INFORMATION TO USERS

This manuscript has been reproduced from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps. Each original is also photographed in one exposure and is included in reduced form at the back of the book.

Photographs included in the original manuscript have been reproduced xerographically in this copy. Higher quality 6" x 9" black and white photographic prints are available for any photographs or illustrations appearing in this copy for an additional charge. Contact UMI directly to order.

UMI
University Microfilms International
A Bell & Howell Information Company
300 North Zeen Road, Ann Arbor, MI 48106-1346 USA
313/761-4700 800:521-0600
Prediction and analysis of wing flutter at transonic speeds

Shieh, Teng-Hua, Ph.D.
The University of Arizona, 1991

Copyright ©1991 by Shieh, Teng-Hua. All rights reserved.
PREDICTION AND ANALYSIS OF WING FLUTTER
AT TRANSONIC SPEEDS

by
Teng-Hua Shieh

Copyright © Teng-Hua Shieh 1991

A Dissertation Submitted to the Faculty of the
DEPARTMENT OF AEROSPACE AND MECHANICAL ENGINEERING
In Partial Fulfillment of the Requirements
For the Degree of
DOCTOR OF PHILOSOPHY
WITH A MAJOR IN MECHANICAL ENGINEERING
In the Graduate College
THE UNIVERSITY OF ARIZONA

1991
As members of the Final Examination Committee, we certify that we have read
the dissertation prepared by Teng-Hua Shieh
entitled Prediction and Analysis of Wing Flutter
at Transonic Speeds

and recommend that it be accepted as fulfilling the dissertation requirement
for the Degree of Doctor of Philosophy

K. Y. Fung
Date 11/22/91

Hermann Fasel
Date 11/21/91

Thomas F. Balsa
Date 11/23/91

Moysey Brio
Date 11/24/91

Final approval and acceptance of this dissertation is contingent upon the
candidate's submission of the final copy of the dissertation to the Graduate
College.

I hereby certify that I have read this dissertation prepared under my
direction and recommend that it be accepted as fulfilling the dissertation
requirement.

Dissertation Director K. Y. Fung
Date 11/22/91
STATEMENT BY AUTHOR

This dissertation has been submitted in partial fulfillment of requirements for an advanced degree at the University of Arizona and is deposited in the University Library to be made available to borrowers under the rules of the Library.

Brief quotations from this dissertation are allowable without special permission, provided that accurate acknowledgement of source is made. Requests for permission for extended quotation from or reproduced of this manuscript in whole or in part may be granted by the copyright holder.

SIGNED: [Signature]

[Signature]

[Signature]
ACKNOWLEDGMENTS

The author would like to thank his advisor Dr. K.-Y. Fung for his advice and guidance in performing this work, as well as his physical and numerical insight. The author would also like to thank his other committee members, Dr. Thomas F. Balsa, Dr. Hermann Fasel and Dr. Moysey Brio for their time and suggestions concerning the dissertation and fundamental concepts.

The author would like to thank his officemates Linda Kral, Jeffrey Schoen, Jeff Currier, Sek-On Man, David Dratler, Peter Dittritch and Frank Brueckner. Their friendship and support were greatly appreciated.

And finally, I would like to thank my wife Lie-Shine for her support and believing in me throughout. Her immense patient and care for Wendy and James allowed me to concentrate on the work, and her encouragement gave me the strength to pass through difficult moments.
DEDICATION

To my parents, whose support and love provided the encouragement to undertake and successfully complete this work.
TABLE OF CONTENTS

LIST OF ILLUSTRATIONS.................................................................................................................. 8

LIST OF TABLES.................................................................................................................................. 14

ABSTRACT .......................................................................................................................................... 15

1. INTRODUCTION.......................................................................................................................... 17
   1.1 Transonic Flow ......................................................................................................................... 18
   1.2 Computational Methods for Transonic Flow ........................................................................... 21
   1.3 Methods for Flutter Prediction ............................................................................................... 25
   1.4 Outline and Objectives of the Present Study .......................................................................... 26

2. MODELLING AND ANALYSIS OF AIRFOIL FLUTTER .......................................................... 31
   2.1 Aerelastic Equations .................................................................................................................. 31
   2.2 Simple Model for Unsteady Transonic Indicial Response ................................................... 36
   2.3 Solution Procedure .................................................................................................................. 39
   2.4 Analysis of Airfoil Flutter ........................................................................................................ 42
   2.5 Simplified Flutter Solution ....................................................................................................... 56
   2.6 On the Transonic Dip ............................................................................................................... 71

3. FORMULATION AND GOVERNING EQUATIONS ...................................................................... 78
   3.1 Aerodynamic Equations .......................................................................................................... 78
      3.1.1 Equation for Transonic Flow ............................................................................................. 78
         3.1.1a Full Potential Equation .................................................................................................. 78
         3.1.1b Boundary Layer Corrections ......................................................................................... 94
         3.1.1c Euler/Navier-Stokes Equations ..................................................................................... 96
      3.1.2 Truncation Error Injection ................................................................................................. 97
   3.2 Structural Responses ................................................................................................................ 98
   3.3 Flutter Analysis ......................................................................................................................... 99
4. DEVELOPMENT AND IMPLEMENTATION OF THE NUMERICAL SCHEME.......................................................................................................................... 103

4.1 Numerical Algorithm........................................................................................................ 103

4.1.1 Discretization of Governing Equation ................................................................... 103

4.1.2 Discretization of Boundary Conditions .............................................................. 109

4.2 Interpolation Scheme .................................................................................................... 111

4.3 Grid Generation .............................................................................................................. 117

5. VERIFICATION OF THE CODE - ZUNAS................................................................ 122

5.1 Effectiveness of T.E.I. for Unsteady Flows............................................................... 122

5.2 Effect of Small Perturbation Boundary Condition and Far-Field Boundary Condition ........................................................................................................ 130

5.3 Two-Dimensional Test Cases ...................................................................................... 134

5.4 Flutter Analysis on the YXX Wing .............................................................................. 146

5.5 Flutter Analysis on the PAPA Wing ............................................................................. 159

6. TRANSONIC DIP OF WINGS ......................................................................................... 166

7. SUMMARY ....................................................................................................................... 170

APPENDIX A: NOMENCLATURE .................................................................................... 172

APPENDIX B: MODE SHAPE INPUT DATA-YXX WING ............................................. 175

REFERENCES .................................................................................................................... 178
### LIST OF ILLUSTRATIONS

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>Flutter speed vs. Mach number curve showing the &quot;transonic dip&quot; - reproduced from Tijdeman's Thesis (1975).</td>
<td>30</td>
</tr>
<tr>
<td>1.2</td>
<td>Comparison of flutter boundaries of a swept wing with supercritical and conventional profiles - from Farmer and Hanson (1976).</td>
<td>30</td>
</tr>
<tr>
<td>2.1</td>
<td>Two-degree-of-freedom airfoil system</td>
<td>35</td>
</tr>
<tr>
<td>2.2</td>
<td>Model for indicial response</td>
<td>36</td>
</tr>
<tr>
<td>2.3</td>
<td>Flow chart for the solution procedure.</td>
<td>41</td>
</tr>
<tr>
<td>2.4</td>
<td>Airfoil configurations</td>
<td>47</td>
</tr>
<tr>
<td>2.5</td>
<td>Comparison of flutter speeds and frequencies computed using present theory and classical method [Yang's (1979)] for various mass density ratios on NACA 64A006.</td>
<td>48</td>
</tr>
<tr>
<td>2.6</td>
<td>Effect of mass center on flutter speed and frequency for NACA 64A006.</td>
<td>49</td>
</tr>
<tr>
<td>2.7</td>
<td>Effect of elastic axis on flutter speed for NACA 64A006.</td>
<td>50</td>
</tr>
<tr>
<td>2.8</td>
<td>Effect of mass density ratio on flutter speed for NACA 64A010.</td>
<td>51</td>
</tr>
<tr>
<td>2.9</td>
<td>Effect of mass density ratio on flutter speed for NACA 64A010.</td>
<td>52</td>
</tr>
<tr>
<td>2.10</td>
<td>Effect of mass density ratio on flutter speed for CAST-7.</td>
<td>53</td>
</tr>
<tr>
<td>2.11</td>
<td>Flutter boundaries on flutter speeds for four airfoil configurations</td>
<td>54</td>
</tr>
<tr>
<td>2.12</td>
<td>Effect of $\mu$ on $Z$ for different airfoils.</td>
<td>55</td>
</tr>
<tr>
<td>2.13a</td>
<td>Converted lift coefficients.</td>
<td>63</td>
</tr>
<tr>
<td>2.13b</td>
<td>Converted time scale of lift response.</td>
<td>63</td>
</tr>
</tbody>
</table>
2.13c Converted moment coefficients ...................................................... 64
2.13d Converted time scale of moment response ........................................ 64
2.14 Comparison of flutter solution predicted using full and simplified formulas for Case 1 .................................................................................................... 65
2.15 Comparison of flutter solutions predicted using full and simplified formulas for Case 2 ....................................................................................... 66
2.16 Comparison of flutter solutions predicted using full and simplified formulas for Case 3 .............................................................................. 67
2.17 Comparison of flutter solutions predicted using full and simplified formulas for Case 4 ................................................................... 68
2.18 Comparison of flutter solutions predicted using full and simplified formulas for Case 5 ............................................................................ 69
2.19 Comparison of flutter solutions predicted using full and simplified formulas for Case 6 .......................................................................................... 70
2.20 Comparison of flutter solutions predicted using full and simplified formulas for Case 7 ...................................................................................... 71
2.21 Comparison of flutter boundary and aerodynamic moment slope for three airfoil configurations ........................................................................ 77
3.1a Entropy contours (upper) and Mach number contours (lower) of a solution of the Euler equations for a supercritical wing at the root plane ................. 83
3.1b Entropy contours (upper) and Mach number contours (lower) of a solution of the Euler equations for a supercritical wing at the root plane ................. 84
3.1c Entropy contours (upper) and Mach number contours (lower) of a solution of the Navier-Stokes equations for a supercritical wing at the root plane........ 85
3.1d Entropy contours (upper) and Mach number contours (lower) of a solution of the Navier-Stokes equations for a supercritical wing at the root plane........ 86
3.2 Definition of the wing deformation function ........................................ 87
<table>
<thead>
<tr>
<th>Section</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.4</td>
<td>Time responses of $C_l$ and $C_m$ to plunging motion at mid-span of the YXX wing model at $M_\infty=0.75$.</td>
</tr>
<tr>
<td>5.5</td>
<td>Phase error and amplification factor of a model implicit upwind difference scheme.</td>
</tr>
<tr>
<td>5.6</td>
<td>The recovery of pressure distribution of a 5% circular arc wing of $AR=3$ computed by applying the S.P.B,C. on a grid generated for a similar wing but with rounded leading edge.</td>
</tr>
<tr>
<td>5.7</td>
<td>Effect of different boundary conditions on the structural response of the YXX wing at $QD=70$ kPa, $M_\infty=0.75$.</td>
</tr>
<tr>
<td>5.8a</td>
<td>Comparison of lift responses computed using full potential, Navier-Stokes, or Euler mean flows of an NACA 0012 airfoil pitching at $M_\infty=0.6$ with $\Delta\alpha=1^\circ$, and $k_c=0.15$.</td>
</tr>
<tr>
<td>5.8b</td>
<td>Comparison of lift responses computed using full potential, Navier-Stokes, or Euler mean flows of an NACA 0012 airfoil pitching at $M_\infty=0.8$ with $\Delta\alpha=1^\circ$, and $k_c=0.15$.</td>
</tr>
<tr>
<td>5.8c</td>
<td>Comparison of lift responses computed using full potential, Navier-Stokes, or Euler mean flows of an NACA 0012 airfoil pitching at $M_\infty=0.85$ with $\Delta\alpha=1^\circ$, and $k_c=0.15$.</td>
</tr>
<tr>
<td>5.8d</td>
<td>Lift and moment responses computed using a full potential mean flow of an NACA 0012 airfoil pitching at $M_\infty=0.85$ and $k_c=0.15$, showing nonzero mean lift increments dependent on initial pitch direction.</td>
</tr>
<tr>
<td>5.9a</td>
<td>Comparison of lift responses computed using full potential, Navier-Stokes, or Euler mean flows of an NACA 64A006 airfoil pitching at $M_\infty=0.80$ with $\Delta\alpha=1^\circ$, and $k_c=0.15$.</td>
</tr>
<tr>
<td>5.9b</td>
<td>Comparison of lift responses computed using full potential, Navier-Stokes, or Euler mean flows of an NACA 64A006 airfoil pitching at $M_\infty=0.875$ with $\Delta\alpha=1^\circ$, and $k_c=0.15$.</td>
</tr>
<tr>
<td>5.9c</td>
<td>Comparison of lift responses computed using full potential, Navier-Stokes, or Euler mean flows of an NACA 64A006 airfoil pitching at $M_\infty=0.90$ with $\Delta\alpha=1^\circ$, and $k_c=0.15$.</td>
</tr>
</tbody>
</table>
5.10a Comparison of mean pressure distributions and magnitudes and phases of lift response computed using Euler/Navier-Stokes mean flows with experimentally measured values of a NLR 7301 airfoil pitching at $M_{\infty}=0.752$, $\alpha=0^\circ$, and a range of frequencies

5.10b Comparison of mean pressure distributions and magnitudes and phases of lift response computed using Euler/Navier-Stokes mean flows with experimentally measured values of a NLR 7301 airfoil pitching at $M_{\infty}=0.807$, $\alpha=0^\circ$, and a range of frequencies

5.11a Time history of the upper surface pressure distribution of a NLR 7301 airfoil pitching at $k_B=0.05, M_{\infty}=0.752$, and $\alpha=0^\circ$, from Davis and Malcolm (1980)

5.11b Time history of the upper surface pressure distribution of a NLR 7301 airfoil pitching at $k_B=0.05, M_{\infty}=0.807$, and $\alpha=0^\circ$, from Davis and Malcolm (1980)

5.12 YXX wing model

5.13a Comparison of steady mean flows of the YXX wing at $M_{\infty}=0.7$ computed using full potential (fp), full potential with viscous effects (fpv), Navier-Stokes (NS), and Euler equations

5.13b Comparison of steady mean flows of the YXX wing at $M_{\infty}=0.825$ computed using full potential (fp), full potential with viscous effects (fpv), Navier-Stokes (NS), and Euler equations

5.14a Comparison of pressure contours of the YXX wing at $M_{\infty}=0.7$ computed using full potential (fp), full potential with viscous effects (fpv), Navier-Stokes (NS), and Euler equations

5.14b Comparison of pressure contours of the YXX wing at $M_{\infty}=0.825$ computed using full potential (fp), full potential with viscous effects (fpv), Navier-Stokes (NS), and Euler equations

5.15 Comparison of flutter boundary computations on the YXX wing

5.16 Comparison of flutter boundary computations on the YXX wing using various mean flows and for $\alpha=0^\circ$
<table>
<thead>
<tr>
<th>Section</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.17</td>
<td>Comparison of flutter boundary computations on the YXX wing using various mean flows and for two ( \alpha )’s.</td>
<td>157</td>
</tr>
<tr>
<td>5.18</td>
<td>Damping rate of the YXX wing at ( \alpha = -1^\circ ) based on Navier-Stokes steady mean flows.</td>
<td>158</td>
</tr>
<tr>
<td>5.19</td>
<td>Planform of the PAPA cambered supercritical wing.</td>
<td>159</td>
</tr>
<tr>
<td>5.20a</td>
<td>Flutter boundary predictions using ZUNAS for the PAPA wing.</td>
<td>162</td>
</tr>
<tr>
<td>5.20b</td>
<td>Flutter boundary predictions using ZUNAS with different upper and lower transition locations for the PAPA wing.</td>
<td>163</td>
</tr>
<tr>
<td>5.20c</td>
<td>Comparison of Flutter boundary predictions computed using ZUNAS with linear theory (FAST) for a wing same as the PAPA wing model except with NACA 64A006 cross sections.</td>
<td>164</td>
</tr>
<tr>
<td>5.21</td>
<td>Comparison of flutter boundary predictions using different numerical and analytic methods.</td>
<td>165</td>
</tr>
<tr>
<td>6.1</td>
<td>Correlation of computed moment slope peaks and dynamic pressure dips on the PAPA wing.</td>
<td>167</td>
</tr>
<tr>
<td>6.2</td>
<td>Correlation of computed moment slope peaks and dynamic pressure dips on the YXX wing.</td>
<td>168</td>
</tr>
<tr>
<td>6.3</td>
<td>Correlation of measured moment slope peaks and dynamic pressure dips on the TF-8A wing.</td>
<td>169</td>
</tr>
</tbody>
</table>
## LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>Comparison of CPU time (CRA Y Y-MP) per time step and per grid node and minimum grid spacing</td>
<td>24</td>
</tr>
<tr>
<td>1.2</td>
<td>Comparison of CPU time for a typical flutter boundary computation (based on 10 Mach numbers)</td>
<td>24</td>
</tr>
<tr>
<td>2.1</td>
<td>Aerodynamic coefficients converted from Yang et al. (1979) according to Fung's model for NACA 64A006 airfoil</td>
<td>45</td>
</tr>
<tr>
<td>2.2</td>
<td>Aerodynamic coefficients converted from Yang et al. (1979) according to Fung's model for NACA 64A010 airfoil</td>
<td>45</td>
</tr>
<tr>
<td>2.3</td>
<td>Aerodynamic coefficients converted from Yang et al. (1979) according to Fung's model for CAST-7 airfoil</td>
<td>46</td>
</tr>
<tr>
<td>2.4</td>
<td>Aerodynamic coefficients converted from Yang et al. (1979) according to Fung's model for Flat plate</td>
<td>46</td>
</tr>
<tr>
<td>2.5</td>
<td>Test Cases</td>
<td>61</td>
</tr>
<tr>
<td>2.6</td>
<td>Derivatives of $F_r$ for NACA 64A006</td>
<td>75</td>
</tr>
<tr>
<td>2.7</td>
<td>Derivatives of $F_r$ for NACA 64A010</td>
<td>75</td>
</tr>
<tr>
<td>2.8</td>
<td>Derivatives of $F_r$ for CAST-7</td>
<td>76</td>
</tr>
<tr>
<td>2.9</td>
<td>Derivatives of $F_r$ for flat plate</td>
<td>76</td>
</tr>
<tr>
<td>5.1</td>
<td>Types of far-field boundary conditions</td>
<td>130</td>
</tr>
<tr>
<td>5.2a</td>
<td>Static generalized forces</td>
<td>158</td>
</tr>
<tr>
<td>5.2b</td>
<td>Structural properties</td>
<td>158</td>
</tr>
<tr>
<td>5.3</td>
<td>Aerodynamic parameters computed from indicial responses</td>
<td>161</td>
</tr>
<tr>
<td>7.1</td>
<td>Comparison of computational resources</td>
<td>171</td>
</tr>
</tbody>
</table>
ABSTRACT

This dissertation deals with the instability, known as flutter, of the lifting and control surfaces of aircraft of advanced design at high altitudes and speeds. Here, an analytic model is constructed for a better understanding of the physics involved, and numerical procedures are developed for accurate, efficient prediction of the occurrence of the instability.

A simple model is used to represent the aerodynamics for flutter analysis of a two-degree-of-freedom airfoil system. This simple aerodynamic model characterizes the aerodynamic forces by the lift and moment slopes and two time constants from the indicial response to pitch, and explicitly relates the phase lags between harmonic motions and their aerodynamic responses. Flutter solutions of this airfoil system are shown to be algebraically homomorphic in that solutions about different elastic axes can be found by mapping them to those about the mid-chord. Algebraic expressions for the flutter speed and frequency are thus obtained. Results using these expressions on typical airfoils show good agreement with those obtained using the classical method. For flutter frequencies that are small compared to the characteristic time constants or for time constants that are close in magnitude, these expressions can be further simplified to uncoupled, explicit expressions. Many characteristics of flutter and their parametric dependency are readily discernible from these expressions, including the condition for flutter and a criterion which relates the location of the lowest flutter speed, the transonic dip, to the maximum of the moment slope $C_{m\alpha}$, a quantity predictable using quasi-steady aerodynamics. Based on this criterion, the transonic dip is simply a manifestation of the variation of $C_{m\alpha}$ with the freestream Mach number $M_{\infty}$. This criterion, derived for airfoils, is found to be valid for wings of advanced design as well.
For the prediction of flutter of a wing at transonic speeds, an accurate and efficient computer code, called ZUNAS, is developed. This code is based on a full potential formulation and stationary body-fitted curvilinear coordinates with a modified density formula to account for the entropy changes in the mean flow. The unique features of this code are the capability of accepting a steady mean flow regardless of its origin, a time dependent perturbation boundary condition for describing wing deformations on the mean surface, and a locally applied three-dimensional far-field boundary condition for minimizing wave reflections from numerical boundaries. The steady mean flow is part of the code input and can be obtained using other codes based on any flow model, full potential, full potential with boundary layer corrections, Euler, or Navier-Stokes. Results for various test cases obtained using this code show good agreement with the experiments and other theories. Nonlinear effects, viscous effects, the effect of the transition point, and other important features in unsteady transonic flow computation are discussed.
CHAPTER 1
INTRODUCTION

This Chapter outlines the objectives of the present study and gives a brief description of the transonic flow, its importance and characteristics. It also reviews the recent numerical developments for solving transonic flow over airfoil or wing geometry, discusses computational methods applicable to the interaction between structural deformation and aerodynamic response, and points out the needs for accurate and efficient methods for flutter prediction.

Under certain flight conditions, the wings, tails, and control surfaces of an airplane may encounter vibration of increasing amplitude. This unsteady phenomenon, called "flutter", is an aeroelastic problem. This problem arises from the interaction between the elastic response of the structure to aerodynamic forcing and the aerodynamic forces due to structural deformations. At flutter conditions, the structure extracts energy from the passing airstream to reinforce the deformations. Usually, the most devastating modes of structural deformation are the first two or three bending and torsional modes. The amplification of these modes leads rapidly to the failure and even disintegration of the structure resulting in the loss of ailerons or tails in flight and sometimes fatal accident.

For a given wing configuration the unsteady aerodynamic force increases with flight speed, while the elastic and inertia forces remain the same. There exists a critical flutter speed, measured in terms of the dynamic pressure \(=\frac{1}{2}\rho_\infty V_\infty^2\), a combined effect of flight speed and air density, above which flutter occurs. The critical flutter speed also varies with the freestream Mach number, \(M_\infty\). The curve that describes the variations of the critical flight speed with \(M_\infty\) is called the flutter boundary. Figure 1.1 shows the flutter boundaries of typical wings at transonic speeds. All of them have a sharp drop at a
Mach number near unity. This phenomenon is called "transonic dip". At the dip, the dynamic pressure reaches the lowest value and hence is the most critical condition for the operation of an aircraft. To prevent flutter, all possible flight paths (flight envelop) of an aircraft must be so designed that they do not cut across the flutter boundary. This requirement is considered a critical design criterion for flight safety of an aircraft. (Bisplinghoff and Ashley, 1962).

Since flutter is a primary concern in the design of an aircraft, a reliable method for the prediction of the flutter characteristics is needed. The vibrational characteristics of a structure can be predicted accurately using the finite elements method, and for a subsonic or supersonic freestream, the unsteady aerodynamics can be predicted reasonably well using methods based on linearized theory. For a transonic flow, linear methods fail to describe transonic effects, which are intrinsically nonlinear. Hence, efficient and reliable methods for the prediction of unsteady transonic aerodynamics are key technologies for the design of advanced aircraft.

1.1 Transonic Flow

The main feature of a transonic flow over an airfoil is the existence of a supersonic zone as the local flow Mach number exceeds unity. This region, commonly referred to as a sonic bubble, is usually terminated by a shock. Although it is possible to design an airfoil so that, at the design condition, the transition from supersonic back to subsonic flow is accomplished without a shock, for all off-design conditions, shocks exist. The location of the shock is very sensitive to perturbations and can change substantially as a result of shock-boundary layer interaction, thereby changing the aerodynamic loadings, lift and moment, on the airfoil.
In unsteady aerodynamics, the dominant phenomenon is the propagation of acoustic waves, which are measured in terms of the chord and span of a wing and freestream Mach number. When the flow is purely subsonic or supersonic, the nonuniformity in the flow field does not affect substantially the propagation of acoustic waves. Hence, linear methods based on potential theory provide accurate and inexpensive predictions despite no account of the flow nonuniformity. When the flow is transonic, the way in which acoustic waves propagate becomes very sensitive to local nonuniformities, especially when a supersonic zone appears in an otherwise subsonic environment. As noted above, this zone is usually terminated by a shock, whose location and strength are sensitive to the body geometry, and can be changed substantially by interactions with the boundary layer. To capture the embedded supersonic zone, the expansion at the leading edge must be accurately resolved. Momentum and energy, in addition to mass, must be conserved and effects of the boundary layer must in some way be taken into account. These requirements severely restrict the use of the transonic small disturbance theory to steady flows of thin airfoil of relatively sharp leading edge and at moderate supercritical Mach numbers. However, that is not to say that the transonic small disturbance theory is not suitable for the prediction of unsteady flows that correspond to small aeroelastic deformations. On the contrary, as demonstrated by Liu, et al. (1988), simplified theories based on the small disturbance approach can yield accurate unsteady flow predictions provided that the mean flow on which the unsteady disturbances propagate is accounted for.

