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Design sensitivity analysis of multibody systems with special reference to four-wheel steering

Lee, Jong-Nyun, Ph.D.

The University of Arizona, 1992
DESIGN SENSITIVITY ANALYSIS
OF MULTIBODY SYSTEMS WITH
SPECIAL REFERENCE TO FOUR WHEEL STEERING

by

Jong-Nyun Lee

A Dissertation Submitted to the Faculty of the
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For the Degree of
DOCTOR OF PHILOSOPHY
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In the Graduate College
THE UNIVERSITY OF ARIZONA

1992
As members of the Final Examination Committee, we certify that we have read the dissertation prepared by Jong-Nyun Lee entitled Design Sensitivity Analysis of Multibody Systems with Special Reference to Four Wheel Steering and recommend that it be accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy.

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SIGNED: ____________________________
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LIST OF SYMBOLS

Nomenclature

Matrices are in boldface upper-case characters.
Column or algebraic vectors are in boldface lower-case characters.
Scalars are in lightface characters.
The meaning of a symbol is described when it first appears. If a symbol has different meanings in different sections, such is clearly stated.

Overscores

\( \sim \) \hspace{1em} 3 \times 3 \text{ anti-symmetric matrix obtained from the components of a vector}
\( \cdot \) \hspace{1em} First derivative with respect to time.
\( \cdot \cdot \) \hspace{1em} Second derivative with respect to time.

Superscripts

\( T \) \hspace{1em} Transpose of a vector or a matrix.
\( \cdot \) \hspace{1em} Components of a vector in a body-fixed coordinate system.
\( * \) \hspace{1em} A vector or a scalar associated with a perturbed component.

Subscripts

\( f \) \hspace{1em} A vector or a scalar associated with a front wheel.
\( r \) \hspace{1em} A vector or a scalar associated with a rear wheel.
\( b \) \hspace{1em} First derivative with respect to a design parameter.
ABSTRACT

Sensitivity analysis methods are investigated for the optimal design of multibody systems. In order to overcome the shortcomings inherent in existing methods, a "mixed" method is developed. The beneficial features of the finite difference and the direct differentiation methods, and equations of motion in the joint coordinates are employed in this method. As a realistic application of the sensitivity analysis, a Four-Wheel-Steering vehicle with complete suspension systems and comprehensive analytical tire model is implemented. This model keeps full nonlinearity in the governing equations of motion for accuracy, and it is simulated using an existing general-purpose multibody dynamics simulation package. However, by using the transient dynamic analysis of the nonlinear model, optimal design parameters are dependent on driving scenarios. Therefore, the transient behavior of the system is represented by a series of steady state configurations. Hence, a steady state analysis procedure which finds a steady state configuration from an arbitrary initial condition is developed. By using the steady state analysis and the sensitivity analysis, the optimal steering ratios between the angles of the front and the rear wheels are obtained over various driving conditions. A steering control strategy is developed for the vehicle simulation to follow a prescribed path. Finally, the simulation results using the optimal steering ratio are compared against the results of the conventional two-wheel steering and the steering ratio based on the linear bicycle model.
1.1 General

High speed digital computers allow engineers to simulate the dynamics of a mechanical system under different input conditions and to modify its design prior to actual production. Ultimately, this capability allows the engineer to pursue the optimal design of mechanical systems. Before the era of computer analysis, a manufacturer had to construct and test a series of prototypes, which was not only time-consuming but also costly.

Equations to represent the motion of a multibody dynamic system are expressed either as differential equations or as mixed differential-algebraic equations, for which a closed form exact solution cannot be found. It is only possible to solve such equations numerically with the help of computers. To obtain better computational efficiency in the process, systematic techniques for formulating the equations of motion of the system and numerical methods for solving them are required. Due to active research, there has been remarkable development in these fields. As a result, various general purpose multibody dynamic analysis codes have been developed and widely used.

The optimal design of a mechanical system requires not only the capability to perform dynamic analysis but it also requires the design sensitivity analysis capability which determines the effect of design modifications. Although active research is being done on the subject of design sensitivity analysis, the methodologies are still too complex.
to be adapted for general usages. Thus, despite the existence of various general purpose multibody dynamic analysis codes, the process of optimal design of multibody systems still largely depends on the experience and the engineering intuition.

One application of dynamic analysis and optimal design of mutibody systems is in vehicle system design. A vehicle is generally characterized as a multibody dynamic system in which the components are interactively connected to each other. The rapid progress of modern technology has resulted in the development of new vehicle sub-systems, such as anti-lock braking system (ABS), traction control system, active suspension system, and Four-Wheel-Steering (4WS) system. The performance of a vehicle incorporating these newly developed systems can be investigated through dynamic (transient) or steady-state analyses, and some improvements can be achieved through the optimal design process. The primary focus of this study will be the dynamic analysis, the steady-state analysis, and the optimal design of the 4WS system.

The 4WS idea first originated in a steam powered four-wheel-steered prototype which was built by a frenchman in 1876 [1]. Thereafter, there were other efforts which brought the benefits of making sharp turns by steering the front wheels in one direction and the rear wheels in the other. Daimler-Benz observed, while testing his 1935 MB 170 VL military car, that severe instability can occur at higher speeds due to a simple four wheel steering system [2].

In recent years, the 4WS system has become a major active research area. At present, the front wheels of a 4WS vehicle are steered by the driver. The rear wheels are controlled by one of several different methods. Each method has utilized different theoretical or experimental background to obtain a proper steering strategy. Most of the
research based on theoretical approach employed approximated linear equations of motion derived by simplifying the 4WS vehicle. Because the results from using these equations cannot provide better vehicular performance under diverse driving conditions occurring in the real world, realistic nonlinear equations of motion are naturally a part of this research. However, problems arise since a nonlinear model can provide different optimal steering strategies corresponding to different driving conditions. Finding several optimal steering strategies also requires a large number of dynamic simulations of the vehicle model. Therefore, a new method must be developed to handle these difficulties. As an alternate approach, optimization of the steady state response of the vehicle in different conditions is proposed. An indepth discussion of this technique is presented in this study.

1.2 Literature Review

Even though design sensitivity studies are plentiful in many engineering fields such as electrical circuit design and structural design, they are sparse in the area of multibody systems. The main reasons are the complexities involved in derivation of the necessary equations and the numerical inefficiency in solving these equations using the existing methods. When equations of motion of a multibody system are formulated using conventional methods, they become too complex and the size of equations becomes too large to be utilized in the sensitivity analysis. Therefore, in order to come up with an efficient optimization method in the design of multibody systems, both the sensitivity analysis techniques and the dynamic analysis techniques need to be improved.
The dynamic analysis of multibody systems involves two stages. The first stage is the formulation process generating a set of equations of motion for a given system. The second stage involves the integration process for solving these equations. For the integration process, many excellent algorithms have been developed such as predictor-corrector algorithms based on Adams-Bashforth and Adams-Moulton methods, and Gear's stiff method. These algorithms focus on reducing the computational time and increasing the accuracy.

Several methods have been developed for formulating equations of motion. One classical method uses a set of generalized coordinates based on the Lagrangian formulation [3]. Though this method gives the minimum number of equations of motion for complex multibody systems, their derivation is cumbersome. Another method uses a large number of Cartesian coordinates (absolute coordinates) with Lagrange multipliers resulting in a mixed set of differential-algebraic equations. Despite these equations producing an inefficient algorithm in the integration process, these are most commonly used in the dynamic analysis due to their simplicity and ease of manipulation [4-6]. In lieu of above methods, two other formulations have been developed. The first method produces Kane's formulation which uses a minimum number of generalized speeds [7]. This formulation has numerical efficiency but lacks simplicity of formulation and ease of manipulation. The second method produces the joint coordinate formulation which uses a minimal set of joint coordinates and is based upon a velocity transformation process. This method can generate a minimal number of equations of motion in a systematic manner [8-10]. For dynamic simulations of real systems, Nikravesh (1988) [11] has proposed a general computer procedure based on generating and integrating the
equations of motion which is simple in formulations and efficient in computations. Hence, the formulation is also expected to exhibit better performances in the sensitivity analysis.

For the sensitivity analysis, researchers have mostly used one of the following three methods: the adjoint variable method, the direct differentiation method, and the finite difference method. Advantages and drawbacks of each method have been reported over the years. Haug and Arora (1979) [12] adopted the use of the adjoint variable method to compute the design sensitivity in the optimization process of structural systems. In later years, the above method was extended to the design sensitivity analysis of multibody dynamic systems by Haug, Wehage, and Barman (1981) [13]. In this method, the cost sensitivity equations were derived by applying the variational method to the cost function, where the equations of motion were described in terms of a large set of absolute Cartesian coordinates. This technique was successfully implemented for a simple planar case. Since the number of sensitivity equations is not dependent on the number of design parameters, the adjoint variable method has computational efficiency when large number of design parameters are involved. However, this method is complicated in generating the sensitivity equations and is inefficient in performing the optimization process for complex and large-scale systems. To overcome these difficulties, Wehage (1984) [14] additionally introduced a generalized coordinate partitioning scheme in the solution of equations of motion. The partitioning scheme was implemented in the dynamic analysis program DADS. Extension of this method to the design sensitivity analysis required analytical derivation of many derivative terms which appear in the sensitivity equations. Furthermore, the method has a major drawback due to a characteristic of the adjoint
system: initial conditions are unknown and terminal conditions are known. Hence, in order to incorporate the sensitivity equations and the equations of motion with the adjoint system, integration processes in forward and backward directions and also additional interpolation steps are inevitable. Therefore, the adjoint variable method is not appropriate for the sensitivity analysis of large scale mechanical systems.

Direct differentiation method uses analytical derivatives of the equations of motion and the cost function to determine the sensitivity equations. Tomovic (1963) [15] introduced this method for performing the first order design sensitivity analysis in the general sense. The first application of this method to multibody dynamic systems was presented by Krishnaswami and Bratti (1984) [16]. Later, in order to reduce numerical errors caused by a kinematically closed loop, Chang and Nikravesh (1985) [17] adopted the constraint violation stabilization technique. As a new approach, Tak and Kim (1989) [18] introduced a recursive and a parallel algorithm to solve the sensitivity equations and the equations of motion in absolute coordinates. Although this approach can achieve numerical efficiency by using a parallel computer, it cannot resolve complexities appearing while formulating the sensitivity equations.

The finite difference method is a simple method which can be used in sensitivity analysis. Tomovic (1963) [15] pointed out that computers make the sensitivity analysis possible for dynamic systems when analytic differentiation is not available due to complexity and high degrees of nonlinearity. However, this method appeared to produce numerical errors due to perturbations in the design parameters, and it may require extensive computational time by repeatedly performing the integration of the equations of motion. Many researchers attempted to reduce numerical errors caused by the finite
difference method [19,20]. Howard, Adelman, and Haftk (1986) [21] carried out comprehensive and systematic research on this subject. Their study concluded that the key source of error is the relationship between the magnitude of perturbation and the tolerance level in a numerical algorithm, and they suggested a method for finding an optimum choice of the perturbation. However, this method requires numerous function evaluations in order to find the optimum choice.

For the study of 4WS systems in the passenger cars, several papers have been published. In one of the theoretical studies, Furukawa (1985) [22] used a highly simplified linear model of a 4WS vehicle, in which he suggested to choose the steering angle ratio of rear wheel to front wheel to provide a zero side-slip angle of the main chassis in a steady turn. The study resulted in a "vehicle speed function based" 4WS system which gave shorter turning radius at lower speeds and reduced lateral acceleration at higher speeds. Later, Shibahata (1986) [23] presented an experimental study which uses a vehicle equipped with the 4WS system. The experimental results showed good agreement with Furukawa's study. On the other hand, Sano (1986) [24] proposed a 4WS system which uses the steering wheel angle in determining the rear wheel steering angle known as "steering wheel function based" 4WS system, and he analyzed its effects on the dynamic response through several experiments. But, Takiguchi (1986) [25] selected the steering angle ratio to equalize the phase lag for yaw rate and lateral acceleration instead of the zero side slip angle criterion, and he obtained some improvement in maneuverability at higher speeds. Furukawa (1989) [26] recently reviewed the published key studies from the perspective of vehicle dynamics and control. He also presented commercially adopted 4WS technologies with theoretical backgrounds and their
mechanisms in detail.

Most of the studies [23-33] in 4WS system used a linear bicycle model which has two degrees of freedom and it contains a simple tire model. As an effort to improve accuracy, Bernard (1988) [28-29] introduced a slightly complex tire model concerning the tire slip angle, in which he assumed that the lateral forces are linear functions of the tire slip angle. Nalecz (1988) [30] presented a vehicle model including kinematic suspension systems and effects of lateral weight transfer on tire properties. However, he neglected nonlinear terms and included some additional assumptions to simplify the equations of motion. With these equations, he tested several rear wheel control strategies, and showed that the rear steering advanced control strategy gave enhancement in vehicle side slip angle and transient response time.

For the study of the directional response of steering input, Segel (1966) [34] developed a five degrees of freedom linear vehicle model, where three degrees of freedom correspond to motions of the main chassis and two degrees represent a quasi-linear steering system with Coulomb friction tire model. Pacejka (1979) [35,36] modified this model by considering the combined effects of tire, suspension, and steering properties, resulting in a two degrees of freedom vehicle model. He presented these linear equations in the phase-plane and introduced a handling diagram by modifying the phase-plane diagram. By using this diagram, he analyzed the regions of stable and unstable motions and the regions of under-steering and over-steering.

In the study of automobile control, MacAdam (1980) [37-39] introduced a two degrees of freedom linear equations of motion with a steering input as the control variable. He solved for an optimal control input which minimized the path error over a current
preview interval. Yeh (1989) [40] introduced a multiple-input steering wheel to decouple the yaw motion and the side slip angle of the vehicle. The control gains of this steering wheel controller were obtained by using three degrees of freedom linear equations of motion in the Laplace domain.

1.3 Objectives

For optimal design of mechanical systems, sensitivity analysis has to be performed. There are a number of existing methods such as finite difference, adjoint variables, and direct differentiation to perform the sensitivity analysis. However, each of these methods has some inherent shortcomings. In the direct differentiation method, the analytical derivation is extremely cumbersome. The computational time is very large in the finite difference method and also numerical errors are encountered. Adjoint variable method has potential errors in the numerical integration process and also some difficulty in the analytical derivation. To overcome these drawbacks, a new "mixed" method will be proposed.

Dynamic analysis of mechanical systems has been made possible by the advent of present day computers. Until now, Newton-Euler method resulting in equations of motion in Cartesian coordinate was used to perform the dynamic analysis and the sensitivity analysis. This method produces a large number of equations of motion which requires large amount of computational time. In contrast, the joint coordinate formulation when combined with the mixed sensitivity analysis method is expected to yield improvement in efficiency over previously developed methods. Another primary objective of this study is to provide a potential method for generating an automated
sensitivity analysis program.

The application of the joint coordinate formulation in the sensitivity analysis of 4WS vehicle is also one of the purposes of this study. Nonlinear equations of motion has been used in place of linear simplified models. By using nonlinear equations of motion, various optimal steering angle ratios or time dependent optimal steering angle ratios which minimize the side-slip angle of the vehicle can be obtained. These optimal steering angle ratios are not useful to be implemented in every driving conditions. Hence, the development of a general optimization methodology for nonlinear systems which is more or less valid under various driving scenarios is required. This problem can be overcome by using steady-state configuration of a vehicle under various conditions. By assuming a transient behavior of the vehicle as a series of steady-state responses as a valid approximation, the optimization of the steady-state can be achieved and therefore can be extended to the transient behavior. Hence, one of the objectives of this study is to develop a general procedure for the steady-state analysis of multibody systems based on the joint coordinate formulation.

Another task of this study is to develop a path control steering strategy to be used in the dynamic simulation of any vehicle model following any arbitrary prescribed path. The newly developed path control strategy is necessary for comparing different types of steering strategies.

1.4 Overview of the Study

Chapter 2 deals with general formulations of equations of motion in multibody dynamics using the absolute coordinates and the joint coordinates. Chapter 3 contains
the steady-state analysis of a nonlinear 4WS model and its applications. Chapter 4 reviews the finite difference method, the direct differentiation method, and the adjoint variable method for sensitivity analysis. The development of a mixed method which includes advantages of finite difference method, direct differentiation method, and joint coordinate formulation of equations of motion is presented. For demonstration purposes, a double A-arm suspension system has been analyzed using this new technique.

General characteristics of a 4WS system are described in Chapter 5. For this 4WS system, the optimal steering angle strategy is obtained using algebraic equations in the linear bicycle model. In the nonlinear 4WS model, the strategy is found numerically using an iterative method applied to the sensitivity analysis at the steady-state configuration. The methodologies are presented in Chapter 6.

Chapter 7 describes the feedback path control steering scheme which uses position and slope errors. In Chapter 8, by using this scheme, various dynamic simulations are performed to compare three steering angle strategies: 2WS, linear bicycle model, nonlinear 4WS model. Overall discussion and conclusion are described in Chapter 9.
CHAPTER 2

EQUATIONS OF MOTION IN MULTIBODY DYNAMICS

This chapter provides a brief review of equations of motion for multibody systems using two different formulations. All of the formulations in the following sections are extracted from Refs. [9-11]. Derivations of these formulations and the components used are presented in detail in these references.

A multibody system can be defined as an assembly of rigid and/or deformable bodies interconnected by kinematic joints acted upon by forces and moments. A schematic diagram of a multibody system is shown in Figure 2.1. The kinematic joints between bodies constrain the relative motion of connected bodies. Commonly used kinematic joints are revolute, universal, spherical, and translational joints. The forces and moments are caused by gravity, actuators, springs, dampers, and other elements. The motion of a multibody system is described by its governing equations of motion.

The equations of motion of a multibody system can be described in terms of different sets of coordinates. If the number of generalized coordinates is greater than the number of degrees of freedom of the system, then algebraic equations are required to show the dependency of the coordinates. One such set of coordinates which leads to defining algebraic constraints for the kinematic joints is the so-called Cartesian (absolute) coordinates. Another set of generalized coordinates which can provide a minimal set of equations is known as the joint coordinates.
Figure 2.1 Schematic diagram of a multibody system [9]
2.1 Equations of Motion in Cartesian Coordinates

In order to specify the position of a rigid body in a global non-moving XYZ coordinates system, it is sufficient to specify the spatial location of the origin (center of mass) and the angular orientation of a body-fixed $\xi\eta\zeta$ coordinate system as shown in Figure 2.2. For the $i$th body in a multibody system, vector $\mathbf{q}_i$ denotes a vector of coordinates which contains a vector of Cartesian translational coordinates $\mathbf{r}_i$ and a set of rotational coordinates. A vector of velocities for body $i$ is defined as $\mathbf{v}_i$, which contains a 3-vector of translational velocities $\mathbf{\dot{r}}_i$ and a 3-vector of angular velocities $\mathbf{\omega}_i$. A vector of accelerations for this body is denoted by $\mathbf{a}_i$, which contains $\mathbf{\ddot{r}}_i$ and $\mathbf{\ddot{\omega}}_i$. For a multibody system containing $b$ bodies, the vector of coordinates $\mathbf{q}_i$, velocities $\mathbf{v}_i$, and accelerations $\mathbf{a}_i$ contain the elements of $\mathbf{q}_i$, $\mathbf{v}_i$, and $\mathbf{a}_i$, respectively, for $i = 1, \ldots, b$.

