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Image restoration for improved spectral unmixing

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The University of Arizona, 1992
IMAGE RESTORATION FOR IMPROVED
SPECTRAL UNMIXING

by
Hsien-Huang Wu

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In the Graduate College
THE UNIVERSITY OF ARIZONA

1992
As members of the Final Examination Committee, we certify that we have read the dissertation prepared by Hsien-Huang Wu entitled "IMAGE RESTORATION FOR IMPROVED SPECTRAL UNMIXING" and recommend that it be accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy.

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Final approval and acceptance of this dissertation is contingent upon the candidate's submission of the final copy of the dissertation to the Graduate College.

I hereby certify that I have read this dissertation prepared under my direction and recommend that it be accepted as fulfilling the dissertation requirement.

Dissertation Director
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To my wife
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ABSTRACT

Because of the resolution limitations in remote sensing, the radiance recorded by the detector at each pixel is the integrated sum of the spectral radiance of all materials within the detector instantaneous-field-of-view (IFOV). If the detector IFOV covers more than one object class, the radiance detected is not characteristic of any single class but a mixture of all classes. These mixed pixels will have spectral signatures that fall within the convex hull formed by the signatures of all the classes. Traditional classifiers are therefore usually left with many misclassified or unclassified pixels. To remedy this problem, unmixing algorithms which decompose each pixel into a combination of several classes have been successfully applied to estimate the percentage of each class inside one pixel. In this dissertation, unmixing error of the least squares unmixing algorithm that is caused by the intrinsic data variance, system PSF blurring, detection noise, and band-to-band misregistration is analyzed and evaluated.

For high unmixing accuracy, image restoration is proposed to remove the PSF blurring degradation. To objectively assess the restoration performance and expedite the design of our application-oriented restoration scheme, an objective criterion based on the measurement of spectral fidelity in frequency domain is suggested. Based on this criterion, a detailed comparison between the conventional Wiener filter and sampled Wiener filter is conducted, which highlights the significance of sampling aliasing and verifies the results obtained visually by other researchers. Our study shows that contrary to restoration for visual purposes, a partial restoration scheme, instead of full restoration, should be used for a better unmixing performance. Also, the sampling aliasing, which is an artifact and should be suppressed in traditional restoration application, is actually a signal component which needs to be restored for unmixing. Under fair SNR conditions (≥ 30dB), the proposed restoration scheme
can reduce the total unmixing error up to 40% to 70% depending on the scene complexity.
CHAPTER 1
INTRODUCTION

1.1 Background

The radiance recorded by a remote sensor at each pixel is the integrated sum of the spectral radiance of all materials within the instantaneous-field-of-view (IFOV). When the IFOV covers the boundary of two large objects or a scene contains objects smaller than the size of one IFOV, the radiance detected will be a mixture from all objects. With the radiation received being a mixture from two or more object classes, the signals generated are not characteristic of any one class. The use of standard multispectral classification techniques on these mixed pixels will result in improper classification which leads to high mean squared error for an estimate of crop proportion [1]. The spectral effect of viewing more than a single object class is illustrated in Figure 1.1. In Figure 1.1(a), the reflectance spectra are depicted for sand and vegetation as they would appear individually [2]. If the sensor is to view both objects simultaneously, the effective reflectance would be quite different. This is shown in Figure 1.1(b) for the linear combinations of 80% vegetable - 20% sand and 50% vegetable - 50% sand. The mixing problem can also be observed spatially. Figure 1.2 illustrates how the mixtures problem may come about when scanning a scene with an IFOV covers class boundaries. It is clear, in this example, that a large number of elements cover areas overlapping the boundaries between adjoining fields. These are the elements which will generate mixtures and would likely be misclassified.

In order to improve the accuracy of classification, which is based on an assumption of one class per pixel, a mixing model was proposed [3, 4] to deal with the mixed pixels problem. These early efforts proved that the mixing model is effective in improving the accuracy of crop acreage estimation. A linear mixing model has since been used
Figure 1.1: (a) Individual reflectance spectra. (b) Mixed reflectance spectra.
Figure 1.2: Spatially mixed pixels

to analyze the mixture of vegetation and rock in arid lands studies [5], to generate fraction images of shade in forest research [6], to estimate the proportion of a given scene devoted to a particular land cover type [7], and to determine the mineral types and abundance in geology surveys [8]. In [9], a fuzzy supervised classifier, which is based on a linear mixing model, is also shown to have higher accuracy than a standard maximum-likelihood classifier.

Recently, the advent of imaging spectrometers [10], which acquire high spectral resolution (hyperspectral) data, encourage a shift from statistical and empirical interpretation techniques to more deterministic and quantitative ones. Individual minerals may be detected and identified based upon their diagnostic spectral responses [11]; however, the IFOV also causes each observation to be a mixed spectrum and obscures the identification. Unmixing algorithms have been developed to find the spatial abundance distribution patterns of the mixing components [12] and to solve for the fractions of so-called endmembers in each pixel [13].

1.2 Motivation and Objectives of the Investigation

Although several methods have been proposed for unmixing analysis, systematic study of unmixing errors is still lacking. Also, even though accuracy in classification, crop acreage and mineral abundance estimation have been increased using mixing models, no attempt has been made to investigate the possibility of improving the
unmixing results by image processing. These two observations generate motivation of our investigation. Our objectives in this dissertation are to understand the causes of unmixing errors, to estimate the amount of unmixing error, and then to propose approaches for improving unmixing accuracy. Study on the first two objectives provides us the information on the accuracy of unmixing results. And considering the large amount of remote sensing imagery, any improvement on the unmixing accuracy will be a contribution to the remote sensing community.

1.3 Organization

In this dissertation, sources of unmixing error are identified and their impact on the unmixing accuracy are examined. Possible improvement schemes will then be studied, developed, and verified. The contents are organized as follows. Chapter 2 describes several unmixing algorithms currently used in the literature. The best one will be chosen for use in quantitative performance evaluation.

Chapter 3 contains the results on the study and analysis of unmixing error.

In chapter 4, we characterize the spatial degradation of remote sensing instruments and examine the unmixing error which it brings. For the purpose of reducing this error, a restoration approach is then proposed.

Chapter 5 discusses and evaluates the restoration algorithms for digital images. In order to choose a proper restoration algorithm and to understand the mechanism behind the restoration, we conduct a comparative study between two restoration methods based on a restoration model for improving visual quality.

Chapter 6 presents a restoration model from the perspective of improved spectral unmixing and explains the difference between restoration for unmixing and restoration for visualization. Depending on the application of an image to be restored, sampling aliasing—a unavoidable effect in digital imaging—can be an artifact degrading the restoration result or it can be a degraded signal that also needs to be restored. This aspect of viewing aliasing makes our restoration approach differ from that of traditional approach. Drawing a conclusion from chapter 5 and based on the current
restoration model, we suggest a proper restoration approach for unmixing purposes and conduct a performance evaluation for this chosen restoration algorithm.

Chapter 7 contains a series of quantitative studies by using simulated agricultural scenes. First, the feasibility of improving the unmixing accuracy by our proposed restoration approach is verified. Second, we demonstrate that an objective criterion, spectral fidelity, is in accordance with unmixing error criterion in the restoration performance. This simplifies the evaluation and prediction of restoration results for unmixing purposes. Third, the effect of misregistration on the unmixing accuracy is evaluated to illustrate the impact of band-to-band misregistration in the multispectral imagery.

Chapter 8 concludes the dissertation with a summary of the results and possible future extension of the work.
CHAPTER 2
LINEAR SPECTRAL UNMIXING

In this chapter, some of the commonly used unmixing techniques are presented. The first section introduces a linear mixing model which we use throughout this dissertation. The second section discusses the maximum likelihood (ML) method, in which unmixing is based on the maximization of a probability density function. The third section describes three contemporary least square (LS) approaches for unmixing. By viewing from a different perspective, we demonstrate that the LS approach is actually a specific case of the ML method.

2.1 Linear Model for the Signature of Mixtures

To estimate the proportions of objects and materials appearing within the IFOV, one requires a method of inverting the mixing process. Depending on the geometry of this spatial mixing either a linear or nonlinear model may be used to describe the resulting spectral mixing. If a single incident photon is multiply scattered and encounters more than one material before being received as reflected light, then a nonlinear spectral mixing results. If no single photon encounters more than one material, then the mixing is linear. Models have been developed for describing both the linear and nonlinear spectral mixing to solve the spectral unmixing. Because the nonlinear model problem eventually is linearized through a change of variables, we will limit our discussion to only the linear mixing model. In the following, we assume that the sensor has m spectral bands and the maximum number of materials inside one pixel is n.

Given an m channels sensor, suppose that the signature of object class $i$, where $1 \leq i \leq n$, can be represented by an m-dimensional Gaussian distribution with mean
vector $\mu_i$ and covariance matrix $\Sigma_i$. If the proportion of mixing for object class $i$ in a resolution cell is $x_i$ then we define $\mathbf{x} \triangleq (x_1, x_2, \ldots, x_n)^t$ as the proportion vector that describes the mixing fraction of all $n$ classes. Suppose that the signature of this mixture of classes has mean vector $\mu_x$ and covariance matrix $\Sigma_x$, then under a linear mixing model, we have

$$\mu_x = \sum_{i=1}^{n} x_i \mu_i \quad (2.1)$$

If the random variables associated with elements from different object classes are statistically independent, then

$$\Sigma_x = \sum_{i=1}^{n} x_i \Sigma_i \quad (2.2)$$

These two equations establish a model for the signature of combinations of object classes in terms of the signatures of the individual object classes. If the radiation spectra generated by objects within the IFOV are linearly independent, then a satisfactory solution will be obtained. Assuming that the combination satisfies the central limit theorem, we will take the distribution of a mixed pixel with proportion vector $\mathbf{x}$ to be Gaussian in the following discussion.

### 2.2 Maximum Likelihood Spectral Unmixing

In order to truly reflect the dependency of unmixing results on the distribution of input data signatures (spectral vectors), Horwitz [3] formulates the unmixing problem directly using above probability model. Given an $m$-band observation vector $\mathbf{y}$, which we assume to be a mixture of object classes with a proportion vector $\mathbf{x}$, the probability density function is

$$p(\mathbf{y}|\mathbf{x}) = \frac{1}{\sqrt{2\pi |\Sigma_x|}} exp[-\frac{1}{2}(\mathbf{y} - \mu_x)^t \Sigma_x^{-1}(\mathbf{y} - \mu_x)] \quad (2.3)$$

The best estimate of the maximum likelihood method (ML) is defined to be the one that maximizes the probability of obtaining the sample actually observed, i.e. maximizes $p(\mathbf{y}|\mathbf{x})$ or minimizes the negative of the natural log of this Gaussian density function. That is,

$$\text{minimize } F_x = \ln |\Sigma_x| + (\mathbf{y} - \mu_x)^t \Sigma_x^{-1} (\mathbf{y} - \mu_x) \quad (2.4)$$
subject to constraints,

\[
\sum_{i=1}^{n} x_i = 1
\]  

(2.5)

and

\[
x_i \geq 0, \quad 1 \leq i \leq n
\]  

(2.6)

Throughout this dissertation, we will assume that the scene under analysis contains no unknown classes. That is, constraint equation (2.5) is always satisfied. We will examine this equation by studying some specific covariance matrices in the following discussion.

2.2.1 Case 1: \( \Sigma_i = \Sigma \)

One simple case arises when the covariance matrices for all classes are identical. That is,

\[
\Sigma_i = \Sigma
\]

and

\[
\Sigma_x = \sum_{i=1}^{n} x_i \Sigma_i = \Sigma
\]

The minimization of \( F_x \) then becomes:

\[
\text{minimize } F_x = \ln|\Sigma| + (y - \mu_x)^t \Sigma^{-1} (y - \mu_x)
\]

Now, since the covariance matrix \( \Sigma \) is independent of the proportion vector \( x \), the equation can be further simplified to:

\[
\text{minimize } F_x = (y - \mu_x)^t \Sigma^{-1} (y - \mu_x)
\]  

(2.7)

If \( \Sigma \) is positive definite, it can be factored into:

\[
\Sigma = LL^t
\]

then we have:

\[
(y - \mu_x)^t \Sigma^{-1} (y - \mu_x)
\]

\[
= (y - \mu_x)^t (LL^t)^{-1} (y - \mu_x)
\]  

(2.8)

\[
= [L^{-1}(y - \mu_x)]^t [L^{-1}(y - \mu_x)]
\]
If we set:

\[
z = L^{-1}y
\]

\[
\nu_x = L^{-1}\mu_x
\]

then equation (2.8) becomes:

\[
\text{minimize } ||z - \nu_x||^2
\]  

(2.9)

Because of the constraints \( \sum_{i=1}^{n} x_i = 1 \) and \( x_i \geq 0, 1 \leq i \leq n \), the vector \( \nu_x = \sum_{i=1}^{n} x_i(L^{-1}\mu_i) \) is a convex combination of \( L^{-1}\mu_i, 1 \leq i \leq n \) [14]. The set of all vectors \( \nu_x \) formed is the convex set referred as the convex hull of the set of \( (L^{-1}\mu_1, \ldots, L^{-1}\mu_n) \).

This minimization problem can then be interpreted geometrically as: find a proportion vector \( x \) such that the convex combination is closest to the transformed observation vector \( z \). In Figure 2.1, which shows the geometric relation for three categories, if \( z \) is inside the convex hull formed by the triangle \( \nu_1\nu_2\nu_3 \) (e.g. \( z_0 \)), then we can always find a unique solution \( x \) such that

\[
z = \sum_{i=1}^{3} x_i\nu_i
\]

If \( z \) is not inside the convex hull of \( \nu_1\nu_2\nu_3 \) (e.g. \( z_1 \)), then an optimum point, which is closest to \( z \) and inside the convex hull, can be found by projecting vector \( z \) onto the convex hull. This is shown as point \( p \) in the figure. The point \( p \) can then be decomposed to find the proportion vector \( x \).
2.2.2 Case 2: $\Sigma_i = \sigma_i^2 \Sigma$

If the $\Sigma_i$ are all scalar multiples of some common matrix $\Sigma$, or $\Sigma_i = \sigma_i^2 \Sigma$, then

$$\Sigma_x = \sum_{i=1}^{n} x_i \sigma_i^2 \Sigma = \sigma_x^2 \Sigma$$  \hspace{1cm} (2.10)

where

$$\sigma_x^2 = \sum_{i=1}^{n} x_i \sigma_i^2$$

and the maximum likelihood estimate of $\mathbf{x}$ is to minimize

$$H_x = \ln \sigma_x^2 + \frac{1}{\sigma_x^2} \|z - \nu_x\|^2$$  \hspace{1cm} (2.11)

Because the function $H_x$ is quasi-convex, a convergence procedure based on the Frank and Wolfe method can be used to approximate the estimate. Details on this method can be found in [3].

2.2.3 Case 3: $\Sigma_i$ arbitrary

In the general case, the covariance matrices are different for each class, and the $F_x$ must be minimized in its original form shown in equation (2.4). There is no better way to solve this problem than a brute force approach which we show below. $F_x$ is minimized over the set of points $\mathbf{x} = (x_1, \ldots, x_n)^t$ such that each component $x_i$ is a positive integral multiple of $1/r$ for some fixed positive integer $r$ and $\sum_{i=1}^{n} x_i = 1$. An $\mathbf{x}$ which minimizes $F_x$ over this set is taken as an estimate. The number of points in the set is $C_{n-1}^{n+r+1}$ and is very large even for moderate values of $n$ and $r$. Because this approach is impractical, the problem is usually simplified by assuming specific $\Sigma_i$.

2.3 Least Square Spectral Unmixing

The current trend in spectral unmixing is to neglect the intra-class variance and assume constant reflectance of class components in each mixed pixel [6, 15, 16]. These representative constant reflectances of all classes are called "library endmembers". The library of mixing endmembers may be either spectra chosen from the data themselves or from laboratory and/or field spectra. If laboratory or field spectra data
are used, the reflectance values should be calibrated to take atmospheric effects into account before the library is used. Given a \( m \)-band multispectral or hyperspectral image and a library of \( n \) endmembers, the spectrum of each observed pixel \( y \) is assumed to be a linear combination of these endmembers:

\[
\begin{align*}
    y_1 &= x_1 r_{11} + x_2 r_{12} + \cdots + x_n r_{1n} + e_1 \\
    y_2 &= x_1 r_{21} + x_2 r_{22} + \cdots + x_n r_{2n} + e_2 \\
    & \quad \vdots \\
    y_m &= x_1 r_{m1} + x_2 r_{m2} + \cdots + x_n r_{mn} + e_m
\end{align*}
\]  

(2.12)

Put in matrix-vector format, we get

\[
y = Rx + e
\]

(2.13)

where \( R = (r_1, r_2, \ldots, r_n) \) and \( r_i = (r_{1i}, r_{2i}, \ldots, r_{mi})^t \) is the reflectance vector (endmember) of class \( i \) from the library spectra, where \( 1 \leq i \leq n \); \( e = (e_1, e_2, \ldots, e_m) \) is the error vector of unmixing.

If \( R \) is a nonsingular square matrix, then \( x \) can be estimated by multiplying the observed vector with the inversion of the reflectance matrix or \( \hat{x} = R^{-1}y \). In real applications, the number of spectral bands is usually larger than the number of classes, or \( m > n \). The system of equations (2.12) then becomes overdetermined and can be solved by a least square (LS) method. To formulate spectral unmixing as an LS problem, rewrite (2.13) as

\[
e = y - Rx
\]

and minimize

\[
||e||^2 = \sum_{i=1}^{n} e_i^2 = ||y - \sum_{i=1}^{n} x_i r_i||^2
\]

(2.14)

Although the ML is a better and natural method in solving the unmixing problem, it also demands much computation time. On the contrary, the LS methods achieve a suboptimal solution by simplifying the modeling and usually also reduce the computation burden. In the next subsection, we will take a look at the relationship between these two approaches.
2.3.1 LS As a Specific Case of the ML

Before we start the discussion of solving the unmixing problem by the LS method, we would like to examine equation (2.14) and compare it with the ML method. Starting with Case 1 of the ML method, we make a further simplification and set the covariance matrix $\Sigma = \sigma^2 I$, i.e. each category has a diagonal hyperspherical covariance matrix. Therefore,

$$\Sigma = \sum_{i=1}^{n} x_i \sigma^2 I = \sigma^2 I$$

and

$$|\Sigma| = \sigma^{2n}, \quad \Sigma^{-1} = \frac{1}{\sigma^2} I$$

The $F_x$ in equation (2.7) becomes:

$$F_x = 2n \ln \sigma + \frac{1}{\sigma^2} \|y - \mu_x\|^2$$

Because $n$ and $\sigma$ are constants, the ML estimate is equivalent to:

$$\text{minimize } F_x = \|y - \mu_x\|^2 = \|y - \sum_{i=1}^{n} x_i \mu_i\|^2$$

(2.15)

Comparing (2.15) with equation (2.14), we notice that if $\mu_i = r_i$, that is, choosing the library endmembers as the mean vector of each class, then the LS method is equivalent to the ML method. It is interesting to find out that even though we ignore the intra-class variance in the LS formulation, it actually has an implicit assumption that all classes have same the covariance matrices in which the variances of the bands are equal.

In the following, we review some of the techniques that have been used for solving the unmixing problem by the LS method and discuss the characteristics of these approaches.

2.3.2 Projection onto the Column Space of $R$

The error $\|e\| = \|y - Rx\|$ can be viewed as the distance from $y$ to the point $Rx$ in the column space of endmember library matrix $R$. Therefore searching for the
least squares solution $\hat{x}$, which minimizes $\|e\|$, is the same as locating the point $Rx$ in the column space $R$ that is closest to $y$. From the projection theorem, the error vector $e$ must be orthogonal to every column of $R$ that is:

$$R'(y - R\hat{x}) = 0$$

or

$$R'R\hat{x} = R'y$$

and we get

$$\hat{x} = (R'R)^{-1}R'y$$

(2.16)

This is also called generalized inverse matrix method (GIMM) by Arai [16].

2.3.3 Singular Value Decomposition (SVD)

Applying the GIMM requires the columns of $R$ (the spectra of endmembers) to be independent; the rank has to be $n$. Otherwise $R'R$ is not invertible and $\hat{x}$ cannot be uniquely determined. If the rank of $R$ is less than $n$, then SVD can be used to obtain a unique solution by choosing the one that has the minimum length [17]. Since we assume the library endmembers are independent, the main motivation of using SVD is that it allows quantitative examination of the separability of these endmembers [12]. The capability of separability evaluation also has been used to predict the unmixing performance under different system parameters setting and to guide the engineering design of the High Resolution Imaging Spectrometer (HIRIS) [15].

Any matrix can be decomposed into the product of two column orthogonal matrices and a diagonal matrix, or

$$R = UWV^t$$

The columns of $U$ ($m$ by $m$) are eigenvectors of $RR^t$ and the columns of $V$ ($n$ by $n$) are eigenvectors of $R'R$. The $k$ singular values on the diagonal of $W$ ($m$ by $n$) are the square roots of the nonzero eigenvalues of both $RR^t$ and $R'R$. Because $U$ and $V$ are orthogonal matrices and $W$ is diagonal, their inverse can be easily obtained.
For an overdetermined set of linear equations

\[ Rx = y \]

the SVD finds a LS solution vector by multiplying the observation with the inverse of \( R \). or

\[ \hat{x} = R^+ y = V[\text{diag}(1/w_j)]U'y \]  

(2.17)

If \( w_j = 0 \), we replace \( 1/w_j \) by zero and this will be the solution vector of smallest length. We call \( R^+ \) a pseudoinverse matrix and \( \hat{x} \) a pseudoinverse solution.

