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The semiclassical limit of the defocusing nonlinear Schrödinger flows

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THE SEMICLASSICAL LIMIT OF THE DEFOCUSING NONLINEAR SCHRÖDINGER FLOWS

by

Shan Jin

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As members of the Final Examination Committee, we certify that we have read the dissertation prepared by Shan Jin

entitled The Semiclassical Limit of the Defocusing Nolinear Schrodinger Flows

and recommend that it be accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy.

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Final approval and acceptance of this dissertation is contingent upon the candidate's submission of the final copy of the dissertation to the Graduate College.

I hereby certify that I have read this dissertation prepared under my direction and recommend that it be accepted as fulfilling the dissertation requirement.

David Levermore
Dissertation Director

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SIGNED: Jim Shaw

DEDICATION

This work is dedicated to my wife

Suguang Li,

my daughter

Sonia

and my parents

Xueqin Chai and **Zhongping Jin**;

for their love, encouragement and support.

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ABSTRACT

The Lax-Levermore strategy for analyzing the zero-dispersion limit of the KdV equation through its inverse scattering transform can be adapted to study the semiclassical limits of the defocusing nonlinear Schrödinger (NLS) equation, which are in fact the limits of corresponding conservation laws.

The weak limits of all conserved densities and their fluxes can be characterized in terms of the solution of a variational problem that in turn can be solved using function theory.

These results rest on a new formula for the N-soliton solutions and a WKB analysis of the semiclassical limit for the direct and inverse Zakharov-Shabat scattering transform.

Moreover, with Levermore's method, one can see that the limiting dynamics explored is hyperbolic and agrees with that obtained by classical nonlinear modulation theory.

The result is extended to the whole NLS hierarchy.

CHAPTER 1
INTRODUCTION

1.1 The NLS Equation

The one dimensional cubic Schrödinger equation :

$$i\psi_T = -\psi_{XX} + C|\psi|^2\psi \tag{1.1.1}$$

is a canonical nonlinear wave model, which arises throughout physics. For example, when the constant $C > 0$, it corresponds to propagation of a wave beam through a defocusing medium. It was first shown to be completely integrable by Zakharov and Shabat (Z-S) [20] and later generated as one of a large class of integrable equations by Ablowitz, Kaup, Newell and Segur (AKNS) [1].

One may be interested in the behavior of the solution $\psi(X,T)$ with a large X and T. If so, let us introduce a small parameter $0 < h << 1$:

$$ihU_t - \frac{1}{2}h^2U_{xx} + |U|^2U = 0. \tag{1.1.2}$$

We rescale T, X with the same small positive parameter h as follows, $T = -t/(2h)$, $X = x/h$ and change the dependent variable ψ with $\psi = U\sqrt{2/C}$. Thus, to study the behavior of solutions of equation (1.1) at big (T,X), we should explore the limits of solutions of equation (1.1.2) as h goes to zero.

Look at equation (1.1.2) again. This is just a usual Schrödinger equation of quantum mechanics with the potential $V(x)$ replaced by the nonlinear function $|U|^2$.

In the case of linear Schrödinger equations of quantum mechanics, the study of semiclassical behavior has a long history with a wealth of literature [17]. Following these classic studies, it could be very interesting to know the semiclassical limits of nonlinear Schrödinger equations (NLS), for example, as of equation (1.1.2).

It is natural to impose a spatially homogeneous far-field boundary condition of the form

$$U(x,t;h)\exp(\frac{-i\kappa^2 t}{h}) \to \kappa\exp(\frac{iS_\pm}{h}) \qquad \text{as} \qquad x \to \pm\infty. \tag{1.1.3}$$

Here κ, S_+, S_-, are real constants with $\kappa > 0$.

The initial condition, as in the usual linear case, is prescribed by two smooth real-valued functions, $A(x)$ and $S(x)$, which are independent of h:

$$U(x,0;h) = A(x)\exp(\frac{iS(x)}{h}).$$

It is assumed that the initial data is consistent with the far-field boundary condition (1.1.3). *i.e.*

$$A(x) \to \kappa \qquad \text{as} \qquad x \to \pm\infty,$$

and

$$S(x) \to \begin{cases} S_+ & (x \to +\infty) \\ S_- & (x \to -\infty). \end{cases}$$

For the requirement of the boundary conditions, it is natural to absorb the explicit time dependence of (1.1.3) into the $U(x,t;h)$, which will then satisfy

$$ihU_t - \frac{1}{2}h^2 U_{xx} - (\kappa^2 - |U|^2)U = 0. \tag{1.1.4}$$

After further absorption of a constant phase factor into $U(x,t;h)$, the boundary condition (1.1.3) can always be brought into the form

$$U(x,t;h) \to \begin{cases} \kappa & (x \to +\infty) \\ \kappa\exp(i\delta/h) & (x \to -\infty) \end{cases}, \tag{1.1.5}$$

where δ is a real constant and $\delta = S_- - S_+$. In fact, the initial function

$$U(x,0;h) = A(x)\exp(i(S(x) - S_+)/h) \stackrel{\text{def}}{=} U_I(x;h) \tag{1.1.6}$$

matchs (1.1.5).

In this paper we will study the semiclassical limits of the defocusing NLS initial value problem as given by (1.1.4) and (1.1.6). We will characterize various properties of these limits.

From now on, we consider the case that $\kappa = 1$ only. Since from (1.1.4), one can get

$$ih\tilde{U}_{\tilde{t}} - \frac{1}{2}h^2\tilde{U}_{\tilde{x}\tilde{x}} - (1 - |\tilde{U}|^2)\tilde{U} = 0,$$

if setting

$$\tilde{t} = t\kappa^2, \qquad \tilde{x} = x\kappa, \qquad \text{and} \qquad \tilde{U} = U/\kappa.$$

Thus $|U| \to \kappa$ if and only if $|\tilde{U}| \to 1$.

1.2 The Semiclassical Limits

The semiclassical limits are the dynamics obtained by letting $h \to 0$ for this initial value problem (1.1.4),(1.1.6). It is not clear directly from the NLS equation (1.1.4) what form such a dynamics might take. Insight into this question can be gained by considering the well-known conservation laws associated with the NLS equation :

$$\begin{cases} \partial_t\rho - \partial_x\mu & = & 0 \\ \partial_t\mu - \partial_x(\frac{\mu^2}{\rho} + \frac{1}{2}\rho^2) & = & -\frac{h^2}{4}\partial_x(\rho\partial_{xx}\log\rho) \\ \partial_t\varepsilon + \partial_x[2(1-\rho)\mu - h^2\mathcal{R}] & = & 0, \end{cases} \tag{1.2.7}$$

where

$$\mathcal{R} = -i\frac{h}{2}(U_x^*U_{xx} - U_xU_{xx}^*). \tag{1.2.8}$$

These conserved densities are related to $U = U(x,t;h)$ through

$$\begin{cases} \rho = |U|^2 \\ \mu = -i\frac{h}{2}(U^*U_x - UU_x^*) \\ \varepsilon = (1 - |U|^2)^2 + h^2|U_x|^2 = (1-\rho)^2 + \frac{\mu^2}{\rho} + h^2\frac{(\rho_x)^2}{4\rho}; \end{cases} \tag{1.2.9}$$

and initialed by

$$\begin{aligned} \rho(x,0) &= A^2(x) \stackrel{\text{def}}{=} \rho(x), \\ \mu(x,0) &= A^2(x)\partial_x S(x) \stackrel{\text{def}}{=} \mu(x). \end{aligned} \tag{1.2.10}$$

They are often treated as particle number density, momentum density and energy density [20].

Notice that (1.2.7) shows a system of perturbed Euler equations if of which all the $O(h^2)$ terms are negligible as $h \to 0$. Thus formally, the limits of densities ρ and μ, say $\bar{\rho}$ and $\bar{\mu}$, satisfy a hyperbolic system

$$\begin{aligned} \partial_t\bar{\rho} - \partial_x(\bar{\mu}) &= 0, \\ \partial_t\bar{\mu} - \partial_x(\bar{\mu}^2/\bar{\rho} + \bar{\rho}^2/2) &= 0; \end{aligned} \tag{1.2.11}$$

with initial data

$$\begin{aligned} \bar{\rho}(x,0) &= \rho(x), \\ \bar{\mu}(x,0) &= \mu(x), \end{aligned} \tag{1.2.12}$$

so long as the classical solution of (1.2.11) exists. The limiting energy density $\bar{\varepsilon}$ is then given by

$$\bar{\varepsilon} = (1-\bar{\rho})^2 + \frac{\bar{\mu}^2}{\bar{\rho}}.$$

As is well-known, by genuine nonlinearity, solutions of (1.2.11) will develop singular behavior (an infinite derivative) at some finite time, say t_b, for all but rarefaction initial data. Compared to (1.2.7), the neglected terms in (1.2.11) are dispersive. Consequently, the impending singularity is then expected to be regularized by development of small wavelength oscillations with an amplitude of order unity. One can confirm this by numerical analysis. Therefore, after the break-time t_b, the conserved densities can be expected to have weak limits at best [15].

It is interesting to see whether the solutions of (1.2.7) converge to those of (1.2.11) before the break-time t_b, and whether the limits of these conserved densities exist after t_b. If these limits exist in whatever sense, do they still satisfy some hyperbolic system?

1.3 The KdV Analogy

When stated this way, the problem has many similarities with that of the small dispersion limit of the Korteweg-de Vries (KdV) equation. In that case, Lax and Levermore [11] studied the limit as $\epsilon \to 0$ of the conserved densities $v(x,t;\epsilon)$, a sequence of real-valued solutions to the initial-value problem

$$\begin{aligned} v_t + 6vv_x + \epsilon^2 v_{xxx} &= 0, \\ v(x,0,\epsilon) &= v(x). \end{aligned}$$

After the break-time, the limit is weak $L^2(\Re)$ due to the development of regularizing small wavelength oscillations with an amplitude of order unity.

In a seminal paper, Gardner, Greene, Kruskal and Miura (GGKM) [5] showed that the KdV equation is completely integrable using the inverse scattering transform associated with a self-adjoint Schrödinger operator

$$L = -\epsilon^2 \partial_{xx} + u. \tag{1.3.13}$$

Lax and Levermore [11] analyzed the limiting behavior of the scattering and inverse scattering transform using a WKB analysis of (1.3.13) and a kind of steepest descent argument [18] to obtain a characterization of the (weak) limits in terms of solutions of a variational problem. The solution of this variational problem was then constructed through solving a Riemann-Hilbert problem. Lax and Levermore's results match the modulation theory of the KdV equation (see [3]) very well.

Moreover, Levermore [12] proved that the small dispersion limit of the KdV equation always satisfied a hyperbolic system even after the break-time t_b.

In the case of wave equations and their limits, research in modulation theory and the related approximate analysis are quite popular. Flaschka, Forest and McLaughlin (FFM) [3] summarized and generated these kinds of results for the KdV equation. Following FFM, Forest and Lee [4] found a similar structure in NLS equations.

Following GGKM, Z-S solved the NLS equation (1.1.1) successfully with the inverse scattering transform [20]. One may expect to use the Lax-Levermore strategy to analyze the semiclassical limits of the NLS initial problem (1.1.4)(1.1.6) since both the KdV and the NLS equation are completely integrable; they converge to hyperbolic systems at least near the initial time; they have real spectrum; and they have similar structures according to the modulation theory. However, it does not appear to be so obvious because the Z-S solution is in a formula of a determinant related to a complex matrix with double dimension and zero eigen-value. This matrix is a more complicated one than that of GGKM's KdV solution, which is related to a real, definite positive and lower dimension matrix.

As a part of our results, we find that the Z-S solution can be simplified to a form the very similar to that of the KdV solution. Thus, we can employ and improve the strategy of Lax and Levermore to analyze semiclassical limits of the NLS equation (1.1.4).

1.4 The NLS Hierarchy

Like the KdV equation, the NLS equation (1.1.1) is a basic member of the AKNS hierarchy. That means the equation (1.1.4) is completely integrable and has an infinite number of conservation laws. Thus, the term "semiclassical limits" should mean the simultaneous limits of all these conserved densities and fluxes. One important aspect of our work, which is not contained in earlier KdV studies, is that we simultaneously treat all of these quantities, which are related to an infinite number of Hamiltonians, or an infinite number of NLS flows (NLS hierarchy).

There are many ways to study these NLS flows. In order to get a uniform formula for all these conserved densities, we start with a linear eigenvalue problem recommended by Zakharov and Shabat [20] :

$$hV_x = PV = \begin{pmatrix} -i\lambda & iU^*(x,t;h) \\ -iU(x,t;h) & i\lambda \end{pmatrix} V, \qquad V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \tag{1.4.14}$$

Following Flaschka, Newell and Ratiu (FNR) [6] [16], the temporal evolution of members of the infinite hierarchy satisfy the relations

$$hV_{t_n} = Q^{(n)}V = \sum_{k=0}^{k=n} \lambda^k Q_{n-k}, \tag{1.4.15}$$

with $n = 1, 2, \cdots$. Here,

$$Q_n = -a_n H + b_n E + b_n^* F,$$

with

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and with $\{a_k, b_k\}$ determined iteratively by

$$\begin{aligned} a_0 &= i, b_0 = 0, \\ b_{k+1} &= i(h\partial_x b_k - i2a_k U^*)/2 \\ ha_{k+1} &= -i\int_{-\infty}^{x} (Ub_{k+1} + U^* b_{k+1}^*)\, dx. \end{aligned} \tag{1.4.16}$$

The *solvability condition* $\partial_x \partial_{t_n} V = \partial_{t_n} \partial_x V$ for (1.4.14) and (1.4.15) leads to

$$hP_{t_n} - hQ_x^{(n)} + [P, Q^{(n)}] = 0, \tag{1.4.17}$$

where $[P, Q]$ is the matrix commutator $PQ - QP$.

The evolution equation for U is then determined by (1.4.17). Different t_n means different time flow and leads to different equations for U, *i.e.*, different members of the defocusing NLS hierarchy. For example, if we choose $n = 2$, and let $t_2 = t$, then the equation for U will be

$$ihU_t = \frac{1}{2}h^2 U_{xx} - (|U|^2 - 1)U.$$

This is the equation found in (1.1.4). We will call it a *t-flow* hereafter.

We display our research on all the NLS flows simultaneously from the following basic knowledge :

(a) All of these equations

$$hP_{t_n} = hQ_x^{(n)} - [P, Q^{(n)}] \qquad \forall n$$

commute with each other.

(b) All of these equations are Hamiltonian. It means we have a sequence of real functionals $\{\mathcal{H}_n\}_0^\infty$ such that

$$-ihU_{t_n} = \frac{\delta \mathcal{H}_n}{\delta U^*}.$$

For example,

$$\mathcal{H}_0 = \int_{-\infty}^{+\infty} |U|^2 \, dx,$$

$$\mathcal{H}_1 = \frac{ih}{2} \int_{-\infty}^{+\infty} (UU_x^* - U^*U_x) \, dx$$

and

$$\mathcal{H}_2 = \frac{1}{2} \int_{-\infty}^{+\infty} (|U|^2 - 1)^2 + h^2 |U_x|^2) \, dx$$

(c) All of these $\{\mathcal{H}_n\}$, $n = 0, 1, 2, ...$ are integrals of certain polynomials, say $\{\rho_{n+1}\}$, in terms of U, U^* and their x derivatives up to n-th order. These $\{\rho_n\}$ are locally conserved because we can always find $\mathcal{G}_n$ such that

$$\partial_t \rho_n = \partial_x \mathcal{G}_n, \qquad n = 1, 2, 3... \, . \tag{1.4.18}$$

It is called the n-th *conservation law* for the t-flow. Here ρ_n is the n-th conserved density, and $\mathcal{G}_n$ its flux.

In general, the n-th conservation law for the m-th flow has the following form

$$\partial_{t_m} \rho_n = \partial_x \mathcal{G}_{nm}, \qquad n, m = 1, 2, 3, ... \, .$$

Here ρ_n is the n-th conserved density, and $\mathcal{G}_{nm}$ its flux.

For example, for the t-flow, the first three ρ_i are $\rho_1 = \rho - 1$, $\rho_2 = \mu$ and $\rho_3 = \varepsilon$. For $n = 1, 2, 3, \cdots$ with $x = t_1$, we have:

$$\begin{aligned} \rho_n &= \int_{-\infty}^{x} \partial_{t_n} \rho_1 \, dx \\ &= \int_{-\infty}^{x} \partial_{t_n} |U|^2 \, dx \\ &= \int_{-\infty}^{x} (U^* \partial_{t_n} U + U \partial_{t_n} U^*) \, dx. \end{aligned}$$

Among these conserved densities, the limits of the first two, i.e., ρ and μ, seem a little more important. We claim that not only because they are simpler, but also because of the following representation :

$$\rho_n(x, t; h) = \int_{-\infty}^{x} \partial_{t_n} \rho_1 \, dx,$$

and

$$\mathcal{G}_n(x, t; h) = \int_{-\infty}^{x} \partial_{t_n} \rho_2 \, dx.$$

Thus, $\{\bar{\rho}_n, \bar{\mathcal{G}}_n\}$ can be constructed by $\{\bar{\rho}, \bar{\mu}\}$, and $\{\bar{\rho}_n, \bar{\mathcal{G}}_n\}$ could break down if $\{\bar{\rho}, \bar{\mu}\}$ do.

1.5 The Main Results

We will show that the limits of ρ and μ exist for all the time as $h \to 0$. First we improve the NLS solution formula of Z-S to the similar one GGKM once derived for the KdV equation. Next we construct formulas, of these limits, which are valid for all the time with the Lax-Levermore strategy. They will satisfy the initial value problem (1.2.11) so long as the classical solutions of this problem exist. They can be described by a hyperbolic system even after the break-time t_b.

Like ρ and μ, the limits of all these conserved densities and their fluxes do exist in the distributional sense and can be constructed. Moreover in some special cases, as we mentioned before, there is a critical time t_b. Before that time, the convergence of these limits will be in the meaning of L^1 strong convergence; but after that time, of L^1 weak convergence. We will discuss that and give a proof for that case.

1.6 The Outline

The paper is organized in the following way :

In Section 2.1, we introduce the Z-S scattering transform to solve the direct scattering problem for given initial data. This scattering problem is then analyzed by WKB techniques. We replace the exact initial data $\{\rho(x), \mu(x)\}$ by an approximate pair $\{\rho(x;h), \mu(x;h)\}$, whose exact scattering data are equal to the WKB approximate data of $\{\rho(x), \mu(x)\}$. As will be shown later,

$$\lim_{h\to 0}\{\rho(x;h), \mu(x;h)\} = \{\rho(x), \mu(x)\}$$

in the *strong* $L^1(\Re)$ sense if both $\rho(x)$ and $\mu(x)$ are constants after a finite domain. By the way, the WKB method shows that the scattering data we will adopt are not valid if $\rho = |U|^2 = 0$. That is why we assume that the initial condition of ρ is never zero and ρ converges to a nonzero constant 1 where $x \to \mp\infty$.

In Section 2.2, the Z-S inverse scattering method is used to obtain an explicit formula of $\{\rho(x,t;h), \mu(x,t;h)\}$ which is initiated by $\{\rho(x;h), \mu(x;h)\}$. This explicit formula describes a solution which is consisting of N solitons through a logarithmic derivative of a determinant. Then we simplify this representation of solutions from a second derivative of the logarithm of a $2N \times 2N$ complex matrix determinant to that of a $N \times N$ real, definite positive matrix determinant. Only by this new formula can one apply the Lax-Levermore strategy to analyze the semiclassical limits of the NLS equation (1.1.4). Moreover, by this new formula, we can compute explicitly for all of these $\{\rho_n, \mathcal{G}_{nm}\}$.

In Section 3.1, we carry out the limit $h \to 0$ from the new formula. We show that

$$\lim_{h\to 0}\{\rho(x,t;h), \mu(x,t;h)\} = \{\bar{\rho}(x,t), \bar{\mu}(x,t)\}$$

exists in the sense of distribution convergence. We also show that, in a natural case, the limit pair exists in the sense of weak convergence in $L^1(\Re)$ with respect to x, and that the weak limit $\{\rho, \mu\}$ can be described as

$$\begin{cases} \bar{\rho} = 1 + \frac{1}{2}\partial_{xx}Q^* \\ \bar{\mu} = \frac{1}{2}\partial_{xt}Q^*, \end{cases} \tag{1.6.19}$$

or generally for the n^{th}-flow,

$$\bar{\rho}_n = \frac{1}{2}\partial_{xt_n}Q^*, \qquad \bar{\mathcal{G}}_n = \frac{1}{2}\partial_{tt_n}Q^*; \tag{1.6.20}$$

where we set $\vec{t} = (t_2, t3, \cdots, t_n, \cdots)$ and assume that $\vec{t}$ belongs to $\Re^\omega$, an infinite dimension real linear space of which each member only has finite nonzero coordinates. Sometimes for convenience we also set $\vec{t} = (t_1, t2, \cdots, t_n, \cdots)$ with $t_1 = x$.

$\bar{\rho}_n$ and $\bar{\mathcal{G}}_n$ exist in the sense of *weak* $L^1(\Re)$ convergence if ρ and μ do.

The function $Q^*(x, \vec{t})$ is determined by solving a variational problem (VP), which is a minimum programming problem with respect to functions belonging to a special admissible set, i.e.,

$$Q^*(x, \vec{t}) = \inf_{0 \le \psi \le \phi} Q(\psi; x, \vec{t}).$$

Here $Q(\psi; x, \vec{t})$ is a quadratic functional of ψ, which depends linearly on the parameters x and $\vec{t}$. The function ϕ is determined by the initial data of the NLS equation. That is :

$$\phi(\sigma) = \Re \int \frac{\sin\sigma}{2} \left(\sqrt{\frac{\cos\hat{\sigma}_1(y) - \cos\sigma}{\cos\hat{\sigma}_2(y) - \cos\sigma}} + \sqrt{\frac{\cos\hat{\sigma}_2(y) - \cos\sigma}{\cos\hat{\sigma}_1(y) - \cos\sigma}} \right) dy, \tag{1.6.21}$$

where $\hat{\sigma}_k$ $_{k=1,2}$ are functions of the initial data, and $\sigma \in (0, \pi)$.

In Section 3.2, we show that Q is continuous in a weak sequential topology and that the space of these admissible functions is compact in that topology. We further show that Q is a strictly convex function. Since the admissible functions form a convex set, this implies that the minimum of Q is taken at a unique function and also is the only one that satisfies the variational condition of VP.

In this section, the variational condition of VP is analyzed. It is then converted to a Riemann-Hilbert problem, to a problem of determining an analytic function which is in H^p and is defined on a unit disc by its boundary data.

In Section 4.1, we solve the Riemann-Hilbert problem for time t that does not exceed t_b, called the 'break time', defined as the time at which the solution of the initial value problem (1.2.11), (1.2.12) breaks down. The break-time t_b can be estimated by the Lax recipe [10]. We show that for $t < t_b$, $\{\rho(x,t;h), \mu(x,t;h)\}$ and all $\{\rho_n(x,t;h), \mathcal{G}_n(x,t;h)\}$ tend to the solution of (1.6.19) or (1.6.20) as h goes to zero. We also show that the convergence of either ρ or μ is a strong convergence in the sense of $L^1(\Re)$ convergence with respect to x, if it is a weak one already.

In Section 4.2, we solve the Riemann-Hilbert problem for any time t. We verify that the solution so obtained satisfies the variational condition, therefore that it solves the minimum problem VP. The structure of limiting solutions $\{\bar{\rho}, \bar{\mu}\}$ determined by (1.6.19) or (1.6.20) also matches modulation theory. That means they have a form suggested by McLaughlin [14] and more generally as in the KdV case described by Flaschka, Forest and McLaughlin [3]. These forms are also recommended for the NLS by Forest and Lee [4].

In sections of Chapter 5, the dynamics beyond the break time is analyzed. Strict hyperbolicity and genuine nonlinearity are proved by using the Levermore strategy [12] for the equations that the NLS semiclassical limits satisfy, wherever the Riemann invariants, say $\hat{\lambda}_k$, remain distinct. More specifically, the characteristic speeds, say S_k, have the opposite ordering of the Riemann invariants

$$\hat{\lambda}_j > \hat{\lambda}_k \ \text{ or } \ (j < k) \qquad \Rightarrow \qquad S_j < S_k,$$

and the derivative of a characteristic speed with respect to its corresponding Riemann invariant is negative

$$\partial_{\hat{\lambda}_k} S_k < 0.$$

A Riccati equation is found for the evolution of $\partial_x \hat{\lambda}_k$ along the k^{th} characteristic at each k, and some implications to breaking are discussed.

CHAPTER 2

SCATTERING DATA AND FORMULA

2.1 Asymptotic Analysis of Scattering Data

2.1.1 Equation and Initial Condition

With Zakharov and Shabat's steps [20], we can solve the following initial value problem with a non-vanished far-field boundary conditions of a defocusing NLS equation :

$$\begin{cases} ihU_t - \frac{1}{2}h^2 U_{xx} - (1-|U|^2)U = 0 \\ U_{t=0} = A(x)e^{i\frac{S(x)-S_+}{h}} \to \begin{cases} 1 & (x \to +\infty) \\ e^{\frac{i\delta}{h}} & (x \to -\infty), \end{cases} \end{cases} \tag{2.1.1}$$

where

$$S(x) \to \begin{cases} S_+ & x \to +\infty \\ S_- & x \to -\infty, \end{cases}$$

and δ, S_+, S_- are constants independent of h as mentioned in (1.1.5).

We assume the initial data $\{A(x), S(x)\}$ tend to constants as $|x| \to \infty$ so fast that (See Figure 1,2)

$$\int_{-\infty}^{+\infty} [1 - A^2(y)](1+y^2)\,dy < \infty$$

and

$$\int_{-\infty}^{+\infty} |\frac{\partial S(y)}{\partial y}|(1+y^2)\,dy < \infty. \tag{2.1.2}$$

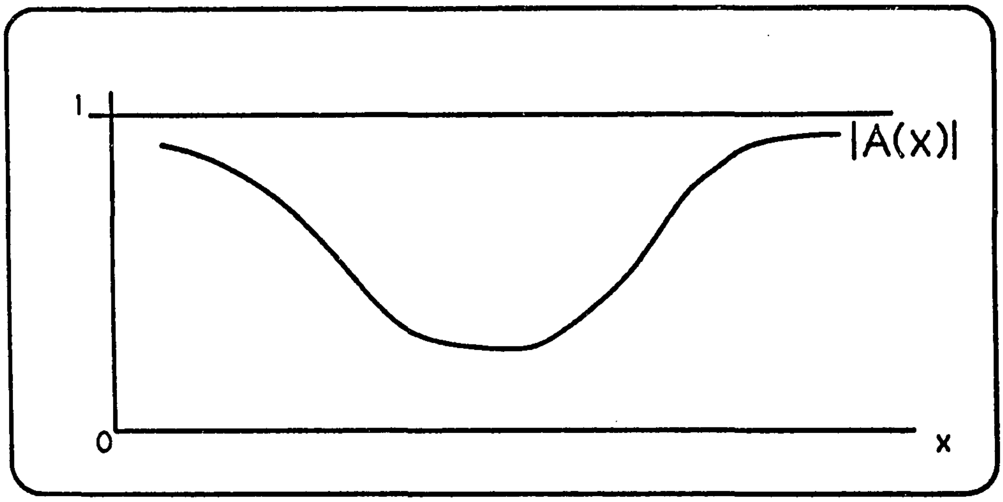

Figure 2.1: The graph of initial function $|A(x)|$

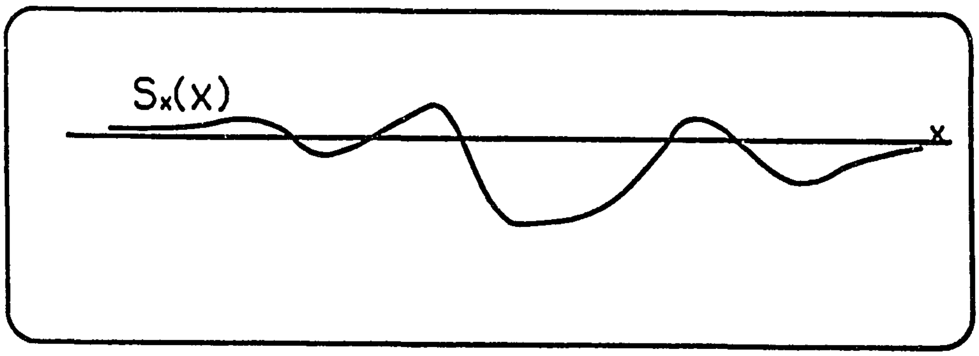

Figure 2.2: The graph of initial function $S_x(x)$

For the latter asymptotic analysis, we need boundness assumption about these initial data. We assume there exist two positive constants c_1, c_2 such that

$$
\begin{aligned}
|A(x)| \geq c_1, |A_x| \leq c_2, |A_{xx}| \leq c_2,& \\
|S_x| \leq c_2, |S_{xx}| \leq c_2.& \qquad (2.1.3)
\end{aligned}
$$

for all x. That means A is bounded away from zero and, the derivatives of A, S up to a second order are uniformly bounded.

2.1.2 Scattering Data and Scattering Transform

Z-S method associates each solution of the NLS equation (2.1.1) with a one parameter family of Schrödinger operators $\mathcal{L}(h)$:

$$
\mathcal{L} = \begin{pmatrix} ih\partial_x & U^*(x,t;h) \\ U(x,t;h) & -ih\partial_x \end{pmatrix}. \qquad (2.1.4)
$$

For the above Schrödinger operator $\mathcal{L}$ whose initial potential U, say $U_I(x;h)$, satisfies (2.1.2) and (2.1.3), one can define scattering data. These data consist of

(i) the *reflection coefficient* $R(k)$,

(ii) the *eigenvalues* λ_j, $_{j=1,\dots,N}$, (the point spectrum) of $\mathcal{L}$ $(-1 < \lambda_j < +1)$,

(iii) the *norming constants* c_j associated with the eigenfunction f^j, $_{j=1,\dots,N}$.

We need to recall the definition of the norming constants. Let f be the eigenfunction of $\mathcal{L}$ associated with eigenvalue λ,

$$
\mathcal{L}f = \begin{pmatrix} ih\partial_x & U_I^*(x;h) \\ U_I(x;h) & -ih\partial_x \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \lambda \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \qquad (2.1.5)
$$

normalized by

$$\int_{-\infty}^{+\infty} |f_1 f_2|\, dx = 1. \tag{2.1.6}$$

Using condition (2.1.2) and (2.1.3), it is easy to show that,

Theorem 2.1

The eigenfunction $f(x)$ decays exponentially as $|x| \to \infty$.

It is an application of the WKB method and the proof will be given later in this section.

More precisely, there is a constant c, such that

$$f(x) \approx c \begin{pmatrix} \lambda - i\nu \\ 1 \end{pmatrix} \exp(-\nu x/h) \qquad \text{as } x \to +\infty, \tag{2.1.7}$$

where $\nu = \sqrt{1-\lambda^2}$. This constant c, chosen to be positive, is the *norming constant.*

We call this operation a *scattering transform*, since it relates the potential U of $\mathcal{L}$ to the scattering data $\{R(k), \lambda_j, c_j\}$. It can be inverted with the aid of the Marchenko equations. This method to solve the NLS equation is first obtained by Zakharov and Shabat [20], and then grows to the well known AKNS hierarchy [1].

Follow them, if the potential U varies with the temporal variable t such that the NLS equation (2.1.1) is satisfied, then the scattering data of U vary in a particular simple manner with t :

(i) $R(k,t) = R(k)\exp\{2ikt\sqrt{k^2-1}/h\}$,

(ii) λ_j is independent of t,

(iii) $c_j(t) = c_j \exp\{-\lambda_j \nu_j t/h\}$, $_{j=1,\ldots,N}$.

Formulas above are derived from those given by Z-S, who treat the case $h = 1$, by the simple rescaling

$$U(x,t) = u\left(\frac{x}{h}, -\frac{t}{2h}\right). \tag{2.1.8}$$

A similar rescaling will be applied at the end of this section to the formulas of Z-S for the inverse scattering transform.

The Z-S method to solve the initial value problem of the NLS equation is first to take the scattering transform of the initial data, then to apply the formulas of $c_j(t)$ to determine the scattering data at time t, and next to invert the scattering transform at that time. (see Appendix).

The temporal forms of the scattering data can be extended by AKNS formula [16] to all the other flows of the NLS hierarchy.

Theorem 2.2

Let the operator L be given by (2.1.4). If the potential U varies with multiple parameter t_n, $n = 1, 2, \dots$ because of the different flows of the defocusing NLS hierarchy, then their scattering data can be determined generally as following :

(i) $R(k, t_n) = R(k)\exp\{2ik^{n-1}\sqrt{k^2-1}\,t_n/h\}$,

(ii) λ_j *is independent of* t_n,

(iii) $c_j(t_n) = c_j \exp\{-\lambda_j^{n-1}\nu_j t_n/h\},\ {}_{j=1,\dots,N}$.

PROOF :

Recall (1.4.15),

$$hV_{t_n} = Q^{(n)}V = \sum_{k=0}^{k=n} \lambda^k Q_{n-k},$$

where the elements of Q are U, U^* or their derivatives or integration of them or only some constants. With the iterating formula (1.4.16), one can choose proper constants such that

$$\lim_{x\to\infty} Q_n = 0, \qquad \text{for } n \geq 2.$$

For example, in case $n = 2$, we may set

$$b_2 = i(h\partial_x b_1 - 2ia_1 U)/2 = -hU_x/2$$

and

$$a_2 = (-\frac{i}{h})\int_{-\infty}^{x}(-\frac{h}{2})|U|_x^2\,dx = \frac{i}{2}(|U|^2 - 1),$$

then

$$\lim_{x\to\infty} a_2 = 0 \qquad \text{and} \qquad \lim_{x\to\infty} b_2 = 0.$$

Now let

$$V_\infty = \lim_{x\to\infty} V,$$

then

$$h\partial_{t_n} V_\infty = (\lambda^n Q_0 + \lambda^{n-1} Q_1)V_\infty = \begin{pmatrix} -i\lambda^n & i\lambda^{n-1}U^* \\ -i\lambda_{n-1}U & i\lambda^n \end{pmatrix} V_\infty. \tag{2.1.9}$$

Comparing to Z-S case (n = 2), the difference is only the order number of the power of λ. So, for t_n $n \geq 3$, by doing same procedure as one did in Z-S case but with different order number of the power of λ, we get (see the following remark)

$$R(k, t_n) = R(k)\exp\{2ik^{n-1}\sqrt{k^2-1}t_n/h\}$$

and

$$c_j(t_n) = c_j \exp\{-\lambda_j^{n-1}\nu_j t_n/h\}.$$

Moreover, λ_j is independent of t_n, for all j and n. If we set $t_2 = t$, and $t_1 = x$, then the above formulas really match the Z-S model as given by (1.1.4). It completes the proof.

REMARK :

A simple way to figure out the forms of the scattering data for general case is to change the equation given by (2.1.9) to the following:

$$h\partial_{\tilde{t}} V_\infty = (\lambda^2 Q_0 + \lambda Q_1)V_\infty = \begin{pmatrix} -i\lambda^2 & i\lambda U^* \\ -i\lambda U & i\lambda^2 \end{pmatrix} V_\infty.$$

with $\tilde{t}$ instead of $\lambda^{n-2}t$.

However, this is exact the relation which Z-S analyzed to get the forms of scattering data for the t-flow. It is then easy to see, for the higher flow, the general forms of the reflection coefficient and the norming constant are exactly as same as the forms of the t-flow except with k^{n-1} instead of k for reflection coefficient or with λ^{n-1} instead of λ for norming constant.

You may also use $\tilde{t} = \lambda^{n-1}t$ to modify (2.1.9) and then get the same question as shown at (1.4.14) except with $\tilde{t}$ instead of x. That is why we call the x-traveling wave as the t_1 flow.

2.1.3 WKB Analysis of Eigenvalue and Eigenfunction

In this paper we will assume that the initial data which satisfy a little further conditions in addition to (2.1.2) :

$$-1 \le \hat{\lambda}_2(x) < \hat{\lambda}_1(x) \le +1 \tag{2.1.10}$$

for all x, and assume that both $\hat{\lambda}_k(x)_{\ k=1,2}$ are C^1 with only a finite number of critical points. Here $\hat{\lambda}_k(x)_{\ k=1,2}$ are defined by

$$\begin{aligned} \hat{\lambda}_1(x) &= \frac{1}{2}\partial_x S(x) + |A(x)|, \\ \hat{\lambda}_2(x) &= \frac{1}{2}\partial_x S(x) - |A(x)|. \end{aligned} \tag{2.1.11}$$

Condition (2.1.2) is to make the machinery of scattering theory work smoothly. Formula (2.1.10) is a genuine restriction that allows us to neglect reflection. Condition that following (2.1.10) is for technical reasons. In fact for simplicity, we will assume that only one single critical point could appear, *i.e.* $\hat{\lambda}_1(x)$ has a single minimum, $\hat{\lambda}_2(x)$ has a single maximum; and moreover $\hat{\lambda}_{2,max} < \hat{\lambda}_{1,min}$. One can see later that how $\{\hat{\lambda}_k,\ {}_{k=1,2}\}$ comes naturally.

Now the asymptotic behavior of the scattering data as $h \to 0$ can be explored by the WKB method :

Theorem 2.3

Let

$$\mathcal{L} = \begin{pmatrix} ih\partial_x & U^* \\ U & -ih\partial_x \end{pmatrix}$$

be a Schrödinger operator as given by (2.1.4). Denote by $N(h)$ the number of eigenvalues of $\mathcal{L}$; then

$$\begin{aligned} N(h) &\approx \tfrac{1}{h\pi}\left(\Re\textstyle\int\sqrt{[1-\hat\lambda_1(y)][1-\hat\lambda_2(y)]}\,dy\right) \\ &+\tfrac{1}{h\pi}\left(\Re\textstyle\int\sqrt{[1+\hat\lambda_1(y)][1+\hat\lambda_2(y)]}\,dy\right) \\ &\approx N(h,1)+N(h,-1). \end{aligned} \tag{2.1.12}$$

More generally, the number $N(h,\lambda)$ of eigenvalues between $(\hat\lambda_{1,min},\lambda)$ or $(\lambda,\hat\lambda_{2,max})$ is

$$N(h,\lambda) \approx \frac{1}{h\pi}(\Re\int\sqrt{[\lambda-\hat\lambda_1(y)][\lambda-\hat\lambda_2(y)]}\,dy). \tag{2.1.13}$$

Since each λ, the eigenvalue, is independent of time, we only need $U = U_I(x;h)$, the initial function, to calculate these λ. We assume this function $U_I(x;h)$ satisfies (2.1.2), (2.1.3) and (2.1.10).

Suppose $\hat\lambda_k(x), k = 1,2$ satisfy (2.1.10) and have a single minimum or maximum respectly. Then, for $\lambda > \hat\lambda_{1,min}$ or $\lambda < \hat\lambda_{2,max}$, there are two functions $x_+(\lambda)$ and $x_-(\lambda)$, with $x_-(\lambda) \le x_+(\lambda)$ defined by (see Figure 3)

$$\hat\lambda_1(x_-) = \hat\lambda_1(x_+) = \lambda, \tag{2.1.14}$$

or

$$\hat\lambda_2(x_-) = \hat\lambda_2(x_+) = \lambda.$$

Obviously $\hat\lambda_2(x) < \hat\lambda_1(x) < \lambda$ or $\hat\lambda_1(x) > \hat\lambda_2(x) > \lambda$ for $x_- < x < x_+$. Thus (2.1.13) can be written as

$$N(h,\lambda) \approx \frac{1}{h\pi}\begin{cases} \Phi(\lambda) & \lambda > \hat\lambda_{1,min} \\ -\Phi(\lambda) & \lambda < \hat\lambda_{2,max}, \end{cases} \tag{2.1.15}$$

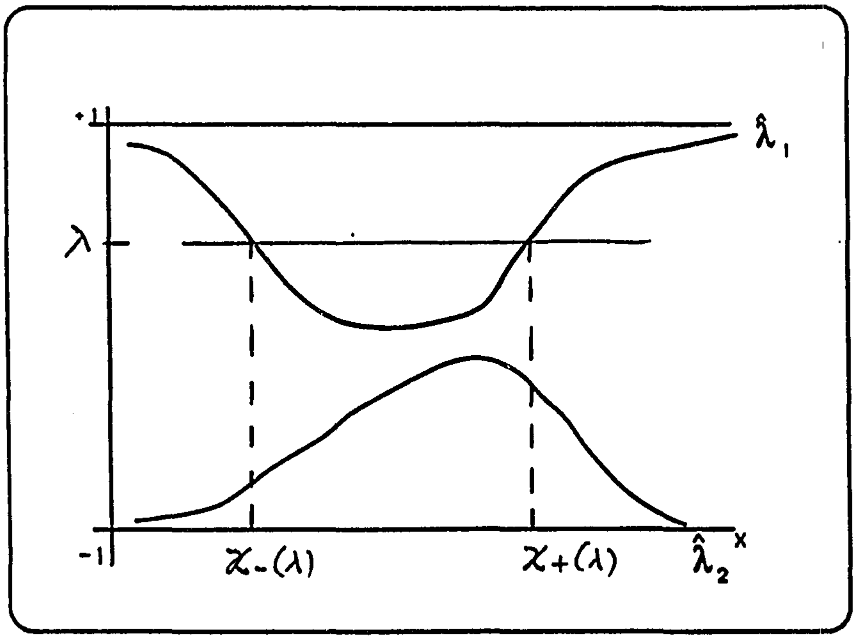

Figure 2.3: **The graph of initial turning points in λ representation**

where

$$\Phi(\lambda) = \begin{cases} \int_{x_-(\lambda)}^{x_+(\lambda)} \sqrt{(\lambda - \hat{\lambda}_1(y))(\lambda - \hat{\lambda}_2(y))}\, dy & \hat{\lambda}_{1,min} < \lambda < 1 \\ 0 & \hat{\lambda}_{2,max} \le \lambda \le \hat{\lambda}_{1,min} \\ -\int_{x_-(\lambda)}^{x_+(\lambda)} \sqrt{(\lambda - \hat{\lambda}_1(y))(\lambda - \hat{\lambda}_2(y))}\, dy & -1 < \lambda < \hat{\lambda}_{2,max} \end{cases} \tag{2.1.16}$$

and the domain of $x_-(\lambda)$ and $x_+(\lambda)$ is $(-1, \hat{\lambda}_{2,max}) \cup (\hat{\lambda}_{2,max}, 1)$.

We begin the proof of Theorem 2.1 and Theorem 2.3.

PROOF :

From (2.1.5), the equations of $f_{1,2}$ should be

$$\begin{cases} \partial_{xx} f_1 - \partial_x f_1 (U_x^*/U^*) + f_1(\lambda^2 - |U|^2 - i\lambda h U_x^*/U^*)h^2 = 0 \\ f_2 = f_1^*. \end{cases}$$

Setting $f_1 = Z\sqrt{U^*}$, we get the equation for Z :

$$\partial_{xx} Z + Z\left(\frac{U_{xx}^*}{2U^*} - \frac{3(U_x^*)^2}{4(U^*)^2} + (\lambda^2 - |U|^2)/h^2 - \frac{i\lambda U_x^*}{hU^*}\right) = 0.$$

At the initial time, with the definition of A and S, we have

$$U_x/U^* = (A_x/A + iS_x/h),$$

$$U^*_{xx}/(2U^*) = A_{xx}/(2A) + iS_{xx}/(2h) + iS_xA_x/(hA) - S_x^2/(2h^2)$$

and

$$3(U_x^*)^2/4(U^*)^2 = (\frac{3}{4})[(A_x/A)^2 + 2iS_xA_x/(hA) - S_x^2/h^2].$$

We get

$$\partial_{xx}Z + Z\{[(\lambda + S_x/2)^2 - |A|^2]/h^2 + M\} = 0,$$

with

$$M = -i[(A_xS_x)/(2A) - S_{xx}/2 - \lambda A_x/A]/h + [A_{xx}/(2A) - 3(A_x/A)^2/4].$$

Comparing with (2.1.3),

$$M = O(1/h).$$

Considering the leading order while $h \to 0$, we get

$$h^2Z_{xx} + Z[(\lambda - \hat{\lambda}_1)(\lambda - \hat{\lambda}_2)] = 0, \tag{2.1.17}$$

where $\hat{\lambda}_{1,2}$ are given by (2.1.11). That shows why we need to introduce $\hat{\lambda}_{1,2}$ and make them very important later.

REMARK :

Here and later in this paper, we fix U, the potential of $\mathcal{L}$, as the solution of equation (1.1.4).

If we want to start with other flows of the NLS hierarchy, say t_n flow $n \neq 2$, then $\hat{\lambda}_1$ and $\hat{\lambda}_2$ will of course be different with which given by (2.1.11). For the t-flow we study now, the time evolution of $\{\hat{\lambda}_1, \hat{\lambda}_2\}$ can be shown equivalent with the time evolution of $\{\bar{\rho}, \bar{\mu}\}$ which is the limit pair of the first two conserved densities. However in general, if one deal with the n-th flow, the time evolution of $\{\hat{\lambda}_1, \hat{\lambda}_2\}$ should be consistent with the time evolution of

$$\{\lim_{h\to 0} \rho_1, \qquad \lim_{h\to 0} \rho_n\}$$

for natural.

Moreover, $\{\hat{\lambda}_1, \hat{\lambda}_2\}$ will be the first set of the Riemann invariants during the time evolution of the semiclassical limits of the NLS hierarchy. We will disclose these facts eventually.

Now, let us go back to the proof of Theorem 2.3 and the equation of Z given by (2.1.17). For bounded and integrable Z, only possible choice of λ, the eigenvalue of $\mathcal{L}$, is that

$$1 > \lambda > \hat{\lambda}_{1,min} \qquad \text{or} \qquad -1 < \lambda < \hat{\lambda}_{2,max}. \tag{2.1.18}$$

If not, for example, $|\lambda| \geq 1$, then the coefficient of Z remains positive sign. That means only oscillation type of Z is possible. Thus Z is bounded but not integrable. If in the opposite case, $\hat{\lambda}_{1,min} > \lambda > \hat{\lambda}_{2,max}$, then the coefficient of Z remains same negative for all x. That means Z is exponential varying. Z will exponential decay at one end but must exponential grow in the other end, and is even boundedless. So, the only choice is (2.1.18). That means the coefficient of Z needs to change sign so that one can get a piecewise Z which is integrable. That Z will oscillate in some finite domain and exponential decay at both far-field.

The points of x which make the coefficient vanish are the singularities of the equation of Z. The WKB method offers a connecting formula of the solutions near these points. It is based on the fact that, in either case of (2.1.18), the possible λ should make the same oscillation part of Z, which is described by two different forms, be equivalent. That is

$$\begin{aligned}\cos\left(\frac{1}{h}\int_{x_-(\lambda)}^{x}\sqrt{(\lambda-\hat{\lambda}_1)(\lambda-\hat{\lambda}_2)}\,dx - \frac{\pi}{4}\right) = \\ (-1)^n\cos\left(\frac{1}{h}\int_{x}^{x_+(\lambda)}\sqrt{(\lambda-\hat{\lambda}_1)(\lambda-\hat{\lambda}_2)}\,dx - \frac{\pi}{4}\right)\end{aligned}$$

for some n. It leads to

$$\int_{x_-(\lambda)}^{x_+(\lambda)}\sqrt{(\lambda-\hat{\lambda}_1)(\lambda-\hat{\lambda}_2)}\,dx = (n+1/2)h\pi, \tag{2.1.19}$$

which is a formula to define eigenvalue of $\mathcal{L}$ with $\lambda = \lambda(n) \stackrel{\text{def}}{=} \lambda_n$. That proves (2.1.13), the key point of Theorem 2.3.

With this connecting formula, the element of the eigenfunction respected, say f_1, can be described as

$$f_1 = \frac{\sqrt{U^*}}{2}\left[\sqrt{(\hat{\lambda}_1 - \lambda)(\lambda - \hat{\lambda}_2)}/h\right]^{-1/2} \cdot \exp\left(-\int_x^{x_-(\lambda)} \sqrt{(\hat{\lambda}_1 - \lambda)(\lambda - \hat{\lambda}_2)}\, dx/h\right)$$

for $x < x_-(\lambda)$; and

$$f_1 = \frac{\sqrt{U^*}}{2}\left[\sqrt{(\hat{\lambda}_1 - \lambda)(\lambda - \hat{\lambda}_2)}/h\right]^{-1/2} \cdot \exp\left(-\int_{x_+(\lambda)}^{x} \sqrt{(\hat{\lambda}_1 - \lambda)(\lambda - \hat{\lambda}_2)}\, dx/h\right)$$

for $x > x_+(\lambda)$.

That means if $|\lambda| < 1$, then the eigenfunction f in deed exponentially decay while $|x| \to \infty$. It proves Theorem 2.1.

The condition (2.1.2) implies that $\sqrt{|(\hat{\lambda}_1 - \lambda)(\lambda - \hat{\lambda}_2)|}$ is integrable, so all the arguments in this proof make sense. We will verify it later.

REMARK :

By (2.1.19), it is clear that

$$\Phi(\lambda_{k+1}) - \Phi(\lambda_k) = O(h).$$

But it is not clear how to estimate $\Phi(1) - \Phi(\lambda_{max})$ and $\Phi(-1) - \Phi(\lambda_{min})$; where $\Phi(\lambda)$ is given by (2.1.16) and both λ_{max}, λ_{min} are defined as following :

$$\lambda_{max} \stackrel{\text{def}}{=} \text{max eigenvalues of } \mathcal{L},$$

$$\lambda_{min} \stackrel{\text{def}}{=} \text{min eigenvalues of } \mathcal{L}.$$

We assume

$$|\Phi(1) - \Phi(\lambda_{max})| = O(h\pi) \tag{2.1.20}$$

and

$$|\Phi(-1) - \Phi(\lambda_{min})| = O(h\pi), \tag{2.1.21}$$

as $h \to 0$.

This assumption makes all the eigenfunction of $\mathcal{L}$ exponentially decay equivalently while $h \to 0$. It will be a key point to prove Lemma 2.4, a basic argument for the proof of weak L^1 convergence. By the way, (2.1.20) and (2.1.21) are consistent with each other for same h because $\hat{\lambda}_1(x)$ and $\hat{\lambda}_2(x)$ are independent of h.

2.1.4 WKB Analysis of Norming Constants

We turn next to the asymptotic determination of the norming constants. With a crude WKB method again, we may represent the eigenfunction in (2.1.5) as

$$f(x) = ce^{-\frac{\Gamma(\lambda,x)}{h}} \begin{pmatrix} (\lambda - i\nu)e^{-i\zeta(x)/h} \\ e^{i\zeta(x)/h} \end{pmatrix}, \tag{2.1.22}$$

where x is big enough and $\zeta(x) = -\frac{1}{2}[S(x) - S_+]$. Substituting this into (2.1.5) and set $h = 0$, it yields

$$\Gamma(\lambda, x) = \nu x + \int_x^\infty \left[\nu - \sqrt{(\hat{\lambda}_1(y) - \lambda)(\lambda - \hat{\lambda}_2(y))}\right] dy. \tag{2.1.23}$$

We shall show later, see inequality (2.2.63) and (2.2.64), that for all $\lambda \in (-1, 1)$

$$\int \sqrt{[1 - A^2(y)] \pm \lambda \partial_y S(y) + \frac{1}{4}(\partial_y S(y))^2}\, dy < \infty. \tag{2.1.24}$$

From this it follows that the integral (2.1.23) converges, and that, as $x \to +\infty$,

$$\Gamma(\lambda, x) = \nu x + O(1). \tag{2.1.25}$$

PROOF :

By (2.1.10) and (2.1.11),

$$\nu \;=\; \sqrt{(1-\lambda)(1+\lambda)}$$

$$
\begin{aligned}
&\geq \sqrt{[\hat{\lambda}_1(x) - \lambda][\lambda - \hat{\lambda}_2(x)]} \\
&= \sqrt{A^2 - (\lambda - S_x/2)^2} \\
&= \sqrt{\nu^2 - [(1 - A^2) - \lambda S_x + (S_x)^2/4]} \\
&\stackrel{\text{def}}{=} \sqrt{\nu^2 - \varsigma^2(x)}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\int_x^\infty \left[\nu - \sqrt{(\hat{\lambda}_1(y) - \lambda)(\lambda - \hat{\lambda}_2(y))}\right] dy \\
= \int_x^\infty [\nu - \sqrt{\nu^2 - \varsigma^2}]\, dy \\
\leq \int_x^\infty \varsigma(y)\, dy.
\end{aligned}
$$

By (2.1.24),

$$
\lim_{x \to \infty} \int_x^\infty \varsigma\, dy = 0.
$$

So, (2.1.25) is true.

