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Horseshoes in the standard map

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HORSESHOES IN THE STANDARD MAP

by

Jian Cheng

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entitled Horseshoes in the Standard Map

and recommend that it be accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy

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I hereby certify that I have read this dissertation prepared under my direction and recommend that it be accepted as fulfilling the dissertation requirement.

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ABSTRACT

We construct the horseshoes for the standard map and show that the Hausdorff dimension of the horseshoes increases to 2 as the parameter goes to infinity. We also show that the Aubry–Mather set of the standard map is a subset of the horseshoes.

§1. Introduction

The fundamental problem in dynamical systems is to describe the long time behavior of orbits. In this paper we are going to study the dynamics of a special 1–parameter family of area–preserving twist maps of the cylinder, the standard map.

The area–preserving twist map, which was introduced and studied by Poincaré in the last century when he was working on the Hamiltonian systems of two degrees of freedom, has been extensively studied recently (see [AuLe] [Kato] [Mat1]). The reason that many researchers are interested in understanding the map is that by studying this map one might be able to explain the behavior of a class of Hamiltonian systems. There is an interesting result by Moser [Mos2] which says that every area–preserving monotone twist map is the " time–one " map of a Hamiltonian system whose hamiltonian is periodic in time and satisfies the Legendre condition.

Area–preserving twist maps are also connected with some special systems, for example the billiard system in a convex domain can be reduced to an area–preserving twist map. For the billiard system in a convex domain Birkhoff showed that for any integers p and q there is a periodic orbit with period q and going around the boundary of the table p times. Such an orbit, which is constructed by a variational method, is called a (p,q)–orbit.

The variational method, which was used to construct the (p,q)–orbits for the billiard system in a convex domain, has been generalized to area–preserving twist maps. Much of the recent work on area–preserving twist maps deals with the existence of orbits which have certain minimality property. These orbits are the counterpart of (p,q)–orbits in the billiard system. Each of these "minimal " (p,q)–orbits of area–preserving twist maps has certain order on the annulus. This order is the same as the order of $\frac{p}{q}$ rotation on the circle. Therefore one can introduce a rotation number for such an orbit and the dynamics of these orbits is simply 1–dimensional. The rotation number of the (p,q)–orbit is just $\frac{p}{q}$. By considering the limit of a sequence of (p,q)–orbits of area–preserving twist map one can prove that for each real number α in a certain interval there is an orbit which has rotation number α. Sometimes orbits with the same rotation number form an invariant circle, sometimes they form a Cantor set. Those invariant circles are the continuation of KAM tori for a small perturbation from a integrable system. The existence of these minimal orbits, which form the Aubry–Mather sets, gives some indication of what could happen to those KAM tori after they are broken. But it is still not clear how the invariant tori become Cantor sets.

The standard map is a 1–parameter family of area–preserving twist maps of the cylinder. For zero value of the parameter the map is integrable. The cylinder is decomposed into layers of invariant circles with different rotation numbers. When the parameter value increases the map begins to lose its invariant circles. And after certain parameter value it was shown that the standard map does not have any invariant circle (see [Mat2]). But for each real number α there is an Aubry–Mather set with rotation number α. And these Aubry–Mather sets are Cantor sets. Katok [Kato] asked the question about the hyperbolicity of the Aubry–Mather sets for general twist maps. It was shown by Le Calvez [LeCa] that for a generic area–preserving twist map and a generic rotation number the Aubry–Mather set of that rotation number is hyperbolic. For the standard map Goroff [Goro] proved that for the parameter value $k > 2\sqrt{1+\pi^2}$ the Aubry–Mather sets are uniformly hyperbolic.

For large values of the parameter the standard map exhibits " chaotic " behavior. There is a well–known conjecture for this map which says that at least for certain values of the parameter there is an ergodic component on the cylinder with positive Lebesgue measure. But it is still not clear how one can prove this. Recently by using a functional analytic method Aubry and Abramovici [AuAb] showed that for large values of the parameter the standard map has a lot of chaotic orbits. Each of these orbits, which is characterized by a sequence of infinitely many symbols, is the fixed point of a contraction mapping on a complete metric space. For the mapping to be a contraction they need a lower bound for the parameter value.

By studying the geometric properties of the standard map for large values of the parameter, we give a much more detailed description of the behavior of the map. We construct the "horseshoes", i.e., hyperbolic basic sets. The horseshoes are described by subshifts of finite type. We also find that the set of all chaotic orbits constructed by Aubry and Abramovici is a subset of our hyperbolic basic sets. Our construction is valid for smaller value of the parameter than allowed by the functional analytic method in [AuAb] and by a general theorem (see [BoRu]) it is easy to see that the Lebesgue measure of the horseshoes is zero. In [AuAb] Aubry and Abramovici made a conjecture which implies that the set of all " chaotic " trajectories which they constructed has Lebesgue measure zero. Our result shows that this conclusion holds.

We also study the relation between the horseshoes and the hyperbolic Aubry–Mather sets. Using the properties of the Aubry–Mather sets described by Bangert [Bang], we prove that for the parameter $k > 2\pi$ the Aubry–Mather sets are the subset of our horseshoes and hence they are uniformly hyperbolic. This improves the result of Goroff [Goro].

Our construction of the hyperbolic basic sets also implies that the " size " of the horseshoes depends on the value of the parameter. One way to define the size of the horseshoes is by its Hausdorff dimension. By applying the relation among Hausdorff dimension, entropy and Lyapunov exponents, which was obtained by Lai–Sang Young [Youn], we estimate the lower bound of the Hausdorff dimension of the horseshoes and show that the Hausdorff dimension of the horseshoes goes to 2 as the parameter approaches to infinity. This result supports the idea that for large values of the parameter the "chaotic" orbits dominate the behavior of the standard map. According to Fathi [Fath] the Hausdorff dimension of the hyperbolic Aubry–Mather sets is zero. Therefore it is clear that the hyperbolic basic sets are much larger than the hyperbolic Aubry–Mather sets.

We also try to find other types of the behavior of the standard map. By a geometric consideration we construct the elliptic periodic orbits of the map. It was proved by Donskaya [Zasl] that for the standard map there is a sequence of values of the parameter approaching to infinity such that the map with these parameter values has elliptic periodic orbits. Our method gives us more than that: there is a subset in the parameter space with infinite Lebesgue measure such that for each element in this set the map has elliptic periodic points with period 2. Actually, we find a sequence of disjoint intervals in the parameter space, each of which has the length $O(\frac{1}{n})$, such that for the parameter in these intervals, the standard map on the cylinder has an elliptic periodic point with period 2.

The following is the outline of this dissertation:

In §2, we introduce the standard map, its generating function and a symmetry of the standard map: reversibility property. We also give the relation between the orbits of the map and the critical points of a special functional.

In §3, we construct the horizontal and vertical strips for the horseshoes, study the relation of these strips and determine the symmetry properties of these strips.

In §4, we use the relation between horizontal and vertical strips to describe our symbolic dynamics for the standard map. We also state the theorem about the existence of the horseshoes in this section.

In §5, in order to follow Moser's scheme we construct the sector bundle on the cylinder, which will be used to show the existence of the horseshoes.

In §6, we show that for each element in our symbolic dynamics there are a sequence of nested horizontal strips and a sequence of nested vertical strips which are needed in Moser's scheme. The existence of these sequences depends on the preservation of the sector bundle constructed in §5.

In order to show the sequence of nested horizontal strips converges to a horizontal curve and the nested vertical strips to a vertical curve we need to show the expansion of the vectors in the sector bundle. In §7 we show that for the area preserving mapping if the sector bundle is strictly preserved then one automatically has the expansion of the vectors in the sector bundle. We derive this conclusion from a more general result for diffeomorphisms on a 2-dimensional symplectic manifold.

In §8, we show that those sequences of nested strips do converge to some curves. In §9, we show that the mapping between the symbolic space and the horseshoes is a homeomorphism. In this section we also obtain some smaller horseshoes which will contain the Aubry–Mather sets.

In §10, we compare the " chaotic " trajectories in [AuAb] with our horseshoes. The coding of " chaotic " trajectories is the same as ours, but our horseshoes contain more orbits.

In §11 and §12 we give the definition of the Aubry–Mather sets and list some properties of these sets. We also show that the Aubry–Mather sets of the standard map are the subset of the horseshoes if the values of the parameter $k > 2\pi$.

In §13 we project the horseshoes to the torus. We also discuss the symbolic dynamics on the torus. In §14 we estimate the Hausdorff dimension of the horseshoes.

In the last section we prove the existence of elliptic periodic points for the standard map.

§2. The Standard Map.

Let us denote by F the standard map, a one parameter family of area–preserving monotone twist maps of the cylinder $\mathbf{S}^1 \times \mathbf{R}$. One lift of this map to the $\mathbf{R}^2$ has the form

$$f(x,y) = (x + y - \frac{k}{2\pi} sin2\pi x,\ y - \frac{k}{2\pi} sin2\pi x\ ),$$

where $0 \leq k < \infty$. It is clear that if $k = 0$, F is integrable. Furthermore, if we let

$$(x',\ y') = f(x,\ y),$$

then

$$y'dx' - ydx$$

is exact. This is so because if we replace y' by $x' - x$ and y by

$$x' - x + \frac{k}{2\pi} sin2\pi x,$$

then we will have

$$y'dx' - ydx = (x' - x)dx' + (x - x' - \frac{k}{2\pi} sin2\pi x)dx.$$

One can easily check that the right hand side of above equation is a differential of a function $h(x, x')$. It is not hard to see that

$$h(x,x') = \frac{1}{2}(x - x')^2 + \frac{k}{4\pi^2} cos2\pi x,$$

where a constant difference is allowed. Therefore, we have

$$y' = \frac{\partial h}{\partial x'}(x,x') \ \text{ and } \ y = -\frac{\partial h}{\partial x}(x,x')$$

and we call $h(x, x')$ the generating function of f.

Let $X : \mathbf{Z} \to \mathbf{R}$ be a two–sided infinite sequence with

$$X(n) = x_n,$$

and

$$\mathbf{S} = \{\ X \mid X : \mathbf{Z} \to \mathbf{R}\ \}$$

the space of all such sequences. Let

$$W(X) = \sum_{n=-\infty}^{\infty} h(x_n,\ x_{n+1}).$$

The functional W may not be well-defined on $\mathbf{S}$, but the critical point of W is well-defined. A sequence $\{x_n\}_{-\infty}^{\infty}$is a critical point of W if

$$\frac{\partial h}{\partial x'}(x_{n-1}, x_n) + \frac{\partial h}{\partial x}(x_n, x_{n+1}) = 0$$

for all n. Combining this with the relation between the map f and the generating function h, we conclude that a sequence $\{(x_n, y_n)\}_{n=-\infty}^{\infty}$ is an orbit of the map f, i.e., $f(x_n, y_n) = (x_{n+1}, y_{n+1})$ for all n, if and only if $\{x_n\}_{n=-\infty}^{\infty}$ is a critical point of the functional W. Therefore, instead of looking for some orbits of the map one can study the critical points of the functional W.

The standard map F is also reversible. The reversibility is described by a linear map T, where

$$T = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix},$$

which satisfies

$$T^2 = I$$

and

$$T \circ f \circ T = f^{-1}.$$

This reversibility will help us to determine the dynamics of the standard map.

§3. The Geometry of the Map F

Let us consider following vertical strips in $\mathbf{R^2}$:

$$S_n = [\frac{n}{2} - \frac{1}{4}, \frac{n}{2} + \frac{1}{4}] \times \mathbf{R}.$$

Again if we let $f(x,y) = (x', y')$, then the image of every vertical line $x = a$ under the map f is a line with the equation $y' = x' - a$, therefore, we construct following parallelograms:

$$A_{n,0} = f(S_n) \cap S_0 \quad \text{and} \quad A_{n-1,1} = f(S_n) \cap S_1,$$

where $-n$ indicates how high the $A_{n,i}$ is and i indicates which of the two vertical strips S_0 and S_1 the $A_{n,i}$ belongs to. We also use these $A_{n,i}$ as a partition of the cylinder (after projecting them into the cylinder), and this partition is shown in following Figure 3.1.

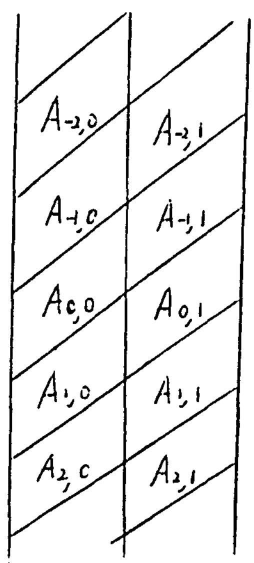

Figure 3.1

Notice that for each $A_{n,i}$ (see Figure 3.2) the four boundary segments

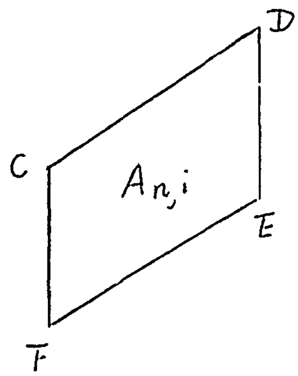

Figure 3.2

CD, DE, EF, FC are line segments with

$$d(C,D) = d(F,E) = \frac{\sqrt{2}}{2} \text{ and } d(D,E) = d(F,C) = \frac{1}{2},$$

where $d(\cdot,\cdot)$ is the distance function. The coordinates of these four vertices C, D, E and F are

$$C = (\frac{i}{2} - \frac{1}{4}, -\frac{n}{2} + \frac{i-1}{2}),\ D = (\frac{i}{2} + \frac{1}{4}, -\frac{n}{2} + \frac{i-1}{2} + \frac{1}{2}),$$

$$F = (\frac{i}{2} - \frac{1}{4}, -\frac{n}{2} + \frac{i-1}{2} - \frac{1}{2}),\ E = (\frac{i}{2} + \frac{1}{4}, -\frac{n}{2} + \frac{i-1}{2}).$$

We consider the pre–image of the $A_{n,i}$, i.e., $f^{-1}(A_{n,i})$. By definition we have

$$f^{-1}(A_{n,i}) \subset S_n$$

and

$$f^{-1}(CD) \subset \{\frac{n}{2} - \frac{1}{4}\} \times \mathbf{R} \text{ and } f^{-1}(FE) \subset \{\frac{n}{2} + \frac{1}{4}\} \times \mathbf{R}.$$

