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Mixed modules and endomorphisms over incomplete discrete valuation rings

Files, Steve Todd, Ph.D.

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MIXED MODULES AND ENDOMORPHISMS OVER INCOMPLETE DISCRETE VALUATION RINGS

by

Steve Todd Files

A Dissertation Submitted to the Faculty of the

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As members of the Final Examination Committee, we certify that we have read the dissertation prepared by Steve Todd Files

entitled Mixed Modules and Endomorphisms over Incomplete Discrete Valuation Rings

and recommend that it be accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy

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Final approval and acceptance of this dissertation is contingent upon the candidate's submission of the final copy of the dissertation to the Graduate College.

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ABSTRACT

Structure theorems are given for several classes of mixed modules over an arbitrary discrete valuation ring R, followed by results on the endomorphism algebras of mixed R-modules.

The opening chapter introduces a fundamental embedding of R-modules into related modules over the p-adic completion of R, and the succeeding two chapters develop generalizations of the theory of simply presented modules of rank one and Warfield modules.

Endomorphism algebras are considered in the penultimate chapter, where it is shown that the related modules over the completion of R are isomorphic if the underlying R-modules possess isomorphic endomorphism algebras. An isomorphism theorem for the endomorphism algebras of Warfield modules is deduced.

Relevant constructions of mixed abelian groups are offered in the final chapter.

CHAPTER 1
INTRODUCTION

1.1. Overview

Throughout this work, R denotes a discrete valuation ring with prime p, and $\hat{R}$ its completion in the p-adic topology. We will usually assume $\hat{R} \neq R$. A *mixed* R-module contains both torsion and torsion-free elements. Our focus in the coming chapters will be on R-modules of this type, along with their algebras of R-endomorphisms. A number of powerful results exist in this setting for the case where $R = \hat{R}$, and some of our results will be generalizations or variations of these for the other case. One advantage here is that all of our results carry over to abelian group theory: the p-localization $G \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ of an abelian group G is a module over the incomplete discrete valuation ring $\mathbb{Z}_{(p)}$ (the localization of $\mathbb{Z}$ at the prime ideal (p)). The reader who is so inclined can replace "R-module" by "p-local abelian group" in the coming chapters without robbing the development of too much vitality.

A persistent theme in our variety of module theory is the description of isomorphism classes of certain kinds of R-modules in terms of numerical invariants. Almost from the outset, we will be playing off one especially far-reaching result of this nature, a theorem for the summands of simply-presented R-modules (the *Warfield* modules):

THEOREM [7, 18]. *Two Warfield R-modules are isomorphic if and only if they have the same Ulm and Warfield invariants.*

In Chapters 2 and 3, we will discover new invariants for mixed R-modules, and ultimately use these new tools to prove structure theorems for entirely new classes. A consequence of our main theorem is an improved version of the last result, which we preview here:

THEOREM. *Corresponding to each finite-dimensional Q-subspace V of $\hat{Q}$ (the quotient field of $\hat{R}$), there is a family $\mathcal{M}(V)$ of R-modules satisfying:*

(1) Two modules in $\mathcal{M}(V)$ are isomorphic if and only if they have the same Ulm and Warfield invariants.

(2) When V has dimension 1, $\mathcal{M}(V)$ contains the reduced Warfield R-modules.

More background on the classification of mixed R- and $\hat{R}$-modules can be found in [1, 2, 12, 13, 14, 17].

In the theory of endomorphism algebras, an important goal remains the description of how the algebra $End_R(M)$ reflects the structure of an R-module M from a given class. In [8], it was shown that for torsion modules M and N, every isomorphism of $End_R(M)$ with $End_R(N)$ is induced by an isomorphism of M with N. Later, a similar result was proved in a mixed setting:

THEOREM [9]. *If M is a reduced R-module of rank 1 with simply-presented torsion and N has rank 1, then every isomorphism of $End_R(M)$ with $End_R(N)$ is induced by an isomorphism of M with N.*

Shortly, we will define a special embedding of R-modules M into modules $\hat{R}M$ over $\hat{R}$, and our central result in Chapter 4 will give conditions for M that guarantee $\hat{R}M \cong \hat{R}N$ whenever $End_R(M) \cong End_R(N)$. Our ensuing corollary then pushes the last theorem beyond rank 1:

COROLLARY. *Let M be a reduced Warfield module of finite rank, and assume M is divisible modulo torsion. If $End_R(M) \cong End_R(N)$ for a Warfield module N, then $M \cong N$.*

More information on theorems of this nature for mixed and torsion-free modules over $\hat{R}$ can be found in [10, 11, 19].