Figure 2.2 Body-fixed and global coordinate systems
The kinematic joints between the connected bodies can be described by \( m \) independent holonomic constraints as

\[
\Phi(q) = 0 \quad (2.1)
\]

The first and the second time derivatives of the constraints yield the kinematic velocity and acceleration equations as

\[
\dot{\Phi} \equiv Dv = 0 \quad (2.2)
\]

\[
\ddot{\Phi} \equiv D\dot{v} - \gamma = 0 \quad (2.3)
\]

where

\[
\gamma = -\dot{D}v
\]

and \( D \) is the modified Jacobian matrix of the constraints.

The translational and rotational equations of motion for a single unconstrained rigid body can be written as

\[
M_i \ddot{\nu}_i = g_i \quad (2.4)
\]

where

\[
M_i = \begin{bmatrix} mI & 0 \\ 0 & J \end{bmatrix}, \quad (2.5)
\]

\[
g_i = \begin{bmatrix} f \\ n - \omega_i J \omega \end{bmatrix}, \quad (2.6)
\]

In the above expressions, \( m_i \) is the mass, \( f_i \) is the resultant force acting at the center of the mass, \( n_i \) is the resultant moment and \( J_i \) is the inertia tensor about the body-fixed coordinate axes expressed in the global coordinate system.
For a system of \( b \) unconstrained bodies, the equation of motion of Eq. (2.4) are repeated for all \( b \) bodies and are written in matrix form as

\[
M\ddot{\mathbf{v}} = \mathbf{g}
\]  

(2.7)

Equation (2.7) represents \( n = 6b \) second order differential equations.

For a system of \( b \) constrained bodies, where the kinematic joints are represented by \( m \) independent algebraic equations, Eq. (2.7) is modified to

\[
M\ddot{\mathbf{v}} - D^T\lambda = \mathbf{g}
\]

(2.8)

Eq. (2.8), similar to Eq. (2.7), represents \( n = 6b \) second-order differential equations. In these equations, \( \lambda \) represents a vector of \( m \) Lagrange multipliers, and the term \( D^T\lambda \) represents the joint reaction forces and moments. Equations (2.1)-(2.3) and (2.8) represent a set of differential-algebraic equations of motion for a constrained multibody system when absolute coordinates are used. The number of degrees of freedom of such a system is \( k = n - m \).

2.2 Equations of Motion in Joint Coordinates

The relative configurations of two adjacent bodies can be defined by one or more joint coordinates which are equal in number to the number of relative degrees of freedom between these bodies. For a multibody system with open-loops as shown in Figure 2.3, the vector of joint coordinates is denoted by \( \theta \) containing all of the joint coordinates and the absolute coordinates of a base body if the base body is not the ground. Therefore,
the vector $\theta$ has $k$ elements, equal to the number of degrees of freedom of the system. The vector of joint velocities is defined as $\dot{\theta}$. It can be shown that there is a linear transformation between $\dot{\theta}$ and $v$ as

$$v = B\dot{\theta}$$  \hspace{1cm} (2.9)

where $B$ is an $n \times k$ matrix and orthogonal to the modified Jacobian matrix $D$. This orthogonality can be proved by using Eqs. (2.9) and (2.2). By substituting Eq. (2.9) into Eq. (2.2), we obtain

$$DB\dot{\theta} = 0$$  \hspace{1cm} (2.10)

Since the joint velocity vector $\dot{\theta}$ is an independent vector, the orthogonal characteristic can be drawn from Eq. (2.10) as

$$DB = 0$$  \hspace{1cm} (2.11)

The time derivative of Eq. (2.9) gives the acceleration of the joint coordinates as

$$\ddot{v} = B\ddot{\theta} + B\dot{\theta}$$  \hspace{1cm} (2.12)

By using Eqs. (2.11) and (2.12), and pre-multiplying Eq. (2.4) by $B^T$, we obtain

$$M\ddot{\theta} = f$$  \hspace{1cm} (2.13)

where

$$M = B^TMB$$

$$f = B^T(g - MB\dot{\theta})$$
Equation (2.13) is the generalized equations of motion for an open-loop multibody system. From this equation it can be noted that the number of selected coordinates is equal to the number of degrees of freedom.

In order to derive the equations of motion for a multibody system containing one or more closed kinematic loops, as shown in Figure 2.4a, each closed-loop is cut at one of the kinematic joints in order to obtain an open-loop system. For this reduced system, a vector of joint coordinate \( \theta \) is defined without defining any joint coordinates for the cut joints as shown in Figure 2.4b. Therefore, the vector of joint coordinates \( \theta \) has a dimension \( k \) greater than the number of degrees freedom of the closed-loop system. If the cut joints are reassembled, the joint coordinates within each loop are no longer independent. Therefore, algebraic constraints between the joint coordinates exist as

\[
\Psi(\theta) = 0 \quad \text{(2.14)}
\]
The first and the second time derivatives of the constraints are

\[ \dot{\Psi} = C\dot{\theta} = 0 \]  
(2.15)

\[ \ddot{\Psi} = C\ddot{\theta} + \dot{C}\dot{\theta} = 0 \]  
(2.13)

where

\[ \tau = -\dot{C}\dot{\theta} \]

and \( C \) is the Jacobian matrix of the constraints. The differential equations of motion of Eq. (2.13) can be modified for a closed-loop system as

\[ M\ddot{\theta} - C^T \nu = f \]  
(2.16)

where \( \nu \) is a vector of Lagrange multipliers.

![Diagram showing a system containing a closed-loop and its reduced open-loop representation.](image-url)

Figure 2.4 A system containing a closed-loop;
(a) a closed-loop,
(b) its reduced open-loop representation.
Equations (2.14)-(2.16) represent a set of differential-algebraic equations for a closed-loop system. These equations can further be reduced to a minimal set of second-order differential equations, equal in number to the number of degrees of freedom of the system. For this purpose, a subset of the joint coordinates vector $\theta$ is selected containing a set of independent joint coordinates $\theta_0$. There is a linear transformation between $\theta$ and $\theta_0$ as

$$\hat{\theta} = E\hat{\theta}_0$$  \hspace{1cm} (2.17)

where $E$ is a velocity transformation matrix. The joint velocities outside the closed-loops and the independent joint velocities within the closed-loops are not affected by this transformation. One characteristic of matrix $E$ is that it is orthogonal to $C$; i.e., $CE = 0$, as similarly observed in Eq. (2.11). The time derivative of Eq. (2.17) gives the acceleration of joint coordinates as

$$\ddot{\theta} = \dot{E}\dot{\theta}_0 + E\ddot{\theta}_0$$  \hspace{1cm} (2.18)

By using the orthogonality and (2.18), and pre-multiplying Eq. (2.16) by $E^T$, we obtain

$$M'\ddot{\theta}_0 = f'$$  \hspace{1cm} (2.19)

where

$$M' = E^TME$$  \hspace{1cm} (2.20)

$$f' = E^T(f - ME\dot{\theta}_0)$$  \hspace{1cm} (2.21)

Equation (2.19) represents the minimal number of equations of motion describing the dynamics of a multibody system containing kinematic closed-loops.
2.3 Summary

In this chapter, the absolute Cartesian coordinate and the relative joint coordinate formulations are reviewed as equations of motion for multibody systems. The derivation of the absolute Cartesian coordinate formulation is simple and the components of the modified Jacobian and vector $\gamma$ for most common constraints have been developed in Ref. [11]. Hence, when the equations of motion of multibody systems use the absolute Cartesian coordinate formulation, the equations can be simply constructed and its computer program can be easily generated. However, this formulation produces a large set of differential-algebraic equations which requires too much computation time and, depending upon the method of solution, may produce numerical error in the process of numerical solution. In order to avoid these inefficiencies, this formulation can be transformed into the joint coordinate formulation by using the joint coordinates and the velocity transformation. Then, the size of equations is reduced and most of the algebraic constraint equations are eliminated. Hence, the computational time and the numerical error can be reduced in the dynamic analysis of multibody systems. Since the equations of motion provide a basis for many types of analyses, these two formulations of equations of motion will be used in order to analyze multibody systems in transient dynamic, steady-state, and sensitivity analysis which will be studied in the following chapters.
CHAPTER 3

STEADY-STATE ANALYSIS

This chapter presents a systematic method for formulating and numerically solving the steady-state equations of multibody systems. In some applications of multibody systems, it may be required to determine the steady-state response of a system under a given condition. For highly simplified problems, it is possible to write and solve the steady-state equations in explicit form. However, for large-scale problems without too many simplifications, finding a closed form solution is not only impossible, but also determining a numerical solution may not be easy if the problem is not properly formulated.

The equations of motion are first written in terms of relative joint coordinates using velocity transformation formulas. Then, conditions for a given steady-state response are introduced into equations of motion in order to obtain the steady-state equations. An example of a nonperiodic steady-state response is: accelerations are generally zero, some of velocities are zero or constants, and specific relationships may exist between some or all coordinates. Such conditions transform the differential equations of motion into a set of nonlinear algebraic equations in which the unknowns are the coordinates and some of the velocities. These equations are solved numerically using the Newton-Raphson method. If the original equations of motion are expressed in terms of a large set of dependent Cartesian or absolute coordinates, the steady-state conditions can not be stated easily. Therefore, the procedure for finding the steady-state configuration can become cumbersome. However, when the equations are expressed in terms of a relative set of joint coordinates, these relationships are rather obvious and simple to introduce.
Vehicle dynamics is one of the areas of application of this methodology. A vehicle moving along a circular path with a constant speed is an example of the steady-state response. In this chapter, a complete multibody model of a vehicle with 4WS system is used. The interacting nonlinear forces and moments between the tires and the road due to the longitudinal slip, lateral slip and camber angle, are incorporated into the model.

3.1 General Formulation

In a steady-state configuration, the relative joint accelerations between bodies become zero. The only nonzero acceleration components normally belong to the body(ies) which has(ve) its acceleration defined with respect to a nonmoving reference frame. However, if these nonzero accelerations are transformed to an alternate coordinate system, it can be shown that the nonzero components of the acceleration are functions of velocities. For example, a point mass moving on a circular planar path with a constant angular velocity has nonzero Cartesian accelerations. But in a cylindrical coordinate system, it has zero tangential acceleration and its normal acceleration is a function of its angular velocity. Therefore, its vector of joint accelerations $\ddot{0}$ can be split into a nonzero set and a zero set as

$$\ddot{0} = \begin{bmatrix} \dot{0}^\theta \\ 0 \end{bmatrix}$$  

(3.1)

where $\dot{0}^\theta$ is a function of $\theta$ and $\dot{\theta}$. 
Equations of motion of a multibody system in terms of a set of joint coordinates are described in Chapter 2. If a system is composed of a set of open-loops, the equations of motion can be rewritten as

\[ f - M\ddot{\theta} = 0 \]  (3.2)

Now, Eq. (3.1) is substituted into Eq. (3.2) to get

\[ f - M^{(o)}\ddot{\theta}^{(o)} = 0 \]  (3.3)

where \( M^{(o)} \) is the portion of \( M \) corresponding to \( \theta^{(o)} \). In a steady-state configuration, one or more algebraic relations can be expressed between the coordinates and the velocities as

\[ \kappa(\theta, \dot{\theta}) = 0 \]  (3.4)

Therefore, for an open-loop system, Eqs. (3.3) and (3.4), using the Lagrange multiplier technique, are written as

\[ \Gamma = \begin{bmatrix} h(\theta, \dot{\theta}, \sigma) = f - M^{(o)}\ddot{\theta}^{(o)} + \Xi^T\sigma = 0 \\ \kappa(\theta, \dot{\theta}) = 0 \end{bmatrix} \]  (3.5)

where \( \Xi \) is the Jacobian matrix for the constraints \( \kappa \), and \( \sigma \) is a vector of Lagrange multipliers.

A vector state variable \( z \) can be defined as containing \( \theta \) and \( \dot{\theta} \). The state variables can be classified into three types; zero state variables, ineffective state variables, and effective state variables and can be represented as

\[ z = \begin{bmatrix} \theta^T \\ \dot{\theta}^T \end{bmatrix} = \begin{bmatrix} z^0 \\ z^1 \\ z^2 \end{bmatrix} \]  (3.6)
The zero state variables $z^0$ are set to zero since they have zero values in a steady-state configuration. The ineffective variables $z'$ can be assigned any arbitrary values since they do not have any influence on the steady-state configuration. The effective state variables $z_e$ uniquely satisfy steady-state configuration and are equal in number to the number of degrees of freedom of the system. The definition of these variables will become clear in the upcoming example. With this classification of the state variables, Eq. (3.6) can be restated as

$$\Gamma(z^e, \sigma) = 0$$ \hspace{1cm} (3.7)

Since the above equations are nonlinear algebraic equations, iterative numerical solution techniques can be used. The linearized Newton-Raphson formula for Eq. (3.7) is

$$\begin{bmatrix} \frac{\partial h}{\partial z^e} & \Xi^T \\ \Xi & 0 \end{bmatrix} \begin{bmatrix} \Delta z^e \\ \Delta \sigma \end{bmatrix} = -\Gamma$$ \hspace{1cm} (3.8)

These equations are solved for the effective state variables $z^e$ and the Lagrange multiplier $\sigma$.

For multibody systems with closed kinematic loops, Eqs. (2.11)-(2.14) can be written for the steady-state as

$$\begin{bmatrix} h(\theta, \dot{\theta}, \sigma) = f - M^{(\theta, \dot{\theta})} + C^T \nu + \Xi^T \sigma = 0 \\ \Psi(\theta) = 0 \\ \kappa(\theta, \dot{\theta}) = 0 \end{bmatrix}$$ \hspace{1cm} (3.9)

or

$$\Gamma(z^e, \nu, \sigma) = 0$$ \hspace{1cm} (3.10)
It is observed that the velocity constraints of Eq. (2.12) do not appear in Eq. (3.9) since the joint velocities within a closed-loop belong to the zero state variable vector $z^0$ and not to $z^e$. Therefore, all of the velocity constraints of Eq. (2.12) are automatically satisfied at the steady-state configuration. The Newton-Raphson formula for Eq. (3.10) is written as

$$
\begin{bmatrix}
\frac{\partial h}{\partial z^e} \\
C \\
\Xi
\end{bmatrix}^T
\begin{bmatrix}
z^e \\
\Delta z \\
\Delta \nu
\end{bmatrix} = -\Gamma
$$

(3.11)

For a closed-loop system, if Eq. (2.16) is used, a set of steady-state equations similar to that of Eq. (3.8) can be written. Here, the vector of effective state variables $z^e$ is partitioned into an independent set and a dependent set as

$$
z^e = \begin{bmatrix} z_{(i)}^e \\ z_{(d)}^e \end{bmatrix}
$$

(3.12)

where the number of independent state variables in $z_{(i)}^e$ is equal to the number of degrees of freedom of the system. Therefore, the steady-state equations are written as

$$
\Gamma(z_{(i)}^e, \sigma) = 0
$$

(3.13)

and with its linearized formula as

$$
\begin{bmatrix}
\frac{\partial h}{\partial z_{(i)}^e} \\
\Xi
\end{bmatrix}^T
\begin{bmatrix}
z_{(i)}^e \\
\Delta z_{(i)}^e \\
\Delta \sigma
\end{bmatrix} = -\Gamma
$$

(3.14)

The coefficient matrices in Eqs. (3.8), (3.10), and (3.14), mainly $\frac{\partial h}{\partial z^e}$ or $\frac{\partial h}{\partial z_{(i)}^e}$ terms, can be computed numerically by the finite difference method. The effective state variables, or each independent state variable in case of Eq. (3.14), is perturbed one at a time in order to
compute its corresponding column of the coefficient matrix. The perturbation is performed as

$$z^{**} = z^e + \Delta z^e$$  \hspace{1cm} (3.15)

where $z^e$ is one of the effective state variable vector $z^e$. Then the corresponding column of the coefficient matrix is computed as

$$\frac{\partial h}{\partial z^e} = \frac{1}{\Delta z^e} [h(z^e, v, \sigma; z^{**}) - h(z^e, v, \sigma; z^e)]$$  \hspace{1cm} (3.16)

Therefore, the deviation of the effective state variables in the steady-state configuration from those in the current configuration is obtained as

$$\begin{bmatrix} \Delta z_{(i)}^e \\ \Delta \sigma \end{bmatrix} = -\begin{bmatrix} \frac{\partial h}{\partial z_{(i)}^e} \\ \Xi \end{bmatrix}^{-1} \Gamma_2 \Gamma_1 \Gamma_3 \cdots \Gamma_n$$  \hspace{1cm} (3.17)

### 3.2 Application in Vehicle Dynamics

As an application of the steady-state analysis, the multibody model of a vehicle with four-wheel steering is presented. The vehicle has four independent double wishbone suspension subsystems as shown in Figure A.2 of Appendix A. Each suspension subsystem forms a closed kinematic loop containing two revolute and spherical joints. Each wheel is connected to its knuckle by a revolute joint. If one spherical joint in each closed-loop is cut, a vector of thirty joint coordinates for the reduced open-loop system is defined as

$$\theta = \begin{bmatrix} r \\ \phi \\ \theta^{(i)} \\ \theta^{(w)} \end{bmatrix}$$

where
The vectors $r$ and $\phi$ contain the translational and the rotational coordinates of the chassis, respectively. The vector $\theta^{(0)}$ contains 12 joint coordinates associated with the suspension subsystems excluding the cut joints, and $\theta^{(w)}$ contains 4 joint coordinates associated with the rolling of the four wheels. 16 constraint equations in the form of Eq. (3.5) can be written for this system: 12 kinematic constraints for the four cut spherical joints and 4 constraints for the steering command on the four wheels. The steering constraints assign values to the joint coordinates associated with the steering degrees of freedom of the wheels. The vector of joint velocities is defined as

$$\dot{\theta} = \begin{bmatrix} \dot{r} \\ \dot{\phi} \\ \dot{\theta}^{(0)} \\ \dot{\theta}^{(w)} \end{bmatrix}$$

where

$$\dot{\phi} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

contains the angular velocity components of the chassis.

The purpose of this analysis is to find the steady-state response of the system when the vehicle is driven along a circular path with an unknown radius $\rho$, and a known constant speed $U_0$. This condition on the speed of the vehicle provides one velocity constraint as

$$\kappa = (x^2 + y^2)^{1/2} - U_0 = 0$$

(3.18)
The circular path equation, $x^2 + y^2 = \rho^2$, provides expressions for two nonzero velocity dependent acceleration components as

$$\begin{align*}
\ddot{x} &= -U_0 \cos(\beta + \phi_z) \omega_z \\
\ddot{y} &= -U_0 \sin(\beta + \phi_z) \omega_z
\end{align*}$$

(3.19)

where $\beta$ is the side-slip angle of the vehicle (refer to Figure 3.1). This side-slip angle is computed as

$$\beta = \arctan\left(\frac{v_n}{v_6}\right) = \arcsin\left(\frac{v_n}{U_0}\right)$$

(3.20)

where $v_6$ and $v_n$ are velocity components of the chassis in the body fixed coordinate $(\xi\eta\zeta)$ and are defined as the longitudinal and lateral velocities, respectively.

