# DISCRETE STOCHASTIC ARBITRAGE MODELS FOR PRICING OPTIONS ON SHORT AND LONG TERM YIELDS: TESTS OF THE HO AND LEE AND THE BLACK, DERMAN AND TOY MODELS.

by

Mahendra Raj

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A Dissertation Submitted to the Faculty of the

COMMITTEE ON BUSINESS ADMINISTRATION

In Partial Fulfillment of the Requirements
For the Degree of

DOCTOR OF PHILOSOPHY

In the Graduate College

THE UNIVERSITY OF ARIZONA

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# Discrete stochastic arbitrage models for pricing options on short and long term yields: Tests of the Ho and Lee and the Black, Derman and Toy models

Raj, Mahendra, Ph.D.

The University of Arizona, 1992

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As members of the Final Examination Committee, we certify that we have read the dissertation prepared by Mahendra Raj

entitled DISCRETE STOCHASTIC ARBITRAGE MODELS FOR PRICING OPTIONS ON SHORT AND LONG TERM YIELDS: TESTS OF THE HO AND LEE AND THE BLACK, DERMAN AND TOY MODELS.

and recommend that it be accepted as fulfilling the dissertation requirement for the Degree of Doctor of PHILOSOPHY

Joel S. Sternberg
Joel S. Sternberg — Date 11/18/92

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Date

Date

Final approval and acceptance of this dissertation is contingent upon the candidate's submission of the final copy of the dissertation to the Graduate College.

I hereby certify that I have read this dissertation prepared under my direction and recommend that it be accepted as fulfilling the dissertation requirement.

Joel S. Sternberg
Dissertation Director Joel S. Sternberg — Date 11/18/92

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# DEDICATION

This dissertation is dedicated to my wife Nandini, who inspired and motivated me at every stage of this effort.

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# ABSTRACT

The Chicago Board of Options Exchange introduced the short-term and the long-term options on interest rates in June, 1989. This paper develops versions of the Ho and Lee and the Black, Derman and Toy models, two of the most popular arbitrage based discrete stochastic term structure models, and conducts joint tests of the models and the option markets. Both parametric and non-parametric tests are employed.

The data consist of the prices of these options and the term structure of interest rates from June, 1989 to June, 1992. The prices of the short and long-term interest options have been obtained from the Wall Street Journals for the period. The term structure of interest rates has been obtained as the prices of treasury bills and the seven, ten and thirty year treasury strips.

It is found that the version of Black, Derman and Toy model used in the study is superior to the Ho and Lee model. The Black, Derman and Toy explains the option prices to some extent as seen from the significant R-square values but it does tend to underprice both the short and long-term options

slightly. However, since the tests are joint tests of the model and the short and long-term interest rate option markets, it may be that though the model is pricing the options correctly, the markets are not efficient. This may be especially true in view of the fact that the trading in these options has been, at times, quite thin.

The Ho and Lee model fails to significantly explain both the short and long-term option prices. The call prices are found to be reasonably close to the actual prices whereas the put prices are grossly over priced. As indicated by the theoretical concepts the unstable and sensitive nature of $\delta$ and the tendency for the interest rates at the extreme nodes to assume unrealistic values causes this problem.

# 4. INTRODUCTION

## 4.1 BACKGROUND

On June 23, 1989 the Chicago Board of Options Exchange introduced two new options on interest rates - an option on short-term interest rates and another on long-term interest rates. These options were unique in that, unlike other interest rate dependent options, there were no assets like bonds underlying them. So, when these options are exercised, settlement delivery is in cash. These options opened new avenues for investors interested in hedging and speculating on interest rate movements. In the present study these markets are analyzed utilizing recent developments on discrete interest rate stochastic processes. Of the various models developed to price interest contingent claims, Ho and Lee (1986) and Black, Derman and Toy (1987) are especially important since these researchers use the entire term structure to derive arbitrage-based models, while still retaining their simplicity. In this study the newly introduced options on interest rates are used to test empirically the Ho and Lee and the Black, Derman and Toy models.

The study finds that both the Black, Derman and Toy and the Ho and Lee models do not do a very good job of explaining the prices of the short- and long-term interest rate options. The Black, Derman and Toy explains the option prices to some extent as seen from the significant R-square values although it exhibits a tendency to underprice both the call and put options. The Ho and Lee model fails to do as good a job of predicting the option prices as the Black, Derman and Toy model. In the Ho and Lee model the interest rates assume unduly high or low values at the extreme points. This results in either the call or the put prices being overpriced if the other is used for the parameter estimation. In the present study, calls were used for the parameter estimation and so, while the model prices the calls reasonably well, the put prices tend to be quite high.

Since the introduction of option trading on bonds and other interest rate dependent assets, much attention has been given to the development of models to price such options. Options on interest rate dependent assets are extremely difficult to price because this involves modeling stochastic interest rate movements. The earliest candidates were obviously the Black - Scholes model and the Binomial model. Both of these models had been successfully applied to price stock options and so it was natural that they were also used for options on bonds. However, it was evident that they would fail, as theoretically both of them were clearly unsuitable.

Options on bonds have some major differences from those on stocks that need to be resolved. First, unlike stocks the price of a bond at maturity is fixed and known with certainty and so the Wiener process typically used to depict stock price movements is inappropriate for bond prices. Second, bond prices are dependent on interest rates which exhibit a complex stochastic behavior. An application of the Black-Scholes model (1973) to bond prices is thus not suitable for evaluating bond options since it assumes constant interest rates, while at the same time it assumes that the bond prices are stochastic - two contradictory assumptions. Moreover, the Black - Scholes model over bond prices assumes that the volatility of the bond price is constant when in fact the volatility of the bond price must decrease with time and it must drop to zero at maturity. The binomial pricing model developed by Cox, Ross and Rubinstein (1979) allows valuation of American options but since the assumptions for the price distribution are the same as that for the B-S model, the same criticisms apply to this model as well.

The second and the more important category of models can be termed the term structure models. In these models, the term structure of interest rates is first modeled. Then, using this the bond prices are estimated and options on bonds priced.

## 4.2 THE TERM STRUCTURE OF INTEREST RATES

The function relating interest rates to the term to maturity of default free debt is referred to as the term structure of interest rates. The shape and movement of the term structure has been one of the most appealing topics for researchers during the past fifty years. The reason for the fascination with this topic stems from the many benefits derived from a better understanding of the term structure of interest rate. It helps in the prediction of future interest rates, analyzing returns of assets with different maturities, arbitraging between bonds of different maturities, providing insights into non-default risk, and in pricing bonds and other interest dependent assets. The last application mentioned above, that is, the pricing of bonds and other interest dependent assets is what we are concerned with in this paper.

Three major theories of term structure have emerged over the past decades. They are:

1. The Expectations Hypothesis,
2. The Liquidity Preference Hypothesis, and
3. The Market Segmentation Hypothesis

Bierwag (1989a), McEnally (1989) and Shiller (1990) are some of the good surveys of the literature in recent years.

Irving Fisher (1896) provides the earliest discussion of the expectations hypothesis and Lutz (1940) elaborated on this. According to the pure form of this hypothesis the forward rate for a future time interval is the rate expected to prevail in the future time period. That is, the forward rate is the current expectation of the future rate. This expected rate may not ensue because of the uncertainty involved. Under this, an upward sloping term structure implies that higher interest rates are expected in future, whereas a downward sloping term structure implies that future interest rates are expected to be lower.

The liquidity preference hypothesis was espoused by Hicks (1946) but became popular after Meiselman (1962). Long-term securities have a greater risk of price fluctuation than short-term securities. So, investors favor short-term securities whereas borrowers favor long-term securities resulting in a disequilibrium in the market. Hence, to induce investors to buy long-term securities, which they believe are riskier, they need to be additionally compensated and the borrowers who prefer long-term debt must provide this compensation in the form of higher interest rates. This results in long-term securities having higher interest rates.

The Market segmentation hypothesis was first advocated by Culbertson (1957). He perceived that institutional investors are constrained to certain time horizons due to regulation. This results in a market consisting of investors with different time preferences. Life insurance companies and Savings and Loan's were interested in long-term securities. Commercial banks and Property and Casualty insurance companies on, the other hand, operated in the short-term market. Other investors have other reasons for their different maturity selections. In such a situation with different investors confined to markets in their desired maturity, the interest rates in these markets are determined by the supply and demand prevailing for each maturity.

Cox, Ingersoll and Ross (1981) developed the modern treatment of expectations hypothesis. They identified five different forms of expectations hypothesis which hold under certainty conditions. They then show that only one of these is valid under uncertainty equilibrium conditions. This, the local expectations hypothesis, states that the one-period expected return of every pure discount bond will be equal to the one-period risk-free return.

## 4.3 TERM STRUCTURE MODELS

During the past couple of decades term structure models incorporating stochastic interest rate movements have been developed to price interest rate dependent assets. These term structure models fall into one of two different groups - equilibrium-based models or arbitrage-based models. Some of the equilibrium-based models are found in Vasicek (1977), Dothan (1978), Richard (1978), Langetieg (1980), Rendelman and Bartter (1980), Brennan and Schwartz (1982), Courtadon (1982), Cox, Ingersoll and Ross (CIR) (1985) and Beekman and Shiu (1988). The arbitrage-based models which are more recent, can be found in Ball and Torous (1983), Schaefer and Schwartz (1987), Ho and Lee (1986), Black, Derman and Toy (1987) and Heath, Jarrow and Morton (1990).

## 4.4 EQUILIBRIUM-BASED MODELS

The equilibrium approaches use a two step procedure for pricing options. The first step involves pricing zero coupon bonds from a finite number of exogenously specified state variables in a continuous trading economy. A Markov process is assumed for bond prices, thus making the bond prices depend only on the current values of the state variables. Then, the prices of all the bonds are shown to depend on these state variables in equilibrium. The bond prices are given by the solution of a partial differential equation for a no arbitrage condition. The second step uses these prices to value options. A major problem with equilibrium models is that they can have preference dependent contingent claim formulae. Equilibrium models also need to be inverted in order to match the initial term structure. This involves a very complicated procedure.

Vasicek (1977) implicitly makes a preference structure assumption by assuming a constant market price of risk and an Ornstein-Uhlenbeck process for the spot rate, which is the only state variable for the entire term structure. The Ornstein- Uhlenbeck process is an elastic random walk wherein the spot rate continually moves in a random manner around a fixed point. However, as shown by Brenner (1990), Vasicek's implied forward rate variance process is free of preference parameters and so closed-form solutions can be obtained that are free of the market price of risk. This model allows for negative interest rates.

Beekman and Shiu (1988) avoid the possibility of negative interest rates inherent in Vasicek's (1977) model by using a Brownian bridge process to modify the Ornstein-Uhlenbeck process of Vasicek's (1977) model.

Dothan (1978) uses an arbitrage argument to derive a partial differential equation to obtain closed form solutions for default free bonds. This solution assumes a continuous stochastic spot interest rate and is preference dependent. This model also prevents negative interest rates.

Richard (1978) presents an arbitrage model using two stochastic state variables - the expected real interest rate and the expected rate of inflation in a continuous economy. This results in two term structures - one for the real interest rate and the other for the inflation - that are both preference dependent.

Langetieg (1980) extends Vasicek's (1977) elastic random walk model by generalizing Richard's (1978) bivariate model into a multivariate model. The model assumes that the bond price depends on an arbitrary number of stochastic factors following a joint elastic random walk. It also assumes that the short-term rate can be expressed as a linear function of the stochastic factors. A third assumption the model makes is that the market price of risk of each stochastic factor is non-stochastic. Under these conditions a closed form solution is obtained.

Rendelman and Bartter (1980) assume that m, the growth rate of the short-term interest rate r, the single state variable for this model, L, the market price of risk and s, the volatility of r are all constant. The movement of r is given by,

$$dr = (m - L.s)r.dt + s.r.dz \tag{1}$$

where dz is a Wiener process. The assumption made about m is invalid and the model is also preference dependent.

Courtadon (1982) presents a preference dependent model for pricing European options on default free bonds using a single state variable - the short rate. No closed form solutions are obtained but numerical techniques such as the finite difference method can be used to obtain prices. The model he used is as shown:

$$m.r = k.(r_0 - r) \tag{2}$$

where m, L and s are as defined above. k, $r_0$, L and s are all constants in this model. Therefore,

$$dr = k.(r_0 - r)dt + s.r.dz \tag{3}$$

The model incorporates a mean reverting drift. This model suffers from the problem that the returns on all bonds are perfectly correlated and that the long-term discount bond yield is constant.

Brennan and Schwartz (1982) is an extension of the Courtadon model where the price of the default free bond is now assumed to depend on both the short rate and the long-term or consol rates. The dynamics of the rates are given by

$$dr = b_r(r,l,t)dt + s(r,l,t)dz_r \tag{4}$$

$$dl = b_l(r,l,t)dt + s(r,l,t)dz_l \tag{5}$$

where $z_r$ and $z_l$ are standardized Wiener process.

The Courtadon as well as the Black-Scholes model can be obtained as special cases of this model. If the mean and variance of the long-term rates are zero, then this reduces to Courtadon model. If short-term rate is assumed constant and the default free bond is of perpetual maturity then this becomes the Black-Scholes model. This model overcomes some of the problems of Courtadon model while retaining the stochastic short rate and the time dependent variance of bond returns.

Cox, Ingersoll and Ross (CIR) (1985) models the term structure using an equilibrium approach giving an analytical solution unlike most of the models above that give numerical solutions. The assumptions of CIR regarding m, L and s are different.

$$m.r = k(r_0 - r) \tag{6}$$

$$s.r = Q.\ \sqrt{r} \tag{7}$$

$$L = W/Q. \sqrt{r} \tag{8}$$

where k, $r_0$, Q and W are constants. The assumption pertaining to m implies a mean reverting process, that is , converges to a long-term value $r_0$ at rate k. Their model is

$$dr = k(r_0 - r)dt + Q. \sqrt{r}.dz \tag{9}$$

So, this model assumes that long-term rates have no volatility. Brown and Dybvig (1986) use the maximum likelihood to test a version of the CIR model and conclude that CIR overestimates short-term interest rates. Longstaff (1990) has extended the CIR model to derive closed form expressions for pricing European call and put options on interest rates.

## 4.5 ARBITRAGE-BASED MODELS

The second group of term structure models are arbitrage-based and are espoused in Ball and Torous (1983), Schaefer and Schwartz (1987), Ho and Lee (1986), Black, Derman and Toy (1987), Bliss and Ronn (1989) and Heath, Jarrow and Morton (1990). These models take the initial bond prices and price processes as given and use these to construct arbitrage-based models.

Ball and Torous (1983) use the price of the bond itself as the state variable to obtain a preference free, closed form formula for European call option price. The constraint of bond prices approaching face value at maturity is incorporated by postulating that the bond prices follow a Brownian bridge process. A Brownian bridge process occurs when the Wiener process is constrained to take a specific value at a specific time which in this case is par value at maturity. The variance of the rate of return of the bond is constant implying that the variability of yield to maturity increases unboundedly as bond approaches maturity, a weakness of the model. Bond prices generated by this process are not consistent with stochastic spot rate process and are not arbitrage free.

Schaefer and Schwartz (1987) derive a single state model where the variable is the price of the underlying bond, but they also assume that the standard deviation of the return of this bond is proportional to the bond's duration. This model does not require estimation of the stochastic process of interest rates, market price of interest risk or bond prices at boundary conditions while at the same time incorporating the changing characteristics of the bond. However, this model suffers from the drawback of assuming a constant short-term rate like the Black - Scholes model.

Ho and Lee (1986) develop a discrete multi-period binomial model of the term structure. They take the initial term structure as given and assume that it moves randomly according to a binomial process. Being a single factor model, bonds of all maturities are perfectly correlated. One special case of the discrete process used for the Ho-Lee model also allows for negative interest rates. If the number of time periods become very large then the interest rates at extreme points assume unrealistic values as shown in Appendix A. In this model all interest rates have the same volatility, whereas actually this may not be the case.

Bliss and Ronn (1989) develop a trinomial model based on Ho - Lee's binomial model. The model incorporates state dependent shifts that are determined by observable state variables. The empirical tests indicate that the state variables affect the variability in the shifts of the term structure. The option prices obtained indicated a systematic deviation from the actual option prices.

Black, Derman and Toy's model (1987) is similar to the Ho-Lee model in approach. However, this incorporates the mean reverting behavior of interest rates. The upward movement is limited and the volatility of interest rates decreases with increases in time period.

Heath, Jarrow and Morton (HJM) (1990) have developed a general arbitrage framework which gives many of the other models, such as the Ho and Lee (1986) and the Black, Derman and Toy (1987), as special cases. This model uses an exogenous initial forward rate curve instead of a zero coupon bond price curve. It assumes a continuous stochastic process for the forward rates. In one special case of the model the constant forward rate volatilities are consistent with a fixed value of the bond at maturity. The inputs required for this model are the initial forward rate curve, volatilities and the contract details. This model subtly avoids the CIR criticism that the arbitrage pricing approach can generate prices inconsistent with equilibrium by matching the initial bond price curve by construction and requiring the existence of a risk neutral measure. Amin (1990) uses a special version of the Ho-Lee model derived from Heath, Jarrow and Morton (1990) to price American options. This is a two variable discrete term structure model that incorporates the lattice framework and is easy to compute. Moreover, it allows the use of the control variate technique to facilitate computation.

Thurston (1992) tested the discrete version of the Heath, Jarrow and Morton model (1992) on the Treasury bill and Treasury strip data. He found that the special version of the HJM model corresponding to the Ho and Lee model performed better than the exponential version. However, neither of the models explained the strip data well and they only did slightly better on the Treasury bill data.

# 5. MODELS FOR PRICING OPTIONS

## 5.1 MODEL DEVELOPMENT

In this study the empirical validity of the Ho-Lee and the Black, Derman and Toy term structure models is tested utilizing the options on short- and long-term interest rates. Both these methods are arbitrage-based and are independent of preference structure. Both use discrete binomial models of the stochastic behavior of interest rates for pricing options on interest rate instruments. The term structure is first modeled and then the future short- and long-term interest rates are estimated from this. The option prices at expiration are calculated from these rates. These prices are then discounted each period utilizing the expected future one-period rates to find the present values of the option prices.

However, there are some differences between these models. One major difference is that the Black, Derman and Toy model incorporates the mean reversion of interest rates and thus prevents the interest rates from assuming very high values. The Ho and Lee model does not do this and so suffers from the drawback that interest rates can become infinite or negative at extreme points. Both use the entire term structure instead of just the short rate unlike the earlier models. The Black, Derman and Toy model considers the prices and the volatilities of the rates together as the term structure, whereas the Ho and Lee model only considers the prices of the discount bonds for the term structure.

In the present study we use these models to price options which do not have a traded asset underlying them. This entails modifying and adapting the models to price these unique options. Since there is no fixed value at maturity as in the case of bonds, the interest rates at maturity have to be first determined using the model. The option data are available only for three years posing additional problems in estimating seven, ten and thirty year rates for the long-term options. However, the models developed in this paper attempt to overcome these problems in pricing the interest rate options.

