambda \delta_{ij} d_k k + 2 \mu \delta_{ij} \]  

(A - 14)

where \( \lambda = \frac{v E_y}{(1+v)(1-2v)} \), \( \mu = \frac{E_y}{2(1+v)} \) and \( E_y \) is Young's modulus, \( v \) is Poisson's ratio and \( \dot{\sigma}_{ij} \) is Jaumann stress rate defined in chapter 3.

Substituting equation (A-11) into equation (A-14) and noting that \( \dot{\sigma}_{ij} = \dot{\epsilon}_{ij} \) (no rotation is involved in homogeneous uniaxial deformation) we obtain

\[ \dot{\sigma}_{rr} = \lambda d_{kk} + 2 \mu \dot{R} = 0 \]  

(A - 15)

\[ \dot{\sigma}_{zz} = \lambda d_{kk} + 2 \mu \dot{Z} \]  

(A - 16)

\[ \dot{\sigma}_{\theta \theta} = \lambda d_{kk} + 2 \mu \frac{\dot{R}}{R} = 0 \]  

(A - 17)

Equations (A-15) and (A-17) are equal to zero since there is no lateral stress increment in this type of loading. Solving the last three equations yields

\[ \frac{\partial \dot{R}}{\partial R} = \frac{\dot{R}}{R} = \frac{-\lambda}{2(\lambda + \mu)} \frac{\partial \dot{Z}}{\partial Z} = -v \frac{\partial \dot{Z}}{\partial Z} \]  

(A - 18)

Thus, \( \epsilon_{rr} = \epsilon_{\theta \theta} = -v \ln \lambda_z \). Substituting equations (A-11) and (A-16) into equation (A-14) we obtain

\[ \dot{\sigma}_{zz} = \lambda(-2v + 1) \frac{\partial \dot{Z}}{\partial Z} + 2 \mu \frac{\partial \dot{Z}}{\partial Z} = E_y \frac{\partial \dot{Z}}{\partial Z} \]  

(A - 19)

Substituting equation (A-13) into (A-19) and integrating we obtain,

\[ \sigma_{zz} = \int E_y \frac{k_0}{z_0 + k_0 t} dt = E_y \ln \lambda_z \]  

(A - 20)

This is exactly the same solution as logarithmic strain solution.
Appendix B: Finite Simple Shear (Axial Symmetry)

Assuming homogeneous deformation, the displacement field of simple shear in axial symmetric coordinates can be described as (Fig 5-5):

\[ R = r \]  \hspace{1cm} (B-1)

\[ Z = z + Kr = z + k(r - r_0)t \]  \hspace{1cm} (B-2)

\[ \Theta = \theta \]  \hspace{1cm} (B-3)

where \( u_z \) and \( u_r \) are displacements in radial and vertical directions respectively, \( R \) and \( Z \) are coordinates after deformation while \( r \) and \( z \) are unstrained coordinates, \( r_0 \) is the reference radius (the radius at point A in Fig 5.5) and \( t \) is time. The notations \( \Theta \) and \( \theta \) are tangential angles in the cylindrical coordinates system. In axial symmetry, these angles are equal. The deformation gradient matrix becomes

\[ [F] = \begin{bmatrix} 1 & 0 & 0 \\ K & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ kt & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]  \hspace{1cm} (B-4)

and the Jacobian is

\[ J = \text{det}[F] = 1 \]  \hspace{1cm} (B-5)

The last equation implies that there is no volumetric change.

It is straightforward to determine that

\[ [L] = [\dot{F}][F]^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]  \hspace{1cm} (B-6)

and

\[ [D] = \frac{1}{2} ([L]^T + [L]) = \begin{bmatrix} 0 & \frac{k}{2} & 0 \\ \frac{k}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]  \hspace{1cm} (B-7)

\[ [\Omega] = \frac{1}{2} ([L]^T - [L]) = \begin{bmatrix} 0 & \frac{k}{2} & 0 \\ -\frac{k}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]  \hspace{1cm} (B-8)

The Jaumann stress rate is:

\[ \dot{\sigma}_{ij}^J = \dot{\sigma}_{ij} - \sigma_{ip}\dot{\Omega}_{pj} - \sigma_{jp}\dot{\Omega}_{pi} \]  \hspace{1cm} (B-9)