For a flat horizontal terrain, since the steady-state equations are position independent, the location of the main chassis in the absolute coordinates can be chosen arbitrarily. However, the $z$-displacement of the center of the mass of the chassis changes the radial deformation of tires and affects the reaction forces between the tire and the terrain. Hence, there must be a unique $z$-coordinate of the main chassis which satisfies Eq. (3.13). This also results in a force equilibrium between the $z$-directional reaction force from the tire and the gravitational force on the system. In order to maintain a constant $z$-coordinate at the steady-state configuration, its velocity needs to be zero.

In order to keep the vehicle in a constant circular motion, the roll and the pitch velocities of the chassis must be zero. A unique yaw velocity needs to be maintained which would corresponds to the constant angular velocity of the circular motion. In the steady-state
configuration, the roll and pitch coordinates of the chassis have unique but unknown values. The remaining orientation coordinate; i.e., the yaw angle, can be chosen arbitrarily since it has no influence on the steady-state configuration of the system.

In the steady-state, the kinematic configuration is unique for each of the four suspension systems corresponding to the four wheels. Hence, the joint coordinates associated with the suspension systems have unique values and their corresponding joint velocities must be zero. In addition, to maintain the constant circular motion of the vehicle, each wheel must have a constant rolling speed. The rolling speed of each wheel is unique and it produces proper longitudinal force, lateral force, and self-aligning yaw moment which satisfy Eq. (3.13). However, since the rotation angle of the wheel along its joint axis does not effect the reaction force on the ground, the orientation of a wheel about its axis can be assigned an arbitrary value.

Based on above discussion, the zero, the ineffective and the effective variable vectors are now defined as

$$z^0 = \begin{bmatrix} \dot{z} \\ \omega_z \\ \omega_n \\ \phi_t^{(i)} \end{bmatrix}, \quad z^i = \begin{bmatrix} x \\ y \\ \phi_t \\ \theta^{(i)} \end{bmatrix}, \quad z^e = \begin{bmatrix} z \\ \phi_z \\ \phi_n \\ \theta^{(i)} \\ \dot{x} \\ \dot{y} \\ \omega_z \\ \dot{\theta}^{(i)} \end{bmatrix}$$

where $z^0$, $z^i$, and $z^e$ have dimensions of 23, 7, and 30 respectively. If Eq. (3.9) is used for the steady-state response of this system, $h$ contains 30 equations, $\Psi$ contains 16 equations,
and $\kappa$ contains 1 equation. This is a total of 47 equations in 47 unknowns; i.e., 30 elements in $z^e$, 16 elements in $v$, and one element in $\sigma$. However, if Eq. (3.13) is used, the number of algebraic equations, and the number of corresponding unknowns, can be reduced to 15.

Any accurate vehicle ride simulation, transient dynamics or steady-state, requires a realistic tire model. For this study, an analytical tire model with comprehensive slip [42-45] has been used. In this model, the interacting forces and moments between the tire and the road surface are determined analytically as functions of the tire orientation, velocity, tire stiffness characteristics and friction parameters. Figure 3.1 is a schematic diagram that shows the force equilibrium of the vehicle at the steady-state configuration.

3.3 Numerical Results

The 4WS vehicle model is subjected to two types of analyses, transient dynamics and steady-state. In the transient dynamic analysis, the model is assigned a constant forward speed of $10 \, \text{m/sec}$ and a steering command as shown in Figure 3.2. The front and rear wheels have initial zero steering angles. At $0.5 \, \text{sec}$ the wheels begin to steer to the left (a positive angle). After $1.5 \, \text{sec}$, the front wheels and the rear wheels reach final steering angles of 4 and 1 degrees, respectively. For the purpose of demonstration, the final steering angles are arbitrarily selected. The transient dynamic response for this model is shown in Figure 3.3-5. The roll, pitch, and yaw velocities of the chassis are shown in Figure 3.3. The spinning speed of the wheels versus time are shown in Figure 3.4, and the vertical displacement of the center of the mass of the chassis is shown in Figure 3.5. The transient response observed during the first $0.2 \, \text{sec}$ is due to the non-static equilibrium state at the start of the simulation. The initial conditions of the position and orientation of the vehicle do not correspond to the
static equilibrium state, which should not affect the final results, as observed. After approximately 1.5 sec, the vehicle reaches its steady-state configuration, following the path of a circle with a radius of \( r = 41.084 \text{ m} \).

For the steady-state analysis, the vehicle forward speed is also assigned to 10.0 m/sec, and the steering angles are set to 4 and 1 degrees for the front and the rear wheels, respectively. The initial estimate for the unknown coordinates was set to be the same as the initial conditions of the dynamic analysis, except for the steering angles. The solution of the steady-state equations for this 4WS vehicle model is presented in Table 3.1. It can be seen that the results are in full agreement with those obtained from the transient dynamic simulation.

The simulation of this example uses the Multi-Body System Simulation computer package (MBOSS) [41]. This is a general purpose computer program for the dynamic analysis of multibody systems which uses the joint coordinate formulation of equations of motion and an comprehensive analytical tire model and its interaction with the ground [42-45]. The steady-state analysis formulation described in this chapter is incorporated in this package.

A comparison between the CPU times for the transient dynamic and steady-state analyses showed that the steady-state analysis requires approximately 4% of the computation time as of that of the dynamic analysis (200 sec). The computer used for the simulations is the Sun SPARC station 1+.
3.4 Summary

A systematic methodology and formulation for determining the steady-state response of multibody systems has been presented. It has been shown that the steady-state equations for a multibody system can be obtained easily if a proper set of relative generalized coordinates is defined. In the steady-state configuration, simple relationships between the relative coordinates and velocities can be stated in order to obtain all of the necessary algebraic equations. The solution of the algebraic steady-state equations requires only a fraction of computational time as compared with the numerical solution of the transient dynamic equations of motion.
<table>
<thead>
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<th>Steady-state</th>
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</thead>
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<tr>
<td></td>
<td>(deg)</td>
<td></td>
</tr>
<tr>
<td>front</td>
<td>4.0</td>
<td>4.0</td>
</tr>
<tr>
<td>rear</td>
<td>1.0</td>
<td>1.0</td>
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</tr>
<tr>
<td>$\omega_n (rad/sec)$</td>
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<td>0.0</td>
</tr>
<tr>
<td>$\omega_t (rad/sec)$</td>
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<td>0.243</td>
</tr>
<tr>
<td>$\phi_t (rad)$</td>
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</tr>
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</tr>
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<tr>
<td>$\rho (m)$</td>
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Figure 3.1  Schematic diagram of a 4WS vehicle
Figure 3.2 Steering angle of the front and the rear wheels versus time.

Figure 3.3 Angular velocity components of the chassis versus time.
Figure 3.4 Spinning speed of the wheels versus time.

Figure 3.5 Vertical displacement of the center of the mass of the chassis versus time.
CHAPTER 4
DESIGN SENSITIVITY ANALYSIS

In multibody dynamic systems, specifically in vehicle dynamics, design parameters optimized in one driving condition may not provide the best performance in other driving conditions, due to the nonlinear nature of the governing equations of motion. In order to obtain generally acceptable optimal design parameters, these parameters need to be checked through several simulations and optimizations with a variety of driving conditions. This results in a large number of iterations of the dynamic analysis as well as the sensitivity analysis. In this respect, the computational speed becomes as important an issue as the accuracy of the sensitivity analysis itself. Furthermore, when large scale multibody dynamic systems, such as ground vehicles, are considered to be optimized, the computational speed in the sensitivity analysis is even more critical since the dynamic analysis itself takes a large amount of computational time. If the sensitivity analysis is purely based on explicit analytical expressions, developing a sensitivity analysis software for ground vehicles also becomes a major effort. Therefore, simplicity in programming and computational speed are critical issues in performing a realistic sensitivity analysis.

This chapter covers several methods for performing sensitivity analysis. Three existing methods will be studied where the equations of motion are expressed in terms of either the absolute Cartesian coordinates or the relative joint coordinates. A mixed method which combines some processes of the two existing sensitivity analysis methods (direct differentiation method and finite difference method) will be introduced to overcome the shortcomings in each of the existing methods.
4.1 Cost Sensitivity Analysis

A general dynamic response optimization problem is defined as a process to minimize a cost function $\psi$ which is based on the equations of motion of a given system. The cost function can be written as a function of system state variables and a set of design parameters as

$$\psi = \psi (b, q(b; t), v(b; t), \ddot{v}(b; t)) \quad (4.1a)$$

where $q$, $v$, and $\ddot{v}$ are vectors of Cartesian coordinates, velocities, and accelerations, respectively. The design parameter vector $b$ is assumed to have $r$ elements which may be bounded as $b_i^L \leq b_i \leq b_i^U$, where $b_i^L$ and $b_i^U$ are the lower and upper bounds of the $i$th design parameter.

If the equations of motion are expressed in terms of a set of joint coordinates, $\theta$, then Eq. (4.1a) can be expressed as

$$\psi = \psi (b, \theta(b; t), \dot{\theta}(b; t), \ddot{\theta}(b; t)) \quad (4.1b)$$

where $\theta$, $\dot{\theta}$, and $\ddot{\theta}$ are vectors of joint coordinates, velocities, and accelerations, respectively.

In certain applications, the cost function may be subjected to some constraints. They can be in the form of equality and/or inequality constraints. The constraints can also be expressed in the general form of Eq. (4.1a) or (4.1b) as function of $q$, $v$, $\ddot{v}$, $b$, and $t$ (or $\theta$, $\dot{\theta}$, $\ddot{\theta}$, $b$, and $t$). Therefore, the term "cost function $\psi$" refers to both the cost function and the constraints. Furthermore, this study is not concerned with any particular optimization algorithm. This study concentrates on methods for evaluating the cost sensitivity which is
a requirement by most optimization algorithms.

The cost sensitivity associated with the $i^{th}$ design parameter is described as the rate of a variation of the cost $\psi$ due to the variation of the associated design parameter as

$$\frac{d\psi}{db_i} = \lim_{\Delta b_i \to 0} \frac{\psi(b_i + \Delta b_i) - \psi(b_i)}{\Delta b_i}$$  \hspace{1cm} (4.2)

This cost sensitivity can be obtained by different methods such as: direct differentiation method, adjoint variable method, finite difference method, and mixed method.

### 4.2 Direct Differentiation Method

This method employs an analytical (direct) differentiation in the derivation of equations required for the cost sensitivity. A direct differentiation of the cost function with respect to design parameters can be written as

$$\frac{d\psi}{db} = \frac{\partial \psi}{\partial b} + \frac{\partial \psi}{\partial q} \frac{dq}{db} + \frac{\partial \psi}{\partial v} \frac{dv}{db}$$ \hspace{1cm} (4.3a)

where for the convenience of notation, $\frac{d}{db}$ denotes the total differentiation with respect to a design parameter vector $b$ for convenience of notation, although $\psi$ is an implicit function of $b$ and $t$. Eq. (4.3a) can also be expressed in a different form as

$$\psi_b = \frac{\partial \psi}{\partial b} + \frac{\partial \psi}{\partial q} q_b + \frac{\partial \psi}{\partial v} v_b$$ \hspace{1cm} (4.3b)

where $q_b$, $v_b$, and $\dot{v}_b$ can be defined as the state sensitivities of the coordinate, velocity vector, and acceleration vectors, respectively. In Eq. (4.3b), $\frac{\partial \psi}{\partial b}$, $\frac{\partial \psi}{\partial q}$, $\frac{\partial \psi}{\partial v}$ and $\frac{\partial \psi}{\partial \dot{v}}$ can be easily obtained through symbolic operation, because $\psi$ is generally defined as a simple
function of \( q, v, \psi, \) and \( b \) in most applications. In a cost sensitivity process, the sensitivity of vectors \( q_b \) and \( v_b \) are normally obtained by numerically integrating \( v_b \) and \( \psi_b \). Therefore, the main focus of the design sensitivity analysis is on the methods for evaluating the acceleration sensitivity \( \psi_b \).

If joint coordinates are used, then Eq. (4.3) is expressed as

\[
\psi_b = \frac{\partial \psi}{\partial b} + \frac{\partial \psi}{\partial \theta} \theta_b + \frac{\partial \psi}{\partial \dot{\theta}} \ddot{\theta}_b.
\]  

(4.3c)

and, therefore, the focus is on methods for evaluating \( \ddot{\theta}_b \).

### 4.2.1 Direct Differentiation with Cartesian Coordinate Formulation

The equations of motion of a multibody system are formulated based on a set of absolute Cartesian coordinates of each body, as given in Eqs. (2.3) and (2.8) or

\[
\begin{bmatrix}
M & D^T \\
D & 0
\end{bmatrix}
\begin{bmatrix}
\dot{v} \\
-\lambda
\end{bmatrix} =
\begin{bmatrix}
g \\
\gamma
\end{bmatrix}
\]  

(4.4)

In order to obtain an expression for the terms \( \psi_b \), the total differentiation of Eq. (4.4) with respect to \( b \) yields

\[
\begin{bmatrix}
M & D^T \\
D & 0
\end{bmatrix}
\begin{bmatrix}
\dot{\psi}_b \\
-\lambda_b
\end{bmatrix} =
\begin{bmatrix}
\frac{\partial}{\partial q} \left( g - M \dot{v} + D^T \lambda \right) q_b + \frac{\partial}{\partial v} v_b + \frac{\partial}{\partial b} \left( g - M \dot{v} + D^T \lambda \right) \\
\frac{\partial}{\partial b} \left( -D \dot{v} + \gamma \right) + \frac{\partial}{\partial q} \left( -D \dot{v} + \gamma \right) q_b + \frac{\partial}{\partial \dot{v}} \dot{\psi}_b
\end{bmatrix}
\]  

(4.5)

Since the design parameter vector \( b \) and the acceleration vector \( \psi \) have \( r \) and \( n \) elements respectively, the state sensitivity \( \psi_b \) is a \( n \times r \) matrix. On the right hand side of Eq. (4.5), the terms corresponding to operations \( \frac{\partial}{\partial q}, \frac{\partial}{\partial v}, \) and \( \frac{\partial}{\partial b} \) are analytically derived. Since the matrix
and vector components $\mathbf{M}$, $\mathbf{D}$, $\mathbf{g}$, $\gamma$ involve multiplication of transformation matrices, geometric vectors, and angular velocities which are functions of $q$, $v$, and $b$, the terms on the right hand side produce a large number of individual terms which are evaluated by using known variables obtained from Eq. (4.4). The state sensitivity terms $q_b$ and $v_b$ are obtained from numerical integration of $v_b$ and $\dot{v}_b$, respectively. Then, the acceleration sensitivity $\ddot{v}_b$ is obtained from Eq. (4.5) by using L-U factorization or Gauss elimination algorithms. Furthermore, the coefficient matrix on the left hand side of Eqs. (4.4) and (4.5) are the same. This means that the same factorized matrices obtained for Eq. (4.4) can be used in Eq. (4.5).

4.2.2 Direct Differentiation with Joint Coordinate Formulation

The generalized equations of motion for an open-loop multibody system is given by Eq. (2.10) as

$$\mathbf{B}^T \mathbf{M} \ddot{\mathbf{b}} = \mathbf{B}^T (\mathbf{g} - \mathbf{M} \dot{\mathbf{b}})$$  (4.6)

The direct differentiation of these equations with respect to $b$ yields the sensitivity equations as

$$\mathbf{B}^T \mathbf{M} \ddot{\mathbf{b}}_b = \frac{\partial}{\partial b} \left( \mathbf{B}^T (\mathbf{g} - \mathbf{M} \dot{\mathbf{b}}) - \mathbf{B}^T \mathbf{M} \dot{\mathbf{b}} \right) + \frac{\partial}{\partial \theta} \left( \mathbf{B}^T (\mathbf{g} - \mathbf{M} \dot{\mathbf{b}}) - \mathbf{B}^T \mathbf{M} \dot{\mathbf{b}} \right) \dot{\theta}_b$$

$$+ \frac{\partial}{\partial \theta} \left( \mathbf{B}^T (\mathbf{g} - \mathbf{M} \dot{\mathbf{b}}) - \mathbf{B}^T \mathbf{M} \dot{\mathbf{b}} \right) \dot{\theta}_b$$  (4.7)

Since the design parameter vector $b$ and the joint coordinates $\theta$ have $r$ and $k$ elements respectively, the state sensitivity $\theta_b$ is a $k \times r$ matrix. On the right hand side of Eq. (4.7), the terms corresponding to operations $\frac{\partial}{\partial \theta}$ and $\frac{\partial}{\partial b}$ are analytically derived. The matrix and
vector components $B$, $\dot{B}$, $M$, and $g$ in Eq. (4.7) are composed of multiplication of transformation matrices, geometric vectors, and angular velocities which are function of $q$, $v$, and $b$. Since the terms in Eq. (4.7) also consist of multiplications of the vectors and matrices, the terms on the right hand side produce a large number of individual terms which are evaluated by using known variables obtained from Eq. (4.6). The state sensitivity terms $\theta_b$ and $\dot{\theta}_b$ are obtained from numerical integration of $\theta_b$ and $\dot{\theta}_b$ respectively as discussed in the previous section.

For a multibody system with one or more closed kinematic loops, the equations of motion become similar to the Cartesian coordinate formulation. From Eqs. (2.13) and (2.14), the equations of motion are described as

$$\begin{bmatrix} B^T MB & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_b \\ -v_b \end{bmatrix} = \begin{bmatrix} B^T (g - MB\dot{\theta}) \\ \tau \end{bmatrix}$$

Equation (4.8)

In order to obtain an expression for the terms $\ddot{\theta}_b$, the total differentiation of Eq. (4.8) with respect to $b$ yields

$$\begin{bmatrix} B^T MB & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} \dddot{\theta}_b \\ -v_b \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial \theta} [B^T (g - MB\dot{\theta}) - M\ddot{\theta} + C^T v] \theta_b + \frac{\partial}{\partial \theta} [B^T (g - MB\dot{\theta})] \dot{\theta}_b + \frac{\partial}{\partial \theta} (B^T (g - MB\dot{\theta}) - M\ddot{\theta} + C^T v) \\ \frac{\partial}{\partial b} (-C\ddot{\theta} + \tau) + \frac{\partial}{\partial \theta} (-C\ddot{\theta} + \tau) \theta_b + \frac{\partial}{\partial \theta} \dot{\theta}_b \end{bmatrix}$$

Equation (4.9)

The solution procedure for the state sensitivity terms $\theta_b$, $\dot{\theta}_b$, and $\ddot{\theta}_b$ is the same as in the open-loop systems.
4.3 Adjoint Variable Method

In the direct differentiation method, the calculation of the state sensitivity matrices is a major drawback in performing the sensitivity analysis. In order to overcome this drawback, an adjoint system of a given system can be used. This method is generally applicable in an integration form of the cost function. The derivation of adjoint variable method based on equations of motion in Cartesian coordinates is presented in Ref.[12,13], where the coefficient matrix of accelerations remains constant in time. The derivation based on equations of motion in joint coordinates is presented here, where the coefficient matrix of accelerations varies with time.