## 5.2. THE HO AND LEE MODEL

The discrete multi-period binomial model of the term structure developed by Ho and Lee (1986) is especially important because, unlike most other approaches, it models movements in the entire term structure. They define discount functions which are the prices of zero coupon bonds with a face value of \$1 and maturing at time t = 0,1,2,...T.... There are a finite number of states at each time period t. The discount function $P^t_u$ (T) describes the price of a bond with T periods to maturity at the Uth state at time t. At time 0, the discount function is $P^0_0$ (T). At time 1, there are 2 possible states - an upstate denoted by $P^1_1$ (T) and a downstate denoted by $P^1_0$ (T). From $P^1_1$ (T) at time 2, there are again 2 possible states $P^2_2$ (T) an upstate and $P^2_1$ (T) a downstate. From $P^1_0$ (T) the two possible states are $P^2_1$ (T) the upstate and $P^2_0$ (T) the downstate. So, the subscript "u" indicates the number of up movements enroute to the present position. Ho and Lee assume that an upstate followed by a downstate is the same as a downstate followed by an upstate. This amounts to assuming that the prices are path independent.

A second assumption that they make is that there are no arbitrage opportunities. In a no-arbitrage world, the one period forward price is defined as,

$$F^1(T-1) = \frac{P(T)}{P(1)} \tag{10}$$

and is the price of a (T-1) period bond one-period from the present in a world with certainty. In the real world there is uncertainty, and this is accounted for by the inclusion of the "Perturbation Functions" $h_u(T-1)$ and $h_d(T-1)$ in the forward rate function. When $h_u(T-1)$ is greater than 1, bond prices will rise and when $h_d(T-1)$ is less than 1, bond prices will fall because,

$$P^{t+1}{}_{u+1}(T-1) = \frac{P^t{}_u(T) \cdot h_u(T-1)}{P^t{}_u(1)} \tag{11}$$

$$P^{t+1}{}_u(T-1) = \frac{P^t{}_u(T) \cdot h_d(T-1)}{P^t{}_u(1)} \tag{12}$$

So, by specifying the perturbation functions and the initial discount function or term structure it is possible to construct a binomial lattice of term structure movement.

If the probabilities of an up movement $h_u(T)$ and a down movement $h_d(T)$ are $\pi$ and $(1-\pi)$, respectively, then since no arbitrage opportunities exist,

$$\pi . h_u(T) + (1-\pi) . h_d(T) = 1 \tag{13}$$

The proof of the above expression is obtained by constructing a risk-free hedge using two discount bonds with different maturities as illustrated in Appendix B. $\pi$ is the implied binomial probability. Under the risk neutral measure, for the term premium to be zero, the implied binomial probability has to be the same as the binomial probability. So, $\pi$ is the risk neutral probability.

The path independent assumption states that an up movement followed by a down movement is the same as a down movement followed by an up movement. So, $P^t_u$ must be the same irrespective of whether in the previous period the price was $P^{t-1}_u$ or $P^{t-1}_{u-1}$. From (11) and (12),

$$P^{t+1}_{u+1}(T) = \frac{P^t_u(T+1) \cdot h_u(T)}{P^t_u(1)} \tag{14}$$

$$P^{t+1}_{u}(T) = \frac{P^t_u(T+1) \cdot h_d(T)}{P^t_u(1)} \tag{15}$$

Therefore,

$$P^{t+2}_{u+1}(T) = \frac{P^{t+1}_u(T+1) \cdot h_u(T)}{P^{t+1}_u(1)} \tag{16}$$

Since,

$$P^{t+1}_{u}(T+1) = \frac{P^t_u(T+2) \cdot h_d(T+1)}{P^t_u(1)} \tag{17}$$

and,

$$P^{t+1}_{u}(1) = \frac{P^t_u(2) \cdot h_d(1)}{P^t_u(1)} \tag{18}$$

substituting (17) and (18) in (16),

$$P^{t+2}{}_{u+1}(T) = \frac{P^t{}_u(T+2) \cdot h_d(T+1) \cdot P^t{}_u(1) \cdot h_u(T)}{P^t{}_u(1) \cdot P^t{}_u(2) \cdot h_d(1)} \tag{19}$$

$$= \frac{P^t{}_u(T+2) \cdot h_d(T+1) \cdot h_u(T)}{P^t{}_u(2) \cdot h_d(1)} \tag{20}$$

Similarly from (14) and (15),

$$P^{t+2}{}_{u+1}(T) = \frac{P^{t+1}{}_{u+1}(T+1) \cdot h_d(T)}{P^{t+1}{}_{u+1}(1)} \tag{21}$$

Since,

$$P^{t+1}{}_{u+1}(T+1) = \frac{P^t{}_u(T+2) \cdot h_u(T+1)}{P^t{}_u(1)} \tag{22}$$

and,

$$P^{t+1}{}_{u+1}(1) = \frac{P^t{}_u(2) \; . \; h_u(1)}{P^t{}_u(1)} \qquad (23)$$

Substituting (22) and (23) in (21)

$$P^{t+2}{}_{u+1}(T) = \frac{P^t{}_u(T+2) \; . \; h_u(T+1) \; . h_d(T)}{P^t{}_u(2) \; . \; h_u(1)} \qquad (24)$$

Therefore from above,

$$P^{t+2}{}_{u+1}(T) = \frac{P^t{}_u(T+2) . \; h_u(T+1) \; . h_d(T)}{P^t{}_u(2) \; . \; h_u(1)} \qquad (25)$$

for an up followed by a down movement

$$P^{t+2}{}_{u+1}(T) = \frac{P^t{}_u(T+2) . \; h_d(T+1) \; . h_u(T)}{P^t{}_u(2) \; . \; h_d(1)} \qquad (20)$$

for a down followed by an up movement.

Equating (20) and (25), eliminating $h_d$ by (13) and substituting $h_d(1) / h_u(1) = \delta$ as shown in Appendix-C,

$$h_u(T) = [\pi + (1-\pi)\delta^T]^{-1}, \qquad T = 0,1,2... \tag{26}$$

$$h_d(T) = \delta^T[\pi + (1-\pi)\delta^T]^{-1}, \qquad T = 0,1,2... \tag{27}$$

The parameters $\pi$ and $\delta$ determine the model.

In order to price bonds or options on bonds it is necessary to obtain the price of a one-period discount bond as this is the short rate to be used for discounting the prices at each node. In our case, since we are trying to price options on yields, we need to obtain the price at time t of a T period bond with i up movements, $P^t_i(T)$ in terms of the initial term structure and the model parameters. That is, we need to express $P^t_i(T)$ in terms of $\delta$, $\pi$ and the initial bond prices $P^0_0(T+t)$ and $P^0_0(t)$. This can be obtained by the backward recursion of equations (11) and (12) as follows.

For $n<T+t$ an $n<K+t$,

$$P^{n+1}{}_{i+1}(T) = \frac{P^{n}{}_{i}(T+1) \cdot h_u(T)}{P^{n}{}_{i}(1)}$$

$$P^{n+1}{}_{i+1}(K) = \frac{P^{n}{}_{i}(K+1) \cdot h_u(K)}{P^{n}{}_{i}(1)}$$

from equation (11).

Therefore,

$$\frac{P^{n+1}{}_{i+1}(T)}{P^{n+1}{}_{i+1}(K)} = \frac{P^{n}{}_{i}(T+1) \cdot h_u(T)}{P^{n}{}_{i}(K+1) \cdot h_u(K)} \qquad (28)$$

Similarly,

$$P^{n+1}{}_{i}(T) = \frac{P^{n}{}_{i}(T+1) \cdot h_d(T)}{P^{n}{}_{i}(1)}$$

$$P^{n+1}{}_i(K) = \frac{P^n{}_i(K+1) \, . \, h_d(K)}{P^n{}_i(1)}$$

from equation (12).

Therefore,

$$\frac{P^{n+1}{}_i(T)}{P^{n+1}{}_i(K)} = \frac{P^n{}_i(T+1) \, . \, h_d(T)}{P^n{}_i(T+1) \, . \, h_d(K)} \tag{29}$$

By repeating (28) i times and (29) n-i times we ~~get~~,

$$P^t{}_i(T) = \frac{P(T+t)}{P(t)} \times \frac{h_d(T+t-1).h_d(T+t-2)\ldots h_d(T+i) .h_u(T+t-1)\ldots h_u(T)}{h_d(t-1).h_d(t-2)\ldots h_d(i).h_u(i-1)\ldots h_u(1)}$$

(30)

Substituting for $h_d(T)$ from equation (27)

$$P^t{}_i(T) = \frac{P(T+t).h_u(T+t-1).h_u(T+t-2)\ldots h_u(T).\delta^{T(t-i)}}{P(t).h_u(t-1).h_u(t-2)\ldots h_u(1)} \tag{31}$$

Equation (31) expresses the discount function at each state time in terms of the initial term structure and the model parameters $\delta$ and the up movements, $h_u$.

For a one-period discount bond with T=1, substituting into equation (31) gives,

$$P^t_i(1) = \frac{P(t+1).h_u(t).h_u(t-1).h_u(t-2)...h_u(1).\ \delta^{(t-i)}}{P(t)\ .h_u(t-1).h_u(t-2)...h_u(1)} \tag{32}$$

$$= \frac{P(t+1).h_u(t).\delta^{t-i}}{P(t)} \tag{33}$$

Substituting for $h_u(t)$ results in the simple expression,

$$P^t_i(1) = \frac{P(t+1)\ .\ \delta^{t-i}}{P(t)\ .\ (\pi + (1-\pi)\ \delta^t)} \tag{34}$$

Similarly for T=2, we get the expression,

$$P^t_i(2) = \frac{P(t+2).h_u(t+1).h_u(t)...h_u(2).\delta^{2(t-i)}}{P(t).h_u(t-1).h_u(t-2)...h_u(1)}$$

$$= \frac{P(t+2).h_u(t+1).h_u(t).\delta^{2(t-i)}}{P(t).h_u(1)} \qquad (35)$$

Instead of representing the term structure by a discount function, it is more convenient in our case to use the yield curve. The yield curve is given as

$$r(T) = \frac{-\ln P(T)}{T} \qquad (36)$$

where r(T) is the continuously compounded yield.

The upstate and the downstate can be represented in terms of the yield curve using equations (11) and (12) as

$$r^1_1(T) = - \frac{1}{T} \frac{\ln P(T+1)}{P(1)} - \frac{1}{T} . \ln [h_u(T)] \qquad (37)$$

in the upstate and

$$r^{t}_{0}(T) = - \frac{1}{T} \frac{\ln P(T+1)}{P(1)} - \frac{1 \, . \, \ln[h_d(T)]}{T} \tag{38}$$

in the down state.

In equation (34) substituting for $P^{t}_{i}(T)$ from (31),

$$r^{t}_{i}(T) = \frac{-\ln P^{t}_{i}(T)}{T}$$

$$= \frac{-\ln}{T} x \frac{[P(T+t).h_u(t+T-1).h_u(t+T-2)\ldots h_u(T).\delta^{T(t-i)}]}{[P(t).h_u(t-1).h_u(t-2)\ldots h_u(1)]} \tag{39}$$

For T=1, equation (37) simplifies to,

$$r^{t}_{i}(1) = -\ln \frac{P(t+1).h_u(t).\delta^{(t-i)}}{P(t)} \tag{40}$$

Bierwag (1989b) gives an interesting development of the interest rates in the Ho-Lee model. The stochastic process followed by the interest rates is given by $R(1-q,h_u,h_d,1+r_0)$ where (1-q) is the probability of an up movement in 1+r~, the interest rate and that implies a downward move in bond prices. He proves that if (1+r) follows the binomial stochastic process $R(1-q,h_u,h_d,1+r_0)$, then local expectations hypothesis implies that the bond prices follow the Ho-Lee process with the implied probability being the same as the binomial probability. However, the reverse does not hold, that is, the Ho-Lee process does not imply the specific binomial process. The implied interest rate process is similar to the binomial process, but the parameters $h_u$ and $h_d$ may be different for different points in time. So, this process would be equivalent to a binomial process only when $h_u = h_{u1} = h_{u2} = \ldots$ and $h_d = h_{d1} = h_{d2} = \ldots$. The Ho-Lee model further requires that $h_{ut}/h_{dt} = \delta$ for $t=1,2,3,\ldots$, and $\delta = h_{d1}/h_{u1}$. Hence, the Ho-Lee process can also be depicted by $HL\{1-q;h_{d1},h_{d2},\ldots;\delta;1+r_0\}$.

The two parameters $\pi$ and $\delta$ must be determined for pricing any interest rate option using this model. The initial discount function can be found using any of the numerous techniques used by McCulloch (1971), Carleton and Cooper

(1976), Vasicek and Fong (1982) and others. Another possibility is to get Treasury strip data. Parameters $\pi$ and $\delta$ can be estimated from prices of the interest rate options traded using an iterative procedure.

It can be seen that in the Ho-Lee model all interest rates have the same volatility, whereas actually long-term interest rates are less volatile than short-term rates. Other models have been developed that overcome some of the shortcomings of the Ho-Lee model. Bierwag (1989) describes a binomial model that does not allow for negative interest rates. Black, Derman and Toy (1987) have formulated a model that limits upward movement and where the volatility of interest rates decreases with increases in maturity.

## 5.3. BLACK, DERMAN AND TOY MODEL

Black, Derman and Toy's one-factor model also uses the initially observed term structure in developing the model. The model uses the data on both bond yields and volatilities, unlike the Ho-Lee model, which uses only the prices and not the volatilities. The term structure is used to estimate the expected means and standard deviations of future short rates. The short rate is the one-factor of this model.

Assumptions:

1. The major assumption of the model is that long-term rates reflect the expectations of the market regarding future short-term rates.

2. The short-term rates are assumed to follow a lognormal distribution.

By the first assumption all interest rate changes are assumed to be caused by changes in short-term rates, thus making the short-term interest rate the one-factor of the model. The second assumption ensures that if the short rate is positive at start then it can never be negative in the future, a major drawback of the Ho-Lee model.

Long-term bond yields have lower volatility than the short-term bond yields. This implies that the short-term rates farther away in future tend to revert to the mean value. That is, the fluctuation of short-term rates around the mean is dampened with time. This is the mean reversion of short-term interest rates which is also used by Cox, Ingersoll and Ross (1985) in their continuous time model of term structure motion. By matching the expected future short rates to today's observed term structure, the model incorporates this tendency of future short rate volatility to decline with time.

Black, Derman and Toy derive a valuation formula which they use to compute long-term bond prices using expected future short-term rates. Consider a one-year zero coupon bond with a face value of \$100. After one year its price, P = 100 with certainty irrespective of the short-term rate. Let today's short-term rate for one year be r. Then, the value of the bond today is obtained by discounting the price by the interest rate, r for one year as,

$$P(1) = \frac{100}{1 + r} \tag{41}$$

Now, consider a similar but two-year, zero coupon bond with face value of $100. One year from now it can have an up value of $P_u$ or a down value of $P_d$ each with probability of 0.5. From $P_u$ it can move up to $P_{uu}$ = 100 or down to $P_{ud}$ = 100 with equal probability. From $P_d$ it can, with equal probability, go up to $P_{ud}$ = 100 or go down to $P_{dd}$ = 100. Let today's one-period interest rate be r. One year from now the one-period interest rate can take an up value of $r_u$ and a down value of $r_d$ each with probability of 0.5. Then, the price one year from now, resulting from an up movement is

$$P_u = \frac{(0.5).100 + (0.5).100}{1 + r_u} \tag{42}$$

Similarly, the price one year from now arising from a down move is

$$P_d = \frac{(0.5).100 + (0.5).100}{1 + r_d} \tag{43}$$

So, the best estimate of today's price is given by

$$P(2) = \frac{(0.5).P_u + (0.5).P_d}{1 + r} \tag{44}$$

This is the model's valuation formula. This can be extended forward into the future to value bonds expiring three or more periods in future, provided the future short-term rates are known.

They also derive a measure of the volatility of the bond price. For the two-year bond from above, the mean of the expected prices one year from today is first calculated as,

$$P_m = (0.5).P_u + (0.5).P_d \tag{45}$$

Then, the volatility as measured by standard deviation, $\sigma_p$ of the expected one-year price is obtained from

$$\sigma_p = [(0.5).(P_u - P_m)^2 + (0.5).(P_d - P_m)^2]^{0.5} \tag{46}$$

In the actual world the one-period interest rates in the future are not known. Today's prices of bonds maturing at various time periods in the future are known. That is, the prices of bonds maturing after one year, two years etc. can be obtained today. Similarly, the volatilities of the prices of the bonds are also known today. Using these prices and volatilities it is possible to estimate the implied future one-period interest rates which are today's expectations of the one-period interest rates expected to occur in the future. This is done by initially guessing the values of the one-period interest rates and then adjusting these rates until the prices and volatilities of the bonds so obtained match the actual prices and volatilities of the bonds as observed today.

The procedure used for estimating future short-term rates is as follows. For a one-year zero coupon bond the price found using the valuation formula should be equal to that obtained from the term structure. This can be solved for the value of r, the one-period rate prevailing now.

$$P(1) = \frac{100}{1 + r} \tag{41}$$

$$\text{Therefore, } r = \frac{100}{P(1)} - 1 \tag{47}$$

For the two-year bond there are two unknown future short rates $r_u$ and $r_d$ that are possible one year from now. Using an approximate guess for the values of $r_u$ and $r_d$ in the valuation formula, the up and down movement prices one year in future can be found as

$$P_u = \frac{(0.5).100 + (0.5).100}{1 + r_u} \tag{42}$$

and,

$$P_d = \frac{(0.5).100 + (0.5).100}{1 + r_d} \tag{43}$$

The prices, $P_u$ and $P_d$ obtained above are then discounted using the one-period interest rate calculated in equation (47) to obtain today's expected price of a two-year bond.

$$P(2) = \frac{(0.5).P_u + (0.5).P_d}{1 + r} \tag{48}$$

The volatility of the price of a two-year bond can be calculated using equations (45) and (46) and the prices $P_u$ and $P_d$ from above as shown below.

$$P_m = (0.5).P_u + (0.5).P_d \tag{45}$$

and,

$$\sigma_p = [(0.5).(P_u - P_m)^2 + (0.5).(P_d - P_m)^2]^{0.5} \tag{46}$$

The price and volatility of the two-year bond estimated above is compared to the observed price and volatility of the two-year bond. The values of $r_u$ and $r_d$ are adjusted until the estimated price and volatility match the observed price and volatility of the two-year bond. The interest rates, $r_u$ and $r_d$ so obtained are the up and down values of the one-period interest rates expected one-period in the future.

The same method can be used to estimate the one-period interest rates two periods in future. In this case consider a three-year bond which at maturity in three years has a face value of 100. One year from today the bond can have an up value of $P_u$ or a down value of $P_d$. From the up value $P_u$ the price can move up to a value $P_{uu}$ or down to a value $P_{ud}$ during the second year. Similarly, from the down value $P_d$ the price can move up to a value $P_{du}$ or down to a value $P_{dd}$. This model also assumes path independence for both prices and interest rates resulting in $P_{ud} = P_{du}$. From $P_{uu}$, $P_{ud}$ and $P_{dd}$ the prices can go to $P_{uuu} = 100$, $P_{uud} = 100$, $P_{udd} = 100$ and $P_{ddd} = 100$.