While recent advances in numerical techniques have made feasible the prediction of steady flow over realistic aircraft geometry for transonic flight conditions, applications of similar techniques to the prediction of unsteady flow have not been equally successful, even when viscous effects can be ignored. The difficulty is due to the disparity in the length and time scales in the physics of the problem. A perturbation analysis suggests the
use of different equation sets for different parts of the flow. For the unsteady flow that corresponds to elastic deformations of the body in a flutter prediction computation, an equation for the conservation of mass should be sufficient, since the length scales are of sizes of the chord and span and the propagation of small disturbances is an isentropic process. The steady flow, however, depends on smaller length scales such as the radius of curvature of the leading edge. Which means numerically, while a grid measured in tenths of the chord is adequate for the unsteady flow, a grid in hundredths is required for the steady flow. For a steady flow, relaxation methods, or methods which allow disturbances of different length and time scales to settle at different rates, are available for a speedy convergence to the steady state, whereas, for an unsteady flow, the advancement of the numerical solution is restricted by the CFL condition to time steps comparable to the smallest grid size. Furthermore, the relation between the flow and the density is non-linear and depends on other flow quantities, which vary according to different length and time scales. For an upstream uniform, steady, adiabatic flow, the total enthalpy is a constant and therefore the density and flow relation is known provided the entropy, which is constant everywhere except in the boundary layer and in the flow behind a shock, is known. For an upstream uniform, unsteady flow, the density flow relation is known only when the flow can be assumed irrotational. For the general case, the momentum and energy equations are needed for a complete description of the rotationality of the boundary layer and behind non-planar shock flows. Numerically, solving the momentum and energy equations implies having to resolve the boundary layer flow on a grid measured in thousandths of the chord, which is unnecessarily fine and impractical for the application of routine flutter predictions.

For a flutter analysis, where the growth of small amplitude oscillations is of primary concern, the unsteady airload due to pressure fluctuations can be assumed small and
linearly proportional to displacement. Therefore, for purely subsonic or supersonic flow, the governing equation and boundary condition allow a solution be decoupled into steady and unsteady parts, and the unsteady part can be solved independent of the steady mean flow. For transonic flows, a perturbation analysis yields equations which contain terms from mean flow components even when the perturbations can be considered small [see Landahl (1961)]. Unlike subsonic or supersonic flow, the unsteady flow field can no longer be treated as independent of the steady mean flow.

1.2 Computational Methods for Transonic Flow

For combat aircraft, the demand for optimal maneuverability at transonic speeds and low wave drag at supersonic speeds implies the use of thin wing. For transport aircraft, the most economical flight regime is transonic and a feature of modern transport aircraft is the employment of high-aspect-ratio shock-free wing of low induced drag and wave drag. These wings are susceptible to aeroelastic instability at transonic speeds. As Farmer and Hanson (1976) have demonstrated, a more pronounced dip (Figure 1.2) is experienced on a wing with modern supercritical profile than one with a conventional profile. An obvious way to elevate the transonic dip is to increase the structural rigidity of the wing, which, unfortunately, decreases the payload of the aircraft. Among the many tradeoffs and tailorings to be made in the design of modern aircraft, aeroelastic stability becomes a critical issue when the range of operation includes transonic conditions.

Due to the high cost and risk involved, it is not practical to conduct extensive aeroelastic flight tests. Also, wind tunnel experiment of unsteady flow is much more expensive than experiment of steady flow. It is hoped that with the aid of accurate computational methods, the overall cost of the development of an aircraft can be considerably reduced.
It was mentioned earlier that linear methods, which yield reliable predictions for subsonic and supersonic flows, are incapable of handling the nonlinearity of a transonic flow. Yang et al. (1979) performed flutter analysis on the NACA 64A006 airfoil using the Kernel function method and found no transonic dip. Isogai (1983) used the doublet-lattice method for a flutter analysis of a high-aspect-ratio supercritical wing and showed no transonic dip. It is quite clear that without accounting for mean flow variations, methods based on linear theory are inadequate for unsteady transonic flow predictions.

Over the last two decades, numerous finite difference methods for the prediction of steady and unsteady transonic flow over airfoil or wing have been proposed. Many significant developments of computer codes for predictions of unsteady transonic flow have been reported since Tijdeman's (1977) review of computational methods for two-dimensional unsteady transonic flow. However, a majority of these codes is based on the transonic small disturbance equation. Traci et al. (1975) developed the STRANS and UTRAN$S$ based on the linearized transonic small perturbation equation. The LTRAN$S$ of Ballhaus and Goorjian (1978), the USTS of Isogai (1980), the XTRAN$S$ of Rizzetta and Borland (1982), and the CAP-TSD of Batina (1988) are all based on the transonic small disturbance equation, TSD. A limitation of the TSD is the requirement of a smallness parameter, such as a small thickness ratio, small incidence angle, or small amplitude of oscillation. For modern supercritical transport wings, which are quite thick and blunt at the leading edge, the TSD formulation is no longer adequate.

Without the smallness assumption, the full potential equation is next in the hierarchy of governing equations for a compressible flow. Isogai's (1982) USTF$S$ is based on a quasi-linear form of the unsteady full potential equation. This code has been used to compute the flutter boundaries of modern transport wings, including the TF-8A wing [Isogai (1983)] and YXX wing [Isogai (1984)]. However, the computed flutter bound-
aries deviate substantially from the corresponding measured values especially at high Mach numbers where stronger shocks exist. Bridgeman et al. (1982) developed the code TUNA, which is based on the ADI scheme proposed by Steger and Caradonna (1980) for the full potential formulation. This code has only been tested for moderately high Mach numbers, and is not expected to remain valid when strong shock appears. Guruswamy (1990) developed the ENSAERO code based on the Beam and Warming (1976) scheme for the Navier-Stokes equations, which remain valid for strong shocks. An inviscid flow calculation is computational more efficient than a viscous flow calculation, since the latter requires a much finer grid to resolve the boundary layer. However, stronger and further aft-positioned shocks are often predicted when viscous effects are ignored. The shock strength and location dominate the flutter characteristics at high Mach numbers.

While there is no dispute about the superiority of the Navier-Stokes equations for modeling fluid flows, a full implementation of the Navier-Stokes equations for flutter analysis is not yet practical. Codes such as ARC2D and ARC3D of Pulliam (1984) and TLNS3D of Vatsa (1987) are based on the Reynolds-averaged Navier-Stokes equations with the thin-layer assumption and an algebraic model for the turbulence eddy viscosity. A numerical solution of the Navier-Stokes equations is meaningful only when the physics of the boundary layer and its interaction with the outer flow are adequately resolved on a fine enough grid, which often is highly stretched from the boundary layer scale measured in thousandths of the chord to distances measured in tens of chords away from the wing. Hence, a solution of the Navier-Stokes equations requires far more computer resources than that of the full potential or the Euler equations.

A comparison of computer resources for obtaining typical steady or unsteady solutions using the TSD, full potential, Euler or Navier-Stokes (NS) equations is shown in Table 1.1, and for a typical flutter boundary prediction in Table 1.2. Even on a super-
computer, the CRAY Y-MP, hundreds of CPU hours would be needed for a flutter boundary calculation using the Navier-Stokes equations.

Table 1.1 Comparison of CPU time (CRAY Y-MP) per time step and per grid node and minimum grid spacing.

<table>
<thead>
<tr>
<th></th>
<th>NS (Steady)</th>
<th>EULER (Steady)</th>
<th>FP (Steady)</th>
<th>FP (Unsteady)</th>
<th>TSD (Unsteady)</th>
</tr>
</thead>
<tbody>
<tr>
<td>CPU†</td>
<td>2.5x10^-5</td>
<td>2.0x10^-5</td>
<td>5.5x10^-6</td>
<td>7.0x10^-6</td>
<td>3.5x10^-6</td>
</tr>
<tr>
<td>Min. grid cell size</td>
<td>1.5x10^-4</td>
<td>1.0x10^-3</td>
<td>2.0x10^-3</td>
<td>6.0x10^-3</td>
<td>6.0x10^-3</td>
</tr>
</tbody>
</table>

†: CPU time per step, per grid node

Table 1.2 Comparison of CPU time for a typical flutter boundary computation (based on 10 Mach numbers)

<table>
<thead>
<tr>
<th>FLOW MODEL</th>
<th>CPU HOURS (CRAY Y-MP)</th>
</tr>
</thead>
<tbody>
<tr>
<td>TSD</td>
<td>2.0 (CAP-TSD)</td>
</tr>
<tr>
<td>F.P.</td>
<td>6.0 (TUNA)</td>
</tr>
<tr>
<td>EULER</td>
<td>25 (ENSAERO)</td>
</tr>
<tr>
<td>NAVIER-STOKES</td>
<td>&gt; 250</td>
</tr>
</tbody>
</table>

On the premise that only limited computer resources are available, the simplifications and assumptions made in the present work of developing an accurate, efficient numerical method to predict flutter boundary will be justified if significant computation time can be saved. An objective of the present work is to develop accurate and efficient numerical method for routine applications in aircraft design. The new algorithm presented in Chapter 3 achieves the same efficiency as methods based on the full potential equations but with accuracy comparable to those on the Navier-Stokes equations.
1.3 Methods for Flutter Prediction

In a classical flutter analysis, the aerodynamic responses are represented by the aerodynamic coefficients, which are computed change of force per unit change of deformation mode. These coefficients are evaluated for a range of reduced frequencies and free stream Mach numbers. The flutter condition is then determined by a search among the computed aerodynamic coefficients for the one that satisfies the coupled system of aerodynamic and structural equations. Yang et al. (1978) used for the aerodynamics the codes STRANS and UTRANS developed by Traci et al. (1975) and LTRAN2 developed by Ballhaus and Goorjian (1978) to study the flutter characteristics of the NACA 64A006 and NACA 64A010 airfoils undergoing harmonic pitching and plunging oscillations. They examined the effects of mass ratio, center of mass location, frequency ratio, etc., on the flutter speed. Similarly, Isogai (1980) used the USTS code for the aerodynamic coefficients and presented a study of the transonic dip mechanism on an NACA 64A010 airfoil.

For complex flows where the evaluation of aerodynamic responses as a function of frequency can be time consuming, Ballhaus and Goorjian (1978) proposed and demonstrated a direct time integration of structural and aerodynamic governing equations for the flutter analysis of a NACA 64A006 airfoil undergoing pitching oscillations using their LTRAN2 code. Rizzetta (1979) extended the technique for the aeroelastic response of a three-degree-of-freedom (pitching, plunging and aileron pitching) system on the NACA 64A010 airfoil. Yang et al. (1979) also reported using this approach for one- or two-degree-of-freedom system on the NACA 64A006 airfoil. Isogai (1983) computed the flutter boundary of a high-aspect-ratio transport supercritical wing (YXX wing model) by direct time integration of coupled systems using his USTF3. Similarly, Guruswamy (1990)
performed the calculation of one flutter point on a 6% circular arc rectangular wing with direct coupling of the structural and aerodynamic equations using his ENSAERO. Unlike the classical method where the flutter frequency is found through an iterative process, direct coupling gives the flutter frequency when the dynamic pressure at flutter (flutter speed) is found by iterations.

In the present work, the classical $U-g$ method is used in Chapter Two for the study of the transonic dip mechanism and direct time integration method is used for flutter predictions of wings.

1.4 Outline and Objectives of the Present Study

The present study focuses on the phenomena of flutter at transonic speeds, in particular, the sudden drop of dynamic pressure. There have been a number of discussions on the possible dip mechanisms. The numerical studies of Yang et al. (1979) and Isogai (1980) shed lights on the complexity of the problem. Ashley (1979) argued how a shock can play a major role in "sub-transonic " flutters. Isogai (1980) showed strong effects of the mass ratio and phase lag on the flutter of a single-degree-of-freedom swept-back wing. These studies showed how the dip might be affected by specific parameters but were not successful in explaining the dip.

In Chapter Two, the two-degree-of-freedom (pitching and plunging) airfoil system is used as a model to study the flutter characteristics of transonic flight. This system is specified by a set of structural parameters and the aerodynamic response to structural deformations, i.e., pitching and plunging. For a subsonic flow, the exact functional representation of the aerodynamic response, e.g. to pitching, is not known and approximations valid for low reduced frequencies are in terms of complex transcendental functions. For a transonic flow, the aerodynamic response function itself is known to exist only as a
numerical approximation to the governing equation. Due to the complexity in the aero-
dynamic response, a flutter analysis of this simple airfoil system using the classical
method entails numerical search for complex roots of algebraic equations involving com-
plex transcendental functions, and amounts to point-wise solution evaluations in a re-
stricted region of the parametric solution space. It will be shown that the aerodynamic
model proposed by Fung (1982) simplifies the aerodynamic representation of low fre-
quency harmonic responses and yields explicit expressions for the flutter speed and fre-
cuency in terms of a set of structural and aerodynamic parameters. The dominant parame-
ter for flutter at transonic speeds is found and a criterion for the Mach number of the tran-
sonic dip is derived from these expressions, providing a qualitative view of flutter at tran-
sonic speeds.

The work explained in the subsequent chapters is prediction of flutter boundaries
of wings and airfoils. The formulation, implementation and verification of an efficient
and accurate method for unsteady transonic flow computation are described in Chapters
Three through Five. In Chapter Three, an approach in essence similar to that of Fung,
Yu, and Seebass (1978) and in theory same as that of Fung and Fu (1987), but is formu-
lated in a framework not restricted to the transonic small disturbance equation and rectilin-
ear grids. An unsteady flow is decomposed into a mean flow and an unsteady flow of
small disturbances. The mean flow is assumed known and available on a curvilinear grid.
The mean flow and the unsteady flow can be computed on different grids, usually a finer
one for the mean flows, and using different flow solvers. The accuracy of the mean flow
is preserved on the coarse grid for unsteady computations by the truncation error injection
 technique of Fung et al. Small disturbances are introduced into the flow at the mean sur-
face of the wing through a time-varying transpiration boundary condition which corre-
sponds to the unsteady elastic deformations of the wing. The propagation of unsteady
disturbances is governed by the continuity equation in terms of a perturbed velocity potential. A coarse body-fitted grid, independent of the grid for the mean flow, is chosen just fine enough to resolve the unsteady flow due to the principal deformation modes and preserve an adequate representation of the steady flow. A far field boundary condition derived from the analytical result of Fung (1981) is applied to direct the outgoing waves. The unsteady flow is assumed irrotational and viscous effects are included in the steady mean flow through a displacement thickness or an entropy correction term in the density formulation. The structural responses are obtained by integrating the aerodynamic force, decomposing it into corresponding components of the structural eigenmodes, and then solving the equations of motion for the elastic deformation of the wing.

The development and implementation of the ideas presented in Chapter Three are discussed in Chapter Four. The discretization of the governing equations for irrotational and rotational flow, the implementation of the viscous/inviscid boundary conditions, and techniques for interpolating solutions on two topologically different grids are discussed.

The time accuracy of the method is established through a study of single mode harmonic oscillations in Chapter Five. A simple one-dimensional model is used to demonstrate the dependence of amplitude and phase errors on the CFL (Courant-Friedrichs-Lewy) number for unconditionally stable implicit schemes for the wave equation. This analysis shows that a finer spatial grid does not necessarily give more accurate result than a coarser grid unless the time steps for the solution on the fine grid are correspondingly reduced, and that a coarse grid for the unsteady flow is preferred as long as it is capable of resolving the principal waves. The overall effectiveness of the present approach is assessed by applying it to flutter boundary predictions of supercritical transport wings. Computed flutter boundaries are compared with available experiments. It will be shown that the present approach yields results which are consistent with linear theory and
in good agreement with wind tunnel experiments, and that the theory developed in Chapter Two, gives satisfactory explanations to the unusual flutter characteristic of a supercritical wing.

In Chapter Six, the criterion developed in Chapter Two for the prediction of the location of transonic dip is extended and tested on three supercritical wings. This simple criterion for the transonic dip, being purely aerodynamic and derived for airfoils, is found to be applicable for supercritical wings as well.

In Chapter Seven, the efficiency of present method is assessed and conclusions are drawn from the studies presented.
Figure 1.1  Flutter speed vs. Mach number curve showing the "transonic dip" - reproduced from Tijdeman's Thesis (1975).

Figure 1.2  Comparison of flutter boundaries of a swept wing with supercritical and conventional profiles - from Farmer and Hanson (1976).
CHAPTER 2
MODELLING AND ANALYSIS OF FLUTTER

In this chapter, the equations of motion for a two-degree-of-freedom airfoil system are derived. It is shown that solutions of this system can be simplified using an algebraic transformation and an aerodynamic model. A solution procedure equivalent to the conventional $U-g$ method is discussed. Simplified expressions for the speed and frequency at flutter are obtained in terms of the structural parameters, the aerodynamic lift and moment slopes, and two time scales which characterize the phase lag between structural deformation and aerodynamic response. These expressions reveal many characteristics of flutter at transonic speeds, including an explanation and criterion for the transonic dip.

2.1 Aeroelastic Equations

Figure 2.1 shows a schematic representation of a binary mass-spring system of an airfoil with two degrees of freedom, flexural displacement $h$ (positive downward) and torsional displacement $\alpha$ (positive nose up). The structural properties of this system are characterized by the mass density ratio $\mu = m(\pi \rho_{c} b^{2})$, natural frequencies $\omega_{h}$ and $\omega_{\alpha}$ (corresponding to different spring constants), moment of inertia $I_{\alpha}$ (about elastic axis), position of the elastic axis $a$, and position of mass center $X_{cg}$. The balance of the aerodynamic, inertia, and elastic forces leads to

\[ m\ddot{h} + S_{\alpha}\ddot{\alpha} + m\omega_{h}^{2}h = -L \]
\[ S_{\alpha}\ddot{h} + I_{\alpha}\ddot{\alpha} + I_{\alpha}\omega_{\alpha}^{2}\alpha = M_{y} \]  

(2.1)

where $L$ and $M_{y}$ are the total lift (positive up) and pitching moment, respectively, (positive nose up) per unit span and $S_{\alpha} = mb(X_{cg} - a)$ is the static unbalance. For harmonic
motions of frequency $\omega$, the corresponding generalized displacements $h$ and $\alpha$ can be approximated as

$$h = h_0 + h_1 e^{i\omega t}$$
$$\alpha = \alpha_0 + \alpha_1 e^{i\omega t} \quad (2.2)$$

The aerodynamic forces $L$ and $M_y$ in terms of the mean and first harmonic aerodynamic coefficients are expressed as

$$\frac{2L}{\rho_\infty U^2_{\infty} c} = C_{L_0} + C_{L_1} e^{i\omega t}$$
$$\frac{2M_y}{\rho_\infty U^2_{\infty} c^2} = C_{M_0} + C_{M_1} e^{i\omega t} \quad (2.3)$$

where

$$C_{L_1} = C_{Lh} \frac{h_1}{c} + C_{L\alpha} \alpha_1$$
$$C_{M_1} = C_{Mh} \frac{h_1}{c} + C_{M\alpha} \alpha_1 \quad (2.4)$$

and $C_{Lh}, C_{L\alpha}, C_{Mh}, C_{M\alpha}$ are lift and moment derivatives with respect to the generalized oscillation amplitudes, $h_1$ and $\alpha_1$. The balance of static force and displacement eliminates the dependence of the dynamic equation on the static lift coefficient, $C_{L_0}$, moment coefficient, $C_{M_0}$, pitch displacement, $\alpha_0$, and plunge displacement $h_0$. After substituting Eqs. (2.2) and Eqs. (2.3) into Eqs. (2.1), and equating the first harmonic components on both sides of Eqs. (2.1), one obtains the following equations:

$$-m\omega^2 h_1 - S_\alpha \omega^2 \alpha_1 + \omega^2 h_1 m_1 = -\frac{1}{2} \rho_\infty U^2_{\infty} c \left( C_{Lh} \frac{h_1}{c} + C_{L\alpha} \alpha_1 \right)$$
$$-S_\alpha \omega^2 h_1 - I_\alpha \omega^2 \alpha_1 + I_\alpha \omega^2 \alpha_1 = \frac{1}{2} \rho_\infty U^2_{\infty} c^2 \left( C_{Mh} \frac{h_1}{c} + C_{M\alpha} \alpha_1 \right) \quad (2.5)$$

or in matrix form
where $A$ is

$$
[A] = \begin{bmatrix}
\mu(1 - R^2Z) + L_1 & \mu(X_{cg} - a) + L_2 \\
\mu(X_{cg} - a) + M_1 & \mu_r^2(1 - Z) + M_2
\end{bmatrix}
$$

(2.7)

$Z$ is a complex form of the structural damping factor $g$ and the harmonic frequency $\omega$, i.e.,

$$
Z = \left(\frac{\omega_\alpha}{\omega}\right)^2 (1 + ig)
$$

(2.8)

$R = \omega_0/\omega_\alpha$ the uncoupled frequency ratio, and $r_\alpha^2 = I_\alpha/(mb^2)$ the dimensionless radius of gyration about the elastic axis. In this chapter, the distance between the center of gravity and elastic axis $(X_{cg} - a)$ may be replaced by $X_b$. $L_1, L_2, M_1$ and $M_2$ are the lift and moment coefficients (in complex form) for plunging and pitching modes, subscripts 1 and 2, respectively. These coefficients are functions of the reduced frequency and Mach number, but independent of $h_1$ and $\alpha_1$. They are defined as

$$
L_1 = -\frac{C_{lh}}{2\pi k_b^2} = -\frac{2C_{lh}}{\pi k_c^2} \\
L_2 = -\frac{C_{l\alpha}}{\pi k_b^2} = -\frac{4C_{l\alpha}}{\pi k_c^2}
$$

(2.9)

$$
M_1 = \frac{C_{mh}}{\pi k_b^2} = \frac{4C_{mh}}{\pi k_c^2} \\
M_2 = \frac{2C_{mix}}{\pi k_c^2} = \frac{8C_{mix}}{\pi k_c^2}
$$

For a flutter analysis, the existence of nontrivial solutions of the homogeneous equations requires that the determinant of $A$ be zero, i.e.,

$$
\text{det}(A) = 0 \quad \text{or} \quad \begin{vmatrix}
\mu(1 - R^2Z) + L_1 & \mu(X_{cg} - a) + L_2 \\
\mu(X_{cg} - a) + M_1 & \mu_r^2(1 - Z) + M_2
\end{vmatrix} = 0
$$

(2.10)
A flutter point is found when the imaginary part of the complex eigenvalue $Z$ is zero, i.e., when the system encounters zero damping. This is the basic formulation of the conventional $U-g$ method.

In general, the structural parameters $X_{cg}$, $R$, $r_{cg}$, and $\mu$ are independent of the pitching axis $a$, whereas the aerodynamic coefficients and moment of inertia about elastic axis are. The radius of gyration is simply related to the mass center and elastic axis by $r_{\alpha}^2 = r_{cg}^2 + a^2$, while the aerodynamic coefficients defined for an arbitrary elastic axis position $a$ are related to those calculated for some particular axis $a'$ by the following transformation:

$$
\begin{pmatrix}
L_1 \\
L_2 \\
M_1 \\
M_2
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 & 0 \\
(a' - a) & 1 & 0 & 0 \\
(a' - a) & 0 & 1 & 0 \\
(a' - a)^2 & a' - a & a' - a & 1
\end{pmatrix}
\begin{pmatrix}
L_1' \\
L_2' \\
M_1' \\
M_2'
\end{pmatrix}

(2.11a)

or, alternatively

$$
\begin{pmatrix}
C_{lh} \\
C_{l\alpha} \\
C_{nh} \\
C_{m\alpha}
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 & 0 \\
-s & 1 & 0 & 0 \\
s & 0 & 1 & 0 \\
-s^2 & s & -s & 1
\end{pmatrix}
\begin{pmatrix}
C_{lh}' \\
C_{l\alpha}' \\
C_{nh}' \\
C_{m\alpha}'
\end{pmatrix}

(2.11b)

where $s = (a - a')/2$. (hereinafter primed quantities are computed about the pitching axis at mid-chord, i.e., $a'=0$ unless otherwise specified)
Figure 2.1 Two-degree-of-freedom airfoil system.
2.2 Simple Model for Unsteady Transonic Indicial Response

The U-g method for a flutter point is to solve Eq. (2.10) for $Z$ with computed or interpolated values of the aerodynamic coefficients over a range of reduced frequencies until the imaginary part of $Z$ is zero. The most time-consuming part of solving Eq. (2.10) is the computation of the aerodynamic coefficients $L_1, L_2, M_1,$ and $M_2$. For each Mach number and frequency, the flow governing equations are solved for as many cycles as needed until the lift and moment responses to harmonic body motion become harmonic and free from effects of the initial condition. Alternatively, these responses can also be composed from the response to a step (indicial) change of a body deformation mode using the Duhamel principle.

After sampling a number of indicial response curves for pitching about mid-chord, Fung (1982) concluded that the response $C(t)$, computed after a sufficiently large time and presumably to have reached its asymptotic value $C_{\text{max}}$, can be characterized by a time scale $\lambda$.