Actually, we have explicit formula for f^{-1}:

$$(x', y') \to (x' - y',\ y' + \frac{k}{2\pi} sin 2\pi(x' - y')).$$

Therefore, $f^{-1}(A_{n,i})$ is given as following Figure 3.3,

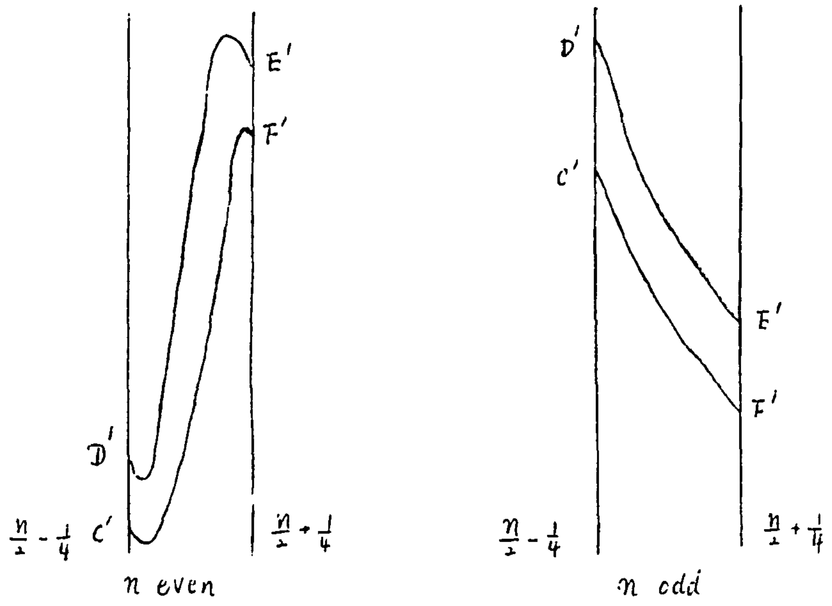

Figure 3.3

where C', D', E' and F' are the pre-images of C, D, E and F respectively, and the curves $D'E'$ and $C'F'$ are given by the graph of following functions

$$y = \frac{i}{2} + \frac{1}{4} - x + \frac{k}{2\pi} sin 2\pi x \qquad x \in [\frac{n}{2} - \frac{1}{4}, \frac{n}{2} + \frac{1}{4}]$$

and

$$y = \frac{i}{2} - \frac{1}{4} - x + \frac{k}{2\pi} sin 2\pi x \qquad x \in [\frac{n}{2} - \frac{1}{4}, \frac{n}{2} + \frac{1}{4}].$$

If we let $k = (2 + b)\pi$ where b is a positive integer, then for n even, the coordinates of C', D', E' and F' are

$$D' = (\frac{n}{2} - \frac{1}{4}, \frac{i+1-n-(2+b)}{2}) \quad E' = (\frac{n}{2} + \frac{1}{4}, \frac{i-n+(2+b)}{2})$$

$$C' = (\frac{n}{2} - \frac{1}{4}, \frac{i-n-(2+b)}{2}) \quad F' = (\frac{n}{2} + \frac{1}{4}, \frac{i-n-1+(2+b)}{2})$$

and for n is odd, the coordinates of these four points are

$$D' = (\frac{n}{2} - \frac{1}{4}, \frac{i+1-n+(2+b)}{2}) \quad E' = (\frac{n}{2} + \frac{1}{4}, \frac{i-n-(2+b)}{2})$$

$$C' = (\frac{n}{2} - \frac{1}{4}, \frac{i-n+(2+b)}{2}) \quad F' = (\frac{n}{2} + \frac{1}{4}, \frac{i-n-1-(2+b)}{2}).$$

Translating $f^{-1}(A_{n,i})$ to $[-\frac{1}{4}, \frac{1}{4}] \times \mathbf{R}$, we have

$$F^{-1}(A_{n,i}) \subset [-\frac{1}{4}, \frac{1}{4}] \times \mathbf{R}$$

if n is even and

$$F^{-1}(A_{n,i}) \subset [\frac{1}{4}, \frac{1}{4}] \times \mathbf{R}$$

if n is odd.

We say $F^{-1}(A_{n,i})$ intersects with $A_{m,j}$ properly if $F^{-1}(A_{n,i})$ and $A_{m,j}$ intersect in one of following two ways.

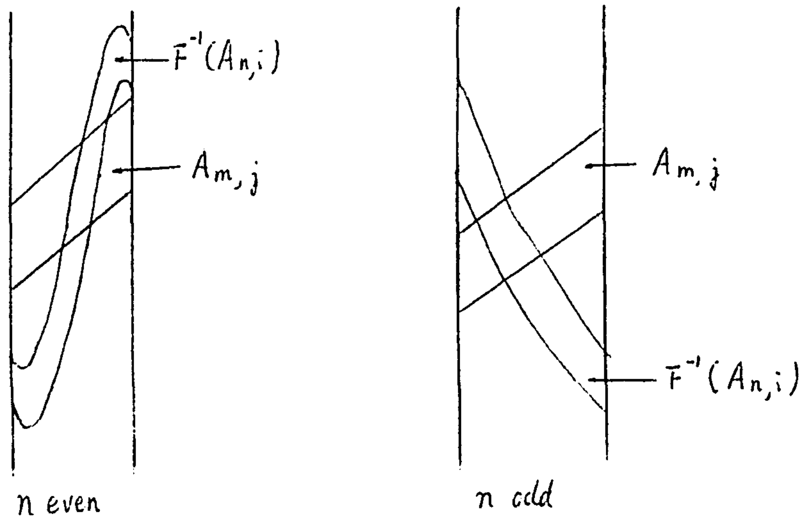

Figure 3.4

It is not hard to derive following two facts:

1. If $n+i$ is even, $F^{-1}(A_{n,i})$ intersects with $A_{m,0}$ properly if and only if

$$|n-m-i| \leq b.$$

2. If $n+i$ is odd, $F^{-1}(A_{n,i})$ intersects with $A_{m,1}$ properly if and only if

$$|n-m+1-i| \leq 2+b.$$

We denote $F^{-1}(A_{n,i})$ by $B_{n,i}$.

Now we are going to consider the reversibility property of those parallelograms $A_{n,i}$. Let us first look at the parallelogram which has the form $A_{2n,0}$ (see Figure 3.5)

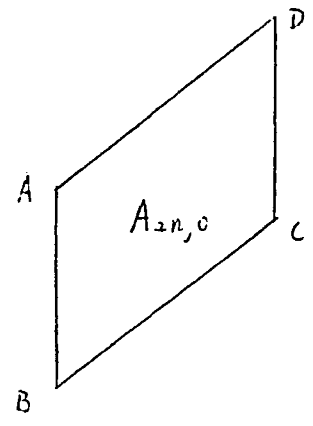

Figure 3.5

where the four vertices A, B, C and D have following coordinates

$$A = (-\frac{1}{4}, -n) \quad D = (\frac{1}{4}, -n + \frac{1}{2})$$

$$B = (-\frac{1}{4}, -n - \frac{1}{2}) \quad C = (\frac{1}{4}, -n).$$

Applying the mapping T on $A_{2n,i}$ the $TA_{2n,0}$ is given by Figure 3.6 as follows

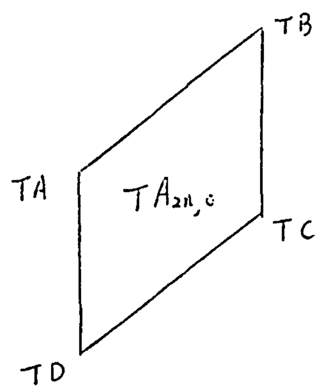

Figure 3.6

where the four vertices TA, TB, TC and TD have the following coordinates

$$TA = (n - \frac{1}{4}, n) \quad TB = (n + \frac{1}{4}, n + \frac{1}{2})$$

$$TD = (n - \frac{1}{4}, n - \frac{1}{2}) \quad TC = (n + \frac{1}{4}, n).$$

Here TA, TB, TC and TD are the images of A, B, C and D. It is easy to see that $TA_{2n,0} = A_{-2n,0}$. Same method can be applied to other types of the parallelograms. One can see that the vertices of the parallelogram change in the same way and the following relations are true:

$$TA_{2n-1,0} = A_{1-2n,1} \quad TA_{2n-1,1} = A_{1-2n,0} \quad TA_{2n,1} = TA_{-2n,1}.$$

§4. The Symbolic Description of the Dynamics and the Main Theorem

Based on the information given in the last section, we are ready to define a special subshift of finite type. We use $D = Z \times \{0,1\}$ as the alphabet set in which each element (n,i) represents the parallelogram $A_{n,i}$. For any positive integer b, we introduce the transition matrix

$$\Pi : D \times D \to \{0,1\}$$

such that

$$\Pi[(n_0, i_0), (n_1, i_1)] = 1$$

if (n_0, i_0) and (n_1, i_1) satisfy one of following two conditions:

C1: $i_0 = i_1, i_0 = n_1 \ mod2$, and $|n_1 - n_0| \le b + 2i_0$.

C2: $i_0 \ne i_1, i_0 = n_1 \ mod2$, and $|n_1 - n_0 + (i_0 - i_1)| \le b + 2i_0$,

and

$$\Pi[(n_0, i_0), (n_1, i_1)] = 0$$

otherwise.

Let Σ_b be the subshift of finite type defined by this transition matrix and

$$\sigma : \ \Sigma_b \to \Sigma_b$$

the left shift on Σ_b.

Remark: One should notice that the topology on

$$\Pi_{n=-\infty}^{\infty} D = \{ \ s = (\cdots, s_{-1}, s_0, s_1, \cdots) \mid s_i \in D, i = 0, \pm 1, \pm 2, \cdots \ \}$$

is induced by a metric on this space. This metric is defined by following way:

Let d be a metric on the space D which satisfies

$$d(s, s') = 1 \text{ if and only if } s \neq s'$$

where $s, s' \in D$. The metric ρ on the space

$$\Pi_{n=-\infty}^{\infty} D$$

is given by

$$\rho(s^1, s^2) = \sum_{n=-\infty}^{\infty} \frac{1}{2^n} d(s_n^1, s_n^2)$$

where

$$s^i = (\ \cdots, s_{-1}^i, s_0^i, s_1^i, \cdots\ ) \in \Pi_{n=-\infty}^{\infty} D, \quad i = 1, 2.$$

By restricting the metric ρ on the space Σ_b we can obtain a topology on this space.

Now our main theorem can be formulated as follows:

Theorem 4.1 *Let b be a positive integer. For all $k > (2 + b)\pi$ and any sequence $s = \{d_j\}_{j=-\infty}^{\infty} \in \Sigma_b$, the intersection*

$$\bigcap_{j=-\infty}^{+\infty} F^{-j} A_{d_j}$$

contains exactly one point of the cylinder $\mathbf{S}^1 \times \mathbf{R}$. Moreover, the mpping $\tau : \Sigma_b \to \mathbf{S}^1 \times \mathbf{R}$ defined by

$$\tau(s) = \bigcap_{j=-\infty}^{\infty} F^{-j} A_{d_j}$$

is a homeomorphism onto its image $\Lambda_k \subset S^1 \times \mathbf{R}$ and the following diagram commutes.

$$
\begin{array}{ccc}
\Sigma_b & \xrightarrow{\sigma} & \Sigma_b \\
\Big\downarrow \tau & & \Big\downarrow \tau \\
\Lambda_k & \xrightarrow{F} & \Lambda_k
\end{array}
$$

§5. The Constant Bundle of Sectors

Our proof of Theorem 4.1 will follow Moser's scheme in [Mos1]. To do this we need to construct a bundle of sectors on the tangent bundle of the cylinder so that it is preserved by the derivative of the map. For the standard map we have a natural bundle of sectors which is determined by the reversibility. For any $p \in \mathbf{S}^1 \times \mathbf{R}$ the sector

$$C^+ = \{(\xi, \eta) \in \mathbf{R}^2 \mid 0 \leq \frac{\eta}{\xi} \leq 2\}$$

in the tangent space T_p has the property that the linear map T (reversibility) maps this sector onto its complement. It is not hard to check that this is the unique sector such that the image of C^+ under the linear map T is its complement.

By reversibility of the standard map one can see that if DF preserves this constant bundle of sectors then DF^{-1} preserves the complement of C^+. This can be checked by differentiating the relation

$$T \circ f \circ T = f^{-1}.$$

After differentiating this relation we have

$$Df^{-1} = TDfT$$

and it is clear that the claim which we made above is true. This will reduce half of our proof since we only need to consider the forward iteration. It is very simple to obtain the condition for which DF preserves C^+. By computing the DF we have

$$DF = \begin{pmatrix} 1 - kcos2\pi x & 1 \\ -kcos2\pi x & 1 \end{pmatrix}.$$

For any vector $(1, v)^T \in C^+$, where $0 \leq v \leq 2$ and $(1, v)^T$ is the transpose of $(1, v)$, the condition for $DF(1, v)^T \in C^+$ is $x \in [\frac{1}{4}, \frac{3}{4}]$ or $kcos2\pi x \geq 4$ provided $x \in [-\frac{1}{4}, \frac{1}{4}]$.

§6. The horizontal and Vertical strips

To proceed our proof we have to define the horizontal and vertical strips which play important roles in the Moser's scheme. Because of the reversibility we only need to define the horizontal strips. The vertical strips can be obtained by applying the linear mapping T on the horizontal ones.

For any parallelogram A_d, where $d \in D$, we say a curve v (see Figure 6.1)

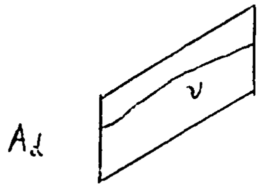

Figure 6.1

is a horizontal curve if v satisfies the following conditions:

(1) The endpoints of the curve v are on the vertical boundaries of A_d.

(2) The curve v is the graph of a function $h(x)$, $x \in I$ where $I = [-\frac{1}{4}, \frac{1}{4}]$ or $[\frac{1}{4}, \frac{3}{4}]$, and the function $h(x) - x$ is a Lipschitz function with Lipschitz constant 1.

We call a region $U \subset A_d$ a horizontal strip (see Figure 6.2)

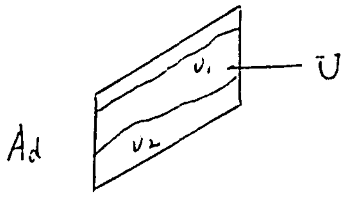

Figure 6.2

if U is bounded by two horizontal curves v_1 and v_2 which are the graphs of functions $h_1(x)$ and $h_2(x)$, $x \in I$, such that

$$h_1(x) > h_2(x), x \in I.$$

More precisely, one can define a horizontal strip U in A_d by

$$U = \{\ (x, y) \in A_d \mid x \in I \text{ and } h_2(x) \leq y \leq h_1(x)\ \}.$$

If U is a horizontal strip in A_d, then we call TU a vertical strip in $A_{d'}$ where $A_{d'} = TA_d$. We use V to represent this vertical strip.