As x decreases from $+\infty$, Γ is real and decreases until x reaches $x_+(\lambda)$. We denote this minimum value of Γ by

$$
\Gamma_+(\lambda) = \nu x_+(\lambda) + \int_{x_+(\lambda)}^\infty \left[\nu - \sqrt{(\hat{\lambda}_1(y) - \lambda)(\lambda - \hat{\lambda}_2(y))}\right] dy. \tag{2.1.26}
$$

We now define

$$
c(\lambda) = \exp(\Gamma_+(\lambda)/h), \tag{2.1.27}
$$

and let f be given by formula (2.1.5), with Γ and c defined by (2.1.23) and (2.1.27). We see that f satisfies the eigenvalue equation (2.1.5) crudly, and the normalization condition (2.1.6) just as crudly. It follows from (2.1.25) that f satisfies the asymptotic relation (2.1.7).

If we apply the WKB method one more again, crude or refined, to calculate the reflection coefficient, we get $R(k) \approx 0$.

2.1.5 Modified Initial Function

Now, we define a modified initial function with its exact scattering data which once appeared as the asymptotic scattering data of the original one.

Definition 2.1 *The modified initial function $U(x;h)$ is defined as the function with following scattering data :*

(i) $R(k) \equiv 0$,

(ii) λ_j *is given by*

$$\Phi(\lambda_j) = \begin{cases} (j - \frac{1}{2})h\pi & j = 1, ..., N^+(h) \\ (j + \frac{1}{2})h\pi & j = -1, ..., -N^-(h), \end{cases}$$

according to the case of $\lambda_j > \hat{\lambda}_{1,min}$ *or of* $\lambda_j < \hat{\lambda}_{2,max}$ *respectively.*

Also the total number of eigenvalues is given by

$$N(h) = [\frac{1}{h\pi}\Phi(1)] + [-\frac{1}{h\pi}\Phi(-1)] \stackrel{\text{def}}{=} N^+(h) + N^-(h).$$

(iii) The norming constants are initialed by

$$c_j = c(\lambda_j), \qquad j = -N^-(h), ..., N^+(h).$$

where $\Phi(\lambda)$ *is the function defined by (2.1.16);* $c(\lambda)$ *is the function defined by (2.1.27) and (2.1.26); and* [ ] *is the "integer part backward" function. For example,* $[P] = 8$ *if* $8 < P \leq 9$.

The above definition about λ_j is the WKB result and is consistent with Theorem 2.3.

In some case, the modified initial data is quite close to the original one. For example, we may assume that there is a positive constant X_b, no matter how large it could be, such that both $\hat{\lambda}_1(x) = 1$ and $\hat{\lambda}_2(x) = -1$ if $|x| \geq X_b$. This hypothesis is often adopted by people doing numerical analysis. From now on, we will only deal with that case for simple. We also assume that (2.1.20) and (2.1.21) are true in that case. We name it as X-case. It will be defined as given by Definition 2.4 for later convenience.

Theorem 2.4

In X-case, the function $U(x;h)$ which is given by the last definition approximates the prescribed initial data $U_I(x;h)$ or $\{A(x), S(x)\}$ in the following sense :

$$\begin{cases} L^1 - \lim_{h\to 0} \rho(x;h) &= \rho(x) \\ \\ L^1 - \lim_{h\to 0} \mu(x;h) &= \mu(x), \end{cases}$$

where $\{\rho(x;h), \mu(x;h)\}$ are defined at $t=0$ by (1.2.9) with $U(x,t;h)$ which is the solution of the NLS equation but initialed by $U(x;h)$ not the original $U_I(x;h)$. $\{\rho(x), \mu(x)\}$ are defined by (1.2.10).

The proof of this theorem will be presented in Section 3.1 (Theorem 4.36).

With above analysis, we can write out the scattering data of $U(x,t;h)$ which is the solution of NLS with initial data $U(x;h)$:

$$\begin{cases} c_j(t) = \exp\{[-\lambda\nu t + \Gamma_+(\lambda)]/h\}, \\ \lambda = \lambda_j, \qquad \nu = \sqrt{1-\lambda^2} \stackrel{\text{def}}{=} \nu_j. \end{cases} \tag{2.1.28}$$

Or generally,

$$c_j(\vec{t}) = \exp\{[-\sum_{n=2} \lambda_j^{n-1}\nu_j t_n + \Gamma_+(\lambda_j)]/h\}. \tag{2.1.29}$$

We now define a further more auxiliary function which will be of use in Section 2.2.

$$\Gamma_-(\lambda) = \nu x_-(\lambda) - \int_{-\infty}^{x_-(\lambda)} \left[\nu - \sqrt{(\hat{\lambda}_1(y) - \lambda)(\lambda - \hat{\lambda}_2(y))}\right] dy. \tag{2.1.30}$$

Theorem 2.5

The functions $\Gamma_+(\lambda)$ and $\Gamma_-(\lambda)$ defined by (2.1.26) and (2.1.30), respectively, are continuous for $\lambda \in [-1,1]$ and satisfy

$$\begin{cases} \Gamma+(1) = \Gamma_-(1) = 0 \\ \Gamma+(-1) = \Gamma_-(-1) = 0. \end{cases} \tag{2.1.31}$$

Moreover

$$\Gamma_-(\lambda) < \nu x_0 < \Gamma_+(\lambda)$$

with either

$$\hat{\lambda}_{1,min} < \lambda < 1$$

or

$$\hat{\lambda}_{2,max} > \lambda > -1,$$

where $\nu = \sqrt{1-\lambda^2}$, and $x_0 = x_\pm(\hat{\lambda}_{1,min})$ or $x_0 = x_\pm(\hat{\lambda}_{2,max})$ respectively.

PROOF :

It follows directly from the definitions of Γ_+ and Γ_- that they are continuous for $\lambda \neq \pm 1$. To show continuity and (2.1.31) at $\lambda = \pm 1$, we argue as follows :

It is easy to find that both $(1-\hat{\lambda}_1)(1-\hat{\lambda}_2)$ and $(1+\hat{\lambda}_1)(1+\hat{\lambda}_2)$ are decreasing functions of $|x|$ for $|x|$ larger by assumption. Thus because $\sqrt{(1-\hat{\lambda}_1)(1-\hat{\lambda}_2)}$ is integrable and $(1-\hat{\lambda}_2) \geq (1-\hat{\lambda}_{2,max}) > 0$ for $x > x_0$, $\sqrt{(1-\hat{\lambda}_1)}$ is a decreasing function of x too for x large enough and is integrable for $x > x_0$ thereafter. It leads to

$$2\int_{x/2}^{x} \sqrt{1-\hat{\lambda}_1(y)}\, dy > x\sqrt{1-\hat{\lambda}_1(x)}$$

for x larger. That is

$$\lim_{x\to+\infty} x\sqrt{1-\hat{\lambda}_1(x)} = 0. \tag{2.1.32}$$

By (2.1.14), $\hat{\lambda}_1(x_+) = \lambda$. With this fact and (2.1.32), we conclude that

$$\lim_{\lambda\to 1} x_+(\lambda)\nu = \lim_{\lambda\to 1} x_+(\lambda)\sqrt{1-\lambda^2}$$

$$= \lim_{\lambda\to 1}[x_+(\lambda)\sqrt{1-\lambda}] \lim_{\lambda\to 1}\sqrt{1+\lambda}$$

$$= \sqrt{2}\lim_{\lambda\to 1} x_+(\lambda)\sqrt{1-\hat{\lambda}_1(x_+(\lambda))}$$

$$= \sqrt{2}\lim_{x\to+\infty} x\sqrt{1-\hat{\lambda}_1(x)} = 0.$$

This proves that the first term in the definition (2.1.26) of $\Gamma_+(\lambda)$ tends to 0 as $\lambda \to 1$.

We now show that the same is true for the second term.

$$\lim_{\lambda\to 1}\int_{x_+(\lambda)}^{\infty}\left[\nu - \sqrt{(\hat{\lambda}_1(y)-\lambda)(\lambda-\hat{\lambda}_2(y))}\right] dy$$

$$= \lim_{\lambda\to 1}\int_{x_+(\lambda)}^{\infty}\left[\nu - \sqrt{(\hat{\lambda}_1^2(y)-\lambda^2)}\right] dy$$

$$+ \lim_{\lambda\to 1}\int_{x_+(\lambda)}^{\infty}\sqrt{\hat{\lambda}_1-\lambda}\left(\sqrt{\hat{\lambda}_1+\lambda}-\sqrt{\lambda-\hat{\lambda}_2}\right) dy$$

$$\stackrel{\text{def}}{=} \lim_{\lambda\to 1}\int_1 + \lim_{\lambda\to 1}\int_2 .$$

But

$$\int_1 = \int_{x_+(\lambda)}^{\infty}\left[\nu - \sqrt{(\hat{\lambda}_1^2(y)-\lambda^2)}\right] dy$$

$$= \int_{\tilde{x}_+(\nu)}^{\infty}\left[\nu - \sqrt{(\nu^2-\hat{\nu}_1^2(y))}\right] dy,$$

where $\tilde{x}_+(\nu)$ is defined as $x = \tilde{x}_+(\nu)$ if $\hat{\nu}_1(x) = \nu$, and $\hat{\nu}_1 = \sqrt{1-\hat{\lambda}_1^2}$. In fact $\tilde{x}_+(\nu) = x_+(\lambda)$. So, as $\lambda \to 1$,

$$0 \leq \lim_{\lambda\to 1}\int_1$$

$$
\begin{aligned}
&= \lim_{\nu\to 0+}\int_{\nu}^{0}(\nu-\sqrt{\nu^2-\varsigma^2})\frac{d\tilde{x}_+}{d\varsigma}d\varsigma \\
&= \lim_{\nu\to 0+}\{(\nu-\sqrt{\nu^2-\varsigma^2})\tilde{x}_+(\varsigma)|_{\varsigma=\nu}^{\varsigma=0} \\
&\quad -\int_{\nu}^{0}\tilde{x}_+(\varsigma)\frac{d}{d\varsigma}[\nu-\sqrt{\nu^2-\varsigma^2}]\} \\
&= \lim_{\nu\to 0+}\lim_{\varsigma\to 0+}(\nu-\sqrt{\nu^2-\varsigma^2})\tilde{x}_+(\varsigma) \\
&\quad -\lim_{\nu\to 0+}\nu\tilde{x}_+(\nu)+\lim_{\nu\to 0+}\int_0^{\nu}\frac{\varsigma\tilde{x}_+(\varsigma)}{\sqrt{\nu^2-\varsigma^2}}d\varsigma \\
&\leq \lim_{\varsigma\to 0+}\varsigma\tilde{x}_+(\varsigma)-\lim_{\nu\to 0+}\nu\tilde{x}_+(\nu) \\
&\quad +\lim_{\nu\to 0+}\{[\max_{0\leq\varsigma\leq\nu}\varsigma\tilde{x}_+(\varsigma)]\int_0^{\nu}\frac{d\varsigma}{\sqrt{\nu^2-\varsigma^2}}\} \\
&= 0.
\end{aligned}
$$

The reason is that

$$\lim_{\varsigma\to 0+}\varsigma\tilde{x}_+(\varsigma)=\lim_{\lambda\to 1}\sqrt{1-\lambda^2}\,x_+(\lambda)=0,$$

and

$$\int_0^{\nu}\frac{d\varsigma}{\sqrt{\nu^2-\varsigma^2}}=\frac{\pi}{2}.$$

Recall the assumption about $S(x)$ (see (2.1.2)) and the relation between $S(x)$ and $\hat{\lambda}_{1,2}(x)$ (see (2.1.11)). If λ is close to 1 enough such that it comes an inequality $|S_+ - S(x_+(\lambda))| \leq 1$, then

$$|\int_2| = |\int_{x_+(\lambda)}^{\infty}\sqrt{\hat{\lambda}_1-\lambda}\left(\sqrt{\hat{\lambda}_1+\lambda}-\sqrt{\lambda-\hat{\lambda}_2}\right)dy|$$

$$= \left| \int_{x_+(\lambda)}^{\infty} \frac{\sqrt{\hat\lambda_1 - \lambda}}{\sqrt{\hat\lambda_1 + \lambda} + \sqrt{\lambda - \hat\lambda_2}} (\hat\lambda_1 + \hat\lambda_2)\, dy \right|$$

$$\leq \frac{\sqrt{1-\lambda}}{c} \left| \int_{x_+(\lambda)}^{\infty} \partial_y S\, dy \right|$$

$$\leq \frac{\sqrt{1-\lambda}}{c} \to 0,$$

where there exists a constant c satisfying $c > 1/(4\sqrt{2}) > 0$ for all λ which are close to 1 enough, because

$$\sqrt{\hat\lambda_1(x) + \lambda} \quad \to \quad \sqrt{2},$$

and

$$\sqrt{\lambda - \hat\lambda_2(x)} \quad \to \quad \sqrt{2},$$

if $\lambda \to 1$ and $x \to +\infty$. Therefore

$$\lim_{\lambda \to 1} \int_{x_+(\lambda)}^{+\infty} \left[\nu - \sqrt{(\hat\lambda_1 - \lambda)(\lambda - \hat\lambda_2)} \right] dy = 0. \tag{2.1.33}$$

It completes the claim that $\Gamma_+(\lambda)$ is continue at $\lambda = 1$ and $\Gamma_+(1) = 0$.

By discussing the same way to $x_-(\lambda)$ and $\Gamma_-(\lambda)$, we obtain $\Gamma_-(1) = 0$. Doing again with $\hat\lambda_2(x)$ where $\lambda \to -1$, we obtain $\Gamma_+(-1) = \Gamma_-(-1) = 0$. It completes the proof of Theorem 2.5.

2.1.6 Zakharov and Shabat Formula

On account of the scattering data, the functions $U(x, t; h)$ are reflectionless potentials of the operator $\mathcal{L}$ because $R(k) = 0$. In that case, Zakharov and Shabat [20] found soliton style solitions of the related Marchenko equations. Their result leads to the following explicit expression of the amplitude square of the potential U. *i.e.* The first conserved quantity ρ can be expressed in terms of the scattering data of U :

$$\rho = |U(x, t; h)|^2 = 1 + \partial_{xx} W, \tag{2.1.34}$$

where

$$W(x,t;h) = -h^2 \log \det(I + \hat{G})_{2N\times 2N}. \tag{2.1.35}$$

Here $\hat{G}$ is a $2N \times 2N$ matrix with the following structure :

$$\hat{G} = \frac{1}{2}h \begin{pmatrix} M & \Theta^* M \\ \Theta M & M \end{pmatrix}, \tag{2.1.36}$$

$$\Theta = \begin{pmatrix} e^{i\sigma_1} & & 0 \\ & * & \\ 0 & & e^{i\sigma_N} \end{pmatrix},$$

$$M = D\Lambda D, \tag{2.1.37}$$

$$\Lambda = \left(\frac{1}{\nu_j + \nu_k}\right)_{N\times N},$$

$$D = \begin{pmatrix} c_1(t)e^{-\frac{\nu_1 x}{h}} & & 0 \\ & * & \\ O & & c_N(t)e^{-\frac{\nu_N x}{h}} \end{pmatrix}_{N\times N},$$

and

$$\nu_j = \sqrt{1-\lambda_j^2}, \quad \sigma_j = \cos^{-1}(\lambda_j).$$

Moreover, with the conservation laws given by (1.6.21) and (1.4.18), it is easy to obtain :

$$\mu = \partial_{xt} W,$$

and in general to obtain

$$\rho_n = \partial_{xt_n} W$$

$$\mathcal{G}_n = \partial_{tt_n} W$$

for $n = 1, 2, \dots$.

Thus, the form of W is particularly important in the analysis for semiclassical limits of the NLS equation. But, because W should be real, the complex elements in (2.1.36) could be removed and the formula (2.1.35) could be improved.

We will show a new formula of W in next section. It simplifies the results of Z-S, and makes Lax-Levermore strategy be applicatable.

2.2 Formula of W and its Property

2.2.1 New Form of W

In this section we simplify Z-S formula for W. The main theorem is

Theorem 2.6

The W given by (2.1.35) can be described as

$$W = -h^2 \log \det(I + G)_{N\times N}, \tag{2.2.38}$$

where we denote

$$G = hDBD,$$

$$B = \left(b_{mn} = \frac{1}{2\sin(\frac{\sigma_m+\sigma_n}{2})}\right),$$

and

$$D = \begin{pmatrix} d_1 & & 0 \\ & * & \\ 0 & & d_N \end{pmatrix}$$

with

$$d_j = c_j(\vec{t})\exp(-x\sin\sigma_j/h) = \exp(-a(\sigma_j; x, \vec{t})).$$

Here

$$a(\sigma; x, \vec{t}) = \sum_{k=1} t_k \sin\sigma \cos^{k-1}\sigma - \Gamma_+(\cos\sigma), \tag{2.2.39}$$

with $t_1 = x$, $t_2 = t$.

Moreover G *is real and positive definite* .

PROOF :

Recall Z-S formula (2.1.35)

$$W = -h^2 \log\det(I + \hat{G})_{2N\times 2N},$$

$$\hat{G} = \frac{1}{2}h \begin{pmatrix} M & \Theta^* M \\ \Theta M & M \end{pmatrix}, \tag{2.2.40}$$

$$\Theta = \begin{pmatrix} e^{i\sigma_1} & & 0 \\ & * & \\ 0 & & e^{i\sigma_N} \end{pmatrix},$$

$$M = D\Delta D, \tag{2.2.41}$$

$$\Delta = \left(\frac{1}{\nu_j + \nu_k}\right)_{N\times N},$$

$$D = \begin{pmatrix} c_1 e^{-\frac{\nu_1 x}{h}} & & 0 \\ & * & \\ O & & c_N e^{-\frac{\nu_N x}{h}} \end{pmatrix}_{N\times N},$$

and

$$\nu_j = \sin\sigma_j \ , \ \nu_j > 0.$$

By (2.2.40) ,

$$
\begin{aligned}
&\det(I + \hat{G})_{2N\times 2N} \\
&= \det\left(\begin{pmatrix} I & -\Theta^* \\ 0 & I \end{pmatrix}\begin{pmatrix} I+\frac{1}{2}hM & \frac{1}{2}h\Theta^* M \\ \frac{1}{2}h\Theta M & I+\frac{1}{2}hM \end{pmatrix}\begin{pmatrix} I & \Theta^* \\ 0 & I \end{pmatrix}\right)_{2N\times 2N} \\
&= \det\left(I + \frac{h}{2}(M + \Theta M\Theta^*)\right)_{N\times N}.
\end{aligned}
$$

We define

$$\hat{\Delta} \ = \ \Delta + \Theta\Delta\Theta^*,$$

and get

$$(\frac{h}{2})(M + \Theta M\Theta^*) = (\frac{h}{2})D\hat{\Delta}D.$$

But

$$
\begin{aligned}
\hat{\Delta} \ &= \ \left(\frac{1 + e^{i(\sigma_j-\sigma_k)}}{\nu_j + \nu_k}\right)_{N\times N} \\
&= \ \hat{\mathcal{O}}(\frac{1}{\sin(\frac{\sigma_j+\sigma_k}{2})})\hat{\mathcal{O}}^*,
\end{aligned}
$$

where

$$\hat{\mathcal{O}} \ = \begin{pmatrix} e^{\frac{i}{2}\sigma_1} & & 0 \\ & * & \\ 0 & & e^{\frac{i}{2}\sigma_N} \end{pmatrix}.$$

Thus

$$\frac{h}{2}(M + \Theta M\Theta^*) \ = \ h(\hat{\mathcal{O}}DBD\hat{\mathcal{O}}^*),$$

where

$$B \ = \ \left(\frac{1}{2\sin(\frac{\sigma_j+\sigma_k}{2})}\right).$$

That is

$$\begin{aligned}
\det(I \; + \; \hat{G})_{2N\times 2N} & \\
&= \det(I + h\hat{O}DBD\hat{O}^{*})_{N\times N} \\
&= \det\hat{O}\det(I + hDBD)\det\hat{O}^{*} \\
&= \det(I \; + \; G)_{N\times N}.
\end{aligned}$$

Obviously,

$$G \; = \; hDBD$$

is real and positive definite. It completes the proof.

So, as we have proved,

$$G = h\left(\frac{\exp\{-(\nu_i x + \nu_j x)/h\}}{2\sin(\frac{\sigma_i+\sigma_j}{2})}c_i c_j\right). \tag{2.2.42}$$

Using (2.1.28) and (2.1.29), we can rewrite G as

$$G = \frac{h}{2}(\frac{g_i g_j}{\sin(\frac{\sigma_i+\sigma_j}{2})}), \tag{2.2.43}$$

where

$$g_j = g_j(x, t; h) = \exp\{-a(\sigma_j; x, t)/h\}, \tag{2.2.44}$$

with

$$a(\sigma; x, t) = x\sin\sigma + t\sin\sigma\cos\sigma - \Gamma_{+}(\cos\sigma), \tag{2.2.45}$$

and $\Gamma_{+}(\lambda = \cos\sigma)$ is defined by (2.1.26).

Or generally,

$$g_j = g_j(x, \vec{t}; h) = \exp\{-a(\sigma_j; x, \vec{t})/h\}, \tag{2.2.46}$$

with

$$a(\sigma; x, \vec{t}) = x\sin\sigma + \sum_{k=2} t_k \sin\sigma\cos^{k-1}\sigma - \Gamma_{+}(\cos\sigma),$$

as given by (2.2.39).

Since G is symmetric and positive definite, we expend $\det(I+G)$ by grouping together all terms which only contain factors from G with indices from a set S. The resulting formula is

$$\det(I+G) = \sum_S \det(G_S), \tag{2.2.47}$$

where G_S is the principal minor of G obtained by striking out all rows and columns whose indexes lie outside S. In the sum (2.2.47), S ranges over all subsets of the N indices and we take $\det(G_S) = 1$ when S is a null set.

Now we can factor $\det G$ with the following results :

Theorem 2.7 *For $N > 1$,*

(1)

$$\det\left(\frac{1}{\sin(\frac{\tau_m+\tau_n}{2})}\right)_{N\times N} = \frac{\prod_{m<n}\sin^2(\frac{\tau_m-\tau_n}{2})}{\prod_{m,n}\sin(\frac{\tau_m+\tau_n}{2})}.$$

(2)

$$\det G = (\frac{h}{2})^N \prod_N \frac{g_j^2}{\sin\sigma_j} \prod_{N\times' N} |\frac{\sin(\frac{\sigma_i-\sigma_j}{2})}{\sin(\frac{\sigma_i+\sigma_j}{2})}|,$$

where $N\times' N$ means no column for $i=j$.

(3)

$$W = -h^2\log(\sum_S \det G_S) \ \ (\ S \text{ over all } N \text{ indices }).$$

Here

$$\det(G_S) = (\frac{h}{2})^{|S|} \prod_S \frac{g_n^2}{\sin\sigma_n} \prod_{S\times' S} |\frac{\sin(\frac{\sigma_m-\sigma_n}{2})}{\sin(\frac{\sigma_m+\sigma_n}{2})}|,$$

if S contains more than one index;

$$\det(G_S) = (\frac{h}{2})\frac{g_j^2}{\sin\sigma_j},$$

if S contains only singular index, say "j"; and $\det(G_S) = 1$ *for empty set S.*

PROOF :

(1)

We define $Z_m = \exp(i\tau_m/2)$; and get

$$\det\left(\frac{1}{\sin(\frac{\tau_m+\tau_n}{2})}\right) = \det\left(\frac{2i}{Z_m Z_n - Z_m^{-1} Z_n^{-1}}\right)$$

$$= (2i)^N \left(\prod_{m=1}^{N} Z_m\right)\left(\prod_{n=1}^{N} Z_n^{-1}\right) \det\left(\frac{1}{Z_m^2 - Z_n^{-2}}\right)$$

$$= (2i)^N \det\left(\frac{1}{Z_m^2 - Z_n^{-2}}\right).$$

By (2.2.48) ,

$$\det\left(\frac{1}{Z_m^2 - Z_n^{-2}}\right) = \frac{\prod_{m<n}(Z_m^2 - Z_n^2)(-Z_m^{-2} + Z_n^{-2})}{\prod_{m,n}(Z_m^2 - Z_n^{-2})}$$

$$= \frac{\prod_{m<n}(Z_m^2 - Z_n^2)^2}{\prod_{m,n}(Z_m^2 Z_n^2 - 1)} \frac{\prod_{m,n}(Z_n^2)}{\prod_{m<n}(Z_m^2 Z_n^2)}$$

$$= \frac{\prod_{m<n}(Z_m^2 - Z_n^2)^2}{\prod_{m,n}(Z_m^2 Z_n^2 - 1)} \frac{N\prod_{n=1}^{N}(Z_n^2)}{\prod_{m=1}^{N-1}\prod_{n=m+1}^{N}(Z_m^2 Z_n^2)}$$

$$= \left(\prod_{n=1}^{N} Z_n^2\right) \frac{\prod_{m<n}(Z_m^2 - Z_n^2)^2}{\prod_{m,n}(Z_m^2 Z_n^2 - 1)}.$$

Now we get

$$\begin{aligned}
\det\left(\frac{1}{\sin(\frac{\tau_m+\tau_n}{2})}\right) &= (2i)^N \prod_{k=1}^{N} Z_k^2 \left(\frac{\prod_{m<n}(Z_m^2 - Z_n^2)^2}{\prod_{m,n}(Z_m^2 Z_n^2 - 1)}\right) \\
&= (2i)^N (\prod_{m=1}^{N} e^{i\tau_m}) \left(\frac{\prod_{m<n}(e^{i\tau_m} - e^{i\tau_n})^2}{\prod_{m,n}(e^{i(\tau_m+\tau_n)} - 1)}\right) \\
&= \frac{\prod_{m<n}\sin^2(\frac{\tau_m-\tau_n}{2})}{\prod_{m,n}\sin(\frac{\tau_m+\tau_n}{2})}.
\end{aligned}$$

(2)

By (2.2.43) ,

$$\begin{aligned}
\det(G) &= \det\left(\frac{h}{2}\left(\frac{g_i g_j}{\sin(\frac{\sigma_i+\sigma_j}{2})}\right)\right) \\
&= (\frac{h}{2})^N \left(\prod_{j=1}^{N} g_j\right)^2 \det\left(\frac{1}{\sin(\frac{\sigma_i+\sigma_j}{2})}\right) \\
&= (\frac{h}{2})^N \prod_{j=1}^{N} \frac{g_j^2}{\sin\sigma_j} \prod_{N\times' N} \left|\frac{\sin(\frac{\sigma_i-\sigma_j}{2})}{\sin(\frac{\sigma_i+\sigma_j}{2})}\right|.
\end{aligned}$$

(3)

By the argument round (2.2.47), the result is trivial.

REMARK :

In the KdV case, the correspond one of the first part of this theorem is the following Cauchy matrix formula :

$$\det\left(\frac{1}{\eta_m+\eta_n}\right) = \frac{\prod_{m<n}(\eta_m-\eta_n)^2}{\prod_{m,n}(\eta_m+\eta_n)}.$$

It is a special example of the general one :

$$\det\left(\frac{1}{\varsigma_m+\nu_n}\right) = \frac{\prod_{m<n}(\varsigma_m-\varsigma_n)(\nu_m-\nu_n)}{\prod_{m,n}(\varsigma_m+\nu_n)}. \tag{2.2.48}$$

We once used it to prove the last theorem.

With these new representation, we can explore some nice properties of W.

2.2.2 Convexity and Boundness of W

Theorem 2.8

W is a differentiable and concave function with respect to x and t_k, $\forall k$.

PROOF :

The differentiability is obvious.

For the concave property, we only need to show that the corresponding Hessian matrix of W is negative definite.

Recall the former result

$$W = -h^2 \log \left(\sum_S \det(G_S) \right),$$

where S over all the subsets of the N indices.

we obtain

$$\det(G_S) = (\frac{h}{2})^{|S|} \exp(-q_h(S; x, \vec{t})/2h^2),$$

where

$$q_h(S; x, \vec{t}) = 4h \sum_{i,j \in S} [a(\sigma_j; x, \vec{t}) + q_h'(\sigma_i, \sigma_j)],$$

and

$$a(\sigma; x, \vec{t}) = x \sin\sigma + \sum_{k=2} t_k \sin\sigma \cos^{k-1}\sigma - \Gamma_+(\sigma).$$

So,

$$\partial_x a = \sin\sigma \stackrel{\text{def}}{=} \nu,$$

$$\partial_x q_h(S; x, t) = 4h \sum_S \nu_j,$$

$$\partial_x(\det(G_S)) = (-\frac{2}{h} \sum_S \nu_j) \det(G_S),$$

and

$$\partial_x W = 2h \frac{\sum_S \det(G_S)(\sum_{j \in s} \nu_j)}{\sum_S \det(G_S)} \stackrel{\text{def}}{=} (2h)\langle\overline{\nu}\rangle > 0;$$

where we set $\sum_{j\in S} \nu_j = 0$, if S is empty.

Thus

$$\begin{aligned}
\partial_{xx} W &= (-4) \left[\frac{\sum_S \det(G_S)(\sum_{j\in S} \nu_j)^2}{\sum_S \det(G_S)} - \langle\overline{\nu}\rangle^2 \right] \\
&= (-4) \frac{\sum_S \det(G_S)(\sum_{j\in S} \nu_j - \langle\overline{\nu}\rangle)^2}{\sum_S \det(G_S)} \\
&\stackrel{\text{def}}{=} (-4)\langle\overline{(\nu - \langle\overline{\nu}\rangle)^2}\rangle < 0.
\end{aligned}$$

Here, for $\lambda = \cos\sigma$, we set $\sum_{j\in S}\nu_j\lambda_j^{k-1} = 0$ if S is empty. In same way, we get

$$\begin{aligned}
\partial_{t_k}W &= 2h\frac{\sum_S \det(G_S)(\sum_{j\in_s}\lambda_j^{k-1}\nu_j)}{\sum_S \det(G_S)} \\
&\stackrel{\text{def}}{=} 2h\langle\overline{\lambda^{k-1}\nu}\rangle,
\end{aligned}$$

$$\begin{aligned}
\partial_{xt_k}W &= (-4)\left[\frac{\sum_S \det(G_S)(\sum_{j\in S}\nu_j)(\sum_{j\in S}\lambda_j^{k-1}\nu_j)}{\sum_S \det(G_S)} - \langle\overline{\nu}\rangle\langle\overline{\lambda^{k-1}\nu}\rangle\right] \\
&= (-4)\left[\frac{\sum_S \det(G_S)(\sum_{j\in S}\nu_j - \langle\overline{\nu}\rangle)(\sum_{j\in S}\lambda_j^{k-1}\nu_j - \langle\overline{\lambda^{k-1}\nu}\rangle)}{\sum_S \det(G_S)}\right] \\
&\stackrel{\text{def}}{=} (-4)\langle\overline{(\nu - \langle\overline{\nu}\rangle)(\lambda^{k-1}\nu - \langle\overline{\lambda^{k-1}\nu}\rangle)}\rangle,
\end{aligned}$$

$$\partial_{tt_k}W = (-4)\langle\overline{(\lambda\nu - \langle\overline{\lambda\nu}\rangle)(\lambda^{k-1}\nu - \langle\overline{\lambda^{k-1}\nu}\rangle)}\rangle,$$

and

$$\partial_{t_kt_k}W = (-4)\langle\overline{(\lambda^{k-1}\nu - \langle\overline{\lambda^{k-1}\nu}\rangle)^2}\rangle < 0.$$

By using the Schwartz inequality,

$$(\partial_{xt_k}W)^2 \leq \partial_{xx}W\partial_{t_kt_k}W. \tag{2.2.49}$$

So, W is concave with respect to x and t_k, $\forall k$.

Similarly, W is concave with respect to t and t_k, $\forall k$, because we have

$$(\partial_{tt_k}W)^2 \leq \partial_{tt}W\partial_{t_kt_k}W.$$

With the convexity of W, we get

Lemma 2.1 *If the limit of W is also a function both differentiable and concave like the member of W family, then the convergence of their first order derivatives are uniformly on compact subset of x and $\vec{t}$ [8].*

The proof will be given at Appendix.

Now, we look for other observable properties of W. They will be useful later. By (2.1.34),

$$0 \leq |\partial_{xx} W| = -\partial_{xx} W = 1 - \rho < 1.$$

That means

Lemma 2.2

W_{xx} is uniformly bounded no matter how h is small.

Look at the following fact :

$$W_{xx}(+\infty) = 0 = W_{xx}(-\infty),$$

$$W_{xt}(+\infty) = 0 = W_{xt}(-\infty),$$

$$W_{xt_k}(+\infty) = 0 = W_{xt_k}(-\infty),$$

and

$$W_{t_j t_k}(+\infty) = 0 = W_{t_j t_k}(-\infty).$$

Thus, for each h, all the second derivatives are integrable in the whole space of x. Moreover, these integration could be uniformly bounded under some condition. One reason to claim them is because of the following fact:

$$W_x(x = +\infty) \;= 0, \qquad W_x(x = -\infty) \;= 2h \sum_{j=1}^{N} \nu_j,$$

$$W_{t_k}(x = +\infty) \;= 0, \qquad W_{t_k}(x = -\infty) \;= 2h \sum_{j=1}^{N} \lambda_j^{k-1} \nu_j.$$

Since $|h \sum \nu_j| < \infty$, $|h \sum \lambda_j^{k-1} \nu_j| < \infty$ uniformly for all small h if we fix k, thus, for any $\vec{t}$ fixed, both W_{xx} and W_{xt_k} are integrable with respect to x. And the integration of them are uniformly bounded for small h. They represent the fact of conservation. That means, all the integrals of these density W_{xx}, W_{xt_k} over the whole space of x are constants with respect to the time variables t or t_j, $\forall j$.

2.2.3 Weak L^1 Compact and Convergence

To explore more properties of W in some special case, we need to introduce "w-L^1 compact" [19].

Definition 2.2 *A family of functions $\{f(x;h)\} \in L^1(\Re)$ is defined weakly L^1 compact if for any sequence of this family, say $\{f_n\}$, there is a subsequence, say $\{f_{n_k}\}$, such that*

$$\lim_{k\to\infty} \int_{-\infty}^{+\infty} f_{n_k} \cdot g \, dx$$

exists for all $g \in L^\infty(\Re)$.

It is equivalent to the following definition [13] because of the Dunford-Pettis Theorem [2]:

Definition 2.3 *A family of functions $\{f(x;h)\}$ is defined weakly L^1 compact if the following properties*

$$\int_{-\infty}^{+\infty} |f(x;h)| \, dx \quad < \infty \tag{2.2.50}$$

$$\lim_{\varsigma(e)\to 0} \int_e |f(x;h)| \, dx \quad = 0 \tag{2.2.51}$$

$$\lim_{k\to\infty} \int_{|x|\geq k} |f(x;h)| \, dx \quad = 0 \tag{2.2.52}$$

are satisfied and uniformly with respect to small h, where $\varsigma(e)$ means the measure of set e.

Theorem 2.9

The family $\{W_{xx}(x, \vec{t};\ h)\}$, as is composed by functions of x with small positive parameter h, is weakly L^1 compact if the assumption of X-case mentioned at Theorem 2.4 is set.

PROOF :

First we prove (2.2.50) for $f = W_{xx}$:

$$\begin{aligned}
&\int_{-\infty}^{+\infty} |W_{xx}(x,t;h)|\,dx \\
&= -\int_{-\infty}^{+\infty} W_{xx}(x,t;h)\,dx \\
&= -\int_{-\infty}^{+\infty} W_{xx}(x,0;h)\,dx \\
&= W_x(-\infty,0;h) - W_x(+\infty,0;h) \\
&= 2h\sum_{j=1}^{N} \nu_j \le 2hN \\
&\le \frac{2}{\pi}[\Phi(0)+\Phi(\pi)].
\end{aligned}$$

So, $W_{xx}(x,t;h)$ is uniformly L^1 bounded independent of h and even independent of t. It matches the first requirement due to weak L^1 convergence.

Next we prove (2.2.51). By Lemma 2.2,

$$\int_e |W_{xx}|\,dx \le \varsigma(e) < \epsilon,$$

for any small measurable set e, if the measure of that set, say $\varsigma(e)$, is less than ϵ.

Now, we try to prove (2.2.52) for W_{xx}. We need to estimate $\partial_x[\det(G_S)]$ a little more explicitly.

Lemma 2.3

Both

$$\sum_{|S|=k-1} \det(G_S) \quad \textit{and} \quad \sum_{|S|=k+1} \det(G_S)$$

can be adopted to estimate the following term :

$$\partial_x[\sum_{|S|=k} \det(G_S)].$$

They are

$$(-\frac{h}{2})\partial_x[\sum_{|S|=k} \det(G_S)]$$

$$\leq (\frac{h}{2}) \sum_{l=1}^{l=N} \exp\{-(2\nu_l/h)[x - (\Gamma_+(\sigma_l)/\nu_l)]\} \sum_{|S|=k-1} \det(G_S) \tag{2.2.53}$$

for $k = 1, 2, \ldots N$; *and*

$$[\sum_{j=1}^{j=N} \nu_j - (-h/2)\partial_x] \sum_{|S|=k} \det(G_S)$$
$$\leq \sum_{l=1}^{l=N} (2\nu_l/h) \exp\{-(2\nu_l^2/h)[-x + (\Gamma_-(\sigma_l)/\nu_l)]\} \sum_{|S|=k+1} \det(G_S) \tag{2.2.54}$$

for $k = 0, 2, \ldots N-1$.

PROOF :

Recall the definition of $\det(G_S)$ given at Theorem 2.7, we have

$$(-\frac{h}{2})\partial_x[\sum_{|S|=k} \det(G_S)] = \sum_{|S|=k} \det(G_S)(\sum_{j\in S} \nu_j)$$

$$= \sum_{|S|=k} \left\{ (\frac{h}{2})^{|S|} \prod_{\substack{S \times' S \\ m,n \in S}} \left|\frac{\sin(\frac{\sigma_m - \sigma_n}{2})}{\sin(\frac{\sigma_m + \sigma_n}{2})}\right| \cdot \sum_{j\in S} g_j^2 \left(\prod_{\substack{k \neq j \\ k \in S}} \frac{g_k^2}{\nu_k} \right) \right\}$$

$$\leq (\frac{h}{2}) \sum_{|S|=k-1} \det(G_S) \sum_{j \notin S} (g_j^2)$$

$$\leq (\frac{h}{2})(\sum_{j=1}^{j=N} g_j^2) \sum_{|S|=k-1} \det(G_S)$$

$$= (\frac{h}{2})\sum_{j=1}^{j=N} \exp\{-2[x\nu_j - \Gamma_+(\sigma_j)]/h\} \sum_{|S|=k-1} \det(G_S).$$

That is (2.2.53). Also

$$\sum_{j=1}^{j=N} \nu_j \sum_{|S|=k} \det(G_S) - (-h/2)\partial_x[\sum_{|S|=k} \det(G_S)]$$

$$= \sum_{|S|=k} \det(G_S)(\sum_{j\notin S} \nu_j)$$

$$= \sum_{|S|=k}\sum_{l\notin S}\left(\frac{h}{2}\right)^k \left[\nu_l^2 \prod_{\substack{S\times' S\\ m,n\in S}} |\frac{\sin(\frac{\sigma_m-\sigma_n}{2})}{\sin(\frac{\sigma_m+\sigma_n}{2})}| \prod_{j\in S} |\frac{\sin(\frac{\sigma_l-\sigma_j}{2})}{\sin(\frac{\sigma_l+\sigma_j}{2})}|^2 \prod_{i\in\{l\}\cup S} \frac{g_i^2}{\nu_i}\right]$$

$$/\left[g_l^2 \prod_{j\in S} |\frac{\sin(\frac{\sigma_l-\sigma_j}{2})}{\sin(\frac{\sigma_l+\sigma_j}{2})}|^2\right]$$

$$= \sum_{|S|=k+1} \det(G_S)\sum_{l\in S}(\frac{2}{h})\nu_l^2 g_l^{-2} \prod_{\substack{j\in S\\ j\neq l}} |\frac{\sin(\frac{\sigma_l-\sigma_j}{2})}{\sin(\frac{\sigma_l+\sigma_j}{2})}|^{-2}$$

$$\leq \sum_{l=1}^{l=N}(\frac{2\nu_l^2}{h})\prod_{j\neq l}|\frac{\sin(\frac{\sigma_l-\sigma_j}{2})}{\sin(\frac{\sigma_l+\sigma_j}{2})}|^{-2} \exp\{2[x\nu_l - \Gamma_+(\sigma_l)]/h\} \sum_{|S|=k+1} \det(G_S)$$

$$\leq \sum_{l=1}^{l=N}(2\nu_l^2/h)\exp\{2[\Gamma_+(\sigma_l) - \Gamma_-(\sigma_l)]/h\}$$

$$\cdot \exp\{2[x\nu_l - \Gamma_+(\sigma_l)]/h\} \sum_{|S|=k+1} \det(G_S)$$

$$\leq \sum_{l=1}^{l=N}(2\nu_l^2/h)\exp\{(2\nu_l/h)[x - (\Gamma_-/\nu_l)]\} \sum_{|S|=k+1} \det(G_S).$$

This is (2.2.54). In the above proof we used a fact :

$$\begin{aligned}
&\prod_{j\neq l}\left|\frac{\sin(\frac{\sigma_l-\sigma_j}{2})}{\sin(\frac{\sigma_l+\sigma_j}{2})}\right|^{-2} \\
&= \exp\{-\sum_{j\neq l}\log\left|\frac{\sin(\frac{\sigma_l-\sigma_j}{2})}{\sin(\frac{\sigma_l+\sigma_j}{2})}\right|^2\} \\
&\leq \exp\{\frac{1}{h\pi}\int_0^{\pi}\left(-\log\left|\frac{\sin(\frac{\sigma_l-\sigma}{2})}{\sin(\frac{\sigma_l+\sigma}{2})}\right|^2\right)\phi(\sigma,h)\,d\sigma\} \\
&\leq \exp\{\frac{1}{h\pi}\int_0^{\pi}\left(-\log\left|\frac{\sin(\frac{\sigma_l-\sigma}{2})}{\sin(\frac{\sigma_l+\sigma}{2})}\right|^2\right)\phi(\sigma)\,d\sigma\} \\
&= \exp\{\frac{2}{h}[-L\phi(\sigma_l)]\} \\
&= \exp\{\frac{2}{h}[\Gamma_+(\sigma_l)-\Gamma_-(\sigma_l)]\}
\end{aligned}$$

With an assumption given at (2.1.20) and (2.1.21), we can estimate ν_l/h by the following lemma.

Lemma 2.4

In the X-case, the ratio (ν_l/h) is bound away from zero for all l. That means there is a positive constant M which is independent of h, suth that

$$\frac{\nu_l}{h} = \frac{\sin\sigma_l}{h} \geq M > 0$$

for all l.

PROOF :

We assume there is an order of all the eigenvalues :

$$-1 < \lambda_N < \lambda_{N-1} < \ldots < \lambda_1 < 1.$$

So we can have

$$0 < \sigma_1 < \sigma_2 < \ldots < \sigma_N < \pi,$$

if we set $\lambda_k = \cos\sigma_k$. Also, we set $\nu_k = \sin\sigma_k$ as before. It is obvious that

$$\lambda_{max} = \lambda_1 \qquad \text{and} \qquad \lambda_{min} = \lambda_N.$$

By (2.1.20), there is a positive constant C_1, such that

$$\begin{aligned}
C_1 h\pi &\le [\Phi(0) - \Phi(\sigma_1)] \\
&= \Big(\int_{x_+(\sigma_1)}^{+\infty} + \int_{-\infty}^{x_-(\sigma_1)}\Big)\sqrt{[1-\hat\lambda_1(x)][1-\hat\lambda_2(x)]}\,dx \\
&+ \int_{x_-(\sigma_1)}^{x_+(\sigma_1)} \{\sqrt{[1-\hat\lambda_1(x)][1-\hat\lambda_2(x)]} - \sqrt{[\lambda_1-\hat\lambda_1(x)][\lambda_1-\hat\lambda_2(x)]}\}\,dx.
\end{aligned}$$

Dividing both sides by ν_1, we have

$$\frac{C_1 h\pi}{\nu_1} \le \Big(\int_{x_+(\sigma_1)}^{X_b} + \int_{-X_b}^{x_-(\sigma_1)}\Big)\sqrt{\frac{1-\hat\lambda_1(x)}{1-\lambda_1}\frac{1-\hat\lambda_2(x)}{1+\lambda_1}}\,dx +$$

$$\int_{x_-(\sigma_1)}^{x_+(\sigma_1)} \frac{\nu_1^2 - (\hat\lambda_1(x)+\hat\lambda_2(x))(1-\lambda_1)}{\nu_1\{\sqrt{[1-\hat\lambda_1(x)][1-\hat\lambda_2(x)]} + \sqrt{[\lambda_1-\hat\lambda_1(x)][\lambda_1-\hat\lambda_2(x)]}\}}\,dx$$

$$\le \sqrt{2}[(X_b - x_+(\sigma_1))][(x_-(\sigma_1) - X_b)]$$

$$+ \int_{x_-(\sigma_1)}^{x_+(\sigma_1)} \left\{ \frac{1 - \frac{\partial_x S(x)}{1+\lambda_1}}{\sqrt{\frac{1-\hat\lambda_1(x)}{1-\lambda_1}\frac{1-\hat\lambda_2(x)}{1+\lambda_1}}} \right\} dx$$

$$\le 2\sqrt{2}X_b + \int_{x_-(\sigma_1)}^{x_+(\sigma_1)} \left(1/\sqrt{\frac{1-\hat\lambda_2(x)}{1+\lambda_1}}\right) dx$$

$$\le 2\sqrt{2}X_b(1 + 1/\sqrt{1-\hat\lambda_{2,max}}) \stackrel{\text{def}}{=} D_1.$$

That is

$$\frac{\nu_1}{h} = \frac{\sin\sigma_1}{h} \ge \frac{C_1\pi}{D_1} \stackrel{\text{def}}{=} M_1 > 0.$$

Similarly, with (2.1.21), there should be positive constants C_2, D_2 such that

$$\frac{\nu_N}{h} = \frac{\sin\sigma_N}{h} \ge \frac{C_2\pi}{D_2} \stackrel{\text{def}}{=} M_2 > 0.$$

Thus, if we set $M = \min\{M_1, M_2\}$, we will have the ratio

$$\frac{\nu_l}{h} \ge \frac{\min\{\nu_1, \nu_N\}}{h} \ge M > 0$$

for all l.

Now, we can go back to the proof of Theorem 2.9 and finish it. With Lemma 2.3,

$$\int_x^{+\infty} |W_{xx}|\, dx = W_x|_{+\infty}^x = W_x(x,t;h)$$

$$= 2h\left[(-h/2)\partial_x \sum_S \det(G_S)\right] / \left[\sum_S \det(G_S)\right]$$

$$\leq (2h)(\frac{h}{2}) \sum_{l=1}^{l=N} \exp\{-(2\nu_l/h)[x - (\Gamma_+(\sigma_l)/\nu_l)]\}$$

$$\leq Nh^2 \exp\{-2M[x - \max_l(\Gamma_+(\sigma_l)/\nu_l)]\},$$

where Lemma 2.4 is applied in the last step. Since

$$\Gamma_+(\sigma_l)/\nu_l$$

$$= x_+(\sigma_l) + \int_{x_+(\sigma_l)}^{X_b} dx \left[1 - \sqrt{\frac{\hat{\lambda}_1(x) - \lambda_l}{1-\lambda_l} \frac{\lambda_l - \hat{\lambda}_2(x)}{1+\lambda_l}}\right]$$

$$\leq X_b + 2X_b = 3X_b,$$

and Nh is uniformly bounded, thus for small h,

$$\int_x^{+\infty} |W_{xx}|\, dx \to 0$$

is exponentially decay and this vanishing is independent of small h, while $x > 3X_b$ and $x \to +\infty$.

Also with Lemma 2.3 and 2.4,

$$\int_{-\infty}^x |W_{xx}| = W_x(x,t;h) - W_x(-\infty,t;h)$$

$$= (2h)\left[\sum_{j=1}^{j=N} \nu_j - \frac{(-h/2)\partial_x \sum_S \det(G_S)}{\sum_S \det(G_S)}\right]$$

$$\leq (2h) \sum_{l=1}^{l=N} (2\nu_l^2/h) \exp\{-(2\nu_l/h)[-x + (\Gamma_-(\sigma_l)/\nu_l)]\}$$

$$\leq (2hN)(2\nu^*/h) \exp\{-(2\nu^*/h)[-x - 3X_b]\},$$

if $(-x - 3X_b) > [1/(2M)]$; where ν^* is defined as $\min_l\{\nu_l\}$. In fact

$$\frac{d}{d\varsigma}[\varsigma e^{-\varsigma y}] = e^{-\varsigma y}(1 - \varsigma y) < 0,$$

if

$$y > \frac{1}{\inf \varsigma} \geq \frac{1}{\varsigma}.$$

It leads that

$$\int_{-\infty}^{x} |W_{xx}| \, dx \to 0$$

is exponentially decay and this vanishing is independent of small h, while $x < -2X_b$ and $x \to -\infty$. Thus W_{xx} matches all the requirement for weak L^1 convergence.

Theorem 2.10

With the requirements met in Theorem 2.9, the family $\{W_{xt_k}(x, \vec{t};\ h)\}$ is weakly L^1 compact too, for all k.

PROOF :

We need the following theorem first. Because of the similar structure between the formula of W_{xx} and of $W_{t_kt_k}$, the L^1 boundness of $W_{t_kt_k}$ could be true. Then, the conclusion of Theorem 2.10 will come out as a direct corollary of it.

Theorem 2.11

$W_{t_kt_k}$ is integrable over the whole x-space and its L^1 norm is uniformly bounded. That means a positive constant C, which is independent on h, exists such that

$$\int_{-\infty}^{+\infty} |W_{t_kt_k}| \, dx < C. \tag{2.2.55}$$

We will prove it by two steps.

First, we show that the family $\{W_{t_k t_k}(x,\vec{t};h)\}$ is uniformly integrable over the whole x-space. That is

Lemma 2.5 *With the requirements met in Theorem 2.9,*

$$\lim_{N\to\infty}\int_{|x|\geq N}|W_{t_k t_k}(x,\vec{t};h)|\,dx = 0 \tag{2.2.56}$$

is uniformly with respect to small h.