![Figure 2.2 Model for indicial response](image)

which is obtained by minimizing the error functional
\[ I(\lambda) = \int_0^T \left( \ddot{C} + \lambda \bar{C} \right)^2 dt \]  
(2.13)

with respect to \( \lambda \) to yield

\[ \lambda = -\frac{\bar{C}^2(T) - \bar{C}^2(0)}{2 \int_0^T \bar{C}^2(t) dt} = \frac{(C_{\text{max}} - C(0))^2}{2 \int_0^T (C_{\text{max}} - C)^2 dt} \]  
(2.14)

Thus, the indicial response is approximated by

\[ C = (C_{\text{max}} - C(0))(1 - e^{-\lambda t}) = \Delta C_{\text{max}}(1 - e^{-\lambda t}) \]  
(2.15)

and the corresponding harmonic response \( \dot{C}(t,k) \) to harmonic motion of frequency \( k \) is obtained through the Duhamel integral, i.e.,

\[ \dot{C}(t,k) = \Delta C_{\text{max}} \int_0^T \left[ 1 - e^{-\lambda (t-t')} \right] \frac{d}{dt} \sin kt = \Lambda \sin(kt - \theta) \]  
(2.16)

where \( \Lambda = \frac{\Delta C_{\text{max}}}{\sqrt{1 + \left( \frac{k}{\lambda} \right)^2}} \), \( \theta = \tan^{-1}\left( \frac{k}{\lambda} \right) \)

This aerodynamic model implies that the amplitude and phase depend on the parameter \( k/\lambda \) (\( k \) and \( \lambda \) hereon are based on full chord). Thus, the lift and moment coefficients with respect to pitching motion can be written as:

\[ C_{\text{l}}(k_c, M_{\infty}) = \frac{C_{\text{l},0}(M_{\infty})}{\left[ 1 + (k_c/\lambda_i)^2 \right]^{1/2}} e^{-i \tan^{-1}\left( \frac{k_c}{\lambda_i} \right)} = \frac{C_{\text{l},0}}{\left[ 1 + \left( \frac{k_c}{\lambda_i} \right)^2 \right]^{1/2}} \left( 1 - \frac{k_c}{\lambda_i} \right) \]  
(2.17)

\[ C_{\text{m}}(k_c, M_{\infty}) = \frac{C_{\text{m},0}(M_{\infty})}{\left[ 1 + (k_c/\lambda_m)^2 \right]^{1/2}} e^{-i \tan^{-1}\left( \frac{k_c}{\lambda_m} \right)} = \frac{C_{\text{m},0}}{\left[ 1 + \left( \frac{k_c}{\lambda_m} \right)^2 \right]^{1/2}} \left( 1 - \frac{k_c}{\lambda_m} \right) \]

Here \( C_{\text{l},0}(=\Delta C_{\text{l},\text{max}}/\Delta \alpha) \) and \( C_{\text{m},0}(=\Delta C_{\text{m},\text{max}}/\Delta \alpha) \) are the lift and moment slopes with respect to change of pitch angle.
For small amplitude pitching and plunging, the displacement function, i.e.,
\[ f = h(t) + (x-x_o)\alpha(t), \]
correspond to the perturbed velocity component,
\[ V = \frac{dh}{dt} + \alpha + \frac{d\alpha}{dt}(x-x_o), \]
normal to the freestream. For low frequency pitching about mid-chord, \((\alpha' = 0)\), the product of \(d\alpha/dt\) and \((x-x_o)\) is small compared to \(\alpha\) and can be neglected. Therefore, the contribution to the perturbed velocity from plunging has the same form as that from pitching. By comparing \(h(t) = -ksinkt = kcos(kt+\pi/2)\) and \(\alpha = cos(kt)\), the aerodynamic coefficients for plunging can then be related to those for pitching, i.e.,
\[
C_{lh} = C_{la}k e^{i\left(\frac{3\pi}{2}\right)}, \quad C_{mh} = C_{ma}k e^{i\left(\frac{3\pi}{2}\right)}
\]
Eq. (2.18) implies that
\[
L_1M_2 - L_2M_1 = L'_1M'_2 - L'_2M'_1 = 0
\]
Using Eq. (2.17) and Eq. (2.18), the aerodynamic forces \(L'_1, L'_2, M'_1,\) and \(M'_2\) can be expressed in real and imaginary parts as:
\[
L'_1 = L'_1,r + iL'_1,i = \frac{2Cla}{\pi \left(1 + \frac{1}{Z\lambda_i^2}\right)} \left[\frac{1}{\lambda_i} \right], \quad \frac{\lambda_i}{k_c}
\]
\[
L'_2 = L'_2,r + iL'_2,i = \frac{4Cla}{\pi \left(1 + \frac{1}{Z\lambda_i^2}\right)} \left[\frac{1}{\lambda_i^2 Z\lambda_i} \right], \quad \frac{-\lambda_i}{k_c}
\]
\[
M'_1 = M'_1,r + iM'_1,i = \frac{4Cma}{\pi \left(1 + \frac{1}{Z\lambda_m^2}\right)} \left[\frac{1}{\lambda_m} \right], \quad \frac{\lambda_m}{k_c}
\]
\[
M'_2 = M'_2,r + iM'_2,i = \frac{8Cma}{\pi \left(1 + \frac{1}{Z\lambda_m^2}\right)} \left[\frac{1}{\lambda_m^2 Z\lambda_m} \right], \quad \frac{-\lambda_m}{k_c}
\]
Here, $C_{lq}$, $C_{m\alpha}$ are the quasi-steady lift and moment slopes, the subscripts $r$, $i$, $l$, $m$ denote real, imaginary, lift and moment, respectively, $1/k_c^2$ has been replaced by $Z \cdot \bar{U}$, where $\bar{U} = (U/c/\omega_\alpha)^2$ is the square of a normalized flutter speed and $Z$, $\bar{U}$, and $k_c^2$ are related by $Z = 1/\left(\bar{U}k_c^2\right)$. It is clear from Eqs. (2.20) that the aerodynamic responses of the airfoil system are now characterized by two time like constants $1/\lambda_l$, $1/\lambda_m$ and two aerodynamic derivatives $C_{lq}$, $C_{m\alpha}$ and that the phase lags between harmonic motions and aerodynamic responses are characterized by $k_c/\lambda_l$ and $k_c/\lambda_m$.

### 2.3 Solution Procedure

The expressions in Eqs. (2.20) with the reduced frequency tied to the aerodynamic coefficients simplify the solution procedure for the conventional $U$-$g$ method. Assuming the structural damping is zero and $Z$ is real, Eq. (2.10) can be decoupled into a real part:

$$
F_r = R^2 \mu^2 \gamma_{\alpha}^2 Z^2 - \left(1 + R^2\right) \mu^2 \gamma_{\alpha}^2 Z + \mu^2 \gamma_{\alpha}^2 - \mu^2 X_b^2 - L_{1,r} \mu R^2 Z - M_{2,r} \mu L_{2,r} \mu X_b - M_{1,r} \mu X_b = 0
$$

(2.21)

and an imaginary part:

$$
F_i = -Z \left(L_{1,i} \mu R^2 + M_{2,i} \mu R^2\right) + L_{1,i} \mu R^2 + M_{2,i} \mu - L_{2,i} \mu X_b - M_{1,i} \mu X_b = 0
$$

(2.22)

For a set of $L^l$, $L^m$, $M^l$, $M^m$ and structural parameters, $Z$ can be found by the following procedure:

(1) Solve Eq. (2.22) for two roots of $Z$ starting with an initial guess $\bar{U}$, i.) if both roots are negative change initial guess and start over,
ii.) if one positive and one negative, choose the positive Z

iii.) if two positive roots, choose the smaller

(2) Examine if the Z found also satisfies Eq. (2.21)

and change the initial guess $\bar{U}$ until $F_r$ is within a small value.

This procedure, schematically shown in Figure 2.3, differs from the conventional $U-g$ method by varying $\bar{U}$ instead of the reduced frequency for a solution. Once $\bar{U}$ and Z are found, the reduced frequency and normalized flutter speed are determined as well. Here, this procedure is called a full solution implementation as opposed to the simplified flutter solution to be introduced later.
Figure 2.3 Flow chart for the solution procedure.
2.4 Analysis of Airfoil Flutter

Three airfoils the NACA 64A006, NACA 64A010, CAST-7(12% thick) and a flat plate, Figure 2.4, were chosen to validate Eq. (2.21) and (2.22). Yang et al. (1979) & (1980) performed a flutter analysis on these airfoils and tabulated the aerodynamic data obtained by both indicial and harmonic approaches. Some of their data is converted according to Fung’s model into parameters $C_{l_{\alpha}}$, $1/\lambda_1$, $C_{m_{\alpha}}$, and $1/\lambda_m$ (denoted as $A_1$, $A_2$, $A_3$, and $A_4$, respectively), and listed in Tables 2.1 to 2.4 for NACA 64A006, NACA 64A010, CAST-7 and the flat plate, respectively. These data are based on an elastic axis located at mid-chord. As pointed out by Fung (1982), his model agrees with direct computation for reduced frequencies up to 0.2, and the cases considered here are within this limitation. With the converted data, Eq. (2.21) and (2.22) yielded flutter boundaries and frequencies, Figure 2.5, in excellent agreement with those shown in Figure 8 of Yang et al. (1979) for the NACA 64A006 airfoil and structural parameter set $X_{cg}=0$, $a=-0.5$, $R=0.1$, $r_{\alpha}=0.5$, and three $\mu$'s at 100, 200, 300.

For this airfoil, Yang et al. (1979) commented that the flutter speed is higher as mass center moves towards the elastic axis, but the location of the transonic dip is insensitive to the mass center, only more profound when the mass center moves toward the elastic axis. Figure 2.5 shows that the flutter speed increases with $\mu$, but the dip location is unaffected, at $M_{\infty}=0.85$, by the changing $\mu$. Figures 2.6 and 2.7 show the effect of the mass center and the elastic axis, respectively, on the flutter speed. From these results, it is clear that the flutter boundary is more sensitive to changes in mass density ratio and mass center than elastic axis.

Figures 2.8 and 2.9 show similar effect of the mass density ratio on the flutter boundary for the NACA 64A006. The transonic dip location is again insensitive to $\mu$, the frequency ratio $R$, or the location of the mass center $X_{cg}$. All this suggests that the flutter
speed and frequency are directly proportional to the mass ratio. No flutter points were found beyond $M_\infty = 0.82$ for the frequency ratio $R = 0.3$, mass center $X_{cg} = -0.25$ and elastic axis at $a = -0.5$. Similar trends are found in the results of Yang et al. (1980) and Isogai (1980) for different sets of structural parameters.

For the CAST-7 airfoil, which represents a typical cross section of a modern supercritical transport wing, the dependence of the flutter boundary on the mass density ratio and frequency ratio is same as those for the two conventional airfoils, but the Mach number at which the flutter dip occurs is different for different airfoils. Figure 2.11 shows how the flutter dip moves from $M_\infty = 0.7$ for the 12% thick CAST-7 supercritical airfoil to 0.85 for the 6% thick conventional NACA 64A006. This dependency on thickness is clearly a transonic effect. The limiting case is that of a flat plate for which the flutter boundary continues to drop as the Mach number increases without the tendency of a dip.

All flutter boundaries predicted using the model agree very well with those of Yang, et al. (1979). The results shown from Figure 2.5 to Figure 2.11 establish that the present approach using a simplified aerodynamic model is equivalent to the conventional $U$-$g$ method using numerically evaluated aerodynamics.

It can be seen from Eq. (2.22) that $Z$ depends weakly on $\mu$, since $\mu$ can be factored out, and the dependence of $L_1, L_2, M_1,$ and $M_2$ on $\mu$ is through the solution of Eq. (2.21) for $k_c$. If $I/\lambda_1, I/\lambda_m$ are the same, the terms $Z \bar{\lambda}_1^2, Z \bar{\lambda}_m^2$ in the expressions for $L_1, L_2,$ and $M_1, M_2$ become common factors, then $Z$ is independent of $\mu$. The $Z$ values for the NACA 64A006, NACA 64A010, CAST-7, and flat plate airfoil and the parameter set $a=0, X_{cg}=0, R=0.1, r_\alpha=0.5$ and different $\mu$'s, are plotted in Figure 2.12 showing a weak dependence on $\mu$. Hence, it is advantageous to replace the conventional $U$ vs $M_\infty$ flutter curve with a $Z$ vs $M_\infty$ curve, which is inversely proportional to $U^2$ but independent
of $\mu$. Since all cases shown in Figure 2.12 are for the same set of structural parameters, the strong dependence of $Z$ on airfoil thickness is due purely to the aerodynamics. In addition, from Figure 2.5 to 2.12 and all data examined and computed for the $M_\infty$ range between 0.6 and 0.9, it is found that $k_c$ has only a weak dependence on $M_\infty$ but varies with $\mu$. 
### Table 2.1  Aerodynamic coefficients converted from Yang et al. (1979) according to Fung's model for NACA 64A006 airfoil

<table>
<thead>
<tr>
<th>$M_\infty$</th>
<th>$1/\lambda_1(=A_2)$</th>
<th>$C_{\alpha}(=A_1)$</th>
<th>$1/\lambda_m(=A_4)$</th>
<th>$C_{m\alpha}(=A_3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.70000</td>
<td>2.1319</td>
<td>9.1513</td>
<td>2.4404</td>
<td>1.9083</td>
</tr>
<tr>
<td>0.72500</td>
<td>2.3407</td>
<td>9.4814</td>
<td>2.6819</td>
<td>2.0129</td>
</tr>
<tr>
<td>0.75000</td>
<td>2.5494</td>
<td>9.8114</td>
<td>2.9234</td>
<td>2.1174</td>
</tr>
<tr>
<td>0.77500</td>
<td>2.7581</td>
<td>10.141</td>
<td>3.1650</td>
<td>2.2220</td>
</tr>
<tr>
<td>0.80000</td>
<td>2.9668</td>
<td>10.472</td>
<td>3.4065</td>
<td>2.3266</td>
</tr>
<tr>
<td>0.82500</td>
<td>3.5880</td>
<td>11.969</td>
<td>4.7188</td>
<td>2.7179</td>
</tr>
<tr>
<td>0.85000</td>
<td>4.6266</td>
<td>14.128</td>
<td>6.0311</td>
<td>3.1093</td>
</tr>
<tr>
<td>0.86250</td>
<td>5.1619</td>
<td>16.277</td>
<td>6.5032</td>
<td>2.5664</td>
</tr>
<tr>
<td>0.87000</td>
<td>5.3553</td>
<td>17.478</td>
<td>4.9933</td>
<td>1.5694</td>
</tr>
</tbody>
</table>

### Table 2.2  Aerodynamic coefficients converted from Yang et al. (1979) according to Fung's model for NACA 64A010 airfoil

<table>
<thead>
<tr>
<th>$M_\infty$</th>
<th>$1/\lambda_1(=A_2)$</th>
<th>$C_{\alpha}(=A_1)$</th>
<th>$1/\lambda_m(=A_4)$</th>
<th>$C_{m\alpha}(=A_3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.72000</td>
<td>1.9266</td>
<td>9.0989</td>
<td>2.1478</td>
<td>1.9997</td>
</tr>
<tr>
<td>0.74000</td>
<td>2.3885</td>
<td>9.7919</td>
<td>2.7046</td>
<td>2.1042</td>
</tr>
<tr>
<td>0.76000</td>
<td>2.8504</td>
<td>10.485</td>
<td>3.2614</td>
<td>2.2686</td>
</tr>
<tr>
<td>0.78000</td>
<td>3.3282</td>
<td>11.715</td>
<td>4.0775</td>
<td>2.5789</td>
</tr>
<tr>
<td>0.80000</td>
<td>3.8552</td>
<td>13.118</td>
<td>4.9014</td>
<td>2.9057</td>
</tr>
<tr>
<td>0.82000</td>
<td>3.9438</td>
<td>11.015</td>
<td>5.2899</td>
<td>1.8909</td>
</tr>
<tr>
<td>0.84000</td>
<td>4.0323</td>
<td>8.9113</td>
<td>5.6783</td>
<td>0.87611</td>
</tr>
</tbody>
</table>
Table 2.3  
Aerodynamic coefficients converted from Yang et al. (1979) according to Fung's model for CAST-7 airfoil

<table>
<thead>
<tr>
<th>$M_{\infty}$</th>
<th>$\lambda_{I}(=A_2)$</th>
<th>$C_{D}(=A_1)$</th>
<th>$\lambda_{m}(=A_4)$</th>
<th>$C_{m_{\infty}}(=A_3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.60000</td>
<td>1.7057</td>
<td>8.1782</td>
<td>1.8986</td>
<td>2.0306</td>
</tr>
<tr>
<td>0.62500</td>
<td>1.8417</td>
<td>8.4825</td>
<td>2.0612</td>
<td>2.1090</td>
</tr>
<tr>
<td>0.65000</td>
<td>2.1139</td>
<td>8.8895</td>
<td>2.4178</td>
<td>2.2236</td>
</tr>
<tr>
<td>0.67500</td>
<td>2.3885</td>
<td>9.5018</td>
<td>2.7801</td>
<td>2.4422</td>
</tr>
<tr>
<td>0.70000</td>
<td>2.9849</td>
<td>11.372</td>
<td>3.6502</td>
<td>2.9197</td>
</tr>
<tr>
<td>0.71000</td>
<td>3.8252</td>
<td>12.532</td>
<td>5.0999</td>
<td>2.9364</td>
</tr>
<tr>
<td>0.72000</td>
<td>5.5851</td>
<td>16.063</td>
<td>7.9916</td>
<td>2.6201</td>
</tr>
</tbody>
</table>

Table 2.4  
Aerodynamic coefficients converted from Yang et al. (1979) according to Fung's model for Flat plate

<table>
<thead>
<tr>
<th>$M_{\infty}$</th>
<th>$\lambda_{I}(=A_2)$</th>
<th>$C_{D}(=A_1)$</th>
<th>$\lambda_{m}(=A_4)$</th>
<th>$C_{m_{\infty}}(=A_3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.70000</td>
<td>2.0548</td>
<td>7.5460</td>
<td>2.8507</td>
<td>1.9399</td>
</tr>
<tr>
<td>0.72500</td>
<td>2.1775</td>
<td>7.7303</td>
<td>2.9996</td>
<td>1.9891</td>
</tr>
<tr>
<td>0.75000</td>
<td>2.3379</td>
<td>7.9569</td>
<td>3.2107</td>
<td>2.0529</td>
</tr>
<tr>
<td>0.77500</td>
<td>2.5360</td>
<td>8.2259</td>
<td>3.4842</td>
<td>2.1312</td>
</tr>
<tr>
<td>0.80000</td>
<td>2.7717</td>
<td>8.5373</td>
<td>3.8199</td>
<td>2.2240</td>
</tr>
<tr>
<td>0.82500</td>
<td>3.0299</td>
<td>8.8789</td>
<td>4.1910</td>
<td>2.3267</td>
</tr>
<tr>
<td>0.85000</td>
<td>3.3563</td>
<td>9.2873</td>
<td>4.6783</td>
<td>2.4533</td>
</tr>
<tr>
<td>0.86250</td>
<td>3.5451</td>
<td>9.5165</td>
<td>4.9655</td>
<td>2.5255</td>
</tr>
<tr>
<td>0.87000</td>
<td>3.6678</td>
<td>9.6648</td>
<td>5.1582</td>
<td>2.5738</td>
</tr>
</tbody>
</table>
Figure 2.4  Airfoil configurations.
Figure 2.5 Comparison of flutter speeds and frequencies computed using present theory and classical method [Yang's (1979)] for various mass density ratios on NACA 64A006.
\( a = -0.5, R = 0.1, r_c = 0.5, \mu = 100 \)

\[
\begin{align*}
U/(b\omega_c) & \quad \text{for } NACA 64A006.
\end{align*}
\]

Figure 2.6   Effect of mass center on flutter speed and frequency for NACA 64A006.
$X_{cg} = 0, R = 0.1, r_\alpha = 0.5, \mu = 100$

Figure 2.7 Effect of elastic axis on flutter speed for NACA 64A006.
Figure 2.8 Effect of mass density ratio on flutter speed for NACA 64A010.
Figure 2.9 Effect of mass density ratio on flutter speed for NACA 64A010.
Figure 2.10  Effect of mass density ratio on flutter speed for CAST-7.
Figure 2.11  Flutter boundaries on flutter speeds for four airfoil configurations.
Figure 2.12  Effect of $\mu$ on $Z$ for different airfoils.
2.5 Simplified Flutter Solutions

The analysis above and the simplicity of Fung's model strongly suggest that simplified algebraic expression for the solution of Eq. (2.10) exist. The flutter determinant, Eq. (2.10) can be written in matrix form as

\[
\begin{bmatrix}
L_1 \\
L_2 \\
M_1 \\
M_2
\end{bmatrix}
= BC - AD
\]  

(2.23)

where

\[A = \mu(1 - R^2Z), \quad B = \mu(X_{cg} - a), \quad C = \mu(X_{cg} - a), \quad D = \mu\alpha^2(1 - Z)\]  

(2.24)

or in terms of the primed quantities, about mid-chord, as

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
-a & 1 & 0 & 0 \\
-a & 0 & 1 & 0 \\
a^2 & -a & -a & 1
\end{bmatrix}
\begin{bmatrix}
L'_1 \\
L'_2 \\
M'_1 \\
M'_2
\end{bmatrix}
= BC - AD
\]  

(2.25)

Notice that the first two matrices of Eq. (2.25) can be combined to form

\[
\Rightarrow \begin{bmatrix}
L'_1 \\
L'_2 \\
M'_1 \\
M'_2
\end{bmatrix}
= B'C' - A'D'
\]  

(2.26)
which is in the same form of Eq. (2.23). Since \( B'C' - A'D' = BC - AD \), the primed and unprimed systems can be made algebraically the same (homomorphic) if the primed and unprimed quantities are related by

\[
D' = D + aC + AB + a^2A
\]
\[
C' = C + aA
\]
\[
B' = B + aA
\]
\[
A' = A
\]  
(2.27)

Since \( B = C \) and \( B' = C' \), equivalently,

\[
A' = \mu' r_\alpha^2 (1 - Z') = \mu_\alpha^2 (1 - Z) + 2a\mu (X_{cg} - a) + a^2 \mu (1 - R^2 Z)
\]  
(2.28a)

\[
B' = \mu' (X_{cg}') = \mu (X_{cg} - a) + \mu a (1 - R^2 Z)
\]  
(2.28b)

\[
C = \mu' (1 - R^2 Z') = \mu (1 - R^2 Z)
\]  
(2.28c)

By defining a new set of structural parameters, \( \mu', R', r'_\alpha, \) and \( X_{cg}' \), a solution \( Z \) of Eq. (2.10) can be found by mapping it to the corresponding solution \( Z' \) if whose dependency on the primed parameters is known.

The real and imaginary part of Eq. (2.10) expressed in primed parameters become

\[
F_r = R^2 \mu^2 r_\alpha^2 Z^{'2} - (1 + R^2) \mu^2 r_\alpha^2 Z' + \mu^2 r_\alpha^2 - \mu^2 X_{cg} - L_{4,r} \mu' r_\alpha^2 Z' - M_{2,r} \mu' R^2 Z' + L_{4,r} \mu' r_\alpha^2 + M_{2,r} \mu' - (L_{2,r} + M_{1,r}) \mu' X_{cg} = 0
\]  
(2.29a)

and

\[
F_i = -Z' \mu' (L_{4,i} r_\alpha^2 + M_{2,i} R^2) + \mu' (L_{4,i} r_\alpha^2 + M_{2,i}) - \mu' X_{cg}' (L_{2,i} + M_{1,i})
\]
\[
= -Z' (L_{4,i} r_\alpha^2 + M_{2,i} R^2) + (L_{4,i} r_\alpha^2 + M_{2,i}) - \left[ \mu (X_{cg} - a) + a (1 - R^2 Z') \right] (L_{2,i} + M_{1,i})
\]  
(2.29b)

\[= 0\]
which are pure algebraic expressions of four aerodynamic and five structural parameters for the solution of the flutter speed $U$ and frequency $k_e$, or equivalently, $\bar{U}'$ and $Z'$. It should be noted that the primed and unprimed systems are identical when $a=0$. Here, $\rho=\mu/\mu'$ as defined later is independent of $\mu$. Without further substitution of the expressions for $L'_1, L'_2, M'_1, M'_2$, Eq. (2.29a) and Eq. (2.29b) are exactly equivalent to Eq. (2.21) and Eq. (2.22), but algebraically simplified through a homomorphic transformation. An explicit expression for $Z'$ is readily available from Eq. (2.29b), i.e.,

$$Z' = \frac{L'_{1,i}r'^2 + M'_{2,i} - \left[p(X_{cg} - a) + a\right](L'_{2,i} + M'_{1,i})}{L'_{1,i}r'^2 + M'_{2,i}R'^2 - aR'^2(L'_{2,i} + M'_{1,i})}$$

(2.30)

which already implies that the circular flutter frequency $\omega = \omega_c/\sqrt{Z'}$ for the case $a=0$ depends only on the imaginary parts of the aerodynamic parameters and very weakly on $\mu$ as mentioned and illustrated in previous section. By further assuming $\lambda_l=\lambda_m$, or $k_c<<\lambda_l$ and $\lambda_m$, the denominators in Eq. (2.20) can be factored out and Eq. (2.29b) reduces to

$$Z' = \frac{A_1r'^2 + 4A_3A_4 + \left[p(X_{cg} - a) + a\right](2A_3 + 2A_1A_2)}{A_1r'^2 + 4A_3A_4R'^2 + aR'^2(2A_3 + 2A_1A_2)}$$

(2.31)

where the aerodynamic parameters $C_{l\alpha}, 1/\lambda_A, C_{m\alpha}, 1/\lambda_m$ have been replaced by $A_1, A_2, A_3, A_4$, respectively for easy distinction from the primed structural parameters, which then can be chosen, when $r'^2 + 2aX_{cg} - a^2 \neq 0$, as $\mu' = \mu$ ($\rho=\mu$),

$$r'^2 = r'^2 + 2aX_{cg} - a^2$$

(2.32a)

$$R'^2 = \frac{r'^2 + 2aX_{cg} - a^2}{r'^2 + a^2R'^2}R'^2$$

(2.32b)

$$Z' = \frac{r'^2 + a^2R'^2}{r'^2 + 2aX_{cg} - a^2}Z$$

(2.32c)
and \( k_c^2 \) can be solved by plugging \( Z' \) and \( X'_{cg} \) into Eq. (2.29a)

\[
k_c^2 = -\frac{8A_3(R'^2Z' - 1) - 4A_4X'_{cg}}{r_\alpha^2 (1 - Z') [\pi \mu' (R'^2Z' - 1) + 2A_1A_2] + \pi \mu' X'_{cg}^2 + 4A_3A_4X'_{cg}}
\] (2.33)

The positive-definiteness of this equation explicitly states the conditions for flutter. If, however, \( r_\alpha^2 + 2aX_{cg} - a^2 = 0 \), one choice to satisfy the conditions of Eqs. (2.28) is

\[
R'^2 = \frac{r_\alpha^2}{r_\alpha^2 + a^2 R^2} R^2
\]

\[
\mu' = \frac{r_\alpha^2 + a^2 R^2}{r_\alpha^2 - r_\alpha^2 R^2 + a^2 R^2} \mu
\] (2.34)

\[
Z = \frac{r_\alpha^2}{r_\alpha^2 - r_\alpha^2 R^2 + a^2 R^2} (Z' - 1)
\]

With the introduction of a \( \beta \) and the relation \( r_\alpha^2 = \beta \cdot r_\alpha^2 \) such that

\[
\begin{cases}
\beta = 1, & r_\alpha^2 - r_\alpha^2 R^2 + a^2 R^2 \neq 0 \\
\beta \neq 1, & r_\alpha^2 - r_\alpha^2 R^2 + a^2 R^2 = 0
\end{cases}
\]

\( Z' \) assumes the form

\[
Z' = \frac{2K_1[R'^2a^2X_{cg} + (1 - R'^2)r_\alpha^2 X_{cg} + R'^2a_\alpha^2 a^2 + (1 + K')(R'^2a^2 + r_\alpha^2) \cdot (1 + K')]}{2K_1(R'^4 a^3 + R'^2a_\alpha^2 a^2) + (1 + K'R'^2)(R'^2a^2 + r_\alpha^2)}
\] (2.35)

Here, \( K' = \frac{4A_3A_4}{A_1r_\alpha^2} \) and \( K_1 = A_2 + \frac{A_3}{A_1} \). It can be shown that when \( \beta \neq 1 \),

\[
Z' - 1 = \frac{(1 - \beta)}{\beta} \cdot \frac{M_{2,i}^2 - X_{cg} \cdot (L_{2,i}^2 + M_{1,i}^2)}{(L_{1,i}r_\alpha^2 + M_{2,i}) - a \cdot (L_{2,i}^2 + M_{1,i})}
\] (2.36)

or
\[
Z = \frac{1}{R^2} \frac{M_{2,i}^2 - X_{cg} \left(L_{2,i}^2 + M_{1,i}^2\right)}{\left(L_{1,i}^2 + M_{2,i}^2\right) - a \left(L_{2,i}^2 + M_{1,i}^2\right)}
\]

(2.37)

which is independent of \(\beta\).