### 2.3.4 Constrained Least Square (CLS) method

Although it is reasonable to have a LS estimation for the proportion vector \( x \), it is not proper to solve \( \sum_{i=1}^{n} x_i = 1 \) in a LS sense. Both of GIMM and SVD did not try to satisfy the constraints \( \sum_{i=1}^{n} x_i = 1 \) and \( x_i \geq 0, \ 1 \leq i \leq n \) exactly. This neglect of constraints will produce proportion values of certain classes to be less than 0 or more than 1! Quadratic programming (QP) is a feasible approach in searching for the solution by minimizing a quadratic objective function under linear constraints. Here we introduce a similar concept of QP used by Shimabukuro [6] in computing the fraction images of forest. In general, this algorithm has the best unmixing results, therefore, we will implement it in a later chapter on the validation of our improvement scheme. Details of this algorithm are given below.

Let us assume the class number \( n \) equals 3 and apply the constraint: \( x_3 = 1 - x_1 - x_2 \), then the error \( e_i \) becomes

\[ e_i = y_i - (x_1 r_{i1} + x_2 r_{i2} + x_3 r_{i3}) \]

\[ = (y_i - r_{i3}) - (r_{i1} - r_{i3})x_1 - (r_{i2} - r_{i3})x_2 \]

and the function to be minimized is:

\[ F_{CLS} = \sum_{i=1}^{3} e_i^2 \]

\[ = \sum_{i=1}^{3}(r_{i1} - r_{i3})^2x_1^2 + \sum_{i=1}^{3}(r_{i2} - r_{i3})^2x_2^2 + 2\sum_{i=1}^{3}(r_{i1} - r_{i3})(r_{i2} - r_{i3})x_1x_2 \]
To solve this problem, a solution vector which minimize \( F_{CLS} \) is searched for in the \( 0 \leq x_1 \leq 1 \) and \( 0 \leq x_2 \leq 1 \) region. Shimabukuro took an indirect approach by solving this constrained minimization problem as an unconstrained one. That is, minimize the \( F \) by setting the partial derivative of \( F_{CLS} \) to zero, or

\[
\frac{\partial F_{CLS}}{\partial x_1} = 0 \\
\frac{\partial F_{CLS}}{\partial x_2} = 0
\]

to get

\[
x_1 = \frac{(A_3A_5 - 2A_2A_4)/(4A_1A_2 - A_3^2)}{4A_1A_2 - A_3^2} \\
x_2 = \frac{(A_3A_4 - 2A_1A_5)/(4A_1A_2 - A_3^2)}{4A_1A_2 - A_3^2} \tag{2.19}
\]

and consider the solution of \( x_1, x_2 \) among five possible outcomes, as we show in Figure 2.2.

1. \((x_1, x_2)\) lies within region I. \( \hat{x} = (x_1, x_2, 1 - x_1 - x_2)^t \).

2. \((x_1, x_2)\) lies within region II. Set \( x_3 = 0, x_2 = 1 - x_1 \) then

\[ F_{CLS} = (A_1 + A_2 - A_3)x_1^2 + (A_3 + A_4 - A_5 - 2A_2)x_1 + (A_2 + A_5 + A_6) \]

Solving \( \partial F_{CLS}/\partial x_1 = 0 \) yields:

\[ x_1 = -(A_3 + A_4 - A_5 - 2A_2)/(A_1 + A_2 - A_3) \]

If \( x_1 > 1 \), then \( \hat{x} = (1, 0, 0)^t \), or if \( x_1 < 0 \), then \( \hat{x} = (0, 1, 0)^t \), otherwise, \( \hat{x} = (x_1, 1 - x_1, 0)^t \).

3. \((x_1, x_2)\) lies within region III. Set \( x_1 = 0, x_3 = 1 - x_2 \), reformulate the minimization to solve \( x_2 \):

\[ x_2 = -A_5/2A_2 \]

If \( x_2 > 1 \), then \( \hat{x} = (0, 1, 0)^t \), or if \( x_2 < 0 \), then \( \hat{x} = (0, 0, 1)^t \), otherwise, \( \hat{x} = (0, x_2, 1 - x_2)^t \).
Figure 2.2: The possible regions of CLS outcomes for 3 classes

(4) \((x_1, x_2)\) lies within region IV. \(\hat{x} = (0, 0, 1)^t\).

(5) \((x_1, x_2)\) lies within region V. Set \(x_2 = 0, x_3 = 1 - x_1\), reformulate the minimization to solve \(x_1\):

\[
x_1 = -A_4/2A_1
\]

If \(x_1 > 1\), then \(\hat{x} = (1, 0, 0)^t\), or if \(x_1 < 0\), then \(\hat{x} = (0, 1, 0)^t\), otherwise, \(\hat{x} = (x_1, 0, 1 - x_1)^t\).

Apparently, the basic strategy in this algorithm is to set the value of a class with negative proportion to be zero and then keep on searching for the optimal solution in the subspace. The same procedures might be applicable to more than 3 classes in the mixture, but the process of formulating a solution in the subspace becomes very tedious. Therefore, this approach is currently being applied only to the application where the class number \(n\) is less than or equal to 4 [18]
CHAPTER 3
ANALYSIS OF UNMIXING ERRORS

Several sources of unmixing errors exist no matter which method we choose for proportion estimation. The first source of error is due to data variance; this includes intra-class data intrinsic variance and detection (or observation) noise. This degradation can mislead the variance of data as a mixture of different class objects and obscure the estimation accuracy. The second source of error results from inter-band misregistration, in which each component of a spectral vector actually represents a different region of interest. This error can be neglected for pixels inside homogeneous areas, but must be taken into account for those pixels close to class boundaries. The third source of error is caused by the sensor system PSF blurring. This degradation makes the estimation of endmember proportions for each pixel actually represent a larger area than one IFOV and therefore increases the unmixing error. The first two unmixing errors are discussed in this chapter, and the error caused by system PSF blurring will be considered in the next chapter. Section one describes the unmixing error for the LS algorithms caused by the data variance and observation noise. The mean squared error are then derived in section two. Section three discusses the unmixing error caused by band misregistration.

3.1 Unmixing Error of Data Variance and Observation Noise

If the remotely sensed signature for each class object is constant, then the proportion of each class in the mixed pixel can be computed exactly. Unfortunately, data of the same class collected by the sensor are rarely constant. The effect of data variance on unmixing can easily be observed by a simple example as we show below. Assume we have a system with band number $m$ equal one, class number $n$ equal two, and a
library matrix $\mathbf{R} = [36, 75]$ built from the individual mean values. Given an observation value of $y = 55$, if each class has constant reflectance equal to its mean value, then the proportion vector $\mathbf{x} = [0.513, 0.487]$. If in this particular pixel, class 2 actually contributes a reflectance value 85 due to the data variance, then the estimated proportion vector $\hat{\mathbf{x}} = [0.612, 0.388]$. Given the proportion vector of the ground truth $\mathbf{x} = (x_1, x_2, \ldots, x_n)'$ and an estimated proportion vector $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)'$, where $n$ is the number of class, the unmixing error vector equals $(\mathbf{x} - \hat{\mathbf{x}})$, and the total unmixing error is defined as:

$$e_{um} \triangleq \frac{1}{2} \sum_{i=1}^{n} |x_i - \hat{x}_i| \text{ (pixel)}$$

(3.1)

where the factor $1/2$ is used to avoid overestimating the error, and the unit pixel represents the area of a detector IFOV. Basing on this error definition, the above example has a 0.1 pixel unmixing error. Because we do not know the exact reflectance for each class in each pixel, this source of error is unavoidable. Apparently, when the data variance increases or the distance between mean vectors decreases, the unmixing error will increase. This error is dependent on a separability measure which we define as the average mean distance divided by the product of standard deviations for all class pairs. That is, for a given $m$-band, $n$-class image, we have

$$\text{Separability (\varepsilon)} \triangleq \frac{1}{m(n-1)!} \sum_{k=1}^{m} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{|\mu_{ki} - \mu_{kj}|}{\sigma_{ki} \sigma_{kj}}$$

(3.2)

where $\mu_{ki}, \sigma_{ki}$ are the mean and standard deviation for the $k$th band of the $i$th class. Given an ideal sensor system, this error sets a limit on the unmixing performance.

In general, the LS method is designed with an assumption that the data matrix $\mathbf{R}$ is constant, and the only error is in the observation vector $\mathbf{y}$. This assumption is usually not true in real applications, and the data variance we consider here is one good example. Recently, this assumption in solving the LS problem has been reconsidered in signal processing fields, and a new technique named total least square (TLS) [19] is currently under intensive investigation [20, 21]. The TLS can be viewed as a generalized LS method for solving $\mathbf{R} \mathbf{x} = \mathbf{y}$ when the reflectance matrix $\mathbf{R}$ and the observation vector $\mathbf{y}$ are both contaminated by data variance, measurement errors.
or noise. If the constraints of the unmixing problem can be formulated into the CLS method, we might be able to get better results than using LS.

In order to compare the performance of the three approaches for LS unmixing that we presented in the last chapter, a more realistic example is shown below. Let us assume that we have a system with:

\[
\mathbf{R} = \begin{bmatrix}
143.0 & 130.0 & 93.9 \\
121.3 & 104.6 & 131.7 \\
37.9 & 30.0 & 50.7
\end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix}
98 \\
129 \\
48
\end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}
\]

where the observed scene is a homogeneous pixel but with data variance and observation noise. Using the GIMM, SVD, and CLS methods for the LS unmixing we get:

\[
\hat{x}_{GIMM} = \begin{bmatrix}
0.4134 \\
-0.2275 \\
0.7277
\end{bmatrix}, \quad \hat{x}_{SVD} = \begin{bmatrix}
0.4134 \\
-0.2278 \\
0.7278
\end{bmatrix}, \quad \hat{x}_{CLS} = \begin{bmatrix}
0.0540 \\
0.0427 \\
0.9032
\end{bmatrix}
\]

There are some points that we can observe from this example:

(1) A slight noisy input observation vector can confuse the algorithm and generate a very unpredictable proportion vector estimation. To avoid this sensitivity, we suggest a hybrid approach that combines a standard classification algorithm with unmixing analysis. With this scheme, a pixel will be classified directly if it is tested to be within a homogeneous region. If this pixel can not be classified unambiguously, then an unmixing algorithm is used to do further analysis. For example, a threshold based on Mahalanobis distance (MD) [22] can be used to decide if a pixel is a 'pure' pixel or if it is a mixed pixel. That is, given a pixel, the MD's of this pixel to all the classes are computed to obtain a minimal MD. If this minimal MD acquired is within the threshold, then this pixel is 'pure' and can be classified directly to the class having the minimal MD. Otherwise, if all the MD's computed are larger than the threshold, then this pixel is assumed to be mixed and unmixing algorithm can be applied to estimate its proportion vector. Applying this scheme will make the pixel in the above example to be directly classified to class 3.

(2) The CLS algorithm is more robust than the other two approaches—GIMM and SVD; this result is empirically very consistent. Also when the reflectance matrix \( \mathbf{R} \) is
not degenerate. GIMM and SVD methods obtain the same solution. In application of GIMM and SVD, the constraint $\sum_{i=1}^{n} x_i = 1$ is used as one of the system equations, or the solution is based on

$$\begin{bmatrix} R \\ 1^t \end{bmatrix}_{(m+1) \times n} \begin{bmatrix} x_n \end{bmatrix}_{1 \times 1} = \begin{bmatrix} y \\ 1 \end{bmatrix}_{(m+1) \times 1} \tag{3.3}$$

Since both GIMM and SVD do not demand the constraint: $x_i \geq 0, 1 \leq i \leq n$ to be strictly satisfied in the solution as the CLS does, a proportion vector with components larger than 1 or smaller than 0 can happen. Although this can be postprocessed by:

$$x_i = \begin{cases} 1.0 & \text{if } x_i > 1 \\ 0.0 & \text{if } x_i < 0 \\ x_i & \text{otherwise} \end{cases}$$

the solution thus obtained is actually quite unpredictable. Contrasting with the GIMM or SVD, the CLS algorithm does not apply these rules to all $x_i$s at the same time but only to one of them at each step in order to search for a suboptimal solution in the constrained region. This more deliberate approach results in a better solution.

3.2 Mean Squared Error of LS Algorithms

While least square methods have been successfully used in spectral unmixing analysis, no attempt has been made to evaluate their performance related to the data variance and detection noise. In this section, an effort will be made on this subject. To simplify the notation, we assume that the constraint $\sum_{i=1}^{n} x_i = 1$ has been inserted into the system equation as in equation (3.3), but the original notations—$y$, $R$, and $e$—are still used to represent the augmented system equations. That is, $y_{(m+1) \times 1} = R_{(m+1) \times n} x_{n \times 1} + e_{(m+1) \times 1}$.

3.2.1 Case 1: Observation error only

If the system contains only observation noise without data variance, then only vector $e$, which is caused by observation noise, will affect the unmixing results. If
we assume the components $e_i$ of $e$ are independently distributed with zero mean and equal variance $\sigma^2_e$, then using the GIMM or SVD estimator,

$$
\hat{x} = (R^tR)^{-1}R^ty \\
= (R^tR)^{-1}R^t(Rx + e) \\
= x + (R^tR)^{-1}R^te
$$

(3.4)

Because of $E(e) = 0$, this gives an unbiased estimate of $x$. The mean squared error of the estimation for proportion vector $x$ is

$$
E[(\hat{x} - x)^t(\hat{x} - x)] = Tr\{E[(\hat{x} - x)(\hat{x} - x)^t]\} \\
= Tr\{E[(R^tR)^{-1}R^te][e^tR(R^tR)^{-1}]\} \\
= Tr\{\sigma^2_e(R^tR)^{-1}\}
$$

(3.5)

where $E(ee^t) = \sigma^2_eI$, $I$ is an identity matrix with dimension $n \times n$.

### 3.2.2 Case 2: Data variance and observation noise

When the data has no variance, the mixed reflectance with a proportion vector $x$ is simply $\sum_{i=1}^n x_i r_i$. But as we mentioned before, the mixed spectrum is actually formed by a varying individual reflectance, or $\sum_{i=1}^n x_i (r_i + \delta_i)$. That is, the data variance of the $i$th class can be modeled by adding an error vector term $\delta_i$ for each library endmember. If we define matrix $\Delta = (\delta_1, \delta_2, \ldots, \delta_n)$, and $R_\delta = R + \Delta$, then the LS estimate becomes

$$
\hat{x}_\delta = (R_\delta^tR_\delta)^{-1}R_\delta^ty
$$

(3.6)

The problem of errors in the element of $R$ (independent variable) is significantly more complicated than that of errors in the measurement (dependent variable). Although we would like to formulate the performance of the LS estimator as a function of the mean vectors and covariance matrices of all class members, it turns out that it is only possible when matrix $\Delta$ satisfies certain conditions.

Assuming that the error in $\Delta$, or intrinsic data variances, are reasonably small and representing the $R_\delta^tR_\delta$ as

$$
R_\delta^tR_\delta = R^tR + (\Delta^tR + R^t\Delta + \Delta^t\Delta) = B + D
$$
then we can expand \((R_\delta 'R_\delta )^{-1}\) as

\[(R_\delta 'R_\delta )^{-1} = B^{-1} - B^{-1}DB^{-1} + B^{-1}DB^{-1}DB^{-1} \ldots \quad (3.7)\]

Substituting equation (3.7) into (3.6), and neglecting terms of order \(\Delta^2\) and higher, the estimation \(\hat{x}_\delta\) becomes [23]

\[\hat{x}_\delta \simeq \hat{x} + (R'RR)^{-1}\Delta'(y - R\hat{x}) - (R'RR)^{-1}R'e + (R'R)^{-1}\Delta R(R'R)^{-1}R'e \quad (3.8)\]

Using \(\hat{x} = x + (R'R)^{-1}R'e\), we then obtain

\[\hat{x}_\delta \simeq x - (R'R)^{-1}R'\Delta x + (R'R)^{-1}R'e + (R'R)^{-1}\Delta R(R'R)^{-1}R'e \quad (3.9)\]

Further assume that \(e_i\) are distributed independently both of each other and of \(\Delta\) with zero mean and variance \(\sigma_i^2\), and that \(E(e_i) = 0\) and

\[E(\Delta^2) = \begin{bmatrix} \sigma_1^2 & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & \sigma_n^2 \end{bmatrix}\]

Hodges [23] showed that the mean square error of \(\hat{x}_\delta\) becomes

\[E\{||\hat{x}_\delta - x||^2\} = Tr\{(\hat{x}_\delta - x)(\hat{x}_\delta - x)^t\} = Tr\left\{\begin{bmatrix} (R'R)^{-1} \sum_{i=1}^{n} x_i^2 \sigma_i^2 + \sigma_e^2 \end{bmatrix}\right\} \quad (3.10)\]

Basically, the error is proportional to the data variance but it is not obvious how the mean vectors would affect the error. As \(\sigma_i^2 \to 0\), equation (3.10) is simplified to Case 1; therefore, the result is consistent. If the assumptions we made above are all satisfied, that is, very small amount of error in matrix \(\Delta\) and all errors are independent, then equation (3.10) can be used to predict the performance of the LS estimation. Unfortunately, data in remote sensing imagery usually have large
variance and high band-to-band correlation; therefore, the above error estimation is useful in a limited way.

The other quantitative measurement for predicting the performance is separability, which we defined in equation (3.2). In a later chapter, where various quantitative performances are evaluated, dependency of the unmixing error on the data separability will be assessed.

3.3 Unmixing Error Arising from Band Misregistration

Band-to-band registration is essential for analysis using multispectral data; for example, classification by multispectral data vector or false color composite for visual image interpretation. But because of the limitation on scanning/detection mechanisms, spectral misregistration is unavoidable. This degradation usually causes deterioration in the classification results or visual interpretation. By using simulated TM images, Swain et al. [24] showed that even a 0.3-pixel error in registration can seriously degrade the classification accuracy. Because unmixing analysis also depends on simultaneous use of multispectral data, apparently, it would be affected by registration accuracy.

Although some research on the effect of misregistration on classification accuracy has been done [24, 25, 26], there is no study on the effect of misregistration on the unmixing performance. In this section, the effect of misregistration on the spectral signature will be studied and some available misregistration measurements are reviewed for reference.

3.3.1 Spectral Effect of Misregistration

As we discussed in the last chapter, ground objects falling in the same IFOV generate a mixed pixel in which spectral signatures of all classes are linearly combined. Viewed in the spectral vector space, this mixing process creates new spectral signatures not belonging to any class and causes misclassification. It turns out that misregistration also creates mixed pixels, although by a different mechanism. Because classification and unmixing both use spectral signatures for data analysis, we
would like to investigate the effect of misregistration from the perspective of spectral signatures.

To simplify the discussion, an example of $m = 2$ bands and $n = 3$ classes with spectral signatures $r_1 = (r_{11}, r_{21})^t$, $r_2 = (r_{12}, r_{22})^t$, $r_3 = (r_{13}, r_{23})^t$, is used for illustration. The results obtained can be extended to higher number of bands and classes. Firstly, suppose there is no band-to-band misregistration then the spectral signature of each pixel sensed can be viewed as a combination of all classes with the same proportion vector in each band. That is,

$$y = x_1 r_1 + x_2 r_2 + x_3 r_3$$

\[\text{constraints: } x_1 + x_2 + x_3 = 1.0 \leq x_1, x_2, x_3 \leq 1\]

where a pure pixel has $x = (1, 0, 0)^t$, $(0, 1, 0)^t$, or $(0, 0, 1)^t$. Because of the constraints on $(x_1, x_2, x_3)$, any new spectral signature created by this mixing process is a convex combination of the individual class signature and is limited to the convex hull formed by vectors $\{r_1, r_2, r_3\}$. Given the data distribution of each original class shown in Figure 3.1(a), this newly created distribution is illustrated as the shaded region in Figure 3.1(b). Although these mixed pixels might be either misclassified or remain unclassified, unmixing analysis can be used to estimate the proportion vector $(x_1, x_2, x_3)^t$ and improve the analysis results.

When there is misregistration, proportion vectors of each pixel in different bands are no longer the same. That is, the spectral signature $y$ of each pixel becomes

$$y_1 = x_1 r_{11} + x_2 r_{12} + x_3 r_{13}$$

$$y_2 = x'_1 r_{21} + x'_2 r_{22} + x'_3 r_{23}$$

\[x_1 + x_2 + x_3 = 1, \quad 0 \leq x_1, x_2, x_3 \leq 1\]

\[x'_1 + x'_2 + x'_3 = 1, \quad 0 \leq x'_1, x'_2, x'_3 \leq 1\]

Because now each band has its own constraints on the proportion vectors, $(x_1, x_2, x_3)$ and $(x'_1, x'_2, x'_3)$, the freedom of possible mixture increases. Without loss of generality,
we assume that \( r_{11} \leq r_{12} \leq r_{13} \), then under the constraint \( x_1 + x_2 + x_3 = 1 \), \( 0 \leq x_1, x_2, x_3 \leq 1 \). we have \( r_{11} \leq y_1 \leq r_{13} \). Similarly, assuming \( r_{21} \leq r_{22} \leq r_{23} \) we get \( r_{21} \leq y_2 \leq r_{23} \). Therefore, the distribution of spectral vectors further disperses and covers the rectangular region bounded by the outermost classes; this is shown as the shaded area in Figure 3.1(c). From the above description, it is obvious that misregistration can also be viewed as a mixing process. Unfortunately, unlike simple spatial mixture due to IFOV in which \( x \) can be recovered by the unmixing inversion, mixture due to misregistration contains false information about its components which will contribute to the unmixing error. An exception is when the misregistration is a known integer multiple of one pixel, and therefore correction can be made before analysis.