PROOF :

Set

$$\begin{aligned} f_S &= \det(G_S) \\ &= \left(\frac{h}{2}\right)^{|S|}\prod_{m,n\in S}\frac{g_n^2}{\sin\sigma_n}\prod_{S\times' S}\left|\frac{\sin(\frac{\sigma_m-\sigma_n}{2})}{\sin(\frac{\sigma_m+\sigma_n}{2})}\right|, \end{aligned}$$

with

$$g_n = \exp\{-[x\sin\sigma_n - \hat{\Gamma}_+(\sigma_n)]/h\}$$

and

$$\hat{\Gamma}_+(\sigma_n) = \Gamma_+(\cos\sigma_n) - \sum_{k=2} t_k \sin\sigma_n\cos^{k-1}\sigma_n.$$

Recall

$$2W_{t_k t_k} = (-4)\frac{\sum_{S,S'} f_S f_{S'}(\sum_{j\in S}\lambda_j^k\nu_j - \sum_{i\in S'}\lambda_i^k\nu_i)^2}{\sum_{S,S'} f_S f_{S'}}$$

and

$$2W_{xx} = (-4)\frac{\sum_{S,S'} f_S f_{S'}(\sum_{j\in S}\nu_j - \sum_{i\in S'}\nu_i)^2}{\sum_{S,S'} f_S f_{S'}},$$

where

$$\lambda_j = \cos\sigma_j, \qquad \nu_j = \sin\sigma_j.$$

Recall $N = O(1/h)$, thus there exists a constant $C_1 > 0$, such that $NH < C_1$.

Set

$$\hat{\epsilon} = \frac{\sum_{S,S'} f_S f_{S'}(\sum_{j\in S_1}\nu_j\sum_{i\in S_2}\nu_i)^2}{\sum_{S,S'} f_S f_{S'}}$$

with

$$S_1 = S - (S \cap S'), \qquad S_2 = S' - (S \cap S').$$

It is obvious that $\hat{\epsilon} > 0$, and

$$(-W_{xx}) + 8\hat{\epsilon} > (-W_{t_k t_k}) > 0.$$

Since we have proved that W_{xx} is weak L^1 compact, then in order to prove Lemma 2.5, it is enough to show that

$$\lim_{N \to \infty} \int_{|x| \geq N} \hat{\epsilon}(x, \vec{t}; h), dx = 0 \tag{2.2.57}$$

uniformly with respect to small h. Actually we can prove more. We will show that $\hat{\epsilon}$ is exponentially decay either as $x \longrightarrow +\infty$ or $x \longrightarrow -\infty$ and that vanishing is independent on small h.

We choose the X-case for simple as we once did with W_{xx}.

Definition 2.4 *The X-case means the following assumption are added to our initial condition :*

(1) There is a positive constant X_b, no matter how large it could be, such that both $\hat{\lambda}_1(x) = 1$ and $\hat{\lambda}_2(x) = -1$ if $|x| \geq X_b$.

(2) There is a positive constant M, no matter how small it could be, such that for all $l = 1, \cdots, N$

$$\frac{\nu_l}{h} \geq M > 0.$$

Here, $\nu_l = \sqrt{1 - \lambda_l^2}$ with the point spectrum $\{\lambda_l\}|_1^N$ of the initial values determined by the inverse scattering transform.

Consider $x \longrightarrow +\infty$ first.

With the proof of Theorem 2.9, we know, for all $l = 1, \cdots, N$,

$$3X_b \geq \Gamma_+(\sigma_l)/\nu_l.$$

Recall Lemma 2.3, we get

$$\begin{aligned}\hat{\epsilon} &\leq \{(-\frac{h}{2})\partial_x[\sum_{|S|=k} f_S]\}^2 \\ &\leq (h^2N^2/4)\exp[-4M(x-3X_b)] \\ &\leq (C_1^2/4)\exp[-4M(x-3X_b)].\end{aligned}$$

Thus, as $x \to +\infty$, $\hat{\epsilon}$ decays exponentially and this vanishing is independent on small h.

Consider $x \to -\infty$ now.

Set

$$S \cap S' = P, \qquad S = S_1 \cup P, \qquad S' = S_2 \cup P.$$

We have

$$1 \leq |S_1| \leq N-1, \qquad 1 \leq |S_2| \leq N-1,$$

$$|S_1| + |S_2| + 2|P| \leq 2N-2,$$

and that $S_1 \cap S_2$ is a null set, where $|S|$ means the number of the indices in S.

Let us estimate $\hat{\epsilon}$ again but with a slightly different view.

$$\begin{aligned}\hat{\epsilon} &= \frac{\sum_{S,S'} f_S f_{S'} \sum_{i\in S_1}\sum_{j\in S_2}(\nu_i\nu_j)}{\sum_{S,S'} f_S f_{S'}} \\ &\leq \frac{\sum_{|P|=0}^{N-1}\sum_{|S_1|=k_1}\sum_{|S_2|=k_2}[f_S(\sum_{j\in S_2}\nu_j)][f_{S'}(\sum_{i\in S_1}\nu_i)]}{\sum_{S,S'} f_S f_{S'}} \\ &\leq \frac{\sum_{|P|=0}^{N-1}\sum_{|S_1|=k_1}\sum_{|S_2|=k_2}[f_S(\sum_{j\notin S}\nu_j)][f_{S'}(\sum_{i\notin S'}\nu_i)]}{\sum_{S,S'} f_S f_{S'}}.\end{aligned}$$

By Lemma 2.3,

$$\hat{\epsilon} \leq \left(\sum_{|P|=0}^{N-1}\sum_{|S_1|=k_1+1}\sum_{|S_2|=k_2+1} f_S f_{S'}\right) / \left(\sum_{S,S'} f_S f_{S'}\right)$$

$$\cdot\{\sum_{j=1}^{j=N}(2\nu_j^2/h)\exp[(2\nu_j/h)(x-\Gamma_-(\sigma_j)/\nu_j)]\}^2$$

$$\leq \left\{ (\frac{h}{2}\sum_{j=1}^{j=N}(2\nu_j/h)^2 \exp[(2\nu_j/h)(x - \Gamma_-(\sigma_j)/\nu_j)] \right\}^2$$

$$\leq \left\{ (\frac{h}{2}\sum_{j=1}^{j=N}(2\nu_j/h)^2 \exp[(2\nu_j/h)(x + 3X_b)] \right\}^2$$

$$\leq \left\{ (\frac{hN}{2})(2M)^2 e^{-M[(-x)-3X_b]} \right\}^2$$

$$\leq 4C_1^2 M^4 e^{-2M[(-x)-3X_b]},$$

if $x < -(1/M) - 3X_b$.

Here, the constant M is given by the assumption of the X-case and we used the following fact because of that case :

$$\begin{aligned} &\Gamma_-(\sigma_j)/\nu_j \\ &\quad = x_-(\sigma_j) - \int_{-X_b}^{x_-(\sigma_j)} dx \left[1 - \sqrt{\frac{\hat{\lambda}_1(x) - \lambda_j}{1-\lambda_j}\frac{\lambda_j - \hat{\lambda}_2(x)}{1+\lambda_j}} \right] \\ &\geq -X_b - 2X_b = -3X_b. \end{aligned}$$

Thus, as $x \longrightarrow -\infty$, $\hat{\epsilon}$ decays exponentially and this vanishing is independent on small h.

With the above results as both $x \longrightarrow +\infty$ and $x \longrightarrow -\infty$, we get (2.2.57) and (2.2.56).

In the next step, we are going to prove the uniformly L^1-boundness of $W_{t_k t_k}$ over any bounded domain of x.

Lemma 2.6 *Assume $E = [a, b]$ is a finite interval of x, then there exists a constant C_2 which is independent on small h such that*

$$\int_E |W_{t_k t_k}(x, \vec{t}; h)|\, dx < C_2.$$

PROOF:

If not, there should be a T and a sequence $\{h_N\}$ which goes to zero as $N \to \infty$ such that as long as $t_k = T$

$$\lim_{N\to\infty} \int_E |W_{t_kt_k}(x, \vec{t}; h_N)|\, dx = \infty.$$

Because of the structure of the form of $W_{t_kt_k}$, there should be a neighborhood of $t_k = T$, in which every value of t_k makes the integral

$$\int_E |W_{t_kt_k}(x, \vec{t}; h)|\, dx$$

be infinite while h goes to zero. One may check it by resetting $\Gamma_\pm(\sigma_j)$ with

$$\tilde{\Gamma}_\pm(\sigma_j) = \Gamma_\pm(\sigma_j) - (t_k - T)\lambda_j^k \nu_j.$$

Therefore, if $t_k = T \in (T_1, T_2)$,

$$\int_{T_1}^{T_2} dt_k \int_E |W_{t_kt_k}(x, \vec{t}; h_N)|\, dx \to \infty, \tag{2.2.58}$$

as $N \to \infty$.

However, $W_{t_kt_k}$ convergence to $Q^*_{t_kt_k}$ in the meaning of distribution. That means the following relation is also true:

$$\begin{aligned} &\lim_{h\to 0}\left(\int_{T_1}^{T_2} dt_k \int_E |W_{t_kt_k}|\, dx\right) \\ &= -\lim_{h\to 0}\left(\int_{T_2}^{T_1} dt_k \int_E W_{t_kt_k}\, dx\right) \\ &= -\int_{T1}^{T_2} dt_k \int_E Q^*_{t_kt_k}\, dx \\ &= \int_a^b [Q^*_{t_k}(x, \vec{t})|_{t_k=T_1} - Q^*_{t_k}(x, \vec{t})|_{t_k=T_2}]\, dx. \end{aligned} \tag{2.2.59}$$

Since $Q^*_{t_k}$ is a continuous function as proved by Theorem 3.25 later, the quantity of (2.2.59) should be finite. It is a contradiction to (2.2.58). Thus, $\int_E |W_{t_kt_k}|\, dx$, for any bounded domain E, should be uniformly bounded with respect to small h. The Lemma 2.6 is approved.

Combining Lemma 2.5 with Lemma 2.6, the proof of Theorem 2.11 is completed.

With Theorem 2.11 and the inequality given by (2.2.49), Theorem 2.10 is quite obvious, since

$$\begin{aligned}&\int |W_{xt_k}(x,t;h)|\,dx\\ &\quad\le [\int |W_{xx}|\,dx]^{1/2}[\int |W_{t_kt_k}(x,t;h)|\,dx]^{1/2}\\ &\quad< \sqrt{C}[\int |W_{xx}|\,dx]^{1/2}.\end{aligned}$$

2.2.4 σ Representation

After all, by taking an insight to the form of W, one can find that it is better to describe the related eigenvalues and all other quantities with σ representation, because $\lambda_j = \cos\sigma_j$ and $\nu_j = \sin\sigma_j$. Thus, we will use σ representation here after.

For later convenience, we review the former results and definitions related with the σ representation.

(1)

For $\sigma \in (-\pi, \hat{\sigma}_2) \cup (\hat{\sigma}_1, \pi)$, there are two functions $x_+(\sigma)$ and $x_-(\sigma)$ defined by (see Figure 4):

$$\hat{\sigma}_1(x_+) = \hat{\sigma}_1(x_-) = \sigma, \tag{2.2.60}$$

or

$$\hat{\sigma}_2(x_+) = \hat{\sigma}_2(x_-) = \sigma.$$

We also define that $x_0 = x_\pm(\hat{\sigma}_{1,max})$ or $x_0 = x_\pm(\hat{\sigma}_{2,min})$.

(2)

Set $N(h,\sigma)$ as the number of all possible eigenvalues $\lambda_j = \cos\sigma_j$ such that σ_j is only chosen from $(\sigma, \hat{\sigma}_{1,max})$ or only from $(\hat{\sigma}_{2,min}, \sigma)$. This N can be figured out by a function Φ.

$$N(h,\sigma) \approx \begin{cases} +\Phi(\sigma) & \text{for } \sigma < \hat{\sigma}_{1,max} \\ -\Phi(\sigma) & \text{for } \sigma > \hat{\sigma}_{2,min}, \end{cases}$$

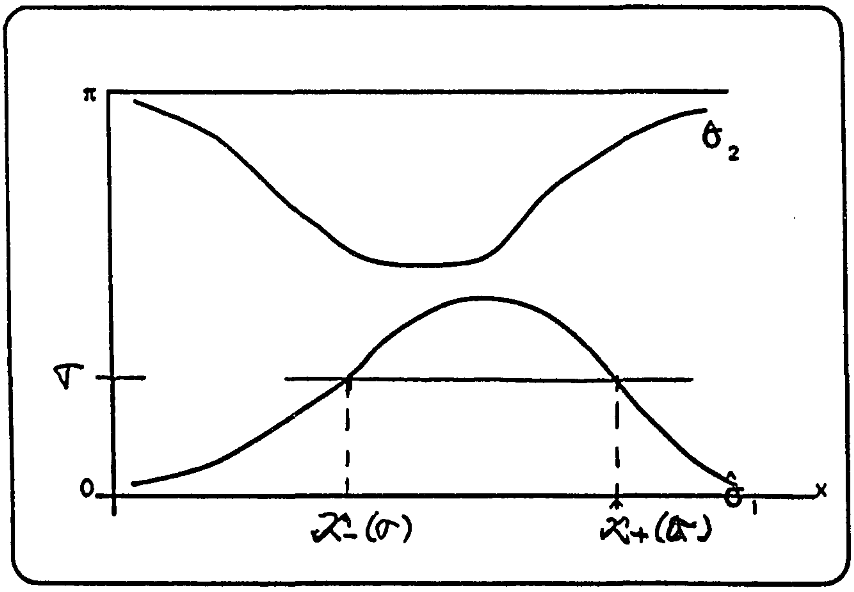

Figure 2.4: **The graph of initial turning points in σ representation**

where

$$\Phi(\sigma) = \begin{cases} -\int_{x_-(\sigma)}^{x_+(\sigma)} \sqrt{(\cos\sigma - \cos\hat{\sigma}_2(y))(\cos\sigma - \cos\hat{\sigma}_1(y))}\, dy & \sigma > \hat{\sigma}_{2,min} \\ 0 & \hat{\sigma}_{2,min} > \sigma > \hat{\sigma}_{1,max} \\ \int_{x_-(\sigma)}^{x_+(\sigma)} \sqrt{(\cos\hat{\sigma}_2(y) - \cos\sigma)(\cos\hat{\sigma}_1(y) - \cos\sigma)}\, dy & \hat{\sigma}_{1,max} > \sigma, \end{cases} \tag{2.2.61}$$

and we only consider $0 < \sigma < \pi$.

(3)

The functions $\Gamma_\pm(\sigma)$ are defined as following :

$$\Gamma_+(\sigma) = \sin\sigma x_+(\sigma) + \int_{x_+(\sigma)}^{+\infty} dy \left[\sin\sigma - \sqrt{(\cos\hat{\sigma}_2(y) - \cos\sigma)(\cos\sigma - \cos\hat{\sigma}_1(y))}\right],$$

and

$$\Gamma_-(\sigma) = \sin\sigma x_-(\sigma) - \int_{-\infty}^{x_-(\sigma)} dy \left[\sin\sigma - \sqrt{(\cos\hat{\sigma}_2(y) - \cos\sigma)(\cos\sigma - \cos\hat{\sigma}_1(y))}\right].$$

They satisfy

$$\Gamma_\pm(0) = \Gamma_\pm(\pi) = 0,$$

and

$$\Gamma_-(\sigma) < (\sin\sigma)x_0 < \Gamma_+(\sigma).$$

Also, they are continuous where $0 \le \sigma \le \hat{\sigma}_{1,max}$ or $\hat{\sigma}_{2,min} \le \sigma \le \pi$.

(4)

The norming constants are

$$c_j(t) = \exp\{[-t\sin\sigma\cos\sigma + \Gamma_+(\sigma)]/h\}.$$

Or, generally,

$$c_j(\vec{t}) = \exp[(\Gamma_+(\sigma) - \sum_{k=2} t_k \sin\sigma\cos^{k-1}\sigma)/h].$$

(5)

We denote the derivative of Φ which was defined at (2.2.61) by $-\phi$:

$$\phi(\sigma) = -\frac{d\Phi}{d\sigma} = \frac{1}{2}\int_{x_-(\sigma)}^{x_+(\sigma)} \sin\sigma \left(\sqrt{\frac{\cos\sigma - \cos\hat{\sigma}_2(y)}{\cos\sigma - \cos\hat{\sigma}_1(y)}} + \sqrt{\frac{\cos\sigma - \cos\hat{\sigma}_1(y)}{\cos\sigma - \cos\hat{\sigma}_2(y)}}\right) dy, \tag{2.2.62}$$

if $\pi > \sigma > \hat{\sigma}_{2,min}$ or $\hat{\sigma}_{1,max} > \sigma > 0$.

Obviously, $\phi(\sigma) = 0$ where $\hat{\sigma}_{2,min} \ge \sigma \ge \hat{\sigma}_{1,max}$.

Lemma 2.7

The function $\phi(\sigma)$ is odd, non-negative and belonging to $L^1(0,\pi)$.

PROOF :

This follows from the fact that Φ defined at (2.2.61) is a bounded and decreasing function of $\sigma \in (0,\pi)$. The decreasing is obvious. In order to verify the boundness we estimate the following

$$\begin{aligned}
0 &\leq \int_{-\infty}^{+\infty} \sqrt{(1 \mp \cos\hat{\sigma}_1(y))(1 \mp \cos\hat{\sigma}_2(y))}\, dy \\
&= \int_{-\infty}^{+\infty} \sqrt{(1 \mp \frac{1}{2}\partial_y S)^2 - A^2(y)}\, dy \\
&= \int_{-\infty}^{+\infty} \sqrt{(1 - A^2(y)) \mp \partial_y S + \frac{1}{4}(\partial_y S)^2}\, dy \\
&\leq \int_{-\infty}^{+\infty} \sqrt{1 - A^2(y)}\, dy + \int_{-\infty}^{+\infty} \sqrt{|\partial_y S(y)|}\, dy \\
&+ \frac{1}{2}\int_{-\infty}^{+\infty} |\partial_y S(y)|\, dy. \qquad (2.2.63)
\end{aligned}$$

If we denote the above quantity as $\int$, then

$$\begin{aligned}
&\int \leq \\
&\left(\int_{-\infty}^{+\infty} \frac{dy}{1+y^2}\right)^{1/2} \left[\int_{-\infty}^{+\infty} (1 - A^2(y))(1+y^2)\, dy\right]^{1/2} + \\
&\left(\int_{-\infty}^{+\infty} \frac{dy}{1+y^2}\right)^{1/2} \left[\int_{-\infty}^{+\infty} |\partial_y S|(1+y^2)\, dy\right]^{1/2} + \\
&\frac{1}{2}\int_{-\infty}^{+\infty} |\partial_y S|(1+y^2)\, dy < \infty, \qquad (2.2.64)
\end{aligned}$$

where we used the Schwartz inequality and the assumption of (2.1.2) again. Thus the boundness of Φ for $\sigma \in [0,\pi]$ follows that

$$\Phi(0) = \int_{-\infty}^{+\infty} \sqrt{[1 - \cos\hat{\sigma}_1(y)][1 - \cos\hat{\sigma}_2(y)]}\, dy,$$

and

$$\Phi(\pi) = \int_{-\infty}^{+\infty} \sqrt{[1 + \cos\hat{\sigma}_1(y)][1 + \cos\hat{\sigma}_2(y)]}\, dy.$$

CHAPTER 3

LIMIT AND VARIATION

3.1 Asymptotic Analysis of W

3.1.1 Limit of W

In this section, we carry out an asymptotic analysis as $h \to 0$ for $W(x, \vec{t}; h)$ which is given by (2.2.38):

$$W = -h^2 \log \det(I + G), \tag{3.1.1}$$

where $G(x, \vec{t}; h)$ is given by (2.2.42). This analysis of W then directly leads to the study of limits of all the conserved densities $\{\rho_k(x, \vec{t}; h)\}$. As examples, we will often mention the first three conserved densities : (see (1.2.9))

$$\rho = |U|^2,$$

$$\mu = -i\frac{h}{2}(U^* U_x - U U_x^*), \tag{3.1.2}$$

$$\varepsilon = (1-\rho)^2 + h^2 |U_x|^2 = (1-\rho)^2 + \frac{\mu^2}{\rho} + h^2 \frac{(\rho_x^2)}{4\rho}.$$

With Theorem 2.7, W can be described by a sum of determinants of G_S which are the principal minors of matrix G. Each determinant of G_S can be factored similarly to the factorization of G. That is

$$\det(I + G) = \sum_S \det(G_S), \tag{3.1.3}$$

and

$$\det G_S = (\frac{h}{2})^{|S|} \prod_S \frac{g_j^2}{\sin \sigma_j} \prod_{S \times' S} |\frac{\sin(\frac{\sigma_i - \sigma_j}{2})}{\sin(\frac{\sigma_i + \sigma_j}{2})}|.$$

With the definition of $\{g_j\}$ given by (2.2.46), we can write $\det G_S$ as

$$\det G_S = \left(\frac{h}{2}\right)^{|S|} \exp\left(-q_h(S; x, \vec{t})/2h^2\right), \tag{3.1.4}$$

where

$$q_h(S; x, \vec{t}) \stackrel{\text{def}}{=} 4h \sum_{j \in S} a(\sigma_j; x, \vec{t}) - h^2 \sum_{\substack{S \times' S \\ m, n \in S}} \log\left[\left(\frac{\sin(\frac{\sigma_m - \sigma_n}{2})}{\sin(\frac{\sigma_m + \sigma_n}{2})}\right)^2\right] + 2h^2 \sum_{j \in S} \log(\sin \sigma_j). \tag{3.1.5}$$

By (2.2.45),

$$a(\sigma; x, \vec{t}) = x \sin \sigma + \sum_{k=2} t_k \sin \sigma \cos^{k-1} \sigma - \Gamma_+(\sigma),$$

or, if we only consider the t-flow, then

$$a(\sigma; x, t) = x \sin \sigma + t \sin \sigma \cos \sigma - \Gamma_+(\sigma).$$

Using (3.1.4) to express (3.1.3) offers

$$\det(I + G) = \sum_{all\ S} (\frac{h}{2})^{|S|} \exp\left(-q_h(S; x, \vec{t})/2h^2\right). \tag{3.1.6}$$

Consider its image under a logarithm mapping. One can find that, for small h, this sum is dominated by its largest term. First we define

$$q^*(x, \vec{t}; h) \stackrel{\text{def}}{=} \min_S \{q_h(S; x, \vec{t})\}. \tag{3.1.7}$$

Since all terms in (3.1.6) are positive, and since there are $2^{N(h)}$ of them, we get the following two-sided inequality :

$$(\frac{h}{2})^N \exp\left(-\frac{q^*}{2h^2}\right) \le \det(I + G) \le 2^N \exp\left(-\frac{q^*}{2h^2}\right).$$

Taking the logarithm of all the sides and multiplying by $2h^2$ gives

$$2h^2 N \log(\frac{h}{2}) - q^* \leq 2h^2 \log\det(I+G) \leq 2h^2 N \log 2 - q^*.$$

By (3.1.1), the center term is just $-2W$. That leads to the following estimate

$$|2W(x,\vec{t};h) - q^*(x,\vec{t};h)| \leq 2h^2 N(h)\log(2/h).$$

According to (2.1.12), $N(h) = O(\frac{1}{h})$, thus we have proved

Theorem 3.12 *That*

$$\lim_{h\to 0}[W(x,\vec{t};h) - \frac{1}{2}q^*(x,\vec{t};h)] = 0, \tag{3.1.8}$$

is uniformly in x and $\vec{t}$.

3.1.2 Limit of q_h^*

The existence of the limit of $q^*(x,\vec{t};h)$ can be verified. This limit can be characterized as a solution of a minimum problem. To find this minimum problem, we first need to write $q_h(S;x,\vec{t})$ as a Stieltjes integral.

For each S we define $\psi_S(\sigma,h)$ as the distribution

$$\psi_S(\sigma,h) = h\pi \sum_{j\in S} \delta(\sigma - \sigma_j). \tag{3.1.9}$$

Then by (3.1.5),

$$\begin{aligned}
& q_h(S;x,\vec{t}) \\
& = \frac{4}{\pi}\int_0^\pi a(\sigma,x,\vec{t})\psi_S(\sigma,h)d\sigma \\
& -\frac{1}{\pi^2}\int_0^\pi\int_0^\pi \log\left[\left(\frac{\sin(\frac{\sigma-\sigma'}{2})}{\sin(\frac{\sigma+\sigma'}{2})}\right)^2\right]\psi_S(\sigma',h)\times'\psi_S(\sigma,h)d\sigma' d\sigma \\
& +\frac{2h}{\pi}\int_0^\pi \log(\sin\sigma)\psi_S(\sigma,h)d\sigma.
\end{aligned} \tag{3.1.10}$$

Here $\psi_S(\sigma', h) \times' \psi_S(\sigma, h)$ is the product of the distributions which minus the diagonal terms:

$$\psi_S(\sigma', h) \times' \psi_S(\sigma, h) = h^2\pi^2 \sum_{\substack{i, j \in S \\ i \neq j}} \delta(\sigma' - \sigma_i)\delta(\sigma - \sigma_j).$$

$\phi(\sigma, h)$ is also a distribution which is defined by

$$\phi(\sigma, h) = h\pi \sum_{j=-N^-}^{j=N^+} \delta(\sigma - \sigma_j), \tag{3.1.11}$$

where N^+ and N^- are given by Definition 2.1. Obviously,

$$0 \leq \psi_S(\sigma, h)d\sigma \leq \phi(\sigma, h)d\sigma \tag{3.1.12}$$

is true for every S in the meaning of measures.

Since, for each j, λ_j is the j_{th} eigen-value and $\lambda_j = \cos\sigma_j$, then, by Theorem 2.3,

$$\begin{array}{rcl} \lim_{h\to 0} \phi(\sigma, h)d\sigma & = & \phi(\sigma)d\sigma, \\ \lim_{h\to 0} \phi(\sigma', h) \times' \phi(\sigma, h)d\sigma' d\sigma & = & \phi(\sigma')\phi(\sigma)d\sigma' d\sigma, \end{array} \tag{3.1.13}$$

in the sense of weak sequential convergence of measures, where $\phi(\sigma)$ is defined by (2.2.62). These considerations suggest a minimum problem satisfied by the limit of $q^*(x, \vec{t}; h)$:

Theorem 3.13 *That*

$$\lim_{h\to 0} q^*(x, \vec{t}; h) = Q^*(x, \vec{t}) \tag{3.1.14}$$

is uniformly on compact subsets of x, $\vec{t}$, *where*

$$Q^*(x, \vec{t}) = \inf\{Q(\psi; x, \vec{t}) : \psi \in A\}; \tag{3.1.15}$$

with the admissible set A *consistings of all Lebesgue measurable functions* ψ *on* $(0, \pi)$, *such that*

$$0 \leq \psi(\sigma) \leq \phi(\sigma), \tag{3.1.16}$$

and with $Q(\psi; x, \vec{t})$ which is a quadratic form of ψ and is defined by

$$
\begin{aligned}
&Q(\psi; x, \vec{t}) \\
&= \frac{4}{\pi} \int_0^\pi a(\sigma, x, \vec{t})\psi(\sigma)\, d\sigma \\
&-\frac{1}{\pi^2} \int_0^\pi \int_0^\pi \log \left(|\frac{\sin(\frac{\sigma-\sigma'}{2})}{\sin(\frac{\sigma+\sigma'}{2})}|^2 \right) \psi(\sigma')\psi(\sigma)\, d\sigma'\, d\sigma. \qquad (3.1.17)
\end{aligned}
$$

The proof is based on a series of lemmas and theorems.

3.1.3 Proof of Theorem 3.1.2

Let A_h denote the set of distributions $\psi_S(\sigma, h)$ defined by (3.1.9) for all possible sets of indices S and for all $h > 0$,

$$A_h = \{\psi_S(\sigma, h) : S, h\}.$$

For the elements ψ of either the admissible sets A_h or A, we assign $\psi\, d\sigma$ as positive measures on $[0, \pi]$. If $\psi_S(\sigma, h) \in A_h$, then (3.1.12) and (3.1.11) imply

$$
\begin{aligned}
&\int_0^\pi \psi_S(\sigma, h)\, d\sigma \leq \int_0^\pi \phi(\sigma, h)\, d\sigma \\
&= h\pi N(h) = h\pi(N^+ + N^-) \\
&\leq \int_{-\infty}^{+\infty} \sqrt{(1-\cos\hat{\sigma}_1)(1-\cos\hat{\sigma}_2)}\, dx + \int_{-\infty}^{+\infty} \sqrt{(1+\cos\hat{\sigma}_1)(1+\cos\hat{\sigma}_2)}\, dx \\
&= \int_{-\infty}^{+\infty} \sqrt{(1-\hat{\lambda}_1)(1-\hat{\lambda}_2)}\, dx + \int_{-\infty}^{+\infty} \sqrt{(1+\hat{\lambda}_1)(1+\hat{\lambda}_2)}\, dx \\
&< \infty; \qquad (3.1.18)
\end{aligned}
$$

while if $\psi \in A$, then (3.1.16) and (2.2.62) imply

$$
\begin{aligned}
&\int_0^\pi \psi(\sigma)\, d\sigma \leq \int_0^\pi \phi(\sigma)\, d\sigma \\
&= -\int_0^\pi \frac{d}{d\sigma}\Phi(\sigma) = \Phi(0) - \Phi(\pi) \\
&= \int_{-\infty}^{+\infty} \sqrt{(1-\hat{\lambda}_1)(1-\hat{\lambda}_2)}\, dx + \int_{-\infty}^{+\infty} \sqrt{(1+\hat{\lambda}_1)(1+\hat{\lambda}_2)}\, dx \\
&< \infty. \qquad (3.1.19)
\end{aligned}
$$

Thus we conclude

Lemma 3.8

The total variations of the measures $\psi \, d\sigma$, ψ in A_h and A are uniformly bounded.

Definition 3.5 *A sequence of distributions ψ_k is defined to be weak convergence to ψ if the corresponding sequence of measures $\psi_k(\sigma)\, d\sigma$ converge to $\psi(\sigma)\, d\sigma$ in the sense of weak sequential convergence of measures, that is, if*

$$\lim_{k\to\infty} (\chi, \psi_k) = (\chi, \psi) \tag{3.1.20}$$

for every continuous function χ. Here the parentheses denote the usual duality on $[0, \pi]$:

$$(\chi, \psi) = \frac{1}{\pi} \int_0^{\pi} \chi(\sigma)\psi(\sigma)\, d\sigma. \tag{3.1.21}$$

We shall abbreviate (3.1.20) as w-convergence and write

$$w\_\lim_{k\to\infty} \psi_k = \psi.$$

The next theorem characterizes the admissible set A as w_limits of the distribution in A_h.

Lemma 3.9

(a) Let ψ_k denote a sequence of elements in A_h, $\psi_k(\sigma) = \psi_{S_k}(\sigma, h_k)$, such that, as $h_k \to 0$,

$$w\_\lim_{k\to\infty} \psi_k(\sigma) = \psi(\sigma); \tag{3.1.22}$$

then ψ belongs to A.

(b) Conversely, for each ψ in A, we can choose $S(h)$ such that

$$w\_\lim_{h\to 0} \psi_{S(h)}(\sigma, h) = \psi(\sigma).$$

(c)

$$w\_\lim_{h\to 0} \phi(\sigma, h) = \phi(\sigma). \tag{3.1.23}$$

PROOF :

By (3.1.12), the ψ_k satisfy

$$0 \le \psi_k(\sigma)\, d\sigma \le \phi(\sigma, h_k)\, d\sigma.$$

Since the inequalities between measures are preserved for their weak limits, it follows from (3.1.22), and (3.1.23) that

$$0 \le \psi(\sigma)\, d\sigma \le \phi(\sigma)\, d\sigma.$$

From this we conclude by the Radon-Nikodym theorem that ψ is a measurable function and that (3.1.16) is satisfied, thus $\psi \in A$.

Part (b) is straightforward and part (c) is contained in (3.1.13).

The main step to prove Theorem 3.13 is by relating $q_h(S; x, \vec{t})$ as given in (3.1.10) to $\mathcal{Q}(\psi; x, \vec{t})$ as given by (3.1.17).

Theorem 3.14

Denote by ψ_k a sequence of elements in A_h, $\psi_k(\sigma) = \psi_{S_k}(\sigma, h_k)$, such that, as $h_k \to 0$,

$$w\text{-}\lim_{k\to\infty} \psi_k(\sigma) = \psi(\sigma);$$

then

$$\lim_{k\to\infty} q_{h_k}(S_k; x, \vec{t}) = \mathcal{Q}(\psi; x, \vec{t}) \tag{3.1.24}$$

for every x and t_n, $n = 2, 3, \ldots$.

The mild technical difficulties in the proof arise in handling the second and third terms of the right-hand side of (3.1.10). This will be accomplished with the aid of lemmas as well as the introduction of an operator L :

$$L\psi(\sigma) = \frac{1}{2\pi}\int_0^\pi \log\left(|\frac{\sin(\frac{\sigma-\sigma'}{2})}{\sin(\frac{\sigma+\sigma'}{2})}|^2\right) \psi(\sigma')\, d\sigma'. \tag{3.1.25}$$

Lemma 3.10

$L\phi$ is a continuous function on $(0,\pi)$ which tends to zero as $\sigma \to 0$ and $\sigma \to \pi$. Moreover,

$$L\phi(\sigma) = \Gamma_-(\sigma) - \Gamma_+(\sigma). \tag{3.1.26}$$

PROOF :

Since it was shown in Lemma 2.5 that both $\Gamma_+(\sigma)$ and $\Gamma_-(\sigma)$ are continuous functions, the continuity of $L\phi$ in $[0,\pi]$ follows from (3.1.26).

Formula (3.1.26) will be derived at the end of Section 3.2.

Corollary 3.1

(a) For every ψ in the admissible set A, $L\psi$ is continuous and

$$L\phi(\sigma) \le L\psi(\sigma) \le 0 \qquad \text{for} \qquad \sigma \in [0,\pi]. \tag{3.1.27}$$

(b) $L\psi$ is an odd function of σ.

PROOF :

(b) is obvious from (3.1.25); thus we may extend $L\psi$, $L\phi$ to $(-\pi,\pi)$ sometimes if convenient. The inequality (3.1.27) of (a) follows from the fact that, by inequality (3.1.16), all the admissible functions are non-negative and the kernel of the integral operator L is negative. The kernel is negative because

$$\frac{\sin^2(\frac{\varsigma-\sigma}{2})}{\sin^2(\frac{\varsigma+\sigma}{2})} = \frac{(\cos\varsigma - \cos\sigma)^2 + (\sin\varsigma - \sin\sigma)^2}{(\cos\varsigma - \cos\sigma)^2 + (\sin\varsigma + \sin\sigma)^2} \le 1$$

for ς and σ both in $(0,\pi)$.

To show the continuity of $L\psi(\sigma)$ where $\sigma \in [0,\pi]$, we first recall (3.1.26). Since both Γ_+ and Γ_- are continuous where $\sigma \in [0,\pi]$, so is $L\phi$. Next, because of (3.1.27) and that if $\sigma_k \to \sigma$ with $\sigma_k \in (0,\pi)$ then the integrands of $L\psi(\sigma_k)$ appearing at (3.1.25) converge almost everywhere to the integrand of $L\psi(\sigma)$ and are bounded below by the integrands of $L\phi(\sigma)$, where a dominated convergence argument is applied. It implies that $L\psi(\sigma_k)$ converges to $L\psi(\sigma)$ and the continuity of $L\psi$ for $\sigma \in [0,\pi]$ follows. Moreover, because of property (b), $L\psi(0) = 0$ and $L\psi$ is continuous for all $\sigma \in [-\pi, 0]$ where $\psi \in A$.

We use the operator L defined by (3.1.25) in conjunction with the duality notation of (3.1.21) to rewrite the functional $\mathcal{Q}(\psi; x, \vec{t})$ of (3.1.25) as

$$\mathcal{Q}(\psi; x, \vec{t}) = 4(a(\sigma, x, \vec{t}), \psi) - 2(L\psi, \psi). \tag{3.1.28}$$

Note that, by Lemma 2.5, the $a(\sigma, x, \vec{t})$ given by (2.2.45) is continuous. In particular, this shows that $\mathcal{Q}(\psi; x, \vec{t})$ is finite for all ψ in the admissible set A.

To prove Theorem 3.14, we break up the functionals $\mathcal{Q}(\psi)$ and $q_h(S)$ into two parts depending on a parameter B, which eventually tends to infinity :

$$\mathcal{Q}(\psi) = \mathcal{Q}^B(\psi) + \mathcal{Q}_B(\psi)$$

and

$$q_h(S) = q_h^B(S) + q_{h,B}(S). \tag{3.1.29}$$

Setting

$$l(\sigma, \sigma') = -\frac{1}{2\pi} \log \left([\frac{\sin(\frac{\sigma - \sigma'}{2})}{\sin(\frac{\sigma + \sigma'}{2})}]^2 \right)$$

and

$$l^B(\sigma, \sigma') = \min\{l(\sigma, \sigma'), B\}, \tag{3.1.30}$$

we define

$$\begin{aligned} \mathcal{Q}^B(\psi) &= \frac{4}{\pi} \int_0^\pi a(\sigma, x, \vec{t}) \psi(\sigma) \, d\sigma \\ &+ \frac{2}{\pi} \int_0^\pi \int_0^\pi l^B(\sigma, \sigma') \psi(\sigma') \psi(\sigma) \, d\sigma' \, d\sigma \end{aligned}$$

and

$$\begin{aligned} q_h^B(S) &= \frac{4}{\pi} \int_0^\pi a(\sigma, x, \vec{t}) \psi_S(\sigma) \, d\sigma \\ &+ \frac{2}{\pi} \int_0^\pi \int_0^\pi l^B(\sigma, \sigma') \psi_S(\sigma') \psi_S(\sigma) \, d\sigma' \, d\sigma. \end{aligned}$$

Lemma 3.11

(a) If the sequence $\psi_k = \psi_{S_k}(\sigma, h_k)$ *satisfies*

$$w\text{-}\lim_{k\to\infty} \psi_k(\sigma) = \psi(\sigma),$$

then

$$\lim_{k\to\infty} q^B_{h_k}(S_k)(h_k) = \mathcal{Q}^B(\psi).$$

(b) For all ψ *in* A*,*

$$|\mathcal{Q}_B(\psi)| < \delta(B), \tag{3.1.31}$$

where

$$\lim_{B\to\infty} \delta(B) = 0. \tag{3.1.32}$$

(c) For all ψ *in* A_h*,*

$$|q_{h,B}(S;\psi)| < \delta(B) + \gamma(h),$$

where

$$\lim_{h\to 0} \gamma(h) = 0.$$

PROOF :

Since both functions $a(\sigma)$ and $l^B(\sigma, \sigma')$ are continuous, part(a) is a direct consequence of the definition of weak convergence (3.1.20).

To prove (3.1.31) and (3.1.32), we write

$$\mathcal{Q}_B(\psi) = \frac{2}{\pi} \int \int_{l>B} (l(\sigma, \sigma') - B)\psi(\sigma')\psi(\sigma)\, d\sigma'\, d\sigma.$$

Since, for admissible ψ,

$$0 \le \psi \le \phi,$$

it follows that

$$|\mathcal{Q}_B(\psi)| \le \mathcal{Q}_B(\phi).$$

This proves (3.1.31) with $\delta(B) = \mathcal{Q}_B(\phi)$ and (3.1.32) follows from the observation that $\mathcal{Q}(\phi) < \infty$.

For the proof of (c), we need some property of $\hat{\sigma}_{k,k=1,2}$ which is from the assumption of the initial data given by (2.1.2). It is devoted by the following lemma.

Lemma 3.12

If $f^2(x)(1+x^2) \in L^1(\Re)$, then both $f(x)$ and $[f(x)\log|f(x)|]$ are functions of $L^1(\Re)$.

PROOF :

It is obvious for $f(x)$ itself if we use the Schwartz inequality:

$$\int_{-\infty}^{+\infty} |f(x)|\, dx \leq$$
$$[\int_{-\infty}^{+\infty} |f(x)|^2(1+x^2)\, dx]^{1/2}[\int_{-\infty}^{+\infty} \frac{dx}{1+x^2}]^{1/2} < \infty.$$

Now we consider $f\log|f|$. For given small $\epsilon > 0$, we always have

$$|f\log|f|| \leq C_\epsilon |f|^{1-\epsilon},$$

where

$$C_\epsilon = \sup_x |f^\epsilon(\log|f|)| > 0.$$

Clearly,$|f|$ is bounded and $|f|^\epsilon \log|f| \to 0$ if $f \to 0$ because $[f^2(1+x^2)] \in L^1(\Re)$. So C_ϵ exists. We try to find a good ϵ so that $|f|^{1-\epsilon}$ can belong to the $L^1(\Re)$. Formally, we have

$$\begin{aligned}
&\int_{-\infty}^{+\infty} |f|^{1-\epsilon}\, dx \\
&= \int_{-\infty}^{+\infty} (|f|\sqrt{1+x^2})^{1-\epsilon}(\frac{1}{\sqrt{1+x^2}})^{1-\epsilon}\, dx \\
&\leq \left\{\int_{-\infty}^{+\infty} [(|f|\sqrt{1+x^2})^{1-\epsilon}]^{p=2/(1-\epsilon)}\, dx\right\}^{\frac{1}{p}} \\
&\cdot \left\{\int_{-\infty}^{+\infty} [(\frac{1}{\sqrt{1+x^2}})^{1-\epsilon}]^q\, dx\right\}^{\frac{1}{q}=1-\frac{1}{p}} \\
&= \left[\int_{-\infty}^{+\infty} f^2(1+x^2)\, dx\right]^{\frac{1-\epsilon}{2}}
\end{aligned}$$

$$\cdot \left[\int_{-\infty}^{+\infty} \left(\frac{1}{1+x^2} \right)^{\frac{1-\epsilon}{1+\epsilon}} dx \right]^{\frac{1+\epsilon}{2}}.$$

So, we may choose

$$\frac{1-\epsilon}{1+\epsilon} \geq \frac{2}{3}$$

to make the above quantity finite. That is $\epsilon \leq 1/3$ and $p = 2/(1-\epsilon) > 1$. It shows that

$$\int_{-\infty}^{+\infty} |f \log(|f|)| \, dx < \infty.$$

With this lemma, the assumption given by (2.1.2) implies the following result.

Lemma 3.13

$\log \sin \sigma$ *is integrable on the domain* $[0, \pi]$ *with the measure* $\phi(\sigma)\, d\sigma$*. That means*

$$\int_0^{\pi} |\log \sin \sigma| \phi(\sigma) \, d\sigma < \infty. \tag{3.1.33}$$

PROOF :

Set

$$I \equiv \int_0^{\pi} |\log \sin \sigma| \phi(\sigma) \, d\sigma.$$

We have

$$\begin{aligned} I &= \left(\int_0^{\hat{\sigma}_{1,max}} + \int_{\hat{\sigma}_{2,min}}^{\pi} \right) (-\log \sin \sigma)(-\frac{d\Phi}{d\sigma}) \, d\sigma \\ &= \left(\int_{\hat{\lambda}_{1,min}}^{1} + \int_{-1}^{\hat{\lambda}_{2,max}} \right) [-\log(\sqrt{1-\lambda^2})](\frac{d\Phi}{d\lambda}) \, d\lambda \\ &= \left(\int_{\hat{\lambda}_{1,min}}^{1} - \int_{-1}^{\hat{\lambda}_{2,max}} \right) [-\log(\sqrt{1-\lambda^2})] \\ &\quad d\left(\int_{x_-(\lambda)}^{x_+(\lambda)} \sqrt{[\lambda - \hat{\lambda}_1(x)][\lambda - \hat{\lambda}_2(x)]} \, dx \right) \\ &\stackrel{\text{def}}{=} I_1 + I_2, \end{aligned}$$

where

$$I_1 =$$
$$\int_{\hat{\lambda}_{1,min}}^{1} [-\log(\sqrt{1-\lambda^2})] \, d\left(\int_{x_-(\lambda)}^{x_+(\lambda)} \sqrt{[\lambda - \hat{\lambda}_1(x)][\lambda - \hat{\lambda}_2(x)]} \, dx \right),$$

and

$$I_2 = - \int_{-1}^{\hat{\lambda}_{2,max}} [-\log(\sqrt{1-\lambda^2})]\, d\left(\int_{x_-(\lambda)}^{x_+(\lambda)} \sqrt{[\lambda - \hat{\lambda}_1(x)][\lambda - \hat{\lambda}_2(x)]}\, dx\right).$$

Do some variables change :

$$\eta = -\lambda, \quad \eta_1(x) = -\hat{\lambda}_2(x), \quad \eta_2(x) = -\hat{\lambda}_1(x),$$

so that

$$\eta_{1,min} = -\hat{\lambda}_{2,max}, \quad \eta_{2,max} = -\hat{\lambda}_{1,min},$$

and

$$x_\pm(\lambda) = x_\pm(\eta).$$

We get

$$I_2 = \int_{\eta_{1,min}}^{1} [-\log(\sqrt{1-\eta^2})]\, d\left(\int_{x_-(\eta)}^{x_+(\eta)} \sqrt{[\eta - \eta_1(x)][\eta - \eta_2(x)]}\, dx\right),$$

which is in a form as same as of I_1. So it is enough to show that I_1 is finite only.

For I_1, possible singularity of the integrand only comes when λ is equal 1. So we need to cut off I_1 into several simpler parts. Set $\lambda_0 = (1 + \hat{\lambda}_{1,min})/2$, then $\lambda_0 > 0$ and there exist x_1 and x_2 such that $\hat{\lambda}_1(x_1) = \lambda_0 = \hat{\lambda}_1(x_2)$ with $x_1 > x_2$. After a change of the order of integration, we have

$$\begin{aligned} I_1 &= \int_{-\infty}^{+\infty} dx \int_{\hat{\lambda}_1(x)}^{1} [-\log(\sqrt{1-\lambda^2})]\, d\sqrt{[\lambda - \hat{\lambda}_1(x)][\lambda - \hat{\lambda}_2(x)]} \\ &= \left[\left(\left(\int_{x_1}^{+\infty} + \int_{-\infty}^{x_2}\right) dx \int_{\hat{\lambda}_1(x)}^{1}\right) + \left(\int_{x_2}^{x_1} dx \int_{\lambda_0}^{1}\right) + \left(\int_{x_2}^{x_1} dx \int_{\hat{\lambda}_1(x)}^{\lambda_0}\right)\right] \\ &\quad \cdot [-\log(\sqrt{1-\lambda^2})]\, d\sqrt{[\lambda - \hat{\lambda}_1(x)][\lambda - \hat{\lambda}_2(x)]} \\ &\stackrel{\text{def}}{=} I_{11} + I_{12} + I_{13}. \end{aligned}$$

We estimate I_{11} first. Before treating I_{11}, we need to estimate

$$d\sqrt{[\lambda - \hat{\lambda}_1(x)][\lambda - \hat{\lambda}_2(x)]}.$$

We have
$$d\sqrt{[\lambda-\hat\lambda_1(x)][\lambda-\hat\lambda_2(x)]}=\frac{\lambda\mathcal{W}}{\sqrt{\lambda^2-\hat\lambda_1^2(x)}},$$
where we define $\mathcal{W}$ and estimate it as following :
$$\begin{aligned}
\mathcal{W}(\lambda;\hat\lambda_1,\hat\lambda_2)&=\frac{\sqrt{\lambda^2-\hat\lambda_1^2(x)}}{\lambda}\left[\frac{2\lambda-\hat\lambda_1(x)-\hat\lambda_2(x)}{2\sqrt{[\lambda-\hat\lambda_1(x)][\lambda-\hat\lambda_2(x)]}}\right]\\
&=\frac{\sqrt{\lambda^2-\hat\lambda_1^2(x)}}{2\lambda}\left[\sqrt{\frac{\lambda-\hat\lambda_1(x)}{\lambda-\hat\lambda_2(x)}}+\sqrt{\frac{\lambda-\hat\lambda_2(x)}{\lambda-\hat\lambda_1(x)}}\right]\\
&<\frac{\sqrt{\lambda^2-\hat\lambda_1^2(x)}}{2\lambda}\left[1+\sqrt{\frac{\lambda-\hat\lambda_2(x)}{\lambda-\hat\lambda_1(x)}}\right]\\
&=\frac{1}{2}\sqrt{1+\frac{\hat\lambda_1(x)}{\lambda}}\left[\sqrt{1-\frac{\hat\lambda_1(x)}{\lambda}}+\sqrt{1-\frac{\hat\lambda_2(x)}{\lambda}}\right]\\
&<(\sqrt{2}/2)\left(1+\sqrt{\frac{1+\lambda_0}{\lambda_0}}\right)\stackrel{\text{def}}{=}C_1.
\end{aligned}$$
At the last step above, we used a fact that in the domain $x>x_1$ or $x<x_2$,
$$\lambda\geq\hat\lambda_1(x)\geq\lambda_0>\hat\lambda_2(x)>-1.$$

Thus,
$$\begin{aligned}
I_{11}&\leq C_1\left(\int_{x_1}^{+\infty}+\int_{-\infty}^{x_2}\right)dx\int_{\hat\lambda_1(x)}^{1}\frac{\lambda[-\log(\sqrt{1-\lambda^2})]}{\sqrt{\lambda^2-\hat\lambda_1^2(x)}}\,d\lambda\\
&=C_1\left(\int_{x_1}^{+\infty}+\int_{-\infty}^{x_2}\right)dx\int_0^{\hat\nu_1(x)}\frac{\nu[-\log(\nu)]}{\sqrt{\hat\nu_1(x)^2-\nu^2}}\,d\nu\\
&=C_1\left(\int_{x_1}^{+\infty}+\int_{-\infty}^{x_2}\right)dx\left(\hat\nu_1(x)\int_0^1\frac{-y\log(y)}{\sqrt{1-y^2}}\,dy\right.\\
&\left.-[\hat\nu_1(x)\log\hat\nu_1(x)+]\int_0^1\frac{y}{\sqrt{1-y^2}}\,dy\right),
\end{aligned}$$
where we set $\nu=\sqrt{1-\lambda^2}$, $\hat\nu_1(x)=\sqrt{1-\hat\lambda_1^2(x)}$, and $y=\nu/\hat\nu_1(x)$. It is easy to see that the integrals over $[0,1]$ interval with respect to y are all finite because
$$\int_0^1\frac{y\,dy}{\sqrt{1-y^2}}=1,$$

and

$$\int_0^1 \frac{y(-\log(y))}{\sqrt{1-y^2}}\,dy = \left(\int_{1/2}^1 + \int_0^{1/2}\right) \frac{y(-\log(y))}{\sqrt{1-y^2}}\,dy$$
$$< [-\log(1/2)] \int_{1/2}^1 d(-\sqrt{1-y^2}) + \int_0^{1/2} \frac{y(-\log(y))}{\sqrt{1-y^2}}\,dy$$
$$\leq \frac{\sqrt{3}}{2}\log 2 + C_2 < \infty.$$

A positive constant C_2 exists since $(y\log(y)/\sqrt{1-y^2})$ is uniformly continuous on the domain $0 \leq y \leq 1/2$. So, we get the last estimate for I_{11} :

$$I_{11} \leq C\left(\int_{x_1}^{+\infty} + \int_{-\infty}^{x_2}\right) dx[\hat{\nu}_1(x) + \hat{\nu}_1(x)|\log(\hat{\nu}_1(x))|],$$

with some constant C. But

$$\begin{aligned} \hat{\nu}_1(x) &= \sqrt{1-\hat{\lambda}_1^2(x)} \\ &= \sqrt{1-(\partial_x S(x)/2 + |A(x)|)^2} \\ &= \sqrt{(1-A^2) - \partial_x S|A| - (\partial_x S)^2/4} \\ &\leq \sqrt{(1-A^2) + |\partial_x S|}. \end{aligned}$$

By assumption (2.1.2), $(1+x^2)\hat{\nu}_1^2(x)$ belongs to $L^1(\Re)$. Thus, by Lemma 3.12, both $\hat{\nu}_1(x)$ and $\hat{\nu}_1(x)\log[\hat{\nu}_1(x)]$ are integrable. So I_{11} is finite.