In a like manner the one-period interest rate can move up to $r_u$ or down to $r_d$ in one year. From $r_u$ it can go to $r_{uu}$ or to $r_{ud}$ and from $r_d$ it can go to $r_{ud}$ or $r_{dd}$ during the second year. Though after two years there are three possible rates, only two of the rates need to be independently guessed. Since the short rate has been assumed to be lognormal with a volatility that depends only on time, the volatilities of $r_u$ and $r_d$ must be equal. That is,

$$\frac{r_{uu}}{r_{ud}} = \frac{r_{ud}}{r_{dd}} \tag{49}$$

Therefore,

$$r^2_{ud} = r_{uu} . r_{dd} \tag{50}$$

Hence, only two one-period rates need to be guessed

As before $P_{uu}$, $P_{ud}$ and $P_{dd}$ are obtained as,

$$P_{uu} = \frac{(0.5).100 + (0.5).100}{1 + r_{uu}} \tag{51}$$

$$P_{ud} = \frac{(0.5).100 + (0.5).100}{1 + r_{ud}} \tag{52}$$

and,

$$P_{dd} = \frac{(0.5).100 + (0.5).100}{1 + r_{dd}} \tag{53}$$

Now using the one-period interest rates expected one year in future as estimated earlier,

$$P_{u} = \frac{(0.5).P_{uu} + (0.5).P_{ud}}{1 + r_{u}} \tag{54}$$

and,

$$P_{d} = \frac{(0.5).P_{ud} + (0.5).P_{dd}}{1 + r_{d}} \tag{55}$$

Finally, applying the valuation formula once more and using the one-period interest rate today,

$$P(3) = \frac{(0.5).P_u + (0.5).P_d}{1 + r} \tag{56}$$

The volatility can be determined just as it was in the case of a two-year bond.

The mean value of the price is

$$P_m = (0.5).P_u + (0.5).P_d \tag{45}$$

and the volatility as measured by the standard deviation is

$$\sigma_p = [(0.5).(P_u - P_m)^2 + (0.5).(P_d - P_m)^2]^{0.5} \tag{46}$$

The price and the volatility of the three-year bond as estimated above is compared to the observed price and volatility of the bond. The values of $r_{uu}$ and $r_{dd}$ are adjusted until the two prices match. The values of the interest rates so obtained are the best estimates of the one-period rates expected two periods in future.

This procedure can be extended to obtain the one-period interest rates three periods or more in future. In all cases only two values of the one-period rates need to be guessed. The other values are not independent and so can be estimated from these two values.

The model can also be used to price options. For this the value of the underlying asset such as a bond, at the expiration date of the option has to be first determined. Then, the price of the call option at expiration date is

$$C = \mathrm{Max}[r - X, 0] \tag{57}$$

This is discounted to the present time using the one-period interest rates estimated earlier to obtain the price of the option today.

This model incorporates the lower volatility of longer term interest rates and limits upward movement. However, it does not eliminate the occurrence of negative interest rates.

## 5.4. OPTIONS ON INTEREST RATES

The short- and long-term rates or yields are simple non-linear functions of bond prices. However, unlike the bonds, the interest rates are not traded assets and so need not follow the local expectations hypothesis of CIR (1981) or have the equivalent martingale measure of Harrison and Kreps (1979) and Harrison and Pliska (1981). This is illustrated by the fact that the price of a call, C, with strike price of zero, on the interest rate, r, is not equal to the interest rate, r, as given by Max[r-0,0]. In the situation of stock options where the price of a call with a strike price of zero on the stock price, s, is equal to the stock price, s, as obtained from Max[s-0,0]. This is due to the absence of self-financing portfolios that could generate arbitrage profits in the case of interest rates.

There are other differences between options on interest rates and options on traded assets like stocks or bonds. Options on assets are bounded above by the price of the underlying asset, unlike interest rate options, which can exceed the current value of the interest rate, r. This is because the future interest rates may be higher than the present rate and though future one-period rates will be higher too, the impact of the higher future interest rates can be greater than that of the higher one-period rates. Options on assets also have a lower bound defined by the intrinsic value so as to prevent arbitrage. Interest rate options on the other hand can have values less than its intrinsic value, Max[r-x,0]. As before, this is due to the two effects of a change in the interest rate on the call price. An increase in the interest rate, r, causes an increase in the payoff of the call as well as the future one-period discount rates. The former effect dominates for small values of the interest rate, r, whereas the latter effect eclipses the former for larger values of the interest rate, r. As the interest rate approaches infinity, the call price converges to zero because the increase in payoff is overshadowed by the increase in the discount rate. This implies that, at some point, the call price will be below its intrinsic value.

The interest rate call option prices do not have to be increasing functions of the term to expiration as in the instance of call options on assets. In fact, as term to expiration, t, increases and approaches infinity, the discount factor converges to zero while the payoff remains bounded. So, the call price can initially increase with t, and later decrease with further increase in the term to expiration, t. Call option prices also need not be increasing functions of the variance, $\sigma^2$. The relationships for put option prices are quite similar to those for call options. An increase in interest rate however, results in a decrease in put price since the payoff is diminished and the discount rate increased.

# 6. METHODOLOGY

## 6.1. DATA

The present study uses six distinct data sets, of which two are option data sets and the remaining four are Treasury securities. The option data used for the study consist of the prices of the options on short- and long-term interest rates from June 1989 to June 1992. These interest rate options began trading at the CBOE on June 23, 1989 and so the prices from that day to June 31, 1992 are used for the present study. The prices of the options on short- and long-term interest rates were collected manually from the Wall Street Journal. There are some inherent limitations regarding the data - they are available only for three years and the trading in these options has sometimes been very thin.

The Treasury securities data obtained for the study consist of the daily prices of Treasury bills and the daily prices of seven-, ten- and thirty-year Treasury strips. The term structure can be obtained from the prices of the Treasury bills and Treasury strip for the relevant period. These data were obtained from the Dow Jones News Retrieval Services, an on-line electronic database and from Barrons.

The short-term interest rate and the long-term interest rate option prices were obtained weekly from June 29, 1989 to June 26, 1992. The monthly prices picked from these weekly data were the prices of the options for the fourth Thursday of each month. Thursdays were chosen because the Treasury bills that are used in the model to price the option, expire on Thursdays. Fridays could have been used instead, since the options expire on the Saturday following the third Friday of each month. So, whenever prices are not available or trading is very thin for a particular Thursday, Friday's prices are used instead of Thursday's. Monthly data is used mainly because pricing options weekly or daily would lead to too many nodes resulting in exploding interest rates in the models. Weekly pricing was tried initially for both the models and led to unrealistic values of interest rates in both cases, and especially in the case of Ho and Lee model.

In the case of the long-term options, since Treasury strips that are used to price them expire on the 15th of February, May, August and November of each year, only options expiring on these four months are priced using the model. Hence, only the monthly prices of options expiring on these four months are required from June 29, 1989 to June 26, 1992.

Longstaff (1990) has used the first six month's trading data on these two options for testing a version of the CIR model. He only used the prices of options which were frequently traded and the present study follows the same procedure.

The new CBOE short-term interest rate options with ticker symbol IRX have payoffs based on the yield of the latest 13 week Treasury bill auctioned. The index for the short-term interest rate option is this yield in percentage terms multiplied by 10. So, if the yield of the most recently auctioned Treasury bill is 6.425%, then the short-term index is obtained as 64.25 (6.425 x 10).

Let an April call with a strike price of 65 be quoted at a premium of 2 1/4. Each premium point represents $100 and so, the cost for buying this call is 2.25 x 100 = $225.

If now the short-term interest rate moves above (6.5 + .225) or 6.725% then, each tenth of a percentage point increase would earn a profit of $100. So, if the short-term rate moved to 6.825% then a profit of $100 would be realized.

The payoffs of the long-term options with ticker symbol LTX are based on the average of the yields of two each of the most recently auctioned seven-year and ten-year Treasury notes and the thirty-year Treasury bonds. The average of these six yields is used to determine the payoffs of these options. The index for this option is this average rate in percentage terms multiplied by 10. That is, if the average interest rate is 7.5%, then the index would be 75. Since each premium point is worth $100, an August call with a strike of 85 quoted at a premium of 1 1/16 would cost

(1.0625 x 100) or $106.25

If the index increases above 86.0625 (85 + 1.0625), then the options would earn $100 for each tenth of a percentage point increase. Therefore, if the long-term index moves to 87.0625, the option would earn, ($8706.25 - $8606.25) = $100.

Another unique feature of these options is that they are European options. They can be exercised only on the date of expiration, not any day prior to it as in the case of American options. This makes these options easier to evaluate. However, as stated earlier these options have major differences from options on traded assets like bonds.

The term structure is obtained from the prices of the Treasury bills and seven-, ten- and thirty-year Treasury strips. The short-term option is dependent on the three-month Treasury bill rate while the long-term option is dependent on the average of the seven-, ten- and thirty-year interest rates. So, the prices of Treasury bills have been obtained from June 29, 1989 to June 26, 1992. For the long-term options the prices of the seven-, ten- and thirty year Treasury strips have been obtained from June 29, 1989 to June 26, 1992. The prices obtained from Dow Jones News Retrieval gives daily prices and so prices on the fourth Thursday of each month are extracted from this. In this case whenever Thursdays' prices are not available Wednesday's prices are used.

The Treasury bill data are used in both the models for both the short- and long-term option pricing. For the short-term option it is used to estimate the future three-month rates and the future one-month rates that are used for discounting. For the long-term option the Treasury bill data are used to obtain the future one-month rates that are used for discounting the option prices. The data gives the daily High, Low and Close. The Ho and Lee model uses the close prices whereas the Black, Derman and Toy model uses all the three prices.

## 6.2. THE HO AND LEE MODEL

Briefly, the procedure involved consists of using a subset of data to estimate the model parameters and then estimating and testing the option prices using the other subset of data. The term structure data, as well as the short- and long-term option data, are each split into two sets. The first data set is used to estimate the parameters of the model. The second data set is used to price the call and put options. The procedure also involves a third step, which is the statistical comparison of the estimated option prices with the observed option prices to test the performance of the model.

First, the parameters of the model have to be estimated. The two parameters for the Ho and Lee model are $\delta$ and $\pi$. Initially, $\delta$ and $\pi$ are assigned arbitrary values. Using the first set of term structure data and the initial values of the parameters, the option prices are calculated from the model. These option prices are then compared to the first set of actual option prices. Using an iterative process the parameter values are adjusted until the estimated values are statistically not different from the actual values.

The parameter values obtained as above and the second set of term structure data are used to price the second set of options. These prices are tested to determine whether they are statistically different from the actual option prices. A number of computer programs were written for the parameter estimation and the option pricing for both the short- and long-term interest rate options. Programs were also written for extracting the monthly data from the daily Treasury bill, and the seven, ten and thirty year Treasury strip data. The statistical testing was done mainly using the SAS statistical package as explained in the section on tests.

In this study the model is used to test both the short- and long-term interest rate options. For the short-term options the term structure data is the Treasury bill data. Starting on June, 1989 the monthly term structure data for six months and initial guesses of $\pi$ and $\delta$ are used to obtain the short-term (three month) rates expected at the option expiration dates one, two and three months in future using equation (39). These short-term rates are then used to estimate the price of the call options at the expiration dates one, two and three months in future. The call option prices are then discounted monthly using the estimated future monthly rates to obtain the option prices on the starting dates. The monthly rates expected to prevail one, two and three months in future are obtained from equation (34) of the model. The estimated prices so obtained are compared to the actual prices observed on that day for the option with the same strike price and expiration date. The values of $\pi$ and $\delta$ are adjusted to estimate call option prices that match the actual call option prices.

The parameters $\delta$ and $\pi$ estimated above are then used in the next six months to estimate the short rate expected one, two and three months in future. So, in the second six-month period, the expected future short rates one, two and three months in future are obtained from the model using the parameter values estimated in the previous six-month period. Once the expected future short rates are obtained, the call and put option prices at expiration are easily calculated for different strike prices and different expiration dates. These prices are then discounted to the present using the one-month rates expected in future. The one-month rates expected in future are as before obtained from the model using equation (34). This procedure is continued until June, 1992.

The long-term options require a slightly different procedure from above. The long-term interest rates depend on the average of the seven-, ten- and thirty-year rates. The term structure data therefore consists of the seven-, ten- and thirty-year Treasury strip prices. Beginning on June, 1989 using initial arbitrary values of $\pi$ and $\delta$, the seven-, ten- and thirty-year rates expected one, two and three months in future are separately estimated for six months. The average of the seven-, ten- and thirty-year rates is then calculated for the six months. The option prices at the expiration dates are calculated using these rates for the different strike prices and expiration dates. These prices are then discounted to the pricing date using the one-period rates expected in future. As in the case of the short-term options the one-period rates are obtained from the model using the Treasury bill data. Thus, on June, the prices of options expiring in July, August and September are estimated for the assigned values of $\pi$ and $\delta$. These prices are then compared to the actual prices for the period. The values of $\pi$ and $\delta$ are adjusted and the procedure repeated until the estimated prices match the actual prices.

The values of the parameters calculated in the first six months are then utilized to estimate the seven-, ten- and thirty-year rates expected in the second six-month period. The average rates are then calculated and the options priced at the expiration dates. These are then discounted as before to get the option prices expected for the period. This procedure is also continued until June, 1992.

Initially, before pricing the options monthly as above, the weekly term structure data were used to price the short-term options weekly. Call prices were calculated using the expected interest rates for three months. However, in this case it was not possible to get values of $\pi$ and $\delta$ that would result in matching estimates of prices for all the call options during the three months or thirteen weeks. So, next, the same procedure using weekly term structure data and initial guesses of $\pi$ and $\delta$ was repeated to obtain on any week the short-term (three-month) rates expected on the option expiration date one, two and three months in future. The call option prices for the different strike prices were calculated at the three expiration dates. Then, as before these call option prices were discounted weekly using estimated future weekly rates. Thus, on a particular week the prices of call options expiring one, two and three months in future were obtained. These prices were compared to the actual option prices. In this case also it was not possible to get values of $\pi$ and $\delta$ that would result in a match of all the estimated and actual option prices. The extremely sensitive nature of $\delta$ again precluded a common value of $\pi$ and $\delta$ for the option prices.

The option prices are extremely sensitive to $\delta$ and so even a minute change in $\delta$ can cause a change of a few dollars in option prices. The sensitive nature of $\delta$ precluded the possibility of a common $\pi$ and $\delta$ for all the option prices on a weekly basis. The highly sensitive behavior of $\delta$ was partly due to the large number of nodes resulting from using weekly data. The number of periods from the starting date to the option expiration date, t, when measured in weeks can have a value from one ...thirteen depending on whether the option is expiring on the first, second or third month. The number of periods in the three-month rate, T, is thirteen if measured weekly. This results in $\delta$ being raised to very high values. On a monthly basis, the number of periods from starting date to the expiration date can have a maximum value of three and the number of periods in the three-month rate, T, is also three. This helps to reduce the explosion of interest rates to a great extent.

The long-term interest rate options are dependent on the average of seven-, ten- and thirty-year rates. So, T, for the seven-, ten- and thirty-year rates are 365, 521 and 1565 respectively if weekly data is used. Hence, this was not tried as no meaningful result would have been obtained. Monthly nodes reduces the T values to 84, 120 and 360. These are still very high, although much lower than before.

## 6.3. THE BLACK, DERMAN AND TOY MODEL

The interest rate options in this paper - both short- and long-term - are different from the other interest dependent options in that there is no asset-like bond underlying these options that will be delivered on exercise. These options depend directly on the level of interest rates. So, instead of estimating future bond prices, we need to estimate future interest rates. In the case of bonds, the price at maturity is fixed and known with certainty. Hence, the price of a bond at any point in time can be found by a backward recursive procedure using the expected future one-period rates.

In the case of the two interest rate options that are being priced in this paper, this is not possible as the interest rates - both short and long - exhibit stochastic motion and do not have a known price and maturity. Since, the short- and long-term options depend on the three-month Treasury bill and the seven-, ten- and thirty-year Treasury bond rates, one possible solution is to make the period equal to the maturity of these securities. This would enable us to estimate these rates at each node. However, given that these options have been trading only for three years, this method is not possible for the seven-, ten- and thirty-year rates. In the case of the three-month Treasury bill this would give us only twelve nodes. So, in this paper an alternative method similar to the binomial model for stock options is used for estimating the interest rates.

The pricing of these short- and long-term interest rate options involves three stages. The first stage involves estimating value of the underlying interest rate in the future. The second stage is the pricing of the option at expiration date. Finally, this price is discounted to the pricing date.

Estimating the value of the interest rate at expiration date is done using a binomial model similar to the CRR model (1979) for stock options. The index rates for the options are obtained along with the option prices from the Wall Street Journal from June, 1989 to June, 1992. These index rates are assumed to move up or down with equal probability. Using six months of the data, the model parameters $u$ and $d$ are evaluated by comparing the estimated rates with the actual rates. The values of $u$ and $d$ are adjusted until the estimated rate matches the actual rate. The parameter values $u$ and $d$ so obtained are used to evaluate the values of the index rates for the next six months of the data. The parameter values for the six months are obtained by adjusting *u* and *d* using an iterative procedure for the past six months rates.

The model used is

$$r^1 = \Sigma^n_{j=0} \left(\frac{n!}{j!\ (n-j)!}\right) \pi^j\ (1-\pi)^{1-j}\ u^j\ d^{1-j}\ r^0 \tag{58}$$

In the next step the future interest rate estimated in equation (58) is used to calculate the option price for the strike price at the expiration date. The call price at expiration is given by,

$$C = \text{Max}\ [r.10 - x,\ 0] \tag{59}$$

and the put price is given by,

$$P = \text{Max}\ [x - r.10,\ 0] \tag{60}$$

The interest rate, r obtained from equation (58) is multiplied by 10 to give the index value. The procedure is the same for both the short- and long-term options.

The final step in the process is to discount these option prices from expiration date to the pricing date. This is done by discounting the prices each period by the one-period rates expected in future. The one-period rates are estimated from the prices and volatilities of the one-, two- and three-period Treasury bills by employing the valuation and sensitivity formulae of the model. The Treasury bills have a fixed value of 100 at maturity. This is discounted utilizing initial forecasts of the up and down values of the short rates back to get the prices at time 0 as in equations (41), (48), (56). The volatilities are estimated utilizing equation (46). These are then compared to the actual prices and volatilities of the Treasury bills. The forecasts of the short rates are adjusted until the prices and volatilities estimated match the actual rates and volatilities. The short rates that enable this are the expected future one-period rates. On any month the one month rates are estimated for one, two and three-months in future using the prices of Treasury bills expiring one, two and three months in future. These rates are then employed for discounting the option prices from one, two and three months in future to the pricing date. Thus, all the options are discounted back to the pricing date.

A number of computer programs were written for estimating this model. Estimating the parameters "*u*" and "*d*" and the future short-term and long-term interest rates required a separate program. The future one-period rates were estimated by another iterative and computer intensive program. A number of other programs were also required for pricing the options and for manipulating the different data sets.

# 7. RESULTS

## 7.1. EMPIRICAL TESTS

The short- and long-term option prices obtained from the Ho and Lee as well as the Black, Derman and Toy models are statistically compared to the actual option prices. Four different statistical tests are performed to test whether the estimated prices explain the actual option prices. Of these, the ordinary least squares regression and comparison of paired means are parametric, whereas the sign test and Wilcoxon signed rank test are non-parametric.