For both classification and unmixing, the errors caused by this degradation are most likely to happen around class boundaries. The actual effect resulting from this anomaly is dependent on the spatial configuration and spectral signature of neighboring pixels. This interplay of geometric and spectral features makes analysis very difficult, if not impossible. Therefore, Swain [24] took a empirical approach and used simulation to evaluate the deterioration of classification accuracy caused by misregistration. In this dissertation, a similar procedure will be used in a later chapter for the assessment of how misregistration affects the unmixing accuracy.

### 3.3.2 Measurements of Misregistration

If the misregistration can be measured and corrected prior to the use of multispectral images, then analysis results would be improved. But this approach is applicable only when the misregistration is larger than one pixel. The relative misregistration estimation can be performed in two different ways; one is visual assessment using digitally zoomed data and the other is cross-correlation of block data from different bands. Using the first approach, Dwivedi [26] reported an estimate of 2 ~ 6 pixels misregistration in MSS imagery. After the band-to-band pixel offsets are removed, a significant reduction in the percentage of the original misclassified and unclassified pixels was shown.
Figure 3.1: Spectral signature distribution with 3 input classes: (a) without mixed pixel (b) with mixed pixels due to IFOV integration (c) with mixed pixels due to misregistration.
Another study done by Bernstein et al. [27] on TM imagery shows agreement between these two methods of misregistration measurement, as presented in Table 3.1. These misregistration values are relative to band 1, and the measurements are averaged from various scene blocks in an image of Chesapeake Bay area (November 2, 1982 Scene E-40109-15140). The data show that all the relative misregistration with respect to band 1 are less than one pixel; therefore, it is unlikely that we can remove the error caused by this degradation.

<table>
<thead>
<tr>
<th>Average Error</th>
<th>Band 1</th>
<th>Band 2</th>
<th>Band 3</th>
<th>Band 4</th>
<th>Band 5</th>
<th>Band 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cross-track</td>
<td>0</td>
<td>0.03</td>
<td>0.07</td>
<td>0.03</td>
<td>0.71</td>
<td>0.69</td>
</tr>
<tr>
<td>Along-track</td>
<td>0</td>
<td>0.03</td>
<td>0.0</td>
<td>0.10</td>
<td>0.29</td>
<td>0.32</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Average Error</th>
<th>Band 1</th>
<th>Band 2</th>
<th>Band 3</th>
<th>Band 4</th>
<th>Band 5</th>
<th>Band 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cross-track</td>
<td>0</td>
<td>0.10</td>
<td>0.14</td>
<td>0.25</td>
<td>0.66</td>
<td>0.44</td>
</tr>
<tr>
<td>Along-track</td>
<td>0</td>
<td>0.09</td>
<td>0.05</td>
<td>0.10</td>
<td>0.44</td>
<td>0.44</td>
</tr>
</tbody>
</table>

Table 3.1: TM band-to-band average misregistration (in pixels) estimation by visual analysis (top) and cross-correlation analysis (bottom).
CHAPTER 4
SAMPLED IMAGING SYSTEM MODELING AND UNMIXING ERROR OF PSF BLURRING

The unavoidable blurring degradation of imaging process in remote sensing is another source of unmixing error in addition to the ones that we discussed in the last chapter. In this chapter, we will investigate the effect of system point spread function (PSF) blurring on the unmixing error. The first section introduces an image observation model commonly used in the image restoration literature to account for the degradation. The second section will then analyze the unmixing error caused by the PSF blurring based on a simplified PSF model. The third section discusses the possibility of using image restoration for reducing the unmixing error caused by PSF-blurring.

4.1 Image Observation Model

With the application of system concepts to optical imaging [28], the imaging system can be thought of as a series of cascaded components, each of them contributes to an overall system PSF. A typical sampled imaging system contains modules of image formation, detection, signal conditioning and sampling. Figure 4.1 illustrates a canonic model for this image formation process.

For a linear space invariant system, the imaging observation model can be expressed as

\[ v(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x - x', y - y') u(x', y') dx' dy' \]  \hspace{1cm} (4.1)

\[ w[m, n] = \{(v(x, y)) + \eta(x, y)\} \Pi(x, y) \]  \hspace{1cm} (4.2)
where \( u(x, y) \) represents the object (original image), \( v(x, y) \) is the observed image, and \( h(x, y) \) is the overall system PSF or impulse response. Function \( \delta(x, y) \) is the ideal two-dimensional sampling function which is composed of an infinite array of Dirac delta functions arranged on a grid of spacing \((\Delta x, \Delta y)\), that is

\[
\delta(x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m\Delta x, y - n\Delta y)
\]

where \( \Delta x, \Delta y \) are the sampling distance at \( x, y \) direction. The term \( \eta(x, y) \) represents the additive noise, which usually consists of a combination of image-dependent and image-independent components. If the sampling aliasing of both signal and noise can be neglected, then the output sampled image \( w[m, n] \) is

\[
w[m, n] = v(m\Delta x, n\Delta y) + \eta(m\Delta x, n\Delta y)
\]

The total system PSF \( h(x, y) \) is defined as a combination of all the modules in the imaging system excluding sampling, that is,

\[
h(x, y) \triangleq h_f(x, y) \otimes h_d(x, y) \otimes h_e(x, y) \tag{4.3}
\]

or in the frequency domain

\[
H(\omega_x, \omega_y) = H_f(\omega_x, \omega_y) \cdot H_d(\omega_x, \omega_y) \cdot H_e(\omega_x, \omega_y) \tag{4.4}
\]

where \( h_f(x, y), h_d(x, y), \) and \( h_e(x, y) \) represent the PSF of image formation, detector, and signal conditioning, respectively; \( H_f(\omega_x, \omega_y), H_d(\omega_x, \omega_y), \) and \( H_e(\omega_x, \omega_y) \) are the corresponding Fourier transforms, or transfer functions (TF). In the following subsections, these three elements of the image observation model together with sampling
and noise will be briefly discussed. The final subsection shows different combinations of PSF components contributing to the total system PSF for several contemporary remote sensors.

4.1.1 Image Formation Module

Given an ideal imaging system with total system PSF $h(x, y)$ given by an infinitesimally thin Dirac delta function, there will be no spatial degradation in the observed image, that is, $v(x, y) = u(x, y)$. Unfortunately, this ideal system can never happen and the observed images always suffer spatial blurring degradation. The extent of blur introduced is determined by the shape and width of $h(x, y)$, and the first component of $h(x, y)$ is the image formation PSF, $h_f(x, y)$.

To separate the cause of this degradation, the image formation PSF $h_f(x, y)$ is further divided into two distinct components; one is the optical system PSF, $h_f^{(o)}(x, y)$, and the other is called the imaging environment PSF, $h_f^{(e)}(x, y)$. In addition, $h_f(x, y) = h_f^{(o)}(x, y) \circledast h_f^{(e)}(x, y)$. The optical system PSF, which includes the lens, mirrors, prisms and the aperture, accounts for the diffraction blur and aberration blur. The degradation caused by $h_f^{(o)}(x, y)$, which can be viewed as characteristic of the system, will appear in all the images taken. Although the degradation can be reduced, it is generally unavoidable. The modulation transfer function (MTF) of a typical $h_f^{(e)}(x, y)$ for a corrected diffraction-limited system [29] is shown in Figure 4.2.

The imaging environment PSF represents other degradation factors which are independent of the optical system itself. This usually includes motion blur, defocus blur and atmospheric turbulence. In contrast to the $h_f^{(o)}(x, y)$, degradations caused by $h_f^{(e)}(x, y)$ can be viewed as a transient event, which is dependent on the external conditions as the image was taken. Generally speaking, the region of support of $h_f^{(o)}(x, y)$ is usually very small for a well-designed system and $h_f^{(e)}(x, y)$ usually can occupy a larger area than that of $h_f^{(o)}(x, y)$.

Although most of the restoration techniques developed in the literature are devoted to the removal of degradations caused by $h_f^{(e)}(x, y)$, our work in this dissertation focuses on the restoration of images whose image formation PSF equals $h_f^{(o)}(x, y)$.
or $h_f(x, y) = h_f^{(o)}(x, y)$. Since we can also model $h(x, y)$ as $h_f^{(o)}(x, y) \otimes h_f^{(e)}(x, y)$ to take extra degradation into account if it happens, the development is actually very general.

### 4.1.2 Detector Module

The detection process is the second module in the image observation model which converts the photons into electrons by the sensing array (e.g. CCD). Each photosensitive element in the array integrates the irradiance over a small area defined as the instantaneous field of view (IFOV). For a given integration time, the detector element must have a certain size in order to accumulate sufficient photons for useful signal level. This finite size of detector contributes to another PSF component in the system response. Figure 4.3 illustrates two possible response profiles [29]: Figure 4.3(a) is the "ideal" uniform rectangular response approximated in arrays designed with a barrier between individual elements. Figure 4.3(b) is typical of detectors for which the sensitive region is defined by the contact at which charge carriers are collected; it gives rise to the trapezoidal response. In the frequency domain, the rectangular profile exhibits the $|\sin x/x|$ MTF as is shown in Figure 4.4. The trapezoidal profile has the $|\sin^2 x/x^2|$ MTF, which has reduced response in the sidelobes beyond the first zero crossing (Figure 4.4) and has significance for decreased aliasing.
4.1.3 Signal Conditioning Module

After the charges collected in the detector element have been converted to a voltage, it is amplified and fed to the signal processing unit. This module usually produces filters for removing high frequency noise in the scan data or limiting the frequency content before the sampling step to reduce the aliasing. These filters are normally low-pass, and therefore cause high frequency degradation in the output images. Furthermore, this component usually causes asymmetry in the total system PSF.

4.1.4 Sampling Module

Even though the input radiance has been spatially sampled by the detector, the actual sampling frequency in both $x$ and $y$ are determined by the detector array geometry and scanning rate. Given a linear array, the detector pitch $P$ and the detector width $W$ are not necessarily equal as is shown in Figure 4.5. The limiting frequency, which equals the first zero crossing of the MTF, is given by

$$\omega_d = f/W = 1/IFOV \ [cycles/\text{rad}] \quad (4.5)$$

where $f$ is the focal length of optical system. The sampling frequency, which is determined by pitch, is given by

$$\omega_s = f/P \ [cycles/\text{rad}] = 1.0 \ [cycles/\text{pixel}] \quad (4.6)$$

where $P$ is pitch, which is defined as the detector center-to-center distance. The folding frequency is defined as $\omega_s/2$ and signal spectrum beyond this frequency produces aliasing. The sampling frequency and the first zero crossing of the MTF are
two independent parameters determined by the focal plane array geometry, and can be made the same if $P = W$.

For a linear array (push-broom) system, the sampling frequency in the cross-track (along-scan) direction is determined by the array geometrical parameter $P$. In the along-track direction, sampling is usually done after the signal has gone through a low-pass filter to limit the aliasing, the sampling frequency can be different from that of cross-track direction. For a whisk-broom system, the sampling frequency in the along-track direction is determined by the parameter $P$, and the sampling in the cross-track direction is dependent on the scanning rate [30].

4.1.5 Noise Model

The ultimate limit to the detection of faint levels of radiant flux is set by system noise, and it is also one of the most important factors in the design of restoration algorithms. The most common noise source associated with electronic imaging systems is thermal noise, which is due to random electron fluctuations in resistive elements of photodetectors or resistors within the signal-conditioning circuit. Thermal noise can be modeled as an additive Gaussian stochastic process independent of the sensed image field. A detailed analysis requires knowledge of the sensor circuitry and will be avoided in our study of restoration.

Photodetector current is not truly constant even when the incident light intensity is constant. The photodetector sensors exhibit a measurement uncertainty due to the quantum-mechanical nature of light. At low light levels the number of electrons emitted by a photodetector is governed by a Poisson probability density. The resultant uncertainty in knowledge of the detector current is called shot noise in photoemissive detectors and generation-recombination noise in photoconductive and photovoltaic detectors. Contrary to the thermal noise, shot noise is signal dependent and is usually assumed to be proportional to the square root of the input signal level. For most imaging sensors, the associated shot noise can be modeled either as a Gaussian or Poisson distributed random process [31]. Because Poisson-distributed noise model is better only when the image light level is extremely small and the detector possesses
a large internal electron amplification, a zero mean Gaussian distributed model will be used for shot noise throughout this dissertation.

The total system noise will be modeled by one Gaussian-distributed random process to represent a combination of two Gaussian-distributed random processes, one is signal-independent and the other is signal-dependent. Calculation of the exact noise mean and variance often entails analysis of complex electronic circuits, and consequently indirect measurement techniques are usually invoked.

4.1.6 System PSF/MTF Model for Remote Sensors

In this section, models of four contemporary remote sensing instruments are considered and compared based on their system MTF components. Due to different array structure and scanning method used, each sensor can be composed of different number of degradation PSF components as are listed below for Landsat MSS and TM, AVHRR, and SPOT respectively [32].

**MSS, AVHRR, TM**

- Cross-track
  1. Image-forming optics
  2. Detector IFOV
  3. Electronics
- Along-track
  1. Image-forming optics
  2. Detector IFOV

**SPOT**

- Cross-track
  1. Image-forming optics
  2. Detector IFOV
  3. Electronic filter
- Along-track
1. Image-forming optics
2. Detector IFOV
3. Sample integration

The MSS, TM, and AVHRR use the whisk-broom scanning method which allows oversampling and includes electronic filter degradation in the cross-track direction. The SPOT system uses a push-broom scanner which contains an electronic filter in the cross-track direction representing the charge transfer inefficiency of the linear CCD array. In the along-track direction, the SPOT has an extra integration time effect which accounts for the sample integration [33]. In this dissertation only TM will be used for simulation of improved unmixing by restoration. Therefore, the MTF and Phase modulation function (PTF) of TM in both directions are illustrated in Figure 4.6 and Figure 4.7 for later reference.

4.2 Unmixing Error of System PSF Blurring

When the proportion vector $x$ is computed, the basic assumption is that the estimate of each pixel is for an area covered by one detector IFOV. If we define the image gathering PSF, $h_a(x, y)$, as the combination of all the PSF components in the observation model except the detector, then for an ideal image gathering PSF ($h_a(x, y) = \delta(x, y)$), the area integrated by one detector cell will satisfy this assumption and is shown in Figure 4.8. From the viewpoint of pixel unmixing (not resolution), an ideal total system PSF is the profile of the detector element. In reality, the other PSF components which include image formation (optics, motion-smear, etc.) and electronic filters are hardly ideal. This nonideal image gathering PSF causes the energy integrated by one detector cell to come from outside of one detector IFOV area, as is shown in Figure 4.9. Because of this blurring, each pixel represents further mixing with the surrounding pixels and causes unmixing error. To make the idea clearer in the following discussion, let us assume that there are no data variance and observation noise, so that the unmixing error can be all attributed to blurring by the sensor PSF.
If a given sensor has an ideal image gathering PSF, then for a given pixel, the proportion vector \( \mathbf{x} \) can be estimated without error, whether it is in a homogeneous region or in a heterogeneous area. For a real sensor system, the situation is more complicated. Although the blurring degradation does not affect the unmixing results if a pixel is situated inside a homogeneous region, the error occurs when a pixel is close to a heterogeneous region. This error is due to the fact that we consider the result as for one detector IFOV when it actually represents a wider region weighted by the image gathering PSF.

To numerically assess unmixing error caused by the sensor PSF blurring, a Gaussian PSF model

\[
h(x, y) = \frac{1}{2\pi \sigma_x \sigma_y} e^{-\left(\frac{x^2}{2\sigma_x^2}\right)} e^{-\left(\frac{y^2}{2\sigma_y^2}\right)}
\]

with \( \sigma_x = \sigma_y = 0.4 \) is used to compute the unmixing error for the simplest heterogeneous case – the boundary of two classes. The use of the Gaussian is justified heuristically by the central limit theorem. With this simulated PSF, 38% of the energy measured by the detector actually comes from outside its IFOV, that is, the blurring degradation causes one detector IFOV to effectively integrate over a 3 x 3 pixel area (see Figure 4.9). In Figure 4.10, we show five possible unmixing results for any given pixel \( p \) close to a class boundary. They are:

(a) \( p \) is class 2, no unmixing error (inside homogeneous region)
(b) \( p \) is class 2, unmixing error from class 1
(c) \( p \) is a mixed pixel, unmixing error occurs in both class proportion estimate
(d) \( p \) is class 1, unmixing error from class 2
(e) \( p \) is class 1, no unmixing error (inside homogeneous region)

Two parameters \( d \) (boundary distance) and \( \theta \) (boundary orientation), which describe the relative position between the center of the pixel under unmixing and the nearby class boundary, are defined in Figure 4.11(a). To compute the unmixing error caused by the image gathering PSF blurring and evaluate the significance of the boundary orientation and distance, these two parameters are varied in the range of \( 0 < \theta < \pi/2 \), \(-1 \leq d/d_{\text{max}} \leq +1\), and the unmixing error computed. Notice that the distance has been normalized by \( d_{\text{max}} \). For a given \( \theta \), \( d_{\text{max}} \) is defined as the maximum value of \(|d|\)
for which the boundary starts to cause unmixing error due to PSF blurring. In the
current example, in which the image gathering PSF has a $3 \times 3$ pixel effective region.
The $d_{max}$ for $\theta = \theta_0 = \pi/4$ is shown in Figure 4.11(b). This distance is dependent
on the effective size of the image gathering PSF, and occurs when the boundary first
touches the effective region. Also, for different $\theta$, $d_{max}$ is different. Because the class
boundary does not cause unmixing error until $|d|/d_{max} \leq 1.0$, and symmetry can be
used to obtain the results for $\theta$ outside $[0, \pi/2]$, the ranges for $d$ and $\theta$ we consider
above actually account for all the possibilities.

Using the image gathering PSF of equation (4.7) and assuming a 2-class case as
above discussion, the results of unmixing error due to PSF blurring are shown in
Figure 4.12 as a function of $d$ and $\theta$. Note that no unmixing algorithm is assumed
under current calculation; the percentage of each class is proportional to the effective
area covered by the PSF and the corresponding PSF value over that area. As we can
see in this figure, the orientation of boundary has no profound effect on the unmixing
error, but the unmixing error is very sensitive to the parameter $d$. For all boundary
orientations, the maximum error occurs when the class boundary touches the border
of the pixel undergoing unmixing. When the class boundary crosses the center of a
pixel, the unmixing error for that pixel resulting from PSF blurring has a minimum
value.

As we can imagine, when the scene becomes more complicated or the number of
class boundary increases, the unmixing error caused by the PSF blurring will increase.
Fortunately, this error can be reduced if the image gathering PSF, which causes the
degradation, is known.

4.3 Proposal for Unmixing Error Reduction

Since the degradation resulting from blurring is a loss of resolution, any improve-
ments in resolution should be able to reduce the error caused by PSF blurring. This
speculation leads us naturally to image restoration as an approach for improving the
unmixing accuracy. While this technique has been used for years in restoring de-
graded imagery, it is always the visual quality which has been of primary interest.
But, does a higher resolution image which is visually appealing really give us a better unmixing result? This is the most important issue if we believe that image restoration would help reduce unmixing error. Therefore, starting from the perspective of better visual quality we investigate the feasibility of our proposal in the next two chapters. Fortunately, the quantitative characteristics of our problem makes the evaluation and comparison of algorithms easier and more natural than for conventional restoration.
Figure 4.4: MTF of a rectangular response detector and of a trapezoidal response detector. Note decreased response in the lobes beyond the first zero crossing of the latter.

Figure 4.5: A linear square-detector array of pitch $P$ and width $W$
Figure 4.6: MTF of TM band-1 (a) Cross-track direction (b) Along-track direction
Figure 4.7: PTF of TM band-1 (a) Cross-track direction (b) Along-track direction
Figure 4.8: Area of integration for an ideal image gathering PSF and square detector PSF.

Figure 4.9: Area of integration for a Gaussian image gathering PSF and square detector PSF.
Figure 4.10: The five possible unmixing conditions when a pixel is close to class boundary
Figure 4.11: The definitions of (a) $d$ (boundary distance) and $\theta$ (boundary orientation) (b) $d_{\text{max}}$ for a given $\theta$ and a 3 x 3 effective PSF.
Figure 4.12: Unmixing error with respect to the boundary orientation and distance
CHAPTER 5
DIGITAL IMAGE RESTORATION FOR SAMPLED IMAGING SYSTEM

Image restoration techniques have been used successfully for reducing image degradation in the last two decades. Although there exist many new algorithms which attack this problem from different aspects in recent literature, for example, regularized iterative approach by the Lagendijk [34], set-theoretic method by Youla and Sezan [35, 36], and recursive Kalman filtering by Woods [37], we will limit our scope to the frequency domain approach. The main reason for this choice is that aliasing, which plays a significant role in our application, is more easily observed and interpreted in the frequency domain. To achieve a meaningful performance evaluation and algorithm comparison, an objective criterion based on the measurement of spectral fidelity is brought in to further the effectiveness of our frequency domain approach. In this chapter, we will focus the discussion on the improvement of visual quality to emphasize the importance of aliasing and bring to attention the difference that our specific application of restoration will bring in the next chapter. Two digital image restoration algorithms taken deliberately from the literature are introduced in the discussion. In section one, we briefly review the derivation of inverse and Wiener filter methods. In the second section, the relevance of aliasing to restoration is taken into consideration in which a new form of Wiener filter results. Section three contains a thorough comparison of these two Wiener filters.
5.1 Digital/Digital Model: Inverse Filter and Conventional Wiener Filter (CWF)

To design an effective digital image restoration system, it is necessary to characterize quantitatively the image degradation effects of the physical imaging system. The basic procedure of restoration is to model the image degradation effect and then perform an inverse operation to obtain a restored image. It should be emphasized that accurate image degradation modeling is often the key to successful restoration. There are two different approaches to the modeling of image degradation effects: a priori modeling and a posteriori modeling. In the former case, measurement can be made on the physical imaging system or system design/test data can be used to determine the degradation PSF even before the degraded images are obtained [38]. The a posteriori approach is to develop the model for the degradation based on the available degraded image. For examples, many algorithms have been developed for estimating the system PSF and MTF for TM [39, 40, 41].