A cost function may be expressed in an integral form as

$$
\psi = \int_0^T L(b, \theta(b; t), \dot{\theta}(b; t)) \, dt
$$

(4.10)

The variation of the cost function can be written as

$$
\delta\psi = \psi_s \delta b
$$

$$
= \int_0^T \left[ \frac{\partial L}{\partial b} \delta b + \frac{\partial L}{\partial \theta} \delta \theta + \frac{\partial L}{\partial \dot{\theta}} \delta \dot{\theta} \right] \, dt
$$

(4.11)

One of the objectives of this method is to eliminate explicit dependencies on $\delta \theta$ and $\delta \dot{\theta}$, and to write $\delta \psi$ explicitly in terms of $\delta b$ in Eq. (4.11). To derive this, an adjoint system of the nominal system can be introduced and derived in the following process. The set of state variables of this adjoint system is denoted by the adjoint variable vector $\mu$. This adjoint variable vector corresponds to the appended state vector $[\theta^T, \dot{\theta}^T]^T$.

The equations of motion of the nominal system can be rearranged for convenience in constructing the adjoint system as
\[
\begin{bmatrix}
I & 0 \\
0 & B^TMB
\end{bmatrix}
\begin{bmatrix}
\dot{\theta} \\
\dot{\theta}
\end{bmatrix}
= 
\begin{bmatrix}
\theta \\
B^T(g - MB\theta)
\end{bmatrix}
\]  

(4.12)

After setting \( z = [\theta^T, \dot{\theta}^T]^T \), Eq. (4.12) can be written as

\[
P(z, b)\dot{z} + h(z, b) = 0
\]  

(4.13)

where

\[
P(z, b) = \begin{bmatrix} I & 0 \\ 0 & B^TMB \end{bmatrix}, \quad h(z, b) = \begin{bmatrix} \theta \\ B^T(g - MB\theta) \end{bmatrix}
\]

Eq. (4.13) can be used as constraint equations of the cost function \( \psi \). The cost function can then be combined the constraint equations and their adjoint variable vector \( \mu \) as

\[
\psi = \int_0^T \{ L(b, z) + \mu^T [P(z, b)\dot{z} + h(z, b)] \} dt
\]  

(4.14)

By performing the first variation of Eq. (4.14), it can be written as

\[
\psi_0\delta b = \int_0^T \left( \frac{\partial L}{\partial b}\delta b + \frac{\partial L}{\partial z}\delta z \right) dt + \int_0^T \mu^T \left( \frac{\partial (Pz)}{\partial b} + \frac{\partial h}{\partial b} \right) \delta b dt + \int_0^T \mu^T P\delta z dt
\]  

(4.15)

where

\[
\int_0^T \mu^T P\delta z dt = \mu^T(T)P\delta z(T) + \mu^T(0)P\delta z(0) - \int_0^T \mu^T P\delta z dt - \int_0^T \mu^T P\delta z dt
\]  

(4.16)

and

\[
\delta z(0) = \frac{\partial z(0)}{\partial b} \delta b, \quad \mu(T) = 0
\]  

(4.17)

After reorganizing Eq. (4.15), the cost sensitivity can be composed of two variational terms as
By choosing the adjoint variable vector \( \mu \) to satisfy the variational term of \( \delta z \) to be zero, the adjoint equations are constructed and the cost sensitivity equations are also simplified as

\[
\psi_b \delta b = \mu^T(0) P \frac{\partial z(0)}{\partial b} \delta b + \int_0^T \left( \frac{\partial L}{\partial b} - \mu^T \left( \frac{\partial (Pz)}{\partial b} + \frac{\partial h}{\partial b} \right) \right) \delta b dt \\
+ \int_0^T \left( \frac{\partial L}{\partial z} + \mu^T P - \mu^T \left[ \frac{\partial (Pz + h)}{\partial z} - \dot{P} \right] \right) \delta z dt
\]  

(4.18)

In Eqs. (4.19) and (4.20), the state sensitivity analysis process which produces a large number of equations for the sensitivity analysis in the direct differentiation method is successfully eliminated. In order to construct the adjoint equations the terms corresponding to operation \( \frac{\partial}{\partial x} \) are derived analytically. Since the terms \( P \) and \( h \) also contains \( B, \dot{B}, M, \) and \( g \) as observed in the direct differentiation method, the analytical derivation also produces a large number of terms.

To solve the cost sensitivity \( \psi_b \) in the integration process, the adjoint variable vector \( \mu \) must be known for the overall time period. Since \( \mu \) is known at the final time \( T \) as shown in Eq. (4.18), \( \mu \) can only be solved through backward integration of Eq. (4.19). This backward integration can be also performed based on the state of the nominal system obtained from the forward integration of Eq. (4.12). Therefore, this method requires a large amount of computer memory to store the data solved through forward and backward integrations.
If different (or variable) time steps are used in each integration process, an interpolation process is also required to calculate the states of the nominal system and the adjoint system at a certain time.

4.4 Finite Difference Method

This method uses a numerical perturbation to perform the sensitivity analysis. The equations of motion and the cost function of perturbed systems are constructed based on the nominal system, where the number of constructed perturbed systems correspond to the number of design parameters. All of the equations of motion and the cost function are simultaneously solved through numerical integration. Then, the sensitivity of the cost function for each design parameter is evaluated as

$$\psi_{b_i} = \frac{\psi(b_i + \Delta b_i) - \psi(b_i)}{\Delta b_i} \quad (4.21)$$

or

$$\psi_{b_i} = \frac{\psi(b_i + \Delta b_i) - \psi(b_i - \Delta b_i)}{2\Delta b_i} \quad (4.22)$$

where Eq. (4.21) and (4.22) are the forward difference scheme and the central difference scheme, respectively. Since in order to compute the cost sensitivity, only the final values of the cost function of the nominal system and the perturbed systems are required, the calculation of the state sensitivity is not required. Therefore, when the forward difference is used, for \( r \) design parameters it only requires the equations of motion to be evaluated \( r + 1 \) times. Moreover, this method does not require any additional programming effort for the sensitivity analysis if the dynamic analysis program is already constructed. The method is illustrated in Figure 4.1.
The performance of this method depends on computational efficiency in solving the equations of motion of the system. When the equations of motion for a large scale system are described in Cartesian coordinates, it takes a large amount of computation time to carry out the sensitivity analysis due to the computational inefficiency in solving these equations. On the other hand, when the equations of motion are described in the joint coordinates, this problem is improved drastically due to the numerical efficiency in solving equations of motion.

4.5 Comparison of Existing Design Sensitivity Methods

In the direct difference method, the cost sensitivity is obtained based on the state and the state sensitivity of the system. The state sensitivity is calculated by solving $r$ sets of sensitivity equations for $r$ design parameters. In the sensitivity equations, $\frac{\partial}{\partial \psi}$ and $\frac{\partial}{\partial \phi}$, or $\frac{\partial}{\partial \psi}$ and $\frac{\partial}{\partial \phi}$ require analytical derivations and an analytical derivation of one single term generates a large number of additional terms. For large scale dynamic systems, such as ground vehicles, the size of the equations of motion is large and these equations may contain a comprehensive tire model or some other nonlinear model of various components. Hence, for the large scale dynamic system, the sensitivity equations requires an enormous amount of effort in analytical derivation and computer programming. When the Cartesian coordinates are used in the sensitivity analysis, the large number of the equations of motion also produces a large number of sensitivity equations. Hence, this method requires extensive computation time. On the other hand, when the joint coordinates are used, the total number of equations in the sensitivity analysis is reduced considerably when compared to the same equations in terms of absolute Cartesian coordinates. However, analytical derivation becomes cumbersome due to complexity in each term in the equations of motion.
The adjoint variable method solves the cost sensitivity based on the state and the adjoint state of the system. Hence, this method reduces the total number of equations which needs to be solved. However, the terms corresponding to operation $\frac{\partial}{\partial x}$ in the adjoint equations requires analytical derivations as observed in the direct differentiation method. Furthermore, the integration process in the adjoint variable method is cumbersome and numerical errors are encountered during this process as discussed in the previous section.

In the finite difference, the cost sensitivity is obtained by solving the equations of motion of the nominal and perturbed systems. The sensitivity analysis for $r$ design parameters requires solutions of $r+1$ sets of equations of motion. When the Cartesian coordinates are used in the sensitivity analysis, the large number of equations of motion also requires extensive computation time. On the other hand, when the joint coordinates are used, the size of equations of motion is reduced. Hence, the computation time is further reduced. However, a well known problem associated with this method is the numerical error caused by an inappropriate perturbation of the design parameter. As the magnitude of perturbation of a design parameter becomes larger, the truncation error becomes dominant. As the magnitude of perturbation of a design parameter becomes smaller, the roundoff error becomes the dominant factor. Furthermore, the equations of motion and the cost functions are solved by a numerical integration process. When a low accuracy numerical integration is used, the numerical error from the integration process may overshadow small changes in the system behavior due to the small perturbation of the design parameter. Therefore, the results of the sensitivity analysis are no longer reliable.
4.6 Mixed Method

The previous three methods have some advantages and disadvantages over one another. In order to make up for these drawbacks, a new approach will be developed which incorporates the beneficial features of the previous methods. The first objective of this method is to avoid the complicated analytical derivation required in the direct differentiation and the adjoint variable methods. Complex analytical derivation requires complicated and lengthy computer programs. The second objective is to improve numerical errors encountered in the finite difference method.

This method uses the sensitivity equations from the formulation of the direct differentiation method, however, the terms in the sensitivity equations are obtained by the finite difference method. Hence, the method can reduce the analytical difficulty encountered in the direct differentiation method and the numerical errors in the finite difference method. A schematic diagram of this method is illustrated in Figure 4.2.

The joint coordinate formulation has advantages in the size of equations and numerical efficiency. Hence, this method uses the sensitivity equations, Eq. (4.7) or (4.9) for the case of open-loop or closed-loop system. For a open-loop system, the sensitivity equation can be rearranged from Eq. (4.7) as

\[ [B^TMB]\hat{\theta}_b = \left[ \frac{\partial \Gamma}{\partial \theta} + \frac{\partial \Gamma}{\partial \theta} \hat{\theta}_b + \frac{\partial \Gamma}{\partial \theta} \hat{\theta}_b \right] \] (4.23)

or

\[ [B^TMB]\hat{\theta}_b = \left[ \Gamma_b + \Gamma_b \hat{\theta}_b + \Gamma_b \hat{\theta}_b \right] \] (4.24)

where
\[
\Gamma = B^T(g - MB\dot{\theta} - MB\ddot{\theta})
\]  

(4.25)

For a closed-loop system, the sensitivity equations can be rearranged similar to an open-loop system. Then, Eq. (4.10) can be written as

\[
\begin{bmatrix}
B^TMB & C^T \\
C & 0
\end{bmatrix}
\begin{bmatrix}
\ddot{b}_i \\
v_i
\end{bmatrix} =
\begin{bmatrix}
\frac{\partial \Gamma}{\partial b_i} + \frac{\partial \Gamma}{\partial \theta_i} \dot{\theta}_i + \frac{\partial \Gamma}{\partial \dot{\theta}_i} \\
\end{bmatrix}
\]  

(4.26)

or

\[
\begin{bmatrix}
B^TMB & C^T \\
C & 0
\end{bmatrix}
\begin{bmatrix}
\ddot{b}_i \\
v_i
\end{bmatrix} =
\begin{bmatrix}
\Gamma_{b_i} + \Gamma_{\theta} \dot{\theta}_i + \Gamma_{\dot{\theta}} \ddot{\theta}_i \\
\end{bmatrix}
\]  

(4.27)

where

\[
\Gamma = \begin{bmatrix}
B^T(g - MB\dot{\theta} - MB\ddot{\theta}) + C^T v \\
-C\dot{\theta} + \tau
\end{bmatrix}
\]  

(4.28)

In the mixed method, the terms \(\Gamma_{\theta}\) and \(\Gamma_{\dot{\theta}}\) are computed by using the finite difference method. The term \(\Gamma_{b_i}\) is obtained numerically or analytically depending on the specific design parameter. When the terms \(\Gamma_{\theta}\) and \(\Gamma_{\dot{\theta}}\) are numerically evaluated, each of the state vectors \(\theta\) and \(\dot{\theta}\) are individually perturbed. Hence, the number of evaluations of \(\Gamma\) is the same as the total number of the state vectors \(\theta\) and \(\dot{\theta}\) in both open- and closed-loop systems. Generally, this number is relatively small in the joint coordinate formulation of the equations of motion compared to the Cartesian coordinate formulation. Furthermore, once the matrices \(\Gamma_{\theta}\) and \(\Gamma_{\dot{\theta}}\) are evaluated, they are used in deriving the sensitivity vectors for other design parameters.

In this method, the integrations of the state sensitivity and the nominal state are completely separated. Thus, different integration algorithms or different time steps in the
same algorithm can be used in the state sensitivity analysis and in the nominal system analysis. For the case of a crude sensitivity analysis, the computation time can be reduced by using a large time step in the same integration routine or a relatively crude integration algorithm in the integration of the state sensitivity matrices. Furthermore, once the coefficient matrix in the equations of motion is evaluated, the same matrix can be used in the sensitivity equations.

4.6.1 Formulation of $\Gamma_0$

The elements in matrix $\Gamma_0$ consist of the column vectors $\Gamma_i$'s, for the $i$th element of the selected joint coordinates. From Eq. (4.24), each column vector $\Gamma_i$ is written as

$$\Gamma_i = \frac{\partial \Gamma}{\partial \theta_i} = \frac{\partial}{\partial \theta_i} [B^T (g - M\theta - B^T M\theta)]$$

(4.29)

where $\theta$ and $\theta'$ are independent from the variation of $\theta$. The operation $\frac{\partial}{\partial \theta_i}$ is performed by using numerical perturbation.

The value of $\Gamma$ for the nominal system is zero and for the perturbed system is the same as the residual of the equations of motion. Hence, the numerical calculation of operation $\frac{\partial}{\partial \theta_i}$ can be performed by using perturbed $\theta$ as

$$\theta_i^* = \theta_i + \Delta \theta_i$$

(4.30)

$$\Gamma_i^* = B^T (g^* - M\theta^* - B^T M\theta^*)$$

(4.31)
where superscript * is a term corresponding to the perturbed $\theta^*_i$. The term $\Gamma_\theta$ can be written as

$$\Gamma_\theta = \frac{\Gamma_i}{\Delta \theta_i}$$  \hspace{1cm} (4.32a)

and this column vector $\Gamma_\theta$ fills the $i$th column of the matrix $\Gamma_\theta$ as

$$\Gamma_\theta = [\Gamma_{\theta_1}, \ldots, \Gamma_{\theta_i}, \ldots, \Gamma_{\theta_n}]$$  \hspace{1cm} (4.32b)

This process can be also applied into a closed-loop system by using the following expression for $\Gamma$ as

$$\Gamma = \begin{bmatrix} B^T (g - MB\theta - MB\dot{\theta}) + C^T v \\ -C\dot{\theta} + \tau \end{bmatrix}$$  \hspace{1cm} (4.33)

### 4.6.2 Formulation of $\Gamma_\theta$

The numerical calculation of this term is performed by using the perturbed $\dot{\theta}$ as

$$\dot{\theta}^*_i = \dot{\theta}_i + \Delta \theta_i$$  \hspace{1cm} (4.34)

Note that the term $\dot{\theta}$ only appears in $g$ and $B$ of $\Gamma$. Therefore, the calculation of $\Gamma_\theta$ is concerned with the terms that are a function of $\dot{\theta}$. These terms can be expressed as

$$c = g - MB\dot{\theta}$$  \hspace{1cm} (4.35)

It can be obtained directly from the calculation of the nominal system. Then, the vector $c$ corresponding to the perturbed $\dot{\theta}_i$ will be obtained from,
\[ \dot{c}_i^* = g^* - MB^* \dot{\theta}^* \]  

(4.36)

where superscript (*) is a term corresponding to the \( \dot{\theta}_i^* \). The term \( \Gamma_{\theta_i} \) can be written as

\[ \Gamma_{\theta_i} = \frac{c_i^* - c}{\Delta \theta_i} \]  

(4.37)

For a closed-loop system, the vector \( c \) can be obtained from the following expression:

\[ c = \begin{bmatrix} g - MB\dot{\theta} \end{bmatrix} \]  

(4.38)

### 4.6.3 Formulation of \( \Gamma_b \)

The most commonly considered design parameters in the design process of multibody systems are inertial properties, spring and damper locations and characteristics, and the locations of the kinematic joint. The inertial properties simply appear as linear factors in the mass matrix \( M \) and the force vector \( g \) (not in the velocity transformation matrix \( B \)). The partial derivatives of \( f \) can be written as

\[ \frac{\partial \Gamma}{\partial m_i} = B_{i(\Theta)} \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} g_{e} - B_{i(\Theta)} \dot{\theta} - B_{i(\Theta)} \ddot{\Theta} \]  

(4.39)

where the subscripts \( i \) and \( (r) \) are the \( i \)th body and its translational coordinates, respectively, and \( B_{i(\Theta)} \) is the corresponding row block submatrix of the velocity transformation matrix \( B \). The gravitational constant \( g_e \) is applied when the given system is in the gravitational field.

Spring and damper coefficients, and their attachment position vectors with respect to a body-fixed coordinate system, are embedded in vector \( g \) in Eq. (4.25). Then, the partial derivatives of \( \Gamma \) with respect to those design parameters can be written as
\[ \frac{\partial \Gamma}{\partial K_{ij}} = [B_i^T, B_j^T] \begin{bmatrix} \mathbf{I} \\ \hat{s}_i \\ -\hat{s}_i \end{bmatrix} (l_{ij} - l_{ij}^0)u_{ij} \]  
(4.40)

\[ \frac{\partial \Gamma}{\partial C_{ij}} = [B_i^T, B_j^T] \begin{bmatrix} \mathbf{I} \\ \hat{s}_i \\ -\hat{s}_i \end{bmatrix} l_{ij}u_{ij} \]  
(4.41)

\[ \frac{\partial \Gamma}{\partial \hat{s}_i} = [B_i^T, B_j^T] \begin{bmatrix} \mathbf{I} \\ \hat{s}_i \\ -\hat{s}_i \end{bmatrix} \begin{bmatrix} \frac{\partial g^i}{\partial \hat{s}_i} & -\frac{\partial g^i}{\partial \hat{s}_i} \\ 0 & 0 \end{bmatrix} \]  
(4.42)

where

\[ g^i = K_{ij}(l_{ij} - l_{ij}^0)u_{ij}, \]

\[ \frac{\partial g^i}{\partial \hat{s}_i} = \frac{\partial g^i}{\partial r_i} A_i = -K_{ij} \left[ \mathbf{I} \left( 1 - \frac{l_{ij}^0}{l_{ij}} \right) + \frac{l_{ij}^0}{l_{ij}} u_{ij} u_{ij}^T \right] A_i \]  
(4.43)

The subscript \( i \) and \( j \) are the two connected bodies by the spring and the damper. The terms \( l \) and \( l_0 \) are the total length and the undeformed length of the spring or the damper, and \( u \) is the direction of the spring or the damper. The prime ('') stands for a vector in a body fixed coordinate system \( (\xi, \eta, \zeta) \), and an illustration is provided in Figure 4.3. For a closed-loop system, since the design parameters do not affect the constraint(s) at the cut joint, no additional calculation is required to obtain \( \Gamma \).