These tests, which compare the estimated option prices to the actual option prices, are all joint tests of the model and the efficiency of the particular market. So, when we compare the short-term option prices determined by the Black, Derman and Toy model to the actual short-term option prices, we are actually conducting joint tests of the effectiveness of the Black, Derman and Toy model and the efficiency of the short-term interest rate option market. The same is true for the tests of the long-term option prices estimated by the Black, Derman and Toy model and the short- and long-term option prices determined by the Ho and Lee model. Hence, the results of the tests need to be interpreted taking this fact into consideration.

The regression is performed to determine whether the estimated option prices are related to the actual option prices. The model used is

$$C_{act} = a + b \cdot C_{est} + e \tag{61}$$

where $C_{est}$, the estimated option price is the independent variable, $C_{act}$, the actual option price is the dependent variable, "*a*" is the intercept term, "*b*" is the coefficient of the independent variable and e is the error term. If the estimated prices fully explain the actual prices the intercept term "*a*" will be zero and "*b*" the coefficient will be equal to one. The SAS statistical package is used for the purpose.

The comparison of paired means is a useful test where each value in one series is related to a particular value in the second series. The test is useful for testing whether the mean of the difference between the pair of values in the two series is different from zero. The two series in our case are the estimated and actual option prices. In this test the difference between the estimated and actual option prices are first determined. Then a t-test is performed to ascertain whether the mean value of the difference is different from zero. This test also makes a normality assumption. The Proc Means procedure in the SAS statistical package is employed for the purpose.

The first non-parametric test performed is a sign test. The main feature of the sign test is that it makes no assumption about the distribution. It is a very simple and quick test and remarkably effective for small samples. This test also enables paired comparison like the t-test on the mean. In a sign test the difference between each value in one series with the corresponding value in the second series is calculated. If there is no significant difference, half of these differences will be positive and the other half negative. If the values in the first series (estimated price) is greater than the values in the second series (actual price), more of the differences will be positive. If the estimated price is less than the actual price, more of the differences will be negative. The null hypothesis is that the mean of the differences is zero. The alternative hypothesis is framed so that if true it implies the smaller of the two differences - positive or negative - in the case of a one-sided test. So, the difference between the estimated and actual price is calculated and the smaller of the positive or negative numbers is the statistic for this test. In a two-sided test the smaller of the two difference numbers is the statistic. If the statistic is less than or equal to the critical value for the number of observations, then the null hypothesis is rejected in favor of the alternative hypothesis.

The critical values for the sign tests for up to 50 observations are usually available from tables. For values of $n$ greater than 50, they can be calculated for the different significance levels using the $z$ values obtained from a normal distribution. The equation that gives these values is,

$$d = 0.5[n + 1 - z.\sqrt{n}] \tag{62}$$

The values of $d$, the critical values were calculated from the above equation for 1%, 5% and 10% significance levels for both one and two sided tests.

The other non-parametric test that is performed in this paper is the Wilcoxon signed rank test. This is a more powerful test than the sign test and does not assume normality. However, it assumes that the distribution is continuous and that the probability function is symmetric. In this test the differences are ranked from the smallest to the largest in terms of the absolute value. Then, the ranks of all the negative and all the positive are separately summed up. The smaller of the two sums is the statistic for this test. The null hypothesis is that the differences are zero and for this to be true the two sums should be the same. In a one sided test the alternative hypothesis implies the smaller sum whereas in a two sided test the smaller number is the statistic. If the statistic is less than or equal to the critical value the null hypothesis is rejected in favor of the alternative hypothesis.

The critical values for the Wilcoxon signed rank test with $n$ greater than 25 can be calculated for the various significance levels. As for the sign test, the $z$ values can be obtained from the normal tables for different significance levels. Then, the critical values are obtained from the equation,

$$w = 0.5\{0.5.n.(n+1) + 1 - z.\sqrt{[n.(n+1)(2.n+1)/6]}\} \qquad (63)$$

The critical values were calculated utilizing the above equation for 1%, 5% and 10% significance levels for both, one and two sided tests.

## 7.2. RESULTS

The two models - Ho and Lee and the Black, Derman and Toy - are tested employing the short-term and long-term interest rate options. The models are utilized to estimate the option prices and then these prices are compared statistically to the actual option prices obtained from the Wall Street Journal. However, as mentioned earlier, these tests are joint tests of the efficiencies of the models and the markets. That is, the tests on each set of estimated option prices are joint tests of the particular model and option market.

Ordinary Least Squares regression is employed to estimate the parameter values of the model in equation (61). If the option pricing models predict the prices exactly, the intercept would be zero and the value of the parameter, "*b*" would be one. OLS regression on the short-term option prices estimated by the Black, Derman and Toy model gives an intercept value of 0.614502 and a "*b*" value of 0.389479. The "t" value for the intercept is 9.166 and for the slope, "*b*", the "t" value under the null that it is one, is -22.280325. This convincingly rejects the null hypothesis that the intercept is zero and that the slope is one. An R-square value of 0.4963 is also obtained using 205 option prices. However, the t-test to determine whether the mean of the difference between the estimated and actual option prices is zero does not reject the null hypothesis that the mean value of the difference is zero. It gives a value of -0.0390034 for the mean but the "t" value is -0.3494022.

Two non-parametric tests that do not make the normality assumption are also performed. The sign test rejects the null hypothesis that the mean of the difference between the estimated and actual option prices is zero. In a one sided test the number of positive values obtained was 46. This is less than the critical value for 205 observations at 1% significance level. Hence, this rejects the null in favor of the alternate hypothesis that the estimated prices are lower than the actual prices. The Wilcoxon signed rank test, which is a more powerful test than the sign test, also rejects the null that the difference between the estimated and actual prices are zero at 1% significance level. The positive signed rank sum of 6,821 is less than the critical value of 8,171 and so the alternate hypothesis of lower estimated prices is supported. A one-sided test is preferred over the two-sided test since the former indicates the direction in which the estimated prices tend to err unlike the two-sided which does not give any clue regarding the direction of error.

So, since the t-test does not reject the null hypothesis though the two non-parametric do and a R-square value of 0.5866 is obtained for the zero intercept regression, it is possible that the Black, Derman and Toy model does explain the option prices to some extent. The model does however slightly underprice the short-term option prices as indicated by the sign and Wilcoxon tests both of which support the alternate hypothesis that the estimated prices are less than the actual prices.

The estimated long-term option prices from applying the Black, Derman and Toy model gives an intercept value of 0.914561 and a parameter value of 0.301641 in an OLS regression. The "t" values of 10.506 and -19.41747 for the intercept and the slope parameter respectively are obtained for 243 option prices under the null hypothesis that the intercept is zero and the slope is one. The R-square value for the regression is 0.2227. So, the null hypothesis is convincingly rejected in this case of the long-term options. The test of the mean of the difference of matched pairs of estimated and actual prices results in a mean of 0.0170728 and a "t" value of 0.1448241. This fails to reject the null hypothesis that the mean difference is zero.

The sign test, however, again rejects the null hypothesis at 1% significance level in favor of the alternate hypothesis that the model underprices the option prices. For 243 observations, a positive number of 72 is observed as against the critical value of 100. The Wilcoxon signed rank test fails to reject the null at 1% but rejects it at 5% significance level in favor of the alternate hypothesis. In this case a positive signed rank sum of 12,211 is less than the 5% critical value of 12,782.

The long-term option strengthens the finding of the short-term options that the Black, Derman and Toy model tends to slightly underprice the options since in both instances the sign and the Wilcoxon signed rank tests reject the null in favor of the alternate hypothesis that the model underprices the options. However, as before, the t-test fails to reject the null and significant values are obtained for the R-square indicating that the model may have some explanatory power. It is also possible that the model is pricing both the short- and long-term options correctly but that both the short- and long-term interest rate option markets are not efficient. The fact that the trading on these options have at times been very thin lends credence to this possibility.

The application of the Ho and Lee model to price the short-term interest rate option gives some interesting results. The OLS estimates of the intercept is 1.398059 with a "t" value of 8.969 whereas the OLS estimate of the parameter is only -0.001695 with "t" value of -0.0059. The t-test on the difference of the estimated and the actual gives a mean value of -8.0670812 with a "t" value -6.3883016.

The sign test gives 48 positive values for the difference of the estimated and the actual price, which is less than the critical value of 66 for 163 observations at 1% significance level. The alternate hypothesis that the actual prices are greater than the estimated may be accepted. The Wilcoxon signed rank test gave a sum of 6,609 for the positive values. This is greater than the critical value of 5,763 at 10% significance level. Hence, the null that the mean of difference between the estimated and the actual prices is zero cannot be rejected.

So, now the t-test and the sign test reject the null whereas the Wilcoxon signed rank test does not. A possible explanation is that in a majority of the observations the estimated prices are less than the actual option prices, but in the few where the estimated price is greater than the actual, the difference is relatively large. So, the overpriced observations get ranked higher in the Wilcoxon signed rank sum test resulting in a high value for the sum of the positive differences. An examination of the estimated prices reveals the problem. The high positive differences arise for the put options whereas in the case of the calls the model underprices only slightly. As explained earlier, the Ho and Lee model allows interest rates to take on unrealistic values as the number of nodes increase. Moreover, the parameter, $\delta$ is by nature very dynamic and unstable and has been estimated from the call prices. Hence, when options far out into the future are estimated employing this model there is a distinct tendency for this model to overprice the puts as the estimated interest rates tend to become small. If the parameter, $\delta$, had been estimated from the put prices, the model would have overpriced the call as the interest rates would then assume unrealistically high values.

Finally, we look at the performance of the Ho and Lee model in pricing the long-term interest rate options. The long-term interest rate option is an extremely complicated and difficult option to price as it depends on not one but three different interest rates - the seven-, ten- and thirty-year rates. The OLS estimation gives a value of 1.195092 for the intercept and 0.015622 for the parameter with "t" values of 10.568 and 3.817 respectively for 240 observations. The R-square value is 0.0537. The t-test on the mean of the paired values of the estimated and actual option prices gives a value of -16.261780 with a "t" value of -12.0268092.

The sign test on the difference of the estimated and the actual option price for 240 observations indicates that 115 observations are negative while 125 are positive. The critical value at 10% significance level is 108 and so the null hypothesis cannot be rejected. The Wilcoxon signed rank sum test gives a value of 21,992 for the sum of the positive differences and a value of 6,928 for the negative differences. The critical value at 1% significance level is 11,438 and so the null is rejected in favor of the alternate hypothesis that difference is positive or that the estimated prices are higher than the actual prices.

The tests on the estimated long-term option prices give mixed results. The OLS estimation does not indicate any significant explanatory power for the model. The t-test implies that the model significantly underprices the options.

However, the sign test is unable to reject the null whereas the Wilcoxon states that the model overprices the options. The explanation for this is similar to the one for the short-term options. The problems of unstable $\delta$ and unrealistic interest rates with greater number of nodes is even more profound in this case as the option depends on the long-term interest rates that require many more nodes for the estimation. As before, the calls tend to be more or less correctly priced while the puts are grossly overpriced.

## 7.3. CONCLUSION

The contribution of the present study to the field of finance is twofold. It tests two of the more popular discrete stochastic arbitrage models of the term structure of interest rates and thus helps fill the void in the finance research literature. Second, it develops and tests models to price the relatively recent options on short- and long-term interest rates and thus helps the investment community to better understand and employ these options for hedging as well as for speculation.

The study indicates that the version of the Black, Derman and Toy model employed in this paper tends to slightly underprice both the short- and long-term options. However, it does explain the option prices to a certain extent as seen by the significant R-square values. Moreover, since the tests were joint tests of the model and the short- and long-term interest rate option markets it is possible that the model is pricing the options correctly but that the markets are not efficient. Given that the trading in these options has been at times very thin, it is highly likely that the markets are inefficient. Also, considering the fact that these options exhibit certain special characteristics that make them more complex and difficult to price, the Black, Derman and Toy model achieves a reasonably good performance. If applied to price options on bonds or other similar interest dependent options that have an asset underlying them, this model would most likely do a better job of predicting the prices than in the case of the options in this paper. However, the trading on options on bonds is very thin and discontinuous and so the more interesting and newer options on yields were used in this paper.

The Ho and Lee model fails to significantly explain the option prices as seen from the results obtained. The call prices are found to be reasonably close to the actual prices whereas the puts are grossly over priced. The problem with the model as indicated by the theoretical concepts is the unstable and sensitive nature of the parameter, $\delta$. As the number of nodes increase the interest rates tend to rise or drop drastically at the extreme nodes. So, when the $\delta$ is estimated from the call prices, the puts are greatly overpriced and vice versa.

It is hoped that this study will contribute, in at least a small way, to the better understanding of both the interest rate options as well as the two term structure models. Though this study only tests two of the many term structure models available to price interest rate dependent assets, future research will try to test the Heath, Jarrow and Morton (1990) and the Cox, Ingersoll and Ross (1985) models using these data with a view to comparing the performance of the various term structure models in estimating option prices.

# BLACK, DERMAN AND TOY MODEL

# SHORT-TERM OPTION

| OLS ESTIMATION | | |
|---|---|---|
| Number of observations = 205 | | R-square = 0.4963 |
| VARIABLE | PARAMETER | T |
| INTERCEPT | 0.614502<br>(0.06704373) | 9.166 |
| $C_{EST}$ | 0.389479<br>(0.02740182) | -22.280325 |

**Table - 1**

# BLACK, DERMAN AND TOY MODEL

# SHORT-TERM OPTION

| MEANS TEST | | |
|---|---|---|
| VARIABLE: $C_{EST}$ - $C_{ACT}$ | | |
| MEAN | STD. ERROR | T |
| -0.0390034 | 0.1116290 | -0.3494022 |

**TABLE - 2**

# BLACK, DERMAN AND TOY MODEL

## SHORT-TERM OPTION

| SIGN TEST | | |
|---|---|---|
| CRITICAL VALUE AT 1% SIGNIFICANCE LEVEL = 83<br>(ONE SIDED TEST) | | |
| POSITIVE | NEGATIVE | N |
| 46 | 159 | 205 |

**TABLE - 3**

# BLACK, DERMAN AND TOY MODEL

# SHORT-TERM OPTION

| WILCOXON RANK SUM TEST | | |
|---|---|---|
| CRITICAL VALUE AT 1% SIGNIFICANCE LEVEL = 8171 (ONE SIDED TEST) | | |
| POSITIVE | NEGATIVE | N |
| 6,821 | 14,294 | 205 |

TABLE - 4

# BLACK, DERMAN AND TOY MODEL

## SHORT-TERM CALL OPTION

| OLS ESTIMATION | | |
|---|---|---|
| Number of observations = 151 | | R-square = 0.6520 |
| VARIABLE | PARAMETER | T |
| INTERCEPT | 0.572744<br>(0.06365179) | 8.998 |
| $C_{EST}$ | -0.459344<br>(0.03219236) | -14.268754 |

**Table - 5**

# BLACK, DERMAN AND TOY MODEL

## SHORT TERM PUT OPTION

| OLS ESTIMATION | | |
|---|---|---|
| Number of observations = 54 | R-square = 0.3796 | |
| VARIABLE | PARAMETER | T |
| INTERCEPT | 0.554675<br>(0.15810201) | 3.508 |
| $C_{EST}$ | -0.733821<br>(0.0460375) | -15.939636 |

**Table - 6**

# BLACK, DERMAN AND TOY MODEL

# LONG-TERM OPTION

| OLS ESTIMATION | | |
|---|---|---|
| Number of observations = 243 | R-square = 0.2227 | |
| VARIABLE | PARAMETER | T |
| INTERCEPT | 0.914561<br>(0.08705176) | 10.506 |
| $C_{EST}$ | 0.301641<br>(0.03596554) | -19.41747 |

**Table - 7**

# BLACK, DERMAN AND TOY MODEL

## LONG-TERM OPTION

| MEANS TEST | | |
|---|---|---|
| VARIABLE: $C_{EST} - C_{ACT}$ | | |
| MEAN | STD ERROR | T |
| 0.0170728 | 0.1178867 | 0.1448241 |

**TABLE - 8**

# BLACK, DERMAN AND TOY MODEL

# LONG-TERM OPTION

| SIGN TEST | | |
|---|---|---|
| CRITICAL VALUE AT 1% SIGNIFICANCE LEVEL = 100 (ONE SIDED TEST) | | |
| POSITIVE | NEGATIVE | N |
| 72 | 171 | 243 |

**TABLE - 9**

# BLACK, DERMAN AND TOY MODEL

# LONG-TERM OPTION

| WILCOXON RANK SUM TEST | | |
|---|---|---|
| CRITICAL VALUE AT 5% SIGNIFICANCE = 12782<br>(ONE SIDED TEST) | | |
| POSITIVE | NEGATIVE | N |
| 12,211 | 17,435 | 243 |

**TABLE - 10**

# BLACK, DERMAN AND TOY MODEL

# LONG TERM CALL OPTION

| OLS ESTIMATION | | |
|---|---|---|
| Number of observations = 121 | | R-square = 0.1987 |
| VARIABLE | PARAMETER | T |
| INTERCEPT | 1.027152<br>(0.11830605) | 8.682 |
| $C_{EST}$ | -0.59179<br>(0.07360086) | -8.0405376 |

**Table - 11**

# BLACK, DERMAN AND TOY MODEL

# LONG TERM PUT OPTION

| OLS ESTIMATION | | |
|---|---|---|
| Number of observations = 122 | R-square = 0.2965 | |
| VARIABLE | PARAMETER | T |
| INTERCEPT | 0.733547<br>(0.12614554) | 5.815 |
| $C_{EST}$ | -0.698564<br>(0.04180237) | -16.711138 |

**Table - 12**

# HO AND LEE MODEL

# SHORT-TERM OPTION

| OLS ESTIMATION | | |
|---|---|---|
| Number of observations = 164 | | R-square = 0.0003 |
| VARIABLE | PARAMETER | T |
| INTERCEPT | 1.398059<br>(0.15588466) | 8.969 |
| $C_{EST}$ | -0.001695<br>(0.0083874) | -119.42855 |

Table - 13

# HO AND LEE MODEL

# SHORT-TERM OPTION

| MEANS TEST<br><br>VARIABLE: $C_{EST} - C_{ACT}$ | | |
|---|---|---|
| MEAN | STD ERROR | T |
| -8.0670812 | 1.2627897 | -6.3883016 |

**TABLE - 14**

# HO AND LEE MODEL

# SHORT-TERM OPTION

| SIGN TEST | | |
|---|---|---|
| CRITICAL VALUE AT 1% SIGNIFICANCE LEVEL = 66 (ONE SIDED TEST) | | |
| POSITIVE | NEGATIVE | N |
| 48 | 116 | 164 |

**TABLE - 15**

# HO AND LEE MODEL

# SHORT-TERM OPTION

| WILCOXON RANK SUM TEST | | |
|---|---|---|
| CRITICAL VALUE AT 10% SIGNIFICANCE = 5,764 (TWO SIDED TEST) | | |
| POSITIVE | NEGATIVE | N |
| 6,609 | 6,921 | 164 |