The most common approach in formulating the restoration problem is to use a digital object/digital image model. To do that, we first need to convert the observation equation into its discrete approximation. Assuming that the input signal is bandlimited, and the sampling frequency is high enough to neglect the sampling aliasing, then from equation (4.1)

\[
v[m, n] = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} h[m - i, n - j]u[m, n]
\]

\[
w[m, n] = v[m, n] + \eta[m, n]
\]

or

\[
w[m, n] = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} h[m - i, n - j]u[m, n] + \eta[m, n]
\]

where \(\eta[m, n]\) is discrete noise field, \(h[m, n]\) is a digitization of the imaging system PSF, and \(u[m, n]\), \(w[m, n]\) are sampled versions of \(u(x, y)\) and \(w(x, y)\) over the sampling grid.
The simplest way to restore a degraded image based on the model of equation (5.3) is to use an inverse filter. Discrete Fourier transforming both sides of (5.3), we get:

\[ W(\omega_x, \omega_y) = H(\omega_x, \omega_y) \cdot U(\omega_x, \omega_y) + N(\omega_x, \omega_y) \]  

(5.4)

Using an inverse filter, the restored image becomes:

\[ \hat{U}(\omega_x, \omega_y) = \frac{W(\omega_x, \omega_y)}{H(\omega_x, \omega_y)} = U(\omega_x, \omega_y) + \frac{N(\omega_x, \omega_y)}{H(\omega_x, \omega_y)} \]  

(5.5)

Because \(1/H(\omega_x, \omega_y)\) is not defined if \(H(\omega_x, \omega_y) = 0\), a pseudoinverse filter defined as

\[ H^{-1}(\omega_x, \omega_y) = \begin{cases} \frac{1}{H(\omega_x, \omega_y)}, & H \neq 0 \\ 0, & H = 0 \end{cases} \]  

(5.6)

is usually used instead. In practice, \(H^{-1}(\omega_x, \omega_y)\) is set to zero whenever \(|H|\) is less than a suitably chosen positive quantity. Although the operation is simple, the result of the inverse filter is dependent on the value of \(N(\omega_x, \omega_y)/H(\omega_x, \omega_y)\) and is very sensitive to noise. To obtain a more satisfactory result, Wiener filtering, which is derived below [42], can be used to restore a degraded noisy image.

Given a degraded image \(w[m, n]\), we would like to obtain an estimate, \(\hat{u}[m, n]\), of \(u[m, n]\) such that the mean square error

\[ \sigma_e^2 = E(u[m, n] - \hat{u}[m, n])^2 \]  

(5.7)

is minimized. If the solution is limited to a linear estimate, then

\[ \hat{u}[m, n] = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} g[m, n; i, j]w[m, n] \]  

(5.8)

and the filter response \(g[m, n; i, j]\) is determined to minimize the mean square error \(\sigma_e^2\). Minimization of (5.7) requires that the orthogonality condition

\[ E(\{u[m, n] - \hat{u}[m, n]\}w[m', n']) = 0, \text{ for all } [m, n], [m', n'] \]  

(5.9)

be satisfied. Defining the cross-correlation as

\[ r_{ab}[m, n; i, j] = E\{a[m, n]b[i, j]\} \]
for two arbitrary random sequences $a[m, n]$ and $b[m, n]$, and given (5.8), the orthogonality condition yields the equation

$$\sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} g[m, n; i, j] r_{uw}[i, j; m', n'] = r_{uw}[m, n; m', n'] \quad (5.10)$$

Equations (5.8) and (5.10) are called the Wiener filter equations. If $u[m, n]$ and $w[m, n]$ can be assumed to be jointly stationary, then (5.10) is simplified to

$$\sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} g[m - i, n - j] r_{uw}[i, j] = r_{uw}[m, n] \quad (5.11)$$

Taking Fourier transform of both sides, and solving for $G(\omega_x, \omega_y)$, we get

$$G(\omega_x, \omega_y) = S_{uw}(\omega_x, \omega_y) S_{uu}^{-1}(\omega_x, \omega_y) \quad (5.12)$$

where $G$, $S_{uw}$, and $S_{uw}$ are the Fourier transform of $g$, $r_{uw}$, and $r_{uw}$ respectively.

Using the observation model equation (5.3), and assuming that the noise $\eta[m, n]$ is a stationary sequence uncorrelated with $u[m, n]$, we get

$$S_{uw}(\omega_x, \omega_y) = |H(\omega_x, \omega_y)|^2 S_{uu}(\omega_x, \omega_y) + S_{\eta\eta}(\omega_x, \omega_y) \quad (5.13)$$

$$S_{uu}(\omega_x, \omega_y) = H^* S_{uu}(\omega_x, \omega_y) \quad (5.14)$$

where $S_{\eta\eta}$ is the power spectral density of noise. This gives

$$G(\omega_x, \omega_y) = \frac{H^* (\omega_x, \omega_y) S_{uu}(\omega_x, \omega_y)}{|H(\omega_x, \omega_y)|^2 S_{uu}(\omega_x, \omega_y) + S_{\eta\eta}(\omega_x, \omega_y)} \quad (5.15)$$

with the mean square error

$$\sigma^2_e = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} S_e(\omega_x, \omega_y) d\omega_x d\omega_y \quad (5.16)$$

and

$$S_e(\omega_x, \omega_y) = \frac{S_{uu}(\omega_x, \omega_y) S_{\eta\eta}(\omega_x, \omega_y)}{|H(\omega_x, \omega_y)|^2 S_{uu}(\omega_x, \omega_y) + S_{\eta\eta}(\omega_x, \omega_y)} \quad (5.17)$$

Because of its commonality in the literature, we call this Wiener filter the conventional Wiener filter (CWF).

Without any blurring, $H = 1$, and the CWF becomes

$$G(\omega_x, \omega_y) = \frac{S_{uu}(\omega_x, \omega_y)}{S_{uu}(\omega_x, \omega_y) + S_{\eta\eta}(\omega_x, \omega_y)} = \frac{S_{\eta\eta}(\omega_x, \omega_y)}{1 + S_{\eta\eta}(\omega_x, \omega_y)} \quad (5.18)$$
where $S_{nr} \triangleq S_{uu}/S_{\eta \eta}$ defines the signal-to-noise ratio at the frequencies $(\omega_x, \omega_y)$ and $G(\omega_x, \omega_y)$ is called the (Wiener) smoothing filter. In the absence of noise, $S_{\eta \eta}(\omega_x, \omega_y) = 0$, the Wiener filter reduces to an inverse filter. Since the blurring process is usually a low-pass filter, the Wiener filter acts as a high-boost filter at low levels of noise and obtains a high degree of restoration, or deblurring. When both noise and blur are present, the CWF achieves a compromise between the low-pass noise smoothing filter and the high-pass inverse filter.

5.2 Continuous/Digital/Continuous (c/d/c) model: Sampled Wiener Filter (SWF)

The conventional formulation of Wiener filter for image restoration, as we reviewed in the last section, has accounted for the blurring and noise that occur in image formation; however, it fails to account for the insufficient sampling that causes aliasing. As we mentioned before, when the noise level is low, the Wiener filter acts as a high-pass filter and achieves a high degree of restoration. Unfortunately, aliasing always occurs in the high frequency region and the neglect of it will cause the degradation of the Wiener filter even under no noise condition.

Although it is permissible to assume sufficient sampling in the design of the Wiener filter for time domain optimal filtering, it is not generally valid when the result is extended to image restoration. The difference between these two applications is related to the difference between the realizabilities of the frequency response of electronic filters and optical apertures. Because the frequency response of an electronic filter can be designed to approximate an ideal low-pass filter, it can be used to limit the bandwidth of the input signal and avoid aliasing. However, the spatial frequency response of an optical aperture to incoherent radiation, which is the autocorrelation of the optical transmittance function, tends to decrease smoothly with increasing frequency. Because insufficient sampling is common in practical digital imaging, this nonideal low-pass response of image formation inevitably leads to the creation of another form of noise — aliasing\(^1\).

\(^1\)Except the sampling frequency is higher than two times of the optical cutoff frequency.
To account for the insufficient sampling that causes aliasing, as well as blurring and noise in image restoration, Fales et al. [43] formulated the restoration filter based on a continuous/digital/continuous end-to-end imaging system model. The image observation module converts a continuous input radiance $u(x, y)$ into a sampled signal $w[m, n]$ and a restoration filter $g[m, n]$ is used to deblur the digital image, then the reconstruction module converts the restored image $q[m, n]$ into a continuous image $r(x, y)$. This conversion process can be represented as

$$w[m, n] = \{u(x, y) \otimes h(x, y) + \eta(x, y)\} \mathbb{I}(x, y) \quad (5.19)$$

$$q[m, n] = w[m, n] \otimes g[m, n] \quad (5.20)$$

and

$$r(x, y) = q[m, n] \otimes h_p(x, y)$$

$$= ([u(x, y) \otimes h(x, y) + \eta(x, y)] \mathbb{I}(x, y) \otimes g[m, n]) \otimes h_p(x, y) \quad (5.21)$$

where $h_p(x, y)$ is the reconstruction filter.

The Wiener filter is obtained by minimizing the mean-squared restoration error (MSRE) $\epsilon^2$ between the incident radiance $u(x, y)$ and the reconstructed image $r(x, y)$, or

$$minimize \epsilon^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E\{|u(x, y) - r(x, y)|^2\} \text{d}x \text{d}y \quad (5.22)$$

Because the purpose of restoration is to make the display (continuous) image as close to the input image (continuous) as possible by processing the sampled image (digital), it is called a continuous/digital/continuous model. A lengthy derivation in the frequency domain based on equations (5.21) and (5.22) leads to the sampled Wiener filter (SWF) [43] $G_s(\omega_x, \omega_y)$, that is

$$G_s(\omega_x, \omega_y) = \frac{H^*(\omega_x, \omega_y)S_{uu}(\omega_x, \omega_y)}{[H(\omega_x, \omega_y)]^2S_{uu}(\omega_x, \omega_y) + S_{\eta\eta}(\omega_x, \omega_y) \otimes \mathbb{I}(\omega_x, \omega_y)} \quad (5.23)$$

where $\mathbb{I}(\omega_x, \omega_y)$ is the Fourier transform of the sampling function $\mathbb{I}(x, y)$. Using the assumption that the sensor noise is bandlimited, then from the computational point of view, the denominator of $G_s(\omega_x, \omega_y)$ can be simplified as

$$\sum_{m,n \neq 0,0} \{[H(\omega_x - m\omega_x, \omega_y - n\omega_y)]^2S_{uu}(\omega_x - m\omega_x, \omega_y - n\omega_y)\}$$

$$+ [H(\omega_x, \omega_y)]^2S_{uu}(\omega_x, \omega_y) + S_{\eta\eta}(\omega_x, \omega_y) \quad (5.24)$$
where \( \omega_x = 2\pi / \Delta x, \omega_y = 2\pi / \Delta y \) are the sampling frequencies at \( x \) and \( y \) direction. Apparently, the last two terms are the same as that of the CWF, and the first term accounts for the sampling aliasing. When the aliasing is negligible, the sampled Wiener filter becomes a conventional Wiener filter. Fidelity improvement using a c/d/c model has proved to be effective. As stated by Huck [44]: “When these transformations (sampling and display) are correctly accounted for by the Wiener filter, the MSRE criterion gained significantly in resolution and sharpness.”

5.3 Performance Comparison of the CWF and SWF

Although restorations performed by the SWF have been shown to be better than the CWF, the existing comparisons are all based on a subjective criterion [43, 45]. Basically, they are all done by passing simulated scenes through the image observation model and restoring the output sampled images by both filters. These restored images are then visually compared to evaluate the performance. While a subjective criterion offers a convenient way for judging the restoration results, it cannot be used to numerically assess the significance of aliasing and the other system parameters. Moreover, as we will show in the next chapter, the (subjective) visual criterion cannot even be applied to our application. To quantitatively evaluate the restoration results and to gain a better understanding of how both filters behave under different conditions, an objective criterion should be used. Since it is the consideration of aliasing that makes a SWF differ from a CWF, a good choice of criterion is the image power spectrum fidelity. This criterion has been used successfully in evaluating the tradeoff between the aliasing error and system MTF degradation on different types of presampling filters in digital imaging system design [31]. In this dissertation, the same criterion will be applied and the restoration performance of the SWF and CWF are compared using the error of the restored spectrum (spectral fidelity).

Given an input radiance with spectrum \( S_{uu}(\omega_x, \omega_y) \) and a sensor with system transfer function \( H(\omega_x, \omega_y) \), the spectrum of the output degraded image before ideal sampling is [46]

\[
S_{uv}(\omega_x, \omega_y) = |H(\omega_x, \omega_y)|^2 S_{uu}(\omega_x, \omega_y)
\]
The filtered image is then spatially sampled by an ideal Dirac delta function at a resolution $\Delta x, \Delta y$. Assuming bandlimited noise, the sampled image spectrum is

$$S_{uw}(\omega_x, \omega_y) = S_{uu}(\omega_x, \omega_y) + S_a(\omega_x, \omega_y) + S_{\eta\eta}(\omega_x, \omega_y)$$

(5.25)

where $S_a(\omega_x, \omega_y)$ is aliased spectrum caused by sampling aliasing. Assuming $u(x, y)$ is wide sense stationary (WSS) random field (ensemble), then [46]

$$S_a(\omega_x, \omega_y) = \sum_{m,n \neq 0,0} S_{uv}(\omega_x - m\omega_x, \omega_y - n\omega_y)$$

(5.26)

Using equation (5.25) and assuming an ideal reconstruction, the spectral error due to the imaging process is defined as

$$S_e(\omega_x, \omega_y) = S_{uw}(\omega_x, \omega_y) - S_{uu}(\omega_x, \omega_y)$$

$$= S_{uu}(\omega_x, \omega_y)(|H(\omega_x, \omega_y)|^2 - 1) + S_a(\omega_x, \omega_y) + S_{\eta\eta}(\omega_x, \omega_y)$$

$$= S_a(\omega_x, \omega_y) + S_a(\omega_x, \omega_y) + S_{\eta\eta}(\omega_x, \omega_y)$$

(5.27)

where $S_a(\omega_x, \omega_y)$ is called system (MTF) degradation error, which is part of the spectral error due to the imaging process. Because $|H(\omega_x, \omega_y)|^2$ is always less than 1, $S_a(\omega_x, \omega_y)$ is also always negative. On the other hand, the aliased spectrum ($S_a(\omega_x, \omega_y)$) as shown in equation (5.26) is a positive quantity which is added to the signal spectrum. To simplify the notation, we will assume a normalized frequency, or $\omega_x = \omega_y = 1$, in the following discussion. Also, the frequency indices $\omega_x, \omega_y$ are dropped for brevity. Given a sampled image, if it is to be restored by the Wiener filter $G(\omega_x, \omega_y)$, the spectral error after the restoration becomes

$$\hat{S}_c = S_{uu}(|GH|^2 - 1) + |G|^2 S_a + |G|^2 S_{\eta\eta}$$

(5.28)

If the CWF is used, this spectral error is equal to

$$\hat{S}_{cCWF} = \frac{S_{uu}S_{\eta\eta}}{|H|^2S_{uu} + S_{\eta\eta}} + \frac{S_{uu}^2S_a}{(|H|^2S_{uu} + S_{\eta\eta})^2}$$

(5.29)

Instead, if the SWF is applied, then the spectral error equals

$$\hat{S}_{cSWF} = \frac{S_{uu}S_{\eta\eta}(|H|^2S_{uu} + S_{\eta\eta} + 2S_a) + S_{uu}S_a(|H|^2S_{uu} + S_a)}{(|H|^2S_{uu} + S_{\eta\eta} + S_a)^2}$$

(5.30)
Without further assumptions on the input signal spectrum and the imaging system transfer function, it is very difficult, if not impossible, to continue the derivation and obtain a meaningful comparison between these two filters. Thus, instead of mathematical analysis, a numerical method will be adopted in the following discussion by using an AR signal model and a Gaussian image gathering MTF. To clarify presentation of the results, a one-dimensional signal and system are used in all the computations.

For a given AR signal, its power spectrum can be modeled as

$$S_{uu}(\omega) = \frac{A}{1 + (\omega/\omega_r)^{2m}}$$

(5.31)

where $A$ is the magnitude constant, $m$ is an integer governing the rate of falloff of the spectrum, and $\omega_r$ is the half-magnitude frequency. We also assume that the MTF of the imaging system has the form

$$H(\omega) = e^{-(\omega/\omega_c)^2} \cdot \frac{\sin(\pi \omega)}{\pi \omega}$$

(5.32)

where $\omega_c$ is the parameter which controls the bandwidth of the imaging system and the $\frac{\sin(\pi \omega)}{\pi \omega}$ accounts for the detector's rectangular PSF. The frequency $\omega$ in both equations are normalized to the sampling frequency, and the unit is cycles/pixel.

To begin the simulation, a set of parameters with $A = 10$, $m = 1$ and $\omega_r = 0.1$ are chosen for the input signal spectrum, and the noise is assumed to be white with spectrum $S_{\eta \eta}(\omega) = 0.01$ (SNR=22 dB). Figure 5.1(a) shows what the input spectrum looks like when $\omega_r$ is varied to change the signal bandwidth under constant SNR.

To see how the imaging system MTF affects performance of the restoration filter, five imaging system with different bandwidths are used. These systems are specified by having parameter $\omega_c = \{1.6, 1.3, 1.0, 0.7, 0.4\}$, and their MTF's are shown in Figure 5.1(b). These five imaging systems degrade the input signal spectrum to different degrees and effectively limit the bandwidth of the input (Figure 5.2(a)) and

---

$^2$SNR is defined as

$$SNR \triangleq 10 \cdot \log\left(\frac{\int_0^1 S_{uu}(\omega)d\omega}{\int_0^1 S_{\eta \eta}(\omega)d\omega}\right)$$
level of the aliasing (Figure 5.2(b)). Using the given parameters for the signal, noise and imaging system, the CWF and SWF are designed and presented in Figure 5.3\(^3\). As can be seen in Figure 5.3(b), the SWF, which accounts for insufficient sampling, shows conservative restoration when it approaches the folding frequency and remains essentially within the sampling passband; however, high boost effect of the CWF (in Figure 5.3(a)) can extend far beyond the sampling passband thereby permitting degradation from high-frequency aliasing artifacts. This difference can also be observed from the combined total throughput MTF as we show in Figure 5.4\(^4\), where the system using the CWF exhibits a very flat response up to the folding frequency, but the system with the SWF acts as a low-pass filter and suppresses the anticipated aliasing.

Assuming that we have an ideal low-pass filter with frequency response

\[
H_p(\omega) = \begin{cases} 
1, & 0 \leq |\omega| < 0.5 \text{ cycles/pixel} \\
0, & \text{otherwise}
\end{cases}
\]  

(5.33)

for image reconstruction (interpolation), we only need to consider the spectral error within the passband frequency range \((0 \leq |\omega| < 0.5, \text{ or equivalently, } 0 \leq \omega < 1.0)\) for performance evaluation. To consider the spectral error, we first look at the system MTF degradation error \((S_d)\) and aliasing error \((S_a)\) as we defined before. These two errors, under different system MTF, are computed and plotted separately in Figure 5.5(a). Obviously, as the bandwidth of the imaging system increases, it also increases the aliasing error, but the MTF degradation error is reduced. If no restoration is applied, the trade-off between the degradation error and aliasing error can be directly taken into consideration in designing an optimal imaging system. But this trade-off would become more difficult to perform if the signal is to be restored for improvement. This difficulty is due to the dependency of the output spectral error on the restoration filter used, as we illustrate below.

\(^3\)Note that the effective region of digital filter implementation is limited to \(0 \leq \omega < 0.5\).

\(^4\)Note that the overall response is shown for an analog filter. For digital filtering, the effective region is limited to \(0 \leq \omega < 0.5\).
To elucidate the actual effects of restoration on the spectral error, the $S_a$ and $S_d$ after restoration are plotted separately for both CWF and SWF in the Figure 5.6(a)-(d). Figure 5.6(a) indicates that the CWF achieves excellent results in restoring the system degradation under various $\omega_c$. The only exception is the one with $\omega_c = 0.4$; its severe degradation on the input, which causes the Wiener filter to have much higher gain at folding frequency (Figure 5.3), makes the restoration more susceptible to noise and left with larger error around folding frequency. It is evident that the system with wider bandwidth has less error, for it causes less degradation and makes restoration more accurate under noisy environment. For the SWF, although the system degradation in the low frequency region was well restored, the errors in the high frequency are much higher than that of the CWF. In addition, contrary to the CWF, the SWF actually performs better as system bandwidth reduces. While these results show that the CWF is better in restoring the system degradation, we need to examine the outcome of aliasing error. In reality, even though the aliasing seems negligible in the original sampled signal (Figure 5.5(a)), they have been amplified by the CWF and become dominant in the restored signal, as is shown in Figure 5.6(c). On the contrary, the aliasing errors are suppressed by the SWF and show counterbalance with the system degradation errors to bring down the total error (Figure 5.5(b), (d)). The total spectral error is the sum of $S_a$ and $S_d$ and is illustrated in Figure 5.7. Comparing this with the original errors in Figure 5.5(b), we see that both filters obtain great improvement in reducing the spectral error within the range of $0 \leq |\omega| < 0.2$. As the frequency goes beyond 0.2, errors of both filters start to increase with the CWF having a higher rate. When they approach the folding frequency, while the SWF still shows improvements, the CWF deteriorates and some systems even have higher error than that of the unrestored signal.