Further simplifications are possible by noting that the expression for \(Z\), Eq. (2.31), depends weakly on \(a\) and \(X_{cg}\) (assumed small), and setting both to zero reduces Eq. (2.31) and (2.33) to

\[
Z = \frac{\tau_2^2 A_1 + 4 A_3 A_4}{\tau_2^2 A_1 + 4 A_3 A_4 R^2} = \frac{1 + K}{1 + KR^2}
\]

(2.38)

\[
k_c^2 = \frac{-2(\tau_2^2 A_1 + 4 A_3 A_4 R^2)}{-\pi \mu_2^2 (1 - R^2) A_4 + 2(\tau_2^2 A_1 + 4 A_3 A_4 R^2) A_2 A_4} = \frac{-2(1 + KR^2) A_1 / A_4}{-\pi \mu (1 - R^2) + 2(1 + KR^2) A_1 A_2}
\]

(2.39)

The condition for flutter, \(k_c^2 > 0\), simply requires that the denominator in Eq. (2.39) be negative and \(R\) be less than one -usually \(A_i's\) are all positive-, and consequently, \(k_c = 1 / \sqrt{\mu}\) for large mass ratio \(\mu\). Since \(Z\) is independent of \(\mu\) and \(Z = 1 / \left(k_c^2 \bar{U}\right)\), the flutter speed \(U = c \omega_d \bar{U}\) must then be proportional to \(\sqrt{\mu}\).

2.5.1 Test Cases

We chose again the published results of Yang, et al. (1979) and Isogai (1980) to verify the formula derived and justify the assumptions made in the previous Section. Several test cases were designed to verify the simplified expressions in comparison with the full expression. Test cases 1 to 5 use the aerodynamic coefficients \(A_1, A_2, A_3, A_4\) converted from Yang's data for the NACA 64A006 airfoil and test case 6 and 7 use those converted from Isogai's data for the NACA 64A010 airfoil. These converted coefficients are shown in Figures 13a, 13b, 13c, and 13d. Test cases 1 to 5 were designed to verify
the formula for different structural parameter sets when \( r_{\alpha}^2 + 2aX_{cg} - a^2 \neq 0 \) - either \( r_{\alpha}^2 > 0 \) or \( r_{\alpha}^2 < 0 \) - for Eq. (2.31) and (2.33) and when \( r_{\alpha}^2 + 2aX_{cg} - a^2 = 0 \) for Eq. (2.35) and (2.33). Case 6 and test case 7, are chosen to be the same as CASE A and CASE B of Isogai (1980). The structural parameters sets, the transformation conditions, as well the airfoils are listed in Table 2.5.

**Table 2.5 Test Cases.**

<table>
<thead>
<tr>
<th>Case No.</th>
<th>( r_{\alpha}^2 + 2aX_{cg} - a^2 )</th>
<th>Airfoil type</th>
<th>( a )</th>
<th>( X_{cg} )</th>
<th>( \mu )</th>
<th>( \frac{r_{\alpha}^2}{2} ) R</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>&gt;0</td>
<td>NACA 64A006</td>
<td>0</td>
<td>0</td>
<td>100</td>
<td>0.25 0.1</td>
</tr>
<tr>
<td>2</td>
<td>&gt;0</td>
<td>NACA 64A006</td>
<td>-0.3</td>
<td>0</td>
<td>100</td>
<td>0.25 0.1</td>
</tr>
<tr>
<td>3</td>
<td>&gt;0</td>
<td>NACA 64A006</td>
<td>0</td>
<td>0.5</td>
<td>100</td>
<td>0.25 0.1</td>
</tr>
<tr>
<td>4</td>
<td>&lt;0</td>
<td>NACA 64A006</td>
<td>0</td>
<td>-0.6</td>
<td>100</td>
<td>0.25 0.1</td>
</tr>
<tr>
<td>5</td>
<td>=0</td>
<td>NACA 64A006</td>
<td>-0.5</td>
<td>0</td>
<td>100</td>
<td>0.25 0.1</td>
</tr>
<tr>
<td>6†</td>
<td>&gt;0</td>
<td>NACA 64A010</td>
<td>-2.0</td>
<td>-0.2</td>
<td>60</td>
<td>3.48 1.0</td>
</tr>
<tr>
<td>7‡</td>
<td>&gt;0</td>
<td>NACA 64A010</td>
<td>-0.3</td>
<td>0.2</td>
<td>60</td>
<td>0.49 0.2</td>
</tr>
</tbody>
</table>

†: Same as Isogai’s CASE A
‡: Same as Isogai’s CASE B

As mentioned before, a full implementation of Eq. (2.21) and (2.22) requires guesses of the normalized flutter speed square \( \bar{\lambda} \) unless \( \lambda_f = \lambda_m \), which in general are of the same order but different, as the values in Figure 2.13b and 2.13d exemplify. Figures 2.14 to 2.20 compare results obtained using the simplified formula Eq. (2.31), (2.33), and (2.34) with those using Eq. (2.21) and (2.22) for test cases 1 to 5, respectively. The agreement is in general good with only quantitative differences near the dips. It is interesting to note that the frequencies predicted using the simplified formulas are substantially higher near the dip, whereas the full implementation keeps them at about the same as those at lower Mach numbers. It seems that the coupling between Eq. (2.21) and (2.22) make the dips shallower and the flutter frequency rather insensitive to \( M_{\infty} \). Setting both \( a \) and \( X_{cg} \) to zero for reasons mentioned in Section 2.5 and taking advantage of insensitive of \( k_e \), the difference between Eq. (2.30) and (2.31), i.e.,
\[ Z_{\text{exact}} - Z_{\text{simp}} = \frac{1 + K F\left(k_c^2, A_2, A_4^2\right)}{1 + K R^2 F\left(k_c^2, A_2, A_4^2\right)} \frac{1 + K}{1 + K R^2} \]

\[ = k_c^2 \frac{(1 - R^2) k \left(A_2^2 - A_4^2\right)}{\left(1 + K R^2 + k_c^2\left(A_2^4 + K R^2 A_4^4\right)\right)\left(1 + K R^2\right)} \]

(2.40)

where \( F\left(k_c^2, A_2, A_4^2\right) = \frac{1 + k_c^2 A_2^2}{1 + k_c^2 A_4^2} \)

shows clearly that the error is proportional to \( K \left(A_2^2 - A_4^2\right)\), which attains its maximum at the dip according to the data on Tables 2.1 to 2.4. Since \( k_c \) is usually small for flutter at transonic speeds and is shown earlier to be proportional to \( 1/\sqrt{\mu} \), the weak sensitivity of \( Z \) to \( \mu \) especially near the dip is evident from the expression for \( Z_{\text{exact}} \).

Another comparison was made, showing good agreement in general, with the \( \text{CASE A} \) and \( \text{CASE B} \) of Isogai (1980). The discrepancies between full and simplified formulas, the bullets and circles, can be attributed to the substantial difference between the two aerodynamic time scales \( A_2 \) and \( A_4 \), the circles and crosses in Figure 2.13b and 2.13d. The difference in the predictions of the location of the dip between the present theory and that of Isogai, is due to the differences in the \( A \)'s converted from different harmonic responses of different frequencies. It is interesting to note that the present theory yields predictions in better agreement for \( \text{CASE B} \) than for \( \text{CASE A} \). It seems that the large distance of elastic axis away from the mid-chord in \( \text{CASE A} \) (\( a=-2.0 \)) magnifies the sensitivity of the aerodynamic response to reduced frequency.

These test cases establish the accuracy of the simplified expressions for flutter solutions.
Figure 2.13a  Converted lift coefficients.

Figure 2.13b  Converted time scale of lift response.
Figure 2.13c  Converted moment coefficients.

Figure 2.13d  Converted time scale of moment response.
Figure 2.14 Comparison of flutter solution predicted using full and simplified formulas for Case 1.
Figure 2.15  Comparison of flutter solutions predicted using full and simplified formulas for Case 2.
Figure 2.16  
Comparison of flutter solutions predicted using full and simplified formulas for Case 3.

\( a=0.5, \ X_{cg} = 0, \ R = 0.1, \ r_{\alpha} = 0.5 \ \mu = 100 \)

\( Z \)

\( M_\infty \)

\( a_{cg} = 0.5 \ a = 0 \ (\text{full}) \)

\( a_{cg} = 0.5 \ a = 0 \ (\text{simp}) \)
Figure 2.17  Comparison of flutter solutions predicted using full and simplified formulas for Case 4.
Comparison of flutter solutions predicted using full and simplified formulas for Case 5.
Comparison of flutter solutions predicted using full and simplified formulas for Case 6.
Comparison of flutter solutions predicted using full and simplified formulas for Case 7.
2.6 On the Transonic Dip

The study above amply demonstrates the validity of Fung's aerodynamic model and the further simplifications made here for the analysis of airfoil flutter at transonic speeds. Equations (2.20), (2.21), and (2.22) contain a set of structural parameters and four aerodynamic parameters $1/\lambda_{Ia}$, $C_{Ia}$, $1/\lambda_{m}$, and $C_{ma}$. Since only the latter are functions of the Mach number, the dependence of the flutter speed on the Mach number can be examined by differentiating $F_r$ with respect to the Mach number, or using the chain rule,

$$\frac{d\bar{U}}{dM_{\infty}} = \left( \frac{\partial F_r}{\partial \lambda_{Ia}^{-1}} \right)_T \cdot \frac{\partial \lambda_{Ia}^{-1}}{\partial M_{\infty}} + \left( \frac{\partial F_r}{\partial C_{Ia}} \right)_T \cdot \frac{\partial C_{Ia}}{\partial M_{\infty}} + \left( \frac{\partial F_r}{\partial \lambda_{m}^{-1}} \right)_T \cdot \frac{\partial \lambda_{m}^{-1}}{\partial M_{\infty}} + \left( \frac{\partial F_r}{\partial C_{ma}} \right)_T \cdot \frac{\partial C_{ma}}{\partial M_{\infty}}$$

(2.41)

where

$$\left( \frac{\partial F_r}{\partial \lambda_{Ia}^{-1}} \right)_T = \left( \frac{\partial F_r}{\partial \lambda_{Ia}^{-1}} + \frac{\partial F_r}{\partial \lambda_{Ia}} \cdot \frac{\partial \lambda_{Ia}}{\partial \lambda_{Ia}^{-1}} \right)$$

$$\left( \frac{\partial F_r}{\partial C_{Ia}} \right)_T = \left( \frac{\partial F_r}{\partial C_{Ia}} + \frac{\partial F_r}{\partial C_{Ia}} \cdot \frac{\partial C_{Ia}}{\partial C_{Ia}} \right)$$

$$\left( \frac{\partial F_r}{\partial \lambda_{m}^{-1}} \right)_T = \left( \frac{\partial F_r}{\partial \lambda_{m}^{-1}} + \frac{\partial F_r}{\partial \lambda_{m}} \cdot \frac{\partial \lambda_{m}}{\partial \lambda_{m}^{-1}} \right)$$

$$\left( \frac{\partial F_r}{\partial C_{ma}} \right)_T = \left( \frac{\partial F_r}{\partial C_{ma}} + \frac{\partial F_r}{\partial C_{ma}} \cdot \frac{\partial C_{ma}}{\partial C_{ma}} \right)$$

Here, $a, b, c$ and $d$ can be evaluated locally by taking the derivatives of the corresponding parameters with respect to $M_{\infty}$, and $A, B, C,$ and $D$ can be found with the help of a symbolic manipulation program such as MACSYMA. The four parameter groups $-A/aE, -B/bE, -C/cE$ and $-D/dE$ represent the parametric dependence of the slope of the flutter boundary on $M_{\infty}$. Tables 2.6 to 2.9 list these values for three Mach numbers around the
dip region for four airfoils and the parameters \( \mu = 100, X_{cg} = 0, R = 0.1, r_{\alpha} = 0.5 \) and \( a = -0.5 \). The following observations can be made from the data listed. The absolute value of \( B \) is small compared to \( A, C \) and \( D \). \( D \) is at least one decade larger than \( A \) or \( C \). \( Aa + Bb + Cc \) combined is of the order of \( D \).

1. \( |D|, |C|, |A| >> |B| \)
   \( |D| > |C|, |A| \)

2. \( Aa + Bb + Cc = O(D) \)

Therefore, it is clear that \( -DdE \) is the dominant term in Eqs. (2.41). Also, from Figures 2.13c and Figure 2.21, it can be observed that \( C_{m_\alpha} \) reaches the maximum as the flutter boundary reaches the minimum for all airfoils except the flat plate. Since \( D \) is the dominant term in Eqs. (2.41), a change of sign in \( d \) changes the slope of the flutter boundary, or the dip occurs at

3. \( \frac{dC_{m_\alpha}}{dM_\infty} = 0 \)

As mentioned before, the conventional \( U \) vs \( M_\infty \) flutter boundary curve can be replaced by \( Z \) vs \( M_\infty \), since \( k_c \) at flutter is rather insensitive to \( M_\infty \), and the dip location is insensitive to \( a \) and \( X_{cg} \), as shown in Figure 2.5 to 2.12. By setting both \( a \) and \( X_{cg} \) to zero in the expression for \( Z \), the dependence of \( Z \) on the parameter group \( K \) is clear and the its variation with \( M_\infty \) is found by

\[
\frac{dZ}{dM_\infty} = \left( \frac{1 - R^2}{1 + KR^2} \right) \frac{dK}{dM_\infty} = \frac{4(1 - R^2)K}{(1 + KR^2)^2} \left[ \frac{1}{A_1 dM_\infty} + \frac{1}{A_2 dM_\infty} + \frac{1}{A_4 dM_\infty} \right] \]

or approximately,
\[
\frac{dZ}{dM_\infty} = \frac{4(1 - R^2)K A_3}{(1 + KR^2)^{1/2} A_3} \frac{dA_3}{dM_\infty} = \frac{4(1 - R^2)K}{(1 + KR^2)^{1/2}} \frac{dC_{ma}}{dM_\infty} \tag{2.43}
\]

since \( A_3 \) is substantially smaller than both \( A_I \) and \( A_4 \) at transonic speeds. Hence, the simple criterion for the location of the flutter dip, at \( \frac{dC_{ma}}{dM_\infty} = 0 \), is confirmed and is predictable using quasi-steady results.

According to the present theory flutter at transonic speeds depends on two structural parameters, \( R \) and \( r_\alpha \), but critically on the aerodynamic parameter group, \( C_{ma} / C_{l_\alpha} \lambda_m \). At supercritical speeds, these values deviate substantially from those at low Mach numbers, or predicted by linearized theory. Due to the increasing time lag, measured by \( 1/\lambda_m \), the flutter speed becomes increasingly dependent on the moment slope, \( C_{ma} \), which increases rapidly as the shock appears and decreases sharply as the shock approaches the trailing edge. The transonic dip is then simply a manifestation of this purely transonic aerodynamic phenomenon.
Table 2.6 Derivatives of $F_r$ for NACA 64A006.

<table>
<thead>
<tr>
<th>$M_\infty$</th>
<th>0.825</th>
<th>0.850</th>
<th>0.870</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>3126.73</td>
<td>4144.96</td>
<td>1912.49</td>
</tr>
<tr>
<td>$B$</td>
<td>-160.59</td>
<td>-180.49</td>
<td>20.70</td>
</tr>
<tr>
<td>$C$</td>
<td>-1527.59</td>
<td>-2114.15</td>
<td>-938.57</td>
</tr>
<tr>
<td>$D$</td>
<td>12414.54</td>
<td>12651.41</td>
<td>12589.44</td>
</tr>
<tr>
<td>$E$</td>
<td>10393.47</td>
<td>14277.65</td>
<td>4035.94</td>
</tr>
<tr>
<td>$a$</td>
<td>34.58</td>
<td>46.29</td>
<td>15.83</td>
</tr>
<tr>
<td>$b$</td>
<td>63.88</td>
<td>146.04</td>
<td>141.43</td>
</tr>
<tr>
<td>$c$</td>
<td>50.17</td>
<td>81.03</td>
<td>-340.95</td>
</tr>
<tr>
<td>$d$</td>
<td>22.68</td>
<td>-8.74</td>
<td>-177.56</td>
</tr>
<tr>
<td>$Aa+Bb+Cc$</td>
<td>21210.80</td>
<td>-5814.20</td>
<td>353200.90</td>
</tr>
<tr>
<td>Slope($d\bar{U} / dM_\infty$)</td>
<td>-29.14</td>
<td>8.15</td>
<td>466.35</td>
</tr>
</tbody>
</table>

Table 2.7 Derivatives of $F_r$ for NACA 64A010.

<table>
<thead>
<tr>
<th>$M_\infty$</th>
<th>0.780</th>
<th>0.800</th>
<th>0.820</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>2625.66</td>
<td>3358.18</td>
<td>2627.54</td>
</tr>
<tr>
<td>$B$</td>
<td>-99.00</td>
<td>-138.24</td>
<td>-174.59</td>
</tr>
<tr>
<td>$C$</td>
<td>-1244.35</td>
<td>-1700.41</td>
<td>-1170.86</td>
</tr>
<tr>
<td>$D$</td>
<td>11592.96</td>
<td>11786.00</td>
<td>15980.89</td>
</tr>
<tr>
<td>$E$</td>
<td>8752.86</td>
<td>11402.04</td>
<td>6579.67</td>
</tr>
<tr>
<td>$a$</td>
<td>28.05</td>
<td>16.27</td>
<td>-0.73</td>
</tr>
<tr>
<td>$b$</td>
<td>91.54</td>
<td>-12.71</td>
<td>-145.71</td>
</tr>
<tr>
<td>$c$</td>
<td>45.42</td>
<td>30.48</td>
<td>14.53</td>
</tr>
<tr>
<td>$d$</td>
<td>25.32</td>
<td>-15.58</td>
<td>-66.19</td>
</tr>
<tr>
<td>$Aa+Bb+Cc$</td>
<td>8061.18</td>
<td>4550.52</td>
<td>6497.31</td>
</tr>
<tr>
<td>Slope($d\bar{U} / dM_\infty$)</td>
<td>-34.46</td>
<td>15.71</td>
<td>159.76</td>
</tr>
</tbody>
</table>
Table 2.8 Derivatives of $F_r$ for CAST-7.

<table>
<thead>
<tr>
<th>$M_{\infty}$</th>
<th>0.700</th>
<th>0.710</th>
<th>0.720</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>2636.11</td>
<td>3533.73</td>
<td>4374.64</td>
</tr>
<tr>
<td>$B$</td>
<td>-94.68</td>
<td>-183.77</td>
<td>-234.08</td>
</tr>
<tr>
<td>$C$</td>
<td>-1250.39</td>
<td>-1769.72</td>
<td>-2149.77</td>
</tr>
<tr>
<td>$D$</td>
<td>10504.30</td>
<td>12382.11</td>
<td>14521.81</td>
</tr>
<tr>
<td>$E$</td>
<td>10040.88</td>
<td>12133.31</td>
<td>11584.05</td>
</tr>
<tr>
<td>$a$</td>
<td>55.30</td>
<td>122.13</td>
<td>236.25</td>
</tr>
<tr>
<td>$b$</td>
<td>77.57</td>
<td>199.93</td>
<td>529.99</td>
</tr>
<tr>
<td>$c$</td>
<td>95.15</td>
<td>206.63</td>
<td>380.73</td>
</tr>
<tr>
<td>$d$</td>
<td>11.86</td>
<td>-11.96</td>
<td>-53.89</td>
</tr>
<tr>
<td>$Aa+Bb+Cc$</td>
<td>19470.30</td>
<td>29155.69</td>
<td>91193.81</td>
</tr>
<tr>
<td>Slope($d\bar{U} / dM_{\infty}$)</td>
<td>-14.35</td>
<td>9.804</td>
<td>59.69</td>
</tr>
</tbody>
</table>

Table 2.9 Derivatives of $F_r$ for flat plate.

<table>
<thead>
<tr>
<th>$M_{\infty}$</th>
<th>0.825</th>
<th>0.850</th>
<th>0.870</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>2549.12</td>
<td>2938.78</td>
<td>3322.41</td>
</tr>
<tr>
<td>$B$</td>
<td>-201.39</td>
<td>-236.76</td>
<td>-271.04</td>
</tr>
<tr>
<td>$C$</td>
<td>-1026.68</td>
<td>-1248.58</td>
<td>-1466.36</td>
</tr>
<tr>
<td>$D$</td>
<td>14701.49</td>
<td>15162.13</td>
<td>15515.69</td>
</tr>
<tr>
<td>$E$</td>
<td>8932.12</td>
<td>10332.25</td>
<td>11732.14</td>
</tr>
<tr>
<td>$a$</td>
<td>11.49</td>
<td>14.43</td>
<td>16.88</td>
</tr>
<tr>
<td>$b$</td>
<td>14.85</td>
<td>17.64</td>
<td>20.40</td>
</tr>
<tr>
<td>$c$</td>
<td>16.84</td>
<td>21.74</td>
<td>26.92</td>
</tr>
<tr>
<td>$d$</td>
<td>4.54</td>
<td>5.50</td>
<td>6.75</td>
</tr>
<tr>
<td>$Aa+Bb+Cc$</td>
<td>9016.17</td>
<td>11092.13</td>
<td>11072.81</td>
</tr>
<tr>
<td>Slope($d\bar{U} / dM_{\infty}$)</td>
<td>-8.48</td>
<td>-9.14</td>
<td>-9.88</td>
</tr>
</tbody>
</table>
Figure 2.21  Comparison of flutter boundary and aerodynamic moment slope for three airfoil configurations.
CHAPTER 3
FORMULATION AND GOVERNING EQUATIONS

In this chapter, the governing equations, far field and body boundary conditions for steady and unsteady flows are derived. The equations of motion for the response of the wing structure to aerodynamic forces, the procedures for determining a flutter point, and techniques for interpolation between solutions on topologically different grids are discussed.

3.1 Aerodynamic Equations

Flutter is a result of the interaction between structural deformations and aerodynamic forces. The airload is obtained by solving the flow governing equation and integrating the pressure difference across the wing surface. The present approach divides a flow field into a steady mean and an unsteady flow about the mean. While the steady mean flow can be obtained by either solving the full potential equation with or without a displacement correction to account for the viscous effects or the Euler/Reynolds averaged Navier-Stokes equations, the unsteady flow is computed by solving the continuity equation in a perturbation potential form.

3.1.1 Equations for Transonic Flow

3.1.1a Full Potential Equation

The three dimensional, unsteady, full potential equation in conservation-law form is

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho \phi_x) + \frac{\partial}{\partial y} (\rho \phi_y) + \frac{\partial}{\partial z} (\rho \phi_z) = 0
\]  (3.1)
where $\phi$ is the total velocity potential and $\rho$ the density is given by the Bernoulli equation,

$$\rho = \left[1 + \frac{\gamma - 1}{2}(M_{\infty}^2 - 2\phi_t - \phi_x^2 - \phi_y^2 - \phi_z^2)\right]^{\frac{1}{\gamma - 1}} = \left[1 + \frac{\gamma - 1}{2}(M_{\infty}^2 - 2\phi_t - |\mathbf{u}|^2)\right]^{\frac{1}{\gamma - 1}} \quad (3.2)$$

Here, the density $\rho$, velocity components $\phi_x$, $\phi_y$, and $\phi_z$ have been respectively nondimensionalized by the freestream density $\rho_{\infty}$ and speed of sound $a_{\infty}$. The coordinates $x$, $y$, and $z$ have been nondimensionalized by the root chord $c$ of the wing and the time $t$ by $u/a_{\infty}$.

Equation (3.1) is an expression of mass conservation for an unsteady, inviscid, isentropic, and irrotational flow.