**TABLE - 16**

# HO AND LEE MODEL

# LONG-TERM OPTION

| OLS ESTIMATION | | |
|---|---|---|
| Number of observations = 240 | R-square = 0.0537 | |
| VARIABLE | PARAMETER | T |
| INTERCEPT | 1.195092<br>(0.11308888) | 10.568 |
| $C_{EST}$ | 0.015622<br>(0.00409255) | -240.53219 |

**Table - 17**

# HO AND LEE MODEL

# LONG-TERM OPTION

| MEANS TEST | | |
|---|---|---|
| VARIABLE: $C_{EST} - C_{ACT}$ | | |
| MEAN | STD ERROR | T |
| -16.26178 | 1.3521275 | -12.0268092 |

**TABLE - 18**

# HO AND LEE MODEL

# LONG-TERM OPTION

| SIGN TEST | | |
|---|---|---|
| CRITICAL VALUE AT 1% SIGNIFICANCE LEVEL = 108 (TWO SIDED TEST) | | |
| POSITIVE | NEGATIVE | N |
| 125 | 115 | 240 |

**TABLE - 19**

# HO AND LEE MODEL

# LONG-TERM OPTION

| WILCOXON RANK SUM TEST | | |
|---|---|---|
| CRITICAL VALUE AT 10% SIGNIFICANCE = 11,438 (ONE SIDED TEST) | | |
| POSITIVE | NEGATIVE | N |
| 21,992 | 6,928 | 240 |

**TABLE - 20**

# HO AND LEE MODEL

# SHORT TERM CALL OPTION

| OLS ESTIMATION | | |
|---|---|---|
| Number of observations = 111 | R-square = 0.018087 | |
| VARIABLE | PARAMETER | T |
| INTERCEPT | 1.259038<br>(1.251395) | 1.006107 |
| $C_{EST}$ | 1.340606<br>(0.946097) | 0.3600117 |

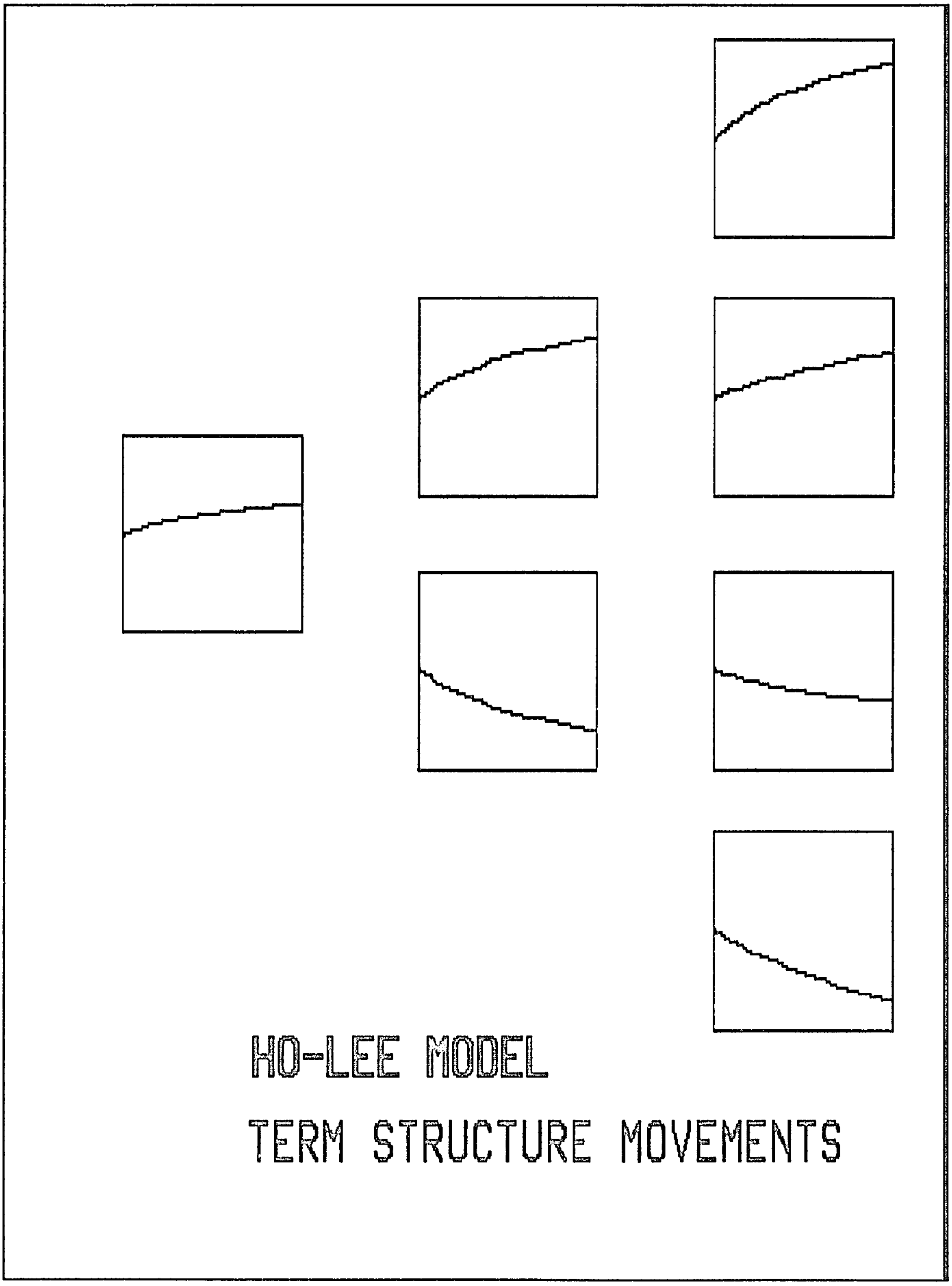
**Table - 21**

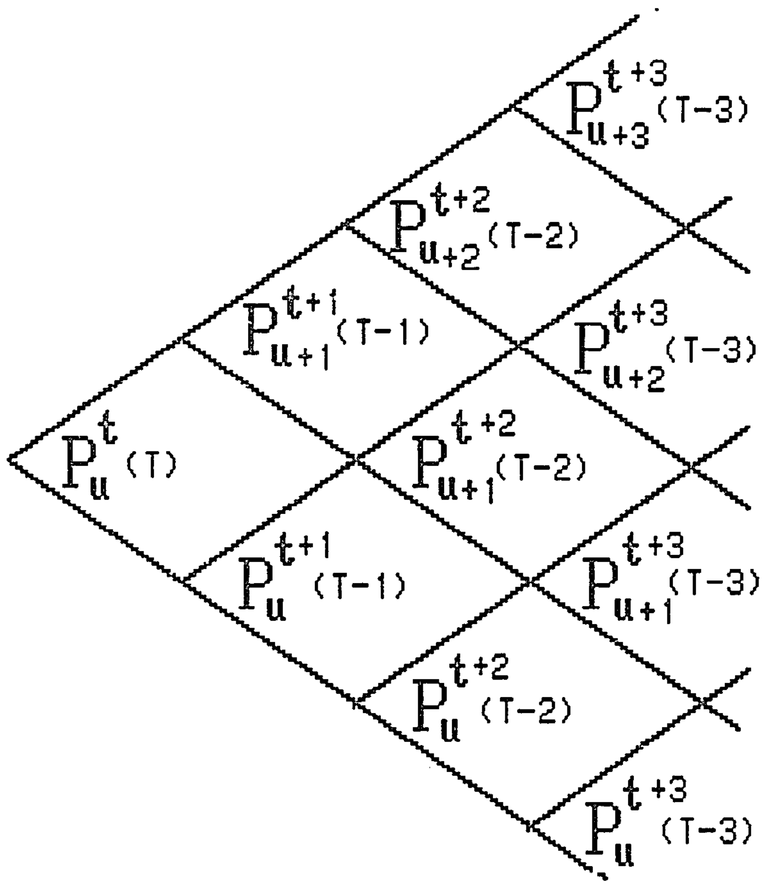
# HO AND LEE MODEL

# LONG TERM CALL OPTION

| OLS ESTIMATION | | |
|---|---|---|
| Number of observations = 134 | | R-square = 0.05862 |
| VARIABLE | PARAMETER | T |
| INTERCEPT | 1.308948 (1.149951) | 1.13826 |
| $C_{EST}$ | -0.19999 (0.069756) | -17.2026 |