Next, we consider I_{12}.

$$I_{12} = \int_{x_2}^{x_1} dx \int_{\lambda_0}^1 [-\log(\sqrt{1-\lambda^2})]\, d\sqrt{[\lambda-\hat{\lambda}_1(x)][\lambda-\hat{\lambda}_2(x)]}.$$

Similarly, we can rewrite

$$d\sqrt{[\lambda-\hat{\lambda}_1(x)][\lambda-\hat{\lambda}_2(x)]} = \frac{\lambda \mathcal{W}_1}{\sqrt{\lambda^2-\lambda_0^2}},$$

with $\mathcal{W}$ defined and estimated as following :

$$\mathcal{W}_1(\lambda; \hat{\lambda}_1, \hat{\lambda}_2, \lambda_0)$$

$$
\begin{aligned}
&= \frac{\sqrt{\lambda^2-\lambda_0^2}}{2\lambda}\left[\frac{2\lambda-\hat{\lambda}_1(x)-\hat{\lambda}_2(x)}{\sqrt{[\lambda-\hat{\lambda}_1(x)][\lambda-\hat{\lambda}_2(x)]}}\right] \\
&= \frac{\sqrt{\lambda^2-\lambda_0^2}}{2\lambda}\left[\sqrt{\frac{\lambda-\hat{\lambda}_1}{\lambda-\hat{\lambda}_2}}+\sqrt{\frac{\lambda-\hat{\lambda}_2}{\lambda-\hat{\lambda}_1}}\right] \\
&< \frac{\sqrt{\lambda^2-\lambda_0^2}}{2\lambda}\left(1+\sqrt{\frac{\lambda-\hat{\lambda}_2}{\lambda-\lambda_0}}\right) \\
&= \frac{1}{2}\sqrt{1+\frac{\lambda_0}{\lambda}}\left(\sqrt{1-\frac{\lambda_0}{\lambda}}+\sqrt{1-\frac{\hat{\lambda}_2(x)}{\lambda}}\right) \\
&< (\sqrt{2}/2)\left(1+\sqrt{1+\frac{1}{\lambda_0}}\right) \stackrel{\text{def}}{=} D_1.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
I_{12} &\leq D_1\int_{x_2}^{x_1} dx \int_{\lambda_0}^{1}[-\log(\sqrt{1-\lambda^2})]\frac{\lambda\, d\lambda}{\sqrt{\lambda^2-\lambda_0^2}} \\
&= D_1\int_{x_2}^{x_1} dx \int_{0}^{\nu_0}[-\log(\nu)]\frac{\nu\, d\nu}{\sqrt{\nu_0^2-\nu^2}} \\
&= D_1\int_{x_2}^{x_1} dx \int_{0}^{\nu_0}\left(\frac{-\nu\log(\nu)}{\sqrt{\nu_0^2-\nu^2}}\right) d\nu,
\end{aligned}
$$

where we define $\nu_0 = \sqrt{1-\lambda_0^2}$. With a variable change : $y = \nu/\nu_0$, we find that I_{12} is bounded by a constant which is to be integrated on a finite domain $x_2 \leq x \leq x_1$, so I_{12} is finite.

Next, we consider I_{13}.

$$
\begin{aligned}
&I_{13} \\
&= \int_{x_2}^{x_1} dx \int_{\hat{\lambda}_1(x)}^{\lambda_0}[-\log(\sqrt{1-\lambda^2})]\, d\sqrt{[\lambda-\hat{\lambda}_1(x)][\lambda-\hat{\lambda}_2(x)]} \\
&\leq D\int_{x_2}^{x_1} dx \int_{\hat{\lambda}_1(x)}^{\lambda_0} d\sqrt{[\lambda-\hat{\lambda}_1(x)][\lambda-\hat{\lambda}_2(x)]} \\
&= D\int_{x_2}^{x_1} dx\, \sqrt{[\lambda_0-\hat{\lambda}_1(x)][\lambda_0-\hat{\lambda}_2(x)]} \\
&< \infty,
\end{aligned}
$$

where the constant D exists because for $x \in [x_2, x_1]$ and $\lambda \in [\hat{\lambda}_1(x), \lambda_0]$, $(1-\lambda^2)$ is bounded away from zero. It completes the proof of Lemma 3.1.33.

Let's go back to the proof of the part (c) of Lemma 3.11. By (3.1.29),

$$
\begin{aligned}
&q_{h,B}(S) \\
&\quad = \frac{2}{\pi} \int\int_{l>B} [l(\sigma,\sigma') - B] \psi_S(\sigma,h) \psi_S(\sigma',h)\, d\sigma\, d\sigma' \\
&\quad + \frac{2h}{\pi} \int_0^{\pi} \log(1 \sin\sigma) \psi_S(\sigma,h)\, d\sigma.
\end{aligned}
$$

Comparing (3.1.18) and (3.1.19), we can estimate it as following :

$$
\begin{aligned}
&|q_{h,B}(S)| \leq \\
&\quad \frac{2}{\pi} \int\int_{l>B} [l(\sigma,\sigma') - B] \phi(\sigma,h) \phi(\sigma',h)\, d\sigma\, d\sigma' + \\
&\quad \frac{2h}{\pi} \int_0^{\pi} (|\log\sin\sigma|) \phi(\sigma,h)\, d\sigma \leq \\
&\quad \frac{2}{\pi} \int\int_{l>B} [l(\sigma,\sigma') - B] \phi(\sigma) \phi(\sigma')\, d\sigma\, d\sigma' + \\
&\quad \frac{2h}{\pi} \int_0^{\pi} (|\log\sin\sigma|) \phi(\sigma)\, d\sigma \leq \\
&\quad Q_B(\phi) + h \left(\frac{2}{\pi} \int_0^{\pi} |\log\sin\sigma| \phi(\sigma)\, d\sigma \right) = \\
&\quad \delta(B) + \gamma(h),
\end{aligned}
$$

where we denote $\gamma(h) = hC$ with a constant C defined as

$$
C = \frac{2}{\pi} \int_0^{\pi} |\log\sin\sigma| \phi(\sigma)\, d\sigma.
$$

A finite C exists because of Lemma 3.1.33. So

$$
\lim_{h \to 0} \gamma(h) = 0.
$$

It completes the proof of Lemma 3.11.

Theorem 3.14 is then an immediate consequence of Lemma 3.11.

The pointwise limit (3.1.24) of Theorem 3.14 is uniform on compact subsets of the x, $\vec{t}$-space. This follows from

Lemma 3.14

Being defined in (3.1.10) as a family of linear functions of x and $\vec{t}$, $\{q_h(S; x, \vec{t}) : S, h\}$ is equicontinuous and equibounded on compact subsets of x and $\vec{t}$.

PROOF :

Since the functions $\{q_h(S; x, \vec{t})\}$ are linear, it suffices to show that the coefficients of x, t_k, $k = 2, 3, \ldots$ and the quantity $q_h(S; 0, 0)$ are uniformly bounded. The coefficients of x and $\{t_k\}$ are uniformly bounded since

$$|h \sum_S \sin \sigma_n| \leq hN,$$

$$|h \sum_S \sin \sigma_n \cos^{k-1} \sigma_n)| \leq hN,$$

respectively, and since, by (2.1.12), hN is bounded. A uniform bound for $q_h(S; 0, 0)$ will be demonstrated by considering the three terms of the right-hand side of (3.1.10) separately. According to Lemma 2.5, $\Gamma_+(\sigma)$ is continuous and therefore bounded, say $\Gamma_+(\sigma) \leq M$. The first term of $q_h(S; 0, 0)$ is then uniformly bounded since

$$h \sum_S \Gamma_+(\sigma_n) \leq hNM$$

and, by (2.1.12), hN is bounded. The uniform bounds of the second and third terms of $q_h(S; 0, 0)$ follow directly from Lemma 3.11.

We now turn to Theorem 3.13.

PROOF :

Formula (3.1.14) of Theorem 3.13 follows from two facts that are proved bellow:

$$\limsup_{h \to 0} q^*(x, \vec{t}; h) \leq \inf\{\mathcal{Q}(\psi; x, \vec{t}) : \psi \in A\} \tag{3.1.34}$$

and

$$\limsup_{h \to 0} q^*(x, \vec{t}; h) = \mathcal{Q}(\psi^*; x, \vec{t}), \tag{3.1.35}$$

for some ψ^* in the admissible set A. Comparing (3.1.34) and (3.1.35) gives

$$\mathcal{Q}(\psi^*; x, \vec{t}) = \inf\{\mathcal{Q}(\psi; x, \vec{t}) : \psi \in A\}.$$

Thus the minimum asserted in the definition (3.1.15) of $\mathcal{Q}^*(x,\vec{t})$ is attained:

$$\mathcal{Q}^*(x,\vec{t}) = \mathcal{Q}(\psi^*; x, \vec{t}),$$

and (3.1.14) follows.

Now, we are going to prove (3.1.34). Let ψ be any element of the admissible set A. By Lemma 3.9, there exists $S(h)$ such that

$$w\text{-}\lim_{h\to 0} \psi_{S(h)}(\sigma, h) = \psi(\sigma). \tag{3.1.36}$$

Let $\{h_k\}$ be a sequence such that $h_k \to 0$ and

$$\lim_{k\to\infty} q^*(x,\vec{t}; h_k) = \limsup_{h\to 0} q^*(x,\vec{t}; h). \tag{3.1.37}$$

By Definition (3.1.7) and Lemma 3.14, the $q^*(x,\vec{t};h)$ are minimas over a subfamily of an equibounded family of functions and so are themselves equibounded. This insures that the limits of (3.1.37) are finite.

By (3.1.7),

$$q^*(x,\vec{t}; h_k) \le q_{h_k}(S_k; x, \vec{t}).$$

Using (3.1.36) and applying Theorem 3.14, we see that the right side above tends to $\mathcal{Q}(\psi; x, \vec{t})$ for that ψ. Using (3.1.37) on the left side, we obtain the limit

$$\limsup_{h\to 0} q^*(x,\vec{t}; h) \le \mathcal{Q}(\psi; x, \vec{t}). \tag{3.1.38}$$

It works for all ψ in A. Since the left-hand side of (3.1.38) is independent of ψ, taking the infimum of the right side gives (3.1.34).

To prove (3.1.35), we choose $S(h)$ for fixed h such that the minimum of (3.1.7) is attained:

$$q_h(S(h); x, \vec{t}) = q^*(x,\vec{t}; h). \tag{3.1.39}$$

Let $\{h_k\}$ be a sequence such that $h_k \to 0$ and

$$\lim_{k\to\infty} q^*(x,\vec{t}; h_k) = \liminf_{h\to 0} q^*(x,\vec{t}; h). \tag{3.1.40}$$

Since by Lemma 3.8 the total variations of the measures $\psi_{S(h_k)}(\sigma, h)\, d\sigma$ are uniformly bounded, we apply the Helly selection theorem and extract a w-convergence subsequence. Passing to this subsequence, we know by Lemma 3.9 that the w-limit ψ^* lies in the admissible set A. Applying Theorem 3.14 to this sequence and using (3.1.39) gives

$$\mathcal{Q}(\psi^*; x, \vec{t}) = \lim_{k\to\infty} q^*(x, \vec{t}; h_k),$$

which, along with (3.1.40) and (3.1.34), proves (3.1.35).

The limit in (3.1.14) is uniform on compact subsets of x and $\vec{t}$, since, by definition (3.1.7) and Lemma 3.14, the $q^*(x, \vec{t}; h)$ are minimas over subfamilies of an equicontinuous family of functions and so are themselves equicontinuous. The result then follows from the Arzela-Ascoli theorem and the proof of Theorem 3.13 is complete.

Combining the limit (3.1.14) of Theorem 3.13 with (3.1.8) of Theorem 3.12 we conclude that

$$\lim_{h\to 0} W(x, \vec{t}; h) = \frac{1}{2}\mathcal{Q}^*(x, \vec{t})$$

uniformly on compact subsets of x and $\vec{t}$.

If only stand by the t-flow, we need to fix the initial value of the other temporal variables. For simplicity, we restrict our consideration at a hyperplane $\{(x, \vec{t}) : t_n = 0,\ n > 2\}$. We often write the projection of function $f(x, \vec{t})$ on that plane as $f(x, t)$.

3.1.4 Limit in the Meaning of Distribution

Let us denote convergence in the sense of distributions of functions of (x, t) by d-lim. Since derivatives of functions of a uniformly convergent sequence converge in the distribution sense, we conclude from (3.1.2) and (3.1.9)

Theorem 3.15

Let $U(x, t; h)$ be the solution of the NLS equation given by (1.1.4) with the modified initial data $U(x; h)$; then for all the conserved density ρ_k and their flux $\mathcal{G}_k$, $k = 1, 2, \ldots$, *we have formulas :*

$$\bar{\rho}_k(x,t) \stackrel{\text{def}}{=} d\_\lim_{h\to 0} \rho_k(x;t;h) = \frac{1}{2}\partial_{xt_k} Q^*(x,\vec{t})\Big|_{\substack{t_n = 0 \\ n>2}}, \tag{3.1.41}$$

and

$$\bar{\mathcal{G}}_k(x,t) \stackrel{\text{def}}{=} d\_\lim_{h\to 0} \mathcal{G}_k(x;t;h) = \frac{1}{2}\partial_{tt_k} Q^*(x,\vec{t})\Big|_{\substack{t_n = 0 \\ n>2}}.$$

For example, for ρ and μ given by (1.2.9),

$$d\_\lim_{h\to 0} \rho(x,t;h) = \bar{\rho}(x,t)$$

exists and

$$\bar{\rho}(x,t) = 1 + \frac{1}{2}\partial_{xx} Q^*(x,t). \tag{3.1.42}$$

Similarly,

$$d\_\lim_{h\to 0} \mu(x,t;h) = \bar{\mu}(x,t)$$

exists and

$$\bar{\mu}(x,t) = \frac{1}{2}\partial_{xt} Q^*(x,t), \tag{3.1.43}$$

where $Q^*(x,t)$ is defined by (3.1.15), with $t_n = 0$ for $n > 2$.

Actually we have proved more; we have shown that $\rho(x,t;h)$ tends to $\bar{\rho}(x,t)$, or generally, $\rho_k(x,t;h)$ tends to $\bar{\rho}_k(x,t)$, in the x-distribution sense. That is, for any C_o^∞ function w of x,

$$\lim_{h\to 0}(\rho_k(t;h), w) = (\bar{\rho}_k(t), w).$$

Moreover, the limit can be reached uniformly over compact subsets of t. We shall strengthen this result in Theorem 3.17.

Exploiting the concept of convergence in the distribution sense, a corollary of Theorem 3.15 is deduced:

Theorem 3.16

Let $\varepsilon(x,t;h)$ be as given by (1.2.9) and U as the same in Theorem 3.15, then

$$d\_\lim_{h\to 0} \varepsilon(x,t;h) = \bar{\varepsilon}(x,t)$$

exists, and

$$\begin{aligned}
&\bar{\varepsilon}(x,t) \\
&= (1-\bar{\rho}) + \frac{1}{2}(1-\bar{\rho})^2 + \frac{1}{2}\partial_{tt} Q^*(x,t) && (3.1.44)\\
&= \frac{1}{2}\partial_{xt_3} Q^*(x,\vec{t})_{t_n=0 \quad n>2}. && (3.1.45)
\end{aligned}$$

PROOF :

(3.1.45) is direct from (3.1.41). We only need to prove (3.1.44). By a little calculation to the definition of ρ, μ and the relation between W and $\{\rho,\mu\}$, we get

$$W_{tt} = \frac{\mu^2}{\rho} + \frac{1}{2}(\rho^2-1) - \frac{h^2}{4}\rho(\log\rho)_{xx}.$$

That is

$$Q^*_{tt} = 2\,d\_\lim_{h\to 0} W_{tt} = d\_\lim_{h\to 0}\left(\frac{2\mu^2}{\rho} + \rho^2 - 1\right). \qquad (3.1.46)$$

Then by (1.2.9), because that the d-lim of $h^2(\rho_x)^2/(4\rho)$ is zero too, we obtain

$$\begin{aligned}
\bar{\varepsilon}(x,t) &= d_\lim_{h\to 0}[(1-\rho)^2 + \mu^2/\rho] \\
&= d_\lim_{h\to 0}[(3/2 - 2\rho + \rho^2/2) + (\rho^2 - 1 + 2\mu^2/\rho] \\
&= d_\lim_{h\to 0}[(3-\rho)(1-\rho)] + Q^*_{tt}/2 \\
&= (1-\bar{\rho}) + \frac{1}{2}(1-\bar{\rho})^2 + \frac{1}{2}\partial_{tt} Q^*(x,t).
\end{aligned}$$

That is (3.1.44).

Moreover, Theorem 3.15 can be strengthened :

Theorem 3.17

Let $U(x,t;h)$ and its conserved quantity $\{\rho,\mu\}$ be as the same in Theorem 3.15. As functions of x, $\{\rho(\cdot,t;h),\mu(\cdot,t;h)\}$ converge to $\{\bar{\rho},\bar{\mu}\}$ weakly in $L^1(\Re)$ as $h\to 0$ and converge uniformly over compact subsets of t.

PROOF :

These are direct results following Theorem 2.9 and Theorem 3.15. Recall

$$1 - \rho(x, t; h) = -W_{xx}$$

and

$$\mu(x, t; h) = W_{xt}.$$

Since both $\{W_{xx}(\cdot\,, t; h)\}$ and $\{W_{xt}(\cdot\,, t; h)\}$ are weakly L^1 compact, so are $\{(1 - \rho)\}$ and $\{\mu\}$. We will show later that Q^*_{xx} and Q^*_{xt} , *i.e.* the distribution limits of $\{(1-\rho)\}$ and $\{\mu\}$, are all in $L^1(\Re)$. Because of the uniqueness of distribution limits and the fact that w-L^1 limit should be distribution limit too, the convergence at Theorem 3.15 are weakly L^1 convergence also. It completes the proof of Theorem 3.17.

3.2 Analysis of the Minimum Problem

3.2.1 Convex Property of Q^*

Theorem 3.15 gives explicit formulas for $\rho_k(x, t)$, $k = 1, 2, \ldots$. The formulas involve the function $Q^*(x, \vec{t})$ characterized by a minimum problem, see (3.1.15), a so-called quadratic programming problem. The quantity $Q(\psi; x, \vec{t})$ to be minimized, see (3.1.28), is a quadratic function of ψ; and the functions ψ of the admissible set A are subject to two linear inequalities given by (3.1.16). In this section we begin to use these formulas to obtain a variety of properties of the limit $\bar{\rho}(x, t)$, and then extend them for $\bar{\mu}(x, t)$ and all other limits $\bar{\rho}_k$.

We start with some observations :

Theorem 3.18
As a function of x and $\vec{t}$, $Q^*(x, \vec{t})$ is

(a) *less than or equal to 0,*

(b) continuous in x, t_k, $k = 1, 2, \ldots$

(c) concave with respect to a pair of variables (x, t_k), $\forall k$.

(d) increasing in x.

PROOF :

Since $0 \in A$, it follows from (3.1.15) that $\mathcal{Q}^*(x, \vec{t}) \leq \mathcal{Q}(0; x, \vec{t}) = 0$; this proves (a). Using the definition (2.2.45) of $a(\sigma, x, \vec{t})$ in the formula (3.1.28) for $\mathcal{Q}(\psi; x, \vec{t})$ gives

$$\begin{aligned}
&\mathcal{Q}(\psi; x, \vec{t}) \\
&\quad = 4(a, \psi) - 2(L\psi, \psi) \\
&\quad = 4(\sin\sigma, \psi)x + 4\sum_{k=2}(\sin\sigma\cos^{k-1}\sigma, \psi)t_k - 4(\Gamma_+, \psi) - 2(L\psi, \psi).
\end{aligned} \tag{3.2.47}$$

Since the admissible ψ satisfy $0 \leq \psi \leq \phi$ and ϕ belongs to $L^1[0, \pi]$, it follows easily from (3.2.47) that $\{\mathcal{Q}(\psi; x, \vec{t}) : \psi \in A\}$ is an equicontinuous family of functions of x and $\vec{t}$. It follows that $\mathcal{Q}^*$, the infimum of an equicontinuous family of functions, is itself continuous.

For each ψ, as is a linear function of x and $\vec{t}$, $\mathcal{Q}(\psi; x, \vec{t})$ has properties (c) and (d). It follows that $\mathcal{Q}^*$, their infimum, has these properties too.

We draw now some conclusions from Theorem 3.12. With the concavity of $\mathcal{Q}^*$, we deduce that the matrix of second derivatives of $\mathcal{Q}^*$ with respect to any pair (x, t_k) is negative in the distribution sense. Using expressions (3.1.42), (3.1.43), (3.1.44) and (3.1.46) for these second derivatives yields

Corollary 3.2

Choose $\vec{t} = t_2 = t$, for any real (ξ, τ), we have following inequality :

$$\begin{aligned}
&2\xi^2(\bar{\rho} - 1) + 4\xi\tau\bar{\mu} + \tau^2(\bar{\rho}^2 + 2\bar{\mu}^2/\bar{\rho} - 1) \\
&\quad = \xi^2\mathcal{Q}^*_{xx} + 2\xi\tau\mathcal{Q}^*_{xt} + \tau^2\mathcal{Q}^*_{tt} \leq 0,
\end{aligned}$$

or

$$2\xi^2(\bar{\rho}-1)+4\xi\tau\bar{\mu}+2\tau^2\left(\varepsilon-(1-\bar{\rho})-[(1-\bar{\rho})^2]/2\right)\leq 0.$$

In particular,

$$\bar{\rho}\leq 1, \tag{3.2.48}$$

and

$$\bar{\rho}^2+2\bar{\mu}^2/\bar{\rho}\leq 1,$$

or

$$\bar{\varepsilon}\leq(1-\bar{\rho})+\frac{1}{2}(1-\bar{\rho})^2. \tag{3.2.49}$$

We remark that no maximum principles like (3.2.48) and (3.2.49) can be valid for (1.2.7), which is the conservation system of the NLS equation.

REMARK :

If the initial value of $\rho(x,t)$, which is the amplitude square of $U(x,t)$ as a solution of the NLS equation, say $|A(x)|^2$, satisfies $\rho(x,0)\leq 1$, one cannot conclude that $\rho(x,t)\leq 1$. This may be seen by noting that if $\rho(x_o,0)=1$, the value of the x-derivative of μ could well be negative. It follows then from the system (1.2.7) that $\rho_t(x_o,0)>0$, and so $\rho(x_o,t)>1$ for small $t>0$. However, the assumption as given by (2.1.10) offers

$$\rho(x)^{\frac{1}{2}}\equiv|A(x)|\leq 1-|\frac{1}{2}\partial_x S(x)|<1.$$

Thus once we show $\bar{\rho}(x,0)=\rho(x)$, then (3.2.48) and (3.2.49) can be interpreted as a kind of maximum principle for the semiclassical limit of solutions of the NLS equation.

3.2.2 Existence and Uniqueness of ψ^*

To derive further information, some properties of the integral operator L defined by (3.1.25) are needed. The first is a strengthening of Corollary 3.1 which asserted that, for every ψ in the admissible set A, $L\psi(\sigma)$ is a continuous function belonging to C_0R that vanishes at zero and $\pm\pi$.

Theorem 3.19

The set of functions $\{L\psi : \psi \in A\}$ *is an equicontinuous subset of* $C_0(\Re)$.

PROOF :

We break up the operator L into two parts depending on a parameter B which will tend to infinity :

$$L = L^B + L_B,$$

where

$$L^B\psi(\sigma) = -\int_0^\pi l^B(\sigma, \sigma')\psi(\sigma')\,d\sigma'$$

and l^B is defined by (3.1.30).

As B tends to infinity, $L^B\phi(\sigma)$ will tend to $L\phi(\sigma)$, monotonically for all $\sigma \in (-\pi, \pi)$. Since these functions are all $\in C[-\pi, \pi]$, it follows from Dini's theorem that the convergence is uniform in σ. That is, if we define

$$\delta(B) = \max_\sigma |L_B(\phi)|, \tag{3.2.50}$$

then

$$\lim_{B\to\infty} \delta(B) = 0.$$

Let ψ be any admissible function. Since admissibility means that

$$0 \le \psi(\sigma) \le \phi(\sigma),$$

and since the kernel of L_B is of one sign, it follows by (3.2.50) that, for any σ,

$$|L_B(\psi)| \le |L_B(\phi)| \le \delta(B). \tag{3.2.51}$$

The kernel of L^B, defined in (3.1.30), is uniformly continuous. It follows that L^B maps any set bounded in the L^1 norm into an equicontinuous set. Since the admissible functions are L^1-bounded, it follows that they are mapped by L^B into an equicontinuous set. Since, by (3.2.51), $L^B(\psi)$ differs from $L\psi$ at most by $\delta(B)$, and since $\delta(B) \to 0$ as $B \to \infty$, Theorem 3.19 follows.

Recalling the definition of w-convergence (3.1.20) for sequences in A, we prove the continuous dependence of L and Q on ψ by the previous result :

Theorem 3.20

If ψ_k is a w-convergent sequence of elements in A :

$$w\_\lim \psi_k = \psi,$$

then

$$\lim_{k\to\infty} L\psi_k(\sigma) = L\psi(\sigma) \tag{3.2.52}$$

uniformly on $[-\pi,\pi]$ of σ, and

$$\lim_{k\to\infty} Q(\psi_k; x, \vec{t}) = Q(\psi; x.\vec{t}), \tag{3.2.53}$$

uniformly on compact subsets of x and $\vec{t}$.

PROOF :

By Theorem 3.19, the functions $L\psi_k$ are equicontinuous. Then by the Arzela-Arscoli Theorem, it is enough to show that the convergence of (3.2.52) is pointwise. Since the kernel of L^B is continuous, it follows from the definition of w-convergence that, for each σ,

$$\lim_{k\to\infty} L^B\psi_k(\sigma) = L^B\psi(\sigma).$$

Since, by Theorem 3.19, $L^B\psi_k$ differs at most by $\delta(B)$ from $L\psi_k$, (3.2.52) follows.

Since the family of $Q(\psi; x, \vec{t})$ is equicontinuous in x and $\vec{t}$, it suffices to show the pointwise convergence of (3.2.53). We consider separately the two terms on the right side of the form of $Q(\psi; x, \vec{t})$ (see (3.1.28)). According to Lemma 2.5, The $a(\sigma, x, \vec{t})$ given by (2.2.44) is a continuous function of σ and thus, by the definition (3.1.20) of w-convergence,

$$\lim_{k\to\infty} (a, \psi_k) = (a, \psi) \tag{3.2.54}$$

for every x and t_n, $n = 2, 3, \ldots$.

Next we consider the identity

$$(L\psi_k, \psi_k) - (L\psi, \psi) = (L\psi_k - L\psi, \psi_k) + (L\psi, \psi_k - \psi). \tag{3.2.55}$$

The first term on the right is bounded in absolute value by

$$\|L\psi_k - L\psi\|_{L^\infty} \|\psi_k\|_{L^1}.$$

Formula (3.2.52) shows that the first factor tends to zero. Since ψ is admissible, the second factor is bounded by $\|\phi\|_{L^1}$. Thus the first term on the right of (3.2.55) vanishes as $k \to \infty$. Corollary 3.1 asserts that $L\psi$ is continuous, hence the second term on the right side of (3.2.55) tends to zero as $k \to 0$ because $\{\psi_k\}$ w-convergence to ψ. This proves that

$$\lim_{k\to\infty} (L\psi_k, \psi_k) = (L\psi, \psi). \tag{3.2.56}$$

By combined (3.2.54) with (3.2.56), the formula (3.1.28) of $Q(\psi; x, \vec{t})$ implies the result (3.2.53) as is asserted in the theorem.

An immediate consequence of (3.2.53) in Theorem 3.20 is

Corollary 3.3

The functional $Q(\psi; x, \vec{t})$ is a continuous function over $A \times R \times R^D$ where the admissible set A is given the weak sequence topology.

PROOF :

This result follows from (3.2.53) and the fact that $Q(\psi; x, \vec{t})$ is jointly continuous in x and $\vec{t}$.

Theorem 3.21

The set A which is defined by (3.1.16) as the set of all the admissible functions, is compact in the weak sequential topology. That means, every sequence in A contains a subsequence which is w-convergent to an element of A.

PROOF :

By Lemma 3.8, the total variation of the measures associated with A are uniformly bounded. It follows then from the Helly selection principle that every sequence of associated measures contains a subsequence $\{\psi_k(\sigma)\,d\sigma\}$ being convergent to a measure $d\varsigma$ in the sense that

$$\lim_{k\to\infty}\frac{1}{\pi}\int_0^\pi \chi(\sigma)\psi_k(\sigma)\,d\sigma = \frac{1}{\pi}\int_0^\pi \chi(\sigma)\,d\varsigma$$

for any continuous function χ. We conclude from (3.1.16) that, for all non-negative χ,

$$0 \le \frac{1}{\pi}\int_0^\pi \chi(\sigma)\,d\varsigma \le \frac{1}{\pi}\int_0^\pi \chi(\sigma)\phi(\sigma)\,d\sigma.$$

This implies that the measure $d\varsigma$ satisfies

$$0 \le d\varsigma \le \phi(\sigma)\,d\sigma$$

from which we conclude by the Radon-Nikodym theorem that $d\varsigma = \psi(\sigma)\,d\sigma$ where ψ satisfies (3.1.16). Thus ψ_k is w-convergent to $\psi \in A$ and the proof of Theorem 3.21 is complete.

Combining the compactness of A asserted by Theorem 3.21 with the continuity of Q stated in Corollary 3.3, it will follow from a general topological principle that, for each x and t, $Q(\psi; x, \bar{t})$ takes on a minimum value over A. This gives a direct argument that the minimum problem (3.1.15) of Theorem 3.13 has a solution.

To prove uniqueness of the solution we need another property of the operator L.

Theorem 3.22

The integral operator L is negative definite over all ψ in $L^1[0,\pi]$ for which $L\psi$ is in $L^\infty[-\pi,\pi]$. This means

$$-(L\psi, \psi) > 0$$

for all $\psi \neq 0$ in that class.

PROOF :

Lax and Levermore [11] have proved a similar result in the KdV case. We now treat our result as a corollary of theirs. In order to do that, we extend the functions ψ, which are originally defined on the interval $[0,\pi]$, to the interval $[-\pi,\pi]$ by $\psi(-\sigma) = -\psi(\sigma)$ and setting an odd function $F(x) = 2\psi(\sigma(x))/(1+x^2)$ with Cayley transform :

$$\sigma = 2\tan^{-1} x, \qquad d\sigma = \frac{2dx}{1+x^2}.$$

This allows us to let the operator L which is defined by (3.1.25) be dominated by a convolution operator :

$$\begin{aligned} L\psi(\sigma) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left[\sin^2(\frac{\sigma-\sigma'}{2})\right]\psi(\sigma')\,d\sigma' \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\log(x-x')^2 - \log(1+x^2) - \log(1+(x')^2)\right]F(x')\,dx' \\ &\leq -\hat{l} * F(x), \end{aligned}$$

where

$$\hat{l}(x) = -\frac{1}{2\pi}\log x^2. \qquad (3.2.57)$$

Since $L\psi(\sigma)$ is nonpositive, we have

$$-(L\psi,\psi) \geq \frac{1}{4\pi}\langle \hat{l} * F, F\rangle, \qquad (3.2.58)$$

where the brackets on the right denote the usual duality on R,

$$\langle \chi, F\rangle = \int_{-\infty}^{\infty} \chi(x)F(x)\,dx.$$

The definiteness of convolution operators is the subject of theorems going back to Carathéodory, Bochner, and L.Schwartz. By (3.2.58), Theorem 3.22 follows from

Lemma 3.15 *(Lax and Levermore)*
If $F \in L^1(\Re)$ is odd with compact support and $\hat{l} * F$ is in $L^\infty(\Re)$, then

$$\langle \hat{l} * F, F\rangle = \int_{-\infty}^{+\infty} \frac{1}{|s|}|\tilde{F}(s)|^2\,ds,$$

where $\tilde{F}(s)$ is the Fourier transform given by

$$\tilde{F}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixs} F(x)\, dx.$$

Now, for any F which is in $L^1(\Re)$, odd and which makes $\hat{l} * F$ be in $L^\infty(\Re)$, we can construct a sequence of functions $\{F_n\}$ which are odd and in L^1 with compact support, such that

$$\lim_{n\to\infty} \|F_n - F\|_{L^1} = 0,$$

and

$$\|\hat{l} * F_n\|_{L^\infty} \leq \|\hat{l} * F\|_{L^\infty}.$$

It implies

$$\lim_{n\to\infty} \langle \hat{l} * F_n, F_n\rangle = \langle \hat{l} * F, F\rangle.$$

Thus Theorem 3.22 follows.

An important consequence of the negative definiteness of L is :

Theorem 3.23

For each given x and $\vec{t}$, the functional $\mathcal{Q}(\psi; x, \vec{t})$ assumes its minimum at exactly one element, denoted $\psi^*(x, \vec{t})$, of the admissible set A.

PROOF :

Since every ψ in A satisfies $0 \leq \psi \leq \phi$ and ϕ is in $L^1[0,\pi]$, then ψ is also in $L^1[0,\pi]$. By Corollary 3.1, $L\psi$ is an element of $C[-\pi,\pi]$ and therefore also of $L^\infty[-\pi,\pi]$. We can then apply Theorem 3.22 to the difference of functions in A to prove that $\mathcal{Q}(\psi; x, \vec{t})$ is strictly convex over the set A. Since A is a convex set, $\mathcal{Q}(\psi)$ assumes its minimum at exactly one admissible ψ.

3.2.3 Regularity of ψ^*, $L\psi^*$ and Q^*

Theorem 3.24

(a) $\psi^*(x, \vec{t})$ depends continuously on x, $\vec{t}$ in the sense of w-convergence.

(b) $L\psi^*(x, \vec{t})$ depends continuously on x, $\vec{t}$ in the sense of maximum norm.

PROOF :

Suppose $(x_n, \vec{t_n}) \to (x, \vec{t})$. According to Theorem 3.21 that A is compact, then the sequence $\psi^*(x_n, \vec{t_n})$ has a cluster point ψ_* in A and a subsequence of $\psi^*(x_n, \vec{t_n})$ exists and w-converges to ψ_*. Since, by Corollary 3.3, $\mathcal{Q}(\psi; x, \vec{t})$ is a continuous function of ψ, and since, by definition, $\mathcal{Q}^*(x_n, \vec{t_n}) = \mathcal{Q}(\psi^*(x_n, \vec{t_n}); x_n, \vec{t_n})$, we conclude that over this subsequence

$$\lim_{n\to\infty} \mathcal{Q}^*(x_n, \vec{t_n}) = \lim_{n\to\infty} \mathcal{Q}(\psi^*(x_n, \vec{t_n}); x_n, \vec{t_n}) = \mathcal{Q}(\psi_*; x, \vec{t}). \tag{3.2.59}$$

On the other hand, since by part(b) of Theorem 3.18, $\mathcal{Q}^*$ is a continuous function of x and $\vec{t}$,

$$\lim_{n\to\infty} \mathcal{Q}^*(x_n, \vec{t_n}) = \mathcal{Q}^*(x, \vec{t}).$$

Comparing this result with (3.2.59) gives

$$\mathcal{Q}^*(x, \vec{t}) = \mathcal{Q}(\psi_*; x, \vec{t}).$$

It shows ψ_* solves the minimum problem at $(x, \vec{t})$ and thus, by the uniqueness of Theorem 3.23, $\psi_* = \psi^*(x, \vec{t})$. Since $\psi^*(x, \vec{t})$ is therefore the only cluster point of $\psi^*(x_n, \vec{t_n})$ and A is compact, we conclude

$$w\_\lim_{n\to\infty} \psi^*(x_n, \vec{t_n}) = \psi^*(x, \vec{t}).$$

This verifies the continuity of ψ^*. The continuity of $L\psi^*$ follows since, by Theorem 3.20, L is a sequentially continuous mapping of A into $C[-\pi, \pi]$. Now the proof is completed.

The argument given in Theorem 3.18 for the continuity of $\mathcal{Q}^*(x, \vec{t})$ shows that $\mathcal{Q}^*$ is Lipschitz continuous in x and t. But now we can show that more is true.

Theorem 3.25

$\mathcal{Q}^*(x, \vec{t})$ is a continuously differentiable function of x and $\vec{t}$ with

$$\begin{aligned} \partial_x \mathcal{Q}^*(x, \vec{t}) &= 4(\sin\sigma, \psi^*(x, \vec{t})), \\ \partial_{t_k} \mathcal{Q}^*(x, \vec{t}) &= 4(\sin\sigma\cos^{k-1}\sigma, \psi^*(x, \vec{t}))\ \forall k. \end{aligned} \tag{3.2.60}$$

PROOF :

We deduce from formula (3.2.47) that

$$\mathcal{Q}(\psi; x, t_k) - \mathcal{Q}(\psi; y, s_k) = 4(\sin\sigma, \psi)(x-y) + 4(\sin\sigma\cos^{k-1}\sigma, \psi)(t_k - s_k) \tag{3.2.61}$$

for all$\psi \in A$. Since ψ^* minimizes $\mathcal{Q}$, we have the two-sided inequality for any k :

$$\begin{aligned}\mathcal{Q}(\psi^*(x,t_k); x, t_k) - \mathcal{Q}(\psi^*(x,t_k); y, s_k) &\le \mathcal{Q}^*(x,t_k) - \mathcal{Q}^*(y,s_k)\\ &\le \mathcal{Q}(\psi^*(y,s_k); x, t_k) - \mathcal{Q}(\psi^*(y,s_k); y, s_k)\end{aligned}$$

which when combined with (3.2.61) gives

$$\begin{aligned}&4(\sin\sigma, \psi^*(x,t_k))(x-y) + 4(\sin\sigma\cos^{k-1}\sigma, \psi^*(x,t_k))(t_k - s_k)\\ &\le \mathcal{Q}^*(x,t_k) - \mathcal{Q}^*(y,s_k)\\ &\le 4(\sin\sigma, \psi^*(y,s_k))(x-y) + 4(\sin\sigma\cos^{k-1}\sigma, \psi^*(y,s_k))(t_k - s_k).\end{aligned}$$

Applying the continuity of ψ^* in the w-topology, which is shown in Theorem 3.24, to both sides of the above two-sided inequality yields the formulas of (3.2.60) and shows that $\mathcal{Q}^*$ is continuously differentiable with respect to x and $\vec{t}$. It completes the proof.

Corollary 3.4

All the first derivatives of $\mathcal{Q}^*_x$ belong to L^1 space with respect to x.

PROOF :

By Theorem 3.25, all the first derivatives of $\mathcal{Q}^*$ are continuous with respect to x. Moreover, we can claim that thay are all functions of bounded variation. By Theorem 3.18, $\mathcal{Q}^*$ is concave with respect to its variables. Thus $\mathcal{Q}^*_x$ is monotonic. It means that $\mathcal{Q}^*_x$ is a function of bounded variation. By Lemma 4.25 in Chapter 4, one can find that ψ^*_x is even nonpositive if it exits. That also shows ψ^* is monotonic and so is $\mathcal{Q}^*_x$. With the formulas given by (3.2.60), the total variation of $\mathcal{Q}^*_{t_k}$ $\forall k$ can be bounded by the total variation of $\mathcal{Q}^*_x$. It comes out that they are all functions of bounded variation. Thus their x-derivatives are local L^1. But $\psi^* \in A$, that is, $0 \leq \psi^* \leq \phi$. So, by (3.2.60), all the first derivatives of $\mathcal{Q}^*$ are uniformly bounded. That proves the corollary.

3.2.4 Variational and Differentiated Variational Condition

Solution of a minimum problem should satisfy certain variational conditions. For a minimum problem with constraints, such as the one we are dealing with, the variational conditions in general take the form (suppressing x and $\vec{t}$) :

$$\frac{d}{d\epsilon}\mathcal{Q}(\psi^* + \epsilon\chi)_{\epsilon=0} \geq 0 \tag{3.2.62}$$

for all functions χ such that $\psi^* + \chi$ is admissible. Since

$$\mathcal{Q}(\psi + \epsilon\chi) = \mathcal{Q}(\psi) + 4\epsilon(a - L\psi, \chi) - 2\epsilon^2(L\chi, \chi),$$

we can restate (3.2.62) as

$$(a - L\psi^*, \chi) \geq 0. \tag{3.2.63}$$

Definition 3.6 *We define that ψ in A satisfies the* variational condition *for minimizing $\mathcal{Q}(\psi)$ in A if*

$$\frac{d}{d\epsilon}\mathcal{Q}(\psi + \epsilon\chi)_{\epsilon=0} = 4(a - L\psi, \chi) \geq 0 \tag{3.2.64}$$

for all χ, such that

$$(\psi + \chi) \in A. \tag{3.2.65}$$

Suppose ψ satisfies (3.2.64); set

$$\chi_+ = \begin{cases} -\psi & \text{where} \quad a - L\psi > 0, \\ 0 & \text{otherwise,} \end{cases}$$
$$\chi_- = \begin{cases} \phi - \psi & \text{where} \quad a - L\psi < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, χ_+ and χ_- satisfy (3.2.65). By construction, $(a - L\psi, \chi_\pm) \leq 0$; this is the opposite of (3.2.64), which implies that

$$(a - L\psi, \chi_\pm) = 0.$$

This yields another criterion :

Corollary 3.5 *If $\psi \in A$ satisfies :*

$$\psi = \begin{cases} 0 & \text{where} \quad a - L\psi > 0, \\ \phi & \text{where} \quad a - L\psi < 0. \end{cases} \tag{3.2.66}$$

then ψ satisfies the variational condition (3.2.64).

Now we can relate the minimum problem and its variational condition together:

Theorem 3.26

If ψ in A satisfies the variational condition, then $\psi = \psi^*$.

PROOF :

Since $\chi = \psi^* - \psi$ satisfies (3.2.65), ψ satisfies (3.2.64) with $\chi = \psi^* - \psi$,

$$(a - L\psi, \psi^* - \psi) \geq 0.$$

Since the minimizing ψ^* also satisfies the variational condition, and $\chi = \psi - \psi^*$ satisfies (3.2.65), thus ψ^* satisfies (3.2.63) :

$$(a - L\psi^*, \psi - \psi^*) \geq 0.$$

We add these inequalities and obtain

$$(L(\psi^* - \psi), \psi^* - \psi) \geq 0.$$

By using the negative definiteness of L, we conclude that $\psi^* - \psi = 0$.

The importance of Theorem 3.26 lies in this: in order to show that ψ^* solves the minimum problem (3.1.15), it suffices to verify that ψ^* satisfies the variational conditions (3.2.66). We shall exploit this idea in Section 4.1 and Section 4.2 .

An easy direct consequence of the variational conditions (3.2.66) that will be needed in later section is

Theorem 3.27

For all x and t_k, $k = 2, 3, \ldots,$

$$\psi^*(x,\vec{t}) = \begin{cases} 0 & \text{where} \quad x\sin\sigma + \sum_{k=2} t_k \sin\sigma\cos^{k-1}\sigma - \Gamma_+(\sigma) > 0 \\ \\ \phi & \text{where} \quad x\sin\sigma + \sum_{k=2} t_k \sin\sigma\cos^{k-1}\sigma - \Gamma_-(\sigma) < 0 \end{cases} \tag{3.2.67}$$

PROOF :

By (3.1.26) of Corollary 3.1,

$$a \leq a - L\psi^* \leq a - L\phi. \tag{3.2.68}$$

Recall Lemma 3.10

$$-L\phi(\sigma) = \Gamma_+(\sigma) - \Gamma_-(\sigma)$$

and definition (2.2.45)

$$a(\sigma, x, \vec{t}) = x\sin\sigma + \sum_{k=2} t_k \sin\sigma\cos^{k-1}\sigma - \Gamma_+(\sigma).$$

The inequality (3.2.68) then can be generally written as following:

$$\begin{aligned} x\sin\sigma + \sum_{k=2} t_k \sin\sigma\cos^{k-1}\sigma - \Gamma_+(\sigma) \\ \leq a - L\psi^* \leq \\ x\sin\sigma + \sum_{k=2} t_k \sin\sigma\cos^{k-1}\sigma - \Gamma_-(\sigma); \end{aligned}$$

or, if one wants to stand by a specific t_n-flow,

$$\begin{aligned} x\sin\sigma + t_n \sin\sigma\cos^{n-1}\sigma - \Gamma_+^{(n)}(\sigma) \\ \leq a - L\psi^* \leq \\ x\sin\sigma + t_n \sin\sigma\cos^{n-1}\sigma - \Gamma_-^{(n)}(\sigma), \end{aligned}$$

where

$$\Gamma_+^{(n)}(\sigma) = \Gamma_+(\sigma) - \sum_{k\neq n} t_k \sin\sigma\cos^{k-1}\sigma$$

and

$$\Gamma_-^{(n)}(\sigma) = \Gamma_-(\sigma) - \sum_{k\neq n} t_k \sin\sigma\cos^{k-1}\sigma.$$

Clearly, (3.2.67) follows from this inequality and the variational conditions (3.2.66); and the theorem is proved.

In the case of t-flow, (3.2.67) comes to be

$$\psi^*(x,t) = \begin{cases} 0 & \text{where} \quad x\sin\sigma + t\sin\sigma\cos\sigma - \Gamma_+(\sigma) > 0 \\ \\ \phi & \text{where} \quad x\sin\sigma + t\sin\sigma\cos\sigma - \Gamma_-(\sigma) < 0 \end{cases},$$

where we omited the index $n = 2$ for both Γ_+ and Γ_-.

Motivated by the variational conditions, we partition $[0,\pi] \times R \times R$ into three disjoint subsets with respect to the t-flow :

$$\begin{aligned} I^0 &= \{(\sigma,x,t) : a - L\psi^* = 0\}, \\ I^+ &= \{(\sigma,x,t) : a - L\psi^* > 0\}, \\ I^- &= \{(\sigma,x,t) : a - L\psi^* < 0\}. \end{aligned} \tag{3.2.69}$$

By theorem 3.24, $(a - L\psi^*)$ is continuous on $[0,\pi] \times R \times R$ and so the sets I^+ and I^- are open while I^0 is closed and separates I^+ and I^-. We denote the (x,t) slices of I^0, I^+, and I^- by $I^0(x,t)$, $I^+(x,t)$ and $I^-(x,t)$, respectively, where, for instance,

$$I^0(x,t) = \{\sigma : (\sigma, x; t) \in I^0\}. \tag{3.2.70}$$

With the sets $I^0(x,t)$, $I^+(x,t)$ and $I^-(x,t)$, one can determine $\psi^*(x,t)$ by:

Theorem 3.28

If $\psi \in L^1[0,\pi]$ and $L\psi \in L^\infty[-\pi,\pi]$, such that ψ satisfies

$$L\psi = a \qquad \text{on} \quad I^0(x,t) \tag{3.2.71}$$

and

$$\begin{aligned} \psi &= 0 \quad \text{on} \quad I^+(x,t), \\ \psi &= \phi \quad \text{on} \quad I^-(x,t), \end{aligned} \tag{3.2.72}$$

then $\psi = \psi^*(x,t)$.

PROOF :

That $\psi^*(x,t)$ satisfies (3.2.71) and (3.2.72) is obvious from the construction (3.2.69) of the sets I^0, I^+, I^-, and from the variational conditions (3.2.66). Now if we have another ψ satisfying (3.2.71) and (3.2.72), then the difference $\psi_o = \psi - \psi^*(x,t)$ satisfies the homogeneous version of (3.2.71) and (3.2.72) :

$$\begin{aligned} L\psi_o &= 0 \quad \text{on} \quad I^0(x,t), \\ \psi_o &= 0 \quad \text{off} \quad I^0(x,t). \end{aligned} \tag{3.2.73}$$

This implies the product $\psi_o \cdot L\psi_o = 0$ for all σ, then

$$(L\psi_o, \psi_o) = 0.$$

According to Theorem 3.22, L is negative definite. We thus conclude $\psi_o = 0$, proving the theorem.

We now establish the last major property of the operator L. We extend the functions ψ defined on $[0,\pi]$ to $[-\pi,\pi]$, such that $\psi(-\sigma) = -\psi(\sigma)$ as we did at the proof of Theorem 3.22. Then, under this odd extension, the admissible functions are all odd, bounded and vanish at $\sigma = 0$, because so is the function ϕ as given by (2.2.62). Therefore they are in $L^p[-\pi,\pi]$ for any p if $1 < p < \infty$.

Theorem 3.29

If ψ is a bounded and odd function of σ, then $L\psi$ is odd, $\in C[-\pi,\pi]$ and

$$L\psi(\sigma) = \int_0^\sigma H\psi(\tau)\,d\tau, \tag{3.2.74}$$

where H is the Hilbert transform [9] with respect to the unit disk :

$$\begin{aligned} H\psi(\sigma) &= \frac{d}{d\sigma}(L\psi(\sigma)) \\ &= \frac{1}{\pi} p.v. \int_{-\pi}^{\pi} \frac{\psi(\tau)}{2\tan(\frac{\sigma-\tau}{2})}\,d\tau. \end{aligned} \tag{3.2.75}$$

PROOF :

Because ψ is bounded on $[0,\pi]$ and oddly extended, ψ is in $L^p[-\pi,\pi]$ for any p, if $1 < p < \infty$.

Since the Hilbert transform takes L^p functions into L^p functions for $1 < p < \infty$ with the form of

$$H\psi(\sigma) = L^p\text{-}\lim_{\epsilon\to 0} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin(\frac{\sigma-\tau}{2})\cos(\frac{\sigma-\tau}{2})}{\sin^2(\frac{\sigma-\tau}{2}) + \epsilon^2} \psi(\tau)\,d\tau,$$

we can conclude that

$$\int_0^\sigma H\psi(\sigma)\,d\sigma = \lim_{\epsilon\to 0} \int_0^\sigma \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin(\frac{\sigma-\tau}{2})\cos(\frac{\sigma-\tau}{2})}{\sin^2(\frac{\sigma-\tau}{2}) + \epsilon^2} \psi(\tau)\,d\tau\,d\sigma.$$

The order of integration can then be exchanged by Fubini's theorem and the σ integration can be carried out to obtain

$$\begin{aligned} &\int_0^\sigma H\psi(\sigma)\,d\sigma = \\ &\lim_{\epsilon\to 0} \frac{1}{2\pi} \int_{-\pi}^{\pi} \log[\sin^2(\frac{\sigma-\tau}{2}) + \epsilon^2]\psi(\tau)\,d\tau = \\ &\lim_{\epsilon\to 0} \frac{1}{2\pi} \int_0^{\pi} \log\left[\frac{\sin^2(\frac{\sigma-\tau}{2}) + \epsilon^2}{\sin^2(\frac{\sigma+\tau}{2}) + \epsilon^2}\right] \psi(\tau)\,d\tau. \end{aligned} \tag{3.2.76}$$

Here the oddness of ψ was used in the last step. Since, for fixed σ,

$$L^q\text{-}\lim_{\epsilon\to 0} \frac{1}{2\pi} \log\left[\frac{\sin^2(\frac{\sigma-\tau}{2}) + \epsilon^2}{\sin^2(\frac{\sigma+\tau}{2}) + \epsilon^2}\right] = \frac{1}{2\pi} \log\left[\left(\frac{\sin(\frac{\sigma-\tau}{2})}{\sin(\frac{\sigma+\tau}{2})}\right)^2\right],$$

where $1/p + 1/q = 1$, we see from (3.2.76) that

$$\int_0^\sigma H\psi(\sigma)\,d\sigma = \frac{1}{2\pi}\int_0^\pi \log\left[\left(\frac{\sin(\frac{\sigma-\tau}{2})}{\sin(\frac{\sigma+\tau}{2})}\right)^2\right]\psi(\sigma)\,d\sigma.$$

By the definition of L, see (3.1.25), it proves (3.2.74).