From this definition, it is easy to see that if v is a horizontal curve then for any two points on the curve the direction of the segment which connects the two points is in the sector C^+. More specifically, we have the following

Lemma 6.1 *Let v be a curve in A_d with the endpoints on the vertical boundaries of A_d. Then v is a horizontal curve if and only if for any $p_1, p_2 \in v$ the vector $p_2 - p_1 \in C^+$.*

Proof: If v is a horizontal curve, we assume that the graph of function $h(x)$ is the curve v. Let $p_1 = (x_1, h(x_1)), p_2 = (x_2, h(x_2))$ where $x_2 > x_1$, then

$$p_2 - p_1 = (x_2 - x_1, h(x_2) - h(x_1)).$$

Since $|h(x_2) - h(x_1) - (x_2 - x_1)| \leq (x_2 - x_1)$, therefore

$$0 \leq \frac{h(x_2) - h(x_1)}{x_2 - x_1} \leq 2.$$

On the other hand, we have

$$-(x_2 - x_1) \leq h(x_2) - h(x_1) - (x_2 - x_1) \leq (x_2 - x_1),$$

this shows that $h(x) - x$ is Lipschitz with Lipschitz constant 1. Q.E.D.

Using this lemma one can describe some property of the horizontal strip under the standard map.

Proposition 6.1 *Let $U \in A_d$ be a horizontal strip and $\Pi(d, d') = 1$ for some $d' \in D$. Then*

1. *$A_{d'} \cap FU$ is a horizontal strip in $A_{d'}$ if $A_d \subset S_1$.*

2. $A_{d'} \cap FU$ is a horizontal strip in $A_{d'}$ if $A_d \subset S_0$ and $k\cos 2\pi x \geq 4$ for any $(x, y) \in U \cap B_{d'}$.

Proof: In both cases DF preserves the the constant bundle of sectors C^+. If v_1 and v_2 are the two horizontal boundary curves then $F(v_i \cap B_{d'})$, $i = 1, 2$, are two horizontal curves in $A_{d'}$. This can be proven by following consideration. For any $p_1', p_1' \in F(v_i \cap B_{d'})$ there are $p_1, P_2 \in v_i \cap B_{d'}$ such that $p_i' = Fp_i, i = 1, 2$. Therefore by mean value Theorem we have

$$p_2' - p_1' = DF_z(p_2 - p_1)$$

where $z \in v_i \cap B_{d'}$. Since DF preserves the constant bundle of the sectors C^+, $A_{d'} \cap U$ is a horizontal strip in $A_{d'}$. Q.E.D.

To construct the nested sequence of horizontal strips, we have to define the horizontal strips of the first generation. We choose the following as the horizontal strips of the first generation. For any $d, d' \in D$ with $\Pi(d, d') = 1$ we define

$$U_{d',d} = A_{d'} \cap FA_d.$$

We need to prove that $U_{d',d}$ is a horizontal strip in $A_{d'}$. To do this we have to consider following two cases.

(1) $A_d \in S_1$. For this case $U_{d',d}$ is a horizontal strip because for any $z \in A_d \cap B_{d'}$ DF_z preserves the sector C^+. By Proposition 5.1 $U_{d',d}$ is a horizontal strip.

(2) $A_d \in S_0$. For this case a careful study needs to be done.

Let $k \geq (2 + b)\pi$ where b is a positive integer. We consider $A_{m,0}$, $A_{n,0}$ and $B_{l,i}$ which have following relative positions (see Figure 6.3):

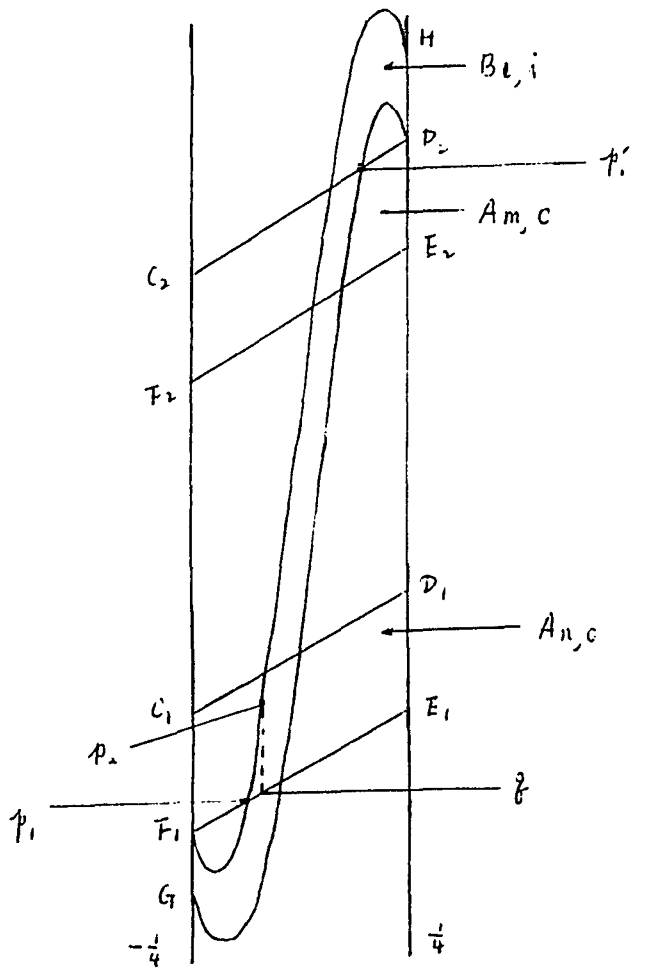

Figure 6.3

where p_1 is the intersection of the curve F_1H and the line segment F_1E_1. And p_1' is the intersection of the curve GD_2 and the line segment C_2D_2. Let x_1 and x_1' be the $x-$coordinates of p_1 and p_1' respectively, then it is easy to see that $x_1 < 0$ and $x_1' = -x_1$. We also let p_2 be a point on the curve F_1H such that if $x_2 \in [-\frac{1}{4}, 0]$ is the $x-$coordinate of p_2 then $kcos2\pi x_2 = 4$. Let q be a point on the line segment F_1E_1 such that p_2q is a vertical segment.

Proposition 6.2 $(2+b)\pi cos2\pi x_1 \geq \sqrt{15}$.

Proof: From the discussion in §3., the curve F_1H and the line segment F_1E_1 are given by

$$y = a - x + \frac{(2+b)}{2} sin 2\pi x \text{ and } y = x - \frac{n}{2} - \frac{1}{4}$$

respectively, where $x \in [-\frac{1}{4}, \frac{1}{4}]$ and $a = -\frac{n}{2} - \frac{1}{2} - \frac{1}{4} + \frac{(2+b)}{2}$. Then x_1 satisfies

$$2x + \frac{1}{2} - \frac{(2+b)}{2} - \frac{(2+b)}{2} sin 2\pi x = 0.$$

Therefore we have

$$4x_1 - (1+b) - (2+b) sin 2\pi x_1 = 0. \tag{1}$$

Let $x_1 = -\frac{1}{4} + \delta_1$, then $0 < \delta_1 < \frac{1}{4}$. Since $sin 2\pi x_1 < 0$, we obtain

$$-\sqrt{1 - cos^2 2\pi x_1} = \frac{4}{(2+b)} \delta_1 - 1. \tag{2}$$

By simple algebraic operation, (2) becomes

$$(2+b)^2 \pi^2 cos^2 2\pi x_1 = 8\pi^2 (2+b)\delta_1 - 16\pi^2 \delta_1^2. \tag{3}$$

On the other hand, using $x_1 = -\frac{1}{4} + \delta_1$, (1) becomes

$$4\delta_1 - (2+b) + (2+b) cos 2\pi \delta_1 = 0$$

or

$$\frac{2\delta_1}{(2+b)} = sin^2 \pi \delta_1.$$

Therefore we have

$$\frac{2\delta_1}{(2+b)} \leq \pi^2 \delta_1^2,$$

so

$$\delta_1 (\frac{2}{\pi^2 (2+b)} - \delta_1) \leq 0.$$

Furthermore, consider function

$$g(\delta) = 8\pi^2(2+b)\delta - 16\pi^2\delta^2, \quad \delta \in [0, \frac{1}{4}].$$

Since $g'(\delta) = 8\pi^2[(2+b) - 4\delta] > 0$,
we have

$$(2+b)^2\pi^2 cos^2 2\pi x_1 = 8\pi^2(2+b)\delta_1 - 16\pi^2\delta_1^2$$

$$\geq 8\pi^2(2+b)\frac{2}{\pi^2(2+b)} - 16\pi^2\frac{4}{\pi^4(2+b)^2}$$

$$16 - \frac{64}{\pi^2(2+b)^2} \geq 15$$

for $b \geq 1$. Q.E.D.

From this proposition we can show that in case (2) $A_{d'} \cap FA_d$ is a horizontal strip. Consider $A_d \cap B_{d'}$ (see Figure 6.4).

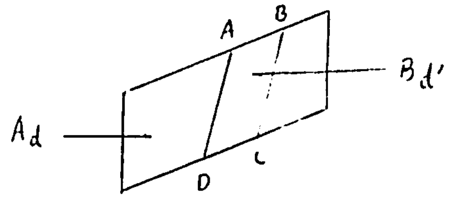

Figure 6.4

Since AB and CD are mapped into the boundaries of the $A_{d'} \cap FA_d$ and they have the slope 1. By Proposition 6.2 we have

$$DF\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2 - kcos2\pi x\\1 - kcos2\pi x\end{pmatrix}$$

and

$$0 \leq \frac{1 - kcos2\pi x}{2 - kcos2\pi x} \leq \frac{\sqrt{15}-1}{\sqrt{15}-2} < 2,$$

this shows that $A_{d'} \cap FA_d$ is a horizontal strip.

To show that

$$U_{d_1,d_2,\cdots,d_n} = \cap_{i=1}^{n} F^{-i+1} A_{d_i}$$

is a horizontal strip where $\Pi(d_i, d_{i+1}) = 1,\ i = 1, \cdots, n-1$, we only need to show that if $\Pi(d_1, d_2) = \Pi(d_2, d_3) = 1$ then

$$U_{d_1,d_2,d_3} = A_{d_1} \cap FA_{d_2} \cap F^2 A_{d_3}$$

is a horizontal strip. From the discussion above, we know that

$$U_{d_2,d_3} = A_{d_2} \cap FA_{d_3}$$

is a horizontal strip, we have to show that for $(x, y) \in B_{d_1} \cap U_{d_2,d_3}$ $DF_{(x,y)}$ preserves the bundle of sectors C^+. This is true if $U_{d_2,d_3} \subset S_1$. We only need to consider the case when $U_{d_2,d_3} \subset S_0$. To do this let us go back to Figure 6.3. What we need to show is that for the worst case the intersection of the curvilinear triangle region with the vertices p_1, p_2 and q, which we denote by Δ, and $A_{d_2} \cap FA_{d_3}$ is empty. For this purpose we need

Lemma 6.2 *For $b \geq 1$ and $k > (2+b)\pi$,*

$$\frac{k}{2\pi}(1 - cos 2\pi\delta_2) - 2\delta_2 - \delta_1 < 0$$

where $x_2 = -\frac{1}{4} + \delta_2$.

Proof: Since $k cos 2\pi x_2 = 4$, by simple trigonometry we have $k sin 2\pi\delta_2 = 4$. Therefore we obtain

$$1 - cos 2\pi\delta_2 = 1 - \sqrt{1 - sin^2 2\pi\delta_2}$$

$$= 1 - \sqrt{1 - \frac{16}{k^2}} = \frac{16}{k(k + \sqrt{k^2 - 16})}$$

and

$$\delta_2 > \frac{4}{2\pi^2(2+b)}.$$

From the proof of Proposition 5.2 we have an estimate that

$$\delta_1 > \frac{2}{\pi^2(2+b)}.$$

The direct computation gives us

$$\frac{k}{2\pi}(1 - cos2\pi\delta_2) - 2\delta_2 - \delta_1 < 0.$$

Q.E.D.

Based on this lemma we can conclude

Proposition 6.3 *Let d_1, d_2 and $d_3 \in D$ with $\Pi(d_1, d_2) = \Pi(d_2, d_3) = 1$. If $A_{d_2} \subset S_0$, then for any $(x, y) \in B_{d_1} \cap A_{d_2} \cap FA_{d_3}$ we have $kcos2\pi x > 4$.*

Proof: We only need to show that for the worst case

$$\Delta \cap A_{d_2} \cap FA_{d_3} = \emptyset.$$

In order to prove this we have to consider two cases:

1. $A_{d_3} \subset S_0$.
2. $A_{d_3} \subset S_1$.

Let us consider first the case 1.

By reversibility we know that $TA_{d_2} \cap TFA_{d_3}$ is a vertical strip in TA_{d_2} and $T\Delta$ is also a curvilinear triangle in TA_{d_2} (see Figure 6.5)

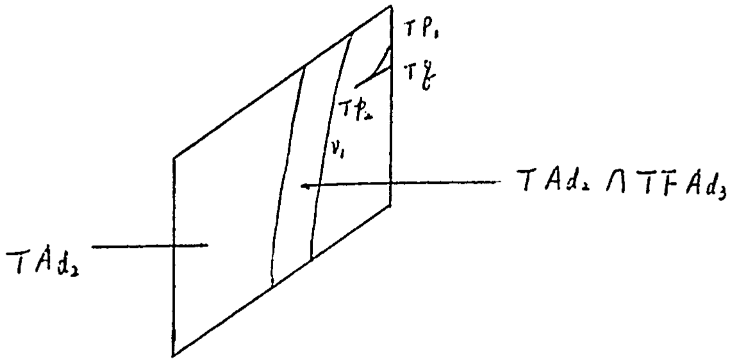

Figure 6.5

Since $A_{d_3} \subset S_0$, this implies that $A_{d_2}, B_{d_2} \subset S_0$. By direct calculation we can claim that $A_{d_2} \subset S_0$. To show that $T\Delta$ and $TA_{d_2} \cap TFA_{d_3}$ do not intersect we need to prove that the Tp_2 is under the curve v_1. For the worst case it suffices to show that the length of $T(p_2q)$ is less than $\sqrt{2}\delta_1$. Equivalently we need to show that the length of p_2q is less than δ_1. Since the curve p_1p_2 is given by the graph of

$$r_1(x) = c - x + \frac{k}{2\pi} sin 2\pi x, \quad x \in [-\frac{1}{4}, \frac{1}{4}]$$

where c is a contant, we have to show $r_1(-\frac{1}{4}+\delta_2) - r_1(-\frac{1}{4}) < \delta_1 + \delta_2$. It is equivalent to prove

$$\frac{k}{2\pi}(1 - cos 2\pi\delta_2) - 2\delta_2 - \delta_1 < 0,$$

by Lemma 6.2, above inequality is true.