1.2. Preliminaries

We recount briefly some definitions and tools that will be used freely after this section. All modules are assumed to be faithful. If M is an R-module, then the *rank* $rk(M)$ of M is the Q-dimension of $M \otimes_R Q$, where Q is the quotient field of R. We denote the torsion submodule of M by tM, and the socle $\{x \in M : px = 0\}$ of M by $M[p]$. By taking the submodules $p^k M$ for finite $k \geq 0$ as a subbase of neighborhoods of $0 \in M$, we obtain the ***p*-*adic topology*** on M. If σ is an ordinal number, we define $p^\sigma M = \cap_{\lambda<\sigma} p^\lambda M$ if σ is a limit ordinal, and $p^\sigma M = p(p^\lambda M)$ if $\sigma = \lambda + 1$ is isolated. M is *divisible* if $pM = M$, and *reduced* if $p^\sigma M = 0$ for some σ. The *length* $\ell(M)$ of M is the least σ such that $p^{\sigma+1}M = p^\sigma M$. If $x \in M$ and $x \in p^\sigma M \backslash p^{\sigma+1} M$ for some σ, we say the *height* of x is σ and write $|x|_M = |x| = \sigma$; if no such σ exists, we say $|x| = \infty$. The *height sequence* of x is the sequence $|x|$, $|px|$, $|p^2x|, \cdots$ of ordinal numbers. This sequence is *equivalent* to the height sequence of $y \in N$ if there exist ℓ, $m \geq 0$ such that $|p^{k+\ell}x|_M = |p^{k+m}y|_N$ for all $k \geq 0$. A submodule P of M is *pure* if $p^k M \cap P = p^k P$ for all $k \geq 0$, and *isotype* if $p^\sigma N \cap P = p^\sigma P$ for all ordinals σ. If N is any submodule of M, its *purification* N_* in M is the submodule $\{x \in M : p^k x \in N \text{ for some } k \geq 0\}$ of M. N is *nice* in M if every coset of N has a representative x such that $|x + N|_{M/N} = |x|_M$.

We will make use of several numerical invariants of an R-module M. If N is any submodule and σ an ordinal, the *σth Ulm invariant* of M *relative to* N is the dimension of the R/pR-space $(p^\sigma M)[p]/\big((p^\sigma M)[p] \cap (N + p^{\sigma+1}M)\big)$, written $U_{M,N}(\sigma)$. The ordered set $\{U_{M,0}(\sigma) : \sigma \text{ an ordinal}\}$ of cardinal numbers will be called the *Ulm invariants* U_M of M. A reduced torsion module T is *totally projective* if it contains a system of nice submodules as in [16]; this is equivalent to T being simply-presented, but we use the former term for contextual reasons. T is totally projective if it contains a totally projective submodule S with T/S

countably-generated. The next result will be used frequently and without further mention, hence we isolate it for easy reference.

THEOREM [16]. *(1) Let M/M_0 and N/N_0 be totally projective for nice submodules M_0 and N_0 such that $U_{M,M_0}(\sigma) = U_{N,N_0}(\sigma)$ for all ordinals σ. Then every isomorphism $M_0 \to N_0$ which does not decrease heights relative to M and N extends to an isomorphism $M \to N$. (2) If M/M_0 is totally projective for a nice submodule M_0, then every homomorphism $M_0 \to N$ which does not decrease heights relative to M and N extends to a homomorphism $M \to N$.*

If $e = \{\sigma_0, \sigma_1, \sigma_2, \cdots\}$ is an increasing sequence of ordinals and M is an R-module, let $M(e) = \{x \in M : |p^k x| \geq \sigma_k \text{ for all } k \geq 0\}$, and let $M(e)^*$ be the submodule generated by $\{x \in M(e) : |p^k x| \neq \sigma_k \text{ for almost all } k\}$. The *Warfield invariant* $W_M(e)$ of M at e is the dimension of the space $M(e)/M(e)^*$ over R/pR.