The continuity of $L\psi$ follows directly from formula (3.2.74). It can be also seen from the definition of L that the kernel

$$\frac{1}{2\pi}\log\left[\left(\frac{\sin(\frac{\sigma-\tau}{2})}{\sin(\frac{\sigma+\tau}{2})}\right)^2\right]$$

tends to zero uniformly over $[0,\pi]$ as σ tends to π or 0. Thus $L\psi$ continuous on $[0,\pi]$ and vanishes at the boundary and the theorem is established.

REMARK :

We can apply Theorem 3.29 to the admissible functions ψ if they are oddly extended. The function ϕ given by (2.2.62) is bounded on $[0,\pi]$, so are all the functions ψ if they belong to A as given by (3.1.16).

Let $H^p(\Omega)$, $1 \le p < \infty$ be the space of all complex functions $f(\omega)$ which satisfy :

(a) $f(\omega)$ is analytic in the unit disc $\Omega = \{\omega : |\omega| < 1\}$.

(b) There exists a constant $C > 0$ such that

$$\int_{-\pi}^{\pi} |f(\omega = \tau e^{i\sigma})|\,d\sigma < C \quad \text{uniformly for all } \tau < 1.$$

(c) $f(\omega)$ has boundary value which is $\in L^p[-\pi,\pi]$ and

$$L^p\text{-}\lim_{\tau\to 1} f(\tau e^{i\sigma}) = f(e^{i\sigma}) \equiv f(\omega)_{\partial\Omega}$$

exists.

The following result is classical [11] and shows a relation between Hilbert transform and analytic functions.

Theorem 3.30

If $\psi(\sigma)$ defined on $[-\pi,\pi]$ is real and in L^p for some p, $1 \le p < \infty$, then there exists *a function $f(\omega) \in H^p(\Omega)$ such that $\psi(\sigma)$ and $H\psi(\sigma)$ are the real and imaginary parts of its boundary values. That is*

$$\begin{aligned} \psi(\sigma) &= \Re f(e^{i\sigma}), \\ H\psi(\sigma) &= \Im f(e^{i\sigma}), \end{aligned} \tag{3.2.77}$$

for some f in H^p. In fact, ψ is in $L^p[-\pi,\pi]$ if it is bounded.

As is well known, this is Riemann-Hilbert theorem. That how to find $f(\omega)$ with $\psi(\sigma)$ is so called a Riemann-Hilbert problem. In the next few paragraphs we will show that the variational condition as given by (3.2.64) and (3.2.65) can be reduced to solve this problem. With this observation, couple of useful representation of the solution of this variational condition will follow in Chapter 4 through solving the Riemann-Hilbert problems respected.

Rather than determining $\psi^*(x,t)$ directly, we determine the partial derivatives, $\partial_x\psi^*$ and $\partial_t\psi^*$ instead. We shall later make assumptions which imply that ψ^* depends differentiably on x and t in the weak topology over the σ-variable with derivatives in $L^p[0,\pi]$ for some p, $1 < p < 2$. This is

Lemma 3.16

For every χ in $C[0,\pi]$ the function $(\chi, \psi^*(x,t))$ is a differentiable function of x and t, and there exist functions $\psi^*_x(x,t)$ and $\psi^*_t(x,t)$ in $L^p[0,\pi]$ such that

$$\begin{aligned} \partial_x(\chi, \psi^*) &= (\chi, \psi^*_x), \\ \partial_t(\chi, \psi^*) &= (\chi, \psi^*_t). \end{aligned} \tag{3.2.78}$$

Whenever the support of χ lies inside the open sets $I^+(x,t)$ or $I^-(x,t)$, the left-hand side of (3.2.78) vanishes because of (3.2.72). Therefore,

$$\psi^*_x = \psi^*_t = 0 \qquad \text{off} \qquad I^0(x,t). \tag{3.2.79}$$

For any continuous function χ, $L\chi$ is also continuous and so, since L is a symmetric operator, we deduce that $(\chi, L\psi^*(x,t)) = (L\chi, \psi^*(x,t))$ is a differentiable function of

x and t. By (3.2.78),

$$\begin{aligned} \partial_x(\chi, L\psi^*) &= (L\chi, \psi_x^*) = (\chi, L\psi_x^*), \\ \partial_t(\chi, L\psi^*) &= (L\chi, \psi_t^*) = (\chi, L\psi_t^*). \end{aligned} \tag{3.2.80}$$

Let I_0 be the interior of the closed set I^0 given by (3.2.69) in $[0,\pi] \times R \times R$ and let $I_0(x,t)$ denote its (x,t) slice defined as in (3.2.70). Choose the support of χ in (3.2.80) to lie inside the open set $I_0(x,t)$; then, by (3.2.71), $L\psi^* = a$ on the support of χ :

$$(\chi, L\psi^*) = (\chi, a).$$

Using the definition (2.2.45) of a, we deduce that

$$\begin{aligned} \partial_x(\chi, L\psi^*) &= (\chi, a_x) = (\chi, \sin\sigma), \\ \partial_t(\chi, L\psi^*) &= (\chi, a_t) = (\chi, \sin\sigma\cos\sigma). \end{aligned} \tag{3.2.81}$$

Combining (3.2.80) and (3.2.81) we find

$$L\psi_x^* = \sin\sigma, \qquad L\psi_t^* = \sin\sigma\cos\sigma, \qquad \text{on } I_0(x,t). \tag{3.2.82}$$

By Theorem 3.29, both sides of (3.2.81) are continuous functions of σ, so the equality may be extended to $\bar{I}_0(x,t)$, the closure of $I_0(x,t)$ in $[0,\pi]$.

Our later assumptions will imply that $I_0(x,t)$ and $I^0(x,t)$ differ by at most a finite number of points. Then $\bar{I}_0(x,t)$ and $I^0(x,t)$ will have a common interior, denoted by $I(x,t)$, on which (3.2.82) holds. Since $I^0(x,t)$ and $I(x,t)$ differ by a set of measure zero, we can replace $I^0(x,t)$ by $I(x,t)$ in (3.2.79). We summarize :

ψ_x^* is in $L^p[0,\pi]$ for some p, $1 < p < \infty$, and satisfies

$$\begin{aligned} L\psi_x^*(x,t) &= \sin\sigma \quad \text{on} \quad I(x,t), \\ \psi_x^*(x,t) &= 0 \qquad \text{off} \quad I(x,t). \end{aligned} \tag{3.2.83}$$

Similarly, ψ_t^* is in $L^p[0,\pi]$ for some p, $1 < p < \infty$, and satisfies

$$\begin{aligned} L\psi_t^*(x,t) &= \sin\sigma\cos\sigma \quad \text{on} \quad I(x,t), \\ \psi_t^*(x,t) &= 0 \qquad \text{off} \quad I(x,t). \end{aligned} \tag{3.2.84}$$

Arguing as in Theorem 3.28, it follows that ψ_x^* and ψ_t^* are the unique L^p solutions of (3.2.83) and (3.2.84), respectively.

We name (3.2.83) and (3.2.84) as *differentiated variational conditions.* In fact here, the inhomogeneous terms on the right side, (*i.e.* $\sin\sigma$ and $\sin\sigma\cos\sigma$) do not depend on the initial data $\{A(x), S(x)\}$.

3.2.5 Corresponding Riemann-Hilbert Problem

If we extend $I(x,t)$ to the whole interval $[-\pi,\pi]$, by reflection about zero, then (3.2.83) and (3.2.84) hold for the extended functions. We differentiate the top equations of (3.2.83) and (3.2.84) with respect to σ on the open set $I(x,t)$. By using (3.2.74) we obtain

$$H\psi_x^* = \cos\sigma \qquad \text{on} \quad I(x,t), \tag{3.2.85}$$

and

$$H\psi_t^* = [\cos^2\sigma - \sin^2\sigma] \qquad \text{on} \quad I(x,t).$$

Denoted by f the function in H^p whose real part is ψ_x^* :

$$\Re(f) = \psi_x^*. \tag{3.2.86}$$

We claim that

$$\begin{aligned} \Im(f) &= \cos\sigma \quad \text{on} \quad I(x,t) \\ \Re(f) &= 0 \quad \text{off} \quad I(x,t). \end{aligned} \tag{3.2.87}$$

The first relation is obtained by combining (3.2.85) and Theorem 3.30; the second relation is obtained from (3.2.83).

Similarly, if g denotes the H^p function whose real part is ψ_t^* :

$$\Re(g) = \psi_t^*, \tag{3.2.88}$$

then

$$\begin{aligned} \Im(g) &= [\cos^2\sigma - \sin^2\sigma] \quad \text{on} \quad I(x,t), \\ \Re(g) &= 0 \qquad \text{off} \quad I(x,t). \end{aligned} \tag{3.2.89}$$

We shall exploit this fact in the next two sections to construct explicit functions $\psi_x^*(x,t)$ and $\psi_t^*(x,t)$ by these differentiated variational conditions. We shall verity that the function ψ^* coming this way satisfies the original variational condition. Then we can compute semiclassical limits by it. For simplicity, we only calculate $\{\bar\rho, \bar\mu, \bar\varepsilon\}$ which are the limits of the first three conserved densities. The whole theory and method work for limits of all the other conserved densities and their fluxes as well.

Now, we show how to express $\bar\rho$, $\bar\mu$ and $\bar\varepsilon$ in terms of ψ_x^* and ψ_t^* :

Theorem 3.31

$$\bar\rho(x,t) = 1 + 2(\sin\sigma, \psi_x^*(x,t));$$

$$\bar\mu(x,t) = 2(\sin\sigma, \psi_t^*(x,t)) = \bar{\mathcal{G}}_1(x,t),$$

or

$$\bar\mu(x,t) = 2(\sin\sigma\cos\sigma, \psi_x^*(x,t)) = \bar\rho_2(x,t);$$

and

$$\bar\varepsilon(x,t) = 2(\sin\sigma\cos^2\sigma, \psi_x^*(x,t)) = \bar\rho_3(x,t),$$

or

$$\bar\varepsilon(x,t) = -2(\sin\sigma, \psi_x^*) + 2(\sin\sigma, \psi_x^*)^2 + 2(\sin\sigma\cos\sigma, \psi_t^*).$$

Generally,

$$\bar\rho_n(x,t) = 2(\sin\sigma\cos^{n-1}\sigma, \psi_x^*(x,t))$$

and

$$\bar{\mathcal{G}}_n(x,t) = 2(\sin\sigma\cos^{n-1}\sigma, \psi_t^*(x,t)).$$

PROOF :

Recall the formulas (3.2.60) of Theorem 3.25 :

$$\partial_x Q^*(x,t) = 4(\sin\sigma, \psi^*(x,t))$$

and

$$\begin{aligned}\partial_{t_n} Q^*(x,t) &= \partial_{t_n} Q^*(x,\bar{t})_{t_n=0,\ n>2}\\ &= 4(\sin\sigma\cos^{n-1}\sigma, \psi^*(x,t)).\end{aligned}$$

With the formula given by (3.2.78), we differentiate them with respect to x, t, and get

$$\partial_{xx} Q^*(x,t) = 4(\sin\sigma, \psi_x^*(x,t)),$$

$$\partial_{xt} Q^*(x,t) = 4(\sin\sigma, \psi_t^*(x,t)),$$

$$\partial_{xt} Q^*(x,t) = 4(\sin\sigma\cos\sigma, \psi_x^*(x,t)),$$

and

$$\partial_{tt} Q^*(x,t) = 4(\sin\sigma\cos\sigma, \psi_t^*(x,t)).$$

Or, generally,

$$\partial_{xt_n} Q^*(x,t) = 4(\sin\sigma\cos^{n-1}\sigma, \psi_x^*(x,t)),$$

and

$$\partial_{tt_n} Q^*(x,t) = 4(\sin\sigma\cos^{n-1}\sigma, \psi_t^*(x,t)).$$

If we compare these formulas with definitions given by (3.1.42), (3.1.43), (3.1.45) and (3.1.44) for $\bar{\rho}$, $\bar{\mu}$ and $\bar{\varepsilon}$, we conclude that Theorem 3.31 holds.

We end this section by proving (3.1.26). Using the definition (2.2.62) of ϕ, the fact that L and $d/\,d\sigma$ commute and the relation (3.2.74), we have

$$L\phi = -L\frac{d}{d\sigma}\Phi = -\frac{d}{d\sigma}(L\Phi) = -H\Phi.$$

According to Theorem 3.2.57, $-L\phi$ is the imaginary part of a function of class $H^p(\Omega)$ whose real part is Φ. This function can be fixed as:

$$f(\omega) = \int_{-\infty}^{\infty}\left\{\sqrt{[\frac{\omega^2+1}{2\omega} - \cos\hat{\sigma}_1(x)][\frac{\omega^2+1}{2\omega} - \cos\hat{\sigma}_2(x)]} + \frac{\omega^2-1}{2\omega}\right\} dx,$$

where

$$
\begin{aligned}
\Re(f_{\omega=e^{i\sigma}}) &= \Re\left(\int_{x_-(\sigma)}^{x_+(\sigma)} \sqrt{[\cos\sigma - \cos\hat{\sigma}_1(x)][\cos\sigma - \cos\hat{\sigma}_2(x)]}\, dx\right) \\
&= \begin{cases} \int_{x_-}^{x_+} \sqrt{[\cos\sigma - \cos\hat{\sigma}_1(x)][\cos\sigma - \cos\hat{\sigma}_2(x)]}\, dx & 0 \leq \sigma < \hat{\sigma}_{1,0} \\ 0 & \hat{\sigma}_{1,0} \leq \sigma \leq \hat{\sigma}_{2,0} \\ -\int_{x_-}^{x_+} \sqrt{[\cos\sigma - \cos\hat{\sigma}_1(x)][\cos\sigma - \cos\hat{\sigma}_2(x)]}\, dx & \hat{\sigma}_{2,0} < \sigma \leq \pi \end{cases} \\
&= \Phi(\sigma),
\end{aligned}
$$

as may be seen from (2.1.16).

Using (2.1.26) and (2.1.22) we see that

$$\Im(f_{\omega=e^{i\sigma}}) = \Gamma_+(\sigma) - \Gamma_-(\sigma).$$

But

$$\Im(f_{\omega=e^{i\sigma}}) = H\Phi.$$

This proves (3.1.26) :

$$L\phi(\sigma) = \Gamma_-(\sigma) - \Gamma_+(\sigma);$$

or, if for the the t_n-flow,

$$L\phi(\sigma) = \Gamma_-^n(\sigma) - \Gamma_+^n(\sigma).$$

CHAPTER 4

CONSTRUCTION OF SOLUTION

4.1 Solution Until Break-time

4.1.1 Limits of the Initial Data

The variational problem (3.1.15)-(3.1.17) contains x and t as parameters. We first investigate the case $t = 0$. We assume, as a *posteriori*, that for $0 \leq t < t_b$, the set I consists of two disjoint intervals :

$$I = (0, \hat{\sigma}_1) \cup (\hat{\sigma}_2, \pi), \tag{4.1.1}$$

where $\hat{\sigma}_1, \hat{\sigma}_2$ are differentiable functions of x, and $0 < \hat{\sigma}_1 < \hat{\sigma}_2 < \pi$ for all x. We also, for later convenience, extend I symmetrically :

$$I = (-\pi, -\hat{\sigma}_2) \cup (-\hat{\sigma}_1, \hat{\sigma}_1) \cup (\hat{\sigma}_2, \pi).$$

We denote $\hat{I}$ as a part of the boundary of the unit disk Ω related to I :

$$\hat{I} = \{e^{i\sigma} : \sigma \in I\}.$$

We will investigate the limits of ρ and μ first, then get the general picture for the other conserved densities.

We claim that the solution of the Riemann-Hilbert problem posed in (3.2.87) is

$$f(\omega) = \frac{i[\omega + (\frac{1-\omega}{1+\omega})(1 + \frac{\cos\hat{\sigma}_1 + \cos\hat{\sigma}_2}{2})]}{\sqrt{[1 - \frac{2\omega(1+\cos\hat{\sigma}_1)}{(1+\omega)^2}][1 - \frac{2\omega(1+\cos\hat{\sigma}_2)}{(1+\omega)^2}]}}$$

$$= \frac{i[\omega + (\frac{1-\omega}{1+\omega})(1 + \frac{\cos\hat{\sigma}_1 + \cos\hat{\sigma}_2}{2})]}{2\sqrt{(1+\cos\hat{\sigma}_1)(1+\cos\hat{\sigma}_2)}R(\omega)}; \tag{4.1.2}$$

where the sign of the square root which defines $R(\omega)$ is chosen to be

$$\begin{cases} R(\omega) > 0 & \text{if} \quad \omega = e^{i\sigma}, \quad 0 < \sigma < \hat{\sigma}_1 \\ iR(\omega) < 0 & \text{if} \quad \omega = e^{i\sigma}, \quad \hat{\sigma}_1 < \sigma < \hat{\sigma}_2, \end{cases} \tag{4.1.3}$$

with

$$R(\omega) = \sqrt{\left[\frac{\omega}{(1+\omega)^2} - \frac{1}{2(1+\cos\hat{\sigma}_1)}\right]\left[\frac{\omega}{(1+\omega)^2} - \frac{1}{2(1+\cos\hat{\sigma}_2)}\right]}. \tag{4.1.4}$$

Clearly, $f(\omega)$ is analytic inside of the unit disc $\Omega = \{\omega : |\omega| < 1\}$. For $\omega \in \partial\Omega$, the singularities of $f(\omega)$ are only at $\omega = \hat{\omega}_1 = e^{i\hat{\sigma}_1}$, or $\omega = \hat{\omega}_2 = e^{i\hat{\sigma}_2}$. They are zeros of $R(\omega)$. On account of (4.1.4),

$$f(\omega) \approx O(\frac{1}{\sqrt{\cos\sigma - \cos\hat{\sigma}_k}}) \qquad \text{when} \quad \omega_{\partial\Omega} \to \hat{\omega}_k \quad k = 1, 2.$$

So, for $1 \le p < 2$, there exists a constant $c > 0$, such that

$$(\int_{-\pi}^{\pi} |f(re^{i\sigma})|^p)^{\frac{1}{p}} < c$$

for all $r \le 1$. This shows that $f \in H^p$ for $1 \le p < 2$.

For $\omega = e^{i\sigma}$, *i.e.* ω on the boundary of that unit disc Ω, it follows from (4.1.2) and (4.1.3), (4.1.4) that

$$\Re(f) = \begin{cases} (-\sin\sigma)\frac{\cos\sigma - \frac{1}{2}(\cos\hat{\sigma}_1 + \cos\hat{\sigma}_2)}{\sqrt{(\cos\sigma - \cos\hat{\sigma}_1)(\cos\sigma - \cos\hat{\sigma}_2)}} & \\ & \text{on } I, \text{ but } -\hat{\sigma}_1 < \sigma < \hat{\sigma}_1 \\ 0 & \text{off } I \\ (\sin\sigma)\frac{\cos\sigma - \frac{1}{2}(\cos\hat{\sigma}_1 + \cos\hat{\sigma}_2)}{\sqrt{(\cos\sigma - \cos\hat{\sigma}_1)(\cos\sigma - \cos\hat{\sigma}_2)}} & \\ & \text{on } I, \text{ but } \begin{cases} \hat{\sigma}_2 < \sigma < \pi \text{ or} \\ -\hat{\sigma}_2 > \sigma > -\pi \end{cases} \end{cases} \tag{4.1.5}$$

and

$$\Im(f) = \begin{cases} \cos\sigma & \text{on } I, \begin{cases} -\hat{\sigma}_1 < \sigma < \hat{\sigma}_1 \\ \hat{\sigma}_2 < \sigma < \pi \\ -\hat{\sigma}_2 > \sigma > -\pi \end{cases} \\ \cos\sigma - \sin\sigma \frac{\cos\sigma - \frac{1}{2}(\cos\hat{\sigma}_1 + \cos\hat{\sigma}_2)}{\sqrt{(\cos\hat{\sigma}_1 - \cos\sigma)(\cos\sigma - \cos\hat{\sigma}_2)}} & \\ & \text{off } I, \begin{cases} \hat{\sigma}_1 < \sigma < \hat{\sigma}_2 \\ or \ -\hat{\sigma}_1 > \sigma > -\hat{\sigma}_2 \end{cases} \end{cases} \tag{4.1.6}$$

where, because of (4.1.3), the square root in (4.1.6) changes sign the same as σ does, and it in (4.1.5) makes the fraction keep positive. These relations show that, the conditions given by (3.2.87) are satisfied, $\Re(f)$ is odd with support in I, and $\Im(f)$ is even. We set (4.1.5) into (3.2.86):

$$\psi_x^* = \begin{cases} (-\sin\sigma)\frac{\cos\sigma - \frac{1}{2}(\cos\hat{\sigma}_1 + \cos\hat{\sigma}_2)}{\sqrt{(\cos\sigma - \cos\hat{\sigma}_1)(\cos\sigma - \cos\hat{\sigma}_2)}} & -\hat{\sigma}_1 < \sigma < \hat{\sigma}_1, \\ 0 & \text{else}, \\ (\sin\sigma)\frac{\cos\sigma - \frac{1}{2}(\cos\hat{\sigma}_1 + \cos\hat{\sigma}_2)}{\sqrt{(\cos\sigma - \cos\hat{\sigma}_1)(\cos\sigma - \cos\hat{\sigma}_2)}} & \begin{cases} \hat{\sigma}_2 < \sigma < \pi \ \ or \\ -\hat{\sigma}_2 > \sigma > -\pi. \end{cases} \end{cases} \tag{4.1.7}$$

Now, using the formula (3.2.74) of Theorem 3.29

$$L\psi(\sigma) = \int_0^\sigma H\psi(\tau)\, d\tau$$

with

$$H\psi_x^* = \Im(f), \qquad \frac{d}{d\sigma}(L\psi_x^*) = H\psi_x^*$$

and $\Im(f)$ given by (4.1.6), we have

$$L\psi_x^* = \begin{cases} \sin\sigma & \text{on } I, \\ \sin\sigma - \int_{\hat{\sigma}_1}^{\sigma} (\sin\varsigma)\frac{[\cos\varsigma - \frac{1}{2}(\cos\hat{\sigma}_1 + \cos\hat{\sigma}_2)]}{\sqrt{(\cos\hat{\sigma}_1 - \cos\varsigma)(\cos\varsigma - \cos\hat{\sigma}_2)}}\, d\varsigma & \text{off } I. \end{cases} \tag{4.1.8}$$

Carrying out the integral in (4.1.8) we obtain

$$L\psi_x^* = \begin{cases} \sin\sigma & \text{on } I, \\ \sin\sigma - \sqrt{[\cos\hat\sigma_1(x) - \cos\sigma][\cos\sigma - \cos\hat\sigma_2(x)]} & \text{off } I. \end{cases} \tag{4.1.9}$$

The values of $\hat\sigma_1, \hat\sigma_2$ are so far undetermined; for that we turn to the formulas given by Theorem 3.31,

$$\begin{aligned} 1 - \bar\rho &= -2(\sin\sigma, \psi_x^*), \\ \bar\mu = 2(\sin\sigma\cos\sigma, \psi_x^*) &= 2(\sin\sigma, \psi_t^*). \end{aligned} \tag{4.1.10}$$

We expect that

$$\begin{aligned} \bar\rho(x,0) \equiv \lim_{h\to 0} |U|^2_{t=0} &= A^2(x), \\ \bar\mu(x,0) \equiv \lim_{h\to 0} \frac{ih}{2}(UU_x^* - U^*U_x)_{t=0} &= A^2(x)\partial_x S(x). \end{aligned} \tag{4.1.11}$$

Combining (4.1.10) with (4.1.11)and expressing ψ_x^* in (4.1.10) by (4.1.7) yields

$$\begin{aligned} &1 - A^2(x) \\ &= -2(\sin\sigma, \psi_x *) \\ &= \frac{2}{\pi}\left(\int_0^{\hat\sigma_1} - \int_{\hat\sigma_2}^{\pi}\right)(\sin^2\sigma)\frac{[\cos\sigma - \frac{1}{2}(\cos\hat\sigma_1 + \cos\hat\sigma_2)]}{\sqrt{(\cos\sigma - \cos\hat\sigma_1)(\cos\sigma - \cos\hat\sigma_2)}}\, d\sigma \\ &= 1 - \frac{1}{4}(\hat\lambda_1 - \hat\lambda_2)^2 \\ &= [1 - \frac{1}{4}(\cos\hat\sigma_1 - \cos\hat\sigma_2)^2] \end{aligned}$$

and

$$\begin{aligned} &A^2(x)\partial_x S(x) \\ &= -2(\sin\sigma\cos\sigma, \psi_x^*) \\ &= (-\frac{2}{\pi})\left(\int_0^{\hat\sigma_1} - \int_{\hat\sigma_2}^{\pi}\right)(\sin^2\sigma\cos\sigma)\frac{[\cos\sigma - \frac{1}{2}(\cos\hat\sigma_1 + \cos\hat\sigma_2)]}{\sqrt{(\cos\sigma - \cos\hat\sigma_1)(\cos\sigma - \cos\hat\sigma_2)}}\, d\sigma \\ &= (\hat\lambda_1 + \hat\lambda_2)A^2(x) \\ &= (\cos\hat\sigma_1 + \cos\hat\sigma_2)A^2(x), \end{aligned}$$

where we define

$$\hat{\lambda}_k = \cos\hat{\sigma}_k \qquad k = 1, 2.$$

Now, we obtain

$$\begin{cases} \hat{\lambda}_1 + \hat{\lambda}_2 = \partial_x S(x) \\ \hat{\lambda}_1 - \hat{\lambda}_2 = 2|A(x)|. \end{cases}$$

Thus we get

$$\begin{cases} \hat{\lambda}_1 = \frac{1}{2}\partial_x S(x) + |A(x)| \\ \hat{\lambda}_2 = \frac{1}{2}\partial_x S(x) - |A(x)| \end{cases} \tag{4.1.12}$$

and

$$\begin{cases} \hat{\sigma}_1 = \cos^{-1}(\hat{\lambda}_1) = \cos^{-1}[\partial_x S(x)/2 + |A(x)|] \\ \hat{\sigma}_2 = \cos^{-1}(\hat{\lambda}_2) = \cos^{-1}[\partial_x S(x)/2 - |A(x)|]. \end{cases} \tag{4.1.13}$$

They are just consistent with the one given by (2.1.11) as expected.

Since $\hat{\sigma}_1(x), \hat{\sigma}_2(x)$ are both larger or both smaller than σ where x is between $x_-(\sigma)$ and $x_+(\sigma)$, and, $\hat{\sigma}_2(x) > \sigma > \hat{\sigma}_1(x)$ where x is outside of (x_-, x_+), (see Figure 4), formula (4.1.7) can be put in the following form

$$\psi_x^*(x,0) = \begin{cases} -(\sin\sigma)\frac{[\cos\sigma - \frac{1}{2}(\cos\hat{\sigma}_1 + \cos\hat{\sigma}_2)]}{\sqrt{(\cos\sigma - \cos\hat{\sigma}_1)(\cos\sigma - \cos\hat{\sigma}_2)}} & \begin{cases} x_-(\sigma) < x < x_+(\sigma) \\ \sigma < \hat{\sigma}_{1,max}, \end{cases} \\ (\sin\sigma)\frac{[\cos\sigma - \frac{1}{2}(\cos\hat{\sigma}_1 + \cos\hat{\sigma}_2)]}{\sqrt{(\cos\sigma - \cos\hat{\sigma}_1)(\cos\sigma - \cos\hat{\sigma}_2)}} & \begin{cases} x_-(\sigma) < x < x_+(\sigma) \\ \hat{\sigma}_{2,min} < \sigma, \end{cases} \\ 0 & x \le x_-(\sigma) \ or \ x \ge x_+(\sigma), \end{cases} \tag{4.1.14}$$

where $\hat{\sigma}_{1,max} = \max\hat{\sigma}_1(x)$; $\hat{\sigma}_{2,min} = \min\hat{\sigma}_2(x)$. This completes the determinations of $\psi_x^*(x,0)$.

To determine $\psi^*(x,0)$ itself we integrate $\psi^*_x(x,0)$ as given by (4.1.14). Theorem (3.27) states that $\psi^*_x(x,0) = 0$ where $x \sin\sigma - \Gamma_+(\sigma) > 0$; in particular, it follows that, for fixed σ, $\psi^*(x,0) = 0$ for x large enough. Thus

$$\psi^*(x,0) = -\int_x^\infty \psi^*_x(y,0)\,dy.$$

That is,

$$\psi^*(x,0) = \begin{cases} 0 & x \geq x_+(\sigma), \\ \int_x^{x_+(\sigma)} (\sin\sigma) \frac{[\cos\sigma - \frac{1}{2}(\cos\hat\sigma_1(y) + \cos\hat\sigma_2(y))]}{\sqrt{(\cos\sigma - \cos\hat\sigma_1(y))(\cos\sigma - \cos\hat\sigma_2(y))}}\,dy & \begin{cases} x_-(\sigma) < x < x_+(\sigma) \\ 0 < \sigma < \hat\sigma_{1,max}, \end{cases} \\ -\int_x^{x_+(\sigma)} (\sin\sigma) \frac{[\cos\sigma - \frac{1}{2}(\cos\hat\sigma_1(y) + \cos\hat\sigma_2(y))]}{\sqrt{(\cos\sigma - \cos\hat\sigma_1(y))(\cos\sigma - \cos\hat\sigma_2(y))}}\,dy & \begin{cases} x_-(\sigma) < x < x_+(\sigma) \\ \hat\sigma_{2,min} < \sigma < \pi, \end{cases} \\ \phi(\sigma) & x \leq x_-(\sigma). \end{cases} \tag{4.1.15}$$

In (4.1.15) we have used the definition (2.2.62) of $\phi(\sigma)$:

$$\phi(\sigma) = \begin{cases} \int_{x_-(\sigma)}^{x_+(\sigma)} (\sin\sigma) \frac{[\cos\sigma - \frac{1}{2}(\cos\hat\sigma_1(y) + \cos\hat\sigma_2(y))]}{\sqrt{(\cos\sigma - \cos\hat\sigma_1(y))(\cos\sigma - \cos\hat\sigma_2(y))}}\,dy & 0 < \sigma < \hat\sigma_{1,max}, \\ 0 & \hat\sigma_{1,max} < \sigma < \hat\sigma_{2,min}, \\ -\int_{x_-(\sigma)}^{x_+(\sigma)} (\sin\sigma) \frac{[\cos\sigma - \frac{1}{2}(\cos\hat\sigma_1(y) + \cos\hat\sigma_2(y))]}{\sqrt{(\cos\sigma - \cos\hat\sigma_1(y))(\cos\sigma - \cos\hat\sigma_2(y))}}\,dy & \hat\sigma_{2,min} < \sigma < \pi. \end{cases}$$

Clearly, $0 \leq \psi^*(x,0) \leq \phi$. So $\psi^*(x,0)$ is in the admissible set A.

We now show that $\psi^*(x,0)$ of (4.1.15) satisfies the variational conditions (3.2.66) and thus justifies our assumption (4.1.1). Using (4.1.14) we may rewrite formula (4.1.9) as following :

$$L\psi^*(x,0) = \begin{cases} \sin\sigma & x_-(\sigma) \le x \le x_+(\sigma), \\ \\ \sin\sigma - \sqrt{(\cos\hat\sigma_1 - \cos\sigma)(\cos\sigma - \cos\hat\sigma_2)} & \\ & \begin{cases} x < x_-(\sigma) \;\; or \\ x > x_+(\sigma). \end{cases} \end{cases} \tag{4.1.16}$$

Theorem 3.27 implies that

$$\lim_{x\to\infty} \psi^*(x,0) = 0,$$

then by Theorem 3.9 and the definition of $\hat\sigma_{1,2}$,

$$\lim_{x\to\infty} L\psi^*(x,0) = 0.$$

We then obtain, by integrating (4.1.16),

$$-L\psi^*(x,0) = \int_x^\infty L\psi_x^*(y,0)\,dy$$

$$= \begin{cases} -x\sin\sigma \;+\; \Gamma_+(\sigma) & \\ \qquad\qquad + \; \int_{x_+}^{x} \sqrt{(\cos\hat\sigma_1(y) - \cos\sigma)(\cos\sigma - \cos\hat\sigma_2(y))}\,dy & \\ & x > x_+(\sigma), \\ \\ -x\sin\sigma \;+\; \Gamma_+(\sigma) & x_-(\sigma) \le x \le x_+(\sigma), \\ \\ -x\sin\sigma \;+\; \Gamma_+(\sigma) & \\ \qquad\qquad + \; \int_{x}^{x_-} \sqrt{(\cos\hat\sigma_1(y) - \cos\sigma)(\cos\sigma - \cos\hat\sigma_2(y))}\,dy & \\ & x < x_-(\sigma). \end{cases} \tag{4.1.17}$$

In (4.1.17) we used the definition (2.1.26) of $\Gamma_+(\sigma)$:

$$\Gamma_+(\sigma) = x_+(\sigma)\sin\sigma$$

$$+\int_{x_+(\sigma)}^{+\infty}\left[\sin\sigma-\sqrt{(\cos\hat{\sigma}_1(y)-\cos\sigma)(\cos\sigma-\cos\hat{\sigma}_2(y))}\right]dy.$$

Since $a(\sigma,x,0)=x\sin\sigma-\Gamma_+(\sigma)$, we get from (4.1.17) that

$$(a-L\psi^*)_{t=0}=$$
$$\begin{cases}\int_{x_+(\sigma)}^{x}\sqrt{(\cos\hat{\sigma}_1(y)-\cos\sigma)(\cos\sigma-\cos\hat{\sigma}_2(y))}\,dy & x>x_+(\sigma)\\ 0 & x_-(\sigma)\le x\le x_+(\sigma)\\ \int_{x}^{x_-(\sigma)}\sqrt{(\cos\hat{\sigma}_1(y)-\cos\sigma)(\cos\sigma-\cos\hat{\sigma}_2(y))}\,dy & x<x_-(\sigma),\end{cases}$$

which clearly shows that $\psi^*(x,0)$ given by (4.1.15) satisfies the variational conditions (3.2.66).

According to Theorem 3.26, the only admissible function that satisfies the variational conditions is ψ^*. Thus we have proved

Theorem 4.32

$\psi^*(x,0)$ which is given by (4.1.15) solves the variational problem (3.1.15)-(3.1.17) at $t=0$.

Formula (4.1.15) indicates that the $\psi^*(x,0)$ given by (4.1.14) depends differentiably on x. As shown in Theorem 3.31, the differentiability of ψ^* implies that (4.1.10) holds; Comparing this with (4.1.14) we conclude from Theorem 3.17:

Theorem 4.33

$$\begin{aligned}\bar{\rho}(x,0) &\equiv \text{weak } L^1\text{-}\lim_{h\to 0}[|U(x,0;h)|^2]\\ &= \frac{[\hat{\lambda}_1(x)-\hat{\lambda}_2(x)]^2}{4}=A^2(x),\end{aligned}$$

$$\begin{aligned}\bar{\mu}(x,0) &\equiv \text{weak } L^1\text{-}\lim_{h\to 0}[\frac{ih}{2}(UU_x^* - U_xU^*)]_{t=0} \\ &= [\hat{\lambda}_1(x)+\hat{\lambda}_2(x)]\frac{[\hat{\lambda}_1(x)-\hat{\lambda}_2(x)]^2}{4} \\ &= A^2(x)\partial_x S(x).\end{aligned}$$

These results match which were expected in (4.1.11) and provide justification for the replacement of the exact scattering data by the asymptotic data. In Theorem 4.36 we shall sharpen Theorem 4.33.

4.1.2 Limits Until Break-time

We now begin our investigation of non-zero values of t. If, in the system of the conserved quantities (1.2.7), one sets $h = 0$ and considers $\bar{\rho}, \bar{\mu}$ in stead of ρ, μ, then

$$\begin{cases} \partial_t\bar{\rho} - \partial_x(\bar{\mu}) = 0 \\ \partial_t\bar{\mu} - \partial_x(\bar{\mu}^2/\bar{\rho} + \frac{1}{2}\bar{\rho}^2) = 0. \end{cases}$$

This hyperbolic system is equivalent to its Riemann invariant form :

$$\partial t\begin{pmatrix}\hat{\lambda}_1 \\ \hat{\lambda}_2\end{pmatrix} - \frac{1}{2}\begin{pmatrix}3\hat{\lambda}_1+\hat{\lambda}_2 & 0 \\ 0 & \hat{\lambda}_1+3\hat{\lambda}_2\end{pmatrix}\partial x\begin{pmatrix}\hat{\lambda}_1 \\ \hat{\lambda}_2\end{pmatrix} = 0, \tag{4.1.18}$$

where

$$\begin{cases} \hat{\lambda}_1(x,t) = \bar{\mu}/(2\bar{\rho}) + \sqrt{\bar{\rho}} \\ \hat{\lambda}_2(x,t) = \bar{\mu}/(2\bar{\rho}) - \sqrt{\bar{\rho}} \end{cases}.$$

In the introduction (Chapter 1), we pointed out that there is a break-time t_b, such that the system given by (4.1.18) has smooth solutions for $t < t_b$ but not beyond. This critical time can be estimated by the initial data of $\hat{\lambda}_1(x,t)$ and $\hat{\lambda}_2(x,t)$. This is a typical example of Lax's general results [10].

Theorem 4.34

The break-time t_b can be estimated in this way :

$$t_b = \min\{t_{1,b}; t_{2,b}\}$$

where for $k = 1, 2$,

$$t_{k,b} \le \inf_{x \in \mathcal{T}_k} \{t : \mathcal{D}_k(x,t) = 0\},$$

with

$$\mathcal{T}_k = \{x : \frac{\partial \hat{\lambda}_k(x)}{\partial x} \le 0\},$$

and

$$\mathcal{D}_k = 1 + \left(\frac{\partial_x \hat{\lambda}_k(x)}{\sqrt{\hat{\lambda}_1(x) - \hat{\lambda}_2(x)}} \right) \int_0^t 3\sqrt{\hat{\lambda}_1(x_k(\tau),\tau) - \hat{\lambda}_2(x_k(\tau),\tau)}\, d\tau,$$

where $x_k(t)$ *is involved by*

$$\frac{dx_k}{dt} + \hat{\lambda}_k(x_k,t) + \frac{1}{2}[\hat{\lambda}_1(x_k,t) + \hat{\lambda}_2(x_k,t)] = 0,$$

$$x_k(0) = x.$$

PROOF :

We use Lax's recipe [10] to estimate t_b. Recall (4.1.18), the Riemann invariant forms which before break-time are coupled and given by

$$\begin{aligned} \partial_t \hat{\lambda}_1(x,t) + S_1(\hat{\lambda}_1, \hat{\lambda}_2)\partial_x \hat{\lambda}_1(x,t) = 0 \\ \partial_t \hat{\lambda}_2(x,t) + S_2(\hat{\lambda}_1, \hat{\lambda}_2)\partial_x \hat{\lambda}_2(x,t) = 0 \end{aligned} \tag{4.1.19}$$

with the characteristic speeds

$$S_1 = -\frac{1}{2}(3\hat{\lambda}_1 + \hat{\lambda}_2), \qquad S_2 = -\frac{1}{2}(\hat{\lambda}_1 + 3\hat{\lambda}_2).$$

Choose characteristic direction S_1 and define

$$\frac{d}{dt} = \frac{\partial}{\partial t} + S_1 \frac{\partial}{\partial x}.$$

They yields

$$\frac{d\hat{\lambda}_1}{dt} = 0,$$

and

$$\frac{d\tau}{dt} - \frac{3}{2}\tau^2 - \frac{1}{2}\tau\frac{\partial\hat{\lambda}_2}{\partial x} = 0, \tag{4.1.20}$$

where

$$\tau = \frac{\partial\hat{\lambda}_1}{\partial x}.$$

From (4.1.19), we get

$$\frac{\partial\hat{\lambda}_2}{\partial x} = \left(\frac{1}{(\hat{\lambda}_2 - \hat{\lambda}_1)}\right)\frac{d\hat{\lambda}_2}{dt}. \tag{4.1.21}$$

Find a function $\tilde{h}(x,t)$, such that

$$\frac{\partial\tilde{h}}{\partial\hat{\lambda}_2} = \frac{1}{(\hat{\lambda}_2 - \hat{\lambda}_1)}.$$

A simple choice is

$$\tilde{h} = \log(\hat{\lambda}_1 - \hat{\lambda}_2). \tag{4.1.22}$$

So,

$$\frac{d\tilde{h}}{dt} = \frac{\partial\hat{\lambda}_2}{\partial x}.$$

Putting (4.1.21), (4.1.22) in (4.1.20) gives

$$0 = \frac{d}{dt}(\tau e^{\tilde{h}}) + 3(\tau e^{\tilde{h}})^2 e^{-\tilde{h}}.$$

Set

$$\tilde{q} = \tau e^{\tilde{h}} = \tau(\hat{\lambda}_1 - \hat{\lambda}_2),$$

we obtain

$$\frac{d\tilde{q}}{dt} + 3\tilde{q}^2(\hat{\lambda}_1 - \hat{\lambda}_2)^{-1} = 0.$$

That is

$$\frac{d}{dt}(\frac{1}{\tilde{q}}) = \frac{3}{\hat{\lambda}_1 - \hat{\lambda}_2}.$$

Integrating it gives

$$\frac{1}{\tilde{q}(t)} - \frac{1}{\tilde{q}(0)} = 3\int_0^t \left(\frac{dt}{\hat{\lambda}_1(x(t),t) - \hat{\lambda}_2(x(t),t)}\right),$$

with $x(t)$ which satisfies

$$\frac{dx}{dt} + \frac{1}{2}[3\hat{\lambda}_1(x,t) + \hat{\lambda}_2(x,t)] = 0.$$

That comes

$$\tilde{q}(x,t) = \frac{\tilde{q}(x,0)}{1 + \tilde{q}(x,0)\vartheta(t)},$$

where

$$\tilde{q}(x,0) = \partial_x\hat{\lambda}_1(x)[\hat{\lambda}_1(x) - \hat{\lambda}_2(x)]$$

and

$$\vartheta(t) = \int_0^t \frac{3\,d\tau}{\hat{\lambda}_1(x(\tau),\tau) - \hat{\lambda}_2(x(\tau),\tau)}.$$

Thus, at initial time, if $\partial_x\hat{\lambda}_1 < 0$ at some x, then $\tilde{q}(x,0) < 0$ at same x. That means $\tilde{q}(x,t)$ must break at some later time t where

$$1 + \tilde{q}(x,0)\vartheta(t) = 0.$$

Therefore we get an estimate for t_b, such that

$$t_b \le \inf_{x_0 \in \mathcal{T}} \{t : \mathcal{D}(x,t) = 0\},$$

with

$$\mathcal{T} = \{x_0 : \partial_x\hat{\lambda}_1(x_0) \le 0\},$$

and

$$\mathcal{D} = 1 + \partial_x\hat{\lambda}_1(x)(\hat{\lambda}_1(x) - \hat{\lambda}_2(x)) \int_0^t \frac{3\,d\tau}{\hat{\lambda}_1(x(\tau),\tau) - \hat{\lambda}_2(x(\tau),\tau)},$$

where $x(t)$ satisfies

$$\frac{dx}{dt} + \frac{1}{2}[3\hat{\lambda}_1(x,t) + \hat{\lambda}_2(x,t)] = 0,$$

$$x(0) = x_0.$$

Similarly, we can estimate t_b by using the other characteristic line

$$\frac{\partial x}{\partial t} = S_2$$

with the aid of

$$\frac{d}{dt} = \frac{\partial}{\partial t} + S_2 \frac{\partial}{\partial x}.$$

This completes the proof of Theorem 4.34.

Hereafter, we denote by

$$\begin{pmatrix} \hat{\lambda}_1(x,t) \\ \hat{\lambda}_2(x,t) \end{pmatrix}$$

the solutions of system (4.1.18) witch dresses the prescribed initial values

$$\begin{pmatrix} \hat{\lambda}_1(x) \\ \hat{\lambda}_2(x) \end{pmatrix}$$

given by (4.1.12).

Theorem 4.35

For $0 \leq t < t_b$*,*

$$\bar{\rho}(x,t) = \frac{1}{4}[\hat{\lambda}_1(x,t) - \hat{\lambda}_2(x,t)]^2$$

$$\bar{\mu}(x,t) = \frac{1}{4}[\hat{\lambda}_1(x,t) - \hat{\lambda}_2(x,t)]^2[\hat{\lambda}_1(x,t) + \hat{\lambda}_2(x,t)] \tag{4.1.23}$$

uniformly in t*, where*

$$\bar{\rho}(x,t) \equiv w\_L^1 \lim_{h\to 0} |U(x,t,h)|^2$$

$$\bar{\mu}(x,t) \equiv w\_L^1 \lim_{h\to 0} \frac{ih}{2}(UU_x^* - U_x U^*).$$

PROOF :

We shall reduce this to Theorem 4.32. As $\hat{\lambda}_1(x,t)$, $\hat{\lambda}_2(x,t)$ evolve according to (4.1.18), it will, until break-time, remain of the same class as the initial data $\hat{\lambda}_1(x)$, $\hat{\lambda}_2(x)$. We now consider how the asymptotic scattering data corresponding to $\hat{\lambda}_1$, $\hat{\lambda}_2$ change with t. Recall the definition of $x_+(\sigma,t)$ and $x_-(\sigma,t)$, they satisfy $x_-(\sigma,t) < x_+(\sigma,t)$ for σ in $(0,\pi)$ and

$$\hat{\sigma}_1(x_-,t) = \hat{\sigma}_1(x_+,t) = \sigma \qquad \text{or}$$

$$\hat{\sigma}_2(x_-,t) = \hat{\sigma}_2(x_+,t) = \sigma. \tag{4.1.24}$$

It is equivalent to

$$\hat{\lambda}_1(x_-,t) = \hat{\lambda}_1(x_+,t) = \lambda \qquad \text{or}$$

$$\hat{\lambda}_2(x_-,t) = \hat{\lambda}_2(x_+,t) = \lambda.$$

Replacing

$$\begin{pmatrix} \hat{\lambda}_1(x) \\ \hat{\lambda}_2(x) \end{pmatrix}$$

by

$$\begin{pmatrix} \hat{\lambda}_1(x,t) \\ \hat{\lambda}_2(x,t) \end{pmatrix}$$

and $x \pm (\sigma)$ by $x \pm (\sigma, t)$ in formulas (2.1.15) and (2.1.26), we define

$$\Phi(\sigma,t) = \begin{cases} -\int_{x_-(\sigma,t)}^{x_+(\sigma,t)} \sqrt{(\cos\hat{\sigma}_1(y,t) - \cos\sigma)(\cos\hat{\sigma}_2(y,t) - \cos\sigma)}\,dy & \pi \geq \sigma > \hat{\sigma}_{2,min}(t) \\ 0 & \sigma_{2,min}(t) \geq \sigma \geq \hat{\sigma}_{1,max}(t) \\ \int_{x_-(\sigma,t)}^{x_+(\sigma,t)} \sqrt{(\cos\hat{\sigma}_1(y,t) - \cos\sigma)(\cos\hat{\sigma}_2(y,t) - \cos\sigma)}\,dy & \hat{\sigma}_{1,max}(t) > \sigma \geq 0, \end{cases} \tag{4.1.25}$$

where

$$\hat{\sigma}_{1,max}(t) = \sup_x\{\hat{\sigma}_1(x,t)\}; \quad \hat{\sigma}_{2,min}(t) = \inf_x\{\hat{\sigma}_1(x,t)\}.$$

Since $t < t_b$, the characteristic lines have no singularity. So, the initial condition

$$\hat{\sigma}_{1,max} < \hat{\sigma}_{2,min}$$

implies

$$\hat{\sigma}_{1,max}(t) < \hat{\sigma}_{2,min}(t), \quad \forall t < t_b.$$

We also define

$$\Gamma_+(\sigma,t) = x_+(\sigma,t)\sin\sigma + \int_{x_+(\sigma,t)}^{\infty} dy \left[\sin\sigma - \sqrt{(\cos\hat{\sigma}_1(y,t) - \cos\sigma)(\cos\sigma - \cos\hat{\sigma}_2(y,t))}\right]. \tag{4.1.26}$$

Lemma 4.17

For $0 \le t < t_b$*,*

$$\Phi(\sigma,t) = \Phi(\sigma), \tag{4.1.27}$$

and

$$\Gamma_+(\sigma,t) = \Gamma_+(\sigma) - t\sin\sigma\cos\sigma. \tag{4.1.28}$$

PROOF:

Differentiating (4.1.24) with respect to t and then using both (4.1.18) and (4.1.24) gives

$$\partial_t x_\pm = \begin{cases} 3\hat{\lambda}_1 + \hat{\lambda}_2 = 3\lambda + \hat{\lambda}_2(x,t) \\ 3\hat{\lambda}_2 + \hat{\lambda}_1 = 3\lambda + \hat{\lambda}_1(x,t) \end{cases} \text{for} \begin{cases} \hat{\lambda}_1(x_+,t) = \hat{\lambda}_1(x_-,t) = \lambda \\ \hat{\lambda}_2(x_+,t) = \hat{\lambda}_2(x_-,t) = \lambda. \end{cases}$$

Differentiating (4.1.25) with respect to t and using (4.1.18), we obtain

(a) If $\hat{\sigma}_{2,min} > \sigma > \hat{\sigma}_{1,max}$, then $\partial_t\Phi(\sigma,t) = 0$.

(b) And if $0 < \sigma < \hat{\sigma}_{1,max}$ or $\hat{\sigma}_{2,min} < \sigma < \pi$, then

$$\partial_t\Phi(\sigma,t)$$

$$= \partial_t\left[\mp\int_{x_-}^{x_+} \sqrt{(\cos\hat{\sigma}_1 - \cos\sigma)(\cos\hat{\sigma}_2 - \cos\sigma)}\,dx\right]$$

$$= \mp \int_{x_-}^{x_+} \partial_t(\sqrt{(\hat\lambda_1 - \lambda)(\hat\lambda_2 - \lambda)})\, dx$$

$$= \mp \int_{x_-}^{x_+} \frac{(\hat\lambda_2 - \lambda)\partial_t\hat\lambda_1 + (\hat\lambda_1 - \lambda)\partial_t\hat\lambda_2}{2\sqrt{(\hat\lambda_1 - \lambda)(\hat\lambda_2 - \lambda)}}\, dx$$

$$= \mp \int_{x_-}^{x_+} \frac{(\hat\lambda_2 - \lambda)(3\hat\lambda_1 + \hat\lambda_2)\partial_x\hat\lambda_1 + (\hat\lambda_1 - \lambda)(3\hat\lambda_2 + \hat\lambda_1)\partial_x\hat\lambda_2}{4\sqrt{(\hat\lambda_1 - \lambda)(\hat\lambda_2 - \lambda)}}\, dx$$

$$= \mp \int_{x_-}^{x_+} \{\frac{(\hat\lambda_1 + \hat\lambda_2 + 2\lambda)[(\hat\lambda_2 - \lambda)\partial_x\hat\lambda_1 + (\hat\lambda_1 - \lambda)\partial_x\hat\lambda_2]}{4\sqrt{(\hat\lambda_1 - \lambda)(\hat\lambda_2 - \lambda)}}$$
$$+ \frac{2(\hat\lambda_1 - \lambda)(\hat\lambda_2 - \lambda)(\partial_x\hat\lambda_1 + \partial_x\hat\lambda_2)}{4\sqrt{(\hat\lambda_1 - \lambda)(\hat\lambda_2 - \lambda)}}\}\, dx$$

$$= \mp(\frac{1}{2}) \int_{x_-}^{x_+} [(\hat\lambda_1 + \hat\lambda_2)\, d\sqrt{(\hat\lambda_1 - \lambda)(\hat\lambda_2 - \lambda)}$$
$$+ 2\lambda\, d\sqrt{(\hat\lambda_1 - \lambda)(\hat\lambda_2 - \lambda)}$$
$$+ \sqrt{(\hat\lambda_1 - \lambda)(\hat\lambda_2 - \lambda)}\, d(\hat\lambda_1 + \hat\lambda_2)] = 0.$$

This proves (4.1.27). Performing the same procedure on (4.1.26) gives

$$\partial_t\Gamma_+(\sigma, t)$$

$$= -\frac{1}{2} \int_{x_+}^{\infty} \frac{\partial_t\hat\lambda_1(\lambda - \hat\lambda_2) + \partial_t\hat\lambda_2(\lambda - \hat\lambda_1)}{\sqrt{(\hat\lambda_1(x,t) - \lambda)(\lambda - \hat\lambda_2(x,t))}}\, dx$$

$$= -\frac{1}{4} \int_{x_+}^{\infty} \frac{(3\hat\lambda_1 + \hat\lambda_2)\partial_x\hat\lambda_1(\lambda - \hat\lambda_2) + (\hat\lambda_1 + 3\hat\lambda_2)\partial_x\hat\lambda_2(\lambda - \hat\lambda_1)}{\sqrt{(\hat\lambda_1(x,t) - \lambda)(\lambda - \hat\lambda_2(x,t))}}\, dx$$

$$= (-\frac{1}{2}) \int_{x_+}^{\infty} [(\hat\lambda_1 + \hat\lambda_2 + 2\lambda)\, d\sqrt{(\hat\lambda_1 - \lambda)(\hat\lambda_2 - \lambda)}$$
$$+ \sqrt{(\hat\lambda_1 - \lambda)(\hat\lambda_2 - \lambda)}\, d(\hat\lambda_1 + \hat\lambda_2)]$$

$$= (-\frac{1}{2})[(\hat{\lambda}_1 + \hat{\lambda}_2) + 2\lambda]\sqrt{(\hat{\lambda}_1 - \lambda)(\hat{\lambda}_2 - \lambda)}|_{x=x_+}^{x=\infty}$$

$$= -\lambda\sqrt{1-\lambda^2} = -\cos\sigma\sin\sigma,$$

which after integrating yields (4.1.28) and establishes Lemma 4.17 .