The use of curvilinear coordinates transforms the body surface into a plane surface on which boundary condition can be conveniently and accurately applied, and allows clustering of grid points to improve solution accuracy. Let $\xi, \eta, \zeta$ be coordinate transformation functions,

$$\xi = \xi(x, y, z), \quad \eta = \eta(x, y, z), \quad \zeta = \zeta(x, y, z), \quad \tau = t \quad (3.3)$$

Eq. (3.1) written in curvilinear coordinates is

$$\frac{\partial (\rho)}{\partial \tau} + \frac{\partial (\rho U)}{\partial \xi} + \frac{\partial (\rho V)}{\partial \eta} + \frac{\partial (\rho W)}{\partial \zeta} = 0 \quad (3.4)$$

and the Bernoulli equation, Eq. (3.2), becomes

$$\rho = \left[1 + \frac{\gamma - 1}{2}(M_{\infty}^2 - 2\phi_t - (U\phi_\xi + V\phi_\eta + W\phi_\zeta))\right]^{\frac{1}{\gamma - 1}} \quad (3.5)$$

where

$$U = A_1\phi_\xi + A_4\phi_\eta + A_5\phi_\zeta, \quad V = A_4\phi_\xi + A_2\phi_\eta + A_6\phi_\zeta, \quad W = A_5\phi_\xi + A_6\phi_\eta + A_3\phi_\zeta \quad (3.6)$$
Here $U$, $V$, and $W$ are the contravariant velocities, $A_1$ to $A_6$ are metric coefficients, and $J$ is the Jacobian of transformation $\partial(\xi, \eta, \zeta)/\partial(x, y, z)$. The metric coefficients in Eqs. (3.7a) and (3.7b) can also be expressed as

$$\xi_x = J(\eta_x \xi - \eta_\zeta \zeta), \quad \xi_y = J(x_\zeta \xi - x_\eta \zeta), \quad \xi_z = J(\eta_\eta \xi - \eta_\zeta \zeta)$$

$$\eta_x = J(\eta_x \eta - \eta_\zeta \zeta), \quad \eta_y = J(x_\zeta \eta - x_\eta \zeta), \quad \eta_z = J(x_\eta \eta - x_\zeta \zeta)$$

$$\zeta_x = J(\eta_x \zeta - \eta_\zeta \zeta), \quad \zeta_y = J(x_\zeta \zeta - x_\eta \zeta), \quad \zeta_z = J(x_\eta \zeta - x_\zeta \zeta)$$

(3.8)

Since one of the objects here is to develop an efficient algorithm for flutter analysis, with the mean flow $\bar{u}_s = \bar{u}_s(x, y, z)$ assumed known and solved using well-established techniques, the velocity field can be split into mean and unsteady parts, i.e.,

$$\bar{u} = \bar{u}_s + \bar{u}_u = \bar{u}_s + \nabla \phi$$

(3.9)

or in curvilinear coordinates

$$U = U_s + U_u = U_s + (A_1 \phi_\xi + A_4 \phi_\eta + A_5 \phi_\zeta)$$

$$V = V_s + V_u = V_s + (A_4 \phi_\xi + A_2 \phi_\eta + A_6 \phi_\zeta)$$

$$W = W_s + W_u = W_s + (A_5 \phi_\xi + A_6 \phi_\eta + A_3 \phi_\zeta)$$

(3.10)

The total velocity can be expressed as

$$|\bar{u}|^2 = |\bar{u}_s|^2 + 2 \bar{u}_s \cdot \nabla \phi + \nabla \phi \cdot \nabla \phi$$

$$= |\bar{u}_s|^2 + 2(U_s \phi_\xi + V_s \phi_\eta + W_s \phi_\zeta) + (U_u \phi_\xi + V_u \phi_\eta + W_u \phi_\zeta)$$

(3.11)
and thus the density formula Eq. (3.2) is expressed as a function of mean velocities and perturbed velocity potential, i.e.,

\[
\rho_i = \left[1 + \frac{\gamma - 1}{2} \left( M_{\infty}^2 - 2 \phi_t - |\vec{a}|^2 \right) \right]^{\frac{1}{\gamma - 1}} = \left[1 + \frac{\gamma - 1}{2} \left( M_{\infty}^2 - 2 \phi_t - |\vec{a}|^2 - 2(U_s \phi_x + V_s \phi_y + W_s \phi_z) - (U_u \phi_x + V_u \phi_y + W_u \phi_z) \right) \right]^{\frac{1}{\gamma - 1}} = \left[1 + \frac{\gamma - 1}{2} \left( M_{\infty}^2 - 2 \phi_t - |\vec{a}|^2 - 2(U_s \phi_x + V_s \phi_y + W_s \phi_z) - (U_u \phi_x + V_u \phi_y + W_u \phi_z) \right) \right]^{\frac{1}{\gamma - 1}}
\]

(3.12)

The subscript \( i \) denotes isentropic relation, and \( s \) denotes steady mean quantities. If the mean quantities are not provided, the total flow is assumed potential. This decomposition of velocity field allows the extension to others than strictly potential flow.

Behind a shock, in the boundary layer or the wake, the steady mean flow is in general not isentropic, irrotational or potential, but it always satisfies the continuity equation. To account for entropy production, the density relation assumes the form

\[
\rho = \rho_i e^{-\Delta s/\gamma} = \rho_i e^{-\Delta \tilde{s}}
\]

(3.13)

where \( \Delta s \) is the total entropy increase from the freestream value. This entropy increase is illustrated in Figure 3.1a,b,c,d, for the flow over a transport-type wing at \( M_{\infty}=0.7 \) and 0.825. For a solution of the Euler equations the entropy changes, Figure 3.1a and Figure 3.1b, are visible only after the shock and much smaller than those inside the boundary layer (cf. Figure 3.1d) for a Navier-Stokes flow, (cf. Figure 3.1c and 3.1d). Given a mean flow, the ratio of the mean flow density to the isentropic density, Eq. (3.12), is stored and used to modify the density formula for unsteady calculations. Strictly speaking, Eq. (3.12) is valid for irrotational flows, and Eq. (3.13) is valid for steady flows.
For unseparated flows and weak shocks, since vorticity is either confined in the thin boundary layer or weak enough not to affect the propagation of acoustic waves, the use of the same entropy correction term for the entire flow can be justified as valid for the largely irrotational part, slightly in error after shocks, and incorrect for the thin area containing the boundary layer and wake. This entropy correction implies that the pressure coefficient can be expressed as

\[ C_p = \frac{2}{\gamma M_\infty^2} \left( \rho_1 \gamma - 1 \right) \]  

(3.14)

for irrotational flow, or for rotational flow as

\[ C_p = \frac{2}{\gamma M_\infty^2} \left( \rho_1 \gamma - 1 \right) = \frac{2}{\gamma M_\infty^2} \left[ \left( \rho e^{\Delta s} \right)^{\gamma} - 1 \right] \]  

(3.15)
Figure 3.1a Entropy contours (upper) and Mach number contours (lower) of a solution of the Euler equations for a supercritical wing at the root plane.
Figure 3.1b  Entropy contours (upper) and Mach number contours (lower) of a solution of the Euler equations for a supercritical wing at the root plane.
Figure 3.1c  Entropy contours (upper) and Mach number contours (lower) of a solution of the Navier-Stokes equations for a supercritical wing at the root plane.
Figure 3.1d  Entropy contours (upper) and Mach number contours (lower) of a solution of the Navier-Stokes equations for a supercritical wing at the root plane.
Boundary Conditions

(1) Small Perturbation Boundary Condition

For a viscous flow, the impermeable and no slip conditions require the velocity to be zero at the body surface. For an inviscid flow, only the velocity $W$ normal to the surface can be specified. For a steady flow, the flow tangency condition is satisfied by setting the contravariant velocity zero on the grid plane representing the body surface, i.e.

$$W = A_5 \phi_5 + A_6 \phi_6 + A_2 \phi_2 = 0$$  \hspace{1cm} (3.16)

Although the same condition applies for unsteady flow as well, its implementation requires a redefinition of the grid in response to the deformations. For a flutter analysis, deviations of the wing from the mean surface are assumed small, and a perturbation boundary condition should be adequate to describe the deformation, which can be viewed as suction and blowing at the mean body surface proportional to $w_u$ in the z-direction (see Figure 3.2)

![Figure 3.2](image)

Figure 3.2 Definition of the wing deformation function.

Let the mean surface function $f$ and unsteady surface function $F$ be defined as:

$$f = z - f_o(x,y)$$
$$F = z - [f_o(x,y) + f_u(x,y)]$$  \hspace{1cm} (3.17)
Here $f_o(x,y)$ represents the geometry of wing and $f_e(x,y,t)$ the deviation of the wing from the mean position. For an inviscid flow over a solid surface, the tangency condition can be written as,

$$\left( \frac{DF}{Dt} \right)_{F=0} = 0 \quad \text{or} \quad \left( \frac{\partial F}{\partial t} \right)_{F=0} + (\vec{V} \cdot \nabla F)_{F=0} = 0 \tag{3.18} $$

where the subscripts in Eq. (3.18) denote evaluation on the true surface ($F=0$). The unsteady surface $F$ can be decomposed into two components by a Taylor series expansion about the mean surface ($f=0$).

$$\left( \frac{DF}{Dt} \right)_{f=0} + \left[ \nabla \left( \frac{DF}{Dt} \right) \cdot \vec{d} \right]_{f=0} = 0 \tag{3.19}$$

Using Eqs. (3.17) and $\vec{V}=(u,v,w)$, Eq. (3.19) can be rearranged as

$$\frac{\partial f_e}{\partial t} + w - u (f_o + f_e)_x - v (f_o + f_e)_y + \nabla \left[ - \frac{\partial f_e}{\partial t} + w - u (f_o + f_e)_x - v (f_o + f_e)_y \right] \cdot \vec{d} = 0 \tag{3.20}$$

The function $f_e(x,y,t)$ can be computed for simple pitching or plunging body motions as demonstrated by Schoen (1988). For the deformation of a flexible wing, this function is determined by the interaction of aerodynamic and structural responses. Assuming the deformations due to aerodynamics are small, i.e., $f_e \sim O(e)$ and $1d_{l} \sim O(e)$, and

$$u = u_s + \epsilon u_u \tag{3.21}$$
$$v = v_s + \epsilon v_u$$
$$w = w_s + \epsilon w_u$$

After substituting Eqs. (3.21) into Eq. (3.20), a collection of terms yields the zero order tangency condition

$$w_s = u_s f_o + v_s f_o, \tag{3.22}$$
and first order, $O(\varepsilon)$, condition

$$ w_u = \frac{\partial f}{\partial t} + u_s f_{e_x} + v_s f_{e_y} + u_u f_{o_x} + v_u f_{o_y} \quad \text{(A)} $$

$$ -\nabla\left( w_s - u_s f_{o_x} - v_s f_{o_y} \right) \cdot \hat{a} \quad \text{(B)} $$

$$ \text{(C)} $$

Equation (3.22), $W_s=0$, is satisfied by the mean flow at the mean body surface. Equation (3.23) contains three parts, $A, B, C$. Part $A$ is due to the mechanical motion of the body, Part $B$ is due to the local change of slope, and Part $C$ is the correction for evaluating on the mean surface rather than the instantaneous surface. Thus, the flow tangency condition is satisfied by specifying the contravariant velocity on the mean surface, i.e.,

$$ W_u = \bar{V}_u \cdot \hat{n} = \zeta_x u_u + \zeta_y v_u + \zeta_z w_u \quad \text{(3.24)} $$

where $u_u, v_u$ are solved by satisfying Eq. (3.1) and $w_u$ computed according to Eq. (3.23). For an inviscid flow, the tangential component of the velocity is in general nonzero and the terms in Part $B$ correspond to local flow divergence due to the change of slope. For an attached viscous flow the velocity decreases rapidly from the freestream value at the edge of the boundary layer to zero at the surface while the pressure is almost a constant. The flow divergence which corresponds to a body slope change is only valid for the inviscid outer flow. Hence, for the implementation of an inviscid perturbation boundary condition on a viscous mean flow, the velocity components just outside the boundary layer are used instead of those at the body surface. For this purpose, the edge of the boundary layer is defined as the location where the entropy is sufficiently small to be considered isentropic. The velocity components of the mean flow at the edge of the boundary layer are also used to determine the unsteady pressure on the surface due to ve-
locity disturbance. These components are scaled so that the corresponding pressure using the isentropic relation is the same as the pressure of the viscous mean flow, i.e.,

$$\left| \vec{u}_e \right|^2 = M_{\infty}^2 - \frac{2}{\gamma-1} \left[ \left( 1 + \frac{\gamma M_{\infty}^2}{2} C_{P,NS} \right)^{(\gamma-1)/\gamma} - 1 \right]$$  \hspace{1cm} (3.25)

where subscript $e$ and sub-subscript NS denote equivalent quantity and value from a Navier-Stokes solution, respectively. The pressure coefficient on the surface for an unsteady flow is then computed using the equivalent mean velocity $U_e$ instead of the zero wall velocity, i.e.,

$$C_p = \frac{2}{\gamma M_{\infty}^2} \left[ \frac{1}{2} \left( M_{\infty}^2 - 2\phi_t - \left| \vec{u}_e \right|^2 - 2 \left( U_e \phi_\xi + V_e \phi_\eta + W_e \phi_\zeta \right) \right) \right]^{\gamma/(\gamma-1)} - 1$$  \hspace{1cm} (3.26)

Equation (3.26) implies that the unsteady flow remains attached and the perturbed flow is inviscid.

(2) Far-Field Boundary Condition

According to Klunker (1971), the potential of a steady flow approaches the freestream, $\phi = M_{\infty} x$ and $\rho = 1$, asymptotically as

$$\phi(x,y,z) = \sum_{\bar{y}=0}^{\bar{y}=s/2} \frac{\bar{y}}{4\pi} \frac{1+x/R(\bar{y})}{(y-\bar{y})^2 + z^2} + \frac{1+x/R(-\bar{y})}{(y+\bar{y})^2 + z^2} \bar{y}$$  \hspace{1cm} (3.27)

where $R(\bar{y}) = [(x-x) + \beta^2(y-\bar{y})^2 + \beta^2(z-\bar{z})^2]^{1/2}$ and $\beta = (1-M_{\infty}^2)^{1/2}$

while $x, y,$ and $z$ are coordinates of a point measured with respect to the wing quarter-chord line. The use of a far-field condition instead of the freestream condition avoids having to compute on a large domain. Similarly, a suitable far-field boundary condition
reduces substantially the otherwise large computational domain and prevent reflection of disturbance waves from numerical boundaries. Fung (1981) showed that a three dimensional outgoing receding wave travels according to phase relation

\[ \tilde{r} - \tilde{r}_0 = \left( \tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2 \right)^{1/2} - \tilde{x} \]

without distortion of the wave form. An analytical expression for the far-field can be obtained by modifying the arguments of Eq. (3.27) and multiplying it with an Heaviside function which turns on the disturbance as it reaches the far-field boundary. Schoen (1988) applied this far field but concluded that its effectiveness is limited and may cause wave reflection if the frequency is not sufficiently low. Here a local representation of Eq. (3.28) is derived and use instead. Let

\[ \bar{x} = x, \quad \bar{y} = \beta y, \quad \bar{z} = \beta z, \quad \bar{r} = k_c t \]

and the velocity potential be decomposed into \( \phi = \phi_s + \phi' \), then,

\[ \phi' \approx \phi' \left\{ k_c t + x - \left[ x^2 + \beta^2 (y^2 + z^2) \right]^{1/2} \right\} \]

where \( \bar{r} = k_c t + x \) \quad \( r = \left[ (x - \bar{x})^2 + \beta^2 (y - \bar{y})^2 + \beta^2 (z - \bar{z})^2 \right]^{1/2} \)

Eq. (3.30) then satisfy the equation

\[ \phi'_x + \phi'_r = 0 \]

(3.31a)

where

\[ \phi'_x = \frac{1}{k_c} \phi'_x + \phi'_x \]

\[ \phi'_r = \phi'_x \frac{\partial x}{\partial r} + \phi'_y \frac{\partial y}{\partial r} + \phi'_z \frac{\partial z}{\partial r} = \frac{x}{r} \phi'_x + \frac{y}{r} \phi'_y + \frac{z}{r} \phi'_z \]

(3.31b)
Combining Eqs. (3.31b) and Eq. (3.31a), one finds that

\[
\frac{1}{k_c} \phi_r + \left( 1 + \frac{z}{r} \right) \phi_x + \frac{y}{r} \phi_y + \frac{z}{r} \phi_z = 0
\] 

(3.32)

(3) Other Boundary Conditions

At the wing symmetry plane (i.e. root section) the contravariant \( V \) is set to zero. Behind the trailing edge, for lifting cases, allowance must be made for a jump of velocity potential

\[
\Gamma = [\phi] = \phi_u - \phi_l
\] 

(3.33)

to convect along a cut representing the wake according to

\[
\Gamma_x + \langle V \rangle \Gamma_\eta + \langle W \rangle \Gamma_\xi = 0
\] 

(3.34)

Here \( \zeta \) and \( \eta \) are directions in the wake plane, and \( \langle V \rangle \) and \( \langle W \rangle \) are average contravariant velocities across the wake, i.e.,

\[
\langle V \rangle = \frac{1}{2}(V_u + V_l)
\]

\[
\langle W \rangle = \frac{1}{2}(W_u + W_l)
\] 

(3.35)

Equation (3.34) corresponds to shedding of vorticies from the wing trailing edge. The jump in velocity potential is determined by an enforcement of the Kutta condition, i.e.,

\[
\phi_x_{\text{lower}} = \phi_x_{\text{upper}}
\]

(3.36)

Here \( \phi \) represents the total velocity potential for steady flow and perturbed velocity potential for unsteady flow. The boundary conditions for an \( O-H \) type grid topology are shown schematically in Figure 3.3.
Boundary conditions on an O-H grid

Figure 3.3

A: WAKE-LIKE CUT
B: WING
C: WING EXTENSION
D: SPANWISE FAR-END
E: SYMMETRICAL PLANE
F: FAR-FIELD
G: TRAILING EDGE
3.1.1b Boundary Layer Corrections

One way to incorporate viscous effects into a solution of the potential equation is to modify the airfoil geometry by the boundary layer displacement thickness. Isogai (1982) found the 2-D Nash & McDonald (1967) turbulence model significantly improved the accuracy of a flutter prediction. For comparison with the results of Isogai, the Nash & McDonald (1967) scheme is also used in the present work. A brief account of this method is given in the following.

This method is based on the momentum integral equation, where the momentum thickness $\theta$ is related to the outer flow velocity $u$, shape factor $H$ and wall shear $\tau_w$ by

$$\frac{d\theta}{dx} = -\left( H + 2 - M^2 \right) \frac{\theta}{u} \frac{du}{dx} + \frac{\tau_w}{\rho u^2} \tag{3.37}$$

and the skin-friction law for a compressible turbulent boundary layer,

$$\frac{\tau_w}{\rho u^2} = \left[ F_c^{1/2} \left\{ 2.4711 \ln \left( \frac{F_R u \theta}{\sqrt{\gamma}} \right) + 4.75 \right\} + 1.5G + \frac{1724}{G^2 + 200} - 16.87 \right]^{-2} \tag{3.38a}$$

The effect of compressibility is accounted for by the auxiliary expressions

$$F_c^{1/2} = 1 + 0.066M^2 - 0.008M^3$$
$$F_R = 1 - 0.134M^2 + 0.027M^3$$

and the effect of pressure gradient accounted for by

$$G = 6.1(\Pi + 1.81)^{1/2} - 1.7 \tag{3.38b}$$

$$\Pi = \frac{\delta^* dp}{\tau_w dx} = \frac{H}{\left( \frac{\tau_w}{\rho u^2} \right)} \frac{\theta}{u} \frac{du}{dx} \tag{3.38c}$$
where \[ H = \frac{\delta^*}{\theta} = \left( \frac{1}{H} + 1 \right) \left( 1 + 0.178 M^2 \right) - 1 \] 

(3.38d)

and \[ H = \left( 1 - G \left( \frac{\tau_w}{\rho u^2} \right)^{1/2} \right)^{-1} \]

(3.38e)

Her \( \delta^* \) is the displacement thickness, \( \Pi \) pressure-gradient parameter, \( M \), local Mach number, and \( p \) the local pressure.

The five unknowns \( \frac{\tau_w}{\rho u^2} \), \( H \), \( H \), \( G \), \( \Pi \) are solved by iteration given a specified transition location, a Reynolds number (based on the freestream velocity and full chord) and the initial guesses of

\[ \frac{u\theta}{V} = 320 \]
\[ G = 6.5 \]

(3.39)

The iteration process stops when the shape factor \( G \) converges to \( |G^n - G^{n-1}| \leq \varepsilon \). Then, the displacement thickness is obtained from the definition, \( \delta^* = H\theta \). The momentum thickness \( \theta \) is obtained by integrating Eq. (3.37) explicitly to a downstream location, and the corresponding displacement thickness for this downstream location is obtained by repeating the above iteration procedure until the trailing edge is reached or separation is detected. Flow separation is assumed when \( -(\theta/u)du/dx = 0.004 \). When shock appears, this method converges slowly or fails to converge to a solution.
3.1.1c Euler/Navier-Stokes Equations

For strong shocks, only the Navier-Stokes equations are capable of describing the shock and its interaction with the boundary layer. For the prediction of a steady mean flow with strong shock, the Euler/Navier-Stokes code TLNS3D developed by Vatsa (1987) is used here. This code uses an explicit multistage Runge-Kutta time-stepping scheme to solve the three-dimensional, compressible, thin-layer Navier-Stokes equation in finite volume, i.e.,

\[
\frac{\partial \bar{U}}{\partial t} + \frac{\partial \bar{F}}{\partial \xi} + \frac{\partial \bar{G}}{\partial \eta} + \frac{\partial \bar{H}}{\partial \zeta} = \frac{\partial \bar{S}}{\partial \eta} \tag{3.40}
\]

where

\[
\bar{U} = J^{-1} U = J^{-1} \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho E \end{bmatrix}, \quad \bar{F} = J^{-1} \begin{bmatrix} \rho \bar{u} \\ \rho \bar{u}u + \xi \bar{p} \\ \rho \bar{u}v + \eta \bar{p} \\ \rho \bar{u}w + \zeta \bar{p} \\ \rho \bar{u} \bar{H} \end{bmatrix} \tag{3.41a,b}
\]

\[
\bar{G} = J^{-1} \begin{bmatrix} \rho \bar{\phi} \\ \rho \bar{\phi}u + \eta \bar{p} \\ \rho \bar{\phi}v + \eta \bar{p} \\ \rho \bar{\phi}w + \eta \bar{p} \\ \rho \bar{\phi} \bar{H} \end{bmatrix}, \quad \bar{H} = J^{-1} \begin{bmatrix} \rho \bar{\psi} \\ \rho \bar{\psi}u + \xi \bar{p} \\ \rho \bar{\psi}v + \eta \bar{p} \\ \rho \bar{\psi}w + \zeta \bar{p} \\ \rho \bar{\psi} \bar{H} \end{bmatrix} \tag{3.41c,d}
\]

\[
\bar{S} = J^{-1} \sqrt{\frac{\gamma M_a}{Re_\infty}} \begin{bmatrix} \mu E \left( \eta_x^2 + \eta_y^2 + \eta_z^2 \right) u_\eta + \frac{\mu E}{3} \left( \eta_x u_\eta + \eta_y v_\eta + \eta_z w_\eta \right) \eta_x \\ \mu E \left( \eta_x^2 + \eta_y^2 + \eta_z^2 \right) u_\eta + \frac{\mu E}{3} \left( \eta_x u_\eta + \eta_y v_\eta + \eta_z w_\eta \right) \eta_x \\ \mu E \left( \eta_x^2 + \eta_y^2 + \eta_z^2 \right) u_\eta + \frac{\mu E}{3} \left( \eta_x u_\eta + \eta_y v_\eta + \eta_z w_\eta \right) \eta_x \\ \frac{\mu E}{2} \left( \eta_x^2 + \eta_y^2 + \eta_z^2 \right) \left( u^2 + v^2 + w^2 \right) \eta_x \\ + \frac{\mu E}{3} \left( \eta_x u + \eta_y v + \eta_z w \right) \left( \eta_x u_\eta + \eta_y v_\eta + \eta_z w_\eta \right) \\ + \frac{\gamma}{\gamma - 1} \frac{\mu E}{\sigma} \left( \eta_x^2 + \eta_y^2 + \eta_z^2 \right) \eta_\eta \end{bmatrix} \tag{3.41e}
\]
These equations have been nondimensionalized by the freestream density, pressure, and viscosity. The spatial dimensions are scaled by the root chord of the wing, and velocities are scaled by the reference velocity \( u_{\infty} = a_{\infty}/\sqrt{\gamma} \). For an ideal gas, the enthalpy is given by

\[
\rho H = \left( \frac{\gamma}{\gamma - 1} \right) p + \rho \left( \frac{u^2 + v^2 + w^2}{2} \right)
\]  
(3.42)

The molecular viscosity \( \mu \) is replaced by an eddy viscosity for turbulent flows, i.e.,

\[
\mu_e = \mu + \mu_t = \mu (1 + \mu_t / \mu) = \mu \bar{\varepsilon}
\]  
(3.43)

Similarity, the molecular conductivity is replaced by an effective turbulent conductivity \( k_e \),

\[
k_e = k + k_t = \frac{C_p \mu}{\sigma} + \frac{C_p \mu_t}{\sigma_t}
\]  
(3.44)

\[
= \frac{C_p \mu}{\sigma} \left( 1 + \frac{\sigma_t}{\sigma} \frac{\mu_t}{\mu} \right) = \frac{C_p \mu}{\sigma} \bar{\varepsilon} = k \bar{\varepsilon}
\]

Here \( \sigma \) is the laminar Prandtl number, and \( \sigma_t \) the turbulent Prandtl number. Notice that this set of equations are not nondimensionalized in the same way as the full potential equation, Equations 3.1 and 3.2.

For two-dimensional mean flow calculations, the Euler/Navier-Stokes algorithm ARC2D developed by Pulliam (1978) is also used. This algorithm is discretized according to a finite difference form of the governing equations. It employs an approximate factorization procedure to speed up the convergence to a solution and the coordinates used are nondimensionalized by the full chord.

3.1.2 Truncation Error Injection

Following Fung and Fu (1987), we cast the governing equation into the form

\[
L_t \phi_h + L_h \phi_h = L_h \phi_f
\]  
(3.45)
Here $\phi_f$ satisfies $L_f \phi_f = 0$, and the operator on the left-hand side of Eq. (3.45) has been split into steady and unsteady operators, $L_h$ and $L_t$, for clarity of discussion. In the absence of unsteady disturbances, Eq. (3.45) forces $\phi_h$ to accept $\phi_f$ as a steady solution, regardless of whether $\phi_f$ satisfies any equation at all. Fung et al. (1988) pointed out that if $L_f$ is the exact operator or a better approximation to the exact operator than $L_h$, the right-hand side of Eq. (3.45) represents the truncation error or an approximate of the truncation error of discretization. Or, if $L_f$ is the operator of a different equation model, the right hand side of Eq. (3.45) represents both the errors due to differences in equation models and discretization. By the inclusion of the right hand side, the injection of the truncation error $T.E.D$, an accurate solution obtained on a fine grid using the most accurate equation model can be recovered on a coarse grid using an approximated equation model.

This decoupling of the unsteady flow from the mean flow allows different parts of the solution to be obtained on different grids using different solvers satisfying different equation models. It preserves the accuracy of the steady mean flow on a grid which is not suitable for the resolution of the mean flow but adequate and efficient for the resolution of unsteady, low frequency perturbations.