Since we will frequently deal with R-submodules of $\hat{R}$-modules, care will need to be taken to avoid confusing one base ring with the other. If M is an R-submodule of an $\hat{R}$-module A and $S \subseteq M$, then $R\langle S\rangle$ and $\hat{R}\langle S\rangle$ denote the submodules of M and A generated by S over R and $\hat{R}$, respectively. When there is minimal likelihood of confusion, we sometimes write simply $\langle S\rangle$ for one or the other of these. Obviously, A/M should be viewed as an R-module, and the rank of A means its rank as an $\hat{R}$-module. To further highlight the distinction between R and $\hat{R}$, we write r, s, etc. for elements of R, and α, β, etc. for elements of $\hat{R}$.

1.3. Embedding in $\hat{R}$-*Mod*

Let R-*Mod* and $\hat{R}$-*Mod* denote the respective categories of R- and $\hat{R}$-modules. An ever-present tool in our development will be a homologically-defined, covariant functor from the reduced modules of R-*Mod* to $\hat{R}$-*Mod*. We begin by noting that if M is an R-module, then the cotorsion hull $M^\bullet = Ext_R(Q/R, M)$ of M has an $\hat{R}$-module structure which extends the usual R-module structure of Ext. In fact, if $0 \neq \alpha \in \hat{R}$, then multiplication by α induces an endomorphism of the torsion R-module Q/R, from which we obtain an endomorphism α^* of $Ext_R(Q/R, M)$ in the standard homological fashion. If $x \in M^\bullet$, we define αx to be $\alpha^*(x)$. Henceforth, when dealing with the cotorsion hull of an R-module, we assume it to be in $\hat{R}$-Mod.

If M is also reduced, then the exact sequence $0 \rightarrow R \rightarrow Q \rightarrow Q/R \rightarrow 0$ induces an exact sequence $Hom_R(Q, M) \rightarrow Hom_R(R, M) \rightarrow Ext_R(Q/R, M) \rightarrow Ext_R(Q, M)$. Clearly $Hom_R(Q, M) = 0$ and $Hom_R(R, M) \cong M$, so M is embedded in $M^\bullet$ with $M^\bullet/M \cong Ext_R(Q, M)$ torsion-free and divisible. In particular, M is an isotype R-submodule of $M^\bullet$ and $tM = tM^\bullet$. Now denote $T = tM$, and assume further that M/T is divisible. Then the exact sequence $0 \rightarrow T \rightarrow M \rightarrow M/T \rightarrow 0$ induces another exact sequence $Hom_R(Q/R, M/T) \rightarrow Ext_R(Q/R, T) \rightarrow Ext_R(Q/R, M) \rightarrow Ext_R(Q/R, M/T)$, and the end terms vanish because M/T is torsion-free and divisible. Thus, $T^\bullet$ is naturally identified with $M^\bullet$ when M/T is divisible. Under these circumstances, we will view $M \subseteq T^\bullet$ with $T^\bullet/M$ torsion-free and divisible. Further homological arguments show that if M and N are reduced R-modules, then every R-homomorphism or isomorphism $\phi : M \rightarrow N$ has a unique extension to a respective $\hat{R}$-homomorphism or isomorphism $\phi^\bullet : M^\bullet \rightarrow N^\bullet$.

Finally, assume M is a reduced R-module, and let $\hat{R}M$ denote the submodule $\hat{R}\langle M\rangle$ of $M^\bullet$. The assignment of M to $\hat{R}M$ is the covariant functor from R-Mod to $\hat{R}$-Mod that was promised earlier. Clearly, $\hat{R}M/M \subseteq M^\bullet/M$ is torsion-free, and is divisible since $p(\hat{R}M/M) = (p\hat{R}M + M)/M \supseteq ((p\hat{R} + R)M)/M = \hat{R}M/M$. In particular, M is an isotype R-submodule of the reduced $\hat{R}$-module $\hat{R}M$, and $tM = t(\hat{R}M)$. An important feature to keep in mind in what follows is that $rk(\hat{R}M) \leq rk(M)$, with the inequality strict in many cases. We launch the next chapter with a more in-depth study of the relation between M and $\hat{R}M$.

CHAPTER 2
SHARP MODULES

2.1 Fundamental Properties

Our first result gives further information about embeddings of R-modules in $\hat{R}$-modules.

PROPOSITION 2.1. *Let M be a reduced R-module.*

(1) $\hat{R}M$ is isomorphic to the reduced part of $\hat{R} \otimes_R M$.