Assuming symmetric stress, we obtain from equation (B-8) and (B-9):

\[ \dot{\sigma}_{11}^J = \dot{\sigma}_{11} + k\sigma_{12} \]
\[ \dot{\sigma}_{22}^J = \dot{\sigma}_{22} - k\sigma_{12} \]
\[ \dot{\sigma}_{33}^J = \dot{\sigma}_{33} \]
\[ \dot{\sigma}_{12}^J = \dot{\sigma}_{12} - \frac{k}{2}(\sigma_{11} - \sigma_{22}) \]  \hspace{1cm} (B-10)

Assume that the material obeys the rate type stress-strain relation (hypo-elasticity):

\[ \dot{\sigma}_{ij}^J = \lambda\delta_{ij}\dot{d}_{kk} + 2\mu\dot{d}_{ij} \]  \hspace{1cm} (B-11)
Since \( d_{kk} \) is equal to zero in simple shear, the above equation is reduced to:

\[
\begin{align*}
\dot{\sigma}_{11} &= 2\mu d_{11} = 0 \\
\dot{\sigma}_{22} &= 2\mu d_{22} = 0 \\
\dot{\sigma}_{33} &= 2\mu d_{33} = 0 \\
\dot{\sigma}_{12} &= 2\mu d_{12} = \mu k
\end{align*}
\]  
(B - 12)

Combining equations (B-10) and (B-12), we obtain

\[
\begin{align*}
\dot{\sigma}_{11} + k\sigma_{12} &= 0 \\
\dot{\sigma}_{22} - k\sigma_{12} &= 0 \\
\dot{\sigma}_{33} &= 0 \\
\dot{\sigma}_{12} - \frac{k}{2}(\sigma_{11} - \sigma_{22}) &= \mu k
\end{align*}
\]  
(B - 13)

Substituting equation (B-12) into (B-13), we have

\[
\dot{\sigma}_{22} = -\dot{\sigma}_{11} = k\sigma_{12}
\]  
(B - 17)

Differentiation of equation (B-16) yields

\[
\dot{\sigma}_{12} - \frac{k}{2}(\dot{\sigma}_{11} - \dot{\sigma}_{22}) = 0
\]  
(B - 18)

Substitution of equation (B-17) into the equation (B-18) results in

\[
\dot{\sigma}_{12} + k^2\sigma_{12} = 0
\]  
(B - 19)

The solution for the above second order differential equation is easily obtained as:

\[
\sigma_{12} = A \cos kt + B \sin kt
\]  
(B - 20)

The constant \( A = 0 \) if \( \sigma_{12} = 0 \) at \( t = 0 \). Substituting \( \sigma_{12} \) into equation (B-17) and integrating it, we obtain

\[
\begin{align*}
\sigma_{11} &= B \cos kt - B \\
\sigma_{22} &= -B \cos kt + B
\end{align*}
\]  
(B - 21)

(B - 22)

provided \( \sigma_{11} \) and \( \sigma_{22} \) are equal to zero at \( t = 0 \). The constants \( B = \mu \) is found by substituting equations (B-20),(B-21) and (B-22) into equation (B-16). Finally, we obtain the stresses as follows:

\[
\begin{align*}
\sigma_{11} &= \mu(\cos kt - 1) \\
\sigma_{22} &= -\mu(\cos kt - 1) \\
\sigma_{33} &= 0 \\
\sigma_{12} &= \mu \sin kt
\end{align*}
\]  
(B - 23)

(B - 24)

(B - 25)

(B - 26)

If the material were subjected to non-zero initial stresses the closed form solution would be

\[
\begin{align*}
\sigma_{11} &= -\sigma_{12}^0 \sin kt + \mu(\cos kt - 1) + \sigma_{11}^0 \\
\sigma_{22} &= \sigma_{12}^0 \sin kt - \mu(\cos kt - 1) + \sigma_{22}^0 \\
\sigma_{33} &= \sigma_{33}^0 \\
\sigma_{12} &= \sigma_{12}^0 \cos kt + \mu \sin kt
\end{align*}
\]  
(B - 27)

(B - 28)

(B - 29)

(B - 30)
Appendix C: Constrained One-dimensional Compression

This type of deformation can be considered as a uniaxial loading with radial deformation constrained. This is the deformation pattern used in the one dimensional consolidation. Adopting the kinematic equations (A-1) to (A-9) except that $R = r$, $u_r = 0$ and $\dot{R} = 0$ due to the constrained radial deformation, the rate of deformation $d_{ij}$ is

$$[D] = [d_{ij}] = \frac{1}{2}([L]^T + [L]) = \begin{bmatrix} 0 & 0 & 0 \\ \frac{\partial u}{\partial r} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(C-1)