### 3.2 Structural Responses

As discussed in Chapter 2, the $U-g$ method is suitable for the analysis of simple, two-dimensional flutter modes, such as pitching and plunging. A more general approach is to couple, through modal decomposition, the structural and aerodynamic responses by solving the structural and aerodynamic equations simultaneously. This approach is explained in the following: The vertical deflections $f_e$ of a wing can be expressed in terms of the structural vibration mode shapes as
Here $\phi_i(x,y)$ are the natural mode shape functions, $q_i(t)$ are the generalized displacements. The equations of motion are then

$$M_i\ddot{q}_i + \omega_i^2 M_i q_i = Q_i \quad (i = 1, \ldots, N)$$

(3.47)

Here $\omega_i$'s are the natural frequencies of the vibration modes. $M_i$'s, the generalized masses, and $Q_i$'s, the generalized aerodynamic forces, are obtained by and integrating the mass distribution of the wing, $m(x,y)$, and the pressure difference $\Delta P(x,y,t)$ across the wing surface, i.e.,

$$M_i = \int_S \phi_i^2(x,y) m(x,y) dxdy$$

(3.48)

$$Q_i = \int_S \Delta P(x,y,t) \phi_i(x,y) dxdy$$

The generalized aerodynamic forces are coupled with the generalized displacements through solving the aerodynamic equations with $q_i(t)$ as input to the boundary condition.

It should be mentioned that $\Delta P$ here is the total pressure difference with the mean airload subtracted, assuming the wing is already at static equilibrium.

### 3.3 Flutter Analysis

The procedure used here for calculating the aeroelastic response of a wing is: 1) compute and store the steady flow for a wing at a static undeformed position using an appropriate solver (full potential, Navier-Stokes, etc...); 2) start the unsteady calculation with some initial deformation, in terms of generalized displacements, and a guess of dynamic pressure; 3) advance the flow in time to obtain the pressure distribution on the wing; 4) decompose the pressure according to Eq. (3.48) to obtain the generalized forces; 5) inte-
grate Eq. (3.47) to find the wing deformations; 6) repeat step 3) to step 5) for three to four cycles of oscillation of the first mode; 7) check the growth or decay of the generalized coordinates, usually the first three modes determine flutter; 8) change to a different dynamic pressure and repeat step 2) to step 7) until the growth becomes decay or vice versa; 9) interpolate to find the dynamic pressure for which one of the modes either grows first or decays last. Typical structural responses are shown in Figure 3.5. A flutter point or flutter dynamic pressure is determined when the logarithmic damping factor, defined as the log of the ratio of two consecutive maxima or minima $h_1$ & $h_2$ of the least damped structural mode, $ln(h_2/h_1)$, is zero. A schematic of this procedure is shown in Figure 3.4. Since structural damping is assumed zero here, the flutter boundary so determined is conservative.
Figure 3.4 Flow chart for the flutter analysis procedure.
Figure 3.5 Typical structural responses.
CHAPTER 4

DEVELOPMENT AND IMPLEMENTATION
OF THE NUMERICAL SCHEME

In this chapter, the numerical scheme for unsteady flow calculations, implementation of the T.E.I technique and the boundary conditions, and the interpolation of solutions between coarse and fine grids are discussed.

4.1 Numerical Algorithm

4.1.1 Discretization of Governing Equation

The aerodynamic formulation here is based on the continuity equation. The numerical algorithm used here is essentially the same as that of Bridgeman, et al. (1982), but with the isentropic density formula corrected using the entropy changes of the mean flow field, i.e., Eq. (3.13). The first order backward-Euler discretization of Eq. (3.4) is

\[ \rho^{n+1} - \rho^n + h \left[ \partial_z (\hat{\rho} U)^{n+1} + \partial_x (\hat{\rho} V)^{n+1} + \partial_y (\hat{\rho} W)^{n+1} \right] = 0 \]  \hspace{1cm} (4.1)

where \( h = \Delta t \), \( \hat{\rho} = \rho / \rho \) and \( U, V, W \), are the contravariant velocity components.

The first order temporal expansion of the density is

\[ \dot{\rho}^{n+1} = \dot{\rho}^n + \frac{\partial}{\partial t} (\dot{\rho}^n) \cdot h \]  \hspace{1cm} (4.2)

From the modified density law, the rate of change of density is related to the change of potential \( \phi \) by

\[ \frac{\partial}{\partial t} (\dot{\rho}^n) = e^{-\Delta t} \frac{-\partial (\dot{\rho}^n)^{2-\gamma}}{\rho} \left[ -\partial x - U \partial \xi - V \partial \eta - W \partial \zeta \right] (\phi^{n+1} - \phi^n) \frac{1}{h} \]  \hspace{1cm} (4.3a)
Applying Eq. (4.3a) for two consecutive temporal density changes yields

$$\dot{\rho}^{n+1} - \dot{\rho}^n = \dot{\rho}^n - \dot{\rho}^{n-1}$$

$$+ \left\{ e^{-\Delta \tau \rho} \frac{2 - \gamma}{J} \left[ \partial_\xi + U^n \partial_\eta + V^n \partial_\xi + W^n \partial_\zeta \right] \left( \phi^{n+1} - \phi^n \right) \right\}$$

$$+ \left\{ e^{-\Delta \tau \rho} \frac{2 - \gamma}{J} \left[ \partial_\xi + U^{n-1} \partial_\eta + V^{n-1} \partial_\xi + W^{n-1} \partial_\zeta \right] \left( \phi^n - \phi^{n-1} \right) \right\}$$

where $\rho_i$ is the density according to the isentropic formula. To facilitate the application of approximate factorization, the cross-derivative terms are lagged in time and the implicit spatial derivatives become

$$\partial_\xi (\dot{\rho}U)^{n+1} = \partial_\xi (\dot{\rho}A_1)^n \partial_\xi (\phi^{n+1} - \phi^n) + \partial_\xi (\dot{\rho}U)^n + O(h)$$

$$\partial_\eta (\dot{\rho}V)^{n+1} = \partial_\eta (\dot{\rho}A_2)^n \partial_\eta (\phi^{n+1} - \phi^n) + \partial_\eta (\dot{\rho}V)^n + O(h)$$

$$\partial_\zeta (\dot{\rho}W)^{n+1} = \partial_\zeta (\dot{\rho}A_3)^n \partial_\zeta (\phi^{n+1} - \phi^n) + \partial_\zeta (\dot{\rho}W)^n + O(h)$$

Equations (4.2) to (4.4) can be combined to form the change of potential $\phi^{n+1} - \phi^n$ at the new time level as

$$\left[ 1 + h (U^n \partial_\xi + V^n \partial_\eta + W^n \partial_\zeta) \right]$$

$$\left\{ \frac{-\beta_1}{\beta_2} e^{-\Delta \tau} \left[ \partial_\xi (\dot{\rho}A_1)^n \partial_\xi + \partial_\eta (\dot{\rho}A_2)^n \partial_\eta + \partial_\zeta (\dot{\rho}A_3)^n \partial_\zeta \right] \right\} \left( \phi^{n+1} - \phi^n \right)$$

$$= \left( \phi^n - \phi^{n-1} \right) + \frac{\beta_1}{\beta_2} \left( \phi^n - 2\phi^{n-1} + \phi^{n-2} \right) + \frac{h}{\beta_3 e^{-\Delta \tau}} \left( \dot{\rho}^{n} - \dot{\rho}^{n-1} \right)$$

$$+ h \frac{\beta_1}{\beta_2} \left( U^{n-1} \partial_\xi + V^{n-1} \partial_\eta + W^{n-1} \partial_\zeta \right) \left( \phi^n - \phi^{n-1} \right)$$

$$+ \frac{h^2}{\beta_3 e^{-\Delta \tau}} \left[ \partial_\xi (\dot{\rho}U)^n + \partial_\eta (\dot{\rho}V)^n + \partial_\zeta (\dot{\rho}W)^n \right]$$

The spatial differences in Eq. (4.5) can be either first or second order accurate in the form
To avoid costly matrix inversion at each time level, Eq. (4.6) is factored into $L_\xi$, $L_\eta$ and $L_\zeta$ operators

\[
\begin{aligned}
&\left[1 + h(U^n \delta_\xi + V^n \delta_\eta + W^n \delta_\zeta)\right] \\
&\left[1 + h^2 \frac{\hat{\beta}^{-1}}{\beta^n} \delta_\xi (\hat{\beta} A_1) \right] \\
&\left[1 + h^2 \frac{\hat{\beta}^{-1}}{\beta^n} \delta_\eta (\hat{\beta} A_2) \right] \\
&\left[1 + h^2 \frac{\hat{\beta}^{-1}}{\beta^n} \delta_\zeta (\hat{\beta} A_3) \right]
\end{aligned}
\]

\[
\begin{aligned}
&\left(\phi^n - \phi^{n-1}\right) + \frac{\hat{\beta}^{-1}}{\beta^n} \left(\phi^n - 2\phi^{n-1} + \phi^{n-2}\right) + \frac{h^2}{\beta^n e^{-\Delta t}} (\hat{\beta} - \hat{\beta}^{n-1}) \\
&+ h \frac{\hat{\beta}^{-1}}{\beta^n} (U^n \delta_\xi + V^n \delta_\eta + W^n \delta_\zeta) (\phi^n - \phi^{n-1}) \\
&+ \frac{h^2}{\beta^n e^{-\Delta t}} \left[\hat{\delta}_\xi (\hat{\beta} U^n \gamma) + \hat{\delta}_\eta (\hat{\beta} V^n \gamma) + \hat{\delta}_\zeta (\hat{\beta} W^n \gamma)\right]
\end{aligned}
\]

\[(4.6)\]

where $\hat{\beta} = \frac{\rho^{2-\gamma}}{J}$.

To avoid costly matrix inversion at each time level, Eq. (4.6) is factored into $L_\xi$, $L_\eta$ and $L_\zeta$ operators

\[
\begin{aligned}
&\left[1 + hU^n \delta_\xi - \frac{h^2}{\beta^n e^{-\Delta t}} \delta_\xi (\hat{\beta} A_1) \right] \\
&\left[1 + hV^n \delta_\eta - \frac{h^2}{\beta^n e^{-\Delta t}} \delta_\eta (\hat{\beta} A_2) \right] \\
&\left[1 + hW^n \delta_\zeta - \frac{h^2}{\beta^n e^{-\Delta t}} \delta_\zeta (\hat{\beta} A_3) \right]
\end{aligned}
\]

\[
\begin{aligned}
&\left(\phi^n - \phi^{n-1}\right) + \frac{\hat{\beta}^{-1}}{\beta^n} \left(\phi^n - 2\phi^{n-1} + \phi^{n-2}\right) + \frac{h}{\beta^n e^{-\Delta t}} (\hat{\beta} - \hat{\beta}^{n-1}) \\
&+ h \frac{\hat{\beta}^{-1}}{\beta^n} (U^n \delta_\xi + V^n \delta_\eta + W^n \delta_\zeta) (\phi^n - \phi^{n-1}) \\
&+ \frac{h^2}{\beta^n e^{-\Delta t}} \left[\delta_\xi (\hat{\beta} U^n \gamma) + \delta_\eta (\hat{\beta} V^n \gamma) + \delta_\zeta (\hat{\beta} W^n \gamma)\right]
\end{aligned}
\]

\[(4.7)\]

where the spatial differences of Eq. (4.7) are
\[
\delta_{\xi}(\hat{\rho}A_{1})^{\alpha_{\xi}}\phi_{j} = \left[ A_{1j+1/2}J_{j+1/2}^{-1}\left( \frac{\rho_{j+1} + \rho_{j}}{2} \right)(\phi_{j+1} - \phi_{j}) \right. \\
\left. - A_{1j-1/2}J_{j-1/2}^{-1}\left( \frac{\rho_{j} + \rho_{j-1}}{2} \right)(\phi_{j} - \phi_{j-1}) \right]
\]

\[
\delta_{\eta}(\hat{\rho}A_{2})^{\alpha_{\eta}}\phi_{k} = \left[ A_{2k+1/2}J_{k+1/2}^{-1}\left( \frac{\rho_{k+1} + \rho_{k}}{2} \right)(\phi_{k+1} - \phi_{k}) \right. \\
\left. - A_{2k-1/2}J_{k-1/2}^{-1}\left( \frac{\rho_{k} + \rho_{k-1}}{2} \right)(\phi_{k} - \phi_{k-1}) \right]
\] (4.8)

\[
\delta_{\xi}(\hat{\rho}A_{3})^{\alpha_{\xi}}\phi_{l} = \left[ A_{3l+1/2}J_{l+1/2}^{-1}\left( \frac{\rho_{l+1} + \rho_{l}}{2} \right)(\phi_{l+1} - \phi_{l}) \right. \\
\left. - A_{3l-1/2}J_{l-1/2}^{-1}\left( \frac{\rho_{l} + \rho_{l-1}}{2} \right)(\phi_{l} - \phi_{l-1}) \right]
\]

\[
\delta_{\xi}(\hat{\rho}J) = J_{j+1/2}\hat{\rho}J_{j+1/2} - J_{j-1/2}\hat{\rho}J_{j-1/2}
\]

and

\[
\delta_{\eta}(\hat{\rho}J) = J_{k+1/2}\hat{\rho}J_{k+1/2} - J_{k-1/2}\hat{\rho}J_{k-1/2}
\]

\[
\delta_{\xi}(\hat{\rho}J) = J_{l+1/2}\hat{\rho}J_{l+1/2} - J_{l-1/2}\hat{\rho}J_{l-1/2}
\] (4.9)

The contravariant velocities at the mesh half-points are computed as

\[
\bar{U}_{j+1/2} = \frac{U_{s_{j+1}} + U_{s_{j}}}{2} + A_{1j+1/2}(\phi_{j+1} - \phi_{j}) \\
+ \frac{1}{2}(A_{4j+1}\delta_{\eta}\phi_{j+1} + A_{4j}\delta_{\eta}\phi_{j}) + \frac{1}{2}(A_{5j+1}\delta_{\xi}\phi_{j+1} + A_{5j}\delta_{\xi}\phi_{j})
\]

\[
\bar{V}_{k+1/2} = \frac{V_{s_{k+1}} + V_{s_{k}}}{2} + A_{2k+1/2}(\phi_{k+1} - \phi_{k}) \\
+ \frac{1}{2}(A_{4k+1}\delta_{\xi}\phi_{k+1} + A_{4k}\delta_{\xi}\phi_{k}) + \frac{1}{2}(A_{5k+1}\delta_{\xi}\phi_{k+1} + A_{5k}\delta_{\xi}\phi_{k})
\] (4.10)

\[
\bar{W}_{l+1/2} = \frac{W_{s_{l+1}} + W_{s_{l}}}{2} + A_{2l+1/2}(\phi_{l+1} - \phi_{l}) \\
+ \frac{1}{2}(A_{5l+1}\delta_{\eta}\phi_{l+1} + A_{5l}\delta_{\eta}\phi_{l}) + \frac{1}{2}(A_{6l+1}\delta_{\eta}\phi_{l+1} + A_{6l}\delta_{\eta}\phi_{l})
with similar treatment for contravariant velocities at mesh half-points \( j-1/2, k-1/2 \) and \( l-1/2 \).

The subscript \( s \) denotes velocity components evaluated from the steady mean flow. When the local flow becomes supersonic, an upwind biased density is used to maintain the stability of the scheme, i.e.,

\[
\tilde{\rho}_{j+1/2} = (1 - v_{j+1/2}) \left( \frac{\rho_{j+1} - \rho_j}{2} \right) + v_{j+1/2} \left[ \frac{(1+\theta)}{2} \rho_{j+r} + \frac{(1-\theta)}{2} \rho_{j-1+3r} \right]
\]  

(4.11)

where \( r = 1 \) or \(-1\) for \( U < \) or \( > \) 0. Equation (4.11) is first order accurate, but for \( \theta = 2 \), it is second order accurate. The switching parameter \( \nu \) is defined as

\[
\nu_{j+1/2} = \begin{cases} 
\max \left[ 1 - \left( \frac{\rho}{\rho^*} \right)^2, 0 \right] & U_{j+1/2} > 0 \\
\max \left[ 1 - \left( \frac{\rho}{\rho^*} \right)^2, 0 \right] & U_{j+1/2} < 0 
\end{cases}
\]

where \( \rho^* \) is the critical density.

Equation (4.6) can be expressed symbolically as

\[
L_\xi L_\eta L_\zeta (\phi^{n+1} - \phi^n) = R
\]

(4.12)

and solved according to

\[
L_\xi \Delta \phi^* = R \\
L_\eta \Delta \phi^{**} = \Delta \phi^* \\
L_\zeta \Delta \phi^n = \Delta \phi^{**} \\
\phi^{n+1} = \phi^n + \Delta \phi^n
\]

(4.13)

which requires only a series of scalar, tridiagonal inversions, three levels of storage for \( \phi \), two levels of storage for \( \rho \), one level of storage for the density ratio between \( \rho \) and \( \rho_i \), and the steady mean velocity components \( U_s, V_s, \) and \( W_s \).
After the velocity potential has been updated, the density field is computed using the isentropic formula and corrected using the entropy factor.

**Mean Flow Computation**

This scheme can also be used for the prediction of a steady mean flow, for which \( C^2 \) is the total velocity potential. For fast convergence to a steady state, the unsteady terms on the right-hand-side and the space-time derivatives on the left-hand-side of Eq. (4.7) are turned off by a program switch to yield the scheme,

\[
\left[ 1 - h\tilde{\delta}_e(\hat{\beta}A_1)^n\tilde{\delta}_e \right] \left[ 1 - h\tilde{\delta}_n(\hat{\beta}A_2)^n\tilde{\delta}_n \right] \left[ 1 - h\tilde{\delta}_c(\hat{\beta}A_3)^n\tilde{\delta}_c \right] (\phi^{n+1} - \phi^n)
= h\left[ \tilde{\delta}_e \left( \frac{\partial U}{J} \right)^n + \tilde{\delta}_n \left( \frac{\partial V}{J} \right)^n + \tilde{\delta}_c \left( \frac{\partial W}{J} \right)^n - R_\infty \right]
\]

where \( R_\infty \) is the freestream subtraction defined as

\[
R_\infty = \tilde{\delta}_e \left( \frac{\bar{U}_\infty}{J} \right) + \tilde{\delta}_n \left( \frac{\bar{V}_\infty}{J} \right) + \tilde{\delta}_c \left( \frac{\bar{W}_\infty}{J} \right)
\]

It is found that this algorithm yields satisfactory predictions for subcritical and slightly critical flows, but becomes increasingly ineffective when a strong shock occurs.

**Implementation of T.E.I.**

For the scheme discussed in Section 3.1.2, the truncation error (TE) is computed by evaluating the right hand side of Eq. (3.45) for a given mean flow and setting all three time levels of perturbed velocity potential, \( \phi^{n-1} \), \( \phi^n \), and \( \phi^{n+1} \) and two levels of density, \( \rho^{n+1} \) and \( \rho^n \), the same, i.e.,

\[
TE = \tilde{\delta}_e \left( \frac{\partial U}{J} \right) + \tilde{\delta}_n \left( \frac{\partial V}{J} \right) + \tilde{\delta}_c \left( \frac{\partial W}{J} \right)
\]
In the absence of forced disturbances, a one time step advancement of $\phi$ and $\rho$ should render the solution unchanged if the implementation is done correctly, since the inclusion of truncation error forces the system to accept any flow field as a steady solution, regardless whether it satisfies the governing equation.

4.1.2 Discretization of Boundary Conditions

The body surface flow tangency condition Eq. (3.24) can also be factorized and approximated by second order differences for the $\xi$, $\eta$ derivatives and three point one-sided differences for the $\zeta$ derivatives as

$$
\Delta \phi_1^{n+1} = \frac{1}{3} (\phi_2^{n+1} - \phi_3^{n+1}) + \frac{2}{3A_3} \left[ -W_u + \left( A_5 \delta_\xi + A_6 \delta_\eta \right) \phi_2^{n+1} \right]
$$

(4.17)

where

$$
\Delta \phi_1^{n+1} = \phi_{l=1}^{n+1} - \phi_{l=2}^{n+1}
$$

(4.18)

Eq. (4.17) is updated by setting $W_u$ to zero for steady calculation and setting $W_u$ according to Eq. (3.24) for unsteady calculation. The symmetry plane condition is satisfied by setting the fluxes on each side of the wall ($k=1$) equal but opposite in signs.
New values of the circulation $\Gamma$ in the wake are obtained by solving Eq. (3.34) after the unsteady velocity potential has been updated in the field and on the body surface. Equation (3.34) is discretized using central differences for derivatives in the wake plane and backward differences for streamwise derivatives. The circulation convection equation written in delta form is
\[
\left(1 + \Delta t \langle W \rangle^n + \Delta t \langle V \rangle^n \right) \Delta \Gamma^n = -\Delta t \left( \langle V \rangle^n \delta_\eta + \langle W \rangle^n \delta_\phi \right) \Gamma^n + \Delta t \langle W \rangle^n \Delta \Gamma^2_{n-1}
\] (4.20)

Implementation of Boundary Layer Correction

For attached flows, viscous effects can be accounted for in an inviscid calculation by adding the displacement thickness $\delta$ to the surface function of Eq. (3.16), i.e.,
\[
f = z - \left[f_0(x,y) + \delta^* \right]
\] (4.21)
or equivalently adding the velocity component normal to the free-stream,
\[
w_b = u_s \delta_x^* + v_s \delta_y^* - V (w_s - u_s - f_0 - v f_0) \cdot \vec{d}
\] (4.22)
to the contravariant velocity
\[
W_b = E x u + E y v + E z w
\] (4.23)
where $W_b$ can be viewed as nonzero transpiration velocities at the mean surface. The inclusion of the displacement thickness changes the mean body and hence the corresponding total velocity potential and density field, which in turn change the displacement thickness. In the solution process for every hundred iterations for the potential of the mean
flow, the boundary layer subroutine is called once to provide an updated displacement thickness until changes of the solution satisfy the set convergence criterion. Once a steady solution is reached, the boundary layer thickness and corresponding contravariant velocity, \( W_b \), are saved, assumed unchanging, and added to the body deformation function for unsteady calculations, i.e., \( W = (W_u) + (W_b) \).

### 4.2 Interpolation Scheme

To compute the truncation error of Eq. (4.16), an interpolation scheme is needed to relate the mean flow solution from the fine grid on which it was computed to the coarse grid for unsteady perturbations. Figure 4.1 shows a mapping of a physical coordinate system \( (x, y) \) to the computational plane \( (\xi, \eta) \). Figure 4.2 shows parts of two curvilinear grids spanning the same space. Since the mapping between the two coordinate systems is in general unknown, a Newton iteration scheme is used to map bicubic overlapping patches. Bicubic patches are used so that the interpolation process is of higher order accuracy than the finite difference scheme for the solution. Brandt (1980) recommended using an interpolation scheme of order \( m+p \) for his multigrid approach, where \( m \) is the order of the highest derivative in the equations and \( p \) is the order of the approximation to the equations. Goble (1988) used the bicubic patch for his work on interacting viscous and inviscid solutions. His scheme has been tested and optimized. His algorithm is used here and outlined in the following.

A bicubic function in two curvilinear dimensions \((\xi, \eta)\) has the form

\[
F = a_1 + a_2 \xi + a_3 \xi^2 + a_4 \xi^3 + a_5 \eta + a_6 \xi \eta + a_7 \xi^2 \eta + a_8 \xi^3 \eta + a_9 \eta^2 + a_{10} \xi \eta^2 + a_{11} \xi^2 \eta^2 + a_{12} \xi^3 \eta^2 + a_{13} \eta^3 + a_{14} \xi \eta^3 + a_{15} \xi^2 \eta^3 + a_{16} \xi^3 \eta^3
\]

(4.25)
where $F$ can be any quantity. A patch of sixteen nodes, four in each dimension, is needed to define the sixteen coefficients $a_1$ to $a_{16}$, which are solved by inversion given a set of sixteen $F_{ij}$ values and sixteen $(\xi_i, \eta_j)$ pairs. For a given physical coordinate pair $(x, y)$, the corresponding computational coordinates $(\xi, \eta)$ are found by first locating the closest node point $x_p = x_{ij}$, $y_p = y_{ij}$ and iteration for the deviations $d\xi$ and $d\eta$ that correspond to the differences in physical coordinates $dx = x - x_p$ and $dy = y - y_p$.

For three dimensional interpolation which involves grids of different topologies, a trilinear interpolation scheme is used. This scheme locates first the computational cell which contains the point of interest and uses the cell coordinates, an eight point computational cube, to define the local transformation and determine the deviations $d\xi$, $d\eta$, $d\zeta$ from the cell center $x_p, y_p, z_p$ for given differences in physical coordinates $dx = x - x_p$, $dy = y - y_p$, $dz = z - z_p$. The trilinear interpolation is satisfactory for passing solutions from a fine grid onto a coarse grid. Figure 4.4 shows a typical grid cell, and Figure 4.5 shows how that cell intersects cells of a different grid.
Figure 4.1  Mapping of a curvilinear coordinate system in the physical space to the computational space.
Figure 4.2  Two curvilinear coordinate systems and a bicubic patch for interpolation.
Figure 4.3a  A four-point patch.

Figure 4.3b  A sixteen-point patch.
Figure 4.4    A typical cell element.

Figure 4.5    Two curvilinear grids and a cell for trilinear interpolation.
4.3 Grid Generation

One of the difficulties in computational aerodynamics is grid generation. The algebraic grid generation code E88 developed by Sobieczky (1983) is used here to generate different types of grid. E88 uses algebraic functions to distribute grid points over different types of aerodynamic bodies. Typical grids of topology types C-H, C-O, and O-H are shown in Figures 4.6a, 4.7a, and 4.8a, and the corresponding computational domains are shown in Figures 4.6b, 4.7b, and 4.8b, respectively. The fine O-H grid, Fig. 4.6a, is used for computation of potential mean flows using the steady option of the algorithm introduced above, the C-H and C-O grids are for computation of mean flows using TLNS3D, and the coarser O-H grid shown in Figure 4.9 is for all unsteady computations.
Figure 4.6a   A 97 x 25 x 17 C-H grid for steady flows.

Figure 4.6b   Computational domain of a C-H grid.
Figure 4.7a  A 97 x 33 x 17 C-O grid for steady flows.