From (4.1.27) of Lemma 4.17 we see that

$$\phi(\sigma) = -\partial_\sigma \Phi(\sigma, t).$$

Using (4.1.28) with the definition (2.2.45) of a then gives

$$a(\sigma, x, t) = a\sin\sigma - \Gamma_+(\sigma, t).$$

Thus the variational problem (3.1.15)-(3.1.17) for $0 \le t < t_b$ is as the same as the variational problem for $t = 0$ except that $\hat{\sigma}_1(x)$, $\hat{\sigma}_2(x)$ has been replaced by $\hat{\sigma}_1(x,t)$, $\hat{\sigma}_2(x,t)$. By Theorem 4.32, its solution is given by formula (4.1.15) with the same replacements, and Theorem 4.35 follows from Theorem 4.33.

The weak limit in formula (4.1.23) is in fact a strong one.

Theorem 4.36

For $0 \le t < t_b$, both

$$\begin{aligned} \bar{\rho}(x,t) &= L^1\text{-}\lim_{h\to 0} |U(x,t,h)|^2 \\ &= \frac{1}{4}[\hat{\lambda}_1(x,t) - \hat{\lambda}_2(x,t)]^2 \end{aligned}$$

and

$$\begin{aligned} \bar{\mu}(x,t) &= L^1\text{-}\lim_{h\to 0} \frac{ih}{2}(UU_x^* - U_xU^*) \\ &= \frac{1}{4}[\hat{\lambda}_1(x,t) + \hat{\lambda}_2(x,t)][\hat{\lambda}_1(x,t) - \hat{\lambda}_2(x,t)]^2 \end{aligned}$$

are uniformly on compact subsets of t.

In particular this proves Theorem 2.4.

We need some lemmas.

Lemma 4.18

Suppose we have a sequence of functions $V_n(x)$ mapping R into R^D, where D is the dimension; and we have a positive convex functional $S(\cdot)$ mapping $L^1(R^D)$ to R^+ with the form of

$$S(f) = \int_{-\infty}^{\infty} s(f(x))\,dx.$$

Assume V_n converges to $\bar{V}$ in the sense of weak L^1 with respect to $x \in R$, then

$$S(\bar{V}) \le \liminf_{n\to\infty} S(V_n),$$

or in equivalent

$$\int_{-\infty}^{\infty} s(\bar{V}(x))\,dx \le \liminf_{n\to\infty} \int_{-\infty}^{\infty} s(V_n(x))\,dx.$$

PROOF :

Since S is convex, then

$$s(\bar{V}) + (V_n - \bar{V}) s_V(\bar{V}) \le s(V_n), \tag{4.1.29}$$

for all V_n. Set

$$\chi_B = \begin{cases} 1 & |\bar{V}| \le B \\ 0 & \text{else} \end{cases},$$

then $\chi_B s_V(\bar{V}) \in L^\infty$. By (4.1.29),

$$\int_{-\infty}^{\infty} [s(\bar{V})\chi_B + (V_n - \bar{V}) s_V(\bar{V})\chi_B]\,dx \le \int_{-\infty}^{\infty} s(V_n)\,dx.$$

Because $\bar{V}$ is the weak L^1 limit of the sequence $\{V_n\}$, the second term of the left side will vanish after taking the limit. We get

$$\int_{-\infty}^{\infty} s(\bar{V})\chi_B\,dx \le \liminf_{n\to\infty} \int_{-\infty}^{\infty} s(V_n)\,dx$$

for all B. *i.e.*

$$S(\bar{V}) \le \liminf_{n\to\infty} S(V_n).$$

Lemma 4.19

Suppose we have functions $\bar{V}(x)$, $\{V_n(x)\}$ and a functional S defined and with the same property as same as in Lemma 4.18, and suppose

$$S(\bar{V}) = \lim_{n\to\infty} S(V_n),$$

then

$$V_n(x) \to V(x)$$

almost everywhere with respect to x. That means $V_n \to V$ in the sense of L^1 strong convergence with respect to x.

PROOF :

Set

$$\Delta_n(x) \equiv \Delta(\bar{V}, Vn) \equiv s(V_n) - s(\bar{V}) - (V_n - \bar{V})s_V(\bar{V}),$$

then $\Delta_n \geq 0$ because S is convex and, by Lemma 4.18,

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \Delta_n(x)\, dx = 0.$$

Set

$$F(B,n) = \int_{-\infty}^{\infty} \chi_B \Delta_n \, dx, \qquad F(B) = \limsup_{n\to\infty} F(B,n),$$

and calculate

$$\begin{aligned} F(B) &= \limsup_{n\to\infty} \int_{-\infty}^{\infty} \chi_B \Delta(\bar{V}, V_n)\, dx \\ &\leq \limsup_{n\to\infty} \int_{-\infty}^{\infty} [s(V_n) - \chi_B s(\bar{V}) - (V_n - \bar{V})\chi_B s_V(V_n)]\, dx \\ &= \int_{-\infty}^{\infty} (1 - \chi_B) s(\bar{V})\, dx. \end{aligned}$$

The right side of the inequality above is decreasing to zero as B goes to infinity while the left side, $F(B)$, is increasing from zero. That means $F(B) = 0$ for all B. It says that, for all B, the term $\chi_B \Delta(\bar{V}, V_n)$ converges to zero almost everywhere with respect to x because it is non-negative and converges under the meaning of L^1 norm already. That is

$$\lim_{n\to\infty} \Delta(\bar{V}, V_n) = 0$$

almost everywhere. By the convexity of the functional S

$$\lim_{n\to\infty} V_n(x) = \bar{V}(x)$$

almost everywhere. Because V_n is convergent in the L^1 weak sense already, V_n is convergent to $\bar{V}$ in the L^1 strong sense.

Now, we go back to the proof of Theorem 4.36. We set

$$V_n = \begin{pmatrix} 1-\rho \\ \mu \end{pmatrix}, \qquad \bar{V} = \begin{pmatrix} 1-\bar{\rho} \\ \bar{\mu} \end{pmatrix},$$

and

$$\begin{aligned} s(V_n) &= (1-\rho)^2 + \frac{\mu^2}{\rho} \\ &= \varepsilon - h^2\frac{(\rho_x)^2}{4\rho} \\ &\stackrel{\text{def}}{=} s(\rho,\mu), \end{aligned}$$

$$s(\bar{V}) = (1-\bar{\rho})^2 + \frac{\bar{\mu}^2}{\bar{\rho}} \stackrel{\text{def}}{=} s(\bar{\rho},\bar{\mu}),$$

where $n \to \infty$ means $h \to 0$, and ε is the Hamiltonian of the NLS :

$$\varepsilon = (1-\rho)^2 + h^2|U_x|^2 = (1-\rho)^2 + \frac{\mu^2}{\rho} + h^2\frac{(\rho_x)^2}{4\rho}.$$

The S, or s, is convex over either the set of $\{\rho(x,t;h), \mu(x,t;h)\}$ or $\{\bar{\rho}(x,t), \bar{\mu}(x,t)\}$ obviously. Moreover, with the definition of the system given by (1.2.11), S is conserved for $\{\bar{\rho},\bar{\mu}\}$ before the break-time. For $\{(\rho(x,t;h), \mu(x,t;h))\}$, S is not conserved, but

$$\mathcal{H}_2 = \int_{-\infty}^{+\infty} \varepsilon(x,t;h)\, dx$$

is conserved and $s \leq \varepsilon$ at all the time. Now we can use these lemmas to prove Theorem 4.36.

$$\liminf_{h\to 0} S(\rho(\cdot,t;h),\mu(\cdot,t;h))$$

$$= \liminf_{h\to 0} \int_{-\infty}^{\infty} s(\rho(x,t;h),\mu(x,t;h))\,dx$$

$$\leq \liminf_{h\to 0} \int_{-\infty}^{\infty} \varepsilon(x,t;h)\,dx$$

$$= \liminf_{h\to 0} \int_{-\infty}^{\infty} \varepsilon(x,0;h)\,dx$$

$$= \liminf_{h\to 0} \int_{-\infty}^{\infty} [(1-\rho(x,0;h))^2 + \frac{\mu(x,0;h)^2}{\rho(x,0;h)}]\,dx$$

$$\leq \int_{-\infty}^{\infty} s(\bar{\rho}(x,0),\bar{\mu}(x,0))\,dx$$

$$= \int_{-\infty}^{\infty} s(\bar{\rho}(x,t),\bar{\mu}(x,t))\,dx$$

$$= S(\bar{\rho}(\cdot,t),\bar{\mu}(\cdot,t)).$$

However, because S is convex, by Lemma 4.18,

$$S(\bar{\rho}(\cdot,t),\bar{\mu}(\cdot,t)) \leq \liminf_{h\to 0} S(\rho(\cdot,t;h),\mu(\cdot,t;h)),$$

which implies

$$\lim_{h\to 0} S(\rho(x,t;h),\mu(x,t;h))\,dx = S(\bar{\rho}(x,t),\bar{\mu}(x,t))\,dx.$$

uniformly on compact subsets of t, if $t < t_b$. Applying Lemma 4.19, the results of Theorem 4.36 follows.

4.1.3 Dynamics Created by the Variational Condition

In the proof of Theorem 4.35, we once relied on the knowledge that

$$\begin{cases} \hat{\lambda}_1(x,t) &= \bar{\mu}/(2\bar{\rho}) + \sqrt{\bar{\rho}}, \\ \\ \hat{\lambda}_2(x,t) &= \bar{\mu}/(2\bar{\rho}) - \sqrt{\bar{\rho}}, \end{cases}$$

satisfy (4.1.18) for $0 \le t < t_b$. Next, we show how (4.1.18) can be deduced directly from the variational conditions.

Assume that the set I consists of two disjoint intervals, see (4.1.1), where $\hat{\sigma}_1$, $\hat{\sigma}_2$ are two differentiable functions of x and t with relation $0 < \hat{\sigma}_1 < \hat{\sigma}_2 < \pi$. This assumption will be justified a *posteriori.* We claim that the solution of the Riemann-Hilbert problem posed in (3.2.89) of Theorem 3.30 is

$$\begin{aligned} g(\omega) = i(\frac{1+\omega^4}{2\omega^2}) - \\ \frac{i[4+2(\cos\hat{\sigma}_1+\cos\hat{\sigma}_2) - \frac{1}{2}(\cos\hat{\sigma}_1-\cos\hat{\sigma}_2)^2](\frac{1-\omega}{1+\omega})}{2\sqrt{[\frac{2\omega(1+\cos\hat{\sigma}_1)}{(1+\omega)^2}-1][\frac{2\omega(1+\cos\hat{\sigma}_2)}{(1+\omega)^2}-1]}} + \\ \frac{i[2+(1+\cos\hat{\sigma}_1)(1+\cos\hat{\sigma}_2)](\frac{1-\omega^2}{\omega})}{2\sqrt{[\frac{2\omega(1+\cos\hat{\sigma}_1)}{(1+\omega)^2}-1][\frac{2\omega(1+\cos\hat{\sigma}_2)}{(1+\omega)^2}-1]}} - \\ \frac{i\frac{(1-\omega^2)(1+\omega)^2}{\omega^2}}{2\sqrt{[\frac{2\omega(1+\cos\hat{\sigma}_1)}{(1+\omega)^2}-1][\frac{2\omega(1+\cos\hat{\sigma}_2)}{(1+\omega)^2}-1]}}, \end{aligned}$$

where ω is defined inside the same unit disc Ω and the sign of the square root is again chosen to satisfy (4.1.3). The same arguments that applied to $f(\omega)$ given by (4.1.2) can now be applied to $g(\omega)$ to show that g belongs to H^p for $1 \leq p < 2$. For $\omega = e^{i\sigma}$

$$\Re(g) = \begin{cases} -\dfrac{\sin\sigma[\cos^2\sigma + (\cos\sigma - \cos\hat{\sigma}_1)(\cos\sigma - \cos\hat{\sigma}_2) - \frac{1}{2}(\cos\hat{\sigma}_1 + \cos\hat{\sigma}_2)^2]}{\sqrt{(\cos\sigma - \cos\hat{\sigma}_1)(\cos\sigma - \cos\hat{\sigma}_2)}} & \text{on } I, \text{ but } -\hat{\sigma}_1 < \sigma < \hat{\sigma}_1 \\ 0 & \text{off } I \\ +\dfrac{\sin\sigma[\cos^2\sigma + (\cos\sigma - \cos\hat{\sigma}_1)(\cos\sigma - \cos\hat{\sigma}_2) - \frac{1}{2}(\cos\hat{\sigma}_1 + \cos\hat{\sigma}_2)^2]}{\sqrt{(\cos\sigma - \cos\hat{\sigma}_1)(\cos\sigma - \cos\hat{\sigma}_2)}} & \text{on } I, \text{ but } \begin{cases} \hat{\sigma}_2 < \sigma < \pi \text{ or} \\ -\hat{\sigma}_2 > \sigma > -\pi \end{cases} \end{cases} \quad (4.1.30)$$

and

$$\Im(g) = \begin{cases} \cos(2\sigma) & \text{on } I, \begin{cases} -\hat{\sigma}_1 < \sigma < \hat{\sigma}_1 \text{ or} \\ \hat{\sigma}_2 < \sigma < \pi \text{ or} \\ -\hat{\sigma}_2 > \sigma > -\pi \end{cases} \\ \cos(2\sigma) - \\ \dfrac{\sin\sigma[\cos^2\sigma + (\cos\sigma - \cos\hat{\sigma}_1)(\cos\sigma - \cos\hat{\sigma}_2) - \frac{1}{2}(\cos\hat{\sigma}_1 + \cos\hat{\sigma}_2)^2]}{\sqrt{(\cos\hat{\sigma}_1 - \cos\sigma)(\cos\sigma - \cos\hat{\sigma}_2)}} & \text{off } I, \begin{cases} \hat{\sigma}_1 < \sigma < \hat{\sigma}_2 \text{ or} \\ -\hat{\sigma}_1 > \sigma > -\hat{\sigma}_2. \end{cases} \end{cases} \quad (4.1.31)$$

As before, the square root in (4.1.31) changes sign the same as σ does, and it in (4.1.30) makes the fraction remain positive. These relations show that, the conditions given by (3.2.89) are satisfied, $\Re(g)$ is odd with support in I, and $\Im(g)$ is even. We put (4.1.30) into (3.2.88) to obtain

$$\psi_i^* = \begin{cases} \dfrac{-\sin\sigma[\cos^2\sigma+(\cos\sigma-\cos\hat{\sigma}_1)(\cos\sigma-\cos\hat{\sigma}_2)-\frac{1}{2}(\cos\hat{\sigma}_1+\cos\hat{\sigma}_2)^2]}{\sqrt{(\cos\sigma-\cos\hat{\sigma}_1)(\cos\sigma-\cos\hat{\sigma}_2)}} & \\ & -\hat{\sigma}_1 < \sigma < \hat{\sigma}_1 \\ 0 & \text{else} \\ \dfrac{\sin\sigma[\cos^2\sigma+(\cos\sigma-\cos\hat{\sigma}_1)(\cos\sigma-\cos\hat{\sigma}_2)-\frac{1}{2}(\cos\hat{\sigma}_1+\cos\hat{\sigma}_2)^2]}{\sqrt{(\cos\sigma-\cos\hat{\sigma}_1)(\cos\sigma-\cos\hat{\sigma}_2)}} & \\ & \begin{cases} \hat{\sigma}_2 < \sigma < \pi \text{ or} \\ -\hat{\sigma}_2 > \sigma > -\pi \end{cases} \end{cases} \tag{4.1.32}$$

Now, by using the formula (3.2.74) of Theorem 3.29

$$L\psi(\sigma) = \int_0^\sigma H\psi(\tau)\, d\tau$$

with

$$H\psi_i^* = \Im(g)$$

and $\Im(g)$ given by (4.1.31), we have

$$L\psi_i^* = \begin{cases} \sin\sigma\cos\sigma & \text{on } I, \\ \sin\sigma\cos\sigma - & \\ \int_{\hat{\sigma}_1}^{\sigma} \dfrac{\sin\varsigma[\cos^2\varsigma+(\cos\varsigma-\cos\hat{\sigma}_1)(\cos\varsigma-\cos\hat{\sigma}_2)-\frac{1}{2}(\cos\hat{\sigma}_1+\cos\hat{\sigma}_2)^2]}{\sqrt{(\cos\hat{\sigma}_1-\cos\varsigma)(\cos\varsigma-\cos\hat{\sigma}_2)}}\, d\varsigma & \text{off } I. \end{cases} \tag{4.1.33}$$

Carrying out the integral in (4.1.33) gives

$$L\psi_i^* =$$

$$\begin{cases} \sin\sigma\cos\sigma & \text{on } I, \\ \sin\sigma\cos\sigma - & \\ [\cos\sigma + (\cos\hat{\sigma}_1 + \cos\hat{\sigma}_2)/2]\sqrt{(\cos\hat{\sigma}_1 - \cos\sigma)(\cos\sigma - \cos\hat{\sigma}_2)} & \text{off } I. \end{cases}$$

Now ψ_x^* given by (4.1.7) and ψ_t^* given by (4.1.32) must satisfy the compatibility condition $\partial_t\psi_x^* = \partial_x\psi_t^*$. This implies

$$0 = \partial_t\left\{\frac{\sin\sigma[\cos\sigma - \frac{1}{2}(\cos\hat{\sigma}_1 + \cos\hat{\sigma}_2)]}{\sqrt{(\cos\sigma - \cos\hat{\sigma}_1)(\cos\sigma - \cos\hat{\sigma}_2)}}\right\} - \partial_x\left\{\frac{\sin\sigma[\cos^2\sigma + (\cos\sigma - \cos\hat{\sigma}_1)(\cos\sigma - \cos\hat{\sigma}_2) - \frac{1}{2}(\cos\hat{\sigma}_1 + \cos\hat{\sigma}_2)]}{\sqrt{(\cos\sigma - \cos\hat{\sigma}_1)(\cos\sigma - \cos\hat{\sigma}_2)}}\right\}, \tag{4.1.34}$$

for $\sigma \in (0, \hat{\sigma}_1)$ and $\sigma \in (\hat{\sigma}_2, \pi)$. From (4.1.34) we conclude that

$$0 = \frac{\sin\sigma}{[(\lambda - \hat{\lambda}_1)(\lambda - \hat{\lambda}_2)]^{3/2}}\mathcal{W}(\lambda; \hat{\lambda}_1, \hat{\lambda}_2, \partial_t\hat{\lambda}_1, \partial_t\hat{\lambda}_2, \partial_x\hat{\lambda}_1, \partial_x\hat{\lambda}_2), \tag{4.1.35}$$

where

$$\begin{aligned} \mathcal{W} = {} & \partial_t\hat{\lambda}_1[-\frac{3}{4}(\lambda - \hat{\lambda}_1)(\lambda - \hat{\lambda}_2) - \frac{1}{4}(\lambda - \hat{\lambda}_2)^2] \\ & +\partial_t\hat{\lambda}_2[-\frac{3}{4}(\lambda - \hat{\lambda}_1)(\lambda - \hat{\lambda}_2) - \frac{1}{4}(\lambda - \hat{\lambda}_1)^2] \\ & -\frac{1}{2}\partial_x\hat{\lambda}_1[-\frac{3}{2}(\lambda - \hat{\lambda}_1)(\lambda - \hat{\lambda}_2)^2 + \frac{1}{2}(\lambda - \hat{\lambda}_2)(\lambda - \hat{\lambda}_1)^2 \\ & -\frac{1}{2}(\lambda^2 - \hat{\lambda}_1^2)(\lambda - \hat{\lambda}_2) - \frac{1}{2}(\lambda^2 - \hat{\lambda}_2^2)(\lambda - \hat{\lambda}_1) \\ & -\frac{1}{4}(\lambda - \hat{\lambda}_2)(2\lambda + \hat{\lambda}_1 + \hat{\lambda}_2)(2\lambda - \hat{\lambda}_1 - \hat{\lambda}_2)] \\ & -\frac{1}{2}\partial_x\hat{\lambda}_1[-\frac{3}{2}(\lambda - \hat{\lambda}_2)(\lambda - \hat{\lambda}_1)^2 + \frac{1}{2}(\lambda - \hat{\lambda}_1)(\lambda - \hat{\lambda}_2)^2 \\ & -\frac{1}{2}(\lambda^2 - \hat{\lambda}_2^2)(\lambda - \hat{\lambda}_1) - \frac{1}{2}(\lambda^2 - \hat{\lambda}_1^2)(\lambda - \hat{\lambda}_2) \\ & -\frac{1}{4}(\lambda - \hat{\lambda}_1)(2\lambda + \hat{\lambda}_1 + \hat{\lambda}_2)(2\lambda - \hat{\lambda}_1 - \hat{\lambda}_2)], \end{aligned}$$

with

$$\lambda = \cos\sigma,\ \hat{\lambda}_1 = \cos\hat{\sigma}_1,\ \hat{\lambda}_2 = \cos\hat{\sigma}_2.$$

Since (4.1.35) is true for all $\sigma \in I$, so is $\mathcal{W} = 0$ for all such σ. Let $\lambda \to \hat{\lambda}_1$ and then $\lambda \to \hat{\lambda}_2$, we get from the equility $\mathcal{W} = 0$ that

$$\begin{cases} \partial_t\hat{\lambda}_1 & - & \frac{1}{2}(3\hat{\lambda}_1 + \hat{\lambda}_2)\partial_x\hat{\lambda}_1 & = & 0 \\ & & & & \\ \partial_t\hat{\lambda}_2 & - & \frac{1}{2}(3\hat{\lambda}_2 + \hat{\lambda}_1)\partial_x\hat{\lambda}_2 & = & 0. \end{cases}$$

This is just the relation between the first couple of Riemann Invariants given by (4.1.18), which is valid for $t < t_b$.

The above procedure will be generalized in the next section to obtain the temporal dependence of solutions of the variational problem for $t > t_b$, and a general system of the Riemann Invariants for all the time.

Similarly, set ψ_x^* given by (4.1.7) and ψ_t^* given by (4.1.32) into the formulas of Theorem 3.31. A brief calculation then shows

$$\begin{aligned} &\bar{\varepsilon}(x,t) = \\ &\quad -2(\sin\sigma, \psi_x^*) + 2(\sin\sigma, \psi_x^*)^2 + 2(\sin\sigma\cos\sigma, \psi_t^*) \\ &\quad = [1 - \frac{1}{4}(\hat{\lambda}_1 - \hat{\lambda}_2)^2]^2 + \frac{1}{4}(\hat{\lambda}_1 + \hat{\lambda}_2)^2(\hat{\lambda}_1 - \hat{\lambda}_2)^2, \end{aligned}$$

or

$$\begin{aligned} &\bar{\varepsilon}(x,t) = \\ &\quad 2(\sin\sigma\cos^2\sigma, \psi_x^*(x,t)) \\ &\quad = [1 - \frac{1}{4}(\hat{\lambda}_1 - \hat{\lambda}_2)^2]^2 + \frac{1}{4}(\hat{\lambda}_1 + \hat{\lambda}_2)^2(\hat{\lambda}_1 - \hat{\lambda}_2)^2. \end{aligned} \tag{4.1.36}$$

4.2 Solution Beyond Break-time

4.2.1 Solving Differentiated Variational Condition

In this section we describe the solution of the VP (3.1.15)-(3.1.17) when t is larger than the break-time t_b. We make the assumption that the set I consists of a finite union of disjoint intervals, say

$$I = (0, \hat{\sigma}_1) \cup (\hat{\sigma}_2, \hat{\sigma}_3) \cup \ldots \cup (\hat{\sigma}_{2n}, \pi), \tag{4.2.37}$$

total $(n+1)$ intervals. We may also use an arc representation :

$$\hat{I} = \{e^{i\sigma} : \sigma \in I\}, \tag{4.2.38}$$

where

$$0 < \hat{\sigma}_1 < \ldots < \hat{\sigma}_{2n} < \pi.$$

So, I is completely determined by its set of positive end points.

$$\Sigma = \{\hat{\sigma}_k\}_{k=1}^{k=2n} \tag{4.2.39}$$

As the first step, we shall construct all possible solutions of the Riemann-Hilbert problems (3.2.87) and (3.2.89) that ψ_x^* and ψ_t^* satisfy. The solutions will be expressed in terms of a function $R(\omega)$ defined by

$$R^2(w) = \prod_{k=1}^{k=2n} [\frac{\omega}{(1+\omega)^2} - \frac{\hat{\omega}_k}{(1+\hat{\omega}_k)^2}], \tag{4.2.40}$$

where

$$\hat{\omega}_k = e^{i\hat{\sigma}_k}.$$

We choose its sign such that

$$\begin{aligned} R(\omega = e^{i\sigma}) &> 0 \quad \text{for} \quad \sigma \in (0, \hat{\sigma}_1), \\ iR(\omega = e^{i\sigma}) &< 0 \quad \text{for} \quad \sigma \in (\hat{\sigma}_1, \hat{\sigma}_2). \end{aligned} \tag{4.2.41}$$

Lemma 4.20

All functions of the form

$$f(\omega) = i(\frac{1+\omega^2}{2\omega}) - \frac{P(\omega)}{R(\omega)} \tag{4.2.42}$$

and

$$g(\omega) = i(\frac{1+\omega^4}{2\omega^2}) + \frac{Q(\omega)}{R(\omega)} \tag{4.2.43}$$

are solutions of the Riemann-Hilbert problems (3.2.87) and (3.2.89), respectively, where $P(\omega)$ and $Q(\omega)$ are rational fractions with pure imagenary coefficients. They are

$$P(\omega) = \frac{i}{2}R(0)\{(\frac{1-\omega^2}{\omega}) + \sum_{m=1}^{m=n} \alpha_m(\frac{1-\omega}{1+\omega})[\frac{\omega}{(1+\omega)^2}]^{m-1}\} \tag{4.2.44}$$

and

$$\begin{aligned} Q(\omega) &= -\frac{i}{2}R(0)\{[\frac{R'(0)}{R(0)} - 2](\frac{1-\omega^2}{\omega}) + \frac{(1-\omega^2)(1+\omega)^2}{\omega^2} \\ &+ \sum_{m=1}^{m=n} \beta_m(\frac{1-\omega}{1+\omega})[\frac{\omega}{(1+\omega)^2}]^{m-1}\}, \end{aligned} \tag{4.2.45}$$

where $\{\alpha_m\}_1^n$ and $\{\beta_m\}_1^n$ are real and determined by $\{\hat{\sigma}_k\}_1^{2n}$; and $R'(0) = \partial_\omega R_{\omega=0}$.

PROOF :

It is not hard to show that these all are solutions, but we prefer to describe how one constructs them.

First, f and g should be complex functions which are analytic inside of the set $\Omega = \{\omega : |\omega| < 1\}$. By (3.2.87) and (3.2.89), the boundry value of f and g should be

$$\begin{cases} \Re[f(e^{i\sigma})] = 0 & \sigma \notin I \\ \Im[f(e^{i\sigma})] = \cos\sigma = (\frac{1+\omega^2}{2\omega})_{\omega\in\partial\Omega} \stackrel{\text{def}}{=} f_2(\omega)_{\omega\in\partial\Omega} & \sigma \in I \end{cases}$$

and

$$\begin{cases} \Re[g(e^{i\sigma})] = 0 & \sigma \notin I \\ \Im[g(e^{i\sigma})] = \cos(2\sigma) = (\frac{1+\omega^4}{2\omega^2})_{\omega\in\partial\Omega} \stackrel{\text{def}}{=} g_2(\omega)_{\omega\in\partial\Omega} & \sigma \in I. \end{cases}$$

It is expected that

$$f(\omega) = -f_1(\omega) + if_2(\omega),$$

$$g(\omega) = g_1(\omega) + ig_2(\omega),$$

where

$$f_1(\omega) = \frac{P(\omega)}{R(\omega)}, \qquad g_1(\omega) = \frac{Q(\omega)}{R(\omega)},$$

are such functions that on the boundary both P and Q remain real values, while R takes real value only on I and takes pure imaginary value out of I.

By the WKB method, when $n = 1$,

$$R(\omega)_{\omega\in\partial\Omega} \approx \sqrt{(\lambda - \hat{\lambda}_1)(\lambda - \hat{\lambda}_2)} = \sqrt{(\cos\sigma - \cos\hat{\sigma}_1)(\cos\sigma - \cos\hat{\sigma}_2)}.$$

Thus we may guess in general case,

$$R(\omega)_{\omega\in\partial\Omega} \approx \sqrt{\prod_{k=1}^{2n}(\cos\sigma - \cos\hat{\sigma}_k)}.$$

That leads to (4.2.40) :

$$R(w) = \sqrt{\prod_{k=1}^{k=2n}[\frac{\omega}{(1+\omega)^2} - \frac{\hat{\omega}_k}{(1+\hat{\omega}_k)^2}]}.$$

It works on the boundary as required and is analytic inside of the disk Ω.

REMARK :

If we do not require the analytic property, then the construction of R is not unique. For example, one may choose

$$R(w) = \sqrt{\prod_{k=1}^{k=2n}(\frac{1+\omega^2}{\omega} - \frac{1+\hat{\omega}_k^2}{\hat{\omega}_k})}.$$

However, it is not good for creating a H^p function.

Next, we construct $P(\omega)$ and $Q(\omega)$. Since both f_2 and g_2 get pole at $\omega = 0$, therefore in order to be balanced, P and Q need pole at $\omega = 0$ too, because we want

f, g to be analytic. We try the forms of P, Q as the following

$$\begin{aligned} P &= f_p + f_a \\ Q &= g_p + g_a, \end{aligned}$$

such that both $f_p(\omega)$ and $g_p(\omega)$ get pole at $\omega = 0$, and both $f_a(\omega)$ and $g_a(\omega)$ are analytic inside of Ω. Set

$$\begin{aligned} f_m &\equiv if_2 - f_p/R \\ g_m &\equiv ig_2 + g_p/R, \end{aligned}$$

they should be analytic inside of Ω as well. Since as $\omega \to 0$

$$f_2 \approx \frac{1}{\omega}, \qquad g_2 \approx \frac{1}{\omega^2},$$

we guess

$$f_p = c_0 \frac{(1-\omega)(1+\omega)}{\omega},$$

$$g_p = d_0 \frac{(1-\omega)(1+\omega)}{\omega} + d_1 \frac{(1-\omega)(1+\omega)^3}{\omega^2}.$$

After a little calculation of $f_m(0)$, $g_m(0)$ and $g'_m(0)$, with the fact that both f_m and g_m should be analytic at $\omega = 0$, we obtain

$$\begin{cases} c_0 = (\frac{i}{2})R(0) \\ d_0 = -(\frac{i}{2})R(0)[\frac{R'(0)}{R(0)} - 2] \\ d_1 = -(\frac{i}{2})R(0). \end{cases}$$

By (4.2.40), it is not difficult to get :

$$\begin{cases} R(0) = 1/\sqrt{\prod_{k=1}^{2n}[2(1+\cos\hat{\sigma}_k)]}, \\ R'(0)/R(0) = \partial_\omega(\log R)_{\omega=0} = -\sum_{k=1}^{2n}(1+\cos\hat{\sigma}_k). \end{cases} \tag{4.2.46}$$

Now, we can go to construct $f_a(\omega)$ and $g_a(\omega)$.

Notice that both $\Im(f)$ and $\Im(g)$ are even functions of σ where they sit on the boundary, so $\Re(f)$ and $\Re(g)$ must be odd functions of σ on the boundary by the Hilbert transform (3.2.74), (3.2.75) and (3.2.77). We try the following form

$$\{(-i\frac{1-\omega}{1+\omega})[\frac{\omega}{(1+\omega)^2}]^k\}_{\partial\Omega} = (\tan\frac{\sigma}{2})[\frac{1}{2(1+\cos\sigma)}]^k$$

for f_a and g_a. Thus the forms of f_a, g_a should be

$$\begin{aligned} f_a &= \frac{i}{2}R(0)\sum_{m=1}^{m=n}\alpha_m(\frac{1-\omega}{1+\omega})[\frac{\omega}{(1+\omega)^2}]^{m-1} \\ g_a &= -\frac{i}{2}R(0)\sum_{m=1}^{m=n}\beta_m(\frac{1-\omega}{1+\omega})[\frac{\omega}{(1+\omega)^2}]^{m-1}. \end{aligned}$$

It completes the construction of f, g. Since they both have singularities at most only of the form $\sqrt{(\cos\sigma - \cos\hat{\sigma}_k)}$ on the boundary, they belong to $H^p(\Omega)$ for $1 \le p < 2$.

For boundary points $\omega = e^{i\sigma}$, the function $R(\omega)$ defined by (4.2.40) is real and even with respect to σ on I; and is pure imaginary and odd where σ off I. Here the I, a union of intervals, is defined by (4.2.37).

The functions $P(e^{i\sigma})$ and $Q(e^{i\sigma})$ are both real and odd with respect to σ. It follows from (4.2.42) and (4.2.43) that

$$\Re(f) = \begin{cases} -\frac{P(e^{i\sigma})}{R(e^{i\sigma})} & \text{on } I \\ 0 & \text{off } I \end{cases} \tag{4.2.47}$$

and

$$\Im(f) = \begin{cases} \cos\sigma & \text{on } I \\ \cos\sigma + i\frac{P(e^{i\sigma})}{R(e^{i\sigma})} & \text{off } I; \end{cases} \tag{4.2.48}$$

while

$$\Re(g) = \begin{cases} \frac{Q(e^{i\sigma})}{R(e^{i\sigma})} & \text{on } I \\ 0 & \text{off } I \end{cases}$$

and

$$\Im(g) = \begin{cases} \cos(2\sigma) & \text{on} \quad I \\ \\ \cos(2\sigma) - i\frac{Q(e^{i\sigma})}{R(e^{i\sigma})} & \text{off} \quad I. \end{cases} \tag{4.2.49}$$

These relations show that $f(\omega)$ and $g(\omega)$ satisfy the conditions (3.2.87) and (3.2.89) as are asserted in Lemma 4.20.

Using (3.2.86) and (3.2.88) gives

$$\psi_x^* = -\Re\left(\frac{P(e^{i\sigma})}{R(e^{i\sigma})}\right) \tag{4.2.50}$$

and

$$\psi_t^* = \Re\left(\frac{Q(e^{i\sigma})}{R(e^{i\sigma})}\right). \tag{4.2.51}$$

We now show how to choose the coefficients α_m and β_m so that the differentiated variational conditions, (3.2.83) and (3.2.84), are satisfied. Using formula (3.2.74) with

$$H\psi_x^* = \Im(f)$$

given by (4.2.48), and with

$$H\psi_t^* = \Im(g)$$

given by (4.2.49), we obtain

$$L\psi_x^* = \sin\sigma - \int_0^\sigma \Im\left(\frac{P(e^{i\varsigma})}{R(e^{i\varsigma})}\right) d\varsigma \tag{4.2.52}$$

and

$$L\psi_t^* = \sin\sigma\cos\sigma + \int_0^\sigma \Im\left(\frac{Q(e^{i\varsigma})}{R(e^{i\varsigma})}\right) d\varsigma. \tag{4.2.53}$$

In order to satisfy the differentiated variational conditions (3.2.83) (3.2.84), the integrals on the right hand sides of (4.2.52) and (4.2.53) must vanish on I. That is

$$\int_{\hat\sigma_{2k-1}}^{\hat\sigma_{2k}} \frac{P(e^{i\sigma})}{R(e^{i\sigma})}\, d\sigma = 0 \tag{4.2.54}$$

and

$$\int_{\hat{\sigma}_{2k-1}}^{\hat{\sigma}_{2k}} \frac{Q(e^{i\sigma})}{R(e^{i\sigma})}\, d\sigma = 0 \tag{4.2.55}$$

for $k = 1, 2, ..., n$. Recalling the form of P and Q in (4.2.44) and (4.2.45), we see that (4.2.54) is a system of n inhomogeneous linear equations for the n real unknowns $\alpha_1, \alpha_2, ..., \alpha_n$ and (4.2.55) is a system of n inhomogeneous linear equations for the n real unknowns $\beta_1, \beta_2, ..., \beta_n$ as well.

Lemma 4.21

The systems of equations (4.2.54) and (4.2.55) have unique solutions.

PROOF :

Because of similarity, we only need to argue with the system of equations (4.2.54). As one did in KdV case, we have two way to prove it.

First Proof :

If this were not true, then there would be two nontrivial solutions of this algebra system, leading to two different f_a's, say $f_a^{(1)}$ and $f_a^{(2)}$. Set

$$N = f_a^{(1)} - f_a^{(2)},$$

and

$$\psi_0(\sigma) = \Re\left(\frac{N(e^{i\sigma})}{R(e^{i\sigma})}\right).$$

So $\psi_0 \not\equiv 0$ and $\psi_0 \in H^p(\Omega)$, $1 \le p < 2$. However, by (4.2.47) and (4.2.52), we obtain

$$L\psi_0 = \int_0^{\sigma} \Im\left(\frac{N(e^{i\sigma})}{R(e^{i\sigma})}\right) = 0 \qquad \text{on } I$$

$$\psi_0 = 0 \qquad \text{off } I.$$

According to the uniqueness argument given for (3.2.73), such a ψ_0 is zero, and thus $N(e^{i\sigma})$ is zero, a contradiction.

Second Proof :

We can write (4.2.54) and (4.2.55) more explicitly as following :

$$0=\int_{\hat{\sigma}_{2k-1}}^{\hat{\sigma}_{2k}}\frac{d\sigma}{R(e^{i\sigma})}[2\sin\sigma+\frac{\sin\sigma}{1+\cos\sigma}\sum_{m=1}^{m=n}\alpha_m(\frac{1}{2(1+\cos\sigma)})^{m-1}], \tag{4.2.56}$$

and

$$0=\int_{\hat{\sigma}_{2k-1}}^{\hat{\sigma}_{2k}}\frac{d\sigma}{R(e^{i\sigma})}$$

$$\cdot\left[(\frac{R'(0)}{R(0)}+2\cos\sigma)\sin\sigma+\frac{\sin\sigma}{2(1+\cos\sigma)}\sum_{m=1}^{m=n}\beta_m\left(\frac{1}{2(1+\cos\sigma)}\right)^{m-1}\right],$$

for $k=1,2,...,n$. Here all the coefficients $\{\alpha_m\}_1^n$, $\{\beta_m\}_1^n$ are real. It makes clear that both P, Q are odd functions of σ. If the claim of Lemma 4.21 were not true, then there would be a nontrivial function $N(\sigma)$ which satisfies

$$0=\int_{\hat{\sigma}_{2k-1}}^{\hat{\sigma}_{2k}}\frac{N(\sigma)}{R(e^{i\sigma})}\left(\frac{\sin\sigma}{1+\cos\sigma}\right)d\sigma,$$

for $k=1,2,...,n$, with a form

$$N(\sigma)=\sum_{m=1}^{m=n}\gamma_m\delta^{m-1}(\sigma), \tag{4.2.57}$$

and

$$\delta=1/[2(1+\cos\sigma)].$$

Since $\sin\sigma/[R(\sigma)(1+\cos\sigma)]$ remains same sign on $(\hat{\sigma}_{2k-1},\hat{\sigma}_{2k})$, for all k, $N(\sigma)$ must have at least one root in each of the n intervals. Since δ is a nonzero monotonic mapping over $(0,\pi)$, thus the degree of $N(\sigma)$, if see it as a polynomial of δ with real coefficients, must be at least n; or $N(\sigma)\equiv 0$. However, by (4.2.57), the degree of $N(\delta)$ is at most $n-1$. It comes a contradiction. The proof of Lemma 4.21 is then completed.

According to (4.2.56),

$$P(e^{i\sigma})=2\sin\sigma[1+\sum_{k=1}^{n}\alpha_k\delta^k(\sigma)]\stackrel{\text{def}}{=}P(\sigma).$$

Since $\sin\sigma$ remains same sign on each $(\hat{\sigma}_{2k-1}, \hat{\sigma}_{2k})$, and $P/(2\sin\sigma)$ is a polynomial of degree n with respect to δ, so it has at most n roots. With same argument applied to $N(\sigma)$, $P(\sigma)$ has at least one root in each of the n intervals $(\hat{\sigma}_{2k-1}, \hat{\sigma}_{2k})$. In fact we conclude

Corollary 4.6
The function $P(\sigma)$, if odd extended to $(-\pi,\pi)$, has one and only one zero inside of each of the intervals $(\hat{\sigma}_{2k-1}, \hat{\sigma}_{2k})$, $(-\hat{\sigma}_{2k}, -\hat{\sigma}_{2k-1})$ for $k = 1, 2, ..., n$; one more zero at $\sigma = 0$; and nowhere else.

By Lemma 4.21, we can solve (4.2.54) and (4.2.55) uniquely for $\{\alpha_k\}_1^n$ and $\{\beta_k\}_1^n$. These solutions can be written as functions of $\hat{\sigma}_k \in \Sigma$ which is given by (4.2.39). Since the coefficients and the inhomogeneous terms appearing in the linear systems (4.2.54) and (4.2.55) are composed of complete hyperelliptic integrals, thus both $\{\alpha_k\}_1^n$ and $\{\beta_k\}_1^n$ are ratios of determinates whose entries are such integrals.

We have now proved

Theorem 4.37
Given the set Σ, the function ψ_x^*, as determined by (4.2.50) and (4.2.54), is the solution of the differentiated variational conditions (3.2.83); similarly ψ_t^*, as determined by (4.2.51) and (4.2.55), is the solution of the differentiated variational conditions (3.2.84).

4.2.2 Dynamics Created by the Variational Condition

The functions ψ_x^* and ψ_t^* need to satisfy the compatibility condition $\partial_t\psi_x^* = \partial_x\psi_t^*$; or equivalently,

$$\partial_t\left(\frac{P(\omega)}{R(\omega)}\right) + \partial_x\left(\frac{Q(\omega)}{R(\omega)}\right) = 0. \tag{4.2.58}$$

We shall see what conditions this imposes on the set Σ.

Rewrite the left side of (4.2.58) as

$$\partial_t \left(\frac{P(\omega)}{R(\omega)}\right) + \partial_x \left(\frac{Q(\omega)}{R(\omega)}\right) = \frac{R^2(\partial_t P + \partial_x Q) - \frac{1}{2}(P\partial_t R^2 + Q\partial_x R^2)}{R^3}. \tag{4.2.59}$$

Since $R^2 = 0$ at $\omega = \hat{\omega}_k$ for $k = 1, 2, ..., n$, the numerator on the right of (4.2.59) vanishes at those points if and only if so does its second term. That second term may be written as

$$\frac{1}{2}(P\partial_t R^2 + Q\partial_x R^2) =$$

$$\sum_{k=1}^{k=2n} \{[P(\omega)\partial_t(\frac{\hat{\omega}_k}{(1+\hat{\omega}_k)^2}) + Q(\omega)\partial_x(\frac{\hat{\omega}_k}{(1+\hat{\omega}_k)^2})]$$

$$\cdot \prod_{j\neq k}[\frac{\hat{\omega}_j}{(1+\hat{\omega}_j)^2} - \frac{\omega}{(1+\omega)^2}]\}.$$

Clearly it vanishes at $\omega = \hat{\omega}_k$ if and only if $\hat{\omega}_k$ satisfies

$$P(\hat{\omega}_k)\partial_t \left[\frac{\hat{\omega}_k}{(1+\hat{\omega}_k)^2}\right] + Q(\hat{\omega}_k)\partial_x \left[\frac{\hat{\omega}_k}{(1+\hat{\omega}_k)^2}\right] = 0. \tag{4.2.60}$$

This evolution of $\hat{\omega}_k$ allows us to factor $R^2(\omega)$ from the numerator of (4.2.59) and to write down

$$\partial_t \left(\frac{P(\omega)}{R(\omega)}\right) + \partial_x \left(\frac{Q(\omega)}{R(\omega)}\right) = \frac{N(\omega)}{R(\omega)}, \tag{4.2.61}$$

where $N(\omega)$ is given by

$$N(\omega) = \partial_t P(\omega) + \partial_x Q(\omega) + \sum_{k=1}^{k=2n} \left\{ \frac{P(\omega) - P(\hat{\omega}_k)}{[\frac{\omega}{(1+\omega)^2} - \frac{\hat{\omega}_k}{(1+\hat{\omega}_k)^2}]}\partial_t \left[\frac{\hat{\omega}_k}{(1+\hat{\omega}_k)^2}\right] + \frac{Q(\omega) - Q(\hat{\omega}_k)}{[\frac{\omega}{(1+\omega)^2} - \frac{\hat{\omega}_k}{(1+\hat{\omega}_k)^2}]}\partial_x \left[\frac{\hat{\omega}_k}{(1+\hat{\omega}_k)^2}\right]\right\}. \tag{4.2.62}$$

We shall show that $N(\omega) \equiv 0$.

By (4.2.61), $N(\omega)$ is analytic inside of Ω except with the pole $\omega = 0$ at most order 2, because so does the left side. In order to show $N(\omega) \equiv 0$, we only need to verify that N vanishes at the boundary of Ω, because it is equivalent to showing that the analytic one $\omega^2 N(\omega)$ vanishes.

For ϵ sufficiently small we employ a fundamental theorem of calculus to obtain

$$\begin{aligned}
&\partial_t \int_{\hat\sigma_{2k-1}+\epsilon}^{\hat\sigma_{2k}-\epsilon} \frac{P(e^{i\sigma})}{R(e^{i\sigma})}\, d\sigma + \partial_x \int_{\hat\sigma_{2k-1}+\epsilon}^{\hat\sigma_{2k}-\epsilon} \frac{Q(e^{i\sigma})}{R(e^{i\sigma})}\, d\sigma \\
&= \int_{\hat\sigma_{2k-1}+\epsilon}^{\hat\sigma_{2k}-\epsilon} [\partial_t \frac{P(e^{i\sigma})}{R(e^{i\sigma})} + \partial_x \frac{Q(e^{i\sigma})}{R(e^{i\sigma})}]\, d\sigma \\
&+ \frac{P(e^{i(\hat\sigma_{2k}-\epsilon)})\partial_t\hat\sigma_{2k} + Q(e^{i(\hat\sigma_{2k}-\epsilon)})\partial_x\hat\sigma_{2k}}{R(e^{i(\hat\sigma_{2k}-\epsilon)})} \\
&- \frac{P(e^{i(\hat\sigma_{2k-1}+\epsilon)})\partial_t\hat\sigma_{2k-1} + Q(e^{i(\hat\sigma_{2k-1}+\epsilon)})\partial_x\hat\sigma_{2k-1}}{R(e^{i(\hat\sigma_{2k-1}+\epsilon)})},
\end{aligned} \tag{4.2.63}$$

for $k = 1, 2, ..., n$. As ϵ goes to zero, the denominator of the second and third terms on the right side of (4.2.63) will, by (4.2.40), vanish like $\epsilon^{1/2}$, but, by (4.2.60), the numerators of the same terms are $O(\epsilon)$; thus the limit of those terms as $\epsilon \longrightarrow 0$ is zero. Using (4.2.61) in the first term, we then obtain

$$\begin{aligned}
&\lim_{\epsilon\to 0}\left[\partial_t \int_{\hat\sigma_{2k-1}+\epsilon}^{\hat\sigma_{2k}-\epsilon} \frac{P(e^{i\sigma})}{R(e^{i\sigma})}\, d\sigma + \partial_x \int_{\hat\sigma_{2k-1}+\epsilon}^{\hat\sigma_{2k}-\epsilon} \frac{Q(e^{i\sigma})}{R(e^{i\sigma})}\, d\sigma\right] \\
&= \int_{\hat\sigma_{2k-1}}^{\hat\sigma_{2k}} \frac{N(e^{i\sigma})}{R(e^{i\sigma})}\, d\sigma,
\end{aligned} \tag{4.2.64}$$

uniformly on compact subsets of x and t. On the other hand, by (4.2.54) and (4.2.55),

$$\begin{aligned}
\lim_{\epsilon\to 0} \textstyle\int_{\hat\sigma_{2k-1}+\epsilon}^{\hat\sigma_{2k}-\epsilon} \frac{P(e^{i\sigma})}{R(e^{i\sigma})}\, d\sigma &= 0 \\
\lim_{\epsilon\to 0} \textstyle\int_{\hat\sigma_{2k-1}+\epsilon}^{\hat\sigma_{2k}-\epsilon} \frac{Q(e^{i\sigma})}{R(e^{i\sigma})}\, d\sigma &= 0,
\end{aligned} \tag{4.2.65}$$

uniformly on compact subsets of x and t. Since the divergence is a closed operator, we conclude from (4.2.64) and (4.2.65) that

$$\int_{\hat\sigma_{2k-1}}^{\hat\sigma_{2k}} \frac{N(e^{i\sigma})}{R(e^{i\sigma})}\, d\sigma = 0. \tag{4.2.66}$$

Recall (4.2.44) and (4.2.45) with $\omega = e^{i\sigma}$, we obtain

$$
\begin{aligned}
&P(e^{i\sigma}) = \\
&\frac{i}{2}R(0)\{-i\sin\sigma + (-i\sin\sigma)\sum_{k=1}^{k=n}\alpha_k[\frac{1}{2(1+\cos\sigma)}]^k\} = \\
&\frac{R(0)}{2}(\frac{2\tan\frac{\sigma}{2}}{1+\tan^2\frac{\sigma}{2}})\left[1+\sum_{k=1}^{k=n}\alpha_k(\frac{1+\tan^2\frac{\sigma}{2}}{4})^k\right] = \\
&\frac{R(0)}{2}(\frac{2z}{1+z^2})\left[1+\sum_{k=1}^{k=n}\alpha_k(\frac{1+z^2}{4})^k\right] \stackrel{\text{def}}{=} \frac{P_1(z)}{1+z^2},
\end{aligned}
\tag{4.2.67}
$$

where we made Cayley transform $z = \tan(\sigma/2)$, a mapping making $(-\pi,\pi)$ onto $(-\infty,\infty)$. It is easy to see that $P_1(z)$ is odd and a polynomial of degree at most $2n+1$. Similarly, we obtain

$$
\begin{aligned}
&Q(e^{i\sigma}) = \\
&(-\frac{i}{2}R(0)\{(-i\sin\sigma)[\frac{R'(0)}{R(0)}-2] + (-i\sin\sigma)(2+2\cos\sigma) + \\
&(-i\sin\sigma)\sum_{k=1}^{k=n}\beta_k[\frac{1}{2(1+\cos\sigma)}]^k\} = \\
&(-\frac{i}{2}R(0)(\frac{2z}{1+z^2})\left[(\frac{R'(0)}{R(0)}-2)+\frac{4}{1+z^2}+\sum_{k=1}^{k=n}\beta_k(\frac{1+z^2}{4})^k\right] \\
&\stackrel{\text{def}}{=} \frac{Q_1(z)}{(1+z^2)^2},
\end{aligned}
\tag{4.2.68}
$$

where $Q_1(z)$ is odd and a polynomial of degree at most $2n+3$. By (4.2.46),

$$
R(0) = \sqrt{\prod_{k=1}^{k=2n}[(1+\hat{z}_k^2)/4]} = 2^{-2n}\prod_{k=1}^{k=2n}(1+\hat{z}_k^2)^{1/2},
$$

and

$$
\begin{aligned}
\frac{R'(0)}{R(0)} - 2 &= [\partial_\omega \log R(\omega)]_{\omega=0} - 2 \\
&= -\sum_{k=1}^{k=2n}(1+\cos\hat{\sigma}_k) - 2 \\
&= -2\sum_{k=1}^{k=2n}(\frac{1}{1+\hat{z}_k^2}) - 2,
\end{aligned}
\tag{4.2.69}
$$

where $\hat{z}_k = \tan(\hat{\sigma}_k/2)$.