If the constitutive equation assumed is

$$\dot{\sigma}_{ij} = \lambda \delta_{ij} + 2\mu d_{ij}$$

(C-2)

we obtain:

$$\dot{\sigma}_{rr} = \lambda d_{kk}$$

(C-3)

$$\dot{\sigma}_{zz} = \lambda d_{kk} + 2\mu \frac{\partial \dot{Z}}{\partial Z}$$

(C-4)

$$\dot{\sigma}_{\theta\theta} = \lambda d_{kk}$$

(C-5)

Substituting equations (A-19) into the last three equations, we obtain:

$$\sigma_{zz} = \int (\lambda + 2\mu) \frac{k_0}{z_0 + k_0t} dt = (\lambda + 2\mu) \ln \lambda$$

(C-6)

$$\sigma_{rr} = \sigma_{\theta\theta} = \int \frac{k_0}{z_0 + k_0t} dt = \lambda \ln \lambda$$

(C-7)

It is defined

$$\beta = \frac{\sigma_{zz}}{E_v}$$

(C-8)

where $\beta$ is positive if $\sigma_{zz}$ is in tension while $\beta$ is negative if $\sigma_{zz}$ is in compression. Substituting equation (C-8) into (C-6), the stretch $\lambda$ can be expressed as:

$$\lambda = \exp \frac{\sigma_{zz}}{E_v} = \exp \frac{\beta (1 + \nu (1 - 2\nu))}{(1 - \nu)}$$

(C-9)
Appendix D : Cavity Expansion

(a) Kinematics:

According to the definition in section 5-4 and the illustration of Fig. 5-10, the field equation at the cavity wall is

\[ R_0 = r_0 + k_0 t \]  \hspace{1cm} (D-1)

The displacement field at an arbitrary radius can be described as

\[ R = r + u_r(r, t) \]  \hspace{1cm} (D-2)

\[ Z = z \]  \hspace{1cm} (D-3)

where \( u_r \) is the displacement in the radial direction, \( R \) and \( Z \) are deformed coordinates (state \( t_{n+1} \)) while \( r \) and \( z \) are unstrained coordinates (state \( t_0 \)).

The deformation gradient matrix becomes

\[ [F] = \begin{bmatrix} \frac{\partial R}{\partial r} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{R}{r} \end{bmatrix} \]  \hspace{1cm} (D-4)

and the Jacobian is

\[ J = \det [F] = \frac{\partial R}{\partial r} \times \frac{R}{r} \]  \hspace{1cm} (D-5)

The condition of zero volumetric change demands \( J = 1 \) such that

\[ \int_{r_o}^r r' dr' = \int_{R_0}^R R' dR' \]  \hspace{1cm} (D-6)

Solving the last equation we have

\[ R^2 - r^2 = R_0^2 - r_0^2 = a^2 \]  \hspace{1cm} (D-7)

where

\[ a^2 = 2k_0r_0t + k_0^2t^2 \]  \hspace{1cm} (D-8)

such that

\[ R = \sqrt{r^2 + a^2} = \sqrt{r^2 + 2k_0r_0t + k_0^2t^2} \]  \hspace{1cm} (D-9)

\[ \dot{R} = \dot{u}_r = \frac{\partial R}{\partial t} = \frac{k_0 R_0}{R} \]  \hspace{1cm} (D-10)

and

\[ u_r = \int \dot{u}_r dt = \int \frac{k_0 (r_0 + k_0 t)}{\sqrt{r^2 + 2k_0r_0t + k_0^2t^2}} dt = R - r \]  \hspace{1cm} (D-11)

By differentiating equation (D-9) with respect to \( r \), we obtain

\[ \frac{\partial R}{\partial r} = \frac{r}{R} \]  \hspace{1cm} (D-12)

such that the deformation gradient becomes

\[ [F] = \begin{bmatrix} \frac{r}{R} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{R}{r} \end{bmatrix} \]  \hspace{1cm} (D-13)
Applying equations (D4), (D-10) to (D-13), we have

\[ [L] = [\hat{F}] [F]^{-1} = \begin{bmatrix} \frac{\partial \hat{u}_s}{\partial r} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\hat{u}_r}{r} \end{bmatrix} \begin{bmatrix} \frac{\partial \hat{u}_s}{\partial r} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\hat{u}_r}{r} \end{bmatrix} = \begin{bmatrix} \frac{\partial \hat{u}_s}{\partial r} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\hat{u}_r}{r} \end{bmatrix} \]  

\[(D - 14)\]