Figure 4.7b  Computational domain of a C-O grid.
Figure 4.8a  
A 89 x 49 x 18 O-H grid for steady flows.

Figure 4.8b  
Computational domain of an O-H grid.
Figure 4.9  
A 45 x 25 x 18 O-H grid for unsteady computations.
CHAPTER 5

VERIFICATION OF THE CODE - ZUNAS

In this chapter, the implementation of computational techniques and boundary condition, the verification of the formulation, the effectiveness of the truncation error injection are examined. A one-dimensional model of the numerical scheme is used to demonstrate the dependency of solution accuracy on the CFL number. Numerical experiments are conducted to evaluate the implementation of the S.P.B.C. and the locally applied far-field boundary condition. The effectiveness of the techniques is evaluated on simple test cases of airfoil in harmonic pitching motion. The result of these development and implementation is an efficient computer code called ZUNAS. The overall effectiveness of ZUNAS is examined through its application to flutter predictions of the supercritical transport wing, YXX, used in Yonemoto's experiment (1984) and the rectangular wing PAPA with supercritical sections used in Farmer's experiment (1988).

5.1 Effectiveness of T.E.I. for Unsteady Flows

The effectiveness of the T.E.I. technique is examined for applications in flutter analysis. Two different O-H grids are chosen. A grid having a resolution of $10^{-3}$ of the chord is often required for accurate representation of a transonic flow around the leading edge of the wing. For the unsteady flow which corresponds to modal deformations such as those shown in Figure 5.1, a grid having a resolution of $10^{-2}$ should be adequate. The use of a coarse grid reduces the total computation time, which is proportional to the number of grid nodes. The fine grid used here is a 89x49x18 grid. Its spacing is approximately twice as fine as the 45x25x18 coarser grid in directions tangential and normal to the wing. An overlay of these two grids is shown in Figure 5.2.
The flow over the YXX wing pitching about the 32% root chord at $M_{\infty}=0.75$, reduced frequency 0.224 and $1^\circ$ amplitude is computed, and the histories of the lift and moment responses at the mid-span station for different time step sizes and on different grids are shown in Figure 5.3. Both the results computed on the coarse grid with a small time step of 0.02 (dotted line) and on the fine grid with a large time step 0.04 (dash-dotted line) are in error when compared to the result computed on the fine grid with the smaller time step 0.02 (dashed line), whereas the result on the coarse grid (solid line) with injected truncation error according to Eq. (4.16) is in excellent agreement, even with the larger time step. The result on the coarse grid alone, the dotted line, shows in phase oscillations, a shift in magnitude and a slightly smaller oscillation amplitude compared with the dashed line. The shift in magnitude is due to the steady flow which is supercritical at the Mach number of 0.75. The result on the fine grid using the larger time step has a phase error and the smallest oscillation amplitude. For flutter prediction, an error in the phase is worse than an error in the amplitude. Here, a factor of eight savings of computation time is achieved due to the use of a coarser grid and larger time steps for the unsteady flow. This decoupling of the steady and unsteady computations will yield even greater savings if the unsteady flow need not be computed on the same grid as for a Navier-Stokes solution of the steady mean flow. Similar results for plunging oscillations are shown in Figure 5.4.

These results support that the accuracy of the unsteady flow computation does not merely depend on the time step or grid size but rather depends on the CFL number and the accuracy of the mean flow as well. This dependency can be demonstrated using a simplified model of Eq. (4.7). Each of the split operators in Eq. (4.7) is of the form $(\delta_{xt} - \delta_{xx})\phi=0$, which is a wave equation for $\phi_x$. The stability of any consistent numerical scheme for such an equation must be dependent on the ratio between the time step and
grid size, or a Courant number. For example, a stability analysis of the following unconditionally stable upwind implicit scheme

\[
\left[ \frac{1}{CFL} - \delta \xi \right] \delta \xi (\phi^{n+1} - \phi^n) = \delta \xi \phi^n
\]  

(5.1)

shows that the waves computed using this scheme is not nondispersive or undamped as they should, but with increasing phase lag (\(\varphi - \theta \cdot CFL\))/\(\pi\) and damping factor \(\rho\) with increasing CFL and wave number \(\theta\) as depicted in Figure 5.5. It is the numerical dispersion relation that makes the time accurate schemes difficult to construct and expensive to run.
Figure 5.1  
Deformation modes of the YXX wing.
Figure 5.2 Overlapping of two grids at root section of the YXX wing.
Comparison of Lift Response

\[ C_l \]

\[ 0 \pi \quad 2 \pi \quad 4 \pi \quad 6 \pi \]

Comparison of Moment Response

\[ C_m \]

\[ 0 \pi \quad 2 \pi \quad 4 \pi \quad 6 \pi \]

\[ M_\infty = 0.75 \quad \alpha = 0^\circ \]
\[ k_c = 0.224 \quad \theta = 1^\circ \]

* C 45x25x18
CT 45x25x18 with T.E.I.
F 89x49x18

Figure 5.3 Time responses of \( C_l \) and \( C_m \) to pitching motion at mid-span of the YXX wing model at \( M_\infty = 0.75 \).
Comparison of Lift Response

Comparison of Moment Response

\[ M_\infty = 0.75 \quad \alpha = 0^\circ \]
\[ k_c = 0.224 \quad \delta = 0.04c \]

Figure 5.4  Time responses of \( C_l \) and \( C_m \) to plunging motion at mid-span of the YXX wing model at \( M_\infty = 0.75 \).
Figure 5.5 Phase error and amplification factor of a model implicit upwind difference scheme.
5.2 Effect of Small Perturbation Boundary Condition and Far-Field Boundary Condition

The validity of the S.P.B.C., for flutter calculation is demonstrated by the recovery of the solution of a 5% circular arc cross section rectangular wing of aspect ratio $AR = 3.0$ on a body fitted grid generated for a similar but with rounded leading edge wing. Figure 5.6 shows that the pressure distribution for the sharp leading edge wing is recovered on a grid generated for a rounded leading edge with S.P.B.C. applied (solid line).

The reflection of outward propagating disturbances at numerical boundaries can cause erroneous flutter point determination. A simple but costly way to avoid reflection is to bring the far-field boundaries so far away that the reflected disturbances do not have time to return to the body to affect the accuracy of the solution. Four types of boundary conditions, Table 5.1, are considered here.

Table 5.1 Types of far-field boundary conditions

<table>
<thead>
<tr>
<th>Type</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>locally applied wave form Eq. (3.32)</td>
</tr>
<tr>
<td>2</td>
<td>Analytical form Schoen (1988)</td>
</tr>
<tr>
<td>3</td>
<td>Dirichlet ($\phi^1 = 0$)</td>
</tr>
<tr>
<td>4</td>
<td>Neumann ($\phi^1 = 0$)</td>
</tr>
</tbody>
</table>

Figure 5.7 shows the structural responses of the YXX wing for these boundary conditions imposed at a grid plane 25 chords away from the center of root chord. The TYPE 2 boundary condition, which corresponds to the wave form of Eq. (3.28), causes a strong growth after three cycles of damped oscillation, Figure 5.7. The responses for TYPE 1 and TYPE 3 are similar while a slower decay is found for TYPE 4. Schoen (1988) reported that the TYPE 2 boundary condition is better than TYPE 3 and 4 for low frequency oscillations and far grid planes closer than 5 chords away from the wing. The reduced
frequency of the YXX wing at flutter is roughly 0.5, which may not be small enough for a direct application of Eq. (3.28) whose derivation requires the low frequency assumption.

For all subsequent applications, TYPE 1 condition is enforced at numerical boundaries 50 chords away.
The recovery of pressure distribution of a 5% circular arc wing of AR=3 computed by applying the S.P.B.C. on a grid generated for a similar wing but with rounded leading edge.
Figure 5.7  Effect of different boundary conditions on the structural response of the YXX wing at $Q_D=70$ kPa, $M_\infty=0.75$. 
5.3 Two-Dimensional Test Cases

Three airfoils, the NACA 64A006, NACA 0012, and NLR 7301, are chosen to examine the limitations, as well as the range of applicability, of the present approach for unsteady aerodynamic prediction.

Figure 5.8a to 5.8c show the lift and moment increment histories of a NACA 0012 airfoil pitching sinusoidally at \( M_\infty = 0.6 \), 0.8 and 0.85, respectively. The mean flows were obtained on a fine grid using the steady option of ZUNAS or the Euler/Navier-Stokes code ARC2D of Pulliam (1984). All time dependent lift responses were computed on the same coarse grid with truncation error injection and velocity decomposition. For the lower Mach number 0.6 and moderate Mach number 0.8, the responses are independent of the base flow. However, for \( M_\infty = 0.85 \), substantial differences are found between the computed response using a potential mean flow and that using an Euler/Navier-Stokes mean flow. The response computed using a full potential mean flow is not only different, but approaches a positive or negative nonzero mean lift increment depending on whether the airfoil was pitched up or down first, Figure 5.8d. This phenomenon is related to the nonuniqueness problem of the full potential formulation first pointed out by Steinhoff and Jameson (1982) and later reported by William et al. (1985) for the same Mach number and airfoil as shown here. It is interesting to note that when Navier-Stokes or Euler mean flows were used, the response oscillated about a zero mean as it should. Here, uniqueness of solution is rendered by the use of a modified density law with an entropy multiplier whenever the computed mean flow contains entropy variations.

Figure 5.9a b and c show the various lift and moment increment histories of an NACA 64A006 airfoil pitching sinusoidally at \( M_\infty = 0.8 \), 0.875 and 0.90, respectively. For the low Mach number 0.8 and higher Mach number 0.875, the responses are again
independent of the mean flow. However, for $M_\infty=0.90$, substantial phase differences, especially in the moment, are found between the response computed using a potential mean flow and that using an Euler/Navier-Stokes mean flow. The difference between the Euler based and Navier-Stokes based responses are negligible, since the flow is unseparated for this Mach number. The fact that the extended density formula Eq. (3.13) is incorrect for the thin viscous layer does not affect the accuracy of the unsteady waves provides the justification for its use here. Furthermore, these results exemplify the importance of a correct base flow to the computation of unsteady transonic flow and the effectiveness of the techniques introduced here.

For a separated mean flow, the present approach is no longer valid. Figure 5.10a and 5.10b show the steady pressure distribution, magnitude and phase of the lift response of the NLR 7301 airfoil to harmonic pitching oscillations at $M_\infty=0.752$ and 0.807 and for a range of reduced frequencies of oscillation. For the lower Mach number, design condition, the results agree very well with the experimental values (Davis and Malcolm [1980]), which linear theory fails to predict. However, for the higher Mach number, the predicted steady and unsteady flows are distinctively different from the measured separated flows. It is evident from the upper-surface pressure histories, Figure 5.11a and 5.11b, that the flow remains attached at the lower Mach number and is separated at the higher Mach number.
Figure 5.8a Comparison of lift responses computed using full potential, Navier-Stokes, or Euler mean flows of an NACA 0012 airfoil pitching at $M_{\infty}=0.6$ with $\Delta \alpha=1^\circ$, and $k_c=0.15$. 
Figure 5.8b  Comparison of lift responses computed using full potential, Navier-Stokes, or Euler mean flows of an NACA 0012 airfoil pitching at $M_\infty=0.8$ with $\Delta \alpha=1^\circ$, and $k_c=0.15$. 
Comparison of lift responses computed using full potential, Navier-Stokes, or Euler mean flows of an NACA 0012 airfoil pitching at $M_{\infty}=0.85$ with $\Delta \alpha=1^\circ$, and $k_c=0.15$. 
Figure 5.8d  Lift and moment responses computed using a full potential mean flow of an NACA 0012 airfoil pitching at $M_o=0.85$ and $k_c=0.15$, showing nonzero mean lift increments dependent on initial pitch direction.
Figure 5.9a  Comparison of lift responses computed using full potential, Navier-Stokes, or Euler mean flows of an NACA 64A006 airfoil pitching at $M_\infty=0.80$ with $\Delta\alpha=1^\circ$, and $k_c=0.15$. 
Figure 5.9b  Comparison of lift responses computed using full potential, Navier-Stokes, or Euler mean flows of an NACA 64A006 airfoil pitching at $M_\infty=0.875$ with $\Delta\alpha=1^\circ$, and $k_c=0.15$. 
Comparison of lift responses computed using full potential, Navier-Stokes, or Euler mean flows of an NACA 64A006 airfoil pitching at $M_\infty=0.90$ with $\Delta \alpha=1^\circ$, and $k_c=0.15$. 

Figure 5.9c
Figure 5.10a Comparison of mean pressure distributions and magnitudes and phases of lift response computed using Euler/Navier-Stokes mean flows with experimentally measured values of a NLR 7301 airfoil pitching at $M_{\infty}=0.752$, $\alpha=0^\circ$, and a range of frequencies.
Figure 5.10b  Comparison of mean pressure distributions and magnitudes and phases of lift response computed using Euler/Navier-Stokes mean flows with experimentally measured values of a NLR 7301 airfoil pitching at $M_{\infty}=0.807$, $\alpha=0^\circ$, and a range of frequencies.
Figure 5.11a  Time history of the upper surface pressure distribution of a NLR 7301 airfoil pitching at $k_b=0.05, M_{\infty} = 0.752$, and $\alpha=0^\circ$, from Davis and Malcolm (1980).

Figure 5.11b  Time history of the upper surface pressure distribution of a NLR 7301 airfoil pitching at $k_b=0.05, M_{\infty} = 0.807$, and $\alpha=0^\circ$, from Davis and Malcolm (1980).
5.4 Flutter Analysis on the YXX Wing

The YXX wing model used in Yonemoto's experiment (1984) is chosen here for an overall evaluation of the numerical code ZUNAS. This wing model has an aspect ratio of 10.0, an 18° quarter chord swept angle and supercritical airfoil sections tapered from 16% thick at the root chord to 12% at the tip. The planform of the YXX wing model is shown in Figure 5.12.

![YXX wing model diagram](image)

Figure 5.12  YXX wing model

This wing is found to be extremely sensitive to the location of the center of pressure. According to Isogai, private communication, a 2% shift in the elastic axis causes a 70% shift in the dynamic pressure at \( M_\infty = 0.70 \). The structural mode shapes for YXX wing are input along with the generalized masses and natural frequencies into ZUNAS for flutter prediction. The input data are listed in Appendix B. The Structural characteristics are computed using the finite element package NASTRAN and assuming the wing is represented by a cantilever flat plate mounted at the root section. The computed mode shapes
and natural frequencies were found to be in good agreement with the experiment [Yonemoto (1984)]. Figure 5.1 shows the computed mode shapes and natural frequencies of the first four vibrational modes of the YXX wing.

Despite the computational advances made in the last two decades, the prediction of the flow over a wing remains a research topic. There are unresolved uncertainties such as transition, turbulence model and grid resolution, etc., which could amount to substantial deviations between predicted and measured flows. The agreement of flows at one Mach or Reynolds number does not imply the agreement at another. It could only be said that in general, solving the Navier-Stokes equations yields better results than solving inviscid flow models if the former is done on a fine enough grid with the additional parameters due to viscosity set properly. The Euler/Navier-Stokes code TLNS3D of Vatsa (1987) was chosen for the Euler or Navier-Stokes mean flow computation, all flutter boundary predictions were done on the same O-H grid using ZUNAS. A trilinear interpolation routine ZCELL was developed to facilitate the T.E.L. process between topologically different grids.

The effect of the steady mean flow on the flutter boundary calculation is studied by comparing the flutter boundaries obtained using mean flows of different origins. Figures 5.13a & 5.13b show the computed pressure distributions at four spanwise locations for two Mach numbers using different methods. For the lower Mach number 0.70 (subcritical case), all predictions agree well with each other everywhere over the wing span except at the tip (not shown in Figure 5.13 but can be seen in the pressure contour plots of Figure 5.14a & 5.14b). For the higher Mach number 0.825 (supercritical case), it is evident that viscous effects play the primary role in the determination of the strength and location of the shock. Both NS (Navier-Stokes) and ZUNAS+B.L. (full potential plus boundary layer correction) computations predict similar shocks whereas the Euler
shock is only a slight improvement over the prediction using ZUNAS (full potential) alone. Another effect of the mean flow on the flutter characteristics is due to entropy changes. Figure 3.1c and Figure 3.1d show that for a Mach-subcritical flow, entropy changes are confined within the boundary layer and wake region, while Figure 3.1a and Figure 3.1b show clearly that for a Mach supercritical flow, entropy changes are found behind the shock as well.

First, the flutter boundaries computed using ZUNAS in comparison with those using Isogai's USTF3 and the experimental values are shown in Figure 5.15. It can be seen that the present method yields predictions that are in better consistency with that of the linear doublet-lattice method (DLM, dash line) than those of USTF3 for subcritical Mach number, less than 0.75. The flutter boundary computed using ZUNAS and a mean flow with boundary layer corrections (-B.L in Figure 5.15) can account for a difference of 12 kPa in the prediction of flutter dynamic pressure at $M_\infty=0.8$. This difference is due to the effect of viscosity on the shock location and strength. For higher Mach numbers where there are strong interactions between the shock and the boundary layer, all predictions using a potential mean flow are poor and the boundary layer code fails to give converged displacement profiles beyond Mach number 0.825, calling for a better treatment of the viscous effect.

Figure 5.16 shows computed flutter boundaries using different mean flows. Due to the further aft-located shock, the flutter boundary predicted using potential mean flow, ZUNAS alone, rises rapidly to a no flutter point (the least damped dynamic pressure since no growth of generalized displacements was predicted) at $M_\infty=0.825$, and that using Euler mean flow is only slightly better compared to the substantially lower flutter dynamic pressure at Mach 0.80 predicted using ZUNAS+B.L. or NS mean flow. For Mach numbers beyond 0.825, only the prediction using NS mean flow yielded flutter points, shown
as the solid line. Figure 5.16 also compares the predicted flutter frequencies using different mean flows. All predictions except that using NS mean flow show little variations in the flutter frequency versus Mach number curves, and the prediction using NS mean flow is substantially closer to the experimental values for lower Mach numbers than the rest.

By now there is no doubt that the deviations in the flutter predictions are due to the variations in the predicted mean flows and mostly attributable to viscous effects. Some attempt has been made to verify the mean flow predictions by TLNS3D with available experimental results. The agreement between computer (on a 97x25x17 C-O grid) and measured pressures was in general good near the wing root but was only fair towards the tip. Although the prediction using ZUNAS with NS mean flow is far better than all other predictions, the deviations from experiment are still substantial. Also, it is less conservative than predicted by the linear theory for Mach numbers higher than 0.80. Table 5.2a shows a rapidly increasing flutter dynamic pressure at $M_\infty=0.8$ as the mean angle increases until no flutter at $2^\circ$, a trend consistent with Isogai's predictions. This trend suggests that even lower flutter boundaries are attainable if the mean angle is further reduced. Figure 5.17 shows three flutter boundaries and frequencies computed using ZUNAS with NS mean flows for $0^\circ$ and $-1^\circ$ mean angles. The solid curve of $-1^\circ$ mean angle is astonishingly close to the experimental values. In an experiment the mean angle is known at the root, but the shapes a flexible wing model will assume under different dynamic pressure are not known unless measurements are made under the experimental conditions. The computed generalized mean forces at $M_\infty=0.80$, Table 5.2a, suggest a nose-down moment and a strong tip-up bending at $2^\circ$ mean angle. As the angle of attack is changed from $2^\circ$ to $-1^\circ$, the bending force dropped from 0.14 to 0.007 while the torsion force varied from -0.015 to -0.100. It is conceivable that the nose down moment would twist the wing to a shape equivalent to reducing the angle of attack. Figure 5.18 shows the
damping rate (change of logarithmic increment per unit change of dynamic pressure) versus $M_\infty$ for a mean angle of $-1^\circ$. The rate drops from 0.08 at Mach 0.7 (subcritical case) to 0.02 at Mach 0.825 (dip regime), showing the low damping phenomenon reported in Yonemoto's experiment (1984). With the uncertainty of the mean wing profiles in the experiment and the inaccuracy in the mean flow computation, the predictions in both flutter speed and frequency at $-1^\circ$ can be considered excellent.
Figure 5.13a  Comparison of steady mean flows of the YXX wing at $M_\infty=0.7$ computed using full potential (fp), full potential with viscous effect (fpv), Navier-Stokes (NS), and Euler equations.
Comparison of steady mean flows of the YXX wing at $M_{\infty}=0.825$ computed using full potential (fp), full potential with viscous effects (fpv), Navier-Stokes (NS), and Euler equations.
Figure 5.14a  Comparison of pressure contours of the YXX wing at $M_\infty=0.7$ computed using full potential (fp), full potential with viscous effects (fpv), Navier-Stokes (NS), and Euler equations.
Figure 5.14b: Comparison of pressure contours of the YXX wing at M∞ = 0.825.
Figure 5.15  Comparison of flutter boundary computations on the YXX wing.
Figure 5.16  Comparison of flutter boundary computations on the YXX wing using various mean flows and for $\alpha=0^\circ$. 
Figure 5.17  Comparison of flutter boundary computations on the YXX wing using various mean flows and for two $\alpha$'s.
Figure 5.18  Damping rate of the YXX wing at $\alpha=-1^\circ$ based on Navier-Stokes steady mean flows.

Table 5.2a  Static generalized forces

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>gafo(1)</th>
<th>gafo(2)</th>
<th>gafo(3)</th>
<th>$QD(kPa)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>0.00679</td>
<td>-0.00201</td>
<td>-0.10040</td>
<td>61.57</td>
</tr>
<tr>
<td>0</td>
<td>0.05214</td>
<td>-0.04334</td>
<td>-0.07130</td>
<td>66.30</td>
</tr>
<tr>
<td>1</td>
<td>0.10142</td>
<td>-0.06787</td>
<td>-0.04335</td>
<td>71.20</td>
</tr>
<tr>
<td>2</td>
<td>0.14257</td>
<td>-0.08587</td>
<td>-0.01478</td>
<td>No flutter</td>
</tr>
</tbody>
</table>

(1): 1st mode  gafo: static generalized force  QD: dynamic pressure
(2): 2nd mode
(3): 3rd mode

Table 5.2b  Structural properties.

<table>
<thead>
<tr>
<th></th>
<th>1st Mode</th>
<th>2nd Mode</th>
<th>3rd Mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>WGM†</td>
<td>7.102E-5</td>
<td>6.846E-6</td>
<td>9.160E-7</td>
</tr>
<tr>
<td>$\Omega$(rad/s)</td>
<td>381.1</td>
<td>1260.1</td>
<td>2697.8</td>
</tr>
</tbody>
</table>

†: Wing generalized mass
5.5 Flutter Analysis on the PAPA Wing

Farmer (1982) designed a wind-tunnel experiment of a rigid wing with a supercritical cross section and two degrees of freedom. The wing is attached and allowed to move with a splitter plate which is supported away from the tunnel wall by a system of rods. The rods flex in such a way that only pitch and plunge oscillations are permitted. The planform and a cross section of this wing called PAPA are shown in Figure 5.19.

![Figure 5.19 Planform of the PAPA cambered supercritical wing.](image)

The wing model has a natural frequency of 3.43 Hz for plunge and 5.44 Hz for pitch. The pitch axis and mass center are both located at mid-chord. The total mass is 5.647 slugs and mass moment of inertia is 2.585 slug ft². The mode shape functions for this wing model are plunge and pitch. The reduced frequency at flutter is expected to be low.

Figure 5.20a shows computed flutter boundaries using ZUNAS. The experiments were conducted in both air and Freon, but only the experiments in air are compared here. The predicted flutter frequencies are all very close to the experimental values. The
flutter boundary predicted using the linear method (FAST) shows an unconventional upward trend with the Mach number. The predicted flutter boundary using inviscid mean flows shows higher flutter dynamic pressures than that using viscous mean flows with upper and lower surface transition points set at locations where the pressure gradient is most adverse, at 80% and 55% respectively. The latter result agrees very well with that of the experiment on a clean wing where transition is unforced and allowed to occur naturally. Figure 5.20b shows computed flutter boundaries for four sets of transition point locations on the upper and lower surfaces. While all predicted frequencies at flutter are close to the experimental values and in good agreement with linear theory, the dynamic pressure is extremely sensitive to the transition locations, more to the upper than to the lower location, however. Same degree of sensitivity to the placement of boundary layer trips was found in the experiments conducted in Freon, but the sensitivity in air is less pronounced.

To verify the unconventional trend predicted by the linear theory, the 16% thick cambered supercritical wing cross section is replaced by the much thinner NACA 64A006 airfoil. The steady mean flows are computed using the inviscid option of TLNS3D. The results for Freon are shown in Figure 5.20c in comparison with linear theory (FAST). The flutter boundary predicted using ZUNAS shows the same upward trend but deviates as expected more from the linear theory FAST as the Mach number increases due to the thickness of the NACA 64A006 airfoil. The flutter frequencies are in excellent agreement for both methods.

These unusual characteristics should be explainable using the theory developed in Chapter Two. Table 5.3 lists values of the aerodynamic parameters $A_1, A_2, A_3,$ and $A_4$ for five different Mach numbers computed for the PAPA wing with NACA 64A006 cross sections. Both the full and simplified formulas, Eq. (2.29a) & (2.29b) and Eq. (2.31) &
(2.33), give excellent predictions compared with those of FAST and ZUNAS. According to the formulas, the flutter speed increases with $\sqrt{\mu}$ whereas the flutter frequency decreases with it. For the experiment in Freon, the density ratio $\mu$ changes from 208 at $M_\infty=0.4$ to 1424 at $M_\infty=0.8$. Unless there is a strong derivation from linear aerodynamics, it is this increase of $\mu$ that causes the increase of flutter speed, as well as the decrease of flutter frequency, with $M_\infty$ for the modified PAPA wing.

Table 5.3 Aerodynamic parameters computed from indicial responses.