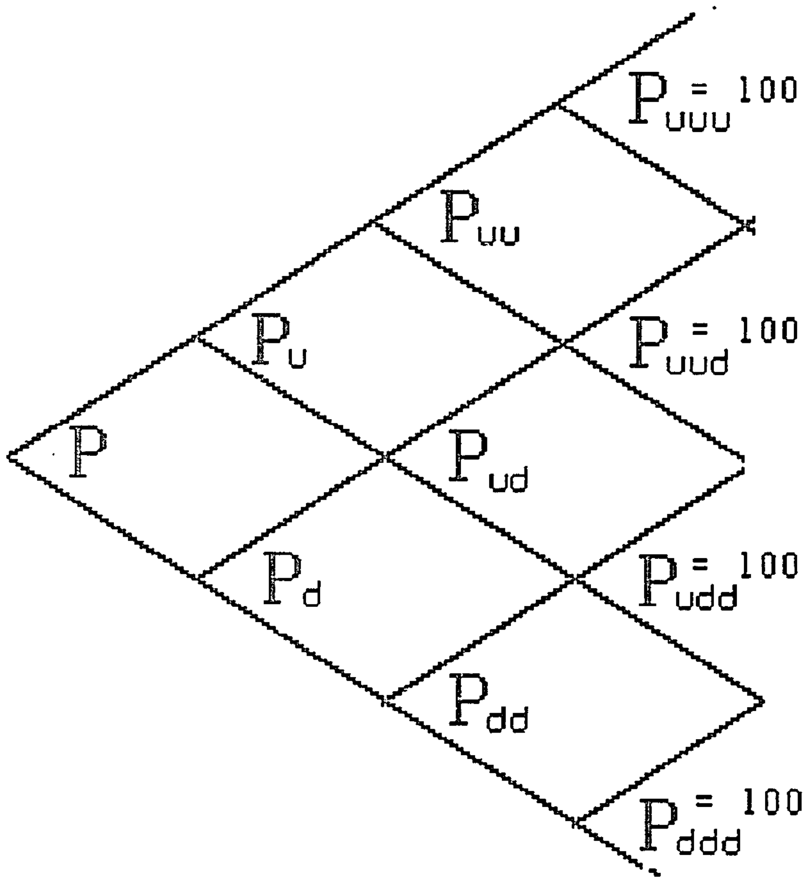
**Table - 22**


HO-LEE MODEL
TERM STRUCTURE MOVEMENTS



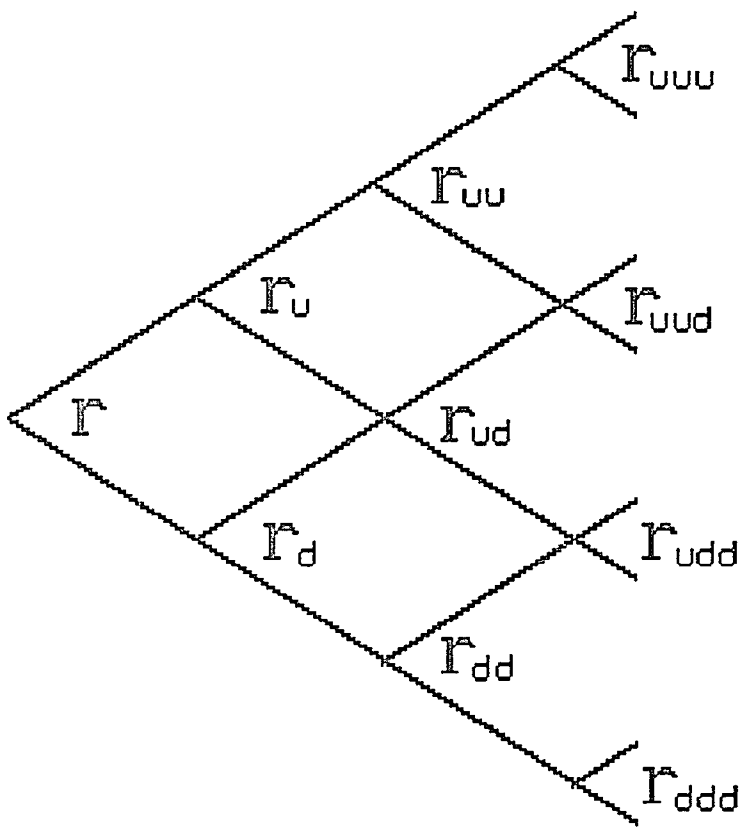

# HO-LEE MODEL

# PRICES




BLACK, DERMAN AND TOY MODEL

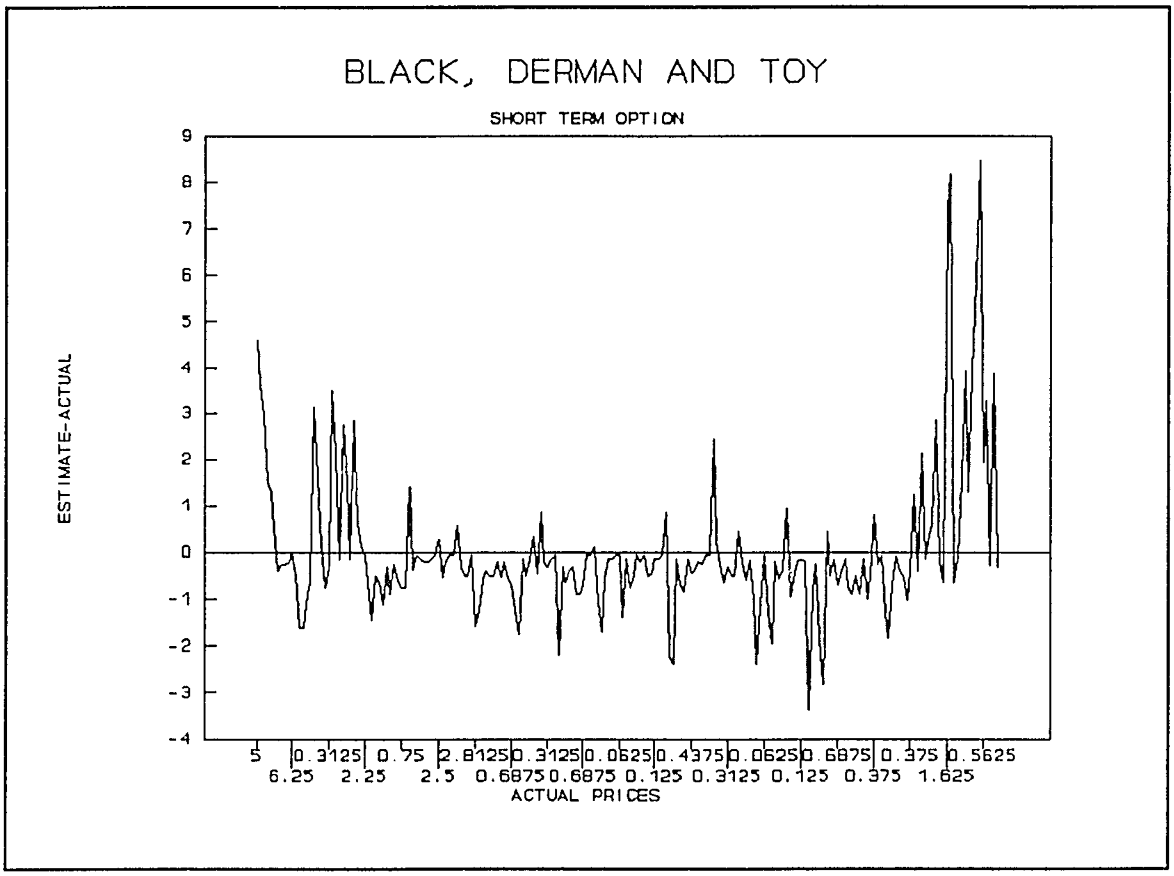
PRICES

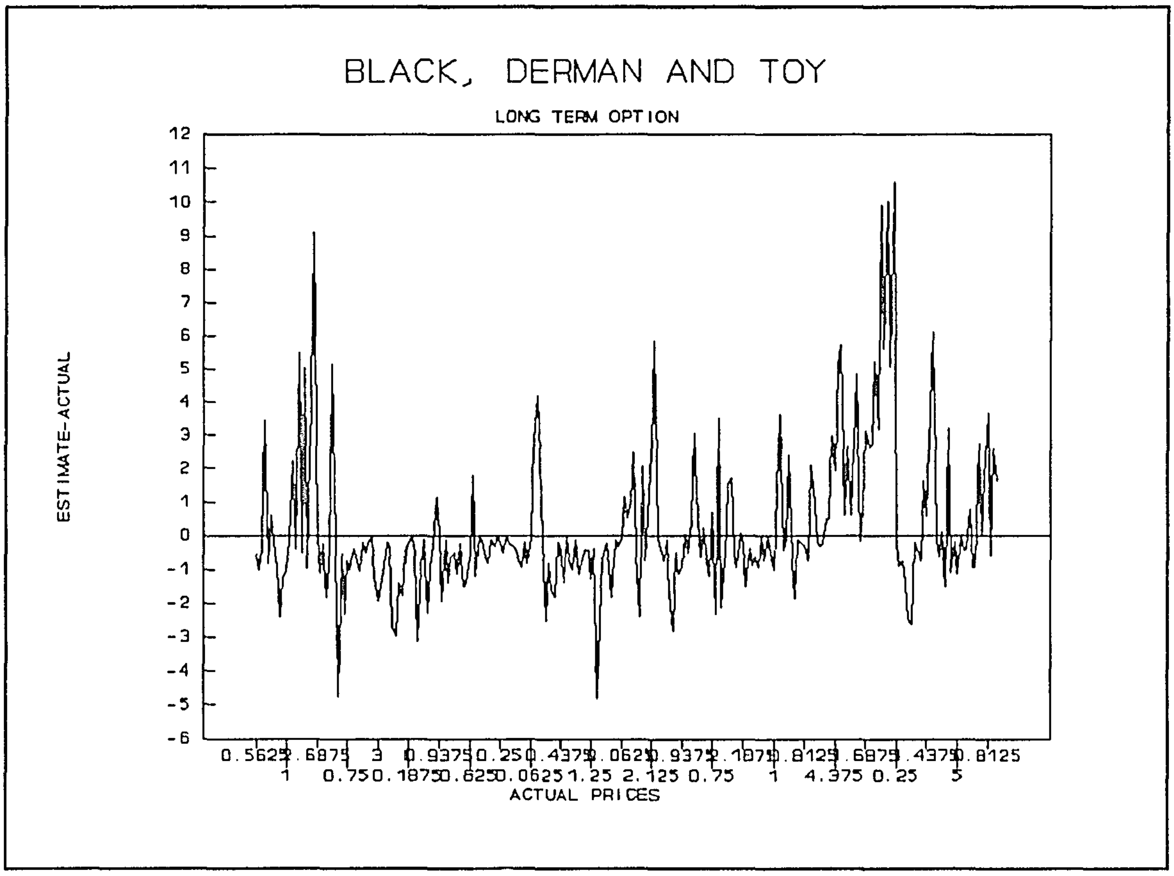


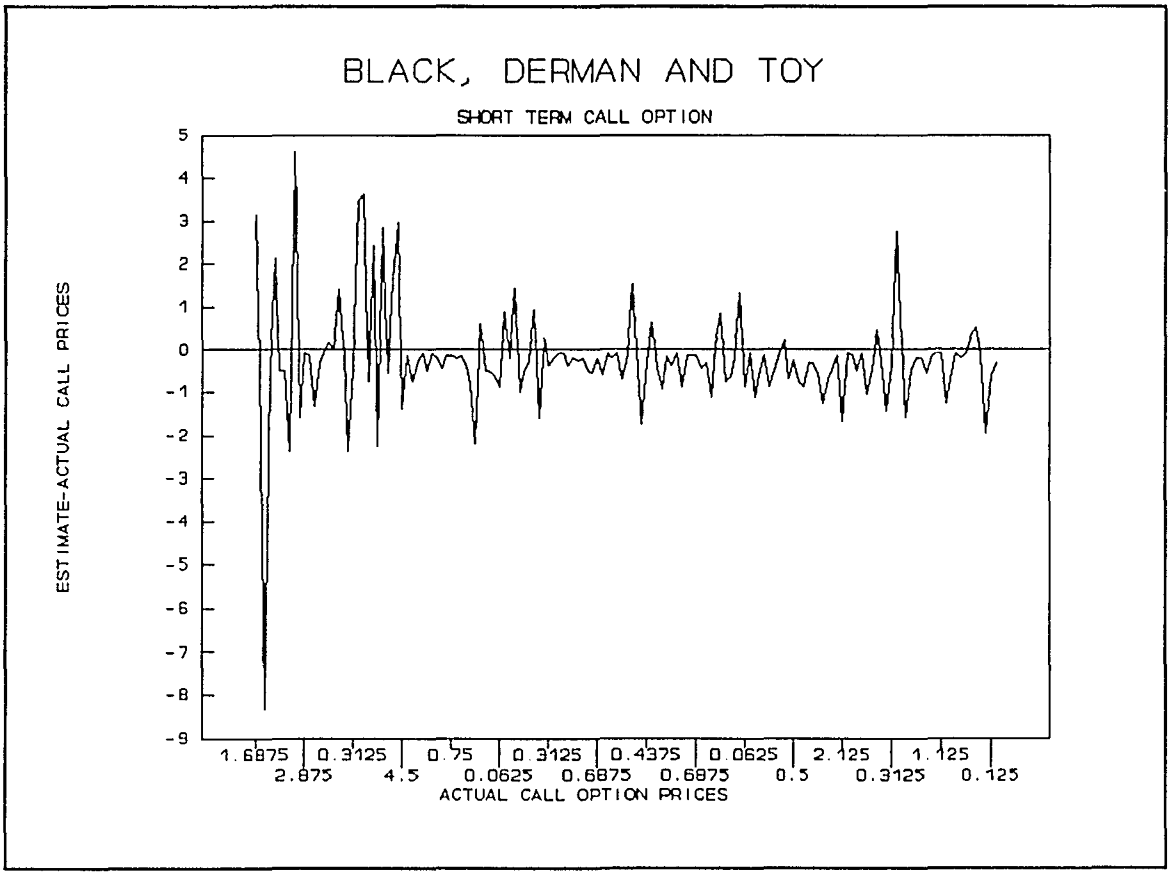

BLACK, DERMAN

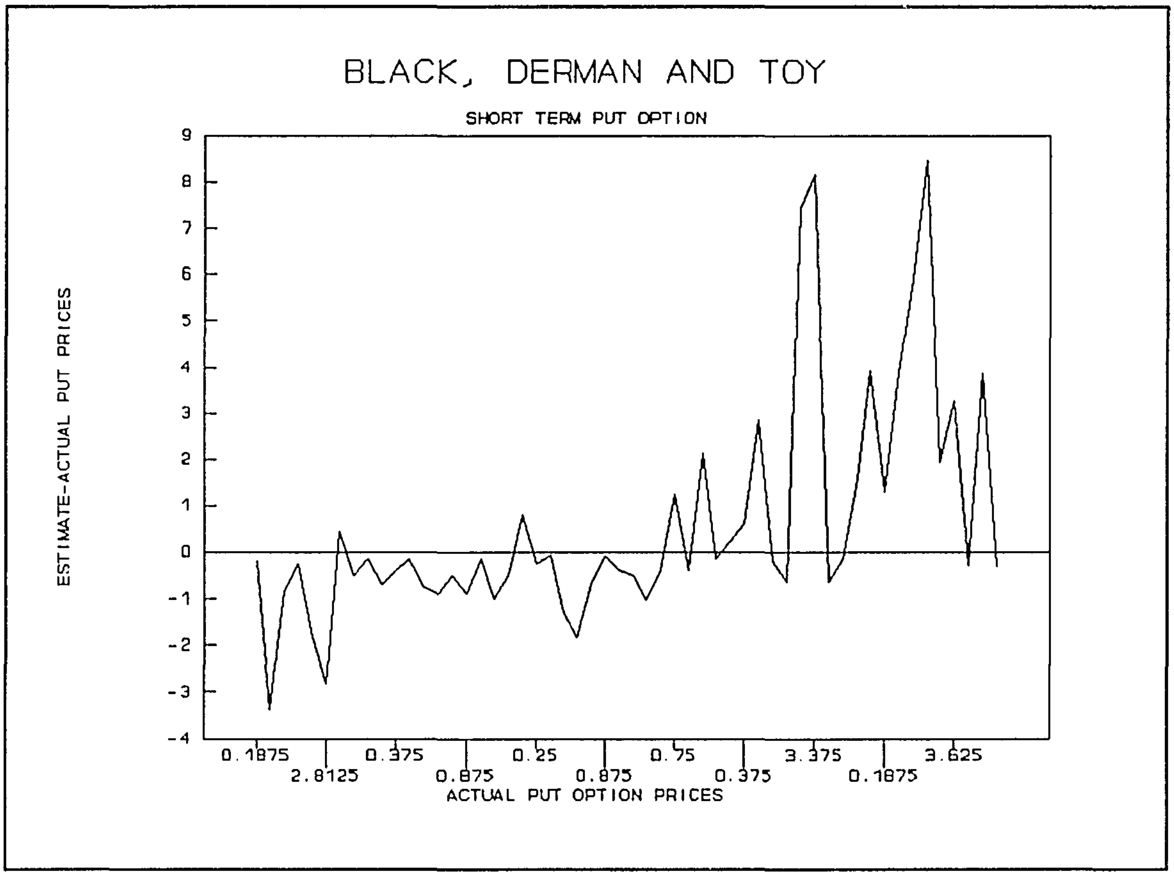
AND TOY MODEL

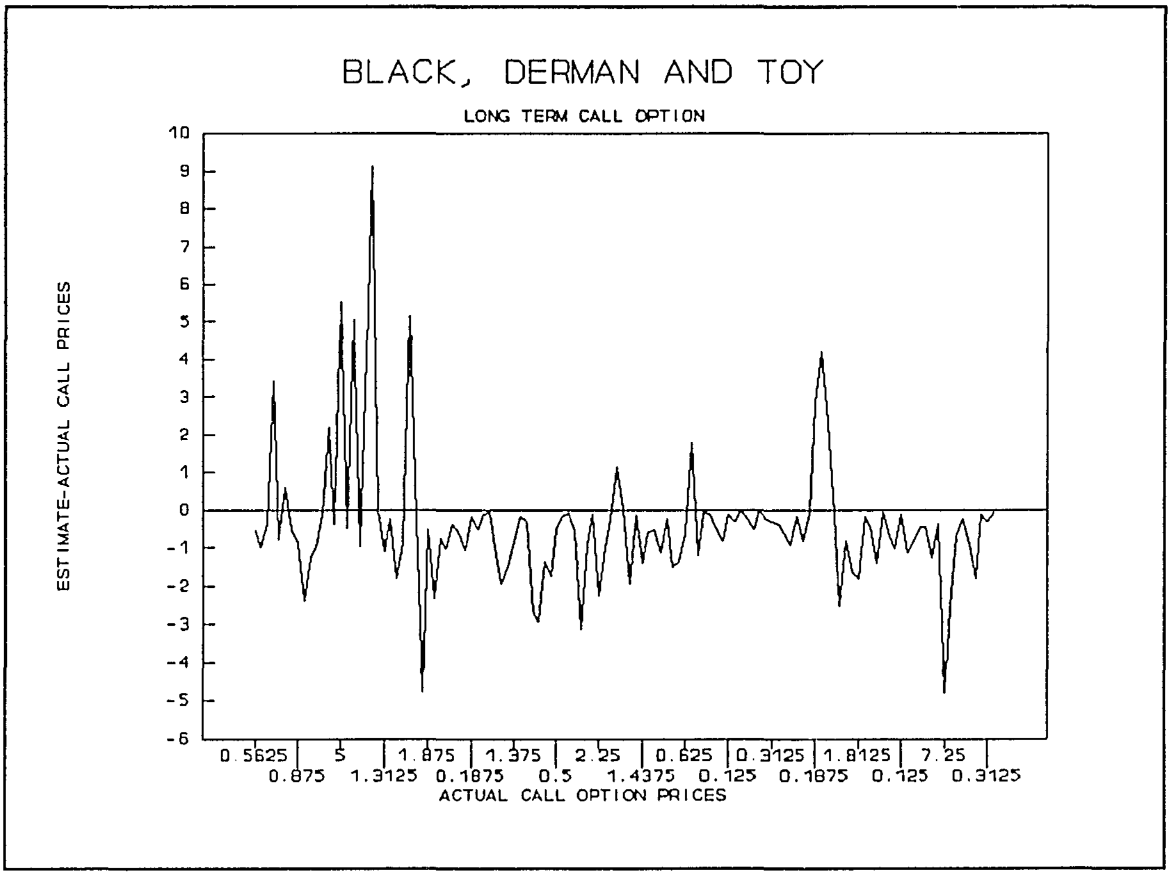
INTEREST RATES

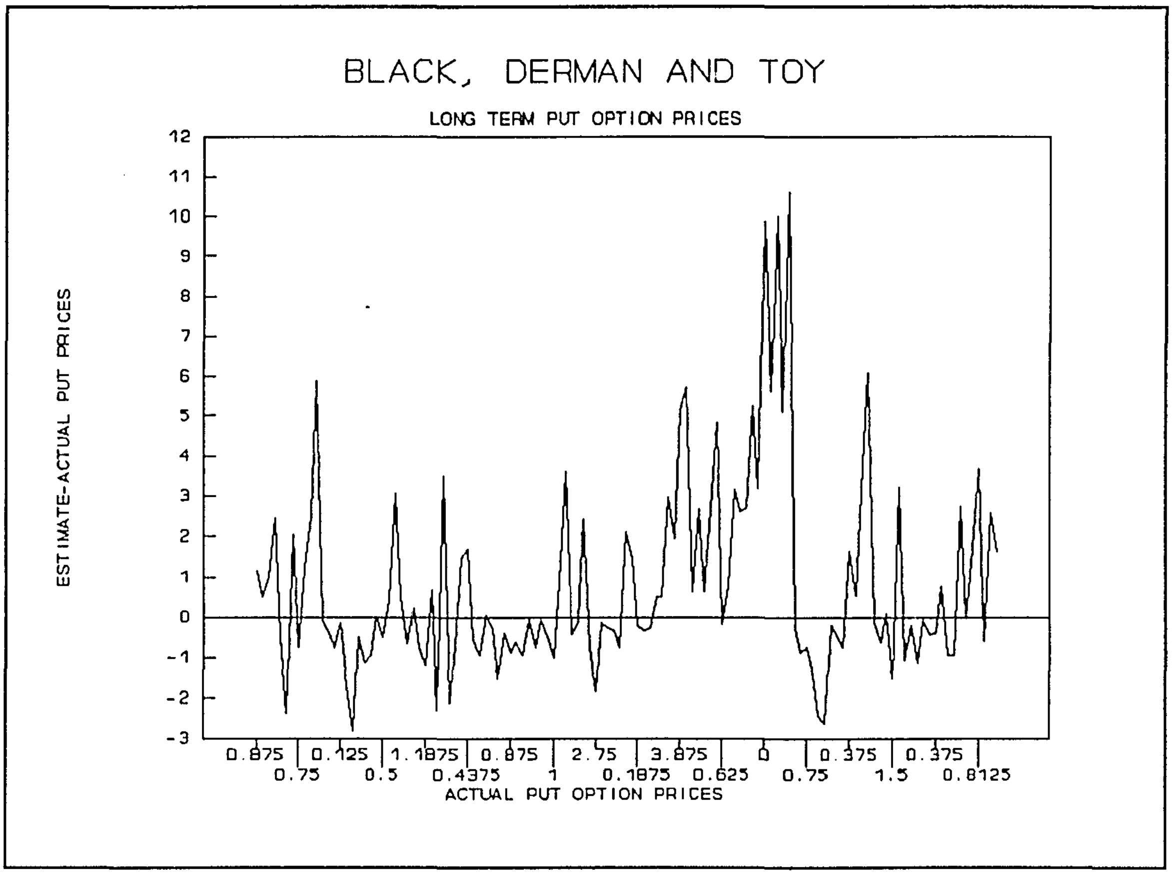

BLACK, DERMAN AND TOY
SHORT TERM OPTION
ESTIMATE-ACTUAL
9
8
7
6
5
4
3
2
1
0
-1
-2
-3
-4
5 0.3125 0.75 2.8125 0.3125 0.0625 0.4375 0.0625 0.6875 0.375 0.5625
6.25 2.25 2.5 0.6875 0.6875 0.125 0.3125 0.125 0.375 1.625
ACTUAL PRICES


BLACK, DERMAN AND TOY
LONG TERM OPTION
ESTIMATE-ACTUAL
ACTUAL PRICES

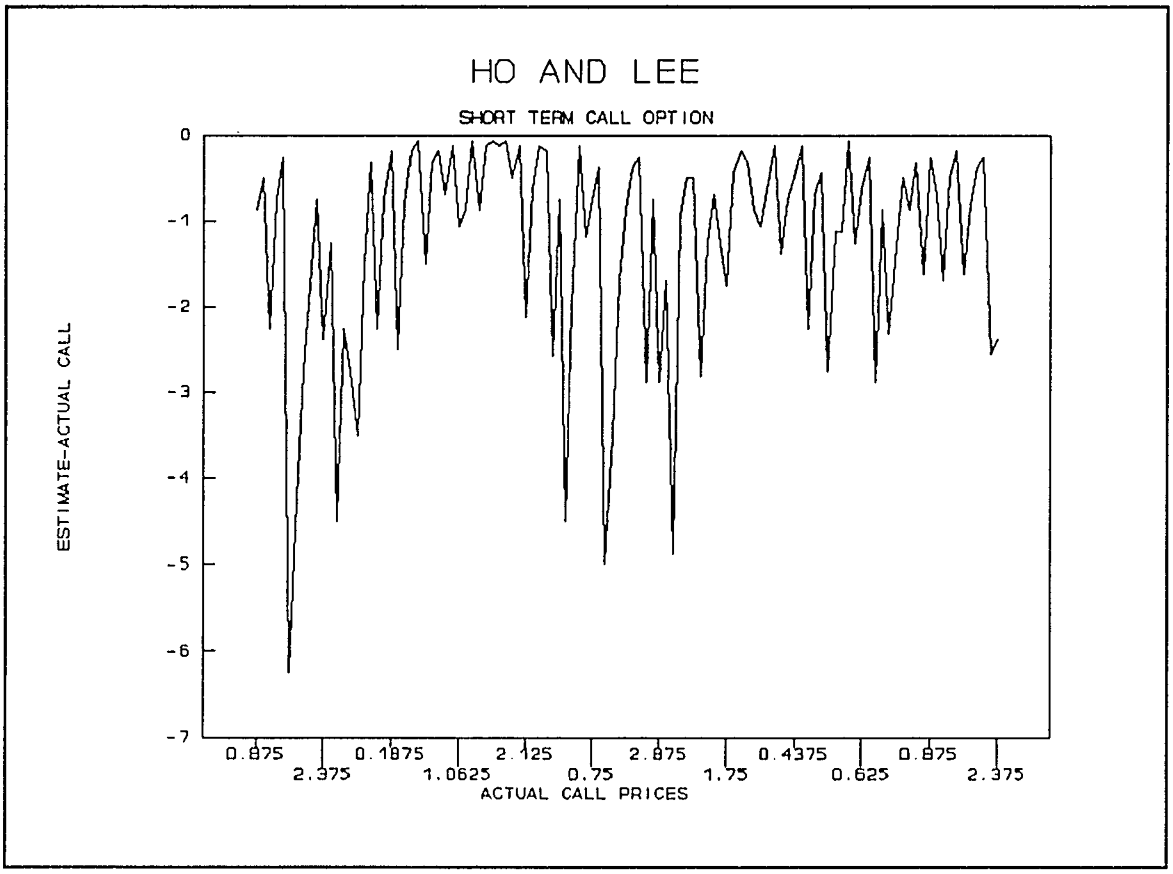

BLACK, DERMAN AND TOY
SHORT TERM CALL OPTION
ESTIMATE-ACTUAL CALL PRICES
5
4
3
2
1
0
-1
-2
-3
-4
-5
-6
-7
-8
-9
1.6875
0.3125
0.75
0.3125
0.4375
0.0625
2.125
1.125
2.875
4.5
0.0625
0.6875
0.6875
0.5
0.3125
0.125
ACTUAL CALL OPTION PRICES