(2) If $\hat{R}M$ has finite rank n and M is contained in a reduced $\hat{R}$-module $\widetilde{M}$ with $rk(\widetilde{M}) \leq n$ and $\widetilde{M}/M$ torsion-free, then $\widetilde{M} \cong \hat{R}M$.

PROOF: (1) The R-bilinear map $(\alpha, m) \longmapsto \alpha m$ from $\hat{R} \times M$ to $\hat{R}M$ induces a surjective $\hat{R}$-homomorphism $\phi : \hat{R} \otimes_R M \rightarrow \hat{R}M$. Let D be the maximal divisible submodule of $\hat{R} \otimes M$. Note $\phi(D) = 0$ since $\hat{R}M$ is reduced. Let S be the submodule of $\hat{R} \otimes M$ generated by elements $\sum \alpha_i \otimes m_i - 1 \otimes \sum \alpha_i m_i$, where $\alpha_i \in \hat{R}$, $m_i \in M$, and $\sum \alpha_i m_i \in M$. Given a generator, write $\alpha_i = p\beta_i + r_i$ $(\beta_i \in \hat{R}, r_i \in R)$ for each i, and observe $\sum \alpha_i \otimes m_i - 1 \otimes \sum \alpha_i m_i$ becomes $p(\sum \beta_i \otimes m_i - \sum \beta_i m_i) \in pS$, so that S is divisible. If $\sum \alpha_i \otimes m_i \in ker(\phi)$, then $\sum \alpha_i \otimes m_i = \sum \alpha_i \otimes m_i - 1 \otimes \sum \alpha_i m_i \in S$, so that $ker(\phi) \subseteq D$. Now $\hat{R}M \cong (\hat{R} \otimes M)/D$ as desired.

(2) The inclusion $M \subseteq \widetilde{M}$ induces an R-homomorphism $M^\bullet \rightarrow (\widetilde{M})^\bullet$ with kernel $Hom_R(Q/R, \widetilde{M}) = 0$, so we can regard $M^\bullet \subseteq (\widetilde{M})^\bullet$ and hence $\hat{R}M \subseteq \hat{R}\widetilde{M} = \widetilde{M}$. Note $\widetilde{M}/\hat{R}M$ is torsion by the condition on ranks. Also, $\widetilde{M}/\hat{R}M \cong (\widetilde{M}/M)/(\hat{R}M/M)$ with $\hat{R}M/M$ divisible, so that $\widetilde{M}/\hat{R}M$ is isomorphic to a summand of $\widetilde{M}/M$ and is torsion-free. Thus, $\widetilde{M}/\hat{R}M = 0$ as desired. $\square$

To obtain our first tractable class of R-modules, we must limit the kinds of torsion and torsion-free submodules they can possess. The efficacy of our first definition will be made clear.

Definition 2.2. A reduced R-module M is *sharp* if $\hat{R}M$ is a Warfield module of rank 1.

PROPOSITION 2.3. *Let M be a reduced, nontorsion R-module. The following are equivalent:*

(1) M is sharp.

(2) The reduced part of $\hat{R} \otimes_R M$ is a rank 1 Warfield $\hat{R}$-module.

(3) M is contained in a rank 1 Warfield $\hat{R}$-module $\widetilde{M}$ with $\widetilde{M}/M$ torsion-free.

PROOF: If $M \subseteq \widetilde{M}$ as in (3), then $\widetilde{M}$ must be reduced and $rk(\hat{R}M) \geq 1$. All equivalences now become clear from Proposition 2.1. $\square$

In Chapter 4, we give a completely intrinsic characterization of sharp modules with totally projective torsion. A result for the general case is elusive. For now, we note some useful properties of these kinds of modules. Henceforth, $\overline{M}$ will be shorthand for the quotient M/tM.

LEMMA 2.4. *(1) Reduced, rank 1 Warfield R-modules are sharp.*

(2) If M is a reduced R-module with $rk(\hat{R}M) = 1$, then $\overline{M}$ is divisible or M splits.

(3) A reduced, torsion-free R-module $M \neq 0$ is sharp if and only if M is isomorphic to a pure R-subalgebra of $\hat{R}$.

(4) R is incomplete if and only if there is a sharp R-module of rank greater than 1.