For the case 2 since $A_{d_3} \subset S_1$, this implies that $B_{d_2} \subset S_1$. Again by simple computation we can claim that $TA_{d_2} \subset S_1$. Also $T\Delta \subset TA_{d_2}$ (see Figure 6.6).

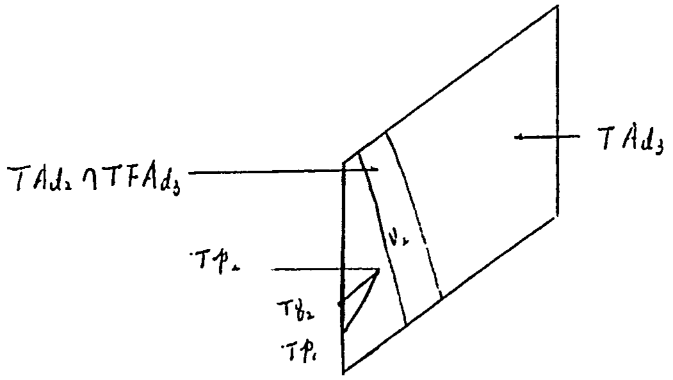

Figure 6.6

It suffices to show that Tp_2 is under the curve v_2. The curve v_2 is given by the graph of a function

$$l(x) = c' - x + \frac{k}{2\pi} sin 2\pi x, \quad x \in [\frac{1}{4}, \frac{3}{4}]$$

where c' is a constant. We need to prove that $l(\frac{1}{4} + \delta_1) - \{l(\frac{1}{4}) - \frac{1}{2} + \delta_2 + \delta_1\} > 0$. This is equivalent to show that

$$\frac{1}{2} - 2\delta_1 - \delta_2 > \frac{k}{2\pi}(1 - cos 2\pi\delta_1).$$

By Lemma 6.2 and the assumption of $\delta_1 < \delta_2$, it suffices to show that $6\delta_2 < \frac{1}{2}$. Since $k cos 2\pi x_2 = 4$, we have $sin 2\pi\delta_2 = \frac{4}{k}$. For $b \geq 1$ we have $\frac{4}{k} \leq \frac{1}{2}$. This implies that $2\pi\delta_2 \leq \frac{\pi}{6}$ and we obtain $\delta_2 \leq \frac{1}{12}$. Q.E.D.

From above discussion we can conclude that if $d_i \in D$ where $i = 1, 2, \cdots$ such that the condition of $\Pi(d_i, d_{i+1}) = 1$ for $i = 1, 2, \cdots$ holds then

$$\{U_{d_1, \cdots, d_n}\}_{n=1}^{\infty},$$

where

$$U_{d_1,\cdots,d_n} = \bigcap_{i=1}^{n} F^{i-1} A_{d_i},$$

is a decreasing sequence of horizontal strips in A_{d_1}. We need to show that this sequence converges to a horizontal curve in the super norm.

§7. Uniform Expansion of the Vectors

In this section we are going to discuss the expansion of the tangent vectors under diffeomorphisms of a 2–dimensional symplectic manifold. What we are interested in here is the condition on diffeomorphisms which guarantees the expansion of the tangent vectors if certain sector bundle is strictly preserved. Our approach is based on [Ruel], [Woj2] and [Woj3]. We give a condition for diffeomorphisms which implies a uniform expansion for the vectors in the sectors.

Let us consider a linear symplectic space $\mathcal{W}$ of dimension 2 with the symplectic form ω. $\mathcal{W} = \mathbf{R} \times \mathbf{R}$ is called the standard linear symplectic space if

$$\omega(w_1, w_2) = \xi_1\eta_2 - \xi_2\eta_1$$

where $w_i = (\xi_i, \eta_i)$, $i = 1, 2$. We call any 1–dimensional linear subspace $V \subset \mathcal{W}$ a line.

Definition 7.1. Given two transversal lines V_1 and V_2 we define the sector between V_1 and V_2 by

$$C = \{\ w \in \mathcal{W} \mid \omega(v_1, v_2) \geq 0 \text{ for } w = v_1 + v_2,\ v_i \in V_i\ i = 1, 2\ \}.$$

Geometrically the sector C between V_1 and V_2 is given by following figure.

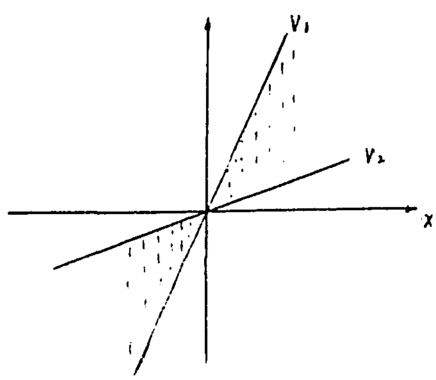

Figure 7.1

We also define a quadratic form associated with this ordered pair of transversal lines V_1 and V_2 by

$$Q(w) = \omega(v_1, v_2)$$

where $w = v_1 + v_2$, $v_i \in V_i$, $i = 1, 2$, is the unique decomposition. This gives us

$$C = \{ \ w \in \mathcal{W} \mid Q(w) \geq 0 \ \}.$$

Since any two pairs of transversal lines are symplectically equivalent, we have a canonical change of coordinate from (ξ, η) to (ξ', η') so that the sector C in the coordinate (ξ', η') is the first and the third quadrants. Also the spaces V_1 and V_2 are ξ' and $\eta'-$axes respectively. In this coordinate system the quadratic form Q has the form

$$Q(w) = \xi'\eta'$$

where $w = (\xi', \eta')$.

We may assume that the sector C in the coordinate system (ξ, η) is given by

$$C = \{(\xi, \eta) \in \mathcal{W} \mid \xi\eta \geq 0 \ \}.$$

Let $A : \mathcal{W} \to \mathcal{W}$ be a linear symplectic map. We say A strictly preserves the sector C if A maps every non–zero vector in C strictly inside C. We assume that in the coordinate (ξ, η) the mapping A is given by the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Therefore we have

$$det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc = 1.$$

Let us consider the sector C and its image AC. In the coordinates (ξ, η), the slopes of the boundaries of C are 0 and $+\infty$. The slopes of the boundaries of AC are

$$\frac{c}{a} \text{ and } \frac{d}{b}.$$

We denote the cross ratio

$$[0, +\infty, \frac{c}{a}, \frac{d}{b}]$$

by ζ. By definition of the cross ratio we have

$$\zeta = \frac{ad}{bc}.$$

Note that this cross ratio is preserved under the linear change of coordinates. It measures how much the sector C is squeezed under the mapping A.

Next lemma says that the strict sector preservation implies the increasing of the quadratic form Q.

Lemma 7.1. *If a linear symplectic map A strictly preserves the sector C, then there is an $\alpha > 1$ such that*

$$Q(Aw) \geq \alpha Q(w)$$

for any $w \in C$.

Proof: Since A maps vectors in C strictly inside C, then the vectors (a, c) and (b, d) are inside the sector C. This implies that $ac,\ bd > 0$. If $a,\ c > 0$ and $b,\ d < 0$, then by continuity there is a vector $w \in C$ such that Aw does not belong to C. Without loss of generality we may assume that a, b, c and d are all positive. Since $ad - bc = 1$, for (ξ, η) with $\xi\eta \geq 0$ we have

$$(a\xi + b\eta)(c\xi + d\eta) = ac\xi^2 + bd\eta^2 + (ad + bc)\xi\eta$$

$$\geq (2\sqrt{adbc} + ad + bc)\xi\eta.$$

This means that

$$Q(A(w) \geq (\sqrt{ad} + \sqrt{bc})^2 Q(w),$$

where $w = (\xi, \eta)$. Since $ad - bc = 1$, we have

$$\sqrt{ad} + \sqrt{bc} = \sqrt{1 + bc} + \sqrt{bc} > 1.$$

Q. E. D.

Note that the constant $(\sqrt{ad} + \sqrt{bc})^2$ derived from above lemma can be expressed by the cross ratio ζ since

$$\sqrt{ad} + \sqrt{bc} = \frac{\sqrt{\zeta} + 1}{\sqrt{\zeta - 1}}.$$

Also one can prove that if there is an $\alpha > 1$ such that for any $w \in C$

$$Q(Aw) \geq \alpha Q(w),$$

then the mapping A strictly preserves the sector C.

In proof of Lemma 7.1 we use the condition $det(A) = 1$. Actually this condition is not necessary. One can impose some extra conditions on the matrix A so that the strict preservation of the sector still implies the uniform increasing the quadratic form Q. Let us consider a matrix B with

$$det(B) = e^2 > 0.$$

For such a matrix we have the following

Lemma 7.2. *If the linear mapping B strictly preserves the sector C, then for any $w \in C$*

$$Q(Bw) \geq \alpha Q(w),$$

where

$$\alpha = det(B)\left(\frac{\sqrt{\zeta}+1}{\sqrt{\zeta}-1}\right)^2.$$

Proof: Let $det(B) = e^2$. We consider

$$A' = \frac{1}{e}(B).$$

It is clear that we have $det(A') = 1$. Also the cross ratio ζ is the same for B and for A'. Then for any $w = (\xi, \eta) \in C$ we have

$$Q(A'w) \geq \left(\frac{\sqrt{\zeta}+1}{\sqrt{\zeta}-1}\right)^2 Q(w).$$

This implies that

$$Q(Bw) = e^2 Q(A'w) \geq e^2 \left(\frac{\sqrt{\zeta}+1}{\sqrt{\zeta}-1}\right)^2 Q((\xi, \eta)).$$

Q. E. D.

Before giving our main theorem in this section we need some notations. We let $(\mathcal{M}, \omega)$ be a 2–dimensional symplectic manifold and $\mathcal{S} : \mathcal{M} \to \mathcal{M}$ an orientation preserving diffeomorphism. Since ω is a volume form, there is a smooth function $j : \mathcal{M} \to \mathbf{R}^+$ such that

$$\mathcal{S}^*\omega = j\omega.$$

The function j is called the Jacobian of the diffeomorphism $\mathcal{S}$. Let $\mathcal{C}$ be a sector bundle on $\mathcal{M}$. If $\mathcal{S}$ strictly preserves the sector bundle, then we can compute the cross ratio and denote it by ζ_z at any point $z \in \mathcal{M}$. Since the cross ratio ζ_z does not depend on the coordinates, it is well-defined. Now we have following

Theorem 7.1 *Let $(\mathcal{M}, \omega)$ be a 2-dimensional symplectic manifold and $< \cdot\, , \cdot >$ a Riemannian metric on $\mathcal{M}$. Let $\mathcal{S} : \mathcal{M} \to \mathcal{M}$ be an orientation preserving diffeomorphism of $\mathcal{M}$. Suppose that $\mathcal{N} \subset \mathcal{M}$ is a compact subset and $\mathcal{C}$ a continuous sector bundle on $\mathcal{N}$ such that if z and $\mathcal{S}z$ are in $\mathcal{N}$ then $D_z\mathcal{S}$ maps vectors in $\mathcal{C}_z$ strictly inside $\mathcal{C}_{\mathcal{S}z}$. Assume that for any $z, \mathcal{S}z \in \mathcal{N}$ the inequality*

$$j(z)\left(\frac{\sqrt{\zeta_z}+1}{\sqrt{\zeta_z-1}}\right)^2 > 1$$

is satisfied.

Then there exist $c > 0$ and $\alpha > 1$ such that if

$$z, \mathcal{S}z, \cdots, \mathcal{S}^n z \in \mathcal{N}$$

then

$$\|D_z\mathcal{S}^n v\|_{\mathcal{S}^n z} \geq c\alpha^{n-1}\|v\|_z$$

for all $v \in \mathcal{C}_z$.

Proof: Let $z, \mathcal{S}z \in \mathcal{N}$. It is well-known that there is a coordinate system (x, y) on the neighborhood of z such that the 2-form ω on this neighborhood has the form

$$\omega = dx \wedge dy.$$

This is also true that near point $\mathcal{S}(z)$ one has a coordinate system (x', y') such that the 2–form ω on the neighborhood has the form

$$\omega = dx' \wedge dy'.$$

Without loss of generality we may assume that the sectors at points z and $\mathcal{S}z$ are given by

$$\mathcal{C}_z = \{\ w = (\xi, \eta) \in T_z\mathcal{M} \mid \mathcal{Q}_z(w) = \xi\eta \geq 0\ \}$$

and

$$\mathcal{C}_{\mathcal{S}z} = \{\ w' = (\xi', \eta') \in T_{\mathcal{S}z}\mathcal{M} \mid \mathcal{Q}_{\mathcal{S}z}(w) = \xi'\eta' \geq 0\ \},$$

where the symmetric 2–form $\mathcal{Q}$ is induced by the 2–form ω as we did in the previous calculation.

In these coordinate systems the map $\mathcal{S}$ can be expressed by the following:

$$x' = x'(x, y),\ \ y' = y'(x, y),$$

and the derivative $D_z\mathcal{S}$ has the form

$$D_z\mathcal{S} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where

$$a = \frac{\partial x'}{\partial x},\ b = \frac{\partial x'}{\partial y},\ c = \frac{\partial y'}{\partial x} \text{ and } d = \frac{\partial y'}{\partial y}.$$

Since $(\mathcal{S}^*\omega)_z = j(z)\omega_z$, this implies that in the coordinate systems (x, y) and (x', y') we have

$$j(z) = ad - bc.$$

Also since $D_z\mathcal{S}$ maps vectors in $\mathcal{C}_z$ strictly inside $\mathcal{C}_{\mathcal{S}z}$, one can conclude that a, b, c and d are non–zero and they have the same sign. Without loss of generality we may assume that a, b, c and d are all positive. At the same time we can compute the cross ratio function ζ_z in these coordinate systems

$$\zeta_z = \frac{ad}{bc}.$$

Let $w = (\xi, \eta) \in \mathcal{C}_z$. We consider $\mathcal{Q}_{\mathcal{S}z}(D_z\mathcal{S}w)$. In the coordinate systems (x, y) and (x', y') we obtain by Lemma 7.2 that

$$\mathcal{Q}_{\mathcal{S}z}(D_z\mathcal{S}w) \geq j(z)\left(\frac{\sqrt{\zeta_z}+1}{\sqrt{\zeta_z}-1}\right)^2 \mathcal{Q}_z(w).$$

Let

$$\mathcal{N}' = \mathcal{S}^{-1}\mathcal{N} \cap \mathcal{N}.$$

It is clear that the set $\mathcal{N}'$ is also a compact subset of $\mathcal{M}$. Since sector bundle $\mathcal{C}$ is continuous, the cross ratio function ζ_z is continuous on $\mathcal{N}'$. Using the condition of

$$j(z)\left(\frac{\sqrt{\zeta_z}+1}{\sqrt{\zeta_z}-1}\right)^2 > 1$$

for $z \in \mathcal{N}'$ one can conclude that there is $\alpha > 1$ such that

$$j(z)\left(\frac{\sqrt{\zeta_z}+1}{\sqrt{\zeta_z}-1}\right)^2 \geq \alpha^2$$

for all $z \in \mathcal{N}'$. This gives us following inequality:

$$\mathcal{Q}_{\mathcal{S}z}(D_z\mathcal{S}w) \geq \alpha^2 \mathcal{Q}_z(w)$$

for all $z \in \mathcal{N}'$ and $w \in \mathcal{C}_z$.