The position vectors with respect to a body-fixed coordinate system is implicitly embedded in matrices \( B \) and \( \dot{B} \). Since, an analytical derivation of the term \( \Gamma_{h_i} \) can be complicated, the term \( \Gamma_{h_i} \) will be evaluated by numerical perturbation as
For a closed-loop system, the term $\Gamma$ is taken from Eq. (4.28).

4.6.4 Numerical Example

In order to demonstrate the feasibility of this method, a simple mechanical system is considered as shown in Figure 4.4. The cost sensitivity results are compared against those obtained from the finite difference method in terms of accuracy and also computational efficiency.

This example system is a quarter car with a double A-arm suspension system. The data for constructing the model are taken from the data of 4WS vehicle presented in Appendix A. For the sake of convenience in the simulation process, the chassis is constrained to have motion in the z-direction. Hence, this system has four degrees of freedom. The tire is mounted on the ground subject to a sine wave given by

$$z_g = A \sin(2\pi f_c t)$$

(4.45)

In this model, the sine wave is assigned with a 0.02 m amplitude and a 3 Hz frequency.

The selected cost function of this system is described as

$$\psi = \int_0^1 \tilde{z}^2 dt$$

(4.46)

where $\tilde{z}$ is the z-directional acceleration of the chassis. The considered design parameters are masses of body 1 and 3, stiffness and damping coefficient of the suspension, radial stiffness of the tire, and the attachment point of the link of joint 5. Hence, the design parameter vector is described as $b^T = [m_1, m_3, K, D, C, s^T_z]$. The unperturbed design
parameters are given as \([200.0, 5.0, 40.0 \times 10^3, 10.0 \times 10^3, 1.5 \times 10^5, 0.0]\).

The cost sensitivity can be obtained analytically from Eq. (4.46) as

\[
\psi_b = \int_0^1 2\ddot{z} \dot{x}_b \, dt \quad (4.47)
\]

where \(\ddot{z}\) is obtained from solving equations of motion and \(\dot{x}_b\) is obtained from solving the sensitivity equations. The simulation of this example uses the Multi-Body System Simulation computer package (MBOSS) [41]. The mixed method for the sensitivity analysis is incorporated in this package. In this method, the Gear integration algorithm is used to solve the nominal system and the Euler integration algorithm is used to solve the state sensitivity and the cost sensitivity.

Table 4.1 shows the sensitivity results from the finite difference method and the mixed method. It can be seen that the sensitivity results from the mixed method are in good agreement with those obtained from the finite difference method. The CPU times for both methods are comparable in the case of six design parameters. However, in order to build confidence in the results from the finite difference method, several trials of sensitivity analysis are required. Therefore, it can be inferred that for an optimal design of a large number of design parameters the mixed method is superior to the finite difference method.

4.7 Steady-State Sensitivity Analysis

In some applications, an optimization problem may require minimization of a cost function \(\psi\) based on the steady-state configuration of the system. The cost function of Eq. (4.1) can then be simplified as a function of system state variables at steady-state, and a set of design parameters as

\[
\psi = \psi(\mathbf{b}, \theta(\mathbf{b}), \dot{\theta}(\mathbf{b}), \ddot{\theta}(\mathbf{b})) \quad (4.47)
\]
By using the steady-state analysis formulation of Chapter 3, the vectors of joint coordinates, velocities, and accelerations at the steady-state can be grouped as zero, ineffective, and effective state variables. Since, the zero and the ineffective state variables are not affected by a change in the design parameters, the cost function can be further simplified to

$$\psi = \psi (b, z^e(b))$$  \hspace{1cm} (4.48)

where $z^e$ is the vector of effective state variables.

The sensitivity analysis methods studied so far in this chapter are based on the integrations of equations of motion and the cost function. However, for the steady-state sensitivity analysis, since the steady-state response is independent of time, the cost sensitivity can be simply evaluated by the finite difference method. Therefore, the cost sensitivity can be written as

$$\psi_{b_i} = \frac{\psi^*(z^e, b; b_i + \Delta b_i) - \psi^*(z^e, b; b_i)}{\Delta b_i}$$  \hspace{1cm} (4.49)

where $\psi^*(z^e, b)$ and $\psi^*(z^e, b; b_i + \Delta b_i)$ are the cost functions evaluated from the steady-state analyses of the unperturbed and the perturbed systems, respectively.

### 4.8 Summary

Various sensitivity analysis methods are studied in this chapter. In the sensitivity analysis of multibody systems in transient dynamics, the mixed method proposed in this chapter provides computational efficiency and numerical accuracy in the solution process. In the sensitivity analysis of multibody systems in the steady-state configuration, the finite difference method can provide computational efficiency and numerical accuracy in the solution process.
Table 4.1. Comparison of results of the finite difference method (F.D.M.) and the mixed method.

<table>
<thead>
<tr>
<th></th>
<th>F. D. M.</th>
<th>Mixed Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{\partial \psi}{\partial m_1}$</td>
<td>-7.40x10^{-2}</td>
<td>-8.06x10^{-2}</td>
</tr>
<tr>
<td>$\frac{\partial \psi}{\partial m_3}$</td>
<td>4.72x10^{-2}</td>
<td>4.32x10^{-2}</td>
</tr>
<tr>
<td>$\frac{\partial \psi}{\partial K}$</td>
<td>-4.45x10^{-6}</td>
<td>-4.14x10^{-6}</td>
</tr>
<tr>
<td>$\frac{\partial \psi}{\partial D}$</td>
<td>1.54x10^{-3}</td>
<td>1.55x10^{-3}</td>
</tr>
<tr>
<td>$\frac{\partial \psi}{\partial C_2}$</td>
<td>-1.09x10^{-5}</td>
<td>-1.00x10^{-5}</td>
</tr>
<tr>
<td>$\frac{\partial \psi}{\partial \delta_2'}$</td>
<td>23.51</td>
<td>23.47</td>
</tr>
<tr>
<td>CPU (sec)</td>
<td>42</td>
<td>40</td>
</tr>
</tbody>
</table>
$$\mathbf{b} = [b_1, b_2, \Delta b_2, b_3, \ldots, b_L]^T$$ Perturbed system

$$\mathbf{b} = [b_1^{+\Delta}, b_2, b_3, \ldots, b_L]^T$$ Perturbed system

$$\mathbf{b} = [b_1, b_2, b_3, \ldots, b_L]^T$$ Nominal system

Figure 4.1 Schematic flow chart of the finite difference method
Figure 4.2 Schematic flow chart of the mixed method
Figure 4.3 Schematic diagram of two bodies with a spring and a damper
Selected design parameters

\[ [m_1, m_3, K, D, C_z, S_z^p] \]

\[ z_g = A \sin (wt) \]

Figure 4.4 Schematic diagram of a quarter car model.
CHAPTER 5
FOUR-WHEEL-STEERING VEHICLES

A four-wheel steering (4WS) is a steering control system which allows rear wheels of a vehicle to be steered unlike the conventional two-wheel steering (2WS) system where the longitudinal directions of rear wheels remain fixed relative to the chassis. Implementation of 4WS systems will enhance overall performance of the vehicle: Ergonomically, it becomes easier to turn the vehicle; Dynamically, it reduces unnecessary yaw motion resulting in improved stability. The above benefits can be derived by properly choosing the steering angle ratio \( K_r \) which is the ratio of the rear wheel steering angle to the front wheel steering angle. The primary focus of the 4WS system study is to determine the proper steering angle ratio. A general review of the characteristics of 4WS systems is presented in this chapter.

5.1 General Characteristics

Vehicle maneuver is accomplished by lateral and longitudinal forces generated at the tires which causes a yaw moment acting on the vehicle. The lateral forces at the tires are due to the side-slip angle of the tire. The side-slip angle \( \alpha \) of the tire is the angle between the moving direction and the longitudinal direction of the tire as shown in Figure 5.1. The yaw moment acting on the vehicle is the sum of the moments of the longitudinal and the lateral forces at four wheels.

In 2WS vehicles, the rear wheels cannot be steered relative to the chassis and therefore the side-slip angles of the rear wheels are dependent on the longitudinal direction of the chassis. However, the rear wheels in a 4WS system can be steered relative to the chassis,
A path of the vehicle

Moving direction
of the vehicle

C.G. of the vehicle

Moving direction
of the tire

Longitudinal direction
of the vehicle

Longitudinal direction
of the tire

$\beta$: Side-slip angle
of the vehicle

$\alpha$: Side-slip angle
of the tire

Figure 5.1 Side-slip angles of the vehicle and tire
and hence, the side-slip angle and the lateral forces of the rear wheels can be controlled by steering the rear wheels. Another important parameter of the vehicle dynamics is the side-slip angle of the vehicle. The side-slip angle ($\beta$) of the vehicle is the angle between the moving direction and the longitudinal direction of the vehicle as shown in Figure 5.1. By reducing the side-slip angle of the vehicle, the vehicle moves constantly in longitudinal direction which in turn makes maneuvering easier and also requires less energy to turn the vehicle.

5.2 Steady Turning Behavior

Steady turning behavior deals with directional behavior of a vehicle while negotiating a curve with a constant radius and a constant forward speed. The behavior of 2WS and 4WS vehicles are compared in low and high speeds under steady turning conditions to demonstrate the benefits of 4WS.

In low speeds, the inertia properties of the tires and the vehicle are not significant. In the absence of inertia properties, the side-slip angle and the lateral force of a tire are small and therefore, the wheel tends to move along its longitudinal direction. Figure 5.2 shows a comparison of steady turning behavior of 2WS and 4WS bicycle models with same turning radius in low speeds. A bicycle model is a simplified representation by joining the right and left wheels of each of the axles into one wheel on the center line of the vehicle body. In Figure 5.2, the vectors represent the linear velocity of the wheels and the chassis.

The 2WS turning behavior shown in Figure 5.2a contains a large side-slip angle of the vehicle. The center of the path circle of the vehicle is the intersection of the perpendiculars to the velocity vectors which coincide with the longitudinal direction of the front and rear wheels in low speeds. By steering the front wheels, the longitudinal direction of the front
Figure 5.2 Turning behaviors of 2WS and 4WS in low speeds; 
a) 2WS, and b) 4WS contra phase and c) 4WS same phase.
and rear wheels are oriented in two different directions. This orients the longitudinal
direction of the vehicle outside the turning path resulting in a large side-slip angle of the
vehicle. To eliminate the side-slip angle of the vehicle, in a 4WS system, the rear wheel is
turned in the opposite direction to that of the front wheel (contra phase) as shown in Figure
5.2b. It can be observed that by properly steering the front and the rear wheels, the side-slip
angle \( \beta \) can be made equal to zero. However, if the rear wheel is steered in the same direction
as that of the front wheel (same phase), in order to maneuver the vehicle along the same
radius of circle, a much larger side-slip angle \( \beta \) is required as shown in Figure 5.2c. Generally,
large values of the side-slip angle of the vehicle will be accompanied by a transient yaw
motion and will produce an undesirable driving condition.

When a vehicle is moving along a curve at high speeds, the influence of centrifugal
forces can no longer be ignored. To maintain a force equilibrium with the presence of
centrifugal forces, proper lateral forces are required at the tires. The sum of the components
in the radial direction of these lateral forces must be equal to the centrifugal forces. These
lateral forces are produced by appropriate side-slip angles of the four tires. Also, a steady
turning of the vehicle is accomplished by producing a zero yaw moment condition. This
condition requires proper magnitude and direction of the lateral force on the wheels.

Figure 5.3 shows the steady turning behavior of 2WS and 4WS vehicles with the same
turning radius in high speeds. To achieve a proper side-slip angle of the rear wheel, the
longitudinal direction of the vehicle must be oriented inside the turning path resulting in a
large side-slip angle of the vehicle as demonstrated in Figure 5.3a. The contra phase in 4WS
system requires larger side-slip angle \( \beta \) than 2WS system to produce proper side-slip angle
and lateral force on the rear wheels as shown in Figure 5.3b. However, by turning the rear
Figure 5.3 Turning behaviors of 2WS and 4WS in high speeds; a) 2WS, and b) 4WS contra phase and c) 4WS same phase.
wheel in the same phase, a lateral force at the rear wheel can be generated without producing a side-slip angle for the chassis as shown in Figure 5.3c. As observed before, the side-slip angle $\beta$ in high speeds also produces transient yaw motion which in turn reduces the overall stability of the vehicle.

5.3 Summary

During vehicle maneuvering along a curve, the 2WS generally produces large side-slip angle of the vehicle. The side-slip angle of the vehicle in a 4WS can be reduced by orienting the rear wheels in contra phase in low speed ranges and in same phase in high speed ranges. This will enhance the overall performance of the vehicle. Therefore, the selection of a proper steering angle ratio $K_r$ is important in 4WS system. The following chapter discusses the strategies for obtaining suitable steering angle ratio in 4WS systems.
CHAPTER 6

OPTIMAL STEERING ANGLE RATIOS IN 4WS VEHICLE

A 4WS vehicle is designed to attain quick and easy maneuverability at low speeds and enhanced stability at high speeds. Achieving improved maneuverability and stability involves finding the optimal steering angle ratio $K_r$ of the front wheel to the rear wheel of the 4WS vehicle in various driving scenarios. For a 4WS system, as discussed in Chapter 5, the side-slip angle of the vehicle is chosen as the cost function in the process of finding the optimal steering angle ratio.

In this chapter, optimal steering angle ratios are found using both the linear bicycle model and the full nonlinear vehicle model. The linear model does not consider the kinematics of the suspension system and the complex dynamic behavior of the tire. Hence, the equations of motion of the linear model have only two degrees of freedom with a simple tire model. The optimal steering angle ratio which provides zero side-slip angle of the vehicle can then be obtained directly from the equations. However, since the nonlinear model considers the kinematics of the suspension system and a comprehensive tire model, the equations of motion are highly nonlinear. Hence, the optimal steering angle ratio needs to be numerically evaluated through the steady-state analysis.

6.1 Linear Model of the 4WS Vehicle

Most of the studies on 4WS systems used a so-called linear bicycle model. The equations of motion of this model are linear and have two degrees of freedom: the yaw
motion and the side-slip angle of the vehicle [28]. The schematic diagram of the linear bicycle model is shown in Figure 6.1, and as it can be observed, this is a planar model in the horizontal plane. The assumptions used in the linear bicycle model include small steering angle, constant speed, and lateral forces being linear functions of the side-slip angle of the tire. Then, the forces acting on the front and rear wheels can be computed as

\[ F_{nf} = -C_{nf} \left( \beta + \frac{a \omega_z}{U_0} - \theta_r \right) \]  

(6.1)
where \( C_{af} \) and \( C_{ar} \) are the cornering stiffness coefficients of the front and rear tires, respectively. The parameters \( a \) and \( b \) are the distances from the center of the mass of the vehicle to the front wheel and to the rear wheel, respectively. The variable \( \omega_r \) is the yaw velocity and the parameter \( U_0 \) is the constant forward speed. The remaining parameters \( \theta_f \) and \( \theta_r \) are the steering angles of the front and rear wheels, respectively.

For a vehicle in plane motion, the equations of motion using \( \eta \) and \( z \) axes in the body fixed coordinates can be written as

\[
M(\dot{y}_\eta + U_0 \omega_r) = F_{ny} \sin \theta_f + F_{nr} \sin \theta_r \tag{6.3}
\]

\[
I_z \ddot{\omega}_z = aF_{ny} \sin \theta_f - bF_{nr} \sin \theta_r \tag{6.4}
\]

where \( M \) and \( I_z \) are the total mass and inertia moment of the vehicle, respectively. By assuming the side-slip angle \( \beta \) being small, \( \dot{y}_\eta \) can be replaced by \( U_0 \beta \) in Eq. (6.3). With the small steering angle assumption, and substituting \( F_{ny} \) and \( F_{nr} \) from Eqs. (6.1) and (6.2) into Eqs. (6.3) and (6.4), it can be found as

\[
\dot{\beta} + \left\{ U_0 + \frac{aC_{af} - bC_{ar}}{MU_0} \right\} \frac{\omega_z}{U_0} + \left\{ \frac{C_{af} + C_{ar}}{MU_0} \right\} \beta = \left\{ \frac{C_{af} + K_r C_{ar}}{MU_0} \right\} \theta_f \tag{6.5}
\]

\[
\dot{\omega}_z + \left\{ \frac{a^2 C_{af} + b^2 C_{ar}}{I_z} \right\} \frac{\omega_z}{U_0} + \left\{ \frac{aC_{af} - bC_{ar}}{I_z} \right\} \beta = \left\{ \frac{aC_{af} - K_r b C_{ar}}{I_z} \right\} \theta_f \tag{6.6}
\]

where the parameter \( K_r \) is the steering angle ratio between \( \theta_r \) and \( \theta_f \).

From Eqs. (6.5) and (6.6), the steady-state solution of the side-slip angle \( \beta \) can be obtained by setting \( \dot{\beta} \) and \( \dot{\omega}_z \) to zero. This yields
where $l = a + b$. It can be observed that for a given rear wheel steering angle $\theta_n$, the side-slip angle $\beta$ is dependent on the speed $U_o$ and the steering angle ratio $K_r$.

When the steering angle ratio $K_r$ is selected as the design parameter, its optimal value can be obtained by setting side-slip angle $\beta$, in Eq. (6.7), to zero; i.e.,

$$
\frac{MU_o^2(aC_{af} - K_r bC_{aw}) - C_{af}C_{aw}l(a + K_r b)}{MU_o^2(aC_{af} - bC_{aw}) + C_{af}C_{aw}l^2} \theta_f = 0
$$

(6.8)

By solving Eq. (6.8), the optimal ratio $K_r$ can be written as a function of the speed $U_o$ as

$$
K_r = \frac{-bC_{aw}C_{af} + aC_{af}MU_o^2}{aC_{aw}C_{af} + bC_{aw}MU_o^2}
$$

(6.9)

In Eq. (6.9) the steering angle ratio is shown to be a function of the forward speed of the vehicle only if the other parameters are assumed constant.