Substituting equation (D-10) into equation (D-14) we obtain

\[ [L] = \begin{bmatrix} \frac{-k_0 R_0}{R^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{k_0 R_0}{R^2} \end{bmatrix} \]  

\[(D - 15)\]

The rate of deformation \(d_{ij}\) can be determined as:

\[ [D] = \frac{1}{2} ([L]^T + [L]) = \begin{bmatrix} \frac{-k_0 R_0}{R^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{k_0 R_0}{R^2} \end{bmatrix} \]  

\[(D - 16)\]

The above equation satisfies the requirement of incompressibility, i.e.,

\[ \text{div} \hat{X} = d_{kk} = 0 \]  

\[(D - 17)\]

The spin tensor \(\Omega_{ij}\) is:

\[ [\Omega] = \frac{1}{2} ([L]^T - [L]) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]  

\[(D - 18)\]

The above equation implies that there is no rotation effect in cavity expansion problems.

(b) Strains:

The two non-zero strains at a given time can be integrated as:

\[ \epsilon_{rr} = -\epsilon_{\theta\theta} = \int_0^t \frac{-k_0 R_0}{R^2} dr = -k_0 \int_0^t \frac{r_0 + k_0 r}{r^2 + 2k_0 r_0 r + k_0^2 r^2} dr = -\ln \frac{R}{r} \]  

\[(D - 19)\]

At \(r = r_0\) (cavity wall) we have

\[ (\epsilon_{rr})_0 = -(\epsilon_{\theta\theta})_0 = \int d_{rr} dt \]

\[ = \int_0^t \frac{-k_0}{R_0} dr = -\ln \frac{R_0}{r_0} \]  

\[(D - 20)\]

According to equation (D-20), the strains at cavity wall are singular if the expansion is started from \(r_0 = 0\).

(c) Effective Stresses - hypo-elasticity:

Assume the constitutive law as

\[ \sigma_{ij} = \lambda \delta_{ij} d_{kk} + 2\mu d_{ij} \]  

\[(D - 21)\]

Since there is no rotation of principal stresses and \(d_{kk} = 0\) we have

\[ \sigma_{rr} = \int 2\mu \frac{-k_0 R_0}{R^2} dt + \sigma_{rr}^0 = -2\mu \ln \frac{R}{r} + \sigma_{rr}^0 \]  

\[(D - 22)\]

\[ \sigma_{\theta\theta} = \int 2\mu \frac{k_0 R_0}{R^2} dt + \sigma_{\theta\theta}^0 = 2\mu \ln \frac{R}{r} + \sigma_{\theta\theta}^0 \]  

\[(D - 23)\]

\[ \sigma_{zz} = \sigma_{zz}^0 \]

\[ \sigma_{rz} = \sigma_{rz}^0 \]  

\[(D - 24)\]

\[(D - 25)\]

where \(\sigma_{rr}^0, \sigma_{\theta\theta}^0, \sigma_{zz}^0\) and \(\sigma_{rz}^0\) are initial stresses. For the above stresses, tension is taken as positive.
(d) Total Stresses and Pore Pressure - hypo-elasticity:

In what follows, we shall adopt the sign conversion that compression is positive. The equilibrium equation for the cavity expansion is

$$\frac{\partial \sigma_{rr}^t}{\partial R} + \frac{\sigma_{rr}^t - \sigma_{\theta \theta}^t}{R} = 0$$  \hspace{1cm} (D - 26)

where the superscript \( t \) denotes total. The difference between the total radial stresses and the total tangential stresses is (see equations (D-22) and (D-23))

$$\sigma_{\theta \theta}^t - \sigma_{rr}^t = \sigma_{\theta \theta}^t - \sigma_{rr}^t = -4\mu \ln \frac{R}{r}$$  \hspace{1cm} (D - 27)

where compressive stresses are taken as positive and the initial effective radial stress and the initial tangential stress are assumed to be equal. The initial total horizontal stresses is \( \sigma_{h}^t(= (\sigma_{rr}^t)_{R = \infty}) \) and the initial vertical total stress is \( \sigma_{v}^t(= (\sigma_{zz}^t)_{R = \infty}) \). The equilibrium equation (D-26) becomes

$$\frac{d\sigma_{rr}^t}{dR} = -4\mu \ln \frac{R \, dR}{r \, R}$$  \hspace{1cm} (D - 28)