For $z \in \mathcal{N}'$ we let

$$m_z = \inf_{w \in \mathcal{C}_z,\ \|w\|_z = 1} \mathcal{Q}_{\mathcal{S}z}(D_z \mathcal{S} w)$$

and

$$M_z = \sup_{w \in \mathcal{C}_z,\ \|w\|_z = 1} \mathcal{Q}_z(w).$$

Since the sector bundle $\mathcal{C}$ is continuous, it is clear that m_z and M_z are continuous on $\mathcal{N}'$ with respect to z. From the fact that the $\mathcal{S}$ preserves the sector bundle $\mathcal{C}$ one can conclude that there are two positive numbers M and m such that for any $z \in \mathcal{N}'$ inequalities

$$m_x \geq m \text{ and } M_z \leq M.$$

are satisfied. If

$$z, \mathcal{S}z, \cdots, \mathcal{S}^n z \in \mathcal{N},$$

for any $w \in \mathcal{C}_z$ we have

$$M\|D_z \mathcal{S}^n w\|^2_{\mathcal{S}^n z} \geq \mathcal{Q}_{\mathcal{S}^n z}(D_z \mathcal{S}^n w) \geq \alpha^2 \mathcal{Q}_{\mathcal{S}^{n-1} z}(D_z \mathcal{S}^{n-1} w)$$

$$\geq \cdots \geq \alpha^{2(n-1)} \mathcal{Q}_{\mathcal{S}z}(D_z \mathcal{S} w) \geq m\alpha^{2(n-1)} \|w\|^2_z.$$

We can complete the proof of this theorem by letting $c = \sqrt{\frac{m}{M}}$. Q. E. D.

Now we consider a special case. Let

$$\mathcal{S} : \mathcal{M} \to \mathcal{M}$$

be a symplectic diffeomorphism. Then we have following

Corollary 7.1. *Let* $(\mathcal{M}, \omega)$ *be a 2–dimensional symplectic manifold and* $< \cdot, \cdot >$ *a Riemannian metric on* $\mathcal{M}$. *Let* $\mathcal{S} : \mathcal{M} \to \mathcal{M}$ *be a symplectic diffeomorphism. Suppose that* $\mathcal{N} \subset \mathcal{M}$ *is a compact subset and* $\mathcal{C}$ *a continuous sector bundle on* $\mathcal{N}$ *such that if* $z, \mathcal{S}z \in \mathcal{N}$ *then* $D_z\mathcal{S}$ *maps* $\mathcal{C}_z$ *strictly inside* $\mathcal{C}_{\mathcal{S}z}$. *Then there exist* $c > 0$ *and* $\alpha > 1$ *such that if*

$$z, \mathcal{S}(z), \cdots, \mathcal{S}^n(z) \in \mathcal{N}$$

then

$$\|D_z\mathcal{S}^n v\|_{\mathcal{S}^n z} \geq c\alpha^{n-1}\|v\|_z.$$

for any $v \in \mathcal{C}_z$.

Proof: Since $\mathcal{S}$ is symplectic, then

$$\mathcal{S}^*\omega = \omega.$$

This implies that the Jacobian $j(z) = 1$ for all $z \in \mathcal{M}$. Also the map S strictly preserves the sector bundle $\mathcal{C}$. This gives us that for $z, \mathcal{S}z \in \mathcal{N}$

$$j(z)\left(\frac{\sqrt{\zeta_z}+1}{\sqrt{\zeta_z}-1}\right)^2 > 1$$

since the cross ratio $\zeta_z > 1$. By Theorem 7.1 we obtain our result. Q. E. D.

§8. Convergence of Horizontal Strips

From §6 we know that for every horizontal strip

$$A_{d_1} \cap FA_{d_2} \subset A_{d_1}$$

and every vertical strip

$$B_{d_0} \cap A_{d_1}$$

DF strictly preserves C^+ on

$$(B_{d_0} \cap A_{d_1}) \cap (A_{d_1} \cap FA_{d_2}) = B_{d_0} \cap A_{d_1} \cap FA_{d_2}.$$

Therefore by reversibility DF^{-1} strictly preserves the complement of C^+ on this intersection since

$$T(B_{d_0} \cap A_{d_1} \cap FA_{d_2}) = TFA_{d_0} \cap TA_{d_1} \cap TFA_{d_2}$$

$$= F^{-1}(TA_{d_0}) \cap TA_{d_1} \cap F(TA_{d_2}).$$

The derivative DF has the form

$$DF = \begin{pmatrix} 1 - kcos2\pi x & 1 \\ -kcos2\pi x & 1 \end{pmatrix}.$$

This implies that there are only finite different

$$B_{d_0} \cap A_{d_1} \cap FA_{d_2}$$

on the cylinder for DF. On the other hand, C^+ is given by

$$C^+ = \{(\xi, \eta) \mid Q_1((\xi, \eta)) \geq 0 \}$$

where

$$Q_1((\xi,\eta)) = (2\xi - \eta)\eta.$$

Hence by compactness we may assume that there is $\alpha > 1$ such that, for any z in those finite different intersections $B_{d_0} \cap A_{d_1} \cap FA_{d_2}$,

$$Q_1(DF_z(\xi,\eta)) \geq \alpha Q_1((\xi,\eta)).$$

By reversibility one can make a similar assumption for DF^{-1}.

For convenience we need another assumption as follows:

For any horizontal strip U and any vertical strip V in A_d the inequality

$$Q_1(DF_z(\xi,\eta)) \geq \alpha Q_1((\xi,\eta))$$

holds for all $z \in U \cap V$.

Similar assumption for DF^{-1} is also needed.

To show the convergence of the sequence of horizontal strips in §**6** we define the diameter of horizontal strip. Let

$$U = \{(x,y) \mid h_1(x) \leq y \leq h_2(x),\ x \in I\ \}$$

be a horizontal strip where $h_1(x) < h_2(x),\ x \in I$. The diameter of U is defined by

$$d(U) = \max_{x \in I}(h_2(x) - h_1(x)).$$

Let $U_i \subset A_{d_i}$, $i = 1, \cdots, n$, be horizontal strips satisfying

$$\Pi(d_i, d_{i+1}) = 1,\ i = 1, 2, \cdots, n-1.$$

We are going to show that there are positive integer n and real $\beta < 1$ such that

$$d(U_1 \cap FU_2 \cap \cdots \cap F^{n-1}U_n) < \beta d(U_n).$$

Suppose that

$$U_1 \cap FU_2 \cap \cdots \cap F^{n-1}U_n$$

is bounded by two horizontal curves v_1, v_2 and

$$d(U_1 \cap FU_2 \cap \cdots \cap F^{n-1}U_n) = |p_1 - p_2|$$

where $p_i \in v_i$, $i = 1, 2$ and p_1p_2 is a vertical segment (see Figure 8.1).

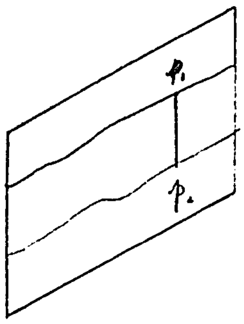

Figure 8.1

Let

$$p(t) = (1-t)p_1 + tp_2, \quad 0 \leq t \leq 1.$$

It is clear that

$$F^{-(n-1)}p(t) = z(t) = (x(t), y(t))$$

is a vertical curve in

$$U_n \cap F^{-(n-1)}U_1 \cap F^{-(n-2)}U_2 \cap \cdots \cap F^{-1}U_{n-1}$$

and $z(0)$ $z(1)$ are the two endpoints of the vertical curve $z(t)$ and they are on the horizontal boundary of U_n. We also have

$$z'(t) = (x'(t), y'(t)) = DF^{-(n-1)}p'(t).$$

Since $z'(t)$ and $p'(t)$ are in the complement of C^+, by the discussion in §7 we have

$$\|z'(t)\| = \|DF^{-(n-1)}p'(t)\| \geq \alpha \|p'(t)\|$$

where $\alpha > 1$ and n large enough. Since $z'(t) = (x'(t), y(t))$ is strictly inside the complement of C^+, there is $\delta > 0$ such that

$$\frac{y'(t)}{x'(t)} \geq 2 + \delta \quad \text{or} \quad -\infty < \frac{y'(t)}{x'(t)} \leq -\delta.$$

This implies that there is constant a such that

$$\|z'(t)\| \leq a|y'(t)|.$$

Therefore we have

$$\|p'(t)\| \leq \frac{a}{\alpha}|y'(t)|.$$

We may assume that $\frac{a}{\alpha} < 1$ (Otherwise we use kn instead of n , then $\frac{a}{\alpha^k} < 1$). Let $\beta = \frac{a}{\alpha}$. This gives us

$$d(U_1 \cap FU_2 \cap \cdots \cap F^{n-1}U_n) = |p_1 - p_2| = \int_0^1 \|p'(t)|dt$$

$$\leq \beta \int_0^1 |y'(t)|dt = \beta |y(1) - y(0)| \leq \beta d(U_n).$$

This estimation implies that the sequence of horizontal strips

$$\{U_{d_1,\cdots,d_n}\}_{n=1}^{\infty}$$

in §**6** converges to a horizontal curve v. Again by the strict preservation of C^+ there is $\delta' > 0$ such that for any $(x_1, y_1)\ (x_2, y_2) \in v$

$$\delta' \leq \frac{y_1 - y_2}{x_1 - x_2} \leq 2 - \delta'.$$

This important fact guarantees that the intersection

$$\bigcap_{n=-\infty}^{\infty} F^{-n} A_{d_n}$$

contains exactly one point since the vertical curve

$$\bigcap_{n=0}^{\infty} F^{-n} A_{d_n}$$

and the horizontal curve

$$\bigcap_{n=0}^{-\infty} F^{-n} A_{d_n}$$

intersect transversely.

§9. The Homeomorphism Property of the Mapping τ and More

From the previous sections it is clear that the mapping τ is well-defined. In this section we are going to show that this mapping τ is a homeomorphism between Σ_b and its image. Let us define the image by Λ_k. It is easy to show that for different s and s' in Σ_b, $\tau(s)$ and $\tau(s')$ are different. Therefore the mapping τ is bijective. We need to show that τ and τ^{-1} are continuous. The topology on the space Σ_b was specified in §4. To show τ is continuous we let $s \in \Sigma_b$ and $z = \tau(s)$. We want to show that τ is continuous at s. For any $\epsilon > 0$ there is a ball $B_\epsilon(z)$ centered at z with radius ϵ. We let $s = \{s_n\}$. Then by the discussion of last section there is $N > 0$ such that

$$\bigcap_{n=-N}^{N} F^{-n} A_{s_n} \subset B_\epsilon(z).$$

It is clear that the set

$$U = \{s' \in \Sigma_b \mid s'_{-N} = s_{-N}, s'_{-N+1} = s_{-N+1}, \cdots, s'_{N-1} = s_{N-1}, s'_N = s_N\}$$

is an open neighborhood of s and

$$\tau(U) \subset \bigcap_{n=-N}^{N} F^{-n} A_{s_n} \subset B_\epsilon(z).$$

This shows that τ is continuous at s. Therefore τ is continuous on Σ_b. To show the continuity of τ^{-1} let us consider τ^{-1} at z. Since $\tau(s) = z$, we have

$$z \in \bigcap_{n=-N}^{N} F^{-n} A_{s_n}.$$

Note that the horizontal boundaries of $A_{d_1} \cap FA_{d_2}$ are inside A_{d_1}. By induction we can conclude that the horizontal boundaries of

$$A_{s_0} \cap FA_{s_{-1}} \cap \cdots \cap F^N A_{s_{-N}}$$

are inside

$$A_{s_0} \cap FA_{s_{-1}} \cap \cdots \cap F^{N-1} A_{s_{-N+1}}.$$

This is also true for vertical strips. These facts imply that z is a interior point of

$$\bigcap_{n=-N}^{N} F^{-n} A_{s_n}$$

for any N. We can conclude that the interior of above set is a neighborhood of z. Then from

$$\tau(U) = \bigcap_{n=-N}^{N} F^{-n} A_{s_n} \bigcap \Lambda_k$$

one can derive that τ^{-1} is continuous at z, hence it is continuous on Λ_k.

The commuting diagram is satisfied automatically. This completes our proof of Theorem 4.1.

To find the connection between the Aubry–Mather sets and the horseshoes of the standard map we need to reduce the the lower bound of the parameter further. But if we let $b = 0$ it is not easy to prove above theorem. Fortunately we can see later that when $b = 0$ the Aubry–Mather sets are completely restricted inside the strip

$$[\frac{1}{4}, \frac{3}{4}] \times \mathbf{R}.$$

Therefore it is necessary to look for the horseshoes inside above strip. We consider the standard map on the vertical strip

$$S_1 = [\frac{1}{4}, \frac{3}{4}] \times \mathbf{R}$$

only, we may let

$$A_n = f(S_{2n-1}) \cap S_1,$$

and Z be the alphabet set in which each element n represents A_n. For any non–negative b, we define the transition matrix as follows:

$$\Pi^* : Z \times Z \to \{0, 1\}$$

$\Pi^*(m, n) = 1$ if $|m - n| \le \frac{b+2}{2}$ and $\Pi^*(m, n) = 0$ otherwise.

Let Σ_b^* be the subshift of finite type defined by this transition matrix and σ the left shift on Σ_b^*, then we have

Proposition 9.1. *Let b be a non–negative integer. for all $k > (2 + b)\pi$ and $s = \{m_n\} \in \Sigma_b^*$, the intersection*

$$\bigcap_{n=-\infty}^{\infty} F^{-n} A_{m_n}$$

contains exactly one point. Moreover, the mapping

$$\tau^* : \Sigma_b^* \to \mathbf{S}^1 \times \mathbf{R}$$

defined by

$$\tau^*(\{m_n\}) = \bigcap_{n=-\infty}^{\infty} F^{-n}(A_{m_n})$$

is a homeomorphism from Σ_b^* onto its image Λ_k^* (Σ_b^* is equipped with the product topology), and the following diagram commutes.

$$\begin{array}{ccc} \Sigma_b^* & \xrightarrow{\sigma} & \Sigma_b^* \\ \Big\downarrow \tau^* & & \Big\downarrow \tau^* \\ \Lambda_k^* & \xrightarrow{F} & \Lambda_b^* \end{array}$$

The proof of this Proposition is much simpler than the proof of Theorem 4.1 and we leave it to the interested readers.