Carefully inspecting the spectral error curves in Figure 5.7 reveals an interesting phenomenon: for any system with $\omega_c \geq 0.6$, as the bandwidth of the system increases, so does the error of the restored signal. This sounds like we can deliberately degrade the input by limiting the bandwidth of an imaging system and then using restoration to recover a better signal from a more degraded one! This is possible, but not unconditionally true; it is due to aliasing and is dependent on both signal, system
bandwidth and the signal-to-noise ratio. In reality, this is actually the same problem
as the trade-off between MTF degradation and aliasing that we mentioned before, and
has been under extensive investigation for designing an optimal end-to-end imaging
system by a group of researchers [44, 47]. For the CWF, if there is no aliasing, then a
system with wider bandwidth has less error, as can be seen from Figure 5.6(a). But
this is not true for the SWF, as can be observed in Figure 5.6(b).

After justifying that under a range of system bandwidths, the SWF always works
better than the CWF does, we would like to examine their performance under differ­
ent input SNR’s. Two different input SNR values—12 dB, 32 dB—with \( \omega_r \) equals 0.1
are used. First, the SNR is decreased to 12 dB. Intuitively, since the SWF suppresses
aliasing, which mostly occurs at high frequencies, it should be less sensitive to noise.
This is because noise is also more significant at high frequencies where the signal
level is usually low. Figure 5.8 justifies this point. Moreover, when the input SNR is
raised to 32 dB, the SWF can still outperform the CWF, as is shown in Figure 5.9.

We might argue that the better restoration obtained by the SWF is probably due
to the bandwidth of input signal. To verify this, an input with a narrower spectrum
(\( \omega_r = 0.02 \)) but the same SNR (22 dB) as that of the first experiment is used for
simulating a much lower level of aliasing. To evaluate the restorability under such a
narrow band signal, errors of the original signal are again shown for comparison in
Figure 5.10. Even for such a narrow-band signal, taking aliasing into account still
gains advantages for the SWF, as is illustrated in Figure 5.11. A comparison between
Figure 5.11 and Figure 5.10(b) reveals that great improvements are obtained in the
low frequency region for both filters. But around the folding frequency, while the
SWF has no harm to the spectrum, the CWF actually increases the error.

Finally, given an existing imaging system, we would like to know the behavior of
both filters under a wide range of input signal bandwidths. In the next computa­
tion, five input signals spectra with \( \omega_r = \{0.2, 0.15, 0.1, 0.05, 0.02\} \), which has been
illustrated in Figure 5.1(a), are used for evaluation. Two imaging systems, one with
a wide bandwidth (\( \omega_c = 1.6 \)) and the other with a narrow bandwidth (\( \omega_c = 0.4 \)), as
are shown in Figure 5.1(b), will be used to study the effect of system bandwidth. In
all the computations, SNR of the input signals are kept at 22 dB. Starting with the
system having $\omega_c = 1.6$, the results are presented in Figure 5.12. Not surprisingly, the SWF obtains better performance for all the input signals. Due to higher aliasing error in the wider band signal, the difference between the CWF and SWF escalates as the input bandwidth increases. When the imaging system has very narrow bandwidth ($\omega_c = 0.4$), the difference in performance of these two filters is approximately the same as that for the wide band system, as is shown in Figure 5.13. Notice that while the narrower system bandwidth ($\omega_c = 0.4$) has higher peak error at folding frequency, its error dispersion has a smaller area. This, again, shows the trade-off between signal degradation and aliasing. That is, when the system bandwidth decreases, so does the aliasing. And the lower level of aliasing narrows the region that is affected when the restoration is performed. But, the overall error level is increased due to the higher degradation of a narrower band system. Although a normal imaging system might not have such a narrow bandwidth ($\omega_c = 0.4$), this can happen when the image suffered severe defocus or motion blurring. Thus, it is deserved to be considered.

In conclusion, under various imaging system parameters, with different signal bandwidth and signal-to-noise ratio, statistically, the SWF always performs better than the CWF does as far as the spectral error is concerned. The investigation also shows that the major differences of the performance between these two filters occur in the high frequency region. Furthermore, the SWF is proved to be more stable and robust than the CWF under all conditions, and this is probably the most important conclusion that we draw from all these comparisons. Therefore, except when the input signal is absolutely band-limited and the sampling frequency is high enough, which is very rare in digital imaging, the SWF should be used for higher spectral fidelity restoration. Although higher spectral fidelity (or lower spectral error) might not correspond to a better visual quality, this study provides us a way to understand the performance of restoration in dealing with degradation and aliasing. In addition, the insights we obtain here will also help us in selecting a suitable restoration scheme to suit our needs in the next chapter.
Figure 5.1: (a) Input signal power spectrum. (b) MTFs with different image formation bandwidths.
Figure 5.2: (a) Power spectrum of input after imaging degradation ($\omega_r = 0.1$). (b) Power spectrum of input after sampling shows aliasing.
Figure 5.3: Wiener filter for corresponding image formation system. (a) Conventional Wiener filter. (b) Sampled Wiener filter.
Figure 5.4: Total system response for corresponding image formation system. (a) Restored conventional Wiener filter. (b) Restored by sampled Wiener filter.
Figure 5.5: Error of blurred image. ($\omega_r = 0.1$, $SNR = 22 \text{ dB}$). (a) Separation of source of error (b) Total error.
Figure 5.6: Comparison of individual error after restoration ($\omega_r = 0.1, SNR = 22 \text{ dB}$). (a) Degradation error of CWF. (b) Degradation error of SWF. (c) Aliasing error of CWF. (d) Aliasing error of SWF.
Figure 5.7: Comparison of total error after restoration ($\omega_r = 0.1$, $SNR = 22\, dB$).  
(a) Restored by conventional Wiener filter. (b) Restored by sampled Wiener filter.
Figure 5.8: Comparison of total error after restoration ($\omega_r = 0.1$, $SNR = 12$ dB). (a) Restored by conventional Wiener filter. (b) Restored by sampled Wiener filter.
Figure 5.9: Comparison of total error after restoration ($\omega_r = 0.1$, $SNR = 32$ dB). (a) Restored by conventional Wiener filter. (b) Restored by sampled Wiener filter.
Figure 5.10: Error of blurred image ($\omega_r = 0.02, SNR = 22 dB$). (a) Separation of source of error (b) Total error.
Figure 5.11: Comparison of total error after restoration ($\omega_r = 0.02$, $SNR = 22 dB$).
(a) Restored by conventional Wiener filter. (b) Restored by sampled Wiener filter.
Figure 5.12: Comparison of total error after restoration with $\omega_c = 1.6$ for various input bandwidth. (a) Restored by conventional Wiener filter. (b) Restored by sampled Wiener filter.
Figure 5.13: Comparison of total error after restoration with $\omega_c = 0.4$ for various input bandwidth. (a) Restored by conventional Wiener filter. (b) Restored by sampled Wiener filter.
CHAPTER 6
IMPROVED SPECTRAL UNMIXING BY IMAGE RESTORATION

From the investigation that we have done in the last chapter, it seems that the SWF is a better choice of restoration scheme for a sampled imaging system. In this chapter, we are going to show that the correctness of this statement is actually dependent on the purpose of the restoration. In the first section, an imaging model with respect to the restoration for improved unmixing is introduced. With the purpose of unmixing, instead of visual appearance, in mind, the second section discusses what an optimal filter for the current application should be. The final section contains a study using the CWF as an suboptimal approach in restoration for improved spectral unmixing.

6.1 Restoration Model for Unmixing Purpose

After a detailed comparison of the CWF and SWF have been done by spectral fidelity criterion for the purpose of improving visual resolution, we would like to investigate a new, somewhat different application of restoration: improved spectral unmixing. In order to apply a correct restoration for this purpose, it is necessary to understand the difference that the new application will bring. As we mentioned before, the digital/digital restoration model (CWF) neglects sampling aliasing and creates large error at high frequency. On the other hand, the continuous/digital/continuous model (SWF), which takes sampling into account, explicitly suppresses the aliasing and obtains better results under all conditions. So, what is the role of aliasing from the perspective of unmixing? Should we neglect or suppress it? These are
the questions that we need to answer before restoration can be applied for improved unmixing.

To answer the above questions, let us look at a simplified imaging model shown in Figure 6.1(a). In this model, we put all degradation factors, as shown in Figure 4.1, except the detector PSF and sampling into a single (image gathering) PSF, that is, \( h_a(x, y) = h_f(x, y) \odot h_e(x, y) \); also, detector cells with square profile are assumed. In the restoration for improved visual resolution, the goal is to process \( w[m, n] \) and make it resemble \( u(x, y) \) as much as possible when it is displayed. But for unmixing purposes, the target signal is not \( u(x, y) \) but \( t[m, n] \), which is generated from an ideal detector as we depict in Figure 6.1(b). Obviously, \( t[m, n] \) is not a signal without degradation. From the aspect of visual resolution, \( t[m, n] \) contains detector IFOV degradation and sampling aliasing. Yet from the viewpoint of unmixing, \( t[m, n] \) is an ideal signal without any flaw. This is because even though aliasing can cause visual artifacts, as long as the signatures of all the objects inside one detector IFOV are all independent, the unmixing analysis can still use multispectral-band information to estimate the proportion for each object. Note that in unmixing, it is not where the object is but how much is present that we are concerned with.

In the following, the same power spectrum fidelity (or spectral fidelity) criterion which used in the last chapter will be used to investigate the necessary modifications that needed to adapt conventional restoration for unmixing. Using the imaging model shown in Figure 6.1, and assuming a bandlimited noise, we have

\[
\begin{align*}
w[m, n] &= \{u(x, y) \odot h_a(x, y) \odot h_d(x, y)\} \lll(x, y) + \eta[m, n] \\
t[m, n] &= \{u(x, y) \odot h_d(x, y)\} \lll(x, y)
\end{align*}
\]

(6.1) (6.2)

These two equations can also be represented by the power spectrum in the frequency domain, that is,

\[
\begin{align*}
S_{ww}(\omega_x, \omega_y) &= \{|H_a(\omega_x, \omega_y)|^2|H_d(\omega_x, \omega_y)|^2S_{uu}(\omega_x, \omega_y)| \odot \lll(\omega_x, \omega_y) \\
S_{tn}(\omega_x, \omega_y) &+ S_{mm}(\omega_x, \omega_y) \\
S_{tt}(\omega_x, \omega_y) &= \{|H_d(\omega_x, \omega_y)|^2S_{uu}(\omega_x, \omega_y)| \odot \lll(\omega_x, \omega_y)
\end{align*}
\]

(6.3) (6.4)
where \(|H_a(\omega_x, \omega_y)|\) and \(|H_d(\omega_x, \omega_y)|\) represent the MTF of the image gathering PSF \((h_a(x, y))\) and detector PSF \((h_d(x, y))\), and \(S_{ww}(\omega_x, \omega_y)\), \(S_{uu}(\omega_x, \omega_y)\), \(S_{tt}(\omega_x, \omega_y)\) and \(S_{\eta\eta}(\omega_x, \omega_y)\) represents the power spectrum of \(w[m, n]\), \(u[m, n]\), \(t[m, n]\) and \(\eta[m, n]\), respectively. Limiting our concern to the sampling passband and dividing the spectrum into two components: signal spectrum and aliased spectrum, we have

\[
S_{ww}(\omega_x, \omega_y) = S_{vv}(\omega_x, \omega_y) + S_{af}(\omega_x, \omega_y) + S_{\eta\eta}(\omega_x, \omega_y)
\]

(6.5)

\[
S_{tt}(\omega_x, \omega_y) = S_{bb}(\omega_x, \omega_y) + S_{at}(\omega_x, \omega_y)
\]

(6.6)

where \(S_{vv}(\omega_x, \omega_y)\) and \(S_{bb}(\omega_x, \omega_y)\) are spectra inside the passband which represent the received (degraded) signal spectrum and the target (ideal) signal spectrum respectively. That is,

\[
S_{vv}(\omega_x, \omega_y) = |H_a(\omega_x, \omega_y)|^2|H_d(\omega_x, \omega_y)|^2S_{uu}(\omega_x, \omega_y)
\]

(6.7)

\[
S_{bb}(\omega_x, \omega_y) = |H_d(\omega_x, \omega_y)|^2S_{uu}(\omega_x, \omega_y)
\]

(6.8)

The other two components, \(S_{af}(\omega_x, \omega_y)\) and \(S_{at}(\omega_x, \omega_y)\), are the aliasing components inside the passband where (in normalized frequency):

\[
S_{af}(\omega_x, \omega_y) = \sum_{m,n \neq 0} |H_a(\omega_x - m, \omega_y - n)|^2|H_d(\omega_x - m, \omega_y - n)|^2S_{uu}(\omega_x - m, \omega_y - n)
\]

(6.9)

\[
0 \leq |\omega_x| < 0.5, \ 0 \leq |\omega_y| < 0.5
\]

is the degraded aliased spectrum and

\[
S_{at}(\omega_x, \omega_y) = \sum_{m,n \neq 0} |H_d(\omega_x - m, \omega_y - n)|^2S_{uu}(\omega_x - m, \omega_y - n)
\]

(6.10)

\[
0 \leq |\omega_x| < 0.5, \ 0 \leq |\omega_y| < 0.5
\]

is the target aliased spectrum. Figure 6.2 contains a model showing the relationship among these four spectra and the input spectrum.

Having defined the degraded and target spectra, the spectral error, \(S_z(\omega_x, \omega_y)\), which is to be reduced by restoration, is taken as the difference between the degraded and target spectra. Considering signal and aliasing separately and limiting
Figure 6.1: Imaging model for (a) real system (b) ideal system for unmixing

Figure 6.2: Simplified model of input signal spectrum ($S_{uu}$), degraded and target signal spectrum ($S_{vv}, S_{bb}$), degraded and target aliased spectrum ($S_{af}, S_{at}$)
the frequency range in the following discussion to \( 0 \leq |\omega_x| < 0.5, 0 \leq |\omega_y| < 0.5 \) we get

\[
S_c(\omega_x, \omega_y) = S_{ww}(\omega_x, \omega_y) - S_{tt}(\omega_x, \omega_y)
\]

\( = (S_{af}(\omega_x, \omega_y) - S_{at}(\omega_x, \omega_y)) + (S_{vv}(\omega_x, \omega_y) - S_{bb}(\omega_x, \omega_y)) + S_{\eta}(\omega_x, \omega_y) \)

\( = \sum_{m,n \neq 0,0} \{ |H_d(\omega_x - m, \omega_y - n)|^2 (|H_a(\omega_x - m, \omega_y - n)|^2 - 1) \}
\cdot S_{uu}(\omega_x - m, \omega_y - n) \}

\( + |H_d(\omega_x, \omega_y)|^2 (|H_a(\omega_x, \omega_y)|^2 - 1) S_{uu}(\omega_x, \omega_y) + S_{\eta}(\omega_x, \omega_y) \) (6.11)

in which the first two items are shown as hatched regions in Figure 6.2. Assuming that the frequency response of the restoration filter is \( G(\omega_x, \omega_y) \), the spectral error after restoration becomes

\[
\tilde{S}_c = |G|^2 S_{ww} - S_{tt}
\]

\( = (|G|^2 S_{af} - S_{at}) + (|G|^2 S_{vv} - S_{bb}) + |G|^2 S_{\eta}
\)

\( = \sum_{m,n \neq 0,0} \{ |H_d(\omega_x - m, \omega_y - n)|^2 (|G(\omega_x, \omega_y)|^2 |H_a(\omega_x - m, \omega_y - n)|^2 - 1) \}
\cdot S_{uu}(\omega_x - m, \omega_y - n) \}

\( + |H_d(\omega_x, \omega_y)|^2 (|G|^2 |H_a|^2 - 1) S_{uu} + |G|^2 S_{\eta} \) (6.12)

Inspection of equation (6.12) suggests our first modification of the restoration filter for the purpose of spectral unmixing. That is, design of the restoration filter, which is to minimize \( \tilde{S}_c \) under a given \( H_d(\omega_x, \omega_y) \), is dependent only on the image gathering MTF, \( H_a(\omega_x, \omega_y) \), not the image system MTF, \( H_a(\omega_x, \omega_y) \cdot H_d(\omega_x, \omega_y) \), as conventional for visual restoration. Therefore, the detector IFOV should not be included in the restoration filter. To distinguish the difference, we call the restoration for visual purposes full restoration and the restoration for unmixing purposes partial restoration. For the other modification, we need to make \( |G(\omega_x, \omega_y)|^2 S_{af}(\omega_x, \omega_y) \) as close to \( S_{at}(\omega_x, \omega_y) \) as possible. That is, we should not suppress \( S_{af}(\omega_x, \omega_y) \), but look at it as signal and restore it from the degradation caused by \( H_a(\omega_x, \omega_y) \).

The above analysis makes us believe that the SWF, which deliberately suppresses the aliased spectrum, is in fact not suitable for unmixing purposes. On the contrary,
the neglect of undersampling in the CWF, which leads to the boost of aliased spectrum, should yield better restoration for improving unmixing accuracy. While we knew the CWF, in this case,

\[ G(\omega_x, \omega_y) = \frac{H_a^*(\omega_x, \omega_y)S_{uu}(\omega_x, \omega_y)}{|H_a(\omega_x, \omega_y)|^2S_{uu}(\omega_x, \omega_y) + S_{\eta\eta}(\omega_x, \omega_y)} \]  

(6.13)

achieves optimal restoration on the passband signal spectrum, \( S_{vv}(\omega_x, \omega_y) \), the performance on the aliased spectrum, \( S_{af}(\omega_x, \omega_y) \), by the same filter is still unknown. In the following sections, an ideal Wiener filter for aliasing restoration will be given and a study will be conducted to investigate the applicability of the CWF in restoring the aliased spectrum.

6.2 Optimal Filter for Aliasing Restoration

Because no attempt has ever been tried to restore aliasing, it is necessary to see how or if this can be done. To get started, we first check equation (6.12) to see what an optimal filter for the aliasing spectrum should be. Because the power spectrum of a sampled image is a periodic replication of the original spectrum, an optimal filter for the original (passband) signal should be also optimal for a replicated (high order sideband) signal centered at any \((m, n)\) or \((m\omega_x, n\omega_y)\), where \(\omega_x\) and \(\omega_y\) are the sampling frequency normalized to 1 at both direction. Therefore, for each replication at \((m, n)\), which may or may not contribute aliasing to the passband, the optimal filter for it is \(G(\omega_x \pm m, \omega_y \pm n)\), where \(G(\omega_x, \omega_y)\) is the CWF defined in equation (6.13).

Therefore, even though the CWF is designed with a neglect of aliasing, it is still optimal if only the signal spectrum and aliased spectrum are separable. To envision this, we first assume that the input spectrum is strictly band-limited and the sampling caused no aliasing. Under this ideal condition, the CWF obtains optimal performance. Then, as we decrease the sampling frequency and generate aliasing, this filter will still be optimal if two conditions are satisfied: (1) aliasing is not an artifact, and (2) the signal spectrum and aliased spectrum are separable. The first condition is not true in visual restoration, but is satisfied in our current application. The second
condition, unfortunately, is not possible for a single given sampled image. However, let us pretend that both conditions are true so that we can formulate the optimal filter in the passband \(0 \leq |\omega_x|, |\omega_y| < 0.5\) as follows:

\[
\begin{align*}
\text{signal spectrum:} & \quad G(\omega_x, \omega_y) \\
\text{aliased spectrum:} & \quad G(\omega_x - m, \omega_y - n), \text{ for all } (m, n)
\end{align*}
\]

(6.14) (6.15)

Apparently, as far as the passband is concerned, the optimal filter for restoring the aliasing is different for each \((m, n)\).

Using equations (6.14) and (6.15), examples of optimal filters for a Gaussian system MTF with \(\omega_c = 0.8\), AR signal with \(\omega_r = 0.1\), and \(SNR = 42\ dB\) are shown in Figure 6.3, where only the nearest sideband aliased spectrum, or \(0 \leq |m|, |n| \leq 1\) are considered. Figure 6.3(a) is the CWF for the passband signal, and Figure 6.3(b), (c) and (d) are the optimal restoration filters for the aliasing inside the passband which is due to the sideband foldover spectrum at \((m, n) = (1, 0), (0, 1)\) and \((1, 1)\) respectively. Evidently, using a single CWF to restore all the aliasing degradation inside the passband is not possible, because each one of them needs a different filter. And, since we cannot separate each aliased component coming from different sideband spectra for individual restoration, a single filter for suboptimal restoration is an indispensable approach. Note that due to the foldover characteristics of aliasing inside the passband, all three filters for aliasing restoration show higher gain at lower frequency and lower gain at higher frequency, which is opposite to the CWF for the passband spectrum.