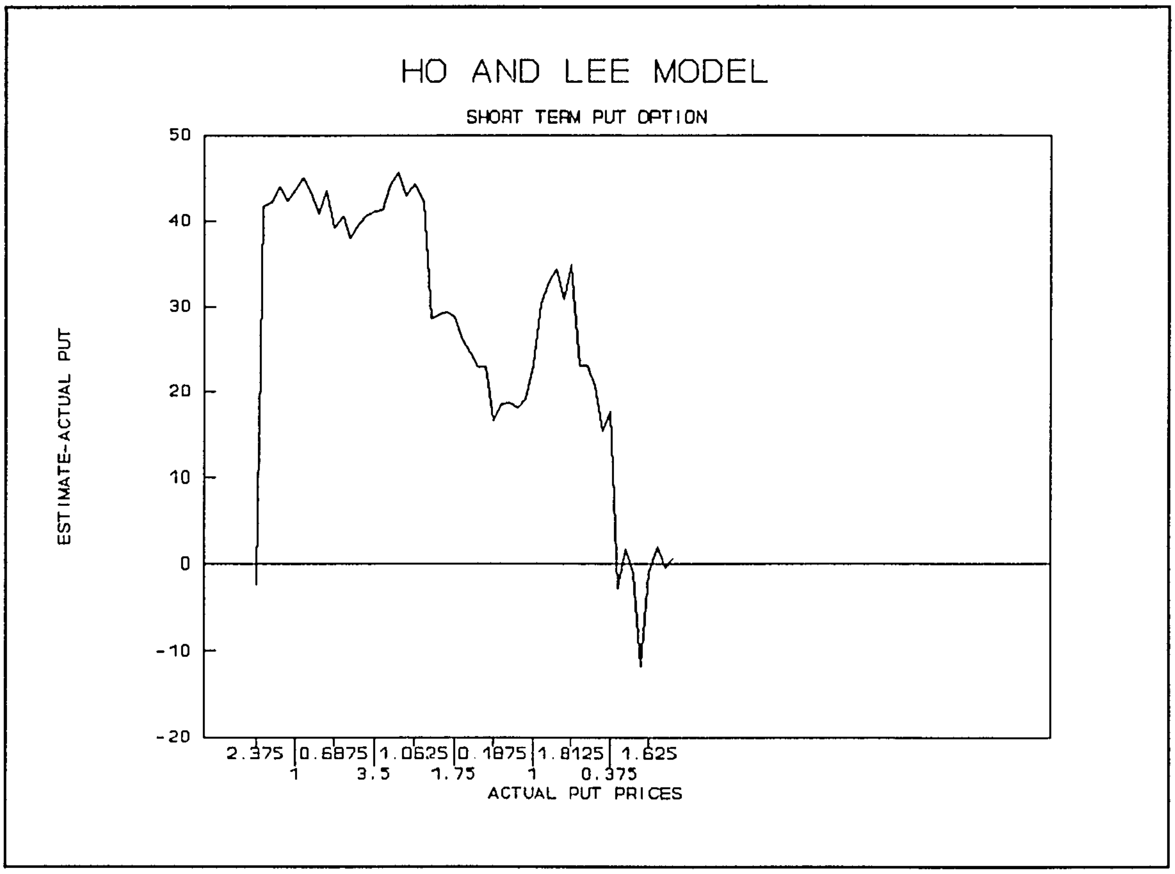

BLACK, DERMAN AND TOY
SHORT TERM PUT OPTION
ESTIMATE-ACTUAL PUT PRICES
9
8
7
6
5
4
3
2
1
0
-1
-2
-3
-4
0.1875
2.8125
0.375
0.875
0.25
0.875
0.75
0.375
3.375
0.1875
3.625
ACTUAL PUT OPTION PRICES

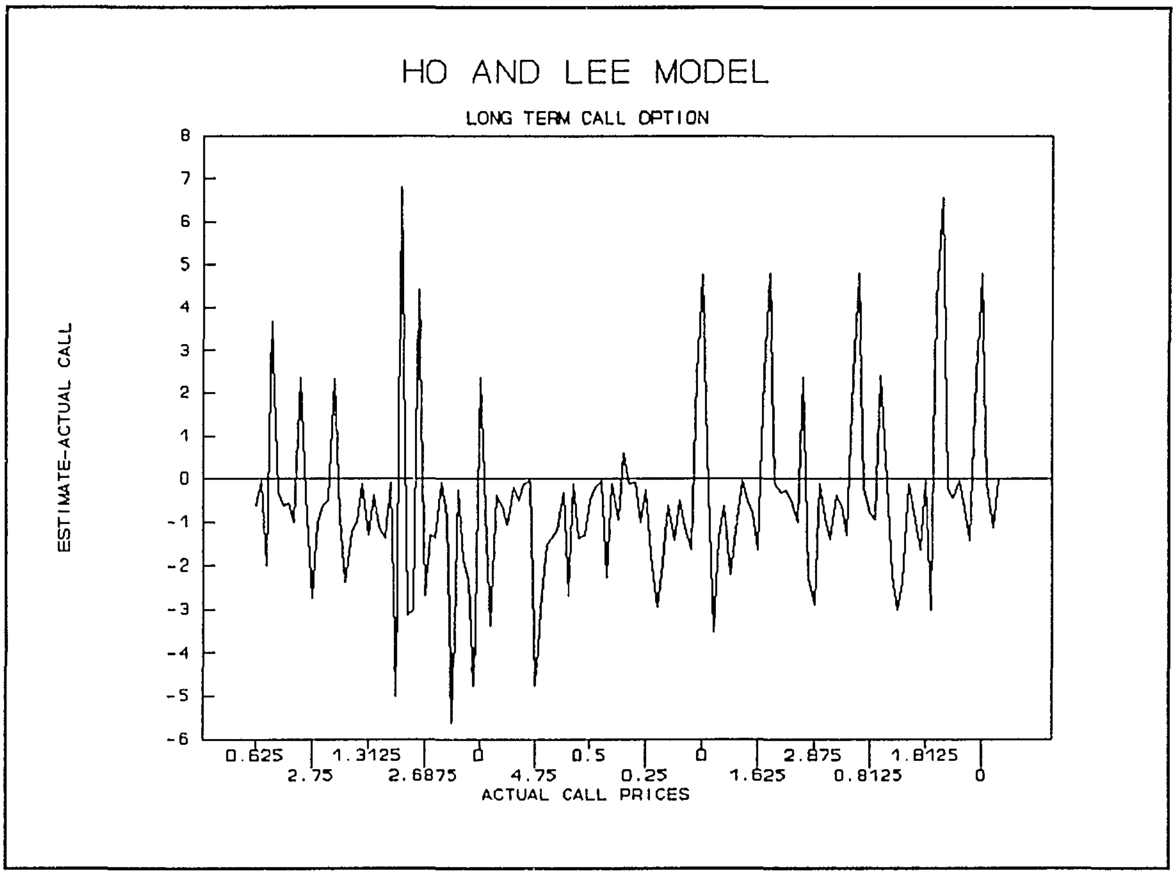

BLACK, DERMAN AND TOY
LONG TERM CALL OPTION
ESTIMATE-ACTUAL CALL PRICES
10
9
8
7
6
5
4
3
2
1
0
-1
-2
-3
-4
-5
-6
0.5625
5
1.875
1.375
2.25
0.625
0.3125
1.8125
7.25
0.875
1.3125
0.1875
0.5
1.4375
0.125
0.1875
0.125
0.3125
ACTUAL CALL OPTION PRICES



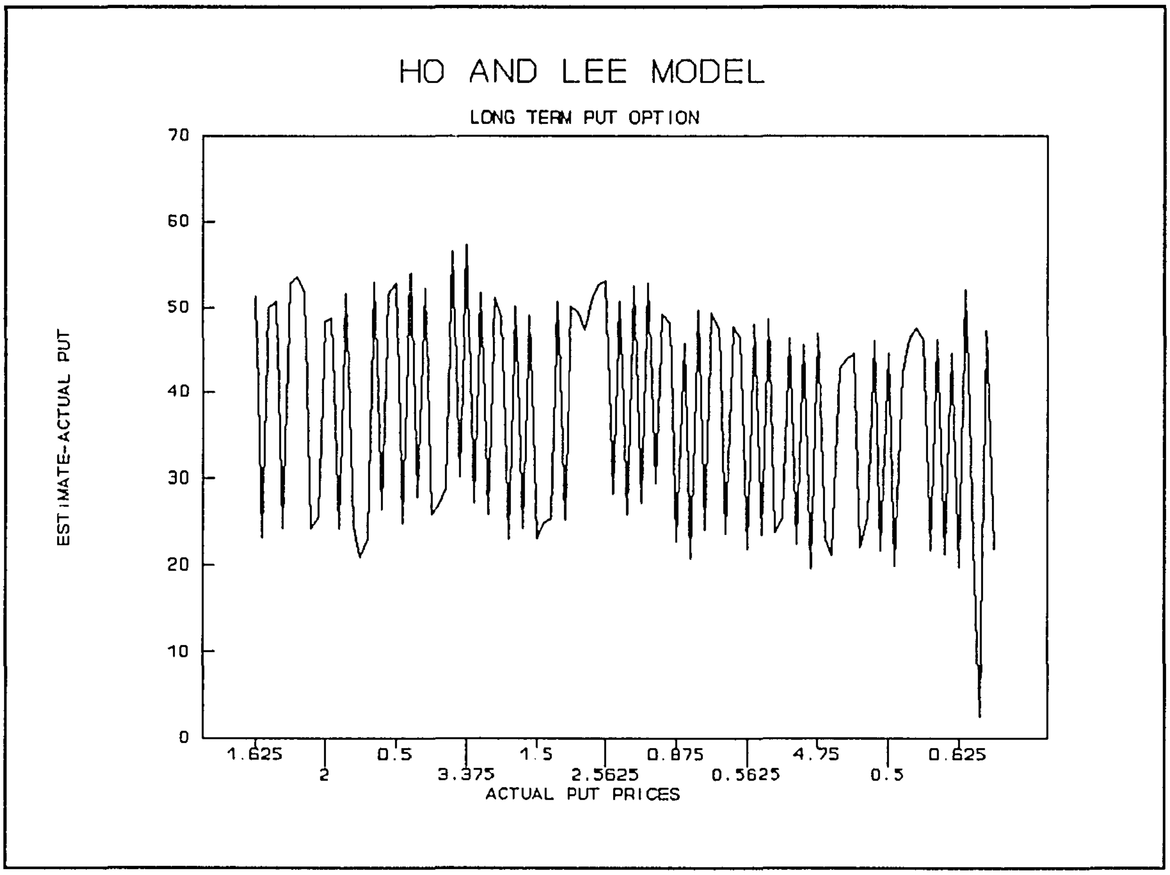
BLACK, DERMAN AND TOY
LONG TERM PUT OPTION PRICES
ESTIMATE-ACTUAL PUT PRICES
12
11
10
9
8
7
6
5
4
3
2
1
0
-1
-2
-3
0.875
0.125
1.1875
0.875
2.75
3.875
0
0.375
0.375
0.75
0.5
0.4375
1
0.1875
0.625
0.75
1.5
0.8125
ACTUAL PUT OPTION PRICES

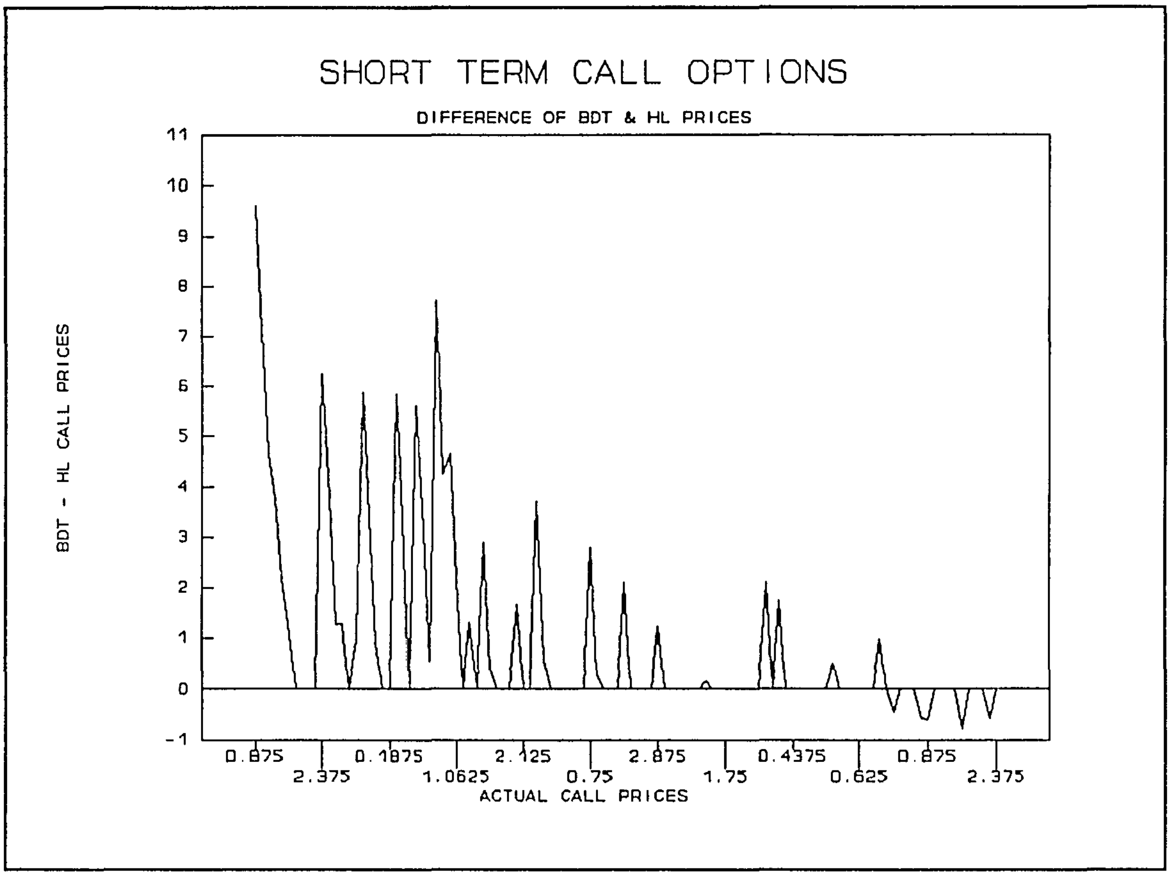

HO AND LEE
SHORT TERM CALL OPTION
ESTIMATE-ACTUAL CALL
0
-1
-2
-3
-4
-5
-6
-7
0.875
2.375
0.1875
1.0625
2.125
0.75
2.875
1.75
0.4375
0.625
0.875
2.375
ACTUAL CALL PRICES

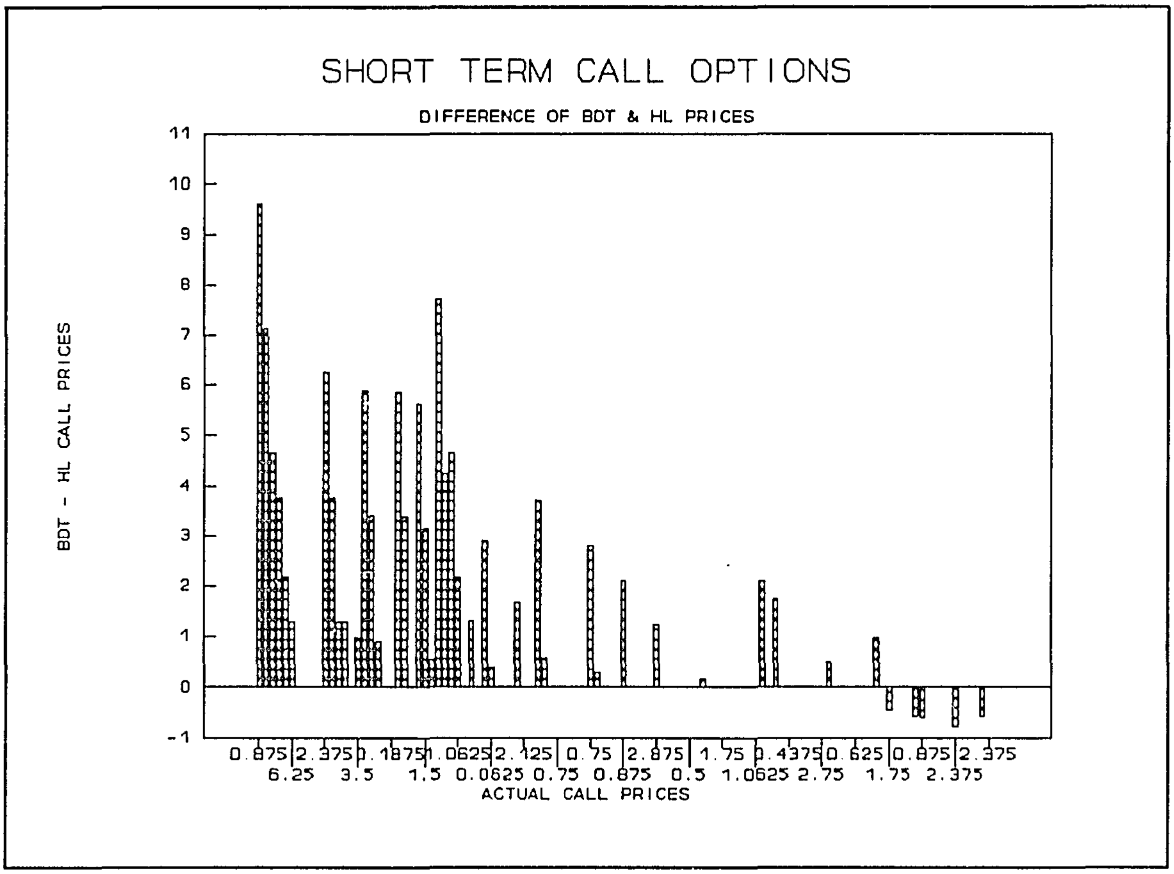

HO AND LEE MODEL
SHORT TERM PUT OPTION
ESTIMATE-ACTUAL PUT
50
40
30
20
10
0
-10
-20
2.375
0.6875
1.0625
0.1875
1.8125
1.625
1
3.5
1.75
1
0.375
ACTUAL PUT PRICES