§10. Chaotic Trajectories of the Standard Map

Aubry and Abramovici [AuAb] give a result of chaotic trajectories of the standard map, which uses the functional analytic method to prove that the existence of those orbits is related to some symbol space. We formulate their results as follows:

Theorem 10.1.(Aubry, Abramovici [AuAb]) *Let b be a positive integer, m_n any sequence of integers satisfying*

$$|m_{n+1} + m_{n-1} - 2m_n| \leq b$$

for all n. Then for

$$k > \sqrt{16 + 4\pi^2(1 + \frac{1}{2}b)^2}$$

there is a unique sequence $\{x_n(k)\}_{n=-\infty}^{\infty}$, which is a critical point of the functional W, such that for all n

$$|x_n(k) - \frac{m_n}{2}| \leq \frac{1}{4}.$$

In addition, this solution depends continuously on the parameter k and we have

$$\lim_{k\to\infty} x_n(k) = \frac{m_n}{2}$$

for all n.
In particular, when the sequence of integer $\{\sigma_n\} = \{m_n - m_{n-1}\}$ is periodic with period s ($\sigma_{n+s} = \sigma_n$), the corresponding trajectory $(x_n \text{ mod } 1\ ,\ p_n = x_n - x_{n-1})$ is represented in the standard mapping as a periodic cycle with the same period s.

Geometrically, it is clear that they are looking for the trajectory (x_n, y_n) of f such that

$$(x_n, y_n) \in S_{m_n}.$$

Consider m_{n+1}, m_n and m_{n-1}, then

$$f(S_{m_{n-1}}) \cap S_{m_n} \quad \text{and} \quad f(S_{m_n}) \cap S_{m_{n+1}}$$

correspond to some $A_{n,i}$'s in our case and

$$f^{-1}(f(S_{m_n}) \cap S_{m_{n+1}}) \subset S_{m_n}$$

corresponds to some $B_{n,i}$'s, therefore if

$$|m_{n+1} + m_{n-1} - 2m_n| \leq b,$$

then $(x_{n-1}, y_{n-1}), (x_n, y_n)$ and (x_{n+1}, y_{n+1}) are the segment of an orbit if and only if

$$f^{-1}(f(S_{m_n}) \cap S_{m_{n+1}}) \quad \text{and} \quad f(S_{m_{n-1}}) \cap S_{m_n}$$

cross each other properly. Also it is easy to see that for any integer sequence $\{m_n\}$ in Theorem 10.1, it corresponds to the sequence $s = \{(n_j, i_j)\}$ in Theorem 4.1 such that

$$n_j = m_{j-1} - m_j, \quad i_j = 0$$

if m_j is even and

$$n_j = m_{j-1} - m_j + 1, \quad i_j = 1$$

if m_j is odd.

It is obvious that our geometric approach not only reduces the lower bound of the parameter, increases the invariant set , it also gives a more clear picture of those chaotic orbits. The relation between two different symbol spaces will also help us to study the Aubry–Mather sets of this standard mapping.

§11. The Aubry–Mather Sets and Their Properties

In this section we are going to give a brief description the Aubry– Mather sets of a measure–preserving twist map and their properties, which one can find in [Bang].

Let $F : S^1 \times \mathbf{R} \to S^1 \times \mathbf{R}$ be an area– preserving twist map and $f : \mathbf{R}^2 \to \mathbf{R}^2$ its lift. Let $h(x, x')$ be the generating function of f. Suppose $\{(x_n, y_n)\}_{n=-\infty}^{\infty}$ is an orbit of f which satisfies that for any segment

$$(x_n, x_{n+1}, \cdots, x_{n+m})$$

in $\{x_n\}$ and any finite sequence $(x'_n, \cdots x'_{n+m})$ with $x_n = x'_n$ and $x_{n+m} = x'_{n+m}$,

$$\sum_{i=n}^{n+m-1} h(x_i, x_{i+1}) \leq \sum_{i=n}^{n+m-1} h(x'_i, x'_{i+1}),$$

we say that this orbit belongs to Aubry–Mather sets which we denote by M. The most important property of an orbit in M is that it is ordered, which is characterized by the following:

Proposition 11.1.(Bangert [Bang]) *Let $\{(x_n, y_n)\}_{n=-\infty}^{\infty}$ be an orbit of f in M, then there exists a continuous map $p : R \to R$ such that*

(1) p is strictly increasing and $p(x+1) = p(x) + 1$.

(2) $p(x_n) = x_{n+1}$.

This Proposition assures that for any orbit in M there is a circle map associated with it. On the other hand, for any circle map, one can define the rotation number of that map, therefore for any orbit in Proposition 11.1, we can define the rotation number which is given by

$$\alpha = \lim_{n \to \infty} \frac{x_n - x_0}{n}.$$

Let us denote by $M(\alpha)$ the subset of M consisting of all the orbits of rotation number α. For different α, $M(\alpha)$ has some special properties:

Proposition 11.2. (Bangert [Bang]) *If $\alpha = \frac{p}{q}$ is rational , then $M(\frac{p}{q})$ has following two kinds of orbits:*

(1) The configuration $\{x_n\}$ is periodic with the period (p, q), i.e.,

$$x_{n+q} = x_n + p.$$

(2) If the configuration $\{x_n\}$ is not periodic in the above sense, then there exist two minimal periodic configuration $\{y_n\}$ and $\{z_n\}$ such that

$$y_n < x_n < z_n$$

for all n and

$$\lim_{n \to -\infty} (x_n - y_n) = 0 \quad \text{and} \quad \lim_{n \to \infty} (z_n - x_n) = 0.$$

Proposition 11.3 (Bangert [Bang]) *If α is irrational, then the set $M(\alpha)$ consists of two kinds of trajectories:*

(1) The configuration $\{x_n\}$ can be approximated by a sequence of configurations $\{x_n^{(j)}\}$ such that the rotation number of $\{x_n^{(j)}\}$ for each j is rational $\frac{p_j}{q_j}$ and

$$\lim_{j \to \infty} \frac{p_j}{q_j} = \alpha.$$

(2) For the $\{x_n\}$ which can not be approximated by those with rational rotation numbers, there exist two orbits in $M(\alpha)$ with the configurations $\{t_n\}$ and $\{z_n\}$, which can be approximated by those with rational rotation numbers, satisfying

$$y_n < x_n < z_n$$

for all n, and

$$\lim_{n \to -\infty} (z_n - y_n) = \lim_{n \to \infty} (z_n - y_n) = 0.$$

These properties will directly help us to find the relation between the Aubry–Mather sets and the hyperbolic invariant set (horseshoes) of the standard mapping.

§12. Hyperbolic Aubry–Mather Sets and the Horseshoes

Let us denote by M_k the Aubry–Mather sets of the standard mapping with parameter k.

Goroff [Goro] proves that for $k > 2\sqrt{1+\pi^2}$, the Aubry–Mather sets of the standard mapping are uniformly hyperbolic. In particular, there is $\delta > 0$ such that if $(x, y) \in M_k$, then

$$-k cos 2\pi x > \delta.$$

In this section, we are going to show that for $k > 2\pi$,

$$M_k \subset \Lambda_k^*,$$

where Λ_k^* is given in a previous section. Let $\{(x_n, y_n)\}_{n=-\infty}^{\infty} \in M_k$ for $k > 2\pi$, then there exists a circle homeomorphism $p : R \to R$ such that

$$p^n(x_0) = x_n \quad n \in Z.$$

Therefore, we have

$$|x_{n+1} + x_{n-1} - 2x_n| \leq 1.$$

If one can show that $cos 2\pi x_n < 0$ for all n, then there are S_{m_n}, where m_n are odd, such that

$$(x_n, y_n) \in S_{m_n} \quad n \in Z,$$

so we have

$$|m_{n+1} + m_{n+1} - 2m_n| \leq 1 \quad n \in Z.$$

Using the relation between the result in [AuAb] and our horseshoes, we conclude that

$$M_k \subset \Lambda_k^*.$$

This assertion can be proved by considering following two cases: rational rotation number and irrational rotation number.
Suppose that $p, q \in Z$, $q \neq 0$ and $(p,q) = 1$. Consider those orbits in M_k whose rotation number is $\frac{p}{q}$. If $\{x_n\}$ is the configuration of such an orbit and it is periodic, i.e.,

$$x_{n+q} = x_n + p,$$

then $(x_0, x_1, \cdots, x_{q-1})$ is a global minimal point of the function

$$H(x_0, x_1, \cdots, x_{q-1}) = \sum_{i=0}^{q-1} h(x_i, x_{i+1})$$

where $x_q = x_0 + p$.

Proposition 12.1. *Suppose $(x_0, x_1, \cdots, x_{q-1})$ is a global minimum of H, then there is an i where $0 \leq i \leq q-1$, such that*

$$cos 2\pi x_i \leq 0.$$

Proof: Suppose for all $i = 0, 1, \cdots, q-1$, $cos 2\pi x_i > 0$.
We consider $(y_0, y_1, \cdots, y_{q-1})$ where

$$y_i = x_i - \frac{1}{2}.$$

Then one obtains $cos 2\pi y_i = -cos 2\pi x_i < 0$, and

$$H(y_0, \cdots, y_{q-1}) = \sum_{i=0}^{q-1} (\frac{1}{2}(x_i - x_{i+1})^2 + \frac{k}{(2\pi)^2} cos 2\pi y_i)$$

$$= \sum_{i=0}^{q-1} (\frac{1}{2}(x_i - x_{i+1})^2 - \frac{k}{(2\pi)^2} cos 2\pi x_i)$$

$$< \sum_{i=0}^{q-1} (\frac{1}{2}(x_i - x_{i+1})^2 + \frac{k}{(2\pi)^2} cos 2\pi x_i) = H(x_0, x_1, \cdots, x_{q-1}).$$

This contradicts the fact that $(x_0, \cdots, x_{q-1})$ is a global minimum of H. Therefore there is an i such that

$$cos 2\pi x_i \leq 0.$$

Q.E.D.

We know that if $(x_0, \cdots, x_{q-1})$ is a global minimal of H, then after a periodic extension, we obtain a minimal configuration of an orbit in $M_k(\frac{p}{q})$ with the rotation number $\frac{p}{q}$. By coosing appropriate lift, we can assume that

$$0 < \frac{p}{q} \leq 1,$$

then we have $0 < x_n - x_{n-1} \leq 1$ for all n. Since there is a circle homeomorphism p such that $p(x_n) = x_{n+1}$, we can conclude

Proposition 12.2. *Suppose $(x_0, \cdots, x_{q-1})$ is a global minimal of H and $k \geq 2\pi$. Then for those x_i such that $cos 2\pi x_i \leq 0$, we have*

$$cos 2\pi x_i < 0.$$

Proof: Suppose there is x_i such that $cos2\pi x_i = 0$. Since $(x_0, \cdots, x_{q-1})$ is a minimal, therefore

$$\frac{\partial H}{\partial x_i} = 2x_i - x_{i-1} - x_{i+1} - \frac{k}{2\pi} sin2\pi x_i = 0,$$

where $sin2\pi x_i = \pm 1$. Let $sin2\pi x_i = 1$, then we have

$$(x_i - x_{i-1}) - (x_{i+1} - x_i) = \frac{k}{2\pi} \geq 1.$$

Because of $x_{i+1} - x_i > 0$, then we obtain $x_i - x_{i-1} > 1$, and this is a contradiction. Q.E.D.

Next result gives us the location of the orbit in M_k by checking one point on the orbit.

Proposition 12.3. *Let $k \geq 2\pi$ and $\{x_n\}$ be the configuration of an orbit in M_k with rotation number $0 < \alpha \leq 1$. Suppose that there is an i such that*

$$cos2\pi x_i < 0,$$

then for any n,

$$cos2\pi x_n < 0.$$

Proof: Without loss of generality we assume

$$cos2\pi x_i \geq 0 \quad \text{and} \quad cos2\pi x_{i-1} < 0.$$

By the minimality property, the segment (x_{i-1}, x_i, x_{i+1}) is a minimal segment so we have

$$2x_i - x_{i+1} - x_{i-1} = \frac{k}{2\pi} sin 2\pi x_i$$

and

$$2 - k cos 2\pi x_i \geq 0.$$

Assume $x_{i-1} \in (-\frac{3}{4}, -\frac{1}{4})$, since

$$cos 2\pi x_i \leq \frac{2}{k},$$

so we obtain

$$|sin 2\pi x_i| \geq \sqrt{1 - \frac{4}{k^2}}.$$

Suppose $sin 2\pi x_i > 0$, then

$$\frac{k}{2\pi} sin 2\pi x_i \geq \sqrt{1 - \frac{1}{\pi^2}} > 0.9,$$

so we have

$$1 > x_i - x_{i-1} > 0.9$$

and

$$x_i \in (-\frac{1}{4}, \frac{1}{4}).$$

This fact indicates that x_{i-1} is very close to $-\frac{3}{4}$, so we can conclude that

$$sin 2\pi x_{i-1} > \frac{1}{2}.$$

consider the segment (x_{i-2}, x_{i-1}, x_i), we also obtain

$$(x_{i-1} - x_{i-2}) - (x_i - x_{i-1}) = \frac{k}{2\pi} sin 2\pi x_{i-1},$$

this gives us $x_{i-1} - x_{i-2} > 1$ and this is a contradiction.

For $sin2\pi x_i < 0$, then $x_i \in (-\frac{1}{4}, 1)$, therefore

$$1 \geq (x_{i+1} - x_i) \geq (x_i - x_{i-1}) + 0.9.$$

This indicates that $x_{i-1} \in (-\frac{1}{2}, -\frac{1}{4})$ and x_{i-1} is very close to x_i. We can get an estimate for $\frac{k}{2\pi} sin2\pi x_{i-1}$ as follows:

$$\frac{k}{2\pi} sin2\pi x_{i-1} \leq \frac{k}{2\pi} sin2\pi(-\frac{1}{4} - 0.1) = -\frac{k}{2\pi} cos\frac{\pi}{5} \leq -\frac{1}{2}.$$

Agian consider the segment (x_{i-2}, x_{i-1}, x_i), we have

$$(x_{i-1} - x_{i-2}) - (x_i - x_{i-1}) = \frac{k}{2\pi} sin2\pi x_{i-1},$$

so we obtain

$$(x_i - x_{i-1}) \geq \frac{1}{2} + (x_{i-1} - x_{i-2}) \geq \frac{1}{2}.$$

This contradicts to the fact $(x_i - x_{i-1}) \leq 0.1$.

Using this idea and induction, we can prove that for all n, $cos2\pi x_n < 0$. Q.E.D.