While the dissimilarity of the optimal filters for \(S_{vv}\) and \(S_{af}\) makes the CWF seem far from ideal for restoring the aliasing, it may still be able to yield a reasonable recovery. The reason is because at the folding frequency, where the level of aliasing is the highest, a CWF for the passband signal actually achieves optimal aliasing restoration. And as the frequency decreases, while the response of a CWF falls off instead of going up as we like it to be, the aliasing value is also less significant for most signals. Figure 6.4 shows an example illustrating this argument using a one-dimensional spectrum. The system parameters used here are the same as that of Figure 6.3. Comparing (a) with (b) in Figure 6.4, not surprisingly, the degraded
passband spectrum is fully restored. The degraded and restored aliased spectra are shown in Figure 6.4(c) and (d). As was mentioned above, the aliasing at the folding frequency is optimally restored. But when the frequency decreases, the error is less significantly reduced. Although no thorough evaluation is given, this simple example does demonstrate qualitatively what a CWF can do for the restoration of degraded aliasing. In the next section, more details on performance assessment of the CWF used as a suboptimal filter for the restoration of two-dimensional signal and aliasing will be given.

6.3 Performance Evaluation of the CWF

Although it might be possible to design a filter which can take both signal spectrum as well as aliasing into account and obtain an optimal restoration for unmixing purposes, currently, we would not make this effort. Instead, we will investigate the capability of CWF in restoring the aliased spectrum and its overall performance on increasing the spectral fidelity. In the following discussion, a separable Gaussian MTF for the image gathering model and an separable power spectrum for the AR signal model, or

\[
P(\omega_x, \omega_y) = e^{-\omega_x^2/\omega_{x,x}^2} \cdot e^{-\omega_y^2/\omega_{y,y}^2}
\]

\[
S_{\text{ss}}(\omega_x, \omega_y) = \frac{A}{1 + (\omega_x/\omega_r)^2} \cdot \frac{A}{1 + (\omega_y/\omega_r)^2}
\]

are used for computation. Examples of these two models are presented in Figures 6.5 and 6.6 (with \(\omega_{x,x} = \omega_{x,y} = \omega_x\)) respectively.

While the power spectrum (error) is convenient for relative performance comparison and evaluation of a restoration filter, this point-by-point (frequency domain) assessment is less suitable for overall quantitative study, especially for multi-dimensional signals. Therefore, signal power, instead of its power spectrum, will be used in the following computation to investigate the performance of a CWF in improving the signal and aliasing fidelity for spectral unmixing. Let's start the investigation with a definition of the power inside the passband, that is,

\[
P_\# \Delta \int_{-0.5}^{+0.5} \int_{-0.5}^{+0.5} S_\#(\omega_x, \omega_y) d\omega_x d\omega_y
\]
Figure 6.3: MTF of the CWF for restoring (a) $S_{vv}$ (b)(c)(d) $S_{af}$ from sideband at (1,0), (0,1) and (1,1).
Figure 6.4: Example showing restoration of (a)(b)$S_{uv}$ and (c)(d)$S_{sf}$ by the CWF.
where \( S_\#(\omega_x, \omega_y) \) represents a power spectrum of \( S_{vv}(\omega_x, \omega_y) \), \( S_{af}(\omega_x, \omega_y) \), \( S_{bb}(\omega_x, \omega_y) \), or \( S_{at}(\omega_x, \omega_y) \), and \( P_\# \) is its corresponding power inside the passband. Passband signal degradation and aliasing power loss resulting from image gathering MTF can then be measured by the difference of power between the target and degraded spectrum, or

\[
\mathcal{E}_s = P_{bb} - P_{vv} \tag{6.19}
\]

\[
\mathcal{E}_a = P_{at} - P_{af} \tag{6.20}
\]

respectively. Restoration by \( G(\omega_x, \omega_y) \) then changes the above two values to

\[
\mathcal{E}'_s = P_{bb} - \int_{-0.5}^{+0.5} \int_{-0.5}^{+0.5} |G(\omega_x, \omega_y)|^2 S_{vv}(\omega_x, \omega_y) \, d\omega_x \, d\omega_y \tag{6.21}
\]

\[
\mathcal{E}'_a = P_{at} - \int_{-0.5}^{+0.5} \int_{-0.5}^{+0.5} |G(\omega_x, \omega_y)|^2 S_{af}(\omega_x, \omega_y) \, d\omega_x \, d\omega_y \tag{6.22}
\]

Notice that these measurements deliberately avoid the inclusion of noise to make the individual evaluation clearer; noise effects will be included when the overall performance is considered.
Figure 6.6: Input $S_{uv}(\omega_x, \omega_y)$ with (a) $\omega_r = 0.1$ (b) $\omega_r = 0.5$. 
Having defined the measurement of degradation before and after restoration, the amount of improvement can then be computed. Two quantities are used for this purpose. The first one, which is defined as

$$\Gamma_s \triangleq 1 - \frac{\varepsilon'_s}{\varepsilon_s}$$
$$\Gamma_a \triangleq 1 - \frac{\varepsilon'_a}{\varepsilon_a}$$

(6.23)
is used to calculate the ratio of reduction in error resulting from restoration for the signal and aliasing degradation independently. This value indicates the effectiveness of a filter in dealing with each individual degradation in a total power sense. The other quantity defined as

$$\Lambda_s \triangleq \frac{\varepsilon_s}{\varepsilon_s + \varepsilon_a}$$
$$\Lambda_a \triangleq \frac{\varepsilon_a}{\varepsilon_s + \varepsilon_a}$$

(6.24)
and

$$\Lambda'_s \triangleq \frac{\varepsilon'_s}{\varepsilon_s + \varepsilon_a}$$
$$\Lambda'_a \triangleq \frac{\varepsilon'_a}{\varepsilon_s + \varepsilon_a}$$

(6.25)
is used to measure the ratio of signal and aliasing degradation with respect to the original total error. These quantities can show the restoration-induced change in the relative significance of the signal and aliasing degradation numerically made by restoration. As we will see later, they make the residue of aliasing error after restoration easier to interpret and reflect the relative significance of both errors before and after restoration.

In order to obtain more insight into the behavior of CWF, various image gathering MTFs and different input signals are used in the computation. As are shown in equations (6.16) and (6.17), parameter $\omega_c$ controls the degradation of the imaging system, $\omega_r$ determines the bandwidth of the input spectrum, and the SNR can be chosen by setting parameter $A$ to a specific value. These three parameters, $\omega_c, \omega_r, A$, are assumed to be the same in the x (cross-track) and y (along-track) directions in the current computation. Six different image gathering MTFs with $\omega_c = \{0.4, 0.5, 0.8, 1.1, 1.4, 1.7\}$ are used for the computation. For each imaging system, the input spectral bandwidth is varied between $0.1 \leq \omega_r \leq 0.5$ under a chosen SNR value, and the performance metrics defined above are computed for evaluation. Notice that for a given input passband signal power (or SNR), a wider band signal
(larger $\omega_r$) has lower $A$ value. That is, for two different bandwidth signals with $\omega_{r1}$ and $\omega_{r2}$, we have

$$\int_0^1 \int_0^1 \frac{A}{1 + (\omega_x/\omega_r)^2} \cdot \frac{A}{1 + (\omega_y/\omega_r)^2} d\omega_x d\omega_y = \text{constant}$$

that is,

$$A_1 \cdot \omega_{r1} \cdot \tan^{-1} \frac{1}{\omega_{r1}} = A_2 \cdot \omega_{r2} \cdot \tan^{-1} \frac{1}{\omega_{r2}}$$  \hspace{1cm} (6.26)

An example for showing this dependency has already been shown in Figure 6.6.

An input signal with $SNR = 30 \text{ dB}$ is used in the first experiment to start our investigation. Figure 6.7 shows the error before and after the restoration. Obviously, the aliasing degradation is dependent on both the imaging system bandwidth and signal bandwidth (Figure 6.7(d)), but the signal degradation is only sensitive to imaging system bandwidth and shows no direct connection to the signal bandwidth (Figure 6.7(b)). The ratio of reduction for both components, $\Gamma_s$ and $\Gamma_a$, are computed and shown in Figure 6.8. Clearly, the degradation in the signal for all the systems is very well restored ($> 99\%$), as is shown in the Figure 6.8(a). On the contrary, Figure 6.8(b) reveals that the aliasing error can only be partly reduced, and it also shows greater variance among different systems in reducing the aliasing error. As the bandwidth of imaging system increases, so is the error reduction ratio; but all the improvements are limited to under 50%. Notice that for both degradations, except for a very narrow band system ($\omega_r$), the improvements are almost independent of the signal bandwidth. Nevertheless, this does not mean that the amount of error is also invariant to signal bandwidth, as we will see from the value of $A'$ below.

The significance of each degradation using the quantities defined in equations (6.24) and (6.25) is also computed and presented in Figure 6.9. Before restoration is applied, the signal degradation is a more significant component, as can be seen by a comparison between Figure 6.9(a) and (c). Especially as the system bandwidth becomes narrower, the discrepancy between the proportion of these two degradations becomes wider. That is, as the imaging system bandwidth decreases, so does the significance of aliasing degradation. Application of restoration then changes the standings of these two degradations, as illustrated in Figure 6.9(b) and (d). Therefore, after restoration, errors of signal degradation is much less important. Instead, the residue error is
all due to aliasing degradation and is proportional to the bandwidth of input signal (Figure 6.9(d)). This proportionality is because a wider band signal has a larger aliasing error but the reduction ratio for aliasing as well as signal, as illustrated in Figure 6.8, is almost constant under different $\omega_r$.

To study the dependency of restoration on the input SNR, input signal level ($A$) is decreased so that SNR reduces to 16 dB. The value of SNR has a more profound effect on the system with a narrower bandwidth, as can be noticed in Figure 6.10. Although values of $A_s$ and $A_a$ are almost the same as those for SNR=30 dB, the restoration does deteriorate the $A_s'$ as we show in Figure 6.11. On the contrary, change of SNR has only a slight effect on the value of $A_a'$.

When the input SNR is increased to 42 dB, both $\Gamma_s$ and $\Gamma_a$ have only slight improvement as compared with that of SNR=30 dB, and are shown in Figure 6.12. The higher SNR also makes the residue of signal degradation become even less important (Figure 6.13(b)) after restoration. Again, values of $A_s'$ remain the same as when the input SNR equals 30 dB. This constancy of $A_s'$ implies that the residue of the aliasing error will be proportional to the signal level. Even though higher SNR can help restoration reduce the signal degradation, it does not do as much to the aliasing degradation. This is a limitation of the CVF and may be improved by redesigning the Wiener filter.

While individual degradation improvement has been given, the overall performance of restoration which includes the effect of noise on the filtering operation has not been discussed yet. In order to achieve this, an overall error reduction ($\Upsilon$) including the noise effect is defined below for final performance evaluation. That is,

$$\Upsilon \triangleq 1 - \frac{\mathcal{E}_s' + \mathcal{E}_a' + \mathcal{N}_o}{\mathcal{E}_s + \mathcal{E}_a + \mathcal{N}_i}$$  \hspace{1cm} (6.27)

where $\mathcal{N}_i$ and $\mathcal{N}_o$ represent the passband noise power before and after restoration. Results based on equation (6.27) for those three different input SNR: 16, 30, 42 dB are computed and presented in Figure 6.14(a), (b) and (c) respectively. Dependency of $\Upsilon$ on the input SNR are apparent only when the SNR is low, as the difference between Figure 6.14(a) and (b) is much high than the difference between Figure 6.14(b) and (c). In general, the overall performance is limited by the aliasing restoration and
therefore shows degradation as the input bandwidth increases. For a medium input bandwidth and reasonable SNR, an 80 – 90\% reduction of power error for improved spectral fidelity is possible.

The results that we obtain above from the perspective of increasing spectral fidelity should also be applicable to the restoration of images for improved unmixing accuracy. This assertion is based on the fact that both applications need high fidelity (similarity) between the restored image and ideal image. This is different from restoration for visual purposes, in which a better result is based on a human visual standard that might not have a higher spectral fidelity. Although the analysis in the spectral domain might not truly reflect the actual amount of the improvement on unmixing, the increase in fidelity that it exhibits will no doubt reduce the error of unmixing caused by the image gathering PSF blurring. In the next chapter, more realistic results using simulated agricultural scenes and direct unmixing analysis will be conducted to evaluate the effectiveness of the restoration approach in improving the unmixing accuracy. Also, the relation between the power error reduction and unmixing error reduction as a function of SNR and $\omega_r$ will be investigated.
Figure 6.7: Power error of (a) degraded signal (b) restored signal (c) degraded aliasing (d) restored aliasing (SNR=30 dB)
Figure 6.8: Error of individual reduction ratio (a) $\Gamma_s$ (b) $\Gamma_a$ (SNR=30 dB)
Figure 6.9: Ratio between the original total error and (a) degraded signal error (b) restored signal error (c) degraded aliasing error (d) restored aliasing error (SNR=30 dB)
Figure 6.10: Error of individual reduction ratio (a) $\Gamma_s$ (b) $\Gamma_a$ (SNR=16 dB)
Figure 6.11: Ratio between the original total error (a) degraded signal error (b) restored signal error (c) degraded aliasing error (d) restored aliasing error (SNR=16 dB)
Figure 6.12: Error of individual reduction ratio (a) $\Gamma_s$ (b) $\Gamma_z$ (SNR=42 dB)
Figure 6.13: Ratio between the original total error (a) degraded signal error (b) restored signal error (c) degraded aliasing error (d) restored aliasing error (SNR=42 dB)
Figure 6.14: The overall error reduction ratio of (a) SNR=16 dB (b) SNR=30 dB (c) SNR=42 dB.
Figure 6.14: (continued)
CHAPTER 7

PERFORMANCE EVALUATION

Although improving the spectral fidelity by image restoration based on an unmixing model has been evaluated in the last chapter, it is the improvement of unmixing results that we are really after. The advantage of the spectral fidelity approach is that the signal and aliasing can be observed separately, so we can understand the limitations of a restoration scheme. In this chapter, in order to evaluate the improvement directly on the unmixing, agricultural type scenes will be used for unmixing analysis. Due to the difficulty of obtaining the ground truth for real scenes, simulated scenes will be used throughout this chapter for performance evaluation. The first section describes the scene modeling procedures that we use for scene generation. The second section presents the results of unmixing improvement by the proposed partial restoration scheme. The third section contains a comparative study to evaluate the similarity between the spectral fidelity improvement and unmixing error reduction. The fourth section conducts a quantitative study of the effect of misregistration on the unmixing results. In the final section, the separability measure ($\xi$) is related to the unmixing error for an examination of the limitation of data characteristics on unmixing accuracy.

7.1 Scene Modeling and Generation

No model can accurately represent all of the complex variations that make up the spectral radiance present at the input of the sensor. However, through the use of various simplifying assumptions, developing such a model becomes a reasonable task. In this section, approaches to modeling the scene variation [48] are discussed.
7.1.1 Reflectance/Radiance Model

A complete procedure for simulating images contains ground reflectance generation, atmospheric effects, and sensor modeling. The most general way of modeling the ground reflectance is to describe it by the Spectral Bidirectional Reflectance Distribution Function (SBRDF). This function is defined [49] as $\beta_\lambda$ in equation (7.1).

$$\beta_\lambda = \frac{dL_\lambda(\theta_v, \phi_v)}{dE_\lambda(\theta_s, \phi_s)}sr^{-1}$$  \hspace{1cm} (7.1)

Where $L_\lambda(\theta_v, \phi_v)$ is the reflected spectral radiance observed at viewing angle $\theta_v, \phi_v$, and $E_\lambda$ is the incident spectral irradiance at solar angles $\theta_s, \phi_s$. The quantities $\theta_s, \theta_v$ are the zenith angles as measured from local vertical, and $\phi_s, \phi_v$ are the azimuthal angles as measured from North.

The SBRDF gives the reflectance of an object from all incidence and view angles and thus is the most complete representation of the surface reflectance. However, the accurate measurement of the SBRDF is a difficult task and few studies have been made. Shibayma [50] and Irons [51] have reported some measurements of aggregate SBRDF, but only for limited crop species and over a few wavelength intervals. While the use of the measured SBRDF is the most complete way of representing the reflectance, it is impractical because of the difficulty in obtaining complete and accurate measurements for various cover types.

Strahler [52] discussed modeling of the scene for land resource remote sensing applications and divided surface models into two types: deterministic canopy models and stochastic image processing models. The term canopy comes about because these models attempt to calculate the SBRDF of vegetation by using radiative transport theory. Differential equations are used to compute the reflectance and transmittance of the several layers of leaves in a vegetative canopy.

Some examples of canopy models are the AGR model [53], the Suits model [54] with extensions for azimuthal [55] and row effects [56], the SAIL model [57], and the models by Park [58], Cooper [59], and Kimes [60]. All of these models are based upon having precise knowledge of the reflectance, transmittance, and orientation of
the leaves in each layer of the canopy. A model that used probability distributions for the orientations of the layers was described in Smith and Oliver [61].

All of these canopy models, however, only consider the reflectance within a single surface cell, assuming the entire area covered by a particular surface type is homogeneous and with no regard to the spatial variability typical of remotely sensed scenes. While they are capable of accurately modeling the SBRDF of a particular surface material, their lack of spatial information limits their applicability for our current study. However, it certainly would be conceivable, if one had the appropriate data to extend a canopy model to include spatial information and thereby develop a accurate surface reflectance model. Unfortunately, this type of detailed database does not exist at the present time.

Image processing models, on the other hand, are not concerned with the reflectance structure within a scene resolution cell, but rather how the radiance vary spatially and spectrally from cell to cell. In these models, the spectral radiance of a surface area is taken to consist of multidimensional (across the spectral domain) random vectors with spectral and spatial correlation. Also, the radiance within each cell is assumed to be independent of illumination or viewing angle. This is known as Lambertian reflectance [49].

In the use of image processing models, two assumptions are usually made about the spectral and spatial variation in the scene. The multispectral radiance vectors are usually assumed to be samples from an m-dimensional multivariate Gaussian probability distribution function. The form of this distribution is shown in equation (7.2).

\[ p(r_1, r_2, \ldots, r_m) = \frac{1}{\sqrt{2\pi} \sqrt{|\Sigma|}} \exp \left\{ -\frac{1}{2} (r - \mu_r)^t \Sigma^{-1} (r - \mu_r) \right\} \]  (7.2)

where \( r = (r_1, r_2, \ldots, r_m) \) is data vector, \( \mu_r \) is the mean vector, and \( \Sigma \) is the covariance matrix. The work that is often cited in justifying this Gaussian assumption is that of Crane [62].

Remotely sensed images have also been shown to have a pixel-to-pixel spatial correlation. Kettig [63] used this fact in development of the ECHO spatial classifier. Also, Mobasseri [64] developed a multispectral spatial model that was a separable
(cross and along track directions) exponential model. This model is specified by its spatial autocorrelation function $R_{jj}(\tau_x, \tau_y)$ for the $j$th band scene radiance $r_j$, or

$$E\{r_j(x + \tau_x, y + \tau_y)r_j(x, y)\} = R_{jj}(\tau_x, \tau_y) = e^{-a_j|\tau_x|}e^{-b_j|\tau_y|} \quad \text{(7.3)}$$

where $a_j$ and $b_j$ are the cross track (x) and along track (y) correlation parameters for band $j$, and $\tau_x, \tau_y$ are the respective scene cell lag values. The coordinates $(x, y)$ are the scene cell location.

Written in terms of the autocorrelation coefficients which are assumed to be constant over different spectral bands, $\rho_x = e^{a_j}$ and $\rho_y = e^{b_j}$, equation (7.3) becomes

$$R_{jj}(\tau_x, \tau_y) = \rho_x^{-|\tau_x|}\rho_y^{-|\tau_y|} \quad \text{(7.4)}$$

This form of autocorrelation for a random field is equivalent to that of a wide-sense Markov random field with the neighbor set consisting of the quarterplane causal neighbors, $\{(0,-1), (-1,0), (-1,-1)\}$ [65]. This is also equivalent to a two-dimensional autoregressive (AR) model [66] as given by

$$r(x, y) = c_1r(x - 1, y) + c_2r(x, y - 1) + c_3r(x - 1, y - 1) + \sigma_u z(x, y) \quad \text{(7.5)}$$

Where,

- $c_1 = \rho_x$
- $c_2 = \rho_y$
- $c_3 = -\rho_x\rho_y$
- $\sigma_u$: standard deviation of Gaussian driving process
- $z(x, y)$: Gaussian random numbers with unit variance and zero mean

Given arbitrary initial conditions, the AR model can generate a radiance array with the desired spatial correlation. Throughout this chapter, this spatial model will be used for scene generation.

### 7.1.2 Procedure of Image Generation

In order to focus on the issues of sensor spatial degradation and image restoration, both atmospheric effects and signal unit conversion inside the sensor are neglected in the simulation procedure. That is, instead of creating a surface reflectance array of
1. Statistics Estimate

- Input a template image containing all the classes of interest.
- Specify areas of each class for statistics computation.
- Obtain spatial correlation coefficients, mean vector, and covariance matrix of each class from chosen areas.

2. Spatial Configuration

- Define input full scene size.
- Generate positions of straight-line class boundaries randomly.
- Assign one class for each region (field) surrounded by class boundaries.

3. For each field, using AR model to generate spatial correlated image for each band with each array being spectrally uncorrelated.

4. Transform each spectral vector to (statistically) have the proper mean vector and covariance matrix for the corresponding class.

Figure 7.1: Sequence in generating a simulated AR image array.

the ground or a radiance map in front of the sensor, a digital image with higher spatial resolution than that of the sensor is generated. Because we are primarily interested in spatial resolution, this simplification can be justified. Although the signal conversion process is then neglected, the radiometric degradation due to conversion (reflectance $\rightarrow$ irradiance $\rightarrow$ photons $\rightarrow$ electrons $\rightarrow$ digital number) will be accounted for by adding noise. Base on the above assumptions, the input simulated scene can be generated by the sequence shown in Figure 7.1. Details on each step will be discussed in the rest of this section.