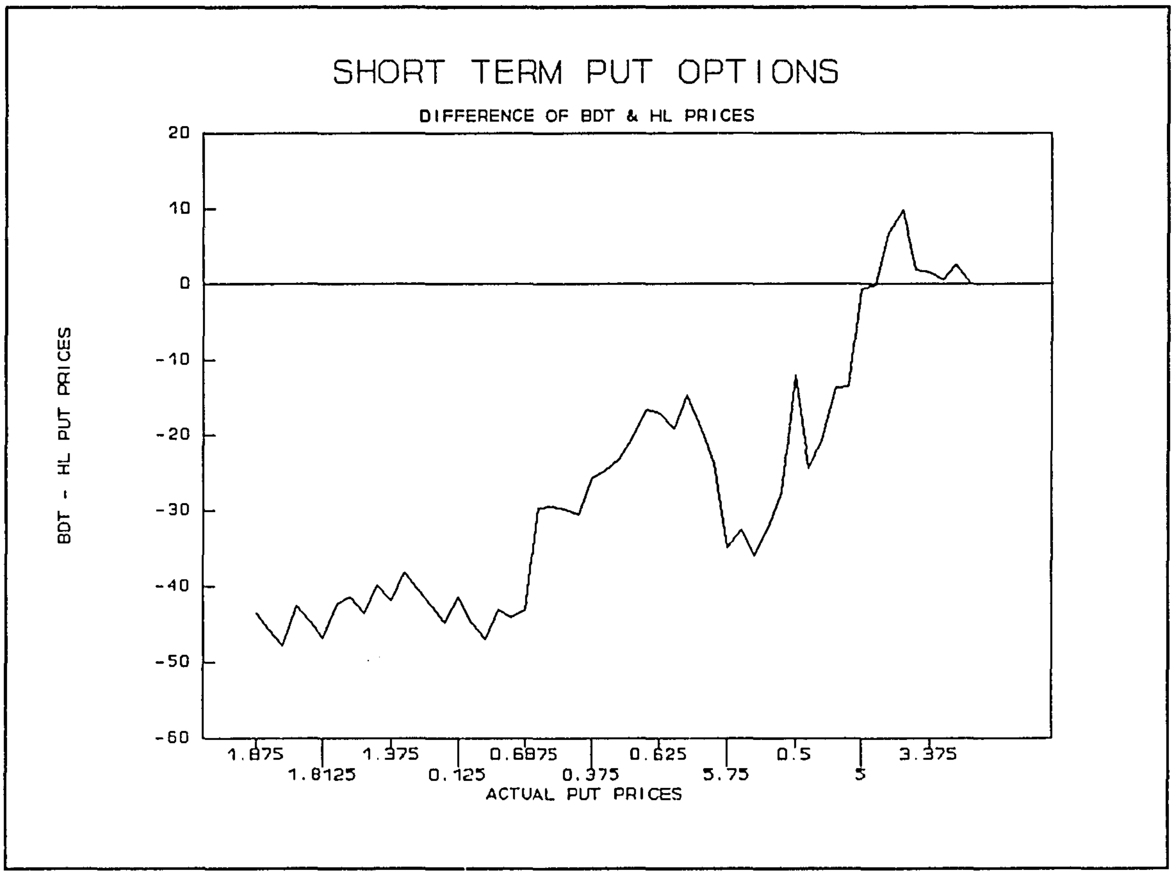


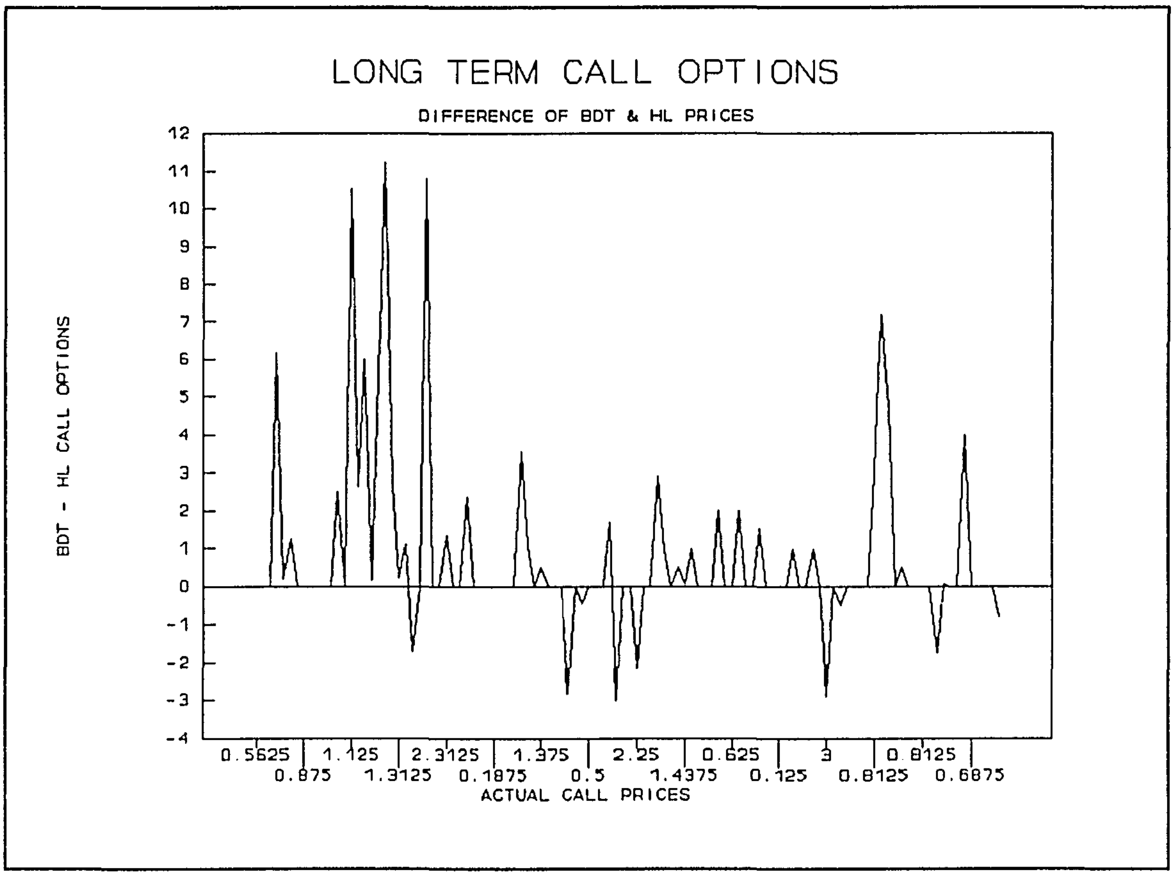
HO AND LEE MODEL
LONG TERM CALL OPTION
ESTIMATE-ACTUAL CALL
8
7
6
5
4
3
2
1
0
-1
-2
-3
-4
-5
-6
0.625
2.75
1.3125
2.6875
0
4.75
0.5
0.25
0
1.625
2.875
0.8125
1.8125
0
ACTUAL CALL PRICES

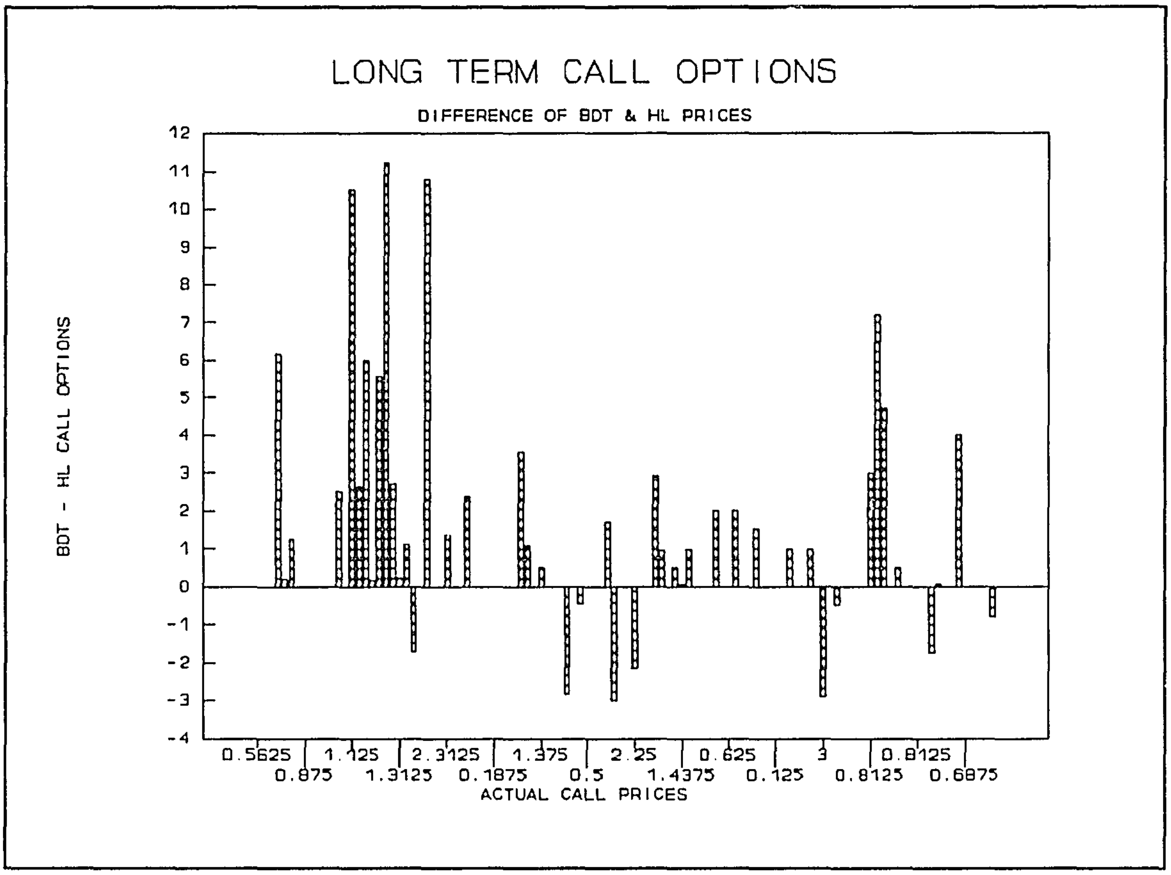

HO AND LEE MODEL
LONG TERM PUT OPTION
ESTIMATE-ACTUAL PUT
70
60
50
40
30
20
10
0
1.625
2
0.5
3.375
1.5
2.5625
0.875
0.5625
4.75
0.5
0.625
ACTUAL PUT PRICES

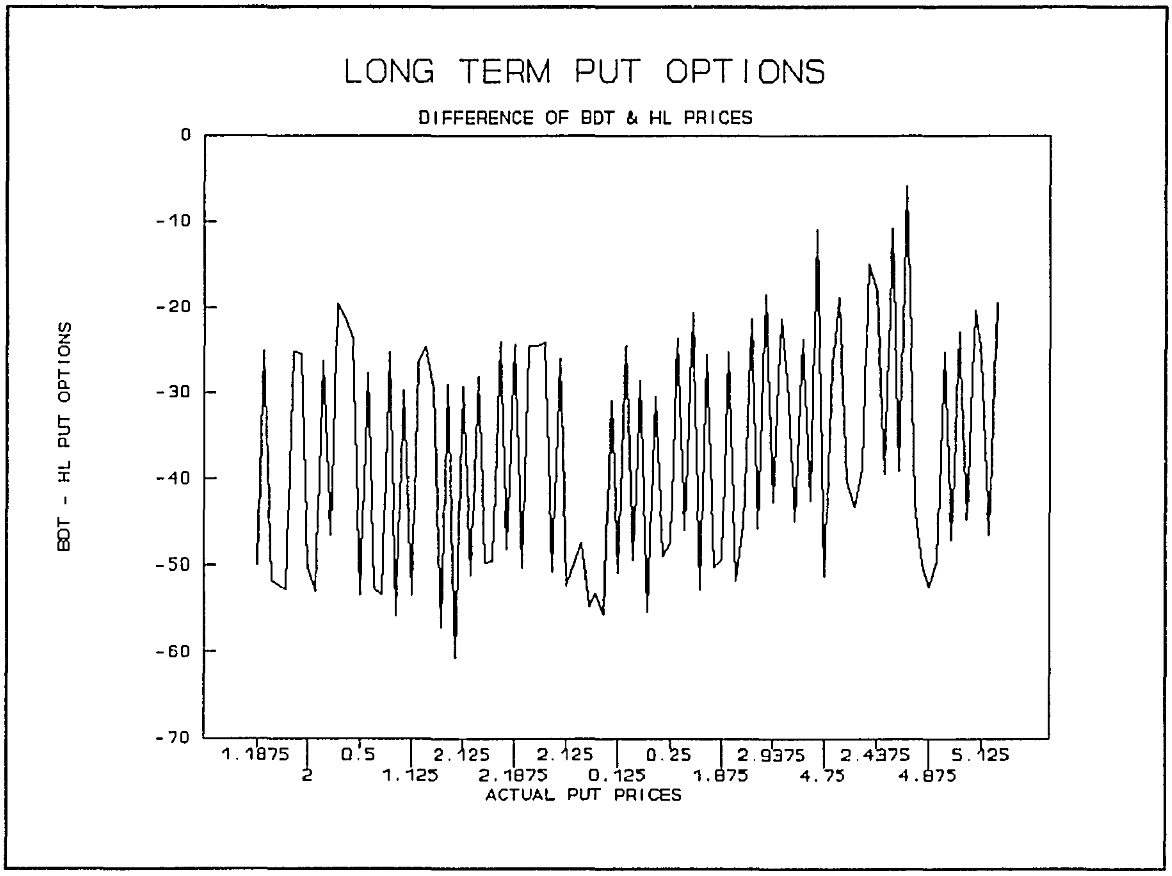

SHORT TERM CALL OPTIONS
DIFFERENCE OF BDT & HL PRICES
BDT - HL CALL PRICES
11
10
9
8
7
6
5
4
3
2
1
0
-1
0.875
2.375
0.1875
1.0625
2.125
0.75
2.875
1.75
0.4375
0.625
0.875
2.375
ACTUAL CALL PRICES


SHORT TERM CALL OPTIONS
DIFFERENCE OF BDT & HL PRICES
BDT - HL CALL PRICES
11
10
9
8
7
6
5
4
3
2
1
0
-1
0.875 2.375 0.1875 1.0625 2.125 0.75 2.875 1.75 0.4375 0.625 0.875 2.375
6.25 3.5 1.5 0.0625 0.75 0.875 0.5 1.0625 2.75 1.75 2.375
ACTUAL CALL PRICES


SHORT TERM PUT OPTIONS
DIFFERENCE OF BDT & HL PRICES
BDT - HL PUT PRICES
20
10
0
-10
-20
-30
-40
-50
-60
1.875
1.8125
1.375
0.125
0.6875
0.375
0.625
5.75
0.5
5
3.375
ACTUAL PUT PRICES


LONG TERM CALL OPTIONS
DIFFERENCE OF BDT & HL PRICES
BDT - HL CALL OPTIONS
12
11
10
9
8
7
6
5
4
3
2
1
0
-1
-2
-3
-4
0.5625
1.125
2.3125
1.375
2.25
0.625
3
0.8125
0.875
1.3125
0.1875
0.5
1.4375
0.125
0.8125
0.6875
ACTUAL CALL PRICES


LONG TERM CALL OPTIONS
DIFFERENCE OF BDT & HL PRICES
BDT - HL CALL OPTIONS
12
11
10
9
8
7
6
5
4
3
2
1
0
-1
-2
-3
-4
0.5625
1.125
2.3125
1.375
2.25
0.625
3
0.8125
0.875
1.3125
0.1875
0.5
1.4375
0.125
0.8125
0.6875
ACTUAL CALL PRICES


LONG TERM PUT OPTIONS
DIFFERENCE OF BDT & HL PRICES
BDT - HL PUT OPTIONS
0
-10
-20
-30
-40
-50
-60
-70
1.1875
2
0.5
1.125
2.125
2.1875
2.125
0.125
0.25
1.875
2.9375
4.75
2.4375
4.875
5.125
ACTUAL PUT PRICES

## APPENDIX-A

Occurrence of infinitely high or negative interest rates in the Ho and Lee Model.

In the Ho-Lee model the future term structure can be expressed in terms of the parameters $\pi$ and $\delta$. At time n and T=1 the expression is

$$P^n_i(1) = \frac{P(n+1)}{P(n)} \frac{\delta^{n-i}}{[\pi - (1-\pi)\delta^n]} \qquad \text{A.1}$$

For an infinite number of down moves

$$\lim_{n\to\infty} P^n_0(1) = \lim_{n\to\infty} \frac{P(n+1)}{P(n)} \frac{\delta^n}{[\pi + (1-\pi)\delta^n} = 0 \qquad \text{A.2}$$

for $0<\delta<1$ and assuming that the initial one-period forward rate is bounded from above for all maturities. For a large number of down moves A.2 implies that short-term rate will tend toward infinity.

Similarly, for an infinite number of up moves,

$$\lim_{n\to\infty} P^n_n(1) = \lim_{n\to\infty} \frac{P(n+1)}{P(n)} \frac{\delta^0}{[\pi + (1-\pi)\delta^n]} = \frac{1}{\pi(1 + f_\infty)} \qquad \text{A.3}$$

where $1 + f_\infty \equiv \lim_{n\to\infty} P(n)/P(n+1)$. If $\pi<1/(1 + f_\infty)$ negative interest rates will occur.

# Appendix B

Proof of Equation (13):

Construct a portfolio of one discount bond maturing T periods later and y discount bonds maturing t periods later. The value of the portfolio is,

$$V = P(T) + yP(t) \tag{B.1}$$

After one time period in an upstate,

$$V_u = [P(T)h_u(T-1) + yP(t)h_u(t-1)]/P(1) \tag{B.2}$$

In a downstate after one-period,

$$V_d = [P(T)h_d(T-1) + yP(t)h_d(t-1)]/P(1) \tag{B.3}$$

By choosing y such that $V_u = V_d$ and using B.2 and B.3 it can be shown that

$$y = P(T)[h_u(T-1) - h_d(T-1)]/P(t)[h_d(t-1) - h_u(t-1)] \tag{B.4}$$

Since arbitrage opportunities cannot exist the return on this portfolio should be the one-period risk-free rate, $1/P(1)$.

Therefore,

$$P(T)h_d(T-1) + yP(t)h_d(t-1) = P(T) + yP(t) \qquad \text{B.5}$$

By substituting B.4 in B.5 it is seen that

$$[1 - h_d(T-1)]/[h_u(T-1) - h_d(T-1)]$$
$$= [1 - h_d(t-1)]/[h_u(t-1) - h_d(t-1)] \qquad \text{B.6}$$

for all T and t>0.

B.6 holds only if there is a constant q such that

$$q = \frac{1 - h_d(T)}{h_u(T) - h_d(T)} \qquad \text{B.7}$$

or by rearranging,

$$\pi.h_u(T) + (1 - \pi).h_d(T) = 1 \quad \text{for } T = 0,1,2 \ldots \qquad \text{B.8}$$

This is the same as equation (13).

## Appendix C

Derivation of Equations (26) and (27):

Equating (20) and (25) we get,

$$\frac{P^t_u(T+2)\,.\,h_u(T+1)\,.\,h_d(T)}{P^t_u(2)\;.\;h_u(1)} = \frac{P^t_u(T+2)\,.\,h_d(T+1)\,.\,h_u(T)}{P^t_u(2)\;.\;h_d(1)} \qquad \text{C.1}$$

or,

$$h_u(T+1)\,.\,h_d(T)\,.\,h_d(1) = h_d(T+1)\,.\,h_u(T)\,.\,h_u(1) \qquad \text{C.2}$$

Therefore,

$$\frac{h_d(T+1)\,.\,h_u(T)}{h_u(T+1)\,.\,h_d(T)} = \frac{h_d(1)}{h_u(1)} = \delta \qquad \text{C.3}$$

So,

$$\frac{h_d(T+1)}{h_u(T+1)} = \frac{h_d(T)\;.\;\delta}{h_u(T)} = \frac{h_d(T-1)\,.\,\delta\,.\,\delta}{h_u(T-1)}$$

$$= \frac{h_d(T-2)\ .\ \delta^3}{h_u(T-2)} = \ldots\ldots = \frac{h_d(0)\ .\ \delta^{T+1}}{h_u(0)} \qquad \text{C.4}$$

Therefore,

$$\frac{h_d(x)}{h_u(x)} = \frac{h_d(0)\ .\ \delta^x}{h_u(0)} = \delta^x \qquad \text{C.5}$$

From (13)

$$\pi . h_u(x) + (1 + \pi) . h_d(x) = 1$$

or,

$$1 - \pi . h_u(x) = (1 - \pi) . h_d(x)$$

Therefore,

$$h_d(x) = \frac{1 - \pi . h_u(x)}{1 - \pi} \qquad \text{C.6}$$

Substituting C.6 into C.5 we get,

$$\frac{1 - \pi . h_u(x)}{(1 - \pi) . h_u(x)} = \delta^x \qquad \text{C.7}$$

or,

$$\frac{1 - \pi.h_u(x)}{(1 - \pi).h_u(x).\delta^x} = 1$$

Therefore,

$$\frac{1}{(1 - \pi).h_u(x).\delta^x} = 1 + \frac{\pi}{(1 - \pi).\delta^x} = \frac{(1 - \pi).\delta^x + \pi}{(1 - \pi).\delta^x}$$

Therefore,

$$h_u(x) = \frac{1}{(1 - \pi).\delta^x + \pi} \qquad \text{C.8}$$

Substituting C.8 into C.6 and simplifying we get,

$$h_d(x) = \frac{\delta^x}{\pi + (1 - \pi).\delta^x} \qquad \text{C.9}$$

C.8 and C.9 are the same as equations (26) and (27).

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