Based on these Propositions and the properties in the previous section, we can prove following several corollaries:

Corollary 12.1. *If $\{x_n\}$ is periodic with the rational rotation number, then for all n, $cos2\pi x_n < 0$.*

Proof: From the Proposition 12.1 and 12.2, there is i such that

$$cos2\pi x_i < 0.$$

From the Proposition 12.3, $cos2\pi x_n < 0$ for all n. Q.E.D.

Corollary 12.2. *For the non-periodic $\{x_n\}$ with rational rotation number, we also have $cos 2\pi x_n < 0$ for all n.*

Proof: Since $\{x_n\}$ is non-periodic with rational rotation number, by Proposition 11.2, there is a periodic configuration $\{z_n\}$ with the same rotation number such that $x_n < z_n$ and

$$\lim_{n\to\infty}(z_n - x_n) = 0.$$

Since $\{z_n\}$ is periodic, then there is $\alpha > 0$ such that

$$cos 2\pi z_n \leq -\alpha$$

for all n. Therefore, there is n_0 such that $cos 2\pi x_{n_0} < 0$. Using Proposition 12.3, we have $cos 2\pi x_n < 0$ for all n. Q.E.D.

Corollary 12.3. *For the $\{x_n\}$ with irrational rotation number, we have*

$$cos 2\pi x_n < 0.$$

Proof: Let α be the rotation number of $\{x_n\}$.

If $\{x_n\}$ can be approximated by a sequence of configurations $\{x_n^{(j)}\}$ such that the rotation number of these configurations are $\frac{p_j}{q_j}$ for each j and

$$\lim_{j\to\infty}\frac{p_j}{q_j} = \alpha,$$

then

$$\lim_{j\to\infty} x_n^{(j)} = x_n$$

for all n. Since $\cos 2\pi x_n^{(j)} < 0$ for all n and j, we have $cos 2\pi x_n \leq 0$ for all n. Applying the idea in the proof of Proposition 12.2, we have

$$cos 2\pi x_n < 0$$

for all n.

If $\{x_n\}$ can not be approximated by those with rational rotation numbers, then by Proposition 11.3 (2), there exist two orbits in $M_k(\alpha)$ with the configurations $\{y_n\}$ and $\{z_n\}$, which can be approximated by those with rational rotation numbers, satisfying

$$y_n < x_n < z_n$$

for all n and

$$\lim_{n \to -\infty} (z_n - y_n) = 0 \quad \text{and} \quad \lim_{n \to \infty} (z_n - y_n) = 0.$$

It is easy to see that there is n_0 such that

$$cos 2\pi x_{n_0} < 0$$

since

$$cos 2\pi y_n < 0 \quad \text{and} \quad cos 2\pi z_n < 0$$

for all n. Q.E.D.

Using the explanation at the beginning of this section, we can conclude that

$$M_k \subset \Lambda_k^*$$

if $k > 2\pi$.

Note that since Λ_k^* has a uniform hyperbolic structure therefore the Aubry–Mather set M_k is uniformly hyperbolic. This improve the Goroff's result about uniformly hyperbolic Aubry–Mather set of the standard map. Also using the condition of $k > 2\pi$ one can show by the same method which Goroff used in [Goro] that the Aubry–Mather set is uniformly hyperbolic.

§13. The Symbolic Description of the Horseshoes on T^2

In Theorem 4.1 we give a symbolic description of horseshoes of the standard map on the cylinder. In this section we want to project the horseshoes into the the torus T^2 and find the symbolic description of the the image of this projection. This information will be used to estimate the Hausdorff dimension of the horseshoes in the next section.

For a fixed positive integer b we let

$$D' = \{(n,i,j) \mid i,j = 0,1. \text{ and } n = 0, \pm 1, \cdots, \pm(b + 2|i-j|)\}$$

be the alphabet set. Geometrically, we use $i = j = 0$ to represent $A_{2l,0}$, $i = 0, j = 1$ to represent $A_{2l-1,0}$, $i = 1, j = 0$ to represent $A_{2l,1}$ and $i = j = 1$ to represent $A_{2l-1,1}$.

We also define a mapping $R : \Sigma_b \to D'^Z$ by the formula

$$R(\{(n_m, l_m)\}) = \{(a_m, i_m, j_m)\}$$

where

$$a_m = n_m - n_{m-1},\ i_m = l_m \text{ and } j_m = n_m\ mod 2.$$

We let

$$\Sigma_b' = R(\Sigma_b).$$

Let σ' be the left shift on D'^Z then we have following

Lemma 13.1 *The mapping $R : \Sigma_b \to \Sigma_b'$ is a covering mapping and the following diagram commutes*

$$\begin{array}{ccc} \Sigma_b & \xrightarrow{\sigma} & \Sigma_b \\ \downarrow R & & \downarrow R \\ \Sigma'_b & \xrightarrow{\sigma'} & \Sigma'_b \end{array}.$$

Proof: By the definition it is clear that R is continuous. It is also easy to see that above diagram commutes.

Let N be a open set in Σ'_b given by

$$N = \{ \ \{(a_m, i_m, l_m)\} \in \Sigma'_b \mid (a_m, i_m, l_m) = (a^*_m, i^*_m, l^*_m), m = p, p+1, \cdots, q \ \}.$$

Then

$$R^{-1}(N) = \bigcup_{k=-\infty}^{\infty} N_k$$

where

$$N_k = \{ \ \{(n_m, i_m)\} \in \Sigma_b \mid (n_{p-1}, i_{p-1}) = (2k + l^*_p - a^*_p, (i^*_p + l^*_p)(mod\ 2)),$$

$$(n_p, i_p) = (2k + l^*_p, i^*_p) \text{ and } (n_m, i_m) = (n_{m-1} + a^*_m, i^*_m) \ m = p+1, p+2, \cdots, q \ \}.$$

Clearly N_k is an open set in Σ_b and

$$N_k \cap N_{k'} = \emptyset$$

if $k \neq k'$. This shows that R is continuous. To show that

$$R : N_k \to N$$

is a homeomorphism for each k one should notice that $R|_{N_k}$ is bijective and continuous. The continuity of $(R|_{N_k})^{-1}$ can be seen from above formula for $(R|_{N_k})^{-1}$. This proves that $R : \Sigma_b \to \Sigma'_b$ is a covering map. Q. E. D.

Let us denote the standard map on the T^2 by $\mathcal{F}$. Then by definition we have the following commuting diagram

$$\begin{array}{ccc} \mathbf{S}^1 \times \mathbf{R} & \xrightarrow{F} & \mathbf{S}^1 \times \mathbf{R} \\ \downarrow \pi & & \downarrow \pi \\ T^2 & \xrightarrow{\mathcal{F}} & T^2 \end{array}$$

where π is the natural projection form the cylinder onto the torus. Next Proposition tells us that there is a continuous mapping from Σ'_b into the torus T^2 which can be reduced from the homeomorphism τ in Theorem 4.1.

Proposition 13.1 *There is a continuous mapping $\tau' : \Sigma'_b \to T^2$ such that following diagram commutes.*

$$\begin{array}{ccc} \Sigma_b & \xrightarrow{R} & \Sigma'_b \\ \downarrow \tau & & \downarrow \tau' \\ \mathbf{S}^1 \times \mathbf{R} & \xrightarrow{\pi} & T^2 \end{array}$$

where π is the natural projection. Moreover if

$$\tau'(\Sigma'_b) = \Lambda'_k,$$

then τ' is a homeomorphism from Σ'_b onto Λ'_k.

To Prove this Proposition we need following

Lemma 13.2 *Let $(\tilde{X}_i,\ p_i)$ be a covering space of X_i, $i = 1, 2$. Suppose $h : \tilde{X}_1 \to \tilde{X}_2$ is a homeomorphism and $h' : X_1 \to X_2$ for which the following diagram commutes*

$$\begin{array}{ccc} \tilde{X}_1 & \xrightarrow{p_1} & X_1 \\ \Big\downarrow h & & \Big\downarrow h' . \\ \tilde{X}_2 & \xrightarrow{p_2} & X_2 \end{array}$$

Then h' is continuous. Moreover $h' : X_1 \to X_2$ is a homeomorphism if and only if h' is bijective.

Proof: It is clear that the map h' is surjective. The second assertion can be proven if we can show the first one. Let

$$U(x_1)$$

be the admissible open set of $x_1 \in X_1$. Then

$$p_1^{-1}U(x_1) = \bigcup_{\alpha} U_\alpha$$

where $U_\alpha \subset \tilde{X}_1$ are disjoint open sets in $\tilde{X}_1$ such that for each α

$$p_1|_{U_\alpha} : U_\alpha \to U(x_1)$$

is a homeomorphism. Obviously for a fixed α we have

$$h'|_{U(x_1)} = p_2 \circ h \circ (p_1|_{U_\alpha})^{-1}.$$

This implies that h' is continuous. Q. E. D.

Proof of Proposition 13.1. : For the existence of the mapping τ' we need to show that for any X and $Y \in \Sigma_b$ if $R(X) = R(Y)$ then

$$\pi \circ \tau(X) = \pi \circ \tau(Y).$$

Let

$$X = \{(n_m, l_m)\} \quad \text{and} \quad Y = \{(n'_m, l'_m)\}.$$

Since $R(X) = R(Y)$, we have

$$n_m - n_{m-1} = n'_m - n'_{m-1},\ l_m = l'_m \text{ and } n_m = n'_m \ mod2.$$

This implies that

$$\pi(A_{n_m, l_m}) = \pi(A_{n'_m, l'_m}).$$

Therefore we obtain that

$$\pi \circ \tau(X) = \pi(\bigcap_{m=-\infty}^{\infty} F^{-m} A_{n_m, l_m})$$

$$= \bigcap_{m=-\infty}^{\infty} \mathcal{F}^{-m} \circ \pi(A_{n_m, l_m}) = \bigcap_{m=-\infty}^{\infty} \mathcal{F}^{-m} \circ \pi(A_{n'_m, l'_m})$$

$$= \pi \circ \tau(Y).$$

So τ' is well-defined. On the other hand we can show that the mapping τ' is injective. Suppose $X, Y \in \Sigma_b$ such that $R(X) \neq R(Y)$. We also assume that X, Y have the same form as above. Without loss of generality we may assume that

$$(n_0 - n_{-1}, i_0, n_0(mod\ 2)) \neq (n'_0 - n'_{-1}, i'_0, n'_0(mod\ 2)).$$

If

$$i_0 \neq i'_0 \ \text{ or } \ n_0 \neq n'_0(mod\ 2)$$

then we have

$$\pi A_{n_0, i_0} \cap \pi A_{n'_0, i'_0} = \emptyset.$$

Therefore we can assume that

$$n_0 - n_{-1} \neq n'_0 - n'_{-1}$$

and

$$i_0 = i'_0, n_0 = n'_0 (mod\ 2).$$

This assumption gives us

$$\pi(A_{n_0,i_0} \cap FA_{n_{-1},i_{-1}}) \cap \pi(A_{n'_0,i'_0} \cap FA_{n'_{-1},i'_{-1}}) = \emptyset.$$

So we obtain that τ' is injective. By Lemma 12. 2 we can see that $\tau' : \Sigma'_b \to \Lambda'_k$ is a homeomorphism. Q. E. D.

Using this Proposition we can show that σ' and $\mathcal{F}$ are conjugate.

Theorem 13.1 *The following diagram commutes.*

$$\begin{array}{ccc} \Sigma'_b & \xrightarrow{\sigma'} & \Sigma'_b \\ \downarrow \tau' & & \downarrow \tau' \\ \Lambda'_k & \xrightarrow{\mathcal{F}} & \Lambda'_k \end{array}$$

Proof: We need to show that this diagram commutes. For any $x \in \Sigma'_b$ there is $X \in \Sigma_b$ such that

$$R(X) = x.$$

Using the relation between F and $\mathcal{F}$, Lemma 13.1 and Proposition 13.1 we have

$$\mathcal{F} \circ \tau'(x) = \mathcal{F} \circ \tau' \circ R(X)) = \mathcal{F} \circ \pi\tau(X)$$

$$= \pi \circ F \circ \tau(X) = \pi \circ \tau \circ \sigma(X) = \tau' \circ R\sigma(X) = \tau' \circ \sigma' \circ R(X) = \tau' \circ \sigma'(x).$$

Q. E. D.

Σ_b' is actually a subshift of finite type. In the rest of this section we are going to determine the transition matrix for the subshift of finite type Σ_b'. Let us first consider the case that b is odd. From the definition of the mapping R it is not hard to see that there are b symbols of the form

$$(2n, 0, 0),$$

$b + 1$ symbols of the form

$$(2n + 1, 1, 1),$$

$b + 2$ symbols of the form

$$(2n + 1, 0, 1),$$

and $b + 3$ symbols of the form

$$(2n, 1, 0).$$

Let us call them symbols of type I, II, III and IV respectively. We can easily check that the type I can be followed by type I and II, type II can be followed by type III and IV, type III can be followed by type I and II, and type IV can be followed by type III and IV. Therefore the transition matrix for the subshift of finite type Σ_b' has the form

$$\Pi' = \begin{pmatrix} A & B & 0 & 0 \\ 0 & 0 & C & D \\ E & F & 0 & 0 \\ 0 & 0 & G & H \end{pmatrix}$$

where A, B, C, D, E, F, G and H are the matrices in which all the entries are equal to 1. Also these matrices have different sizes: A is $b \times b$ matrix, B is $b \times (b+1)$, C is $(b+1) \times (b+2)$, D is $(b+1) \times (b+3)$, E is $(b+2) \times b$, F is $(b+2) \times (b+1)$, G is $(b+3) \times (b+2)$, and H is $(b+3) \times (b+3)$.

For even b the transition matrix Π' has the same structure except that the sizes of the matices A, B, $\cdots$, H are different. These sizes can be obtained by exchanging b and $b+1$, $b+2$ and $b+3$.

For the purpose of estimate the Hausdorff dimension of the horseshoes the following description of he symbolic dynamics is also useful. Let us consider only the case of odd b. We use vertices to represent transitions of above symbolic dynamics and edges to states of the system. This representation can be expressed by following graph.

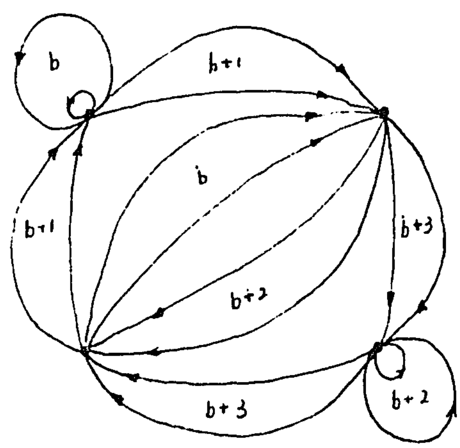

Figure 13. 1

The incidence matrix of this graph is

$$Q = \begin{pmatrix} b & b+1 & 0 & 0 \\ 0 & 0 & b+2 & b+3 \\ b+1 & b & 0 & 0 \\ 0 & 0 & b+3 & b+2 \end{pmatrix}.$$

Similarly for the even b the incidence matrix is

$$Q = \begin{pmatrix} b+1 & b & 0 & 0 \\ 0 & 0 & b+3 & b+2 \\ b & b+1 & 0 & 0 \\ 0 & 0 & b+2 & b+3 \end{pmatrix}.$$

Also for the purpose of the next section let us consider the largest eigenvalue L of the matrix Q. The next proposition gives a estimate of L.