The constant parameters in Eq. (6.9), obtained from the data of the 4WS presented in Appendix A, are: $a = 1.305m$, $b = 0.965m$, $l = 2.27m$, $M = 1024Kg$, $C_{af} = 7.0 \times 10^4N/rad$ and $C_{aw} = 7.0 \times 10^4N/rad$. Substituting these values in Eq. (6.9), the optimal value of the steering angle ratio $K_r$ is obtained and it is presented in Figure 6.2. It can be observed that the steering angle ratio in Figure 6.2 is the function of the forwarding speed, in which the negative (contra phase) and the positive (same phase) steering ratios are obtained in low and high speeds, respectively. At a speed of approximately $U_o = 14m/ \text{sec}$, the steering ratio is zero; i.e., the vehicle acts as a 2WS system.
6.2 Nonlinear Model of the 4WS vehicle

The 4WS vehicle can be represented by a set of nonlinear equations of motion with the front wheel steering angle $\theta_f$ and the vehicle speed $U_0$ as the input, and the lateral velocity $\dot{y}'$ of the vehicle as the output. The side-slip angle $\beta$ can then be obtained from the lateral velocity. The lateral velocity and the side-slip angle of the vehicle are dependent on the steering ratio $K_r$ for a given front wheel steering angle and a given speed. Therefore, the optimization problem is to determine the optimal steering ratio which produces the least side-slip angle or the least lateral velocity of the vehicle.
If an optimization process is to be performed on the transient dynamic response of the vehicle, the cost function considering the side-slip angle of the vehicle can be described in two forms as

$$\psi = \text{Max}[\beta(t)], \quad t_0 < t < t_f$$

(6.10)

or

$$\psi = \int_{t_0}^{t_f} [\beta(t)]^2 dt$$

(6.11)

where $\beta(t)$ is an implicit function of the steering angle ratio $K_r$. The cost function of Eq. (6.10) considers a maximum side-slip angle occurring in a given time interval, where the cost function in Eq. (6.11) is defined as a quadratic sum of the side-slip angle in a given time interval. The cost function is evaluated by solving the equations of motion and the sensitivity equations. Eventually, the optimal steering angle ratio is found through an optimization process using this cost sensitivity. Since this is a single design parameter optimization case, the Newton-Raphson method is best suited and will be used in this process.

In the study of vehicle dynamics, a variety of driving scenarios can be considered. One optimal steering angle ratio found in one scenario may not be effective in the other. It is therefore difficult to optimize the steering angle ratio using Eq. (6.10) or (6.11) in the transient state analysis. Furthermore, if the optimization is performed using the sensitivity analysis in the transient state, evaluation of the cost function can only be performed through the transient analysis over a certain time period. The optimization using the cost sensitivity requires several evaluations of the cost function, which in turn requires an extensive com-
putational time. Hence, the cost function evaluated at the steady-state configuration can be used in the sensitivity analysis to obtain the optimal steering angle ratio, and it can then be used in the transient analysis.

The transient behavior of a system can be represented as a series of steady-state configurations. Therefore, a set of optimal parameters obtained from various steady-state configurations may be more effective in various scenarios than those obtained through a transient state. For the nonlinear 4WS vehicle, the optimal steering angle ratio is obtained by performing both the sensitivity and the steady-state analyses. The cost function at the steady-state configuration can be directly evaluated by using the steady-state analysis procedure discussed in Chapter 3. The steady-state analysis is performed through a numerical method to solve a set of nonlinear algebraic equations. The finite difference method will be used for the sensitivity analysis as explained later in this section.

In the sensitivity analysis of the steady-state response, the cost function of Eqs. (6.10) and (6.11) find a much simpler form as

\[ \psi = \beta(K_r) \quad \text{at the steady-state} \quad (6.12) \]

Using this cost function, the optimization process can be described in a schematic diagram as shown in Figure 6.3. For a given forward speed and a given front wheel steering angle, the rear wheel steering angle is determined based on the initial approximate value of the steering ratio. After the approximate steady-state configuration of the system is set up, the lateral velocity of the vehicle at the steady-state is obtained using the steady-state analysis. Since the cost function is defined in terms of the side-slip angle of the vehicle at the steady-state, the cost of nominal system is obtained from the lateral velocity and Eq. (3.20).
In order to determine the cost sensitivity, the rear wheel steering angle is perturbed by a small variation. For the perturbed cost, the side-slip angle of the perturbed system is also obtained at the steady-state configuration. Hence, the sensitivity of the side-slip angle is computed from the nominal and the perturbed costs as

$$\beta_{kr} = \frac{\partial \beta}{\partial K_r} = \frac{\beta^* - \beta}{\Delta K_r} \quad (6.13)$$

For the next trial value, the steering angle ratio increment and the new steering angle ratio is obtained as

$$\Delta K_r = -\frac{\beta}{\beta_{kr}} \quad (6.14)$$

$$K_{r}^{i+1} = K_{r}^{i} + \Delta K_r \quad (6.15)$$
The process is repeated until the side-slip angle $\beta$ approaches zero. Then, the optimal steering angle ratio is found for the given speed and the given front wheel steering angle.

The above process can be repeated for different speeds, $U_o$, and different front wheel steering angles, $\theta_f$, to obtain the optimal ratios under different driving conditions. This process yields the results as presented in Figure 6.4. It can be observed that the optimal steering angle ratios are dependent on both the vehicle speed and the steering angle.

Figure 6.4 Optimal steering ratio curves from the nonlinear 4WS model
One important point that needs to be addressed here is that for certain driving conditions; i.e., certain steering angle ratio, front wheel steering angle, and speed, a steady-state configuration with a zero side-slip angle of the vehicle may not be found. In other words, for a given front wheel steering angle and a given speed, there is no steering angle ratio for which the steady-state response has a zero side-slip angle. In such cases, the optimization process determined the steering angle ratio for which the side-slip angle of the vehicle is a minimum (instead of zero) and the vehicle is still in a stable steady-state configuration. These values of the steering angle ratio are shown with broken lines in Figure 6.4.

6.3 Summary

Two different optimal steering ratios are obtained by using the linear bicycle model and the full nonlinear vehicle model in this chapter. The optimal steering ratio from the linear bicycle model is a function of the forward speed of the vehicle only. However, the optimal steering ratio from the nonlinear 4WS model is the function of the forward speed and the front wheel steering angle. Hence, multiple curves corresponding to various front wheel steering angles are obtained in the plot of the optimal steering ratio versus forward speed from the nonlinear model. These two optimal steering ratio strategies will be compared through lane change and J-turn simulations using the 4WS vehicle model presented in Appendix A. For its objective comparison, a path control strategy will be developed in the following chapter.
CHAPTER 7
PATH CONTROL OF NONLINEAR VEHICLES

In this chapter, path control methods for maneuvering a vehicle will be discussed. By using these control methods, various dynamic simulations of vehicles can be performed. Typical simulations are for a vehicle to follow a given path. Effectiveness of optimal steering ratios in the 4WS system obtained in Chapter 6 can be then investigated by performing several scenarios of vehicle maneuvers.

In order to control a vehicle to follow a given path, a steering control strategy is required. An open loop path control method and a closed loop path control method can be introduced into the equations of motion of the vehicle. The developed path control methods will be tested a nonlinear bicycle model. For demonstration purpose, this study uses the nonlinear bicycle model which is a simplified one half version of a full nonlinear vehicle model but without suspension systems. However, since equations of motion of this system represent the three dimensional dynamics and the nonlinear analytical comprehensive tire model, the methodology is completely applicable to any complex vehicle model.

7.1 Steering Control in Equations of Motion

In order to establish a control law, the steering angle needs to be investigated as to how it works in the equations of motion. To achieve this, a nonlinear bicycle model is introduced. The bicycle model can be described as consisting of a main chassis, a front wheel and a rear wheel. The front wheel is connected to the main chassis by a revolute-revolute joint and the rear wheel is connected to the main chassis by a revolute joint. Hence,
this system has nine degrees of freedom. The configuration of the system is shown in Figure 7.1. The necessary data to formulate the equations of motion are taken from Appendix A by considering just one half of the 4WS vehicle without the suspension system.

![Nonlinear bicycle model with joint coordinates](image)

**Figure 7.1** Nonlinear bicycle model with joint coordinates

In order to consider the equations of motion of this system, a set of joint coordinates can be defined as

\[
\theta = \begin{bmatrix}
\theta_1 \\
\theta_{(o)} \\
\theta^{(o)}_{r} \\
\theta^{(o)}_{f}
\end{bmatrix}
\]  

(7.1)

where \(\theta_1\) contains six absolute coordinates of the chassis. The equations of motion in joint coordinates are written in compact form as

\[
\begin{bmatrix}
M_a & m_{af}^T \\
m_{af} & m_f
\end{bmatrix}
\begin{bmatrix}
\ddot{\theta}_a \\
\ddot{\theta}_f
\end{bmatrix} = \begin{bmatrix}
f_a \\
f_f
\end{bmatrix}
\]  

(7.2)

where \(\ddot{\theta}_f\) is the acceleration of the steering angle in the joint coordinates which will be used as a control input, and \(\ddot{\theta}_a\) contains other eight joint accelerations. The terms \(M_a\) and \(f_a\) are the generalized mass matrix and the generalized force vector corresponding to \(\ddot{\theta}_a\). The terms
$m_f$ and $f_f$ are the generalized mass and the generalized force corresponding to the steering angle $\theta_f$. The term $m_{gf}$ is a generalized mass vector corresponding to coupling between the steering angle and the remaining joint coordinates. The equations of motion can then be reorganized for path control as

$$M_s \dot{\theta}_s = f_s - m_{gf} \ddot{\theta}_f$$  \hspace{1cm} (7.3)$$

where $\dot{\theta}_s$ is the system state vector dependent on $\ddot{\theta}_f$ which is the input variable of the system. This means that a vehicle can be controlled to follow a given path using the input $\ddot{\theta}_f$ in Eq. (7.3).

### 7.2 Open Loop Path Control

Open loop path control on a system is a control scheme in which the output of a system has no influence on the input. The magnitudes of the control input parameters are usually decided by studying the behavior of the system. This is generally achieved through experience.

In the open loop path control, a vehicle can follow a given path with a prescribed steering angle input shown in Figure 7.2. The steering angle input consists of specified parameters: $\Delta \theta$'s for increments of the steering angle, $\Delta t$'s for the time intervals. The transient steering angle between the point $A$ to the point $B$ in Figure 7.2.a can be produced using a fourth order polynomial function for a smooth change of the steering input [45]. The polynomial function of the steering angle is

$$\theta_f(t) = \theta_f^{j} + \frac{6\Delta \theta_f}{\Delta t^2} (t - t_j)^2 - \frac{8\Delta \theta_f}{\Delta t^3} (t - t_j)^3 + \frac{3\Delta \theta_f}{\Delta t^4} (t - t_j)^4$$  \hspace{1cm} (7.4)$$
where, $t_1 \leq t \leq t_f$, $\Delta t = t_f - t_i$, and $\Delta \theta_j = \theta_j - \theta_j'$. The second derivative of Eq. (7.4) with respect to time is applied to Eq. (7.3) as the steering input command:

$$\dot{\theta}_j(t) = \frac{12 \Delta \theta_j}{\Delta t^2} - \frac{48 \Delta \theta_j}{\Delta t^3} (t - t_i) + \frac{36 \Delta \theta_j}{\Delta t^4} (t - t_i)^2$$

(7.5)

For the lane change maneuver, the required parameters are $\Delta \theta_1$, $\Delta \theta_2$, $\Delta T_1$, $\ldots$, and $\Delta T_5$. The example of the open loop control input is given in Figure 7.2a. A change of each parameter results in a different path of the vehicle. Furthermore, two different vehicle models will result in two different paths even though the same set of control parameters are applied. Therefore, when two vehicles are compared through dynamic simulations, it is hard to find two sets of open loop control parameters which results in two vehicles to follow the same path. If a deviation between a given path and a desired path occurs, it can only be corrected by adjusting the control parameters. For this purpose, a closed loop path control is desirable.

Four open loop path control simulations are performed based on the nonlinear bicycle model. The performed simulations are lane change maneuvers with a constant vehicle speed at 10 m/sec, and applied steering angle inputs are 2, 3, 4 and 5 degrees. Figure 7.3a shows the front wheel steering angle history for the four simulations. Figure 7.3b shows the corresponding trajectories of the mass center of chassis.

For the circular path maneuver, the required parameters are $\Delta \theta_1$, $\Delta T_1$, $\Delta T_2$, and $\Delta T_3$, as shown in Figure 7.2b. The applied constant vehicle speed is 10 m/sec and the steering input is 3 degrees. Figure 7.4a shows the steering angle history and Figure 7.4b shows the corresponding trajectory.
Figure 7.2 Open loop control inputs;
   
a) for lane change maneuver,
   b) for circular path maneuver.
Figure 7.3 Lane change maneuver with the open loop controller;
(a) Steering angle history, (b) Trajectory,
with $\Delta T_1 = \Delta T_5 = 0.5$ sec, $\Delta T_2 = \Delta T_3 = \Delta T_4 = 1.0$ sec,
Speed of the vehicle = 10 m/sec
Figure 7.4 Circular path maneuver with the open loop path control;
(a) Steering angle history, (b) Trajectory,
with \( \Delta t_1 = \Delta t_3 = 0.5 \text{ sec}, \Delta t_2 = 8.0 \text{ sec}, \Delta \Theta_r = 3.0^\circ \),
Speed of the vehicle = 10 m/sec.
7.3 Closed Loop Path Control

In a closed loop path control, the steering angle command can be determined by monitoring one or more error measurements between a trajectory of the vehicle and the given path during a maneuver. The errors are a position error and a moving direction error (later it will be referred to as the slope error) which will be utilized in the feedback path control. The moving direction error can be presented as the error between the moving direction of the vehicle to the tangential direction at the given path. The feedback control method is studied first in the lane change maneuver for convenience. Then, the feedback control method can be applied to the circular maneuver.

7.3.1 Lane Change Maneuver

The lane change maneuver is the most frequent maneuver occurring in real driving situations. It can be described as the shifting of a vehicle from one lane to another lane within a certain distance at a given speed. The intermediate path of the vehicle may not be of importance. However, for the feedback control, a prescribed path is required as a reference to measure errors. Hence, the prescribed path needs to be given as a continuous explicit function which connects the start and the end points.

A polynomial function is introduced to represent the path of the lane change. In order to avoid the sharp turns at the start or the end point of the path, tangential slopes at the start and the end points can be specified to zero. For smoother path, the derivatives of the tangential slope at both points are also required to be zero. These conditions can be written as
The fourth order polynomial function used in Eq. (7.4) does not satisfy one of the conditions in Eq. (7.6), since it has a sharp slope increment at the beginning of the function. However, it has a smooth slope increment at the rear end. In order to avoid the sharp increment slope in the prescribed path, two fourth order polynomial functions can be combined, which will have smooth slope increments at both ends. The polynomial function which satisfies the conditions in Eq. (7.6) can then be written as

\[
Y(X) = Y_0 + A_1(X - X_0)^3 + A_2(X - X_0)^4 \quad \text{for} \quad X_0 < X < X_0 + \frac{\Delta X}{2} \quad (7.7a)
\]

\[
Y(X) = Y_f - A_1(X_f - X)^3 - A_2(X_f - X)^4 \quad \text{for} \quad X_0 + \frac{\Delta X}{2} < X < X_f \quad (7.7b)
\]

Then, the tangential slopes along the path can be obtained as

\[
\frac{dY}{dX}(X) = 3A_1(X - X_0)^2 + 4A_2(X - X_0)^3 \quad \text{for} \quad X_0 < X < X_0 + \frac{\Delta X}{2} \quad (7.8a)
\]

\[
\frac{dY}{dX}(X) = 3A_1(X_f - X)^2 + 4A_2(X_f - X)^3 \quad \text{for} \quad X_0 + \frac{\Delta X}{2} < X < X_f \quad (7.8b)
\]

where, \(\Delta X = X_f - X_0\), \(\Delta Y = Y_f - Y_0\)

\[
A_1 = 8 \frac{\Delta Y}{\Delta X^3} \quad \text{and} \quad A_2 = 8 \frac{\Delta Y}{\Delta X^4}.
\]

A plot of the polynomial function is illustrated in Figure 7.5.
Figure 7.5 A fourth order polynomial of lane change path
The feedback controller generally requires that the position and the slope errors be based on a reference path. The position error can be computed by comparing the $Y$ coordinate of the center of the mass of the vehicle ($y$) against the $Y$ coordinate of the prescribed path corresponding to the $X$ coordinate of the center of mass of the vehicle ($x$) as shown in Figure 7.6. This position error can be written as

$$e_1(t) = y(t) - Y(X=x(t))$$

(7.9)
The slope error is computed by comparing the moving direction of the vehicle to the corresponding tangential slope at the prescribed path. This error can be written as

\[ e_2(t) = \frac{y(t)}{x(t)} - \frac{dY}{dX}(X = x(t)) \]  

Therefore, the feedback control law can be set as

\[ \hat{\theta}_f = K_1 e_1 + K_2 e_2 \]  

where \( K_1 \) and \( K_2 \) are the gains for the feedback control. Because of the nonlinearity of the system, proper gains can only be determined through simulations.

Since the lateral response of the vehicle has a time delay from the moment the steering command is initiated, the feedback control of the position and the tangential slope errors at the current position can not enforce the vehicle to follow a given path. An example is illustrated in Figure 7.7a-b. As can be observed, when the vehicle approaches the start point of the curve, there is a delay in the lateral response. This delay in response will result in a slope and a position error [Figure 7.7a] despite the increase in the steering angle [Figure 7.7b]. These errors further cause a larger steering angle. Later the vehicle may approach the prescribed path resulting in a large slope error. The slope error and the large steering angle in turn increase the position error in the opposite direction. Finally, the feedback controller fails in controlling the vehicle to follow the prescribed path.

This example clearly shows that some steering control action need to be applied in advance before the error is detected. The feedback control of the current errors is not sufficient for the vehicle to follow a prescribed path due to the time delay in lateral response of the steering input. In reality, an actual driver observes a path and initiates an action in
advance. Hence, the position and slope errors estimated in advance are added to the feedback control as

\[ \dot{\theta}_f = K_1 e_1 + K_2 e_2 + K_3 e_3 + K_4 e_4 \]  

(7.12)

where \( e_3 \) and \( e_4 \) are the error of the positions and the error of the slopes at an advanced time \( \Delta T \), which are illustrated in Figure 7.6. The corresponding gains are \( K_3 \) and \( K_4 \). In other words this control scheme assumes the driver looks ahead a distance of \( U_0 \Delta T \), where \( U_0 \) is the speed of vehicle in the direction of its motion.

For better feedback control, the current acceleration can also be accounted in error estimation under the assumption that the acceleration changes of the center of mass of the chassis in \( x \) and \( y \) directions are smooth. By considering the acceleration, the position and the velocity of the center of mass of the chassis at the advanced time \( \Delta T \) are written as

\[
x(t + \Delta T) = x(t) + \dot{x}(t)\Delta T + \frac{1}{2} \ddot{x}(t)\Delta T^2
\]

(7.13)

\[
y(t + \Delta T) = y(t) + \dot{y}(t)\Delta T + \frac{1}{2} \ddot{y}(t)\Delta T^2
\]

(7.14)

\[
\dot{x}(t + \Delta T) = \dot{x}(t) + \ddot{x}(t)\Delta T
\]

(7.15)

\[
\dot{y}(t + \Delta T) = \dot{y}(t) + \ddot{y}(t)\Delta T
\]

(7.16)

Then Eq. (7.12) can be written as

\[
\dot{\theta}_f = K_1 [y - Y(X = x)]_t + K_2 \left[ \frac{\ddot{y}}{k} \frac{dY}{dX}(X = x) \right]_t
\]

\[
+ K_3 [y - Y(X = x)]_{t+\Delta T} + K_4 \left[ \frac{\ddot{y}}{k} \frac{dY}{dX}(X = x) \right]_{t+\Delta T}
\]

(7.17)
Figure 7.7 Lane change maneuver with a feedback control;
(a) Trajectory, (b) Steering angle history,
with $K_1 = 20$, $K_2 = 20$, Speed of the vehicle = 10 m/sec.
In the vehicle feedback path control, there is often difficulty in finding a proper set of control parameters which can produce a prescribed path. This may be because of a rapid change of the steering angle due to a relatively large feedback control input. Since, the response of the steering wheel has a time delay, a rapid change of the steering angle does not improve the current error but may increase it to a potentially large error in the future. Moreover, a rapid change of the steering angle may cause an excessive lateral slip of tires and may not control the vehicle but may increase its instability. Hence, the feedback controller may lose capability to control the path.