The total radial stresses (\( \sigma_{rr}^t \)) at an arbitrary radius \( r \) can be determined by integrating the last equation from \( R \) to infinity, i.e.,

$$\sigma_{rr}^t = \sigma_{h}^t + \int_{R}^{\infty} 4\mu \ln \frac{R' \, dR'}{r \, R'} = \sigma_{h}^t + I$$  \hspace{1cm} (D - 29)

The value of \( I \) can be integrated numerically. The total tangential stress calculated by equations (D-27) and (D-29) is:

$$\sigma_{\theta \theta}^t = \sigma_{h}^t + I - 4\mu \ln \frac{R}{r}$$  \hspace{1cm} (D - 30)

The excess porewater pressure, obtained by subtracting the effective stresses from the total stresses, is

$$\Delta u_w = I - 2\mu \ln \frac{R}{r}$$  \hspace{1cm} (D - 31)

and, since there is no increase in effective vertical stress (see equation (D-24)), we have

$$\sigma_{zz}^t = \sigma_{v}^t + \Delta u_w = \sigma_{v}^t + I - 2\mu \ln \frac{R}{r}$$  \hspace{1cm} (D - 32)

(e) Hypo Elastic-Perfect Plasticity:

For an elasto-perfect plastic material, since the yield stress and the radius at the interface (\( R_b \) or \( r_b \)) between elastic zone and plastic zone can be solved a priori, a similar scheme as described above can be applied except the integration has to be separated into two parts - elastic zone and plastic zone. For a material obeying the Tresca yield criteria we have

$$\sigma_{rr}^t - \sigma_{\theta \theta}^t = \sigma_{rr}^t - \sigma_{\theta \theta}^t = 2c$$  \hspace{1cm} (D - 33)

where \( c \) is the undrained cohesion. The substitution of the elastic solution of stresses at \( R_b \) into the above equation yields

$$2c = 4\mu \ln \frac{R_b}{r_b}$$  \hspace{1cm} (D - 34)
Solving the last equation for \( r_b \) and \( R_b \), we obtain

\[
r_b = \sqrt{\frac{a^2}{e^{\frac{2}{3}} - 1}} \quad (D - 35a)
\]

or

\[
R_b = \sqrt{\frac{a^2 e^{\frac{2}{3}}}{e^{\frac{2}{3}} - 1}} \quad (D - 35b)
\]

By integrating equation (D-28) from \( R_b \) to infinity, we obtain the total radial stress at \( R_b \) as

\[
(s_{rr})_b = s^I_b + I_b = s^I_b + \int_{R_b}^{\infty} 4\mu \ln \frac{R dR}{r} \quad (D - 36)
\]

and the excess pore water pressure \((\Delta (u_w)_b)\) at \( R_b \) is

\[
\Delta (u_w)_b = I_b - 2\mu \ln \frac{R_b}{r_b} = I_b - c \quad (D - 37)
\]

The total stresses inside the plastic region \((R_0 \leq R \leq R_b)\) are (from equations (D-26) and (D-33))

\[
\sigma^I_{rr} = (s^I_b + \Delta (u_w)_b + c) + 2\mu \ln \frac{R_b}{R} \quad (D - 38)
\]

\[
\sigma^I_{\theta \theta} = (s^I_b + \Delta (u_w)_b - c) + 2\mu \ln \frac{R_b}{R} \quad (D - 39)
\]

\[
\sigma^I_{zz} = s^I_b + \Delta (u_w)_b + 2\mu \ln \frac{R_b}{R} \quad (D - 40)
\]

\[
\Delta u_w = \Delta (u_w)_b + 2\mu \ln \frac{R_b}{R} \quad (D - 41)
\]
Appendix E: Undrained Triaxial Compression (or Extension)

The undrained triaxial compression test is usually used to determine the undrained shear strength of a soil. The test is done by applying a vertical loading or a horizontal loading on a cylindrical sample. The term "undrained" means the loading is applied very fast such that no volumetric change is allowed. That is, the condition of incompressibility applies. In here, we provide finite strain solutions of a hypo elastic material subjected to undrained vertical compression and undrained radial compression.