Proposition 13.2 *Let b be any positive integer. Then the following inequality holds:*

$$L \geq 2b+1.$$

Proof: We consider a subshift of finite type $\Sigma_b'' \subset \Sigma_b'$ defined by the incidence matrix

$$Q_1 = \begin{pmatrix} b & b+1 & 0 & 0 \\ 0 & 0 & b & b+1 \\ b+1 & b & 0 & 0 \\ 0 & 0 & b+1 & b \end{pmatrix}$$

where b is odd. Let L^s be the largest eigenvalue of Q_1. Since $\Sigma_b'' \subset \Sigma_b'$, then the topological entropy of previous one is less than later, therefore we have $L \geq L^s$. It is easy to see that $2b+1$ is the eigenvalue of Q_1. For b even, one can obtain the similar result. Q. E. D.

This inequality gives us an estimate of the topological entropy of the horseshoes, which is also important for the estimation of Hausdorff dimension.

§14. Hausdorff Dimension of Λ'_k .

Let us begin by the definition of the Hausdorff dimension.

Definition 14.1. Let X be a metric space and $Y \subset X$ a subset of X, for any positive number p, the $p-$dimensional Hausdorff measure of Y is

$$m_p(Y) = \sup_{\epsilon} \ \inf \ \sum_{i=1}^{\infty}[diamA_i]^p$$

where inf is taken over all open covers $\{A_i\}$ of Y such that $diamA_i \leq \epsilon$ for all i. The Hausdorff dimension of Y is

$$HD(Y) = \sup\{p \mid m_p(Y) = \infty \} = \inf\{p \mid m_p(Y) = 0 \}.$$

To estimate the Hausdorff dimension of the hyperbolic basic set Λ_k we are going to use the formula proven by Lai–Sang Young [Youn]. We formulate the Theorem as follows:

Theorem 14.1. *Let $f : M \to M$ be a $C^{1+\alpha}$ $\alpha > 0$ diffeomorphism of a compact two dimensional Riemannian manifold M and let μ be a $f-$invariant ergodic Borel probability measure with exponents $\lambda_1 \geq \lambda_2$. Then*

$$HD(\mu) = h_\mu(f)[\frac{1}{\lambda_1} - \frac{1}{\lambda_2}]$$

whenever the right side of this equation is not equal to $\frac{0}{0}$.

Note that in above formula $HD(\mu)$ is the Hausdorff dimension of the measure μ which is defined by

$$HD(\mu) = \inf_{Y \subset M, \ \mu(Y)=1} HD(Y)$$

and $h_\mu(f)$ is the measure theoretic entropy of f with respect to measure μ.

Based on above information the estimation of $HD(\Lambda_k')$ can be done by the following way: We consider the standard map on the torus T^2, $\mathcal{F} : T^2 \to T^2$. For any $\mathcal{F}$–invariant ergodic Borel probability measure μ with $\mu(\Lambda_k') = 1$ we have

$$HD(\Lambda_k') \geq HD(\mu).$$

By Theorem 14.1 we obtain an estimate as follows:

$$HD(\Lambda_k') \geq h_\mu(\mathcal{F})(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}).$$

Let us consider first some uniform estimate of the Lyapunov exponents λ_1, λ_2 for the standard map $\mathcal{F}$.

Lemma 14.1.

$$(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}) \geq \frac{2}{\log N}$$

where

$$N = 2k + 1.$$

Moreover this estimate is independent of the invariant measure.

Proof: Since $det(DF) = 1$, then by the general theory one has

$$\lambda_1 + \lambda_2 = 0 \quad a.e.$$

for any $\mathcal{F}$–invariant measure μ. Let us assume that $\lambda_1 \geq 0$. Now we only need to show that

$$\lambda_1 \leq \log N.$$

We let

$$A(x) = DF|_z = \begin{pmatrix} 1 - kcos2\pi x & 1 \\ -kcos2\pi x & 1 \end{pmatrix}.$$

It is clear that

$$A(x)A^*(x)$$

is positive definite, where A^* is the transpose of the matrix A. If $a_1 \geq a_2 > 0$ are the two eigenvalues of $A(x)A^*(x)$, then

$$\|A(x)\| = \sqrt{a_1}.$$

Again by the condition of $det(DF) = 1$ we have

$$\|DF|_z\| \leq \sqrt{tr(A(x)A^*(x))} = \sqrt{(1 - kcos2\pi x)^2 + (kcos2\pi x)^2 + 2}$$

$$\leq \sqrt{(1+k)^2 + k^2 + 2} \leq 2k + 1 = N,$$

where trA is the trace of the matrix A. This implies that for any $\mathcal{F}$–invariant probability measure μ the Lyapunov exponent λ_1 satisfies

$$\lambda_1 \leq \log N.$$

Q. E. D.

To estimate the measure theoretic entropy $h_\mu(\mathcal{F})$ we should go back to the dynamical system $(\Sigma_b', \sigma', \mu')$, where μ' is σ'–invariant Borel probability measure on Σ_b'.

Since we have

$$\begin{array}{ccc} \Sigma_b' & \xrightarrow{\sigma'} & \Sigma_b' \\ \downarrow \tau' & & \downarrow \tau', \\ \Lambda_k' & \xrightarrow{\mathcal{F}} & \Lambda_k' \end{array}$$

for any σ'–invariant Borel probability measure μ' on Σ_b' one can derive a $\mathcal{F}$–invariant Borel probability measure μ on Λ_k' by

$$\mu = \mu' \circ \tau'^{-1}.$$

Therefore we have a $\mathcal{F}$–invariant Borel probability measure μ on T^2. Note that if μ' is ergodic, then μ is ergodic. Furthermore we have

$$h_\mu(\mathcal{F}) = h_{\mu'}(\sigma').$$

Let

$$P = (p_{ij})$$

be a $(4b+6) \times (4b+6)$ stochastic matrix satisfying

$$p_{ij} > 0 \quad \text{if and only if} \quad \pi_{ij} = 1,$$

where π_{ij} is the (ij)–entry of the transition matrix Π' for Σ_b'. It is easy to check that (p_{ij}) is irreducible, therefore the system (Σ_b', σ') with the invariant measure μ', which is induced by the stochastic matrix, is ergodic. Actually one can show that

$$(\Sigma_b', \sigma', \mu')$$

is mixing (see [Walt]).

Let $p = (p_i)$ be a (row) probability vector such that

$$pP = p.$$

Then by [Walt] the measure theoretic entropy of this system is

$$h_{\mu'}(\sigma') = -\sum_{i,j} p_i p_{ij} \log p_{ij}.$$

One can rewrite this formula by

$$h_{\mu'}(\sigma') = \sum_i p_i \left(-\sum_j p_{ij} \log p_{ij} \right).$$

If we maximize this entropy we obtain

$$-\sum_j p_{ij} \log p_{ij} \geq \log(2b+1)$$

since for fixed i there are at least $(2b+1)$ non–zero p_{ij}. From the fact

$$\sum_i p_i = 1$$

we obtain the following: There is a σ'–invariant measure μ', which is induced by some stochastic matrix (p_{ij}), such that

$$h_{\mu'}(\sigma') \geq \log(2b+1).$$

Note that by using the Proposition 13.2 one can derive that the topological entropy of Σ_b' can be estimated from below by $\log(2b+1)$.

Combining these estimate we have

Theorem 14.1. *Let $b = [\frac{k}{\pi}] - 2$. Then we have*

$$\lim_{k \to \infty} HD(\Lambda_k') = 2.$$

Proof: Clearly we have

$$HD(\Lambda'_k) \leq 2.$$

Also we have

$$HD(\Lambda'_k) \geq h_{\mu'}(\sigma')(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}) \geq \frac{2\log(2b+1)}{\log(2k+1)}.$$

Since we have $b \to \infty$ as $k \to \infty$ and $k \leq (3+b)\pi$, it is easy to see that this theorem is true. Q. E. D.

Remark: Similar method can be applied to show that

$$\lim_{k\to\infty} HD(\Lambda^*_k) = 2.$$

§15. Elliptic Periodic Points of the Standard Map.

In the previous sections we study the hyperbolic properties of the standard map. It seems that the hyperbolic orbits dominate the behavior of the standard map with the large values of parameter. In this section we are going to show that for certain values of the parameter the map has elliptic periodic orbits on the cylinder and the set of all these values has infinite measure. First we give a lemma which is important for us to find the elliptic periodic orbits.

Lemma 15.1 *If a matrix A has following form*

$$A = \begin{pmatrix} 1-p & 1 \\ -p & 1 \end{pmatrix}$$

where $0 < p < 4$, then we have

$$-2 < \mathrm{tr}A^2 < 2$$

where $\mathrm{tr}B$ *is the trace of matrix B.*

Proof: It is easy to compute the A^2

$$A^2 = \begin{pmatrix} (1-p)^2 - p & 2-p \\ -p(2-p) & 1-p \end{pmatrix}.$$

Then

$$\mathrm{tr}A^2 = (1-p)^2 - 2p + 1 = (p-2)^2 - 2.$$

Since $0 < p < 4$, then we have $-2 < (p-2)^2 - 2 < 2$. Q.E.D.

We Look for the periodic points with period 2 of the standard map on the cylinder. We require that those orbits have the form

$$F(a, 2a) = (-a, -n-2a) \quad \text{and} \quad F(-a, -n-2a) = (a, 2a)$$

where $0 < a < \frac{1}{4}$ and n is a positive integer. Now we are looking for the condition for existence of such periodic orbits.

Since

$$f(a, 2a) = (a + 2a - \frac{k}{2\pi} sin 2\pi a, 2a - \frac{k}{2\pi} sin 2\pi a)$$

and

$$f(-a, -n - 2a) = (-a - n - 2a - \frac{k}{2\pi} sin 2\pi(-a), -n - 2a - \frac{k}{2\pi} sin 2\pi(-a)\ ).$$

It is not hard to see that the condition for these two points to be a periodic orbit with period 2 on the cylinder is that

$$2a - \frac{k}{2\pi} sin 2\pi a = -n - 2a.$$

On the other hand, by Lemma 14.1 this orbit is elliptic if and only if

$$0 < k cos 2\pi a < 4.$$

Let $u = k cos 2\pi a$. We define a function G_n which depends on u and a

$$G_n(u, a) = n + 4a - \frac{u}{2\pi} tan 2\pi a.$$

Therefore, for every $(u, a) \in (0, 4) \times (0, \frac{1}{4})$ such that $G_n(u, a) = 0$, $(a, 2a)$ is an elliptic periodic point with period 2. It is easy to see that such (u, a) always exists for every positive integer n. In other words, for any $u \in (0, 4)$ there is $a \in (0, \frac{1}{4})$ such that $G_n(u, a) = 0$. By estimating the range of the parameter k, we have

Theorem 15.1 *There is a sequence of pairwise disjoint intervals $I_n \subset \mathbf{R}$ such that if $k \in \cup_n I_n$ then the standard map with this value of parameter has elliptic periodic points with period 2. Moreover, we have*

$$m(\cup_n I_n) = \infty$$

where m is the Lebesgue measure on the real line.

Proof: Since

$$\frac{\partial G_n}{\partial a} = 4 - usec^2 2\pi a,$$

for those (u, a) such that $G_(u, a) = 0$ we have

$$\frac{\partial G_n}{\partial a} = 4 - \frac{2\pi(n+4a)}{sin 2\pi a cos 2\pi a} < 0.$$

By Implicit Function Theorem, a is a function of u, so is k since $k = \frac{u}{cos 2\pi a}$. We define

$$k = g_n(u) \quad 0 < u < 4,$$

then

$$\frac{dk}{du} = \frac{4-u}{cos 2\pi a \frac{\partial G_n}{\partial a}} < 0.$$

This implies that $g_n(u)$ is a decreasing function of u on $(0, 4)$. From the equation $G_n(u, a) = 0$ it is clear that $a \to \frac{1}{4}$ as $u \to 0$. So we have

$$\lim_{u \to 0+} g_n(u) = \lim_{a \to \frac{1}{4}} \frac{2\pi}{sin 2\pi a} = 2\pi(n+1).$$

If we let $a_n \in (0, \frac{1}{4})$ such that $G_n(4, a_n) = 0$. Then by continuity we have

$$\lim_{u \to 4-} g_n(u) = \frac{2\pi(n+4a_n)}{sin 2\pi a_n} = \frac{4}{cos 2\pi a_n}.$$

We need to estimate

$$g_n(0+0) - g_n(4-0) = 2\pi(n+1) - \frac{4}{cos2\pi a_n} = 2\pi(n+1) - \frac{2\pi(n+4a_n)}{sin2\pi a_n}$$

from below. Since $2\pi(n+4a_n)cos2\pi a_n = 4sin2\pi a_n$, we have

$$cos2\pi a_n = \frac{4}{\sqrt{16+4\pi^2(n+4a_n)^2}}.$$

This gives us

$$g_n(0+0) - g_n(4-0) = 2\pi(n+1) - \sqrt{16+4\pi^2(n+4a_n)^2}$$

$$= \frac{16\{\pi^2(2n+4a_n+1)(\frac{1}{4}-a_n)-1\}}{2\pi(n+1)+\sqrt{16+4\pi^2(n+4a_n)^2}}.$$

Note that $a_n \to \frac{1}{4}$ as $n \to \infty$. This implies that for any $\epsilon > 0$ there is a $N > 0$ such that for $n > N$ we have

$$\frac{1}{4} - a_n \geq (\frac{1}{2\pi} - \epsilon)tan2\pi(\frac{1}{4} - a_n)$$

$$= (\frac{1}{2\pi} - \epsilon)\frac{1}{tan2\pi a_n}.$$

Since $n + 4a_n = \frac{4}{2\pi}tan2\pi a_n$, we have

$$\frac{1}{4} - a_n \geq (\frac{1}{2\pi} - \epsilon)\frac{4}{2\pi(n+4a_n)}.$$

This inequality implies that for large n there is constant c such that

$$g_n(0+0) - g_n(4-0) \geq \frac{c}{n}.$$

Therefore we obtain that

$$\sum_{n=1}^{+\infty}(g_n(0+0) - g_n(4-0))$$

converges to the infinity. Q.E.D.

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