PROOF: (1) Let M be as stated and choose $x \in M$ so that $M/R\langle x\rangle$ is totally projective. Note $M \cap \hat{R}\langle x\rangle = R\langle x\rangle$ since $rk(M) = 1$. Now $(M + \hat{R}\langle x\rangle)/\hat{R}\langle x\rangle \cong M/R\langle x\rangle$ is totally projective and $\hat{R}M/\hat{R}\langle x\rangle$ is a countably-generated extension of the former, hence $\hat{R}M/\hat{R}\langle x\rangle$ is totally projective and it follows that M is sharp.

(2) The purity of M in $\hat{R}M$ implies $\overline{M}$ is divisible if $(\hat{R}M)^-$ is. If $\hat{R}M = tM \oplus A$ splits, then M splits as $tM \oplus (M \cap A)$. When $rk(\hat{R}M) = 1$ we have $(\hat{R}M)^- \cong \hat{Q}$ or $\hat{R}$, hence (2) follows.

(3) Note that a reduced, torsion-free R-module is sharp exactly when $\hat{R}M \cong \hat{R}$, and that M is pure in $\hat{R}M$.

(4) If $R \neq \hat{R}$, choose an R-independent set $\{\alpha_i : i \in I\} \subseteq \hat{R}$ of cardinality greater than 1. Let T be an unbounded, totally projective module and x a torsion-free element of $T^\bullet$ (preview Lemma 4.1). Let M be the purification of the free module $\sum_{i \in I} R\langle \alpha_i x \rangle$ in $T^\bullet$. Then M is a (nonsplit) sharp module with torsion-free basis $\{\alpha_i x : i \in I\}$. Conversely, if $R = \hat{R}$ note $\hat{R}M = M$ for all reduced R-modules M. $\square$

When $R = \mathbb{Z}_{(p)}$, it is well known that $\hat{R}$ has transcendence degree $2^{\aleph_0}$ over R, so we can choose I of cardinality $2^{\aleph_0}$ in the proof of part (4) above to obtain sharp, p-local abelian groups of uncountable rank. We will return to the setting of abelian groups in Chapter 5.

2.2 Two Structure Theorems

A key fact in the theory of Warfield modules is the existence of certain nice, free submodules (decomposition bases). The next result shows that our approach here must be along different lines.

PROPOSITION 2.5. *If M is a reduced R-module with $rk(\hat{R}M) = 1$ and $rk(M) \geq 2$, then no nonzero, free submodule of M is nice.*

PROOF: Let $F \neq 0$ be a free submodule of M and choose R-independent elements $x \in F$ and $y \in M$. There exist nonzero $r \in R$ and $\alpha \in \hat{R} \backslash R$ such that $ry = \alpha x$ in $\hat{R}M$. $(M \cap \hat{R}\langle x \rangle)/R\langle x \rangle$ is divisible and nonzero since $\hat{R}/R$ is divisible and $x \in (M \cap \hat{R}\langle x \rangle) \backslash R\langle x \rangle$. Hence $M/F \cong (M/R\langle x \rangle)/(F/R\langle x \rangle)$ has a nonzero divisible submodule since $F/R\langle x\rangle$ is reduced. M is reduced, so F cannot be nice in M. $\square$

COROLLARY 2.6. *No sharp R-module of rank greater than 1 is a Warfield module.*

A useful category for working with mixed modules is *WARF*. Its objects are modules M and P, with morphisms from M to P represented by ordinary homomorphisms $\phi : N \to P$ such that M/N is torsion and ϕ does not decrease heights relative to M and P. M and P are *WARF*-isomorphic if they possess height-isomorphic, full rank submodules, and such an isomorphism implies equality of the Warfield invariants of M and P. Thus, two Warfield modules are isomorphic when they are *WARF*-isomorphic and their Ulm invariants coincide. Since we will encounter nonisomorphic sharp modules with equal Ulm and Warfield invariants, our first theorem reveals more of the potency of *WARF*-isomorphism.

THEOREM 2.7. *Two sharp R-modules are isomorphic if and only if they are WARF-isomorphic and have equal Ulm invariants.*