In step 1, statistics data for each class used in the simulation can be obtained from available remotely sensed images. For each chosen material (class $i$), these data include mean vector ($\mu_i$), covariance matrix ($\Sigma_i$), and correlation coefficients in cross and along track directions($\rho_{i,x}, \rho_{i,y}$). In step 2, after the scene size is chosen, the fields inside are assumed to be contiguous rectangles whose sides are parallel to some axes
(\(x', y'\)). The transitions along each axis obey the Poisson probability density function with a mean separation (\(\lambda_r\)) for controlling signal bandwidth. Each rectangular field is then randomly assigned one of the classes under consideration. The third step is the AR scene generation for each field which is now designated to be a specific class. Here, the correlation coefficients computed in step 1 are used in equation (7.5). Parameter \(\sigma_u\) in this equation is chosen to make the scene have unit variance. For an \(m\) band sensor, this step creates \(m\) spatially-correlated and spectrally-independent image arrays.

The last step is to convert the arrays generated, which have zero mean vectors and identity covariance matrices for all classes, to have the targeted mean vectors and covariance matrices obtained in step 1. In order to do this, pixels at \((x, y)\) from each individual band are arranged as a spectral vector, or

\[
\mathbf{r}(x, y) = [r_1(x, y), r_2(x, y), \ldots, r_m(x, y)]^t
\]

For a given class \(i, 1 \leq i \leq n\), eigenvalues and eigenvectors of the targeted covariance matrix are computed and arranged as diagonal matrix \(\Lambda_i\) and column matrix \(\Phi_i\) respectively. Because the covariance matrix is real and symmetric, it then can be decomposed \([17]\) as

\[
\Sigma_i = \Phi_i \Lambda_i \Phi_i^t = (\Phi_i \Lambda_i^{\frac{1}{2}})(\Phi_i \Lambda_i^{\frac{1}{2}})^t
\]

in which \(\Phi_i\) is an orthonormal matrix\(^1\). Since \(\sigma_u\) in equation (7.5) was chosen to produce unit variance and there is no correlation between different bands, we have \(E\{r(x, y)r'(x, y)\} = I\), where \(I\) is identity matrix. Using this fact and assuming \(r'(x, y)\) is the spectral vector of the converted image arrays, equation (7.6) can then be rewritten as:

\[
\Sigma_i = E\{[\Phi_i \Lambda_i^{\frac{1}{2}} \mathbf{r}(x, y)][\Phi_i \Lambda_i^{\frac{1}{2}} \mathbf{r}(x, y)]^t\} \quad (7.7)
\]

\[
= E\{[\mathbf{r}'(x, y) - \mu_i][\mathbf{r}'(x, y) - \mu_i]^t\} \quad (7.8)
\]

Therefore, spectral vector transformation with:

\[
\mathbf{r}'(x, y) = \Phi_i \Lambda_i^{\frac{1}{2}} \mathbf{r}(x, y) + \mu_i \quad (7.9)
\]

\(^1\)Basically, this is the same as principal component analysis.
will result in a set of image arrays which are multivariate Gaussian and whose mean vectors and covariance matrices of all classes are equal to those of the targeted class statistics.

Does the transformation in equation (7.9) affect the spatial correlation existing in $r(x, y)$? The answer is no as we show below. Firstly, the correlation coefficient of the $j$th band for a given class in the original array (zero mean) is defined as:

$$\rho_j(x, y) = \frac{E[r_j(x, y)r_j(x + x_r, y + y_r)]}{E[r_j(x, y)r_j(x, y)]}$$

(7.10)

To compute the correlation coefficient of the $j$th band for the same class in the transformed array ($\rho'_j$), we assume the mean vector, eigenvalue diagonal matrix, and eigenvector column matrix for the chosen class are $\mu$, $\Lambda$, and $\Phi$ respectively. If we also define

$$\rho'_j(x, y) \triangleq \frac{E[[r'_j(x, y) - \mu_j][r'_j(x + x_r, y + y_r) - \mu_j]]}{E[[r'_j(x, y) - \mu_j][r'_j(x, y) - \mu_j]]}$$

(7.11)

then $\rho'_j(x, y)$ can be derived indirectly from:

$$E[[r'(x, y) - \mu]'[r'(x + x_r, y + y_r) - \mu]]$$

(7.12)

Using transformation equation (7.9), (7.12) is equivalent to:

$$E[[\Phi^{1/2} \mathbf{r}(x, y)][\Phi^{1/2} \mathbf{r}(x + x_r, y + y_r)]]$$

$$= E[[\mathbf{r}'(x, y)\Lambda^{1/2}\Phi\Phi^{1/2} \mathbf{r}(x + x_r, y + y_r)]$$

$$= E[[\mathbf{r}'(x, y)\Lambda \mathbf{r}(x + x_r, y + y_r)]]$$

(7.13)

Here, because $\Phi$ is an orthonormal matrix, $\Phi^t\Phi = \mathbf{I}$. Comp equation (7.13), the spatial correlation in the $j$th band is equal to

$$\lambda_j \cdot E[r_j(x, y)r_j(x + x_r, y + y_r)]$$

(7.14)

where $\lambda_j$ is the $j$th component of diagonal matrix $\Lambda$. From definition, the correlation coefficient of the $j$th band for any given class in the transformed array becomes

$$\rho'_j(x, y) = \frac{\lambda_j \cdot E[r_j(x, y)r_j(x + x_r, y + y_r)]}{\lambda_j \cdot E[r_j(x, y)r_j(x, y)]}$$

(7.15)
Table 7.1: Correlation Coefficients of Exemplar Crops.

Comparing equation (7.15) with (7.10), we have proven that \( \rho_j = \rho_j' \) for \( 1 \leq j \leq m \); that is, the transformation in equation (7.9) does not change the spatial correlation of the original spectrally-uncorrelated array. Therefore, the procedure can now be used to generate a multispectral Gaussian distribution image with the given spatial correlation coefficients, mean vectors and covariance matrices for each corresponding class.

### 7.1.3 Generation of Simulated Images

In this section, an image which contains several types of crop with known ground truth is used as a template image for computation of statistics data for image generation. This template image is a segment (169 lines x 169 pixels x 6 bands) of a Thematic Mapper (TM) scene of Tippecanoe County, Indiana gathered on July 1, 1988. Because the ground truth for this image is available, three types of crop are chosen for statistics computation and scene generation: soybean, corn, and wheat. The computed correlation coefficients and mean vectors are shown in Table 7.1 and Table 7.2; Tables 7.3 contain the covariance matrices for these three crops. The separability (\( \xi \)), defined in equation (3.2), is also computed for these statistics, and the value is 30. The scene size used in all the simulations is 512 x 512 pixels x 3 bands with a cell size (resolution) of 7.5m x 7.5m, in which only bands 4, 5, and 7 are generated. Figure 7.2 shows an example simulated scene with \( \lambda_r \) (mean separation distance of field) equal 375m/side (or 50 resolution cells/side).

The simulated scene is then used to create two 8-bit images: a real image and an ideal image; the procedure is shown in Figure 7.3. The real image, which represents
<table>
<thead>
<tr>
<th>Crop Type</th>
<th>Band 1</th>
<th>Band 2</th>
<th>Band 3</th>
<th>Band 4</th>
<th>Band 5</th>
<th>Band 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Soybean</td>
<td>85.0</td>
<td>34.2</td>
<td>28.6</td>
<td>143.4</td>
<td>121.3</td>
<td>37.9</td>
</tr>
<tr>
<td>Corn</td>
<td>85.4</td>
<td>32.9</td>
<td>29.4</td>
<td>130.0</td>
<td>74.6</td>
<td>20.0</td>
</tr>
<tr>
<td>Wheat</td>
<td>95.5</td>
<td>42.8</td>
<td>47.8</td>
<td>93.9</td>
<td>131.7</td>
<td>50.7</td>
</tr>
</tbody>
</table>

Table 7.2: Mean Vectors of Exemplar Crops.

<table>
<thead>
<tr>
<th>Crop Type</th>
<th>Band 1</th>
<th>Band 2</th>
<th>Band 3</th>
<th>Band 4</th>
<th>Band 5</th>
<th>Band 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Soybean</td>
<td>3.57</td>
<td>0.97</td>
<td>1.86</td>
<td>-2.70</td>
<td>4.09</td>
<td>2.92</td>
</tr>
<tr>
<td>Corn</td>
<td>2.09</td>
<td>0.23</td>
<td>0.13</td>
<td>0.64</td>
<td>0.22</td>
<td>-0.02</td>
</tr>
<tr>
<td>Wheat</td>
<td>2.85</td>
<td>0.79</td>
<td>5.76</td>
<td>-4.45</td>
<td>6.16</td>
<td>3.74</td>
</tr>
</tbody>
</table>

Table 7.3: Covariance Matrices of Three Exemplar Crops.
the actual image that the sensor will produce, is generated by convolving the scene with the TM system PSF and subsampling the output by a factor of 4 to give 30 m pixels. The subsampled output is then added to Gaussian random noise with a specified power level \( \sigma_n^2 \) for a chosen SNR. The *ideal* image, which is the goal of our restoration, is created by the same procedure as the real image, except the PSF of the detector IFOV is used as the system PSF and no noise is added (an ideal sensor system). Both output images have a size of 128 x 128 x 3 with a resolution of 30 m/pixel. Examples of the ideal image and real images with various SNRs using Figure 7.2 as the input scene, are shown in Figure 7.4.

### 7.2 Performance Evaluation of Restoration for Improved Unmixing

In all the simulations, the proposed partial restoration with a Wiener filter is used to test the feasibility of our approach. The image gathering PSF, \( h_a \), of TM is presented in Figure 7.6, which is derived from the system PSF shown in Figure 7.5. The filtering operation is implemented in the frequency domain and the partial Wiener filter, \( G_p(\omega_x, \omega_y) \), which contains no detector IFOV component is formulated as:

\[
G_p(\omega_x, \omega_y) = \frac{H_a^*(\omega_x, \omega_y)S_{uu}(\omega_x, \omega_y)}{|H_a(\omega_x, \omega_y)|^2S_{uu}(\omega_x, \omega_y) + S_{\eta\eta}(\omega_x, \omega_y)}
= \frac{1}{H_a(\omega_x, \omega_y)} \cdot \frac{1}{1 + \frac{S_{\eta\eta}(\omega_x, \omega_y)}{|H_a(\omega_x, \omega_y)|^2S_{uu}(\omega_x, \omega_y)}}
\]  

(7.16)

Without assuming knowledge of the input signal power spectrum, \( S_{uu}(\omega_x, \omega_y) \), this filter is actually implemented as:

\[
G_p(\omega_x, \omega_y) \approx \frac{1}{H_a(\omega_x, \omega_y)} \cdot \frac{1}{1 + \sigma_n^2 |W(\omega_x, \omega_y)|^2}
\]  

(7.17)

where \( W(\omega_x, \omega_y) \) is Fourier transform of the output image \( w(x, y) \) (as seen in equation (6.1), and \( |W(\omega_x, \omega_y)|^2 \) is used to approximate \( |H_a(\omega_x, \omega_y)|^2S_{uu}(\omega_x, \omega_y) \). Furthermore, the noise power spectrum, \( S_{\eta\eta}(\omega_x, \omega_y) \), is assumed to be constant and equal \( \sigma_n^2 \). The advantage of this approximation is that, only the noise power level, \( \sigma_n^2 \), and the image gathering TF, \( H_a(\omega_x, \omega_y) \), are assumed to be known.
Figure 7.2: Image generated by AR modeling procedure presented in Figure 7.1 with $\lambda_r = 375\text{m}$. Only band 4 is shown. Modeling parameters are taken from Tables 7.1-7.3 of bands 4, 5, and 7.
Figure 7.3: Flow diagram showing image generation for performance evaluation of improved spectral unmixing by restoration.
Figure 7.4: Simulated sensor output image (band 4) from input scene Figure 7.2: (a) ideal-sensed image (b) (c) (d) real images simulated using TM PSF with SNR equals 40dB, 20dB, and 6dB, respectively.
7.2.1 Notation of Unmixing Error

Using the CLS algorithm, fraction images which contain proportions of a specific class are computed for the real, ideal, and restored images, which are shown in Figure 7.3 as $\hat{X}_{\text{real}}$, $\hat{X}_{\text{ideal}}$, and $\hat{X}_{\text{rest}}$ respectively. Here, vector $\hat{X} = [\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_n]$ contains the estimated proportion of each class in a certain pixel. Because we have “ground truth” (x) fraction images for each simulated scene, the unmixing error for these three fraction images can be obtained. They are, the unmixing error ($e_b$) before restoration, which is the error in the real (blurred) image; the error caused by the data variance ($e_v$), which is the error in the ideal image; and the error of the restored image ($e_r$), which is the error after restoration has been applied to the real image.

Unmixing error for each class inside one pixel is computed as the absolute value of the difference between the estimated and true abundances, $|x - \hat{x}|$, where $x_i$ is the $i$th band (component) of $X_{\text{real}}$, $X_{\text{ideal}}$, or $X_{\text{rest}}$. Following this definition, the total unmixing error, in pixels, is then accumulated for all the classes and all the pixels. Under current simulation settings, they are:

$$
e_b = \frac{1}{2} \sum_{j=1}^{128} \sum_{k=1}^{128} \sum_{i=1}^{3} |x_{i,j,k} - \hat{x}_{i,\text{real}(j,k)}| \\ e_v = \frac{1}{2} \sum_{j=1}^{128} \sum_{k=1}^{128} \sum_{i=1}^{3} |x_{i,j,k} - \hat{x}_{i,\text{ideal}(j,k)}| \\ e_r = \frac{1}{2} \sum_{j=1}^{128} \sum_{k=1}^{128} \sum_{i=1}^{3} |x_{i,j,k} - \hat{x}_{i,\text{rest}(j,k)}|$$

(7.18)  
(7.19)  
(7.20)

where index $i$ represents class, and $(j, k)$ specifies the pixel position. All the error estimates in the simulation are then computed by a 10 times ensemble average over different noise simulation to reduce the bias of random noise generation.

From these errors, we define the total error reduction ratio (TER) as

$$\text{TER} = \frac{(e_b - e_r)}{e_b} \cdot 100 \, \text{(%)}$$

which describes the overall reduction of unmixing error achieved by the restoration. If TER > 0, then the unmixing error is reduced by the restoration; otherwise, $e_r > e_b$, and the restoration increases the unmixing error.
7.2.2 Objective (Quantitative) Evaluation

In this subsection, four experiments were conducted for the study of unmixing improvements by restoration. Image size and class members specified in section 7.1.3 are used in all the simulations. In the following, the unit of unmixing error, pixel, represents TM pixel or an area of 30m x 30m.

1) Average field size of \( x = 375 \text{ m/side} \) with signal-to-noise ratio (SNR)\(^2\) of 30 dB in all three bands for four different separability values\(^3\), \( \xi = 10, 20, 30, 40 \).

Based on an absolute unit (pixel), this experiment is used to examine the error caused by different data separabilities and to evaluate the improvements obtained using restoration. The total unmixing errors for the ideal, real and restored images are compared in Figure 7.7. The error of the ideal image represents a lower limit for this unmixing algorithm, which is set by the separability. As the separability increases, the error of all three images decreases, and the improvement gained by restoration increases (in number of pixels).

2) Same as 1) with six different SNR (in dB): 40, 30, 20, 10, 6, 0.

To see how effective the restoration is under various class separabilities and SNR’s, the TERR is computed under six different SNR and four \( \xi \); the results are shown in Figure 7.8. Obviously, under various SNR, when the separability increases so does the TERR. Although the restoration actually degrades the results when the noise level is high, it always improves the results under reasonable conditions, i.e. SNR \( \geq 6 \text{ dB} \).

When the separability increases, the proportion of unmixing error caused by PSF blurring \( (e_b) \) becomes more significant with respect to \( e_v \); this makes the restoration approach more effective. The restoration approach is clearly most effective when the SNR and separability are both high.

---

\(^2\)The SNR is defined as the a.c. power ratio of output noiseless image and noise.

\(^3\)The value of \( \xi \) is changed by modifying the mean vector of each class. Therefore, the materials do not represent the original three classes.
3) Eight different average field sizes—15, 25, 37.5, 75, 112.5, 150, 375, 750 m/side with \( \xi = 30 \) and SNR=30 dB.

This experiment is used to assess the dependence of restoration on field size (scene complexity) and to evaluate the significance of unmixing error caused by PSF blurring. The error of ideal images, which depends only on separability, is almost constant for all different field sizes (Figure 7.9(a)) and amounts to only about 3.5% of all the pixels in the image. However, the error of real images increases as the scene becomes more complex (smaller field size). This is because when the scene becomes more complex, unmixing error caused by PSF blurring starts to dominate, and the percentage of error increases to 6.4% ~ 30.7%. The increase of unmixing error with scene complexity continues until the field size is less than a pixel. This decrease of error when field size becomes smaller than one pixel is because under this condition every pixel is a mixed pixel, therefore, the blurring is more likely to include some of the same class material into the pixel undergoing unmixing from its surroundings and makes PSF-blurring error slightly lower. The restoration helps decrease the error for all field sizes, which brings the error percentage down to 4.3% ~ 15.4%.

4) Same as 3) with six different SNR (in dB): 40, 30, 20, 10, 6, 0.

To further evaluate the performance of restoration under different field size and system noise level (or SNR), two measurements were conducted. One is comparison on unmixing error in percentage of the total pixel number; the other is the TERR. For this purpose, eight different field sizes and six different SNR are used. The total unmixing errors in the real image for all conditions are computed and presented in Figure 7.9(b). Partial restoration by Winer filter successfully reduces the amount of error, as shown in Figure 7.10(a). Errors of the ideal image, which is the goal of the restoration, is presented in Figure 7.10(b) for reference. Based on the results in Figure 7.9(b) and Figure 7.10(a), the TERR's are computed and shown in Figure 7.11(a). When the SNR is higher than 10 dB, the TERR increases as the scene becomes more complex until the field size is smaller than 75 m/side (about 2.5 TM pixels). Under fairly low SNR conditions (≤ 10 dB), Figure 7.11(a) shows that the Wiener filter can still improve the unmixing results. This is an advantage of the Wiener filter;
that is, under high SNR, it acts as a high-boost filter and reduces the error caused by blurring. On the other hand, when the SNR is low, the Wiener filter actually becomes a smoothing filter and reduces the error caused by noise. This self-adjusting capability of the Wiener filter in our partial restoration scheme is quite attractive. Because the range of field sizes used here is representative of remote sensing imagery, we contend that image restoration is beneficial for all kinds of scenes under a wide variety of SNR as far as unmixing is concerned.

For comparison, the same images used for partial restoration to obtain the TERR in Figure 7.11(a) are also restored by full restoration and the TERR are computed and shown in Figure 7.11(b). Obviously, the partial restoration obtains better results under all conditions. In contrast, full restoration, which takes all the degradation factors into account, can provide a visually sharper image than partial restoration, as we demonstrate next.

Figure 7.5: System PSF of TM (all three bands are assumed to be the same).
Figure 7.6: Image gathering PSF of TM.

Figure 7.7: Unmixing errors for ideal, real, and restored images under different separabilities ($\lambda_s = 375$ m/side, SNR = 30 dB, total number of pixel: 14400).
7.2.3 Subjective (Visual) Evaluation

Improvement in unmixing error can also be observed visually. In Figure 7.12, we show an ideal image, (a), with $\lambda_r = 375\text{m}$ which is accompanied by a corresponding real image (b), fully-restored image (c), and partially-restored image (d). The real image has an SNR of 30 dB. Because TM has a fairly small PSF kernel, the differences among these four images are not significant. But, visually, the fully-restored image appears to be the sharpest; it is even sharper than the ideal image. The reason for this is because full restoration, which is a traditional restoration approach, takes detector IFOV blurring into account and creates the most appealing result to human eyes. On the other hand, while the result of our partial restoration approach does not look as sharp as that of full restoration, it gives a better performance from the viewpoint of unmixing. This has been quantitatively demonstrated in the last subsection.

The CLS algorithm is applied to the above four images in Figure 7.12 for unmixing, and the unmixing errors for each class are shown in Figure 7.13, 7.14, and 7.15.
Figure 7.9: (a) Unmixing errors for ideal, real, and restored images under different \( \lambda_r \) (\( \xi = 30 \), \( \text{SNR} = 30 \) dB, total number of pixel: 14400). (b) Unmixing error of real image in percentage of total pixel number.
Figure 7.10: Unmixing error in percentage of total pixel number under different field size and SNR. (a) restored image (b) ideal image.
Figure 7.11: The TERR under different $\lambda_r$ and SNR with (a) partial restoration (b) full restoration.
for class 1 (soybean), class 2 (corn), and class 3 (wheat), respectively. Brightness of pixels shown in these figures is proportional to unmixing error. In these three figures, (a) shows the unmixing error of the ideal image, and (b)-(d) show the error of real (blurred), fully-restored, and partially-restored image, respectively. Without system PSF blurring, most of the errors occur inside the fields due to within-class variance ((a)). Because soybean has the highest data variance (see Table 7.3, bands 4, 5, and 7), unmixing errors in Figure 7.13(a) are apparently higher than those in Figure 7.14(a) and Figure 7.15(a). In the real image, PSF blurring introduces errors on the class boundaries, as clearly shown in (b) of these three figures. With our partial restoration approach, the errors around boundaries are greatly reduced ((d)), which in this case gives 47% of TERR; however, a visually favorable fully-restored image, with only 17% of TERR, is still left with unmixing errors on the class boundaries ((c)).

The above presentation visually shows the usefulness of restoration in reducing the unmixing error caused by PSF blurring. Even a full restoration can reduce the unmixing errors on the class boundaries. But, for better improvement a correct restoration scheme—partial restoration—should be used.