(a) Undrained Vertical Compression:

Adopting the kinematic equations (A-1) to (A-9), we obtain

\[ [F] = \begin{bmatrix} \frac{dR}{R} & 0 & 0 \\ 0 & \lambda_z & 0 \\ 0 & 0 & \lambda_r \end{bmatrix} \]  
\[ (E-1) \]

and the Jacobian

\[ J = \det[F] = \frac{\partial R}{\partial r} \lambda_r \lambda_z \]  
\[ (E-2) \]

\[ [D] = \begin{bmatrix} \frac{d^2 R}{d^2 R} & 0 & 0 \\ 0 & \frac{d^2 R}{d^2 R} & 0 \\ 0 & 0 & \frac{d^2 R}{d^2 R} \end{bmatrix} \]  
\[ (E-3) \]

The incompressibility demands \( J = 1 \) such that

\[ \lambda_z R dR = r dr \]  
\[ (E-4) \]

Solving the last equation yields

\[ \frac{R^2}{r^2} = \lambda_z^{-1} \]  
\[ (E-5) \]

or

\[ R = r \lambda_z^{\frac{1}{2}} \]  
\[ (E-6) \]

Applying equation (E-6) and (A-13), we obtain

\[ \frac{\dot{R}}{R} = \frac{\partial \dot{R}}{\partial R} = \frac{-1}{2} \frac{\partial \dot{Z}}{\partial Z} = \frac{-1}{2} \frac{k_0}{z_0 + k_0 t} \]  
\[ (E-7) \]

The last equation satisfies the condition \( d_{kk} = 0 \) for an incompressible material. The total strains are obtained as follows:

\[ \epsilon_{rr} = \epsilon_{\theta \theta} = \frac{-1}{2} \int \frac{\partial \dot{Z}}{\partial Z} dt = \frac{-1}{2} \ln \lambda_z \]  
\[ (E-9) \]

\[ \epsilon_{zz} = \int \frac{\partial \dot{Z}}{\partial Z} dt = \ln \lambda_z \]  
\[ (E-8) \]

Clearly, the radial strain and tangential strain are one half of vertical strain.

If the material obeys the constitutive law:

\[ \dot{\sigma}_{ij} = \lambda \delta_{ij} d_{kk} + 2 \mu d_{ij} \]  
\[ (E-10) \]
the stresses are easily obtained as follows:

\[ \sigma_{rr} = \sigma_{\theta\theta} = -\mu \ln \lambda_z \]  \hspace{1cm} (E-11)

\[ \sigma_{zz} = 2\mu \ln \lambda_z \]  \hspace{1cm} (E-12)

In the undrained triaxial compression test, the confining compression is held constant. To satisfy the equilibrium, the excess pore water pressure (\( \Delta u_w \)) is generated which is equal to \( \mu \ln \lambda_z = (-\sigma_{rr}) \). The increment of total mean stress \( \Delta p' = \Delta u_w = \mu \ln \lambda_z \).

(b) Undrained Radial Compression:

For a material subjected to homogeneously radial compression, the kinematic equations can be assumed as:

\[ R_0 = r_0 + k_0 t \]  \hspace{1cm} (E-13)

\[ R = r + u_r = r + \frac{k_0 r}{r_0} t = r\left(1 + \frac{k_0 t}{r_0}\right) \]  \hspace{1cm} (E-14)

\[ Z = z + u_z \]  \hspace{1cm} (E-15)

The condition of incompressibility also gives the relationship \( \frac{k_0}{r_0} = \lambda_z^{-1} \). \( \lambda_z = \lambda_z^{-2} \).

The derivations of strains and stresses are similar to the procedures described and will not be repeated here. The strains obtained are:

\[ \epsilon_{rr} = \epsilon_{\theta\theta} = \int \frac{\partial R}{\partial \lambda} dt = \ln \lambda_r \]  \hspace{1cm} (E-16)

\[ \epsilon_{zz} = -2\ln \lambda_r \]  \hspace{1cm} (E-17)

The stresses (with constitutive law in equation (E-10)) obtained are:

\[ \sigma_{rr} = \sigma_{\theta\theta} = 2\mu \ln \lambda_r (= -\mu \ln \lambda_z) \]  \hspace{1cm} (E-18)

\[ \sigma_{zz} = -4\mu \ln \lambda_r (= 2\mu \ln \lambda_z) \]  \hspace{1cm} (E-19)

\[ \Delta u_w = \Delta p' = 4\mu \ln \lambda_r (= -2\mu \ln \lambda_z) \]  \hspace{1cm} (E-20)

It is of interest that (1) the vertical strain is also twice of radial strain and tangential strain as in the case of vertical undrained compression; (2) the effective stresses obtained are the same as undrained vertical compression but excess pore water pressure is twice.
List of References


56. Hvorslev, M.J., "Subsurface Exploration and Sampling of Soils for Civil Engineering Purposes," Waterways Experiment Stations, Vicksburg, Miss. (1949)