Now we go back to the definition of $N(e^{i\sigma})$, which will be denoted as $N(\sigma)$ here after for short. By (4.2.62), if (4.2.60) is true, $N(\sigma)$ is an odd function of σ since so are both P and Q. Combine with (4.2.67), (4.2.68) and (4.2.69), we obtain

$$
\begin{aligned}
&N(\sigma) = \\
&\partial_t P(e^{i\sigma}) + \partial_x Q(e^{i\sigma}) \\
&+\frac{1}{2}\sum_{k=1}^{k=2n}\left[\frac{P(e^{i\sigma}) - P(e^{i\hat{\sigma}_k})}{\frac{1}{2(1+\cos\sigma)} - \frac{1}{2(1+\cos\hat{\sigma}_k)}}\partial_t\left(\frac{1}{2(1+\cos\hat{\sigma}_k)}\right)\right. \\
&\left.+\frac{Q(e^{i\sigma}) - Q(e^{i\hat{\sigma}_k})}{\frac{1}{2(1+\cos\sigma)} - \frac{1}{2(1+\cos\hat{\sigma}_k)}}\partial_x\left(\frac{1}{2(1+\cos\hat{\sigma}_k)}\right)\right] \\
&= \partial_t\left(\frac{P_1(z)}{1+z^2}\right) + \partial_x\left(\frac{Q_1(z)}{(1+z^2)^2}\right) \\
&+\frac{1}{2}\sum_{k=1}^{k=2n}\left[\frac{\frac{P_1(z)}{1+z^2} - \frac{P_1(\hat{z}_k)}{1+\hat{z}_k^2}}{\frac{1+z^2}{4} - \frac{1+\hat{z}_k^2}{4}}\partial_t\left(\frac{1+\hat{z}_k^2}{4}\right)\right. \\
&\left.+\frac{\frac{Q_1(z)}{(1+z^2)^2} - \frac{Q_1(\hat{z}_k)}{(1+\hat{z}_k^2)^2}}{\frac{1+z^2}{4} - \frac{1+\hat{z}_k^2}{4}}\partial_x\left(\frac{1+\hat{z}_k^2}{4}\right)\right] \\
&= \frac{1}{(1+z^2)^2}\left\{\partial_t[(1+z^2)P_1(z)] + \partial_x Q_1(z)\right. \\
&+\sum_{k=1}^{k=2n}\left[\frac{(1+z^2)P_1(z) - \frac{(1+z^2)^2}{1+\hat{z}_k^2}P_1(\hat{z}_k)}{z^2 - \hat{z}_k^2}\hat{z}_k\partial_t\hat{z}_k\right. \\
&\left.\left.+\frac{Q_1(z) - (\frac{1+z^2}{1+\hat{z}_k^2})^2 Q_1(\hat{z}_k)}{z^2 - \hat{z}_k^2}\hat{z}_k\partial_x\hat{z}_k\right]\right\} \\
&\stackrel{\text{def}}{=} \frac{1}{(1+z^2)^2}N_1(z).
\end{aligned} \tag{4.2.70}
$$

Obviously, $N_1(z)$ is a polynomial of degree at most of order $(2n+3)$. It is easy to see that $N_1(\pm i) = 0$ and $\partial_z N_1(\pm i) = 0$. That means, as a function of z, $N(\sigma)$ is a polynomial too, and with degree at most of order $(2n-1)$. We denote it as $\hat{N}(z)$.

Recall (4.2.66), in each interval $(\hat{\sigma}_{2k-1}, \hat{\sigma}_{2k})$, $N(\sigma)$ changes sign at least one time. That means N has at least n roots, say $\{\sigma_m\}_1^n$, on the domain $(0,\pi)$ with one in each

$(\hat{\sigma}_{2k-1}, \hat{\sigma}_{2k})$. Because N is an odd function of σ, zero and $\{-\sigma_m\}_1^n$ are roots of N as well. Thus, as a function of z, $\hat{N}$ get roots of $z = 0$ and $z = \pm z_m \equiv \tan(\pm\sigma_m/2)$, where $m = 1, 2, ..., n$. So $\hat{N}(z)$ has degree at least of order $(2n+1)$. It contradicts with (4.2.70). That means $\hat{N}(z)$ or $N(\sigma)$ must be zero. In fact, we have shown

Theorem 4.38

If all the $\hat{\sigma}_k$ in Σ satisfy (4.2.60), then ψ_x^* given by (4.2.50) and ψ_t^* given by (4.2.51) are compatible.

We deduce from Corollary 4.6 that

$$P(e^{i\hat{\sigma}_k}) \neq 0 \qquad k = 1, ..., 2n.$$

Dividing (4.2.60) by $P(e^{i\hat{\sigma}_k})$ gives

$$\partial_t\left(\frac{1}{2(1+\cos\hat{\sigma}_k)}\right) + \hat{V}(\hat{\sigma}_k;\Sigma)\partial_x\left(\frac{1}{2(1+\cos\hat{\sigma}_k)}\right) = 0, \tag{4.2.71}$$

where

$$\hat{V}(\hat{\sigma}_k;\Sigma) \equiv \frac{Q(e^{i\hat{\sigma}_k})}{P(e^{i\hat{\sigma}_k})} \qquad k = 1, ...2n. \tag{4.2.72}$$

It indicates the dependence of $\hat{V}$ on the set Σ through the dependence of $\{\alpha_m\}_1^n$ and $\{\beta_m\}_1^n$ on Σ.

Formula (4.2.71) is equivalent to

$$\partial_t\hat{\lambda}_k + V(\hat{\lambda}_k;\Lambda)\partial_x\hat{\lambda}_k = 0, \qquad k = 1, ...2n; \tag{4.2.73}$$

where for $k = 1, ..., 2n$

$$\hat{\lambda}_k \equiv \cos\hat{\sigma}_k,$$

$$\Lambda \equiv \{\hat{\lambda} = \cos\hat{\sigma} : \hat{\sigma} \in \Sigma\}$$

and

$$V(\hat{\lambda}_k;\Lambda) \equiv \hat{V}(\hat{\sigma}_k;\Sigma) \stackrel{\text{def}}{=} \hat{S}_k(\Sigma). \tag{4.2.74}$$

Formula (4.2.73) is a coupled system of $2n$ partial differential equations, where each $\hat{\lambda} \in \Lambda$ is a Riemann invariant with related characteristic velocity $V(\hat{\lambda}; \Lambda)$. The set Λ evolves according to (4.2.73) so long as the solutions remain regular and n does not change.

We have not said yet how the set Λ, or Σ, and its cardinality change when a singularity is encountered, nor have we addressed how $\Lambda(x,t)$, or $\Sigma(x,t)$, is related to the initial data $\hat{\lambda}_1(x)$ and $\hat{\lambda}_2(x)$, or equivalently, to the $A(x)$ and $S(x)$. We shall deal with these issues by introducing a pair of functions $x_+(\sigma, t)$ and $x_-(\sigma, t)$ for $0 \leq \sigma \leq \pi$, which were already defined in Section 3.2 for $t < t_b$ and from now will be extended to all t. We shall extend the set Σ as well in terms of these functions.

Definition 4.7 *A pair of functions $x_+(\cdot)$, $x_-(\cdot)$ are defined to* cross *the value x at σ if the union of the images of any neighborhood of σ under $x_+(\cdot)$ and $x_-(\cdot)$ is a neighborhood of x. This σ is then defined as a* crossing point *of the pair at x.*

Note that if $0 < \sigma < \pi$ and $x_+(\sigma) = x$ or $x_-(\sigma) = x$ or both $x_\pm(\sigma) = x$, then σ is a crossing point at x unless $x_\pm(\cdot)$ has a local extremum at σ.

For a given pair of functions $x_+(\sigma)$, $x_-(\sigma)$ and a given value x, we define the set $\Sigma(x)$ by

$$\Sigma(x) = \{\sigma : \text{crossing point of } x_+(\cdot), x_-(\cdot) \text{ at } x\}.$$

Recall that

$$U(x;h) = A(x)e^{\frac{i}{\hbar}S(x)}$$

is our initial function and $x_\pm(\sigma)$ initially are defined by (2.2.60), then $\Sigma(x)$ consists of the pair points defined by (4.1.13) :

$$\begin{aligned}
\hat{\sigma}_1(x) &= \cos^{-1}(\hat{\lambda}_1) &= \cos^{-1}(\partial_x S/2 + |A|) \\
\hat{\sigma}_2(x) &= \cos^{-1}(\hat{\lambda}_2) &= \cos^{-1}(\partial_x S/2 - |A|).
\end{aligned}$$

Suppose the functions $x_\pm = x_\pm(\cdot, t)$ depend on a parameter t; then, the set Σ defined above depends on t also:

$$\Sigma(x,t) = \{\hat{\sigma} : \text{crossing point of } x_\pm(\cdot,t) \text{ at } x\}. \tag{4.2.75}$$

For the functions $x_+(\sigma,t)$ and $x_-(\sigma,t)$, we formulate the following initial value problem :

$$\begin{cases} \frac{dx_+}{dt} &= \hat{V}(\sigma; \Sigma(x_+,t)) \\ \\ \frac{dx_-}{dt} &= \hat{V}(\sigma; \Sigma(x_-,t)), \end{cases} \tag{4.2.76}$$

with

$$\begin{cases} x_+(\sigma,0) &= x_+(\sigma) \\ \\ x_-(\sigma,0) &= x_-(\sigma). \end{cases} \tag{4.2.77}$$

Here $x_+(\sigma)$ and $x_-(\sigma)$ are defined by (2.2.60). Note that the set $\Sigma(x,t)$ depends on the functions x_+ and x_-. As long as that dependence is Lipschiz continuous in a suitable norm for $x_+(\sigma)$ and $x_-(\sigma)$, (4.2.76)-(4.2.77) has a unique solution.

We assume that the initial value problem (4.2.76)-(4.2.77) has a unique pair of solutions x_+ and x_- such that

(a) As functions of σ and t, x_+ and x_- are C^1 in $(0, \hat{\sigma}_{1,max}) \cup (\hat{\sigma}_{2,min}, \pi) \times R$ and continuous in $(0, \hat{\sigma}_{1,max}] \cup [\hat{\sigma}_{2,min}, \pi) \times R$.

(b) The limits $\lim_{\sigma \to 0,\pi} x_\pm(\sigma,t)$ exist, possibly as $\pm\infty$. If finite we call them the boundary value of $x_\pm$ at $\sigma = 0, \pi$, denote them by $x_\pm(0,t)$, $x_\pm(\pi,t)$, and assume that they are c^1 with respect to t.

(c) The number of critical points of $x_+(\cdot,t)$ and $x_-(\cdot,t)$ is finite for all t.

(d) If for some σ, $x_+(\sigma) = x_-(\sigma)$, then $x_+(\sigma,t) = x_-(\sigma,t)$ for all t.

Note that (c) implies that the cardinality of $\Sigma(x,t)$ is bounded. Since

$$\begin{cases} x_+(\hat{\sigma}_{1,max}) &= x_-(\hat{\sigma}_{1,max}) \\ \\ x_+(\hat{\sigma}_{2,min}) &= x_-(\hat{\sigma}_{2,min}), \end{cases}$$

it follows that

$$\begin{cases} x_+(\hat{\sigma}_{1,max},t) &= x_-(\hat{\sigma}_{1,max},t) \\ \\ x_+(\hat{\sigma}_{2,min},t) &= x_-(\hat{\sigma}_{2,min},t). \end{cases}$$

We denote this common value by $x_{0,1}(t)$, $x_{0,2}(t)$; or in general $x_0(t)$. From (d) and the fact that $x_-(\sigma) < x_+(\sigma)$ for $\sigma \in (0,\pi)$, we conclude that $x_-(\sigma,t) < x_+(\sigma,t)$ as well.

Note that, according to the definition of crossing, $\sigma = \hat{\sigma}_{1,max}$ and $\sigma = \hat{\sigma}_{2,min}$ are both crossing points at $x = x_{0,1}(t)$ and $x = x_{0,2}(t)$ respectly if, for σ near these points, the following inequality is always true :

$$x_-(\sigma,t) < x_0(t) < x_+(\sigma,t).$$

Lemma 4.22

Suppose that $\bar{x}$ is not a critical value of $x_+(\cdot,\bar{t})$ or $x_-(\cdot,\bar{t})$ nor a boundary value of $x_\pm(\cdot,\bar{t})$ (i.e. $\bar{x} \neq x_\pm(0,t)$, $\bar{x} \neq x_\pm(\pi,t)$, or $\bar{x} \neq x_{0,1}(t)$, $\bar{x} \neq x_{0,2}(t)$), then, for all x, t in a neighborhood of that $\bar{x}$, $\bar{t}$,

(a) the cardinality of $\Sigma(x,t)$ is constant;

(b) $\hat{\sigma}(x,t)$, if it is in Σ, is a differentiable function of x and t.

PROOF :

By the assumptions given above, we can find a neighborhood of $(\bar{x},\bar{t})$ such that $\partial_\sigma x_\pm$ are bounded away from zero and x is bounded away from the boundary values of $x_\pm(\cdot,t)$. By the implicit function theorem, there exists a (maybe smaller) neighborhood $\mathcal{N}$ of $\bar{x}$, $\bar{t}$ such that every solution of $x = x_\pm(\sigma,t)$ is given by $\sigma = \hat{\sigma}(x,t)$, where $\hat{\sigma}(\cdot,\cdot)$ is a C^1 function in $\mathcal{N}$. The conclusions of Lemma 4.22 readily follow.

Now we construct $I(x,t)$ from $\Sigma(x,t)$ according to (4.2.37).

Lemma 4.23

For any x, t and any $\sigma \in (0,\pi)$, if $x_-(\sigma,t) < x < x_+(\sigma,t)$, then $\sigma \in I(x,t)$; otherwise, $\sigma \notin I(x,t)$ if $x < x_-(\sigma,t)$ or $x > x_+(\sigma,t)$.

PROOF :

By (4.2.37), if σ is not in $\Sigma(x,t)$, then σ is in $I(x,t)$ if and only if the number of elements of $\Sigma(x,t)$ in $[\sigma,\pi]$ is even. If $x \neq x_0(t)$, then by (4.2.75), the number of elements of $\Sigma(x,t)$ in $[\sigma,\pi]$ is equal to the number of times that $x_+(\varsigma,t)$ or $x_-(\varsigma,t)$ crosses the value x for ς in $[\sigma,\pi]$. Consider the case $x_-(\sigma,t) < x < x_+(\sigma,t)$; if $\sigma < \hat{\sigma}_{1,max}$ and $x < x_{0,1}(t)$, then $x_+(\varsigma,t)$ must not cross the value x while $x_-(\varsigma,t)$ must cross x at one time for $\sigma < \varsigma < \pi$. Thus, the total number of crossings being even, we conclude $\sigma \in I(s,t)$. The argument is essentially the same for all the cases. In some case one must consider the behavior of x_+ and x_- near $\varsigma = \hat{\sigma}_{1,max}$ or near $\varsigma = \hat{\sigma}_{2,min}$ but it is just easy and we omite here.

Now we use formulas (4.2.50) and (4.2.51) to construct $\psi_x^*(x,t)$ and $\psi_t^*(x,t)$ from $\Sigma(x,t)$.

Lemma 4.24

If $\bar{x}$ is not a critical or boundary value of $x_+(\cdot,\bar{t})$ or $x_-(\cdot,\bar{t})$, then, in a neighborhood of $(\bar{x},\bar{t})$,

(a) both $\psi_x^*(x,t)$ and $\psi_t^*(x,t)$ are differentiable functions of x and t.

(b) the compatibility condition $\partial_t\psi_x^* = \partial_x\psi_t^*$ is satisfied.

PROOF :

The coefficients α_k and β_k of P and Q are functions of $\hat{\sigma}_1,...,\hat{\sigma}_n$. They are expressible as rational functions of complete hyperelliptic integrals. Therefore they are analytic functions of $\hat{\sigma}_1,...,\hat{\sigma}_n$ except if two $\hat{\sigma}_k$ are equal. It follows from Lemma 4.22 that this does not happen under the hypothesis of Lemma 4.24. It then follows from Lemma 4.22 that all $\hat{\sigma}_k$ are differentiable functions of x and t. Part (a) is a consequence of these two observations.

To prove part (b), we differentiate with respect to x, and then t, the following relation

$$x = x_\pm(\hat{\sigma}, t), \qquad \hat{\sigma} = \hat{\sigma}(x, t).$$

we get

$$1 = \partial_{\hat{\sigma}} x_\pm \partial_x \hat{\sigma}, \qquad 0 = \partial_{\hat{\sigma}} x_\pm \partial_t \hat{\sigma} + \partial_t x_\pm.$$

Using equation (4.2.76) to express $\partial_t x_\pm$ as $\hat{V}$ gives

$$\partial_t \hat{\sigma} = -\frac{\partial_t x_\pm}{\partial_{\hat{\sigma}} x_\pm} = -\hat{V}(\hat{\sigma}; \Sigma(x,t))\partial_x \hat{\sigma}.$$

This is just the equation (4.2.71) and is equivalent to (4.2.60). We appeal then to Theorem 4.38 to conclude that (b) holds.

4.2.3 Solution of the Variational Problem

By Theorem 3.31, the following relation should be true:

$$\begin{aligned} 1 \geq \bar{\rho}(x,t) &= 1 + 2(\sin\sigma, \psi_x^*) \\ &= 1 - \frac{2}{\pi}\int_I \sin\sigma \frac{P(\sigma)}{R(\sigma)}\, d\sigma. \end{aligned}$$

Thus, before doing the construction of $\psi^*(x,t)$, *i.e.* the solution of the variational problem (3.1.15)-(3.1.17), we need to make sure that

Lemma 4.25

For $\sigma \in (0, \pi)$,

$$\Re\left(\frac{P(\sigma)}{R(\sigma)}\right) \geq 0, \tag{4.2.78}$$

and

$$\int_0^\sigma \Im\left(\frac{P(\varsigma)}{R(\varsigma)}\right) d\varsigma \geq 0. \tag{4.2.79}$$

PROOF :

By (4.2.47), $\Re(P(\sigma)/R(\sigma))$ is non-zero only on the set I. By Corollary 4.6, the sign of $P(\sigma)$ alternates over the intervals that make up I. On the other hand, by (4.2.40) and (4.2.41), $R(\sigma)$ has the same property. That means $\Re(P(\sigma)/R(\sigma))$ keeps no change of sign. So (4.2.78) follows if $P(\sigma) > 0$ for $\sigma \in (0, \hat{\sigma}_1)$. Here we have assumed $R(\sigma) > 0$ for $\sigma \in (0, \hat{\sigma}_1)$.

Condition (4.2.54) ensures that $\int_0^\sigma \Im(P(\varsigma)/R(\varsigma))\, d\varsigma$ is zero on the set I. By Corollary 4.6, that $P(\sigma)$ alternatively crosses from positive to negative or from negative to positive over the intervals $(\hat{\sigma}_{2k-1}, \hat{\sigma}_{2k})$ is determined in such a way that whether $P(\sigma)$ is positive for σ inside of $(0, \hat{\sigma}_1)$. We will show that $P(\sigma) > 0$ is true at this beginning interval. Then by (4.2.40) and (4.2.41), the sign of $(iR(\varsigma))$ alternates on those intervals and is negative on $(\hat{\sigma}_1, \hat{\sigma}_2)$, we conclude that $\Im(P(\varsigma)/R(\varsigma))$ crosses from negative to positive on the intervals $(\hat{\sigma}_{2k-1}, \hat{\sigma}_{2k})$ for all $k = 1, ..., n$. This fact implies (4.2.79), completing the proof of the lemma.

Now we show that $P(\sigma) > 0$ for all $\sigma \in (0, \hat{\sigma}_1)$. By (4.2.67), it is equivalent to show that

$$h(e^{i\sigma}) > 0, \qquad \text{for } \sigma \in (0, \hat{\sigma}_1),$$

where

$$h(\omega) = 1 + \sum_{k=1}^{k=n} \alpha_k [\frac{\omega}{(1+\omega)^2}]^k.$$

It is clear that $h(\omega)$ is analytic inside of the unit disc Ω and takes real value on $\hat{I}_1$ without any singularity. Here $\hat{I}_1$ is set as a part of the boundary of the disk Ω such that

$$\hat{I}_1 = \{\omega = e^{i\sigma} : -\hat{\sigma}_1 < \sigma < \hat{\sigma}_1\}$$

Thus we can extend $h(\omega)$ through $\hat{I}_1$ by this way :

$$H(\omega) = \begin{cases} h(\omega) & |\omega| < 1 \quad -\hat{\sigma}_1 < \sigma < \hat{\sigma}_1 \\ h(\omega^*) & |\omega| > 1 \quad -\hat{\sigma}_1 < \sigma < \hat{\sigma}_1, \end{cases}$$

with $\omega\omega^* = 1$.

Consider $\omega = re^{i\sigma}$ with $\sigma = 0$, we obtain

$$H(r) = 1 + \sum_{k=1}^{k=n} \alpha_k [\frac{r}{(1+r)^2}]^k. \tag{4.2.80}$$

For our purpose, it is enough to show that $H(1) > 0$. Since if it is true, then $H(r)$ will be positive on a neighborhood of $r = 1$. That means $h(e^{i\sigma})$ or $P(e^{i\sigma})$ is positive on somewhere inside of $(0, \hat{\sigma}_1)$, if σ is close to 0. Since $P(e^{i\sigma})$ is real and remains same sign where $\sigma \in (0, \hat{\sigma}_1)$, so P is positive for all σ which are in this interval.

By (4.2.80), $H(0) = 1$ and $H(\infty) = 1$, therefore that $H(1) > 0$ is true if $H(r)$ has no real roots wherever r remains real and nonnegative.

Recall the condition which $\{\alpha_k\}_1^n$ should be satisfied, see (4.2.56). They are found to build up a matrix equality :

$$AY = C,$$

with

$$A = (A_{ik})_{n \times n}, \qquad Y = \begin{pmatrix} Y_1 \\ * \\ Y_n \end{pmatrix}, \qquad C = \begin{pmatrix} C_1 \\ * \\ C_n \end{pmatrix};$$

such that

$$Y_k = \frac{\alpha_k}{2^k}, \qquad C_k = -\int_{\tau_{2k-1}}^{\tau_{2k}} \frac{(1+\tau)^n}{R(\tau)} d\tau,$$

and

$$A_{ik} = -\int_{\tau_{2i-1}}^{\tau_{2i}} \frac{(1+\tau)^{n-k}}{R(\tau)} d\tau,$$

where

$$-1 < \tau_k < 1, \qquad \text{for} \qquad k = 1, 2, ..., 2n.$$

That implies the following polynomial

$$(1+\tau)^n + \sum_{k=1}^{k=n} \frac{\alpha_k}{2^k}(1+\tau)^{n-k} \tag{4.2.81}$$

has roots in each of the intervals (τ_{2k-1}, τ_{2k}) $k = 1, \ldots n$, say $\bar{\tau}_k$, and no more else. Compare (4.2.81) with (4.2.80), all the possible roots, say $\bar{r}$, of $H(r)$ should satisfy

$$\frac{\bar{r}}{(1+\bar{r})^2} = \frac{1}{2(1+\bar{\tau}_k)},$$

for some k between 1 and $2n$. That is, these $\bar{r}$'s should satisfy the following equation

$$(1+\bar{r})^2 = 2\bar{r}(1+\bar{\tau}_k),$$

for some $\bar{\tau}_k$. *i.e.*

$$\bar{r}^2 - 2\bar{r}\bar{\tau}_k + 1 = 0.$$

Now it is easy to see that no real value of $\bar{r}$ satisfy the above relation with any such $\bar{\tau}_k,\ _{k=1,\ldots,2n}$, because $|\bar{\tau}_k| < 1$ for all k and it makes a negative discriminant :

$$\Delta = 4\bar{\tau}_k^2 - 4 < 0.$$

That means $H(1)$ occupies same sign as $H(0)$ and $H(\infty)$ do, completing the proof of the lemma.

In order to construct $\psi^*(x,t)$ which is the solution of the variational problem (3.1.15)-(3.1.17), we integrate $\psi_x^*(x,t)$ as given by (4.2.50). Theorem 3.27 requires that $\psi^*(x,t) = 0$, where $x \sin\sigma - t\sin(2\sigma) - \Gamma_+(\sigma) > 0$. In particular, it follows that, for fixed σ and t, $\psi^*(x,t) = 0$ for x large enough. Thus we let

$$\psi^*(x,t) = -\int_x^\infty \psi_x^*(y,t)\,dy \equiv \int_x^\infty \Re\left(\frac{P(\sigma)}{R(\sigma)}\right)\,dy. \tag{4.2.82}$$

Lemma 4.23 shows that, for fixed σ and t, the integrand is zero for x large enough. By Lemma 4.24 we get that $\partial_t\psi^*(x,t)$ calculated from (4.2.82) equals $\psi_t^*(x,t)$ given by (4.2.51). Inequality (4.2.78) of Lemma 4.25 implies that ψ^* is a positive and decreasing function of x. In the x, t-region, where $x < x_-(\sigma,t)$, we have by Lemma 4.23 and (4.2.50), (4.2.51) that

$$\psi^* = \int_{x_-(\sigma,t)}^{x_+(\sigma,t)} \frac{P(\sigma)}{R(\sigma)}\,d\sigma$$

and

$$\partial_x \psi^* = \partial_t \psi^* = 0;$$

where and hereafter, for short, we use $F(\sigma)$ in stead of $F(e^{i\sigma})$. Since by (4.1.15), $\psi^*(x,0) = \phi$ for $x < x_-(\sigma)$, we conclude that

$$\int_{x_-(\sigma,t)}^{x_+(\sigma,t)} \frac{P(\sigma)}{R(\sigma)}\, dy = \phi(\sigma). \tag{4.2.83}$$

Finally, from the above conclusion, the $\psi^*(x,t)$ given by (4.2.82) is a positive, decreasing function of x. It follows from (4.2.83) that $0 \leq \psi^* \leq \phi$. *i.e.* ψ^* belongs to the admissible set A.

The main result of this part is

Theorem 4.39

The solution of the variational problem (3.1.15)-(3.1.17) is given by

$$\psi^*(x,t) = \begin{cases} 0 & \text{for } \ x_+(\sigma,t) \leq x \\ \\ \int_x^{x_+(\sigma,t)} \frac{P(\sigma)}{R(\sigma)}\, dy & \text{for } \ x_-(\sigma,t) < x < x_+(\sigma,t) \\ \\ \phi(\sigma) & \text{for } \ x \leq x_-(\sigma,t). \\ \end{cases} \tag{4.2.84}$$

PROOF :

We shall show that $\psi^*(x,t)$ satisfies the variational conditions (3.2.66). Introducing

$$\Gamma_+(\sigma,t) = x_+(\sigma,t)\sin\sigma - L\psi^*(x_+(\sigma,t),t), \tag{4.2.85}$$

and integrating (4.2.52) with respect to x gives

$$L\psi^*(x,t) =$$

$$
\begin{cases}
x\sin\sigma - \Gamma_+(\sigma,t) & -\int_{x_+(\sigma,t)}^{x}\int_0^{\sigma}\Im\left(\frac{P(\varsigma)}{R(\varsigma)}\right)d\varsigma\,dy \\
 & x_+(\sigma,t) < x \\
x\sin\sigma - \Gamma_+(\sigma,t) & x_-(\sigma,t) \le x \le x_+(\sigma,t) \\
x\sin\sigma - \Gamma_+(\sigma,t) & +\int_{x}^{x_-(\sigma,t)}\int_0^{\sigma}\Im\left(\frac{P(\varsigma)}{R(\varsigma)}\right)d\varsigma\,dy \\
 & x < x_-(\sigma,t).
\end{cases}
\tag{4.2.86}
$$

Differentiating (4.2.85) and using (3.2.83) (3.2.84) gives

$$
\begin{aligned}
\partial_t\Gamma_+ &= \sin\sigma\partial_t x_+ - L\psi_x^*(x_+,t)\partial_t x_+ - L\psi_t^*(x_+,t) \\
&= -\sin\sigma\cos\sigma.
\end{aligned}
\tag{4.2.87}
$$

We know from (4.1.17) that

$$
\begin{aligned}
\Gamma_+(\sigma,0) &= \sin\sigma x_+(\sigma,0) - L\psi^*(x_+(\sigma,0),0) \\
&= \sin\sigma x_+(\sigma) - L\psi^*(x_+(\sigma),0) \\
&= \Gamma_+(\sigma).
\end{aligned}
$$

Therefore, by integrating (4.2.87) we obtain

$$
\Gamma_+(\sigma,t) = -t\sin\sigma\cos\sigma + \Gamma_+(\sigma).
$$

With the definition (2.2.45) of a, it comes out

$$
a(\sigma,x,t) = x\sin\sigma - \Gamma_+(\sigma,t). \tag{4.2.88}
$$

Thus, with (4.2.88), formula (4.2.86) becomes

$$a - L\psi^* = \begin{cases} \int_{x_+(\sigma,t)}^{x} \int_0^{\sigma} \Im(\frac{P(\varsigma)}{R(\varsigma)})\, d\varsigma\, dy & \text{for } x_+(\sigma,t) < x \\ 0 & \text{for } x_-(\sigma,t) \le x \le x_+(\sigma,t) \\ -\int_x^{x_-(\sigma,t)} \int_0^{\sigma} \Im(\frac{P(\varsigma)}{R(\varsigma)})\, d\varsigma\, dy & \text{for } x < x_-(\sigma,t) \end{cases}.$$

By connecting (4.2.79) of Lemma 4.25, we conclude that the $\psi^*(x,t)$ given by (4.2.84) satisfies the variational conditions (3.2.66). This proves Theorem 4.39.

4.2.4 Formulas for the Weak Limits

We can now get explicit formulas for $\bar{\rho}$, $\bar{\mu}$ and $\bar{\varepsilon}$ by substituting ψ_x^* given by (4.2.50) and ψ_t^* given by (4.2.51) into the formulas of Theorem 3.31 :

$$\bar{\rho} = 1 + 2(\sin\sigma, \psi_x^*) = 1 - \frac{2}{\pi}\int_I \sin\sigma \frac{P(\sigma)}{R(\sigma)}\, d\sigma, \tag{4.2.89}$$

$$\bar{\mu} = (\sin(2\sigma), \psi_x^*) = -\frac{1}{\pi}\int_I \sin(2\sigma)\frac{P(\sigma)}{R(\sigma)}\, d\sigma,$$

or

$$\bar{\mu} = 2(\sin\sigma, \psi_t^*) = \frac{2}{\pi}\int_I \sin\sigma \frac{Q(\sigma)}{R(\sigma)}\, d\sigma;$$

and

$$\begin{aligned}\bar{\varepsilon} &= 2(\sin\sigma\cos^2\sigma, \psi_x^*)\\ &= -\frac{2}{\pi}\int_I \sin\sigma\cos^2\sigma \frac{P(\sigma)}{R(\sigma)}\, d\sigma,\end{aligned}$$

or

$$\begin{aligned}\bar{\varepsilon} &= -2(\sin\sigma, \psi_x^*) + 2(\sin\sigma, \psi_x^*)^2 + (\sin(2\sigma), \psi_t^*)\\ &= \frac{2}{\pi}\int_I \sin\sigma \frac{P(\sigma)}{R(\sigma)}\, d\sigma + \frac{2}{\pi^2}[\int_I \sin\sigma\frac{P(\sigma)}{R(\sigma)}\, d\sigma]^2\\ &+\frac{1}{\pi}\int_I \sin(2\sigma)\frac{Q(\sigma)}{R(\sigma)}\, d\sigma.\end{aligned} \tag{4.2.90}$$

Generally,

$$\bar{\rho}_n(x,t) = 2(\sin\sigma\cos^{n-1}\sigma, \psi_x^*)$$
$$= -\frac{2}{\pi}\int_I \sin\sigma\cos^{n-1}\sigma\frac{P(\sigma)}{R(\sigma)}\,d\sigma,$$

and

$$\bar{\mathcal{G}}_n(x,t) = 2(\sin\sigma\cos^{n-1}\sigma, \psi_t^*)$$
$$= \frac{2}{\pi}\int_I \sin\sigma\cos^{n-1}\sigma\frac{Q(\sigma)}{R(\sigma)}\,d\sigma.$$

These integrals can be evaluated by contour integration. Slit the plane along the arcs of the set $\hat{I}$ given by (4.2.38); the integral along each single arc is then equal to half the integral of a clockwise contour around its slit. By Cauchy's integral theorem, we can deform the contour to a large clockwise circle around the origin containing I and to a small anticlockwise circle only around the origin. Then letting the radius of this big one go to infinity and the radius of this small one go to zero, we can evaluate these integrals by computing their residues of both infinity and origin. In fact, the residue of infinity is equal to the residue of the origin. Thus we only need to consider the residue of the origin.

The following is the calculation of such residues.

By (4.2.40), we have

$$\begin{aligned} R^2(\omega) &= \prod_{k=1}^{k=2n}\left[\frac{\omega}{(\omega+1)^2} - \frac{\hat{\omega}_k}{(\hat{\omega}_k+1)^2}\right] \\ &= (\omega+1)^{-4n}\prod_{k=1}^{k=2n}\left[\omega - \frac{(\omega+1)^2}{2(1+\cos\hat{\sigma}_k)}\right] \\ &= \frac{R^2(0)}{(\omega+1)^{4n}}\prod_{k=1}^{k=2n}[1-2\omega\cos\hat{\sigma}_k+\omega^2]. \end{aligned} \tag{4.2.91}$$

For small $|\omega|$, by (4.2.91), we can estimate

$$\frac{1}{R(\omega)} =$$

$$\frac{(1+\omega)^{2n}}{R(0)}\prod_{k=1}^{k=2n}[1-2\omega\cos\hat{\sigma}_k+\omega^2]^{-1/2}$$
$$=\frac{(1+\omega)^{2n}}{R(0)}\{1+\omega(\sum_1^{2n}\cos\hat{\sigma}_k)$$
$$+\omega^2[\sum_1^{2n}\cos^2\hat{\sigma}_k+\frac{1}{2}(\sum_1^{2n}\cos\hat{\sigma}_k)^2-n]$$
$$+\omega^3[2\sum_1^{2n}\cos^3\hat{\sigma}_k+\frac{1}{2}(\sum_1^{2n}\cos\hat{\sigma}_k)^3$$
$$-3\sum_{i<j<k}\cos\hat{\sigma}_i\cos\hat{\sigma}_j\cos\hat{\sigma}_k-(n+1)\sum_1^{2n}\cos\hat{\sigma}_k]$$
$$+O(\omega^4).$$

From (4.2.44) and (4.2.45) we have

$$P(\omega)=\frac{i}{2}R(0)\{\frac{1-\omega^2}{\omega}+\frac{1-\omega}{1+\omega}\sum_{k=1}^{k=n}\alpha_k[\frac{\omega}{(1+\omega)^2}]^{k-1}\}$$

and

$$Q(\omega) = (-\frac{i}{2})R(0)\{[(\frac{R'(0)}{R(0)}-2)\frac{1-\omega^2}{\omega}+\frac{(1-\omega^2)(1+\omega)^2}{\omega^2}]$$
$$+\frac{1-\omega}{1+\omega}\sum_{k=1}^{k=n}\beta_k[\frac{\omega}{(1+\omega)^2}]^{k-1}\},$$

where

$$\frac{R'(0)}{R(0)}=-\sum_{k=1}^{k=2n}(1+\cos\hat{\sigma}_k).$$

Computing $P(\omega)/R(\omega)$ and $Q(\omega)/R(\omega)$ with $\sin\sigma=(\omega^2-1)/(2i\omega)$, $\cos\sigma=(\omega^2+1)/(2\omega)$ and $d\sigma=(d\omega)/(i\omega)$to find residues, we obtain from (4.2.89)-(4.2.90):

$$\bar{\rho}=1-\frac{1}{2\pi}\oint\left(\frac{\omega^2-1}{2i\omega}\right)\frac{P(\omega)}{R(\omega)}\frac{d\omega}{i\omega}$$
$$=[n^2-n+\alpha_1(n-1+\frac{1}{2}\sum_1^{2n}\cos\hat{\sigma}_k)+\frac{1}{2}\alpha_2$$
$$+n\sum_1^{2n}\cos\hat{\sigma}_k+\frac{1}{4}(\sum_1^{2n}\cos\hat{\sigma}_k)^2+\frac{1}{2}\sum_1^{2n}\cos^2\hat{\sigma}_k], \tag{4.2.92}$$

$$\bar{\mu} = \frac{1}{\pi} \oint \left(\frac{\omega^2 - 1}{2i\omega} \right) \frac{Q(\omega)}{R(\omega)} \frac{d\omega}{i\omega}$$

$$= [\beta_2 + 2(n-1)\beta_1 + \beta_1 \sum_1^{2n} \cos\hat{\sigma}_k$$

$$+(2 \sum_1^{2n} \cos^3\hat{\sigma}_k - 3 \sum_{i<j<k} \cos\hat{\sigma}_i \cos\hat{\sigma}_j \cos\hat{\sigma}_k)$$

$$-(\sum_1^{2n} \cos\hat{\sigma}_k)(\sum_1^{2n} \cos^2\hat{\sigma}_k) - 2n(\sum_1^{2n} \cos\hat{\sigma}_k)^2$$

$$-4n^2 \sum_1^{2n} \cos\hat{\sigma}_k + \frac{8}{3}n - \frac{8}{3}n^3], \tag{4.2.93}$$

and

$$\bar{\mu} = -\frac{1}{2\pi} \oint \left(\frac{\omega^4 - 1}{4i\omega^2} \right) \frac{P(\omega)}{R(\omega)} \frac{d\omega}{i\omega}$$

$$= \{(\frac{2}{3}n^3 - 2n^2 - \frac{2}{3}n) + \frac{1}{2}\alpha_3 + \alpha_2(n-2) + \alpha_1(n^2 - 3n + 1)$$

$$+(n + \frac{1}{2}\alpha_1)[\sum_1^{2n} \cos^2\hat{\sigma}_k + \frac{1}{2}(\sum_1^{2n} \cos\hat{\sigma}_k)^2]$$

$$+[n^2 - n - 1 + (n-1)\alpha_1 + \alpha_2] \sum_1^{2n} \cos\hat{\sigma}_k + \frac{5}{4} \sum_1^{2n} \cos^3\hat{\sigma}_k$$

$$+\frac{3}{4} \sum_{k<j} [\cos\hat{\sigma}_k \cos\hat{\sigma}_j (\cos\hat{\sigma}_k + \cos\hat{\sigma}_j)]\}. \tag{4.2.94}$$

Now, comparing (4.2.93) with (4.2.94) yields an algebraic identity, which is a relation among α_1, α_2, α_3, β_1 and β_2 :

$$\beta_2 + (2n - 2 + \sum_1^{2n} \cos\hat{\sigma}_k)\beta_1 - \frac{1}{2}\alpha_3 - (n - 2 + \frac{1}{2} \sum_1^{2n} \cos\hat{\sigma}_k)\alpha_2$$

$$-\alpha_1[n^2 - 3n + 1 + (n-1) \sum_1^{2n} \cos\hat{\sigma}_k + \frac{1}{4}(\sum_1^{2n} \cos\hat{\sigma}_k)^2 + \frac{1}{2} \sum_1^{2n} \cos^2\hat{\sigma}_k]$$

$$= [\frac{2}{3}\sum_{i<j<k} \cos\hat{\sigma}_i \cos\hat{\sigma}_j \cos\hat{\sigma}_k + \frac{1}{4}(\sum_1^{2n} \cos\hat{\sigma}_k)^3 - \sum_1^{2n} \cos^3\hat{\sigma}_k]$$

$$+(\sum_1^{2n} \cos^2\hat{\sigma}_k)(\sum_1^{2n} \cos\hat{\sigma}_k) + n\sum_1^{2n} \cos^2\hat{\sigma}_k + \frac{5}{2}n(\sum_1^{2n} \cos\hat{\sigma}_k)^2$$

$$+(5n^2 - n - 1)\sum_1^{2n} \cos\hat{\sigma}_k + \frac{10}{3}n^3 - 2n^2 - \frac{10}{3}n.$$

As a typical example, we consider the region of x and t where n, the cardinality number of $\Sigma(x,t)$, is equal two. In that case, the formulas (4.2.92)-(4.2.94) become

$$\begin{aligned} \bar{\rho} &= \tfrac{1}{4}(\cos\hat{\sigma}_1 - \cos\hat{\sigma}_2)^2 &&= \tfrac{1}{4}(\hat{\lambda}_1 - \hat{\lambda}_2)^2, \\ \bar{\mu} &= \tfrac{1}{4}(\cos\hat{\sigma}_1 - \cos\hat{\sigma}_2)^2(\cos\hat{\sigma}_1 + \cos\hat{\sigma}_2) &&= \tfrac{1}{4}(\hat{\lambda}_1 - \hat{\lambda}_2)^2(\hat{\lambda}_1 + \hat{\lambda}_2), \end{aligned}$$

which is just (4.1.23).

Generally, for the k_{th} conserved density,

$$\bar{\rho}_k(x,t) = -\frac{1}{2\pi}\oint(\frac{\omega^2-1}{2i\omega})(\frac{\omega^2+1}{2\omega})^{k-1}\frac{P(\omega)}{R(\omega)}\frac{d\omega}{i\omega}. \tag{4.2.95}$$

For example if $k = 3$,

$$\begin{aligned} \bar{\varepsilon}(x,t) &= \bar{\rho}_3(x,t) \\ &= -\frac{1}{2\pi}\oint(\frac{\omega^2-1}{2i\omega})(\frac{\omega^2+1}{2\omega})^2\frac{P(\omega)}{R(\omega)}\frac{d\omega}{i\omega}. \end{aligned} \tag{4.2.96}$$

It is equivalent to (4.1.36), when $t < t_b$.

The cardinality number, say n, is valid only for certain time period. That means the Riemann invariants $\{\hat{\lambda}_l\}_1^{2n}$ will break again when reaching a new break-time t_{b_n}. After that, either the members of Λ will get different order or the cardinality number will increase to $n + 2$. We will discuss them deeply in the following chapter.

CHAPTER 5

HYPERBOLIC NATURE AND LOCAL DYNAMICS

5.1 The Theorems

In this section, we study the dynamics of these semiclassical limits. The strategy we adopt here is the one offered by Levermore [12].

For the t-flow given by (1.1.4), the strict hyperbolicity and genuine nonlinearity of evolution of these limits can be proved wherever the correspond Riemann invariants, which are defined as $\hat{\lambda}_k \in \Lambda$ given by (4.2.71), remain distinct. In fact, it has been enough to show the hyperbolic nature of those Riemann invariants, because all the semiclassical limits were described by them as shown at (4.2.92)-(4.2.94), (4.2.96) and (4.2.95).

By former study, we have known some local dynamics already. i.e. In a neighborhood of almost all (x, t), there exists a non-negative integer n, such that these weak limits are determined by a $(2n)$-dimensional real vector-valued function $\vec{V} = (\hat{\lambda}_1, ..., \hat{\lambda}_{2n})$ where $\hat{\lambda}_k \in \Lambda$ with the following ordering condition :

$$1 > \hat{\lambda}_1 > \hat{\lambda}_2 > ... > \hat{\lambda}_{2n} > -1. \tag{5.1.1}$$

They satisfy a system given by (4.2.73) :

$$\partial_t \hat{\lambda}_k + S_k(\vec{V}) \partial_x \hat{\lambda}_k = 0, \qquad k = 1, ...2n; \tag{5.1.2}$$

where $S_k(\vec{V}) \equiv V(\hat{\lambda}_k; \Lambda)$, the characteristic speeds, are real and have a functional form depending on n as given by (4.2.72). Nevertheless, it is not the total story. We can disclose more nice properties of them. The main results of this chapter are the following :

Theorem 5.40

Subject to the ordering condition (5.1.1), the system (5.1.2) is strictly hyperbolic with the speeds ordered as

$$S_1(\vec{V}) < S_2(\vec{V}) < \dots < S_{2n}(\vec{V}), \tag{5.1.3}$$

and is genuinely nonlinear with

$$\partial_{\hat{\lambda}_k} S_k(\vec{V}) < 0 \tag{5.1.4}$$

for all $k = 1, \dots, 2n$.

Assertions(5.1.3) and (5.1.4) will be proved in Section 5.3 and 5.5 respectively, with some useful identities proved in Section 5.4.

Since the system (5.1.2) is genuinely nonlinear, a Riemann invariant, say $\hat{\lambda}_k$, may in a finite time develop an infinite spatial derivative at a point while maintaining the ordering condition (5.1.1). When this happens, $\hat{\lambda}_k$ is said to break down. The second main result of this section relates to the onset of breaking.

Theorem 5.41

Let $\vec{V}(x,t)$ *be a solution of (5.1.2) subject to the ordering condition (5.1.1). For any* $k = 1, \dots, 2n$, *there exists* $W_k(\vec{V}) > 0$, *such that* $\xi_k = \partial_x \hat{\lambda}_k$ *satisfies the Riccati equation*

$$(\partial_t + S_k(\vec{V})\partial_x)\left(\frac{\xi_k}{W_k(\vec{V})}\right) + (W_k(\vec{V})\partial_{\hat{\lambda}_k} S_k(\vec{V}))\left(\frac{\xi_k}{W_k(\vec{V})}\right)^2 = 0. \tag{5.1.5}$$

This theorem will be proved in Section 5.6 where the explicit formulas for the functions $W_k(\vec{V})$ will be constructed by again making use of same identities once applied for Theorem 5.40.

5.2 Speed Formulas

In order to prove these theorems, we need to recall the formulas of characteristic speeds $S_k(\vec{V})$ and some related functions. By (4.2.74),

$$S_k(\vec{V}) = \hat{S}_k(\Sigma) \equiv \hat{V}(\hat{\sigma}_k;\Sigma) \equiv \frac{Q(e^{i\hat{\sigma}_k})}{P(e^{i\hat{\sigma}_k})} \tag{5.2.6}$$

for $k = 1, ...2n$; where

$$\Sigma = \{\hat{\sigma}_k\}_{k=1}^{k=2n}.$$

By (4.2.67) and (4.2.68),

$$P(e^{i\sigma}) = \frac{1}{2}R(0)\sin\sigma\left\{1 + \sum_{m=1}^{m=n}\alpha_m\left[\frac{1}{2(1+\cos\sigma)}\right]^m\right\}, \tag{5.2.7}$$

and

$$Q(e^{i\sigma}) = -\frac{1}{2}R(0)\sin\sigma\left\{\left[\frac{R'(0)}{R(0)} - 2\right] + 2(1+\cos\sigma) + \sum_{m=1}^{m=n}\beta_m\left[\frac{1}{2(1+\cos\sigma)}\right]^m\right\}. \tag{5.2.8}$$

Here, the coefficients $\{\alpha_m\}_1^n$ and $\{\beta_m\}_1^n$ are determined by (4.2.54) and (4.2.55) respectly. That is

$$\int_{\hat{\sigma}_{2k-1}}^{\hat{\sigma}_{2k}} \frac{P(e^{i\sigma})}{R(e^{i\sigma})}\,d\sigma = 0 \tag{5.2.9}$$

and

$$\int_{\hat{\sigma}_{2k-1}}^{\hat{\sigma}_{2k}} \frac{Q(e^{i\sigma})}{R(e^{i\sigma})}\,d\sigma = 0 \tag{5.2.10}$$

for $k = 1, 2, ..., n$.

With (4.2.46), it is easy to calculate that

$$R(0) = 1\Big/\sqrt{\prod_{k=1}^{2n}[2(1+\cos\hat{\sigma}_k)]}$$

and

$$\frac{R'(0)}{R(0)} - 2 = -\sum_{k=1}^{2n}(1 + \cos\hat{\sigma}_k) - 2.$$

For some application later, we need the following lemma

Lemma 5.26

If $\tilde{Z}(\delta)$ is a real polynomial such that

$$\int_{\delta_{2l}}^{\delta_{2l-1}} \frac{\tilde{Z}(\delta)}{\tilde{R}(\delta)}\, d\delta = 0, \tag{5.2.11}$$

for $l = 1, 2, ..., n$ with $\delta_1 > \delta_2 > ... > \delta_{2n}$ and

$$\tilde{R} = \prod_{m=1}^{m=2n} (\delta - \delta_m)^{1/2},$$

then $\tilde{Z}(\delta)$ must have at least one zero in each of the intervals $(\delta_{2l}, \delta_{2l-1})$.

PROOF :

The reason of this claim is simply the observation that the integrals cannot be zero if the purely imaginary integrand does not go to zero somewhere. Since $\tilde{Z}(\delta)$ must, therefore, vanish on at least n points, we can also conclude

Corollary 5.7

If $\tilde{Z}(\delta)$ is a real polynomial of degree less than n and satisfying (5.2.11), then $\tilde{Z}(\delta) \equiv 0$.

5.3 Speed Ordering

Proof of the speed ordering assertion (5.1.3) of Theorem 5.40 begins by defining the function

$$Q_k(\sigma) \equiv Q(e^{i\sigma}) - S_k P(e^{i\sigma})$$

for any $k = 1, ..., 2n$. By (5.2.9) and (5.2.10),

$$\int_{\hat{\sigma}_{2l-1}}^{\hat{\sigma}_{2l}} \frac{Q_k(\sigma)}{R(e^{i\sigma})}\, d\sigma = 0, \qquad l = 1, 2, ..., n.$$

By using an argument once we did for Corollary 4.6, we have

Corollary 5.8

The function $Q_k(\sigma)$, if extended to $(-\pi,\pi)$, has zero at $\sigma = 0$, $\sigma = \pm\sigma_k$, one and only one zero inside of each of the intervals $(\hat{\sigma}_{2k-1}, \hat{\sigma}_{2k})$, $(-\hat{\sigma}_{2k}, -\hat{\sigma}_{2k-1})$ for $k = 1, 2, ..., n$ and nowhere else.