<table>
<thead>
<tr>
<th>$M_\infty$</th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$A_3$</th>
<th>$A_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4</td>
<td>7.5794</td>
<td>0.8868</td>
<td>1.9020</td>
<td>0.5839</td>
</tr>
<tr>
<td>0.5</td>
<td>7.8615</td>
<td>1.2056</td>
<td>1.9853</td>
<td>0.7446</td>
</tr>
<tr>
<td>0.6</td>
<td>8.2571</td>
<td>1.6350</td>
<td>2.0821</td>
<td>0.9083</td>
</tr>
<tr>
<td>0.7</td>
<td>8.8419</td>
<td>2.1746</td>
<td>2.2236</td>
<td>1.1108</td>
</tr>
<tr>
<td>0.8</td>
<td>9.8356</td>
<td>2.9061</td>
<td>2.4288</td>
<td>1.4306</td>
</tr>
</tbody>
</table>
Figure 5.20a  Flutter boundary predictions using ZUNAS for the PAPA wing.
Figure 5.20b  Flutter boundary predictions using ZUNAS with different upper and lower transition locations for the PAPA wing.
Comparison of Flutter boundary predictions computed using ZUNAS with linear theory (FAST) for a wing same as the PAPA wing model except with NACA 64A006 cross sections.
Figure 5.21 Comparison of flutter boundary predictions using different numerical and analytic methods.
CHAPTER 6

TRANSONIC DIP OF WINGS

In this chapter, the criterion for the transonic dip location derived in Chapter Two is examined for extension of its application to wings.

The model described in Chapter Two provides a criterion for the location of the lowest flutter speed of a two-degree-of-freedom airfoil system. It was shown that the minimum flutter speed corresponds to the maximum of $C_{m\alpha}$. Figure 6.1 shows the variations of $C_{M\alpha}$ (total integrated moment instead of sectional moment coefficient $C_{m\alpha}$) versus $M_\infty$ and the corresponding flutter boundaries using either inviscid or viscous mean flows. The location predicted by the criterion is in excellent agreement with the flutter dip computed using viscous mean flows, but is off by 0.03 Mach number when flutter was predicted using inviscid mean flows.

Figure 6.2 shows the variations of $C_{M\alpha}$ versus $M_\infty$ for the YXX wing. The dip locations predicted by the criterion are all in agreement with the computed values regardless of the substantial shift of dip location due to using viscous mean flows.

These results exemplify the validity of this simple criterion for the prediction of the transonic dip location using quasi-steady aerodynamics for both airfoil and wing. However, these studies have been based on numerical solutions. To further extend the validity of this criterion, the experimental results on the TF-SA wing of Farmer and Hanson (1976) are examined here. The flutter boundary measured in the experiment of [Farmer and Hanson (1976)] and that predicted by [Yates et al. (1981)] are plotted in Figure 6.3 along with the $C_{M\alpha}$ values extracted and integrated from the static pressure data published in Yates et al. (1981). The consistency between the dip and the $C_{M\alpha}$ peak further support the application of this simple criterion to wing flutter.
Figure 6.1  Correlation of computed moment slope peaks and dynamic pressure dips on the PAPA wing.
Figure 6.2  Correlation of computed moment slope peaks and dynamic pressure dips on the YXX wing.
Figure 6.3  Correlation of measured moment slope peaks and dynamic pressure dips on the TF-8A wing.
CHAPTER 7

SUMMARY

The modeling of a two-degree-of-freedom airfoil system provides a simple means by which the effects of the structural and aerodynamic parameters on the flutter characteristics can be understood. Results using the model are in excellent agreement with those of Yang et al. (1979). The analytical expressions derived from this simple model relate explicitly the flutter speed and frequency to quasi-steady aerodynamic derivatives, and a simple criterion from these expressions relates the transonic dip to the moment slope $C_{m\alpha}$. This simple criterion is found to be applicable for supercritical wings as well, swept or unswept. These results can become useful tools for primary designs of aircraft.

An accurate, efficient and robust code, ZUNAS, for the prediction of wing flutter in the transonic regime has been developed. Table 7.1 compares the efficiency of ZUNAS with that of USTF3 for a typical flutter boundary prediction. For the calculation of an unsteady flow, ZUNAS is twenty-four times faster than USTF3, and requires 1.6 times the amount of computing time for a mean flow on the fine grid as opposed to six times the amount for USTF3 on the same grid. The better accuracy of the current method is attributed to the separation of the steady and unsteady computations via the injection of truncation error. For a flutter boundary prediction the overall efficient of ZUNAS alone is roughly ten times faster than USTF3. Even with steady mean flows computed using TLNS3D at the average cost of 0.6 hours per Mach number, the total cost for a flutter boundary prediction using ZUNAS is only 5.2 hours on the CRAY-YMP.

It is shown that for accurate computation of acoustic disturbances, the limitation of the CFL number on the time steps is of primary importance. Hence, a finer spatial grid
does not necessarily yield more accurate results unless correspondingly finer time steps are used. Substantial savings in computational resources can be achieved by decoupling the computation of the mean flow from that of the small disturbances.

Table 7.1 Comparison of computational resources.

<table>
<thead>
<tr>
<th></th>
<th>ZUNAS</th>
<th>USTF3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steady mean</td>
<td>0.236 hrs/Mach Number</td>
<td>0.694 hrs/Mach Number</td>
</tr>
<tr>
<td></td>
<td>Grid 89x49x18(78498) 2000 iterations</td>
<td>Grid 47x31x23 (33511) 2000 iterations</td>
</tr>
<tr>
<td>Unsteady</td>
<td>0.047 hrs/Mach Number</td>
<td>1.167 hrs/Mach Number</td>
</tr>
<tr>
<td></td>
<td>Grid 45x25x18(20250) with T.E.I.</td>
<td>Grid 47x31x23 (3351)</td>
</tr>
<tr>
<td></td>
<td>1200 steps x 0.14 sec/step</td>
<td>3000 steps x 1.4 sec/step</td>
</tr>
<tr>
<td>Flutter point</td>
<td>0.377 hrs</td>
<td>4.20 hrs</td>
</tr>
<tr>
<td></td>
<td>3 trials for each Mach Number</td>
<td>3 trials for each Mach Number</td>
</tr>
<tr>
<td>Flutter boundary</td>
<td>2.64 hrs</td>
<td>29.40 hrs</td>
</tr>
<tr>
<td></td>
<td>based on 7 Mach Numbers</td>
<td>based on 7 Mach Numbers</td>
</tr>
</tbody>
</table>

** CPU times are based on a CRAY-YMP machine

The accuracy of a flutter prediction depends on the accuracy of the mean flow. For an aft-loaded supercritical transport wing such as YXX, the shock when occurs is close to the tailing edge. The interaction between the shock and the boundary layer determines the mean shock location, which determines the flutter characteristics of the wing at transonic speeds. Hence, a reliable prediction method must take into account the effect of viscosity. Since the mean flow depends sensitively on transition location, the flutter characteristics can be significantly affected even for Mach-subcritical flow, as shown in the study on the PAPA wing.
APPENDIX A

NOMENCLATURE

\(a\) = distance of elastic axis behind mid-chord, local sound speed
\(a_\infty\) = freestream sound speed
\(A_1\) to \(A_6\) = metrics of curvilinear coordinate, Chapter 3 to Chapter 4
\(b\) = semi-chord
\(c\) = chord length
\(Cl_{\alpha}, A_1\) = quasi-steady lift slope
\(C_m_{\alpha}, A_3\) = quasi-steady moment slope
\(g\) = structural damping factor
\(G, H\) = shape factors
\(h\) = plunging displacement (positive downward)
\(i\) = \(\sqrt{-1}\), complex constant
\(J\) = Jacobian
\(k, k_c\) = \(\omega c / U_\infty\) reduced frequency (based on full chord \(c\))
\(k_b\) = \(\omega b / U_\infty\) reduced frequency (based on semi chord \(b\))
\(k_s\) = molecular conductivity
\(K\) = \(4A_2A_4/(r_\alpha^2A_0)\), a key parameter for flutter
\(K_h\) = spring constant for plunging
\(K_\alpha\) = spring constant for pitching
\(l\) = reference length
\(L\) = total lift per unit span (positive up)
\(L_1\) = aerodynamic lift coefficient due to plunging
\(L_2\) = aerodynamic lift coefficient due to pitching
\(M\) = local Mach number
\(M_\infty\) = freestream Mach number
\(M_y\) = pitching moment per unit span about elastic axis (positive nose up)
\(M_1\) = aerodynamic moment coefficient due to plunging
\(M_2\) = aerodynamic moment coefficient due to pitching
\(m\) = mass per unit span
\(r_\alpha\) = dimensionless radius of gyration about elastic axis
$R$ = uncoupled frequency ratio ($R = \omega_R/\omega_0$), residual

$R_\infty$ = freestream subtraction

$t$ = dimensionless time measured by chord lengths traveled

$T$ = truncation error

$U$ = flutter speed, contravariant velocity

$U_\infty$ = freestream velocity

$\bar{U}$ = $(U/\bar{c}_0)^2$

$V$ = contravariant velocity

$W$ = contravariant velocity

$x,y,z$ = Cartesian coordinates

$x_b$ = distance in semi chord from center of gravity to elastic axis

$x_{cg}$ = distance in semi chord from mass center position to mid-chord

$Z$ = $(\omega_0/\omega)^2(1+ig)$, flutter solution

$\alpha$ = pitching displacement, angle of attack

$\beta$ = arbitrary constant, $\sqrt{(1-M^2)}$

$\gamma$ = specific heat ratio for ideal gas

$\delta$ = spatial operator, displacement thickness

$\Gamma$ = circulation

$\theta$ = momentum thickness

$\Pi$ = pressure gradient parameter

$\tau_0$ = skin friction

$\xi, \eta, \zeta$ = computational coordinates

$1/\lambda_l, A_2$ = time constant for lift of indicial response to pitch

$1/\lambda_m, A_4$ = time constant for moment of indicial response to pitch

$\mu$ = mass ratio $(m/\rho_b b^2)$, viscosity

$\rho$ = air density, ratio of $\mu/\mu'$

$\rho_\infty$ = freestream density

$\phi$ = total velocity potential (Section 3.1.1a), perturbed velocity potential (Section 3.1.1a)

$\phi'$ = perturbed velocity potential (Section 3.1.1a)

$\phi_s$ = velocity potential of a steady flow

$\omega$ = flutter frequency

$\omega_h, \omega_\alpha$ = uncoupled natural frequency of the airfoil in plunge and pitch, respectively

$\omega_0, \omega_r$ = circular frequency at flutter
**Subscripts**

- \( b \) = quantities with boundary layer correction
- \( u \) = quantities related to small perturbation boundary condition S.P.B.C
- \( r \) = real part of a complex quantity
- \( i \) = imaginary part of a complex quantity, isentropic quantities
- \( l \) = lower trailing edge
- \( u \) = upper trailing edge
- \( \cdot \) = freestream condition

**Superscripts**

- \( \cdot \) = quantities for pitching axis at mid-chord (Chapter 2)
APPENDIX B

MODE SHAPE INPUT DATA-YXX WING

GRM: = generalized mass (normalized by the square of the largest component of mode shape function for each mode)

MM: = mode number

YS: = spanwise location (normalized by root chord length)

XS: = chordwise location (normalized by root chord length)

Generalized Mass

<table>
<thead>
<tr>
<th>MM</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>GRM</td>
<td>2.2426E-3</td>
<td>1.7628E-3</td>
<td>1.2288E-3</td>
<td>1.4831E-3</td>
<td>8.7494E-4</td>
<td>1.1712E-3</td>
</tr>
</tbody>
</table>

Mode Shape

YS=0.0

<table>
<thead>
<tr>
<th>XS</th>
<th>MM=1</th>
<th>MM=2</th>
<th>MM=3</th>
<th>MM=4</th>
<th>MM=5</th>
<th>MM=6</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.250</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.500</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.750</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>1.000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>XS</td>
<td>MM=1</td>
<td>MM=2</td>
<td>MM=3</td>
<td>MM=4</td>
<td>MM=5</td>
<td>MM=6</td>
</tr>
<tr>
<td>------</td>
<td>-------</td>
<td>-------</td>
<td>-------</td>
<td>-------</td>
<td>-------</td>
<td>-------</td>
</tr>
<tr>
<td>0.000</td>
<td>-0.0004</td>
<td>0.0017</td>
<td>0.0010</td>
<td>-0.0038</td>
<td>-0.0026</td>
<td>0.0064</td>
</tr>
<tr>
<td>0.250</td>
<td>-0.0001</td>
<td>0.0006</td>
<td>0.0003</td>
<td>-0.0013</td>
<td>-0.0009</td>
<td>0.0021</td>
</tr>
<tr>
<td>0.500</td>
<td>0.0001</td>
<td>-0.0005</td>
<td>-0.0003</td>
<td>0.0012</td>
<td>0.0008</td>
<td>-0.0021</td>
</tr>
<tr>
<td>0.750</td>
<td>0.0004</td>
<td>-0.0016</td>
<td>-0.0009</td>
<td>0.0037</td>
<td>0.0025</td>
<td>-0.0063</td>
</tr>
<tr>
<td>1.000</td>
<td>0.0007</td>
<td>-0.0028</td>
<td>-0.0016</td>
<td>0.0062</td>
<td>0.0042</td>
<td>-0.0105</td>
</tr>
</tbody>
</table>

YS = 0.33135

<table>
<thead>
<tr>
<th>XS</th>
<th>MM=1</th>
<th>MM=2</th>
<th>MM=3</th>
<th>MM=4</th>
<th>MM=5</th>
<th>MM=6</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.125</td>
<td>0.0015</td>
<td>-0.0033</td>
<td>0.0766</td>
<td>0.0233</td>
<td>-0.1728</td>
<td>-0.0691</td>
</tr>
<tr>
<td>0.358</td>
<td>0.0033</td>
<td>-0.0114</td>
<td>0.0212</td>
<td>0.0295</td>
<td>-0.0490</td>
<td>-0.0571</td>
</tr>
<tr>
<td>0.591</td>
<td>0.0051</td>
<td>-0.0195</td>
<td>-0.0342</td>
<td>0.0357</td>
<td>0.0749</td>
<td>-0.0450</td>
</tr>
<tr>
<td>0.824</td>
<td>0.0069</td>
<td>-0.0277</td>
<td>-0.0896</td>
<td>0.0418</td>
<td>0.1987</td>
<td>-0.0330</td>
</tr>
<tr>
<td>1.057</td>
<td>0.0088</td>
<td>-0.0358</td>
<td>-0.1450</td>
<td>0.0480</td>
<td>0.3225</td>
<td>-0.0209</td>
</tr>
</tbody>
</table>

YS = 0.6627

<table>
<thead>
<tr>
<th>XS</th>
<th>MM=1</th>
<th>MM=2</th>
<th>MM=3</th>
<th>MM=4</th>
<th>MM=5</th>
<th>MM=6</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.249</td>
<td>0.0131</td>
<td>-0.0409</td>
<td>0.1391</td>
<td>0.1186</td>
<td>-0.2100</td>
<td>-0.2219</td>
</tr>
<tr>
<td>0.465</td>
<td>0.0172</td>
<td>-0.0569</td>
<td>0.0322</td>
<td>0.1205</td>
<td>-0.0509</td>
<td>-0.1829</td>
</tr>
<tr>
<td>0.682</td>
<td>0.0214</td>
<td>-0.0729</td>
<td>-0.0747</td>
<td>0.1223</td>
<td>0.1083</td>
<td>-0.1439</td>
</tr>
<tr>
<td>0.898</td>
<td>0.0255</td>
<td>-0.0889</td>
<td>-0.1816</td>
<td>0.1242</td>
<td>0.2674</td>
<td>-0.1049</td>
</tr>
<tr>
<td>1.114</td>
<td>0.0296</td>
<td>-0.1048</td>
<td>-0.2886</td>
<td>0.1261</td>
<td>0.4266</td>
<td>-0.0659</td>
</tr>
</tbody>
</table>

YS = 0.99405

<table>
<thead>
<tr>
<th>XS</th>
<th>MM=1</th>
<th>MM=2</th>
<th>MM=3</th>
<th>MM=4</th>
<th>MM=5</th>
<th>MM=6</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.374</td>
<td>0.0397</td>
<td>-0.1153</td>
<td>0.2065</td>
<td>0.2523</td>
<td>-0.2201</td>
<td>-0.3459</td>
</tr>
<tr>
<td>0.573</td>
<td>0.0470</td>
<td>-0.1378</td>
<td>0.0471</td>
<td>0.2361</td>
<td>-0.0610</td>
<td>-0.2624</td>
</tr>
<tr>
<td>0.772</td>
<td>0.0542</td>
<td>-0.1603</td>
<td>-0.1124</td>
<td>0.2198</td>
<td>0.0981</td>
<td>-0.1790</td>
</tr>
<tr>
<td>0.972</td>
<td>0.0614</td>
<td>-0.1828</td>
<td>-0.2718</td>
<td>0.2036</td>
<td>0.2571</td>
<td>-0.0956</td>
</tr>
<tr>
<td>1.171</td>
<td>0.0686</td>
<td>-0.2053</td>
<td>-0.4313</td>
<td>0.1873</td>
<td>0.4162</td>
<td>-0.0122</td>
</tr>
</tbody>
</table>

YS = 1.3254

<table>
<thead>
<tr>
<th>XS</th>
<th>MM=1</th>
<th>MM=2</th>
<th>MM=3</th>
<th>MM=4</th>
<th>MM=5</th>
<th>MM=6</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.498</td>
<td>0.0888</td>
<td>-0.2173</td>
<td>0.3187</td>
<td>0.3629</td>
<td>-0.1973</td>
<td>-0.2994</td>
</tr>
<tr>
<td>0.681</td>
<td>0.0993</td>
<td>-0.2426</td>
<td>0.0912</td>
<td>0.3041</td>
<td>-0.0847</td>
<td>-0.1659</td>
</tr>
<tr>
<td>0.863</td>
<td>0.1099</td>
<td>-0.2679</td>
<td>-0.1363</td>
<td>0.2453</td>
<td>0.0279</td>
<td>-0.0323</td>
</tr>
<tr>
<td>1.046</td>
<td>0.1204</td>
<td>-0.2931</td>
<td>-0.3639</td>
<td>0.1865</td>
<td>0.1405</td>
<td>0.1012</td>
</tr>
<tr>
<td>1.228</td>
<td>0.1310</td>
<td>-0.3184</td>
<td>-0.5914</td>
<td>0.1277</td>
<td>0.2530</td>
<td>0.2348</td>
</tr>
</tbody>
</table>

YS = 1.65675

<table>
<thead>
<tr>
<th>XS</th>
<th>MM=1</th>
<th>MM=2</th>
<th>MM=3</th>
<th>MM=4</th>
<th>MM=5</th>
<th>MM=6</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.623</td>
<td>0.1646</td>
<td>-0.3185</td>
<td>0.4531</td>
<td>0.3525</td>
<td>-0.1224</td>
<td>-0.0299</td>
</tr>
<tr>
<td>0.788</td>
<td>0.1783</td>
<td>-0.3388</td>
<td>0.1661</td>
<td>0.2413</td>
<td>-0.0917</td>
<td>0.0935</td>
</tr>
<tr>
<td>0.954</td>
<td>0.1920</td>
<td>-0.3591</td>
<td>-0.1209</td>
<td>0.1301</td>
<td>-0.0611</td>
<td>0.2169</td>
</tr>
<tr>
<td>1.119</td>
<td>0.2056</td>
<td>-0.3794</td>
<td>-0.4078</td>
<td>0.0189</td>
<td>-0.0304</td>
<td>0.3404</td>
</tr>
<tr>
<td>1.285</td>
<td>0.2193</td>
<td>-0.3998</td>
<td>-0.6948</td>
<td>-0.0923</td>
<td>0.0003</td>
<td>0.4638</td>
</tr>
</tbody>
</table>

YS = 1.9881
<table>
<thead>
<tr>
<th>YS = 2.31945</th>
<th>XS</th>
<th>MM=1</th>
<th>MM=2</th>
<th>MM=3</th>
<th>MM=4</th>
<th>MM=5</th>
<th>MM=6</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.747</td>
<td>0.2713</td>
<td>-0.3583</td>
<td>0.5967</td>
<td>0.1708</td>
<td>0.0597</td>
<td>0.2503</td>
<td></td>
</tr>
<tr>
<td>0.896</td>
<td>0.2877</td>
<td>-0.3668</td>
<td>0.2589</td>
<td>0.0160</td>
<td>-0.0172</td>
<td>0.2695</td>
<td></td>
</tr>
<tr>
<td>1.045</td>
<td>0.3041</td>
<td>-0.3754</td>
<td>-0.0789</td>
<td>-0.1388</td>
<td>-0.0940</td>
<td>0.2887</td>
<td></td>
</tr>
<tr>
<td>1.193</td>
<td>0.3204</td>
<td>-0.3839</td>
<td>-0.4167</td>
<td>-0.2935</td>
<td>-0.1709</td>
<td>0.3079</td>
<td></td>
</tr>
<tr>
<td>1.342</td>
<td>0.3368</td>
<td>-0.3924</td>
<td>-0.7546</td>
<td>-0.4483</td>
<td>-0.2478</td>
<td>0.3270</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>YS = 2.6508</th>
<th>XS</th>
<th>MM=1</th>
<th>MM=2</th>
<th>MM=3</th>
<th>MM=4</th>
<th>MM=5</th>
<th>MM=6</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.872</td>
<td>0.4119</td>
<td>-0.2798</td>
<td>0.7352</td>
<td>-0.0911</td>
<td>0.3409</td>
<td>0.2386</td>
<td></td>
</tr>
<tr>
<td>1.004</td>
<td>0.4296</td>
<td>-0.2723</td>
<td>0.3518</td>
<td>-0.2537</td>
<td>0.1531</td>
<td>0.1067</td>
<td></td>
</tr>
<tr>
<td>1.135</td>
<td>0.4474</td>
<td>-0.2648</td>
<td>-0.0317</td>
<td>-0.4164</td>
<td>-0.0348</td>
<td>-0.0253</td>
<td></td>
</tr>
<tr>
<td>1.267</td>
<td>0.4652</td>
<td>-0.2572</td>
<td>-0.4151</td>
<td>-0.5791</td>
<td>-0.2226</td>
<td>-0.1572</td>
<td></td>
</tr>
<tr>
<td>1.399</td>
<td>0.4829</td>
<td>-0.2497</td>
<td>-0.7986</td>
<td>-0.7418</td>
<td>-0.4105</td>
<td>-0.2892</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>YS = 2.98215</th>
<th>XS</th>
<th>MM=1</th>
<th>MM=2</th>
<th>MM=3</th>
<th>MM=4</th>
<th>MM=5</th>
<th>MM=6</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.997</td>
<td>0.5803</td>
<td>-0.0232</td>
<td>0.6179</td>
<td>-0.1712</td>
<td>0.6104</td>
<td>-0.0045</td>
<td></td>
</tr>
<tr>
<td>1.111</td>
<td>0.5983</td>
<td>0.0007</td>
<td>0.2419</td>
<td>-0.2999</td>
<td>0.3164</td>
<td>-0.2264</td>
<td></td>
</tr>
<tr>
<td>1.226</td>
<td>0.6163</td>
<td>0.0246</td>
<td>-0.1341</td>
<td>-0.4285</td>
<td>0.0224</td>
<td>-0.4483</td>
<td></td>
</tr>
<tr>
<td>1.341</td>
<td>0.6343</td>
<td>0.0485</td>
<td>-0.5101</td>
<td>-0.5571</td>
<td>-0.2715</td>
<td>-0.6701</td>
<td></td>
</tr>
<tr>
<td>1.456</td>
<td>0.6523</td>
<td>0.0724</td>
<td>-0.8861</td>
<td>-0.6857</td>
<td>-0.5655</td>
<td>-0.8920</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>YS = 3.3135</th>
<th>XS</th>
<th>MM=1</th>
<th>MM=2</th>
<th>MM=3</th>
<th>MM=4</th>
<th>MM=5</th>
<th>MM=6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.121</td>
<td>0.7792</td>
<td>0.4412</td>
<td>0.5760</td>
<td>0.2543</td>
<td>0.6024</td>
<td>0.1979</td>
<td></td>
</tr>
<tr>
<td>1.219</td>
<td>0.7963</td>
<td>0.4742</td>
<td>0.1991</td>
<td>0.1699</td>
<td>0.2444</td>
<td>-0.0168</td>
<td></td>
</tr>
<tr>
<td>1.317</td>
<td>0.8134</td>
<td>0.5072</td>
<td>-0.1778</td>
<td>0.0855</td>
<td>-0.1137</td>
<td>-0.2316</td>
<td></td>
</tr>
<tr>
<td>1.415</td>
<td>0.8304</td>
<td>0.5402</td>
<td>-0.5546</td>
<td>0.0010</td>
<td>-0.4717</td>
<td>-0.4463</td>
<td></td>
</tr>
<tr>
<td>1.513</td>
<td>0.8475</td>
<td>0.5731</td>
<td>-0.9315</td>
<td>-0.0834</td>
<td>-0.8298</td>
<td>-0.6610</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>YS = 3.7420</th>
<th>XS</th>
<th>MM=1</th>
<th>MM=2</th>
<th>MM=3</th>
<th>MM=4</th>
<th>MM=5</th>
<th>MM=6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.246</td>
<td>0.9713</td>
<td>0.9664</td>
<td>0.2509</td>
<td>1.0000</td>
<td>0.2637</td>
<td>1.0000</td>
<td></td>
</tr>
<tr>
<td>1.327</td>
<td>0.9785</td>
<td>0.9748</td>
<td>-0.0618</td>
<td>0.9023</td>
<td>-0.0522</td>
<td>0.7738</td>
<td></td>
</tr>
<tr>
<td>1.408</td>
<td>0.9856</td>
<td>0.9832</td>
<td>-0.3745</td>
<td>0.8045</td>
<td>-0.3681</td>
<td>0.5477</td>
<td></td>
</tr>
<tr>
<td>1.489</td>
<td>0.9928</td>
<td>0.9916</td>
<td>-0.6873</td>
<td>0.7068</td>
<td>-0.6841</td>
<td>0.3215</td>
<td></td>
</tr>
<tr>
<td>1.570</td>
<td>1.0000</td>
<td>1.0000</td>
<td>-1.0000</td>
<td>0.6090</td>
<td>-1.0000</td>
<td>0.0954</td>
<td></td>
</tr>
</tbody>
</table>

*Mode shapes is normalized to unit value of the largest component for each mode.*
REFERENCES