Any undesirable rapid changes in steering angle can be eliminated by introducing an artificial damper in the feedback path controller. The artificial damper reduces the magnitude of acceleration of the steering angle command and prevents any rapid changes. Hence, the change of the steering angle will be smooth and will produce a proper lateral force from the tire which will result in a smooth path control of the vehicle. Therefore, this smooth path control due to the artificial damper expands a range of acceptable feedback control parameters for a given curved path. The artificial damper is then added in the steering angle controller as

\[ \dot{\theta}_j = K_1 e_1 + K_2 e_2 + K_3 e_3 + K_4 e_4 - K_5 \dot{\theta}_j \]  \hspace{1cm} (7.18)

By using the nonlinear bicycle model described in section 7.1, the results of the closed loop control simulations are illustrated in Figure 7.8. The speed of the vehicle is 10.0 m/sec. The feedback gains \( K_1, K_2, K_3, K_4 \) and \( K_5 \) are 10, 2, 10, 2 and 10, respectively. The advanced observation time \( \Delta T \) is 0.6 second. Figure 7.8a shows that the vehicle successfully follows the prescribed path and Figure 7.8b shows the corresponding steering angle input history.
Figure 7.8 Lane change maneuver with a feedback controller;
(a) Trajectory, (b) Steering angle history,
with $K_1 = 10$, $K_2 = 2$, $K_3 = 100$, $K_4 = 5$, $K_5 = 10$, $\Delta K_1 = 0.6$,
Speed of the vehicle = 10 m/sec.
7.3.2 Circular Path Maneuver

The circular path maneuver can also be considered to use the feedback control of the position and the tangential slope errors as mentioned in the previous section. However, when a circular path is described using a position and a tangential slope in a Cartesian X-Y reference frame, the magnitude of tangential slope along the circle would vary from zero to infinity. Hence, the control method using the tangential slope fails in controlling the vehicle to follow a prescribed circular path. In order to avoid this difficulty, a polar reference frame can be used in circular path maneuvers.

In a polar reference frame, the measurements used in the path control are the radius and the angle. The radius can be described as a constant parameter of a function of the angle. For convenience of the study, the radius is assumed to be constant. Then, the position errors $e_1$ and $e_3$ can be obtained from the distance between the center of the circular path and radius of the prescribed circular path as

$$e_1 = \sqrt{(x - x_c)^2 + (y - y_c)^2} - R$$  \hspace{1cm} (7.19)

$$e_3 = \sqrt{(x(T + \Delta T) - x_c)^2 + (y(T + \Delta T) - y_c)^2} - R$$  \hspace{1cm} (7.20)

where $x_c$ and $y_c$ are the center of circular path in X-Y coordinates shown in Figure 7.9.

For the slope errors, the angle in polar coordinates is used since this angle has uniform increment along the circle. Hence, the slope errors $e_2$ and $e_4$ are obtained from the angles at the current and advanced time as

$$e_2 = \alpha_i(T) - \alpha_p(T)$$  \hspace{1cm} (7.21)
Figure 7.9 Schematic diagram of the circular path
where \( \alpha_t \) is the angle at the current path and \( \alpha_p \) is the corresponding angle at the prescribed path as shown in Figure 7.9. For the calculations of \( \theta_t \) and \( \theta_p \) from X-Y coordinates, the special function \( \text{Atan2} \) is used since this function has the continuity of the angle along the circle. Hence, the angles \( \theta_t \) and \( \theta_p \) can be written as

\[
\alpha_p = \text{Atan2}(x, Y_c - y) \tag{7.23}
\]
\[
\alpha_t = \text{Atan2}(\dot{x}, \dot{y}) \tag{7.24}
\]

A J-turn maneuver in the circular path can be performed by specifying a final exit angle, \( \alpha_F \), as shown in Figure 7.10, where the errors computation is schematically illustrated. Beyond the final angle, the position error and the angle error can be computed as

\[
e_1 = \sqrt{(x - X_e)^2 + (y - Y_e)^2} \cos(\alpha_p - \alpha_p) - R \tag{7.22}
\]
\[
e_2 = \alpha_t - \alpha_F \tag{7.23}
\]

The feedback control for a circular path is same as that of Eq. (7.17), however, the magnitude of the feedback control parameters \( K_1, K_2, K_3, K_4, K_5 \) and \( \Delta T \) may be different than the parameters used in the lane change maneuver.

Simulation results for the closed loop control of J-turn maneuver is illustrated in Figure 7.11. The speed of the vehicle is 10.0 m/sec. The feedback gains \( K_1, K_2, K_3, K_4 \) and \( K_5 \) are 10, 2, 200, 20 and 10, respectively. The advanced observation time \( \Delta T \) is 0.6 second. Figure 7.11a shows that the vehicle successfully follows the prescribed path and Figure 7.11b shows
the corresponding steering angle input history. In the steering angle history, a slight overshooting is observed at the initial and final stages before the steering angle input reaches the steady values.

7.4 Summary

Path control methods for vehicle maneuvers are developed in this chapter. These control methods use the feedback control of errors in positions and slopes at the current and at an advanced time. For reference of the error measurement, a fourth order polynomial function is introduced in the lane change maneuver, and a circular function is used in the J-turn maneuver. Simulations for the validating the steering angle control are performed using the nonlinear bicycle model. Since the model uses full nonlinear factors appearing in the equations of motion, this control strategy is fully applicable to any other nonlinear model of vehicles. By using this control method, the comparison of the nonlinear 4WS model, the linear bicycle model, and 2WS model applied to the nonlinear vehicle model are perform in following chapter.
Figure 7.10 Schematic diagram of the J-turn maneuvering
Figure 7.11  Circular path maneuver with a feedback control;
(a) Trajectory, (b) Steering angle history,
with $K_1 = 10, K_2 = 2, K_3 = 200, K_4 = 20, K_5 = 10, \Delta K_1 = 0.6$,
Speed of the vehicle = 10 m/sec
CHAPTER 8
DYNAMIC SIMULATIONS AND RESULTS

In Chapter 6, two optimal steering angle ratio strategies for the 4WS vehicles were obtained based on a linear and a nonlinear model. In this chapter, the effectiveness of these strategies is compared through various dynamic simulations using the 4WS vehicle model presented in Appendix A. The dynamic simulations consist of lane change and J-turn maneuver cases which are performed by using the closed loop path control method discussed in the previous chapter. In each case, the two optimal steering angle ratio strategies and a zero steering angle ratio for the rear wheel steering angle will be considered. The zero steering angle ratio corresponds to the conventional front wheel steering (2WS) vehicle where the rear wheel steering angle is fixed at zero. To compare the three different simulations in each case, the lateral velocities and the steering angle histories are observed.

8.1 Interpolation of Steering Angle Ratio

The optimal steering angle ratio strategies presented in Chapter 6 can be applied to simulate general maneuvers of any nonlinear vehicle model. In this chapter lane change and J-turn maneuvers of the 4WS vehicle model are studied. During these maneuvers, the vehicle speed is assumed to remain constant for convenience of comparison. Under this assumption, the steering angle ratio based on the linear bicycle model also remains constant. However, based on the nonlinear 4WS model, the steering angle ratio is a function of the front wheel steering angle. For a given speed and a given front wheel steering angle, the
corresponding steering angle ratio may be found from the interpolation or extrapolation of the obtained optimal ratio data presented in Figure 6.4. Since the data can be best approximated by second order polynomials, the steering ratios can be described as

\[ K_r = C_1 + C_2 \theta_f^2 \quad \text{at a given speed } U_0 \]  

(8.1)

By applying the least square curve fitting, the constants \( C_1 \) and \( C_2 \) corresponding to various speeds can be obtained as shown in Table 8.1.

<table>
<thead>
<tr>
<th>Speed \ Constants</th>
<th>( C_1 )</th>
<th>( C_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>6 m/sec</td>
<td>-0.5855</td>
<td>0.0008</td>
</tr>
<tr>
<td>12 m/sec</td>
<td>-0.2291</td>
<td>0.0121</td>
</tr>
<tr>
<td>16 m/sec</td>
<td>0.045</td>
<td>0.0285</td>
</tr>
<tr>
<td>20 m/sec</td>
<td>0.2605</td>
<td>0.0524</td>
</tr>
<tr>
<td>25 m/sec</td>
<td>0.479</td>
<td>0.0632</td>
</tr>
<tr>
<td>30 m/sec</td>
<td>0.6416</td>
<td>0.0651</td>
</tr>
</tbody>
</table>
8.2 Control of the Rear Wheel Steering Angle

In order to apply the optimal steering angle ratio strategy into the 4WS vehicle model, the acceleration of the rear wheel steering angle is required since the rear wheel steering angle can only be changed through the steering angle acceleration in the equations of motion. The front steering angle acceleration can be obtained by employing a path control scheme. Then, the rear wheel steering angle acceleration is obtained from the steering angle ratio between the front and the rear wheels.

For the optimal steering angle ratio based on the linear bicycle model, the steering angle ratio $K_r$ is constant under constant speed of the vehicle. Hence, the rear wheel steering angle is described as

$$\theta_r = K_r \theta_f$$  \hspace{1cm} (8.2)

where $\theta_f$ and $\theta$, are the steering angles of the front and the rear wheels. The first and second derivatives of the rear wheel steering angle are written as

$$\dot{\theta}_r = K_r \dot{\theta}_f$$  \hspace{1cm} (8.3)

and

$$\ddot{\theta}_r = K_r \ddot{\theta}_f$$  \hspace{1cm} (8.4)

where $\ddot{\theta}_f$ is obtained from the feedback path controller described in Chapter 7. In order to apply an accurate rear wheel steering angle, a feedback controller can be added in the acceleration of steering angle, which can compensate the error between the desired rear wheel steering angle $\theta^*_r$ and the current rear wheel steering angle $\theta_r$. Therefore, the acceleration of the rear wheel steering angle can be written as...
\[ \dot{\theta}_f = K_r \dot{\theta}_r + G_1(\theta_r^* - \theta_r) + G_2(\dot{\theta}_r^* - \dot{\theta}_r) \]  
(8.5)

where, \( G_1 \) and \( G_2 \) are gains of controller corresponding to the steering angle error and its derivative.

For the optimal steering angle ratio based on the nonlinear 4WS model, the steering angle ratio can be expressed as a second order polynomial function of the front wheel steering angle as Eq. (8.1). Hence, the rear wheel steering angle is obtained as

\[ \theta_r = (C_1 + C_2 \theta_r^*) \dot{\theta}_r \]  
(8.6)

The derivative of the rear wheel steering angle can be written as

\[ \dot{\theta}_r = (C_1 + 3C_2 \theta_r^*) \ddot{\theta}_r \]  
(8.7)

The acceleration of the rear wheel steering angle is obtained by performing the time derivative of Eq. (8.7), and with the two additional gains \( G_1 \) and \( G_2 \) discussed earlier as

\[ \ddot{\theta}_r = (C_1 + 3C_2 \theta_r^*) \dddot{\theta}_r + 5C_2 \theta_r \dot{\theta}_r^* + G_1(\theta_r^* - \theta_r) + G_2(\dot{\theta}_r^* - \dot{\theta}_r) \]  
(8.8)

Then, Eq. (8.8) will be incorporated into the equations of motion to produce a proper rear wheel steering angle corresponding to the optimal steering angle ratio based on the nonlinear 4WS model.

### 8.3 Lane Change Simulations

Various lane change simulations are performed under different driving conditions with the 4WS vehicle model. Each driving condition in a lane change maneuver is described by specifying a longitudinal distance \( \Delta X \), a lateral-shift distance \( \Delta Y \) and a vehicle speed \( U_v \), which are illustrated in Figure 8.1. Three sets of driving conditions and their path control parameters are tabulated in Tables 8.1, 8.2, 8.3, and 8.4. Each table is accompanied by a set of five figures [Table 8.1, Figure 8.2-4; Table 8.2, Figure 8.5-7; Table 8.3, Figure 8.8-10]
The first figure shows the trajectory of the maneuver. The second and the third figures show the history of the steering angles applied to the front wheel and the rear wheel, respectively, and the steering angle ratio is shown in the fourth figure. The fifth figure shows the lateral velocity of the chassis. Each figure contains results of three simulations corresponding to three different steering angle ratio models: zero steering angle ratio (2WS), optimal steering angle ratio from the linear bicycle model, and optimal steering angle ratio from the nonlinear 4WS model.

The first set of driving conditions is the lane change maneuver consisting of an 8.0 m lateral shift during a 40.0 m longitudinal motion with a 16.0 m/sec constant vehicle speed. Control parameters for this condition are given in Table 8.1. In Figure 8.2, all the three trajectories show that smooth lane changes have taken place and their paths are identical due to the feedback control method described in Chapter 7. The corresponding steering angle histories are shown in Figures 8.3. Here, the nonlinear 4WS model has the smoothest front steering angle history. On the other hand, the 2WS model has the least smooth front wheel steering angle history exhibiting difficulty in controlling the path due to the fixed (zero) rear wheel steering angle (Figure 8.3b). The linear bicycle model and the nonlinear 4WS model produce nonzero rear wheel steering angle histories. For the linear bicycle model, the steering angle ratio is a constant 0.082 throughout the simulation time period. However, the steering angle ratio of the nonlinear 4WS model increases from 0.05 to 0.21 as the front wheel steering increases from 0.0 to approximately 2.4 degrees as shown in Figure 8.3c. Hence, for a small front steering angle there is only a slight difference between the two models in magnitude of the rear wheel steering angle. For a large steering angle the nonlinear model produces a larger rear wheel steering angle than the linear model. This is the reason that the maximum of the lateral velocity of the nonlinear model is reduced to one fifth of that of the linear model, as shown in Figure 8.4.
The second driving condition represented is a lane change maneuver of a 4.0 m lateral shift during a 30.0 m longitudinal motion with a 20.0 m/sec constant vehicle speed. The control parameters for this condition are given in Table 8.2. In Figure 8.5, all three trajectories exhibit smooth lane changes and their paths are identical. The corresponding steering angle histories are shown in Figures 8.6. Here the nonlinear 4WS model has the smoothest front steering angle history and the 2WS model has the least smooth front wheel steering angle history as observed in the first driving condition set. The steering angle ratio of the nonlinear model increases from 0.26 to 0.40 as the front wheel steering increases from 0.0 to 1.6 degrees while the ratio of the linear model is 0.295 for the entire time period. Hence, for a small front steering angle, there is a slight difference in magnitude for the rear wheel steering angles of the two models. For a large steering angle the nonlinear model produces a larger rear wheel steering angle than the linear model. Due to the larger rear wheel steering angle in the nonlinear model, the peak value of the lateral velocity of the nonlinear model is also reduced to one fifth of that of the linear model, as shown in Figure 8.7.

The third driving condition set is the lane change maneuver consisting of a 4.0 m lateral shift during a 40.0 m longitudinal motion with a 25.0 m/sec constant vehicle speed. The control parameters for this condition are given in Table 8.3. Figure 8.8 contains three trajectories which are smooth and identical. The corresponding steering angle histories are shown in Figure 8.9. The steering angle ratio of the nonlinear model increases from 0.46 to 0.58 as the front wheel steering increases from 0.0 to approximately 1.3 degrees while the ratio of the linear model is 0.48 for the entire time period. Lateral velocity is also reduced significantly as shown in Figure 8.10. Hence, in the overall speed ranges, the nonlinear steering angle ratio strategy model reduces the lateral velocity more effectively than the linear model.
Figure 8.1 Schematic diagram of a lane change maneuver
### Table 8.2  Driving conditions and control parameters

<table>
<thead>
<tr>
<th>$U_0$ (m/sec)</th>
<th>$\Delta X$ (m)</th>
<th>$\Delta Y$ (m)</th>
<th>$K_1/K_2/K_3/K_4/K_5/G_1/G_2/\Delta T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>40</td>
<td>8</td>
<td>10/100/40/400/50/10/10/0.65</td>
</tr>
</tbody>
</table>

#### Figure 8.2 Trajectory of vehicle models
Figure 8.3 History of steering angles for a lane change maneuver;
(a) Front wheel steering angle, (b) Rear wheel steering angle,
(c) Steering angle ratio history.
Figure 8.4 Lateral velocities of a lane change maneuver
Table 8.3  Driving conditions and control parameters

<table>
<thead>
<tr>
<th>$U_0$ (m/sec)</th>
<th>$\Delta X$ (m)</th>
<th>$\Delta Y$ (m)</th>
<th>$K_1/K_2/K_3/K_4/K_5/G_1/G_2/\Delta T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>30</td>
<td>4</td>
<td>25/250/70/700/50/20/20/0.70</td>
</tr>
</tbody>
</table>

Figure 8.5 Trajectory of vehicle models
Figure 8.6 History of steering angles for a lane change maneuver;
(a) Front wheel steering angle, (b) Rear wheel steering angle,
(c) Steering angle ratio history.
Figure 8.7 Lateral velocities of a lane change maneuver
Table 8.4  Driving conditions and control parameters

<table>
<thead>
<tr>
<th>$U_0$ (m/sec)</th>
<th>$\Delta X$ (m)</th>
<th>$\Delta Y$ (m)</th>
<th>$K_1/K_2/K_3/K_4/K_5/G_1/G_2/\Delta T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>40</td>
<td>4</td>
<td>25/250/70/700/50/20/20/0.70</td>
</tr>
</tbody>
</table>

Figure 8.8 Trajectory of vehicle models
Figure 8.9 History of steering angles for a lane change maneuver;
(a) Front wheel steering angle, (b) Rear wheel steering angle,
(c) Steering angle ratio history.
Figure 8.10 Lateral velocities of a lane change maneuver
### 8.4 J-turn Simulations

J-turn maneuvers of the 4WS vehicle model are also performed under different driving conditions. A driving condition in the J-turn maneuver is described by specifying a circular radius $R$, a final exit angle $\theta_f$, and a vehicle speed $U_o$, as illustrated in Figure 8.11. The two sets of driving conditions and their path control parameters are tabulated in Tables 8.4 and 8.5. Each table is accompanied by a set of five figures [Table 8.4, Figure 8.12-14; Table 8.5, Figure 8.15-17]. The progression from figure to figure is the same as that of the lane change maneuver.