PROOF :

It is obvious that $Q_k(\sigma)$ has zero at $\sigma = 0$, $\sigma = \pm\sigma_k$, and at least one zero inside of each of the intervals $(\hat{\sigma}_{2k-1}, \hat{\sigma}_{2k})$, $(-\hat{\sigma}_{2k}, -\hat{\sigma}_{2k-1})$ for $k = 1, 2, ..., n$. We only need to show that $Q_k(\sigma)$ have no more zeroes. Since Q_k is an odd function of σ, It is enough to show that for $0 < \sigma < \pi$, Q_k have at most $n + 1$ zeroes.

Set $\delta = 2(1 + \cos\sigma)$. It is a monotonic mapping where $0 < \sigma < \pi$, so the number of zeroes of $F(\sigma)$ equals the number of zeroes of $f(\delta)$ if we define $F = f(\delta(\sigma))$. Then, by (5.2.7) and (5.2.8), we have

$$P(e^{i\sigma}) = \frac{1}{2}[\prod_{l=1}^{l=2n}(\delta_l^{-1})]^{1/2}\sin\sigma(1 + \sum_{m=1}^{m=n}\alpha_m\delta^{-m}) \tag{5.3.12}$$

and

$$Q(e^{i\sigma}) = -\frac{1}{2}[\prod_{l=1}^{l=2n}(\delta_l^{-1})]^{1/2}\sin\sigma[(-\frac{1}{2}\sum_{l=1}^{l=2n}\delta_l - 2) + \delta + \sum_{m=1}^{m=n}\beta_m\delta^{-m}]. \tag{5.3.13}$$

By (4.2.40), for $\omega = e^{i\sigma}$, we have

$$\begin{aligned} R^2(e^{i\sigma}) &= \prod_{l=1}^{l=2n}\left[\frac{1}{2(1+\cos\sigma)} - \frac{1}{2(1+\cos\hat{\sigma}_l)}\right] \\ &= \prod_{l=1}^{l=2n}\left(\frac{\delta_l - \delta}{\delta\delta_l}\right). \end{aligned} \tag{5.3.14}$$

So

$$R^{-1}(e^{i\sigma}) = \delta^n\left(\prod_{l=1}^{l=2n}\delta_l\right)^{1/2} / \left[\prod_{l=1}^{l=2n}(\delta - \delta_l)\right]^{1/2}.$$

Thus, for $j = 1, ..., n$,

$$0 = \int_{\hat{\sigma}_{2j-1}}^{\hat{\sigma}_{2j}} \frac{P(e^{i\sigma})}{R(e^{i\sigma})} d\sigma$$
$$= \int_{\delta_{2j}}^{\delta_{2j-1}} \frac{1}{4} \left(\delta^n + \sum_{m=1}^{m=n} \alpha_m \delta^{n-m} \right) / \sqrt{\prod_{l=1}^{l=2n} (\delta - \delta_l)} \; d\delta$$
$$\stackrel{\text{def}}{=} \int_{\delta_{2j}}^{\delta_{2j-1}} \frac{\tilde{P}(\delta)}{\tilde{R}(\delta)} \; d\delta \qquad (5.3.15)$$

and

$$0 = \int_{\hat{\sigma}_{2j-1}}^{\hat{\sigma}_{2j}} \frac{Q(e^{i\sigma})}{R(e^{i\sigma})} d\sigma$$
$$= \int_{\delta_{2j}}^{\delta_{2j-1}} -(\frac{1}{4}) \left[\delta^{n+1} - \delta^n (2 + \frac{1}{2} \sum_{l=1}^{l=2n} \delta_l) + \sum_{m=1}^{m=n} \beta_m \delta^{n-m} \right]$$
$$/ \sqrt{\prod_{l=1}^{l=2n} (\delta - \delta_l)} \; d\delta$$
$$\stackrel{\text{def}}{=} \int_{\delta_{2j}}^{\delta_{2j-1}} \frac{\tilde{Q}(\delta)}{\tilde{R}(\delta)} \; d\delta. \qquad (5.3.16)$$

Then for $j = 1, 2, ..., n$,

$$\begin{aligned} 0 &= \int_{\hat{\sigma}_{2j-1}}^{\hat{\sigma}_{2j}} \frac{Q_k(\sigma)}{R(e^{i\sigma})} d\sigma \\ &= \int_{\delta_{2j}}^{\delta_{2j-1}} \frac{\tilde{Q}_k(\delta)}{\tilde{R}(\delta)} d\delta, \end{aligned} \qquad (5.3.17)$$

where we defined

$$\tilde{Q}_k = \tilde{P} - S_k \tilde{Q}.$$

Thus, by Lemma 5.26, $\tilde{Q}_k$ has $n+1$ zeroes. Anyway, $\tilde{Q}_k$ is only a $(n+1)^{\text{th}}$-degree polynomial, thus, no more other zeroes. It is obvious that the number of zeroes of $\tilde{Q}_k(\delta)$ while on the domain $4 > \delta > 0$ equals the number of zeroes of $Q_k(\sigma)$ while on the domain $0 < \sigma < \pi$. So $Q_k(\sigma)$ has at most $n+1$ zeroes too. That means $Q_k(\sigma)$ has exactly one simple zero in each of the n intervals $(\hat{\sigma}_{2l-1}, \hat{\sigma}_{2l})$, $l = 1, 2, ..., n$; as well as a simple zero at $\hat{\sigma}_k$. It completes the proof.

We now consider a fractional function of σ :

$$\frac{Q_k(\sigma)}{P(e^{i\sigma})} = \frac{Q(e^{i\sigma}) - S_k P(e^{i\sigma})}{P(e^{i\sigma})} = \hat{V}(\sigma, \Sigma) - \hat{S}_k(\Sigma) \stackrel{\text{def}}{=} \mathcal{F}_k(\sigma),$$

and analyze its sign changes.

By Corollary (4.6) and Corollary (5.8), neither $P(e^{i\sigma})$ nor $Q_k(\sigma)$ change sign on the intervals $(\hat{\sigma}_{2l}, \hat{\sigma}_{2l-1})$ for $l = 0, 1, ..., n$ with definition $\hat{\sigma}_0 = 0$ and $\hat{\sigma}_{2n+1} = \pi$. Thus the fractional function $\mathcal{F}_k$ cannot change sign either. On each of the intervals $(\hat{\sigma}_{2l-1}, \hat{\sigma}_{2l})$ for $l = 0, 1, ..., n$, both $P(e^{i\sigma})$ and $Q_k(\sigma)$ change sign exactly once at their respective simple zeroes; thus the function $\mathcal{F}_k$ will change sign exactly twice or possibly not at all in the case where the two zeroes happened to coincide. The only other sign change is at the point $\sigma = \hat{\sigma}_k$.

By the above accounting, $\mathcal{F}_k(\hat{\sigma}_j)$ will take the same sign of $\mathcal{F}_k(\sigma)$ where $0 < \sigma < \hat{\sigma}_j$ for all such $\hat{\sigma}_j$ with $j < k$, or equivalently for $\hat{\sigma}_j < \hat{\sigma}_k$. It reduce that $\mathcal{F}_k(\hat{\sigma}_j) < 0$ if $j < k$, because we have the following lemma :

Lemma 5.27

For all $k = 1, 2, ..., 2n$, and $\sigma \in (0, \hat{\sigma}_1)$,

$$\mathcal{F}_k(\sigma) < 0.$$

PROOF :

By Lemma 4.25, $P(e^{i\sigma}) > 0$ if $\sigma \in (0, \hat{\sigma}_1)$. By (5.2.7) and (5.2.8), for $\sigma \in (0, \hat{\sigma}_1)$

$$\begin{aligned} &\frac{(-2)Q_k(\sigma)}{R(0)\sin\sigma} = \\ &\quad \left(\frac{R'(0)}{R(0)} - 2 + S_k\right) + 2(1+\cos\sigma) + \\ &\quad \sum_{m=1}^{m=n} [\beta_m + \alpha_m S_k] \left[\frac{1}{2(1+\cos\sigma)}\right]^m \\ &\quad \stackrel{\text{def}}{=} C_1 + 2(1+\cos\sigma) + \sum_{m=1}^{m=n} \gamma_m \left[\frac{1}{2(1+\cos\sigma)}\right]^m \\ &\quad \stackrel{\text{def}}{=} h(e^{i\sigma}). \end{aligned} \tag{5.3.18}$$

So, in order to prove $\mathcal{F}_k(\sigma) < 0$ for $\sigma \in (0, \hat{\sigma}_1)$, we only need to prove $h(e^{i\sigma}) > 0$ for that σ. It is clear that $h(\omega)$ is analytic inside of the unit disc Ω except at $\omega = 0$ and takes real value on $\hat{I}_1$ without any singularity. Here we set $\hat{I}_1$ as a part of the boundary of the disk Ω such that

$$\hat{I}_1 = \{\omega = e^{i\sigma} : -\hat{\sigma}_1 < \sigma < \hat{\sigma}_1\}$$

Thus we can extend $h(\omega)$ through $\hat{I}_1$ from one side to another side by this way :

$$H(\omega) = \begin{cases} h(\omega) & |\omega| < 1 \quad -\hat{\sigma}_1 < \sigma < \hat{\sigma}_1 \\ h(\omega^*) & |\omega| > 1 \quad -\hat{\sigma}_1 < \sigma < \hat{\sigma}_1, \end{cases}$$

with $\omega\omega^* = 1$.

Consider $\omega = re^{i\sigma}$ with $\sigma = 0$, we obtain

$$H(r) = C_1 + \frac{(1+r^2)}{r} + \sum_{m=1}^{m=n} \gamma_m \left[\frac{r}{(1+r)^2}\right]^m . \tag{5.3.19}$$

For our purpose, it is enough to show that $H(1) > 0$. If it is true, then $H(r)$ will be positive on a neighborhood of $r = 1$. That means $h(e^{i\sigma})$ or $[-Q_k(\sigma)]$ is positive on somewhere inside of $(0, \hat{\sigma}_1)$, if σ is close to 0. Since by Corollary 5.8, $Q_k(\sigma)$ is real and remains opposite sign where $\sigma \in (0, \hat{\sigma}_1)$, so $Q_k(\sigma)$ is negative for all σ which are in this interval. Thus $\mathcal{F}_k(\sigma) < 0$ for $\sigma \in (0, \hat{\sigma}_1)$ and any k.

By (5.3.19), $H(0) = +\infty$ and $H(+\infty) = +\infty$, therefore that $H(1) > 0$ is true if $H(r)$ has no real roots wherever r remains real and positive.

We can rewrite (5.3.19) as a $(n+1)$-th degree polynomial of δ :

$$\begin{aligned} \frac{r}{(1+r^2)} H(r) &= 1 + C_1 \frac{r}{(1+r^2)} + \sum_{m=1}^{m=n} \gamma_m \left[\frac{r}{(1+r)^2}\right]^{m+1} \\ &\stackrel{\text{def}}{=} 1 + C_1\delta + \sum_{m=1}^{m=n} \gamma_m \delta^{m+1} \\ &\stackrel{\text{def}}{=} \tilde{H}(\delta), \end{aligned} \tag{5.3.20}$$

if we set

$$\delta = \frac{r}{(1+r)^2}.$$

Thus $H(r)$ has no real roots wherever r remains real and positive if and only if $\tilde{H}(\delta)$ has no real roots wherever δ remains real and positive.

Comparing (5.3.20) with (5.3.18), one way to get roots of $\tilde{H}(\delta)$, say δ_m, is to try a nonzero monotonic mapping :

$$\delta_m = \frac{1}{2(1+\cos\tilde{\sigma}_m)} \qquad \text{for } m = 0, 1, \ldots \tag{5.3.21}$$

while $\{\sigma = \tilde{\sigma}_m\}$ are roots of $Q_k(\sigma)$.

By Corollary 5.8, $Q_k(\sigma)$ has $n+1$ simple roots with which one at $\sigma = \hat{\sigma}_k \overset{\text{def}}{=} \tilde{\sigma}_0$ and n others in each of the interval $(\hat{\sigma}_{2m-1}, \hat{\sigma}_{2m})$ for $m = 1, 2, \ldots n$, say $\sigma = \tilde{\sigma}_m$. However $\tilde{H}$ can have at most $n+1$ different roots, so that by (5.3.21) and (5.3.20), each possible root of $H(r)$, say $\bar{r}$, should satisfy

$$\frac{\bar{r}}{(1+\bar{r})^2} = \delta_m = \frac{1}{2(1+\cos\tilde{\sigma}_m)},$$

for some m between 0 and n. That means

$$\bar{r}^2 - 2\bar{r}\cos\tilde{\sigma}_m + 1 = 0.$$

Nevertheless, it makes a negative discriminant :

$$\Delta = 4\cos^2\tilde{\sigma}_m - 4 < 0,$$

for any such m. Thus, as we wished, there are no real roots of $H(r)$. It means $H(1) > 0$, completing the proof of Lemma 5.27.

With this lemma and the argument of sign change of $\mathcal{F}_k(\sigma)$, we conclude that the fractional function $\mathcal{F}_k$ is negative at $\sigma = \hat{\sigma}_j$ if $j < k$ or positive if $j > k$. That is

$$S_j(\vec{V}) - S_k(\vec{V}) = \mathcal{F}_k(\hat{\sigma}_j) \begin{cases} > 0 & \text{if } j > k \\ < 0 & \text{if } j < k \end{cases},$$

leading to the fact of (5.1.3).

More information can be gained by analyzing the simple zero of the fractional function at $\sigma = \hat{\sigma}_k$. Since

$$\mathcal{F}_k(\sigma) = \hat{V}(\sigma, \Sigma) - \hat{S}_k(\Sigma) \begin{cases} > 0 & \text{if } \sigma > \hat{\sigma}_k \\ < 0 & \text{if } \sigma < \hat{\sigma}_k \end{cases},$$

we find that this function is positive to the immediate right and negative to the immediate left of $\hat{\sigma}_k$. Thus, the derivative of $\mathcal{F}_k(\sigma)$ is possible positive at $\hat{\sigma}_k$. However, this derivative is just $\partial_\sigma \hat{V}(\sigma, \Sigma)$, so we have almost proved :

Lemma 5.28

$$\partial_\sigma \hat{V}(\sigma, \Sigma)_{\sigma=\hat{\sigma}_k} > 0.$$

PROOF :

We only need to show $\partial_\sigma \mathcal{F}_k(\sigma)_{\sigma=\hat{\sigma}_k} \neq 0$. If it is not the case, then $\hat{\sigma}_k$ is at least a double root of $\mathcal{F}_k$. Since $Q_k(\hat{\sigma}_k) = 0$, $\hat{\sigma}_k$ should be at least a double root of $Q_k(\sigma)$ too. It is a contradiction against Corollary 5.8.

Lemma 5.28 will play an important role in the proof of the genuine nonlinearity assertion (5.1.4) which will be given later after following identities.

5.4 The Identities

In this section we derive useful identitics relating $P(e^{i\sigma})$ and $Q(e^{i\sigma})$, which depend on Σ through their coefficients determined by the relations (5.2.9) and (5.2.10), to their derivatives with respect to $\hat{\sigma}_k$, say $P_{\hat{\sigma}_k}(e^{i\sigma})$ and $Q_{\hat{\sigma}_k}(e^{i\sigma})$.

For some technical reason, we need a variable change as before :

$$\delta = 2(1 + \cos\sigma) \tag{5.4.22}$$

with

$$\delta_l = 2(1 + \cos\hat{\sigma}_l)$$

for $l = 1, 2, ..., 2n$. With it we can rewrite P, Q, R, Q_k and some integrals into simpler forms as given by (5.3.12)-(5.3.17).

Taking the derivative of (5.3.17) with respect to $\hat{\sigma}_k$ and noting the fact there can be no endpoint contribution since the integrand after variable change of (5.4.22) vanishes as $|\delta - \delta_j|^{1/2}$ at $\delta = \delta_j$, we find

$$\begin{aligned} 0 &= \partial_{\hat{\sigma}_k} \int_{\hat{\sigma}_{2j-1}}^{\hat{\sigma}_{2j}} \frac{Q_k(\sigma)}{R(e^{i\sigma})} d\sigma \\ &= \partial_{\delta_k} \left[\int_{\delta_{2j}}^{\delta_{2j-1}} \frac{\tilde{Q} - S_k \tilde{P}}{\tilde{R}} d\delta, \right] (-2 \sin \hat{\sigma}_k), \end{aligned}$$

thus

$$\begin{aligned} 0 = \partial_{\delta_k} &\left[\int_{\delta_{2j}}^{\delta_{2j-1}} \frac{\tilde{Q} - S_k \tilde{P}}{\tilde{R}} d\delta, \right] \\ = \int_{\delta_{2j}}^{\delta_{2j-1}} &\left[\frac{\tilde{Q}_{\delta_k} - S_k \tilde{P}_{\delta_k} - \tilde{P} \partial_{\delta_k} S_k}{\tilde{R}} - \frac{1}{2} \frac{(\tilde{Q} - S_k \tilde{P}) \partial_{\delta_k} \tilde{R}^2}{\tilde{R}^3} \right] d\delta, \end{aligned} \tag{5.4.23}$$

for $j = 1, 2, ..., n$. The form of $\tilde{R}^2$ given by (5.3.14) offers

$$\partial_{\delta_k} \tilde{R}^2 = -\tilde{R}^2 / (\delta - \delta_k).$$

With that and the fact $\tilde{P}$ satisfies (5.3.15), we can write (5.4.23) as

$$\int_{\delta_{2j}}^{\delta_{2j-1}} \frac{\tilde{Z}_k(\delta)}{\tilde{R}(\delta)} d\delta = 0,$$

for $j = 1, 2, ..., n$; where

$$\tilde{Z}_k(\delta) = \tilde{Q}_{\delta_k} - S_k \tilde{P}_{\delta_k}(\delta) + \frac{\tilde{Q}(\delta) - S_k \tilde{P}(\delta)}{2(\delta - \delta_k)}. \tag{5.4.24}$$

Since $\tilde{Q}(\delta) - S_k \tilde{P}(\delta)$ is a polynomial in δ that vanishes of $\delta = \delta_k$, we know $\tilde{Z}_k(\delta)$ is also a polynomial in δ. From the definition (5.3.15) of $\tilde{P}(\delta)$ and (5.3.16) of $\tilde{Q}(\delta)$ we can have a large δ asymptotics:

$$\tilde{P}_{\delta_k}(\delta) = O(\delta^{n-1}),$$

$$\tilde{Q}_{\delta_k}(\delta) = \frac{1}{8} \delta^n + O(\delta^{n-1}),$$

$$\frac{\tilde{Q}(\delta) - S_k \tilde{P}(\delta)}{\delta - \delta_k} = -\frac{1}{4} \delta^n + O(\delta^{n-1}),$$

which, to be combined together in (5.4.24), shows that

$$\tilde{Z}_k(\delta) = O(\delta^{n-1}).$$

Therefore, as a polynomial that satisfies (5.2.11) but of degree less than n, $\tilde{Z}_k(\delta)$ must vanish by Corollary 5.7. It shows

Proposition 5.1

For every $k = 1, 2, ..., 2n$; one has the identity

$$\tilde{Q}_{\delta_k} - S_k \tilde{P}_{\delta_k}(\delta) = -\frac{1}{2}\frac{\tilde{Q}(\delta) - S_k \tilde{P}(\delta)}{(\delta - \delta_k)}.$$

5.5 Genuine Nonlinearity

The proof of assertion (5.1.4) of Theorem 5.40 relating to genuine nonlinearity now can be proceeded as a fairly direct calculation from the definition (4.2.74) of $\hat{S}_k(\Sigma)$:

$$\begin{aligned} &\partial_{\hat{\sigma}_k} \hat{S}_k(\Sigma) \\ &= \partial_{\delta_k}\left(\frac{\tilde{Q}(\delta_k)}{\tilde{P}(\delta_k)}\right) \cdot \partial_{\hat{\sigma}_k}\delta_k \\ &= \left[\partial_\delta\left(\frac{\tilde{Q}(\delta)}{\tilde{P}(\delta)}\right) + \partial_{\delta_k}\left(\frac{\tilde{Q}(\delta)}{\tilde{P}(\delta)}\right)\right]_{\delta=\delta_k}(-2\sin\hat{\sigma}_k). \end{aligned} \tag{5.5.25}$$

By using Proposition 5.1 in the third step, we see the last term inside of the brackets becomes

$$\begin{aligned} &\partial_{\delta_k}\left(\frac{\tilde{Q}(\delta)}{\tilde{P}(\delta)}\right)_{\delta=\delta_k} \\ &= \left[\frac{\tilde{P}(\delta)\tilde{Q}_{\delta_k}(\delta) - \tilde{Q}(\delta)\tilde{P}_{\delta_k}(\delta)}{(\tilde{P}(\delta))^2}\right]_{\delta=\delta_k} \\ &= \left[\frac{\tilde{Q}_{\delta_k}(\delta) - S_k\tilde{P}_{\delta_k}(\delta)}{\tilde{P}(\delta)}\right]_{\delta=\delta_k} \\ &= -\frac{1}{2}\left[\frac{\tilde{Q}(\delta) - S_k\tilde{P}(\delta)}{(\delta-\delta_k)\tilde{P}(\delta)}\right]_{\delta=\delta_k} \end{aligned}$$

$$= -\frac{1}{2}\left[\frac{\tilde{V}(\delta) - S_k}{\delta - \delta_k}\right]_{\delta=\delta_k}$$
$$= -\frac{1}{2}\partial_\delta \tilde{V}(\delta)_{\delta=\delta_k}. \qquad (5.5.26)$$

Here we set $\tilde{V}(\delta) = \tilde{Q}(\delta)/\tilde{P}(\delta)$ and use the definition of the derivative to evaluate the quotient in the last step. Substituting the result of (5.5.26) into (5.5.25) gives

Lemma 5.29

For every $k = 1, 2, ..., 2n$,

$$\partial_{\hat{\sigma}_k} \hat{S}_k(\Sigma) = \frac{1}{2}\partial_\sigma \hat{V}(\sigma, \Sigma)_{\sigma=\hat{\sigma}_k}. \qquad (5.5.27)$$

PROOF :

By (5.5.25) and (5.5.26),

$$\begin{aligned}
&\partial_{\hat{\sigma}_k} \hat{S}_k(\Sigma) \\
&= \left[\partial_\delta \left(\frac{\tilde{Q}(\delta)}{\tilde{P}(\delta)}\right) + \partial_{\delta_k}\left(\frac{\tilde{Q}(\delta)}{\tilde{P}(\delta)}\right)\right]_{\delta=\delta_k} (-2\sin\hat{\sigma}_k) \\
&= [\partial_\delta \tilde{V}(\delta) - \frac{1}{2}\partial_\delta \tilde{V}(\delta)]_{\delta=\delta_k}(\frac{\partial\delta}{\partial\sigma})_{\sigma=\hat{\sigma}_k} \\
&= \frac{1}{2}\partial_\sigma \left(fracQ(\sigma)P(\sigma)\right)_{\sigma=\hat{\sigma}_k} = \frac{1}{2}\partial_\sigma \hat{V}(\sigma, \Sigma)_{\sigma=\hat{\sigma}_k}.
\end{aligned}$$

It proves (5.5.27). By Lemma 5.28, the right-hand side of (5.5.27) is positive, thus we have

$$\partial_{\hat{\lambda}_k} S_k(\vec{V}) =$$
$$[\partial_{\hat{\sigma}_k} \hat{S}_k(\Sigma)]/(\partial_{\hat{\sigma}_k}\hat{\lambda}_k) < 0,$$

since

$$(\partial_{\hat{\sigma}_k}\hat{\lambda}_k) = -\sin\hat{\sigma}_k < 0.$$

It proves the assertion (5.1.4) and completes the proof of Theorem 5.40.

5.6 The Breaking

In order to study the breaking of solutions to the Riemann invariant equations given by (5.1.2), we can show that $\xi_k = \partial_x \hat{\lambda}_k$ satisfies the Riccati equation (5.1.5) along the k^{th} characteristic. This is achieved by the same way Levermore used for zero dispersion limit of KdV [12].

Now fix a $k = 1, 2, ..., 2n$. Take ∂_x of the k^{th} Riemann invariant equation (5.1.2) to get an equation for ξ_k,

$$\begin{aligned}
0 &= \partial_x(\partial_t \hat{\lambda}_k + S_k(\vec{V})\partial_x \hat{\lambda}_k) \\
&= \partial_t(\partial_x \hat{\lambda}_k) + [\partial_x S_k(\vec{V})]\partial_x \hat{\lambda}_k + S_k(\vec{V})\partial_x(\partial_x \hat{\lambda}_k) \\
&= \partial_\tau \xi_k + [\sum_j (\partial_{\hat{\lambda}_j} S_k)\partial_x \hat{\lambda}_j]\partial_x \hat{\lambda}_k \\
&= \partial_\tau \xi_k + (\partial_{\hat{\lambda}_k} S_k)\xi_k^2 + [\sum_{j\neq k} (\partial_{\hat{\lambda}_j} S_k)\xi_j]\xi_k, \qquad (5.6.28)
\end{aligned}$$

where $\partial_\tau = \partial_t + S_k \partial_x$ is the advective derivative along the k^{th} characteristic. In this notation the Riemann invariant equations (5.1.2) can be written as

$$\partial_\tau \hat{\lambda}_j + (S_j - S_k)\xi_j = 0, \qquad (5.6.29)$$

for $j = 1, ..., 2n$.

Working toward the simplification of the last term of equation (5.6.28), we start with a direct calculation of $\partial_{\hat{\lambda}_j} S_k$ for $j \neq k$, by using the definition (5.2.6) of $S_k(\vec{V})$ and a variable change $\delta_j = 2(1 + \cos \hat{\sigma}_j) = 2(1 + \hat{\lambda}_j)$:

$$\begin{aligned}
\partial_{\hat{\lambda}_j} S_k &= (\partial_{\hat{\lambda}_j} \delta_j)\partial_{\delta_j} S_k \\
&= (\partial_{\hat{\lambda}_j} \delta_j)\left[\frac{\tilde{Q}_{\delta_j}(\delta_k) - S_k \tilde{P}_{\delta_j}(\delta_k)}{\tilde{P}(\delta_k)}\right].
\end{aligned}$$

Using the identity of Proposition 5.1 to eliminate $\tilde{Q}_{\delta_j}(\delta_k)$ gives

$$\begin{aligned}
&\partial_{\hat{\lambda}_j} S_k \\
&= (\partial_{\hat{\lambda}_j} \delta_j)\left[(S_j - S_k)\frac{\tilde{P}_{\delta_j}(\delta_k)}{\tilde{P}(\delta_k)} - \frac{1}{2}\frac{S_j - S_k}{\delta_j - \delta k}\right]
\end{aligned}$$

$$
\begin{aligned}
&= (\partial_{\hat{\lambda}_j}\delta_j)\left[(S_j - S_k)\partial_{\delta_j}\log\left(\frac{\tilde{P}(\delta_k)}{\tilde{R}_k}\right)\right] \\
&= (S_j - S_k)\partial_{\hat{\lambda}_j}\log\left(\frac{\tilde{P}(\delta_k)}{\tilde{R}_k}\right) \\
&\overset{\text{def}}{=} (S_j - S_k)\partial_{\hat{\lambda}_j}\log(W_k).
\end{aligned}
\tag{5.6.30}
$$

Here $\tilde{R}_k$ is defined by

$$\tilde{R}_k^2 = (-1)^{2n-k}\prod_{j\neq k}(\delta_j - \delta_k),$$

$$\operatorname{sgn}\tilde{R}_k = \operatorname{sgn}\tilde{P}(\delta_k);$$

and $W_k(\vec{V})$ is defined by

$$W_k = \frac{\tilde{P}(\delta_k)}{\tilde{R}_k},$$

with $\delta_j = 2(1 + \hat{\lambda}_j)$ for $j = 1, ..., 2n$. Note the way that $\tilde{R}_k^2$ is defined so that W_k is always positive.

Since for each k, $\hat{\sigma}_k$ is a continues function of (x, t), so is δ_k by definition. Moreover since the set $\{\hat{\sigma}_k\}_1^{2n}$ is in order for any fixed (x, t), then the set $\{\delta_k\}_1^{2n}$ is in order too and no δ_k will be zero. Thus $\tilde{R}_k$ will be not zero in that case as well. Also $\tilde{P}(\delta_k)$ never be zero since $P(e^{i\hat{\sigma}_k}) \neq 0$ for any k. That means for each k and any fixed (x, t), $W_k \neq 0$. So, with a continuity argument, we get

Proposition 5.2

For any compact (x, t) subset over which the ordering condition holds, the positive function W_k is uniformly bounded both above, away from infinity, and below, away from zero.

Using first (5.6.30) and then (5.6.29) gives the relation

$$\sum_{j\neq k}(\partial_{\hat{\lambda}_j}S_k)\xi_j =$$

$$\sum_{j\neq k}(\partial_{\hat\lambda_j}\log W_k)(S_j - S_k)\xi_j =$$
$$-\sum_{j\neq k}(\partial_{\hat\lambda_j}\log W_k)\partial_\tau\hat\lambda_j = -\partial_\tau \log W_k, \tag{5.6.31}$$

where in the last step one used the fact $\partial_\tau\hat\lambda_k = 0$. Substituting (5.6.31) into the last term of (5.6.28) leads to the Riccati equation

$$\begin{aligned} 0 &= \partial_\tau\xi_k + (\partial_{\hat\lambda_k}S_k)\xi_k^2 - \xi_k\partial_\tau \log W_k \\ &= W_k^{-1}[\partial_\tau\xi_k - (\xi_k\partial_\tau W_k)/w_k + (W_k^2\partial_{\hat\lambda_k}S_k)(\xi_k/W_k)^2] \\ &= \partial_\tau\left(\frac{\xi_k}{W_k}\right) + (W_k\partial_{\hat\lambda_k}S_k)\left(\frac{\xi_k}{W_k}\right)^2 . \end{aligned} \tag{5.6.32}$$

Expanding the advective derivative in (5.6.32) gives equation (5.1.5) and proves Theorem 5.41.

By the genuine nonlinearity assertion (5.1.4) of Theorem 5.40, the coefficient of the quadratic term in the Riccati equation (5.6.32) is negative. So ξ_k/W_k is a strictly increasing quantity along the characteristic provided $\xi_k \neq 0$. Now if $\xi_k < 0$, the only way to have $\xi_k \to +\infty$ in finite time is for $W_k \to \infty$ such that ξ_k/W_k is increasing. However, if the ordering condition (5.1.1) holds, the continuity implies that W_k must be bounded above. Thus we conclude :

Corollary 5.9

If $|\partial_x\hat\lambda_k|$ becomes unbounded in finite timc along the k^{th} characteristic, then : either

(a)

$$\partial_x\hat\lambda_k > 0,$$

or

(b)

The ordering condition (5.1.1) is violated in finite time.

APPENDIX A

PROOF OF LEMMA 2.1

Suppose $X \subset \Re$ is a compact set, both $\{f_n\}$ and f are convex, $C^1(X)$ functions and $f_n \to f$ uniformly as $n \to \infty$ for $x \in X$, then by convexity,

$$\frac{\partial f_n}{\partial x} \leq \frac{f_n(x+h) - f_n(x)}{h},$$

and

$$\frac{\partial f_n}{\partial x} \geq \frac{f_n(x) - f_n(x-h)}{h}.$$

Moreover, we have the following lemma:

Lemma A.30 *If f is convex and $f' = \frac{\partial f}{\partial x}$ is continuous, then, as $h \to 0$,*

$$\frac{f(x+h) - f(x)}{h} \to f'(x)$$

is uniformly on X.

PROOF :

Set

$$g_h(x) = \frac{f(x+h) - f(x)}{h} - f'(x).$$

By convexity of f, $g_h(x) \searrow 0$ as $h \searrow 0$. Since both f and f' are continuous, so is $g_h(x)$. That means for any $\epsilon > 0$, there exist a $\hat{h}_{x,\epsilon}$ and a neighborhood O_x for each x, such that if $h < \hat{h}_{x,\epsilon}$, then $0 \leq g_h(y) < \epsilon$ for all $y \in O_x$. However, because X is compact, a finite number of these O_x can cover the whole X and then the total number of such necessary $\hat{h}$ is finite. That means there exists a H_ϵ as the minimum of these $\hat{h}$ such that if $h < H_\epsilon$, then $0 \le g_h(x) < \epsilon$ for all $x \in X$. That is $g_h(x) \searrow 0$ as $h \searrow 0$ uniformly on X.

Similarly, we can prove $g_h(x) \nearrow 0$ as $h \nearrow 0$ uniformly on X. It completes the proof of Lemma A.30.

Let us go back to the proof of Lemma 2.1. Because of Lemma A.30, we can for any $\epsilon > 0$ pick $h > 0$ such that

$$|\frac{f(x+h)-f(x)}{h} - f'| \le \frac{\epsilon}{2}$$

and

$$|\frac{f(x)-f(x-h)}{h} - f'| \le \frac{\epsilon}{2}$$

for all $x \in X$.

Because of the uniform convergence of $\{f_n\}$, for the same h, we can pick N such that if $n \ge N$, then

$$|f_n(x) - f(x)| \le \epsilon h/4,$$

$$|f_n(x+h) - f(x+h)| \le \epsilon h/4$$

and

$$|f_n(x-h) - f(x-h)| \le \epsilon h/4.$$

Thus, by convexity, for all $x \in X$,

$$\begin{aligned}
(f_n' - f') &\le \frac{f_n(x+h)-f_n(x)}{h} - f' \\
&= [\frac{f_n(x+h)-f_n(x)}{h} - \frac{f(x+h)-f(x)}{h}] \\
&+ [\frac{f(x+h)-f(x)}{h} - f'] \\
&\le \epsilon/4 + \epsilon/4 + \epsilon/2 = \epsilon
\end{aligned}$$

and

$$(f_n' - f') \ge \frac{f_n(x)-f_n(x-h)}{h} - f'$$

$$
\begin{aligned}
&= [\frac{f_n(x) - f_n(x-h)}{h} - \frac{f(x) - f(x-h)}{h}] \\
&+ [\frac{f(x) - f(x-h)}{h} - f'] \\
&\geq -\epsilon/4 - \epsilon/4 - \epsilon/2 = -\epsilon.
\end{aligned}
$$

That means $f_n' \to f'$ uniformly as $n \to \infty$. The proof of Lemma 2.1 is then completed.

APPENDIX B

WKB METHOD

The following brief of WKB method is created by [7].

The asymptotic solution of the ordinary linear differential equation of the second order

$$\frac{d^2y}{dx^2} + k^2P(x)y = 0 \tag{B.0.1}$$

for $k \to \infty$ was first treated systematically by G.B.Jeffreys (1925).

However, G.Wentzel, H.Kramers and L.Brillouin (1926) studied similar solutions of the Schrödinger equation independently.

Their method is therefore called the *W.K.B. method* or the *Jeffrey's method.*

When we put

$$z = \int P^{1/2}dx, \qquad\qquad y = P^{-1/4}w,$$

(B.0.1) reduces to the form :

$$\frac{d^2w}{dz^2} + (k^2 - Q(z))w = 0, \tag{B.0.2}$$

where

$$\begin{aligned} Q &= P^{-1/4}\frac{d^2P^{1/4}}{dz^2} \\ &= -P^{-3/4}\frac{d^2P^{-1/4}}{dx^2}. \end{aligned} \tag{B.0.3}$$

From (B.0.2), we can obtain an asymptotic solution by the trigonometric formula

$$w \sim Ae^{ikz} + Be^{-ikz}, \tag{B.0.4}$$

or by the exponential formula

$$w \sim A' e^{k|z|} + B' e^{-k|z|}, \tag{B.0.5}$$

according as $P > 0$ or $P < 0$.

A point x_0 at which $P(x)$ vanishes is called a *turning point.* Since Q tends to infinity at a turning point, we need certain devices to continue the solutions (B.0.4) and (B.0.5) analytically beyond a turning point. When x_0 is a single zero of $P(x)$, the *connection formula* connecting the two solutions is given by

$$P^{-1/4} \cos(kz - \frac{\pi}{4} + y) \rightarrow \sin y |P|^{-1/4} e^{k|z|}, \quad y \neq 0, \tag{B.0.6}$$

$$P^{-1/4} \cos(kz - \frac{\pi}{4}) \leftarrow \frac{1}{2} |P|^{-1/4} e^{-k|z|}, \tag{B.0.7}$$

where an arrow means that, for example in (B.0.6), the solution approximated by the left-hand side when $P > 0$ is approximated by the right-hand side when $P < 0$.

APPENDIX C

ZAKHAROV-SHABAT METHOD

C.1 Linearization of the NLS equation

Define operator L and A :

$$L = ih \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_x + \begin{pmatrix} 0 & \bar{U} \\ U & 0 \end{pmatrix}$$

and

$$\begin{aligned} A = {} & (2\sqrt{2}h^2\partial_{xx} + 4i\lambda h\partial_x) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ & -2 \begin{pmatrix} \frac{\lambda^2}{\sqrt{2}} + (\sqrt{2}-1)|U|^2 + 1 & ih\bar{U}_x(\sqrt{2}+1) \\ -ihU_x(\sqrt{2}-1) & \frac{\lambda^2}{\sqrt{2}} + (\sqrt{2}+1)|U|^2 - 1 \end{pmatrix}, \end{aligned} \tag{C.1.1}$$

where $\bar{U}$ means the complex conjugate of U.

Lemma C.31 *The NLS PDE*

$$ihU_t - \frac{1}{2}h^2 U_{xx} - (1 - |U|^2)U = 0 \tag{C.1.2}$$

can be linearized by the following ODE :

$$\begin{cases} LV &= \lambda V \\ ih\partial_t V &= AV \end{cases} \qquad V \in R^2.$$

with

$$\partial_t \lambda = 0.$$

PROOF :

(1)

Define operator $\hat{L}$ and $\hat{A}$:

$$\hat{L} = \begin{pmatrix} 1+\sqrt{2} & 0 \\ 0 & 1-\sqrt{2} \end{pmatrix} ih\partial_x + \begin{pmatrix} 0 & \bar{U} \\ U & 0 \end{pmatrix}, \tag{C.1.3}$$

and

$$\hat{A} = 2\sqrt{2}h^2\partial_{xx} - 2\begin{pmatrix} \frac{|U|^2}{\sqrt{2}+1}+1 & ih\bar{U}_x \\ -ihU_x & \frac{|U|^2}{\sqrt{2}-1}-1 \end{pmatrix}. \tag{C.1.4}$$

By a little calculation, one can find $\{\hat{L}, \hat{A}\}$ is just the Lax pair of equation (C.1.2). That means if χ and $\hat{E}$ are set as eigenfunction and eigenvalue of $\hat{L}$, then

$$-ih\hat{L}_t = (\hat{L}\hat{A} - \hat{A}\hat{L}) \stackrel{\text{def}}{=} [\hat{L}, \hat{A}] \tag{C.1.5}$$

is nessary for the following simultaneous facts :

$$\hat{E}_t = 0.$$

and

$$ih\chi_t = \hat{A}\chi.$$

(2) Set

$$\lambda = \sqrt{2}\hat{E}$$

and

$$\chi = e^{i\hat{E}x/h}\begin{pmatrix} (\sqrt{2}-1)^{1/2} & 0 \\ 0 & (\sqrt{2}+1)^{1/2} \end{pmatrix} V,$$

then

$$ih\chi_t = \hat{A} \Leftrightarrow ihV_t = AV$$

and

$$\hat{L}\chi = \hat{E}\chi \Leftrightarrow LV = \lambda V.$$

Therefore with respect to the NLS equation (C.1.2), we will name $LV = \lambda V$ as the ***associated eigenvalue equation*** and $ihV_t = AV$ as the ***associated time-evolution equation.***

C.2 Time evolution of scattering data

Lemma C.32 *The scattering data of the NLS equation (C.1.2) vary with time t in the following manner:*

(i) $R(k,t) = R(k)\exp\{2ikt\sqrt{k^2-1}/h\}$,

(ii) λ_j *is independent of* t,

(iii) $c_j(t) = c_j \exp\{-\lambda_j \nu_j t/h\}$, $_{j=1,\ldots,N}$.

PROOF :

Recall

$$\mathcal{L}V = LV - \lambda V = \begin{pmatrix} ih\partial_x - \lambda & \bar{U} \\ U & -ih\partial_x - \lambda \end{pmatrix} V,$$

and set

$$\mathcal{L}_o V = L_o V - \lambda V = \begin{pmatrix} ih\partial_x - \lambda & 1 \\ 1 & -ih\partial_x - \lambda \end{pmatrix} V.$$

We get

$$\mathcal{L}V \to \mathcal{L}_o V \qquad (x \to \infty)$$

and

$$V \to \alpha_1 e^{-i\varsigma x/h} + \alpha_2 e^{i\varsigma x/h} \quad (x \to +\infty)$$

with $\varsigma = \sqrt{\lambda^2 - 1}$.

We assume the Jost functions of NLS equation (C.1.2) are ψ_1, ψ_2, ϕ_1 and ϕ_2. That means they are not only solutions of $\mathcal{L}V = 0$ but also with the following boundary conditions.

$$\psi_j \to X_j^+ \qquad (x \to +\infty)$$

and

$$\phi_j \to X_j^- \qquad (x \to -\infty)$$

for $j = 1, 2$. Here

$$X_1^+ = \begin{pmatrix} 1 \\ \lambda - \varsigma \end{pmatrix} e^{-i\varsigma x/h}, \quad X_2^+ = \begin{pmatrix} \lambda - \varsigma \\ 1 \end{pmatrix} e^{i\varsigma x/h}$$

and

$$X_1^- = \begin{pmatrix} 1 \\ e^{i\varrho}(\lambda - \varsigma) \end{pmatrix} e^{-i\varsigma x/h}, \quad X_2^- = \begin{pmatrix} e^{i\varrho}(\lambda - \varsigma) \\ 1 \end{pmatrix} e^{i\varsigma x/h}$$

with ϱ from the boundary condition of U:

$$U \to \begin{cases} 1 & (x \to +\infty) \\ e^{i\varrho/h} & (x \to -\infty) \end{cases} .$$

Thus the Jost functions could have the following forms:

$$\psi_j(x, \lambda) = X_j^+(x, \lambda) - \int_x^{+\infty} \Psi(x, s) X_j^+(s, \lambda)\, ds$$

and

$$\phi_j(x, \lambda) = X_j^-(x, \lambda) + \int_{-\infty}^{x} \Phi(x, s) X_j^-(s, \lambda)\, ds,$$

where

$$\Psi(x, y) = \begin{pmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{pmatrix}$$

with

$$\Psi(x, y) \to 0 \quad (y \to +\infty)$$

and

$$\Phi(x, y) \to 0 \quad (y \to -\infty).$$

With the relation of that

$$L\Psi_j = \lambda \Psi_j \qquad and \qquad L\Psi_j \to L_o \Psi_j$$

for $j = 1, 2$, one can find :

$$(\partial_x + \partial_y)\begin{pmatrix} \Psi_{11} \\ \Psi_{22} \end{pmatrix} = -\frac{i}{h}\begin{pmatrix} 1 & -\bar{U} \\ U & -1 \end{pmatrix}\begin{pmatrix} \Psi_{12} \\ \Psi_{21} \end{pmatrix} \tag{C.2.6}$$

and

$$(\partial_x - \partial_y)\begin{pmatrix} \Psi_{12} \\ \Psi_{21} \end{pmatrix} = -\frac{i}{h}\begin{pmatrix} 1 & -\bar{U} \\ U & -1 \end{pmatrix}\begin{pmatrix} \Psi_{11} \\ \Psi_{22} \end{pmatrix} \tag{C.2.7}$$

with

$$\Psi_{22} = \bar{\Psi}_{11}, \quad \Psi_{21} = \bar{\Psi}_{12}.$$

The associated boundary condition of system (C.2.6)-(C.2.7) is

$$\begin{aligned} &\Psi_{12}(x,x) = i(\bar{U}(x) - 1)/(2h) \\ &\Psi_{jk}(x,y) \to 0 \;\; (y \to \infty). \end{aligned} \tag{C.2.8}$$

Since all the Jost functions are solutions of $\mathcal{L}V = 0$, there should exist a matrix

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix},$$

such that

$$(\psi_1, \psi_2)S = (\phi_1, \phi_2).$$

Since L is self-adjoint, then $\tilde{V}$ is also an eigenfunction of L with same eigenvalue if V is already an eigenfunction, where the relation between V and $\tilde{V}$ is given by

$$V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \quad \tilde{V} = \begin{pmatrix} \bar{V}_2 \\ \bar{V}_1 \end{pmatrix}.$$

Thus

$$S_{11} = \bar{S}_{22} \overset{\text{def}}{=} a$$

and

$$S_{21} = \bar{S}_{12} \overset{\text{def}}{=} b.$$

We can write ϕ_1 as a linear combination of ψ_j :

$$\phi_1 = a\psi_1 + b\psi_2.$$

Usually, $T = 1/a$ is named as the *transmission coefficient* and $R = b/a$ is named as the *reflection coefficient.*

Recall

$$ih\partial_t V = AV$$

with operator A given by (C.1.1). Consider $x \to +\infty$ or $x \to -\infty$, we get

$$\partial_t a = 0, \qquad \partial_t b = i2(k\sqrt{k^2-1})b/h.$$

Here we set $k = \lambda$ if $\lambda \geq 1$.

Follow these results, it comes that

$$R(k,t) = R(k)\exp\{2ikt\sqrt{k^2-1}/h\}.$$

Now we only consider $x \to +\infty$.

Suppose

$$V \propto \psi_2 = \begin{pmatrix} (\lambda - i\nu)\tau(x) \\ \bar{\tau}(x) \end{pmatrix} \to \begin{pmatrix} \lambda - i\nu \\ 1 \end{pmatrix} e^{-\nu x/h}$$

with $\nu = \sqrt{1-\lambda^2}$.

The norming constant $c(\lambda, t)$ can then be defined as satisfying :

$$\int_{-\infty}^{+\infty} c^2|\tau(x)|^2\, dx = 1.$$

It comes out that

$$c^2 = -b/(h\nu\partial_\lambda a).$$

We get

$$\frac{d[\log(c^2)]}{dt} = \frac{d(\log b)}{dt}$$

and

$$c(\lambda, t) = c(\lambda)\exp(-\lambda\nu t/h).$$

C.3 Inverse scattering transform and soliton solution

Recall

$$\phi_1 = a\psi_1 + b\psi_2.$$

That is

$$(\phi_1/a - X_1^+)e^{i\varsigma y/h}/(2\pi\varsigma) = (\psi_1 - X_1^+ + b\psi_2/a)e^{i\varsigma y/h}/(2\pi\varsigma)$$

with

$$y > x, \qquad \varsigma^2 + 1 = \lambda^2.$$

Since the left-hand-side is analytic while $\Im\varsigma > 0$ except simple poles $\{\lambda_j\}_1^N$ (zeros of a), the conter integral can be applied for both two sides.

For the left side,

$$\oint_{L.H.S} = \sum_{j=1}^{j=N} 2\mu_j\psi_2(x,\lambda_j)e^{-\nu_j y/h},$$

with

$$\nu_j = \sqrt{1-\lambda_j^2} > 0$$

and

$$\mu_j = -hc_j^2/2,$$

where c_j is the norming constant.

It could have a form which is more symmetrical :

$$\oint_{L.H.S} = 2\begin{pmatrix} -F_1^{(2)}(x+y) + i(F_2^{(2)})'(x+y) \\ F_2^{(2)}(x+y) \end{pmatrix}$$

$$-2\int_x^{+\infty} \Psi(x,s)\begin{pmatrix} -F_1^{(2)}(s+y) + i(F_2^{(2)})'(s+y) \\ F_2^{(2)}(s+y) \end{pmatrix} ds$$

with

$$F_1^{(2)}(z) = -\sum_{j=1}^{j=N} \mu_j\lambda_j e^{-\nu_j z/h},$$

$$F_2^{(2)}(z) = -\sum_{j=1}^{j=N} \mu_j e^{-\nu_j z/h}$$

and

$$F'(z) = h\partial_z F.$$

Similarly, we can also write out

$$\oint_{R.H.S} = -2h\Psi(x,y)\begin{pmatrix} 0 \\ 1 \end{pmatrix} + 2\begin{pmatrix} F_1^{(1)}(x+y) - i(F_2^{(1)})'(x+y) \\ -F_2^{(1)}(x+y) \end{pmatrix}$$
$$-2\int_x^{+\infty} \Psi(x,s)\begin{pmatrix} F_1^{(1)}(s+y) - i(F_2^{(1)})'(s+y) \\ -F_2^{(1)}(s+y) \end{pmatrix} ds$$

with

$$F_1^{(1)}(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} R_1(\varsigma)e^{i\varsigma z/h}\, d\varsigma$$

and

$$F_2^{(1)}(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} R_2(\varsigma)e^{i\varsigma z/h}\, d\varsigma.$$

Here

$$R_1(\varsigma) = [R(\lambda,\varsigma) + R(-\lambda,\varsigma)]/2$$

and

$$R_2(\varsigma) = [R(\lambda,\varsigma) - R(-\lambda,\varsigma)]/2$$

with

$$R(\lambda,\varsigma) = b(\lambda,\varsigma)/a(\lambda,\varsigma).$$

That is just the Marchenko equation pair:

$$\begin{aligned} &h\Psi_{12}(x,y) + \\ &\int_x^{\infty}\{\Psi_{12}(x,s)[-F_2(s+y)] + \Psi_{11}(x,s)[F_1(s+y) - iF_2'(s+y)]\}\, ds \\ &= F_1(x+y) - iF_2'(x+y) \end{aligned} \tag{C.3.9}$$

$$\begin{aligned} &h\Psi_{11}(x,y) + \\ &\int_x^{\infty}\{\Psi_{11}(x,s)[-F_2(s+y)] + \Psi_{12}(x,s)[F_1(s+y) + iF_2'(s+y)]\}\, ds \\ &= -F_2(x+y), \end{aligned}$$

with

$$F_1(z) = F_1^1(z) + F_1^{(2)}(z)$$

$$F_2(z) = F_2^1(z) + F_2^{(2)}(z).$$

Consider reflectionless case. That is

$$b(\lambda, \varsigma) \equiv 0 \text{ or } R(k) \equiv 0.$$

Then the Marchenko equation pair is reduced to the same form of (C.3.9) but in which only

$$F_1(z) = -\sum_{j=1}^{j=N} \mu_j \lambda_j e^{-\nu_j z/h}$$

and

$$F_2(z) = \sum_{j=1}^{j=N} \mu_j e^{-\nu_j z/h}.$$

Therefore it is easy to write

$$\Psi_{12} = (f, \beta) \quad \Psi_{11} = (g, \beta)$$

as inner product of vectors with

$$f = \begin{pmatrix} * \\ f_j(x) \\ * \end{pmatrix}, \quad g = \begin{pmatrix} * \\ g_j(x) \\ * \end{pmatrix}, \quad \beta(y,t) = \begin{pmatrix} * \\ 2^{-1/2} c_j(t) \exp(-\nu_j y/h) \\ * \end{pmatrix},$$

and norming constants $\{c_j\}_1^N$.

Then the Marchenko equation (C.3.9) can be converted to an algebra system and solved for:

$$\Psi_{11} = -(1/2)\partial_x[\log\det(I + \hat{G})_{2N\times 2N}]$$

with $\hat{G}$ defined by (2.1.36).

Recall (C.2.6)-(C.2.8) about the equations and the boundary condition of Jost functions, one can get the relation between Ψ_{11} and U :

$$\partial_x \Psi_{11}(x,x) = (\partial_x + \partial_y)\Psi_{11}(x,y)_{y=x}/2 = (|U|^2 - 1)/(2h^2).$$

Thus we get the Zakharov and Shabat formula for the solution of NLS equation (C.1.2) without reflection. This is a N-soliton type solution.

$$\rho = |U|^2 = 1 - h^2 \log \det(I + \hat{G}).$$

The above process is often named as doing an inverse scattering transform.

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