7.3 Spectral Fidelity and Unmixing Error

At the end of the previous chapter, we stated that the results obtained from the perspective of increasing spectral fidelity by restoration can be applied to the restoration of images for improved unmixing accuracy. Now since we have simulated scenes available for unmixing evaluation, this statement will be verified for its correctness. In this section, we will compare these two criteria for restoration performance using the two different signal formats discussed previously: AR power spectrum model (in spectral error) and field partition/AR signal generating model (in unmixing error). While spectral error is directly related to restoration scheme, unmixing error is further dependent on the unmixing algorithm. Therefore, a qualitative contrast between these two error measures will be used instead of a precise quantitative comparison.
Figure 7.12: (a) Ideal image (b) real image (c) fully-restored of real image (d) partially-restored of real image ($\lambda_e = 375$ m/side, SNR = 30 dB, band 4).
Figure 7.13: Unmixing error of soybean with (a) ideal image (b) real image (c) fully-restored image (d) partially-restored image ($\lambda_s = 375$ m/side, SNR = 30 dB).
Figure 7.14: Unmixing error of corn with (a) ideal image (b) real image (c) fully-restored image (d) partially-restored image ($\lambda_r = 375$ m/side, SNR = 30 dB).
Figure 7.15: Unmixing error of wheat with (a) ideal image (b) real image (c) fully-restored image (d) partially-restored image ($\lambda_x = 375$ m/side, SNR = 30 dB).
The spectral error, which is due to signal-blurred degradation and noise, can be used as a direct indicator of restoration results. For unmixing error, in addition to these two error factors, there is one other error caused by intrinsic data variance. In the last section, the total error reduction ratio (TERR), which includes all three error sources, is used to account for the overall restoration performance. But for a comparison with spectral error, we need to define a new performance measure which will avoid the influence of data variance. To do this, we assume that the unmixing error caused by data variance is not changed by restoration. That is, the original error resulting from PSF blurring and noise is equal to \( e_b - e_v \), and the restoration reduces that amount to \( e_r - e_v \). Basing on this assumption, we define the error reduction ratio (ERR) due to restoration as

\[
ERR \triangleq \left\{ 1 - \frac{e_r - e_v}{e_b - e_v} \right\} \cdot 100 \% 
\]

and compare it with the overall spectral error reduction ratio \( \Upsilon \) that we defined in equation (6.27).

### 7.3.1 System Parameters and Signal Bandwidth

Because the parameters for specifying imaging system MTF and signal bandwidth are different in spectral error and unmixing error studies, some clarification needs to be done before comparison can be made.

Firstly, comparing the image gathering MTF of TM in Figure 4.6 with a Gaussian MTF model in Figure 6.5, we find that the TM system can be approximately modeled by \( \omega_{c,x} = 0.6 \) (cross-track), \( \omega_{c,y} = 0.8 \) (along-track) in the equation (6.16). Therefore, if TM is chosen as system model for unmixing assessment, the above two \( \omega_c \) values should be used in the Gaussian MTF model for use in spectral error computation.

Secondly, in the study of spectral error, the input signal is modeled in the frequency domain as the AR power spectrum specified by parameter \( \omega_r \), as shown in equation (6.17). In the simulated scene that we generated, an AR model is also used for field composition in the spatial domain, but the signal bandwidth is also dependent on the field size parameter \( \lambda_r \). In order to examine their relationship, power spectra of the simulated scene with various \( \lambda_r \) are plotted in one-dimension and fitted
by the AR spectrum model. The results are shown in Figure 7.16(a)-(h) in which the d.c. values are normalized. The following performance criterion study are based on these eight pairs of \((\lambda_r, \omega_r)\).

### 7.3.2 Qualitative Comparison

Since the system MTF parameters are equal and common signal bandwidths are determined, the performance of partial Wiener filter in reducing the spectral error and unmixing error can be compared. Using the error reduction ratio as defined above, the comparative results are shown in Figures 7.17(a) and (b).

Inspection of these two figures reveals some common characteristics in these two measures of error. The first is that both ERR are lower when the signal bandwidth increases. This is because higher aliasing content from the wider bandwidth makes the restoration less satisfactory. Comparison also shows that ERR of unmixing error has a sharper falling rate. The second point is that, in most cases, a better ERR in spectral error does imply a better ERR in unmixing error. Because direct unmixing performance measurement is a time consuming process, restoration evaluation can benefit from this coherence. That is, study of restoration filter performance for improved unmixing can be conducted from the perspective of spectral error instead of unmixing error.

Due to the simplicity of signal/system modeling and error measurement from the aspect of spectral fidelity, the ERR of different system MTFs with different input bandwidths can easily be appraised and compared. For any given system and input signal models, this indirect measurement would make a relative comparison of different filtering schemes possible even before an unmixing algorithm is applied. Although there is no exact relation to link these two ERR's, we can infer from the above results that as far as partial restoration is concerned, a better filter for spectral fidelity improvement is also a better filter for unmixing accuracy. Therefore, we claim that the spectral fidelity criterion is an expedient approach in the development of restoration filters for improved spectral unmixing.
Figure 7.16: Matching of field size $\lambda_r$ to AR model parameter $\omega_r$. 
Figure 7.16: (continued)
Figure 7.17: Comparison of spectral error reduction ratio and unmixing error reduction ratio.
7.4  Quantitative Evaluation of Effect of Misregistration on Spectral Unmixing

In section 3.3, while band-to-band misregistration was shown to create more mixed pixels from the perspective of spectral signature distribution, no evaluation was done to quantify its effect on unmixing accuracy. If classification is used for information extraction, intuitively, the increase in mixed pixels definitely means lower classification accuracy. Nevertheless, simulation conducted by Swain [24] discovered that although this statement is generally true, there are some crop types that can be classified more accurately under certain misregistration conditions. If unmixing analysis is chosen for information extraction, while we can make the same general statement about the impact of misregistration, the actual effect might also be unpredictable. In this section, the simulated images that we used in previous sections will be utilized for the study of the effect of misregistration on the unmixing accuracy. For this purpose, two approaches are used: one is qualitative description of data distribution, and the other is quantitative evaluation of unmixing results.

7.4.1  Spectral Signature Distribution

We claimed in section 3.3 that misregistration extends the data distribution to cover the rectangular region bounded by the outermost classes. To verify this, and also to observe the effect caused by spatial IFOV integration and sensor PSF blurring, spectral signature distributions of simulated images are computed and displayed as scattergrams for inspection. The scattergram is based on band 4 and band 5 of a given noiseless multispectral image with $\lambda_r=375$ m.

In Figure 7.18(a), we show the data distribution of the input scene, which contains only pure pixels. Comparing it with Figure 7.18(b), which is the distribution of an ideal image derived from the input scene by IFOV integration, the effect of spatial mixing is obvious. That is, while dispersion of data signatures in (a) is due to intrinsic data variance within each class, the detector IFOV integration generates pixels with real pixel mixture. Because of the constraints imposed on the proportion vector, these new signatures are limited to the convex hull region formed by all classes. Sensor
PSF blurring further increases the amount of mixing, as presented in Figure 7.18(c), where the density of mixed pixels is higher than in (b). Because most pixel mixture in the image is due to two-class mixing, these newly generated spectral signatures occur most often on the boundary of convex hull.

If misregistration exists, the distribution deviates from the limitation of the convex hull, as we demonstrate in Figure 7.18(d) in which a 1.25 pixel shift in band 4 with respect to band 5 was simulated to model the misregistration. Apparently, the distribution does follow our prediction to form a rectangular region. To further examine the effect of misregistration, different amounts of misregistration are simulated in band 4 and the results are illustrated in Figures 7.19(a)–(d) for shifts of 0.5, 1.0, 1.5, and 2.0 pixels, respectively. This sequence of scattergrams show that as misregistration increases, the region of distribution extends. This extension clearly exhibits a closeness to the limited rectangular region as the amount of misregistration increases.

7.4.2 Unmixing Changes and Errors

Two measures are used to quantify the effect of misregistration on the unmixing. First, using the unmixing results for the case of no band-to-band misregistration as a reference, the percentage of pixels that are unmixed differently in the misregistered image compared to the reference image was tabulated as a function of the degree of misregistration. Second, unmixing errors of images with and without registration are computed to show the overall impact of this degradation. In order to understand the combined effect of sensor PSF-blurring and misregistration, results of both ideal and real images are compared.

Empirically, for a certain crop type, the unmixing change or degradation is usually dependent on the band which is misregistered. To acquire a general measurement without specifying the misregistered band, the results are therefore averaged over all the bands. One way to do it is that every band is shifted independently with respect to the other bands for modeling the misregistration, and the unmixing changes and errors are measured; then the final result is obtained from the average of these
measurements. Also, to study the effect of scene complexity, a set of noiseless images with $\lambda_r = \{375\text{m}, 150\text{m}, 75\text{m}, 37.5\text{m}, 15\text{m}\}$ are used in the following computations.

Results of the first study on unmixing change are presented in Figures 7.20 and 7.21 for cross-track offset and along-track offset respectively. Note that the change in the number of pixel is normalized by the total number of pixel (14400) to obtain a percentage change. Comparison of these two figures on both ideal (a) and real (b) images reveals that the percentage of change is related to the amount of misregistration but is not dependent on the direction of offset that causes the misregistration. Also, positive and negative offsets incur the same amount of changes in all the images, with smaller $\lambda_r$ having a higher percentage of change, except when the field size is smaller than one pixel.

As for a contrast between ideal and real image, we will use Figure 7.20 as an example. When values of $\lambda_r$ are high (low scene complexity), these two images have almost the same amount of changes under every misregistration condition. But when the scene becomes more complex (lower $\lambda_r$ value), the ideal image shows a higher percentage of change than that of the real image; the smaller the $\lambda_r$ is, the larger the difference becomes. The reason why the real image has less change can be attributed to the data correlation. It is known that the spatial blurring caused by the sensor PSF can increase the spatial correlation inside each band. This correlation increase no doubt will decrease the impact of misregistration, for higher correlation means less amount of contrast change and is therefore less sensitive to the misregistration.

Notice that the greatest increase in the percentage of pixel affected occurs at small misregistration, as shown by the steeper slope near zero-misregistration in the curves of all figures. Because of the higher intra-band correlation, real image (a) has a lower increasing rate in this region than that of ideal image (b).

In the second part of this quantitative investigation, the unmixing errors are determined as a function of the degree of misregistration in both cross- and along-track direction and the results are shown in Figures 7.22 and 7.23. While the ideal images show a direct relationship between unmixing error and misregistration, surprisingly,

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4The unit of abscissa, pixel, is TM pixel.
the real images have a deviation at the minimal error. That is, the minimal unmixing error occurs not at zero-misregistration point but around -0.75 pixel offset at cross-track or -0.5 pixel offset at along-track direction. More simulations using different field angles does not affect this phenomenon. In his study of the effect of misregistration on the classification accuracy, Swain [24] also reported the same anomalous behavior which shows that the best classification results occur at -0.3 pixel misregistration.

If the misregistration is less than one pixel, which is true for TM images as described in section 3.3, then the increase of unmixing errors in the ideal images fall within 2% ~ 14%. The error increase caused by cross-track misregistration is the same as that caused by along-track misregistration. The increase is also independent of the sign of the offset. Within the same range of misregistration (less than one pixel), unmixing errors in the real image increase 2% ~ 8% with positive misregistration and -5% ~ 4% with negative misregistration. Also, the cross-track misregistration has higher percentage of error increase than that of along-track.

In concluding this section, the reader is cautioned not to associated too much significance with the absolute level of unmixing results achieved in these experiments. The objective is to quantify the relative impact of various amounts of misregistration on a typical application of TM data.

7.5 Limitation of Unmixing Accuracy

All the unmixing algorithms, no matter it is based on the ML or LS method, is limited by the data variance, mean vector distance between classes, and system noise in performance. As we analyzed in section 3.1, the mean squared error of LS estimation is proportional to the data and noise variances. For the mean vectors, the effect is not obvious as it involves inverse matrix. Given a data set available for unmixing analysis, we would like to predict its results as a function of data variance and mean vectors of the class members. In this section, the relation between unmixing error and separability ($\xi$), which is defined in equation (3.2) a function of both data variance and mean vectors, will be investigated for this purpose.
The most direct way for this evaluation is to inspect the unmixing error as the separability changes. In Figures 7.24(a) and (b), we show the results of several scenes with $\lambda_r=375m$ and various $\xi$ for the ideal and real images respectively. The separability is changed by varying the mean vector of each class, and the unmixing analysis is done by the CLS algorithm. Under different SNR with a reasonable separability ($\geq 20$), the unmixing error and separability shows a linear relationship in both ideal and real images. That is, when separability decreases, the unmixing error proportionally increases. Also, the increase of error rate in ideal images is higher than that of real images. The reason is because PSF blurring, which has a smoothing effect, can decrease the data variance and effectively increases the separability. But the total errors in the real images are higher due to PSF blurring, which has been discussed previously.

Overall, the separability seems to be a good indicator of the unmixing performance. While it might not be able to predict an absolute level of error, it can be used to check if a certain data set is suitable for unmixing analysis. Also, given two different data sets, it can used to evaluate their relative performance.
Figure 7.18: Scattergrams of band 4 and 5 for (a) input scene (b) ideal image (c) real image (d) real image with 1.25 pixel misregistration in band 4.
Figure 7.19: Scattergrams of band 4 and 5 with different misregistration in band 4: (a) 0.5 (b) 1.0 (c) 1.5 (d) 2.0 pixel.
Figure 7.20: Change of unmixing results as a function of misregistration in cross-track direction (a) ideal image (b) real image.
Figure 7.21: Change of unmixing results as a function of misregistration in along-track direction (a) ideal image (b) real image.
Figure 7.22: Unmixing error as a function of misregistration in cross-track direction (a) ideal image (b) real image.
Figure 7.23: Unmixing error as a function of misregistration in along-track direction (a) ideal image (b) real image.
Figure 7.24: Unmixing error as a function of noise, data variance and mean vector of class members (i.e. separability) for (a) ideal image with noise (b) real image.
CHAPTER 8
SUMMARY AND CONCLUSIONS

In this final chapter, we summarize this dissertation study on a chapter-by-chapter basis to highlight the results obtained. Conclusions are then drawn and some recommendations are provided for future extension of the work.

8.1 Summary

In chapter 1, spatial and spectral mixture of individual classes is used to explain the generation of mixed pixels which show no characteristics of any individual class. Because of the highly complicated scenes, and therefore large percentage of mixed pixels, in remotely-sensed imagery, this ‘mixing’ process makes conventional classification algorithms less than unsatisfactory. Instead of classification, an inversion procedure is used to estimate the proportion of each class inside a pixel. In spite of its prevalence in remote sensing, error analysis of the proportion estimation algorithm, and possible improvement of estimation accuracy by image processing have never been investigated. These two observations are our motivation for conducting this study.

In the second chapter, modeling of the mixing effect as a linear combination of individual spectral signatures is presented. In order to extract more information from a mixed pixel instead of a per-pixel classification, an inversion of the mixing process is introduced. This procedure is called spectral unmixing and is used to compute the proportion of each material (class) inside a mixed pixel. Two unmixing approaches currently used are reviewed. One is based on a probabilistic model, which assumes a multivariate Gaussian data distribution and leads to the maximum likelihood (ML) algorithm. The other can be viewed as a deterministic model, which
implicitly assumes a constant feature vector for each class and results in a least square (LS) solution. We show that if the covariance matrices of all classes are equal to a multiple of the identity matrix, then these two approaches are equivalent, or the LS method is actually a special case of the ML solution. Because of the higher cost in ML computation, LS methods are of primary interest. Moreover, the constrained least square (CLS) algorithm, due to its robustness in dealing with the constraints on the proportion vector, is used throughout this dissertation.

Chapter 3 describes some possible sources of degradation which usually cause the unmixing error. Included in the discussion are data variance, observation noise, band-to-band misregistration, and PSF blurring. Except for the last source, the effects of these degradations on the unmixing accuracy are analyzed. For the LS unmixing algorithm, if the data variance of each class is small, the error is shown to be proportional to the data and noise variances. From the viewpoint of spectral signature distribution, misregistration is shown to effectively cause mixed pixels which spread the signature distribution into the region bounded by the outermost classes. But unlike the mixture caused by IFOV spatial integration, this mixing is not invertible by unmixing analysis and therefore might further increase the unmixing error. This chapter is concluded by a review of some results on the measurements of misregistration in MSS and TM imagery for future reference.

Chapter 4 was devoted to modeling of the image formation system of remote sensors and its effect on the unmixing accuracy. The PSF of each component in a imaging system is described, and then the system model of several contemporary remote sensing instruments are introduced. After defining the ideal and real image gathering PSF, its effect on the unmixing error is elucidated. Based on a Gaussian image gathering PSF model and a simple 2-class case, the unmixing errors under various geometrical conditions are computed. The results show that unmixing error is sensitive to the closeness of a pixel to a boundary but is almost irrelevant to the orientation of a boundary. Also shown is that when a simple 2-class boundary is present, the PSF blurring alone can cause unmixing error up to 12% of a pixel. If the scene becomes more complicated, it is obvious that PSF blurring will incur even
higher percentage of error. Fortunately, this error can be reduced by improving the image resolution, and image restoration is proposed for this purpose.

Chapter 5 begins the digital image restoration discussion with a review of two popular filtering schemes: conventional Wiener filter (CWF) and sampled Wiener filter (SWF). These two filters, which attack the restoration problem in the frequency domain, differ in the way they handle the sampling aliasing, and are chosen to highlight the importance of aliasing in our application. Although visual comparison between these two filters have been done in the past, here we take a new look at their performance and conduct a detailed evaluation on both filters based on the spectral fidelity criterion. This new criterion not only furnishes us objective comparison results to clearly show how aliasing affects the restoration performance, it also demonstrates agreement with the visual superiority of the SWF scheme.

In chapter 6, using the same spectral fidelity criterion, we develop an application-oriented restoration scheme which is used for spectral unmixing. This new application brings changes to the prevalent understanding about restoration filter design. That is, unlike restoration for visual purposes, the sampling aliasing is no longer a noise (or artifact) to be suppressed but a degraded signal that needs to be restored. Also, while a full restoration can achieve the sharpest restored image, it is a partial restoration that is most suitable for improving unmixing. These changes make the SWF inferior to the CWF in our application, and also alter the kernel PSF in filter design. While an optimal Wiener filter for restoration is conceived, due to the inseparability of signal and aliasing, it can not be realized. Instead of looking for a compromise between the signal and aliasing restoration to obtain the best achievable spectral fidelity for unmixing, we use the CWF for our 'suboptimal' partial restoration scheme and conduct a detailed study on its performance.

Finally, in chapter 7 we conduct a detailed simulation study to provide empirical information for support and verification of the discussions in the previous chapters. Due to the difficulty in acquiring ground truth data for any given remotely-sensed image, this investigation is carried out by using simulated scenes and images. Therefore, a procedure for generating AR multispectral image arrays is developed. Using these simulated images, the results of our proposed partial restoration filter in improving
the unmixing accuracy are thoroughly examined. Then the correlation between two
criteria for restoration evaluation, spectral fidelity and unmixing accuracy, is exam­
ined. The agreement found in these two criteria confirms the usefulness of the spectral
fidelity measurement, and makes the evaluation of a restoration scheme much easier.
Quantitative effects of the other two sources of unmixing error, misregistration and
data variance, are also examined to conclude our investigation.

8.2 Conclusions

A number of conclusions can be drawn from the results of this research:

1. The proposed spectral fidelity criterion for restoration performance evaluation
   is shown to be valuable in algorithm comparison and in our partial restoration
   filter development. It also is in accord with the subjective criterion for visual
   usage and the objective criterion for unmixing purpose. A quantitative compar­
   ison between the SWF and CWF based on spectral fidelity confirms the results
   obtained visually by other researchers.

2. Image restoration, if applied properly, can reduce the unmixing error caused
   by PSF blurring up to 70%. It is shown that a restoration scheme for better
   visual results is not necessarily suitable for improved unmixing. On the con­
   trary, a partial restoration scheme, which does not yield the sharpest image,
   can achieve better improvement in unmixing performance. Also, the sampling
   aliasing cannot be viewed as noise but should be treated as degraded signal and
   be restored.

3. The misregistration study reveals that it will cause further spread of the spec­
   tral signature distribution which is bounded by the outermost classes. This
   degradation has the most profound effect on unmixing when the misregistr­
   ation is less than one pixel. While it always increases unmixing error when the
   misregistration is positive, it shows some curious improvement when the mis­
   registration is negative. These results exhibit a similarity to other studies of
   the effect of misregistration on classification.
4. The error analysis on the LS algorithm is difficult except when the image data has small variance. When the data have large variance, the empirical study shows a linear relationship between unmixing error and data separability. This relation is very useful for unmixing performance prediction and evaluation.

8.3 Future Research

While many significant facts were revealed in this dissertation, some aspects of the problems in both unmixing analysis and restoration design need further investigation. The first is the sensitivity study of unmixing improvement due to modeling error in the system PSF. In the current investigation, the system PSF is assumed to be known exactly; it would be interesting to know how the partial restoration behaves under an inaccurate PSF. While a partial restoration by the CWF can achieve an unmixing error reduction up to 70%, it only focuses on the restoration of signal degradation. Further improvement can be obtained by redesigning the filter to make a tradeoff between the restoration of signal and aliasing.

Further research is also necessary in order to understand why the unmixing error decreases when the multispectral image suffers a negative misregistration. This answer might also be applicable to the same situation that happens in the classification.

Finally, current least squares algorithms fail to account for the real data variance in the unmixing analysis. This might be solved by the total least squares algorithm to achieve a higher unmixing accuracy.
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