The driving condition of the first set is the J-turn maneuver in which the vehicle moves along a circle of a $30.0$ m radius and exits at $90.0$ degrees with a $12.0$ m/sec constant vehicle speed. The control parameters for this condition are given in Table 8.4. Figure 8.12 shows the three trajectories of smooth J-turns which are identical to the desired path. Figure 8.13 shows the corresponding steering angle histories. For the linear model, the steering angle ratio remains constant at $-0.20$. However, the steering angle ratio of the nonlinear model varies from $-0.23$ to $0.05$ as the front wheel steering angle varies from $0.0$ to $4.2$ degrees. Hence for a small front steering angle there is a slight difference in magnitude of the contra phase rear wheel steering angles of the two models. Since the linear model has a constant negative steering angle ratio, as the front wheel steering angle increases, its rear wheel steering angle becomes larger. Thus this model has a larger lateral velocity than the others. However, the nonlinear model produces a smaller rear wheel steering angle as the steering angle ratio approaches zero for the larger front wheel steering angle. Because the nonlinear model has smaller rear wheel steering angle, the peak lateral velocity of the nonlinear model is reduced drastically.
The second driving condition set describes the J-turn maneuver, in which the vehicle moves along a circle of a 100.0 m radius and exits at 90.0 degrees with a 20.0 m/sec constant vehicle speed. The control parameters for this condition are given in Table 8.5. Figure 8.15 show the three trajectories of smooth J-turns which are identical. The corresponding steering angle histories are shown in Figures 8.16. For the linear model, the steering angle ratio remains constant at 0.295 throughout the entire simulation time period. However, the steering angle ratio of the nonlinear model varies from 0.26 to 0.42 as the front wheel steering varies from 0.0 to 1.9 degrees. Hence for a small front steering angle there is a slight difference in magnitude of the rear wheel steering angles of the two models. For a large steering angle, the nonlinear model produces a larger rear wheel steering angle than the linear model. Because the nonlinear model has a larger rear wheel steering angle, the peak lateral velocity of the nonlinear model is reduced considerably compared to the others.

8.5 Summary

In this chapter, various dynamic simulations are performed by using the nonlinear 4WS vehicle model with three steering angle ratio strategies: 2WS, linear bicycle model, and nonlinear 4WS model. The steering angle ratio strategy from the nonlinear 4WS model uses a second order polynomial function of the front wheel steering angle to obtain the rear wheel steering angle. However, the steering angle ratio of the linear bicycle model is constant and that of the 2WS model is zero. The steering angle ratios of the linear bicycle model and 2WS remains a constant and zero, respectively. For comparison of these three strategies, the closed loop path control scheme developed in Chapter 7 is also incorporated into their dynamic simulations. The dynamic simulations are for lane change and J-turn maneuvers in various speeds. Due to the path control scheme, the three steering strategies produce
identical paths, following the desired path, in each simulation. Through the lane change and the J-turn maneuvers with several speeds, it is observed that the optimal steering angle ratio strategy based on the nonlinear model reduces the lateral velocity of the vehicle more effectively than that based with the linear model.
Figure 8.11 Schematic diagram of a J-turn maneuver
Table 8.5  Driving conditions and control parameters

<table>
<thead>
<tr>
<th>$U_0$ (m/sec)</th>
<th>R (m)</th>
<th>$\theta_F$ (deg)</th>
<th>$K_1$/$K_2$/$K_3$/$K_4$/$K_5$/$G_1$/$G_2$/(\Delta T)</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>30</td>
<td>90</td>
<td>10/2/200/20/15/10/10/0.60</td>
</tr>
</tbody>
</table>

![Figure 8.12 Trajectory of vehicle models](image)

Figure 8.12 Trajectory of vehicle models
Figure 8.13 History of steering angles for a J-turn maneuver;
(a) Front wheel steering angle, (b) Rear wheel steering angle,
(c) Steering angle ratio history.
Figure 8.14 Lateral velocities of a J-turn maneuver
Table 8.6  Driving conditions and control parameters

<table>
<thead>
<tr>
<th>$U_0$ (m/sec)</th>
<th>R (m)</th>
<th>$\theta_f$ (deg)</th>
<th>$K_1$/$K_2$/$K_3$/$K_4$/$K_5$/$G_1$/$G_2$/$\Delta T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>100</td>
<td>90</td>
<td>20/4/400/40/20/20/20/0.7</td>
</tr>
</tbody>
</table>

Figure 8.15  Trajectories of a vehicle
Figure 8.16 History of steering angles of a J-turn maneuvering;
(a) Front wheel steering angle, (b) Rear wheel steering angle,
(c) Steering angle ratio history.
Figure 8.17 Lateral velocities of a J-turn maneuvering
CHAPTER 9
DISCUSSION AND CONCLUSIONS

For sensitivity analysis of large-scale dynamic systems, it is not a simple task to derive the sensitivity equations from the equations of motion and then to generate the corresponding computer program. Furthermore, any optimal design process, particularly for multibody systems, requires a large amount of computational time due to the inherent iterative process. By introducing the joint coordinate formulation of equations of motion and the finite difference method in the sensitivity analysis, the total computational time is reduced. However, the mixed method based on the joint coordinate formulation can reduce computational time but also the numerical error. Numerical accuracy and computational time can also be adjusted by choosing different integration routines. The corresponding computer program can be easily developed from the transient dynamic analysis program.

One example of the sensitivity analysis employs a quarter car with a double A-arm suspension system. The analysis is performed using both the mixed method and the finite difference method in order to contrast the strengths and weaknesses of the two methods. In this example, overall deviation of the sensitivity results of the two methods is less than 10\% and computational times are comparable. However, when more than six design parameters are involved in the sensitivity analysis, the computational time in the mixed method becomes substantially smaller than that of the finite difference method. Moreover, in order to check the results obtained from the finite difference method, more than once of the sensitivity evaluations are required. Hence, the finite difference method requires larger computational time.
Transient behavior of a given system can be represented as a series of steady state configurations. Each steady state configuration can be obtained from an arbitrary configuration by employing steady state analysis formulation. The analysis can be performed by using a numerical iteration process to solve a set of nonlinear algebraic equations representing the steady state of the system. Numerical iteration for this approach requires relatively small computational time. However, when the steady state configuration is obtained through the transient dynamic analysis by solving the nonlinear differential equations, large computational time is required. The results from steady state analysis are verified by simulating the 4WS vehicle model. At 2.0 seconds, the results from the transient analysis coincide with those of the steady state analysis. The computation time for the steady state analysis is less than 4 \% of that of the transient analysis.

For vehicle systems, the steady state configuration depends on the vehicle speed and the steering input, and it can be obtained through several iterations in the steady state analysis. When the new driving speed and the new steering input are not apart from the previously obtained steady state configuration, the new steady state configuration can be found in only a few iterations. Hence, various steady state configurations corresponding to various driving conditions can be obtained quickly by incrementally changing the driving conditions. General tendency of the system can be easily observed by examining various steady state configurations. Therefore, this analysis is effective in parameter optimization in large-scale multibody systems.

The path control steering strategy used in the dynamic simulations of any vehicle model following a prescribed path is developed based on position errors and slope errors at a current position and at an estimated position. In the lane change maneuver, the prescribed
path is defined by a fourth order polynomial function which achieves a smoothly connected path from a starting-point to an end point. In the J-turn maneuver, the prescribed path is described by a circle and an angle of position in the circle was used as the slope function. Additionally, by suppressing an abrupt change of steering angle, smooth path control was performed. The feedback path control method effectively permits comparison of several different types of steering systems.

By using the nonlinear 4WS model and steady state analysis, optimal steering angle ratios are easily found for various driving conditions. The optimal steering ratio obtained in the nonlinear 4WS model depends not only on vehicle speed but also on steering angle while that of the linear 4WS model is dependent on vehicle speed only. The steering angle ratios are applied in several transient dynamic simulations such as lane change and J-turn maneuvers. In order to show the generality of the comparison, the simulations were performed in various ranges of vehicle speed. The overall results from the simulations show that the optimal steering ratio from the nonlinear model produces the least lateral velocity during a simulation period. The above comparisons show that the optimal parameters in the steady state are valid during the transient state.

In closing, the mixed method has advantages in the sensitivity analysis, such as easy programming, efficient computation and less numerical error. Hence, it is a proper method to be used in the sensitivity analysis of complex large-scale dynamic systems. The steady state analysis with nonlinear equations of motion can also give great efficiency in other research areas such as the optimal design of suspension systems and tires. The feedback path control strategy can also be used in the control of unmanned vehicles.
APPENDIX A
FOUR-WHEEL-STEERING VEHICLE MODEL DESCRIPTION

In this appendix, a multibody model of a 4WS vehicle is described. The model consists of the main chassis, the complete suspension system, and four wheels [30]. All the four wheels are connected to the main chassis by A-arm double-wishbones suspension systems as shown in Figure A.1. A suspension spring and a shock absorber are included in each suspension subsystem. The governing equations of motion of the vehicle model employs thirty joint coordinates and sixteen constraints. Twelve constraints represent the loop closure of the double-wishbone suspension systems. The remaining four constraints are corresponding to the steering of the four wheels. This system has fourteen degrees of freedom. The main chassis has six degrees of freedom as a floating base body: three for the translational motions and three for the rotational motion. Four A-arm suspensions have one degree of freedom each and four wheels have four degrees of freedom corresponding to the rolling motion. The multibody model is presented in Figure A.2. The inertia characteristics of the vehicle are presented in Table A.1. The initial position of each body is also presented in Table A.1. The kinematic joints used in the model are described in Table A.2. Each joint links two adjacent bodies. The body fixed coordinates (local coordinates) of the joints are also presented. One more set of local coordinates on the joint axis is added for the revolute joint in Table A.2, which will be used for generating a unit vector along the revolute joint axis. The characteristics of the suspension springs and dampers are presented in Table A.3.

The steering angle of a wheel is defined as the relative angle between the main chassis and the longitudinal direction of the wheel. Mechanically the steering angle can be achieved
through a steering system with a steering wheel, a gear box, a steering rack, a steering tie rod, and a knuckle. From numerical point of view, a given steering angle command needs to be applied to the governing equations of motion of the vehicle. However, in the present form of the equations of motion, the current configuration of the system comes from integration of the accelerations, therefore the steering angle can not be enforced directly into the solution of the equations of motion. Therefore, a steering command is enforced in terms of the steering acceleration in the constraint part of the equations of motion.

The steering constraint of a wheel is modeled by using an angle constraint between two vectors \([45]\),

\[
\Phi^{(\text{steer})} = u_i^T u_j - \cos \left( \theta_f + \frac{\pi}{2} \right) = 0
\]  

(A.1)

where, the subscripts \(i\) and \(j\) stand for the main chassis, and one of the wheels, respectively. \(u_i\) and \(u_j\) are the unit vectors for the longitudinal direction of the vehicle and the lateral direction of the wheel, respectively. The relationship of the vectors is presented in Figure (A.3).

The Jacobian matrix and the right hand side of the acceleration equation are stated as

\[
\Phi_{\dot{u}_i}^{(\text{steer})} = [0^T, -u_i^T \dot{u}_j]
\]  

(A.2)

\[
\Phi_{\dot{u}_j}^{(\text{steer})} = [0^T, -u_i^T \dot{u}_j]
\]  

(A.3)

and

\[
\gamma = -2 \ddot{u}_i^T \dot{u}_j - u_j^T \omega_i \dot{u}_j - u_i^T \omega_j \dot{u}_j - \ddot{\theta}_f \sin \theta_f - \dot{\theta}_f \cos \theta_f
\]  

(A.4)

where, the steering command is enforced by specifying acceleration \(\dot{\theta}_f\) form. This acceleration of the steering command can be given either by an open-loop controller or a
closed-loop controller.

Each tire is modeled as a torus shape for the geometrical description. For the mechanical description, the tire is modeled to be equivalent to a three-dimensional spring system which has three translational spring elements along the three principal axes such as radial, longitudinal, and lateral directions. The radial spring element normally has damping characteristics. The kinematic and dynamic properties are derived as a functions of the tire geometry, orientation, velocity and experimental data. Lateral forces in each tire, due to elastic deformations for steering, are included. The concept of a friction-ellipse is employed to calculate traction and braking forces. The tire model used in this study is based upon a comprehensive development for pneumatic tires in Ref. [42-45]. The characteristics of this tire model are presented in Table A.4.
Table A.1. Description of Rigid Bodies

<table>
<thead>
<tr>
<th>Body #</th>
<th>Description</th>
<th>Mass (Kg)</th>
<th>Inertia (Kg/m²)</th>
<th>Initial position of c.m. x/y/z (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Main Chassis</td>
<td>800</td>
<td>250/1000/1100</td>
<td>0/0/0</td>
</tr>
<tr>
<td>2</td>
<td>Right-front-lower A-arm</td>
<td>0.5</td>
<td>0.5/0.5/0.5</td>
<td>1.316/-0.419/-0.415</td>
</tr>
<tr>
<td>3</td>
<td>Right-front knuckle A-arm</td>
<td>5.0</td>
<td>1.0/1.0/1.0</td>
<td>1.305/-0.593/-0.349</td>
</tr>
<tr>
<td>4</td>
<td>Right-front-upper A-arm</td>
<td>0.5</td>
<td>0.5/0.5/0.5</td>
<td>1.283/-0.448/-0.201</td>
</tr>
<tr>
<td>8</td>
<td>Right-rear-lower A-arm</td>
<td>0.5</td>
<td>0.5/0.5/0.5</td>
<td>-0.943/-0.414/-0.441</td>
</tr>
<tr>
<td>9</td>
<td>Right-rear knuckle A-arm</td>
<td>5.0</td>
<td>1.0/1.0/1.0</td>
<td>-0.965/-0.598/-0.351</td>
</tr>
<tr>
<td>10</td>
<td>Right-rear-upper A-arm</td>
<td>0.5</td>
<td>0.5/0.5/0.5</td>
<td>-0.946/-0.444/-0.224</td>
</tr>
<tr>
<td>14</td>
<td>Right-front wheel</td>
<td>20</td>
<td>1.0/1.9/1.0</td>
<td>1.305/0.594/-0.349</td>
</tr>
<tr>
<td>16</td>
<td>Right-rear wheel</td>
<td>30</td>
<td>1.0/1.9/1.0</td>
<td>-0.965/-0.598/-0.351</td>
</tr>
</tbody>
</table>

* Bodies in the left-hand-side are symmetric to those in the right-hand-side.
Table A.2. Description of Kinematic Joints

<table>
<thead>
<tr>
<th>Joint # &amp; type</th>
<th>Connected bodies</th>
<th>Position of the joint of reference point 1 $\xi/\eta/\zeta$</th>
<th>Position of the joint of reference point 2 $\xi/\eta/\zeta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (RVLT)</td>
<td>(1,2)</td>
<td>(1.310/-0.295/-0.385, 0.0/0.132/0.017)</td>
<td>(1.410/-0.295/-0.385, 0.10/0.132/0.017)</td>
</tr>
<tr>
<td>2 (SPHR)</td>
<td>(2,3)</td>
<td>(0.000/-0.312/-0.017, 0.01/0.040/-0.120)</td>
<td></td>
</tr>
<tr>
<td>3 (SPHR)</td>
<td>(4,3)</td>
<td>(0.0/-0.105/0.027, 0.020/0.050/0.140)</td>
<td></td>
</tr>
<tr>
<td>4 (RVLT)</td>
<td>(1,4)</td>
<td>(1.280/-0.34/-0.215, 0.0/0.105/-0.062)</td>
<td>(1.380/-0.340/-0.215, 0.100/0.105/-0.062)</td>
</tr>
<tr>
<td>9 (RVLT)</td>
<td>(1,8)</td>
<td>(-0.950/-0.25/-0.370, 0.0/0.170/0.040)</td>
<td>(-0.850/-0.25/-0.370, 0.100/0.170/0.040)</td>
</tr>
<tr>
<td>10 (SPHR)</td>
<td>(8,9)</td>
<td>(0.0/-0.170/-0.040, 0.020/0.010/-0.150)</td>
<td></td>
</tr>
<tr>
<td>11 (SPHR)</td>
<td>(10,9)</td>
<td>(0.0/-0.140/0.042, 0.020/0.020/0.150)</td>
<td></td>
</tr>
<tr>
<td>12 (RVLT)</td>
<td>(1,10)</td>
<td>(-0.950/-0.30/-0.235, 0.0/0.140/-0.042)</td>
<td>(-0.850/-0.30/-0.235, 0.100/0.140/-0.042)</td>
</tr>
<tr>
<td>17 (RVLT)</td>
<td>(3,14)</td>
<td>(0.0/0.0/0.0, 0.0/0.0/0.0)</td>
<td>(0.0/1.0/0.0, 0.0/1.0/0.0)</td>
</tr>
<tr>
<td>19 (RVLT)</td>
<td>(9,16)</td>
<td>(0.0/0.0/0.0, 0.0/0.0/0.0)</td>
<td>(0.0/1.0/0.0, 0.0/1.0/0.0)</td>
</tr>
</tbody>
</table>

* Joints in the left-hand-side are symmetric to those in the right-hand-side.
Table A.3. Suspension Spring and Damper Characteristics

<table>
<thead>
<tr>
<th>No.</th>
<th>Connected bodies</th>
<th>K (N/m)</th>
<th>D (Nsec/m)</th>
<th>l₀ (m)</th>
<th>Connected position</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(1,2)</td>
<td>9.16x10⁴</td>
<td>1.44x10⁴</td>
<td>0.344</td>
<td>(1.31/-0.395/-0.10, 0.0/0.0/0.0)</td>
</tr>
<tr>
<td>2</td>
<td>(1,5)</td>
<td>9.16x10⁴</td>
<td>1.44x10⁴</td>
<td>0.344</td>
<td>(1.31/0.395/-0.10, 0.0/0.0/0.0)</td>
</tr>
<tr>
<td>3</td>
<td>(1,8)</td>
<td>9.16x10⁴</td>
<td>1.44x10⁴</td>
<td>0.362</td>
<td>(-0.97/-0.40/-0.10, 0.0/0.0/0.0)</td>
</tr>
<tr>
<td>4</td>
<td>(1,11)</td>
<td>9.16x10⁴</td>
<td>1.44x10⁴</td>
<td>0.362</td>
<td>(-0.97/0.40/-0.10, 0.0/0.0/0.0)</td>
</tr>
</tbody>
</table>

Table A.4. Tire Characteristics

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radius</td>
<td>0.30 (m)</td>
</tr>
<tr>
<td>Radial stiffness Cᵣ</td>
<td>1.5x10⁵ (N/m)</td>
</tr>
<tr>
<td>Longitudinal stiffness Cₓ</td>
<td>8.0x10⁴ (N/slip)</td>
</tr>
<tr>
<td>Cornering stiffness Cₓ</td>
<td>7.0x10⁴ (N/rad)</td>
</tr>
<tr>
<td>Max. friction coeff. μₑₓ</td>
<td>0.80</td>
</tr>
<tr>
<td>Min. friction coeff. μₑₘ</td>
<td>0.60</td>
</tr>
</tbody>
</table>
Figure A.1 Double A-arm suspension system
Figure A.2 Schematic view of the 4WS vehicle model with body numbers, joint types, and joint velocity variables.
Figure A.3  Schematic diagram of a steering angle of a wheel
REFERENCES