PROOF: Necessity is clear. To prove sufficiency, let M and N be *WARF*-isomorphic sharp modules with $U_M = U_N$. Clearly $U_{\hat{R}M} = U_{\hat{R}N}$, and the two $\hat{R}$-modules are *WARF*-isomorphic since they have rank 1 and contain M and N as isotype R-submodules. Thus, there is an isomorphism $\hat{R}M \to \hat{R}N$, and by replacing M by its isomorphic image we may assume M and N are contained in $\hat{R}N$ with $\hat{R}N/M$ and $\hat{R}N/N$ torsion-free. By hypothesis, there exist full-rank, free submodules $F_M = \oplus_{i\in I}\langle x_i\rangle$ of M and $F_N = \oplus_{i\in I}\langle y_i\rangle$ of N such that $|\sum r_i x_i| = |\sum r_i y_i|$ for all $r_i \in R$. Fix $i_0 \in I$. There are nonzero r_0, $s_0 \in R$ and a unit $\gamma \in \hat{R}$ such that $r_0 x_{i_0} = s_0 \gamma y_{i_0}$ in $\hat{R}N$. Let $N' = \gamma N \cong N$

and note that M and N' are the purifications of F_M and γF_N in $\hat{R}N$, respectively. Let $i \in I$ and choose nonzero $r \in R$, α, $\beta \in \hat{R}$ so that $rx_i = \alpha x_{i_0}$ and $ry_i = \beta y_{i_0}$. For $k \geq 1$, choose $r_k \in R$ so that p^k divides $r_k - \alpha$. For all k, we have $|(r_k - \beta)y_{i_0}| = |r_k y_{i_0} - \beta y_{i_0}| = |r_k y_{i_0} - ry_i| = |r_k x_{i_0} - rx_i| = |r_k x_{i_0} - \alpha x_{i_0}| = |(r_k - \alpha)x_{i_0}| \geq |p^k x_{i_0}| = |p^k y_{i_0}|$. Since $\hat{R}N$ is reduced, p^k divides $r_k - \beta$ for all k, and we have $\alpha = \beta$. Now $(r_0 r)x_i = r_0(\alpha x_{i_0}) = r_0(\beta x_{i_0}) = \beta(r_0 x_{i_0}) = \beta(s_0 \gamma y_{i_0}) = (s_0\gamma)\beta y_{i_0} = (s_0 r)\gamma y_i$, and it is evident that the purifications of F_M and γF_N must coincide. Thus, $M = N' \cong N$. □

However, Theorem 2.7 is not wholly satisfactory as a working classification theorem because it can be very difficult to determine if two modules are *WARF*-isomorphic. An important tool in the next chapters will be an alternate set of invariants which is easier to work with. After some spadework, we will formulate this new result in a theorem.

The next definition first appeared in [5] in a different setting. Here, we view $\hat{Q}$ as a vector space over Q.

Definition 2.8. Two subspaces V and W of $\hat{Q}$ are *$\hat{Q}$-equivalent* if there is a nonzero element α of $\hat{Q}$ such that $W = \alpha V$.

$\hat{Q}$-equivalence is an equivalence relation, and by absorbing integral powers of p we can take α to be a unit of $\hat{R}$. There is a natural way to obtain a subspace of $\hat{Q}$ from any reduced R-module M with $rk(\hat{R}M) = 1$, as follows. By Lemma 2.4(2), $\overline{M}$ is divisible or M splits. In the first case, we induce a $\hat{Q}$-isomorphism of $(\hat{R}M)^-$ with $\hat{Q}$ by sending a fixed, nonzero coset to 1; the corresponding image of M is a Q-subspace of $\hat{Q}$. In the second case, $\overline{M}$ is a torsion-free sharp R-module and Lemma 2.4(3) implies $\overline{M}$ is isomorphic to a pure R-submodule A of $\hat{R}$; here, choose the subspace QA of $\hat{Q}$ generated by A.

Lemma 2.9. *Let M and N be reduced R-modules with $rk(\hat{R}M) = rk(\hat{R}N) = 1$.*

(1) The $\hat{Q}$-equivalence class of the subspace obtained from M is independent of the choice of isomorphism.

(2) If M is contained in $\hat{R}N$ with $\hat{R}N/M$ torsion-free, then the subspaces obtained from M and N are $\hat{Q}$-equivalent if and only if there is a unit $\alpha \in \hat{R}$ such that $N = \alpha M$.

Proof: (1) Suppose $\overline{x_1}$ and $\overline{x_2}$ are nonzero elements of $(\hat{R}M)^-$. In case one above, let f and g denote the respective isomorphisms of $(\hat{R}M)^-$ with $\hat{Q}$ induced by sending $\overline{x_1}$ and $\overline{x_2}$ to 1. There are nonzero α, $\beta \in \hat{R}$ such that $\alpha x_1 = \beta x_2$. Then $\beta f = \alpha g$, and we see that $f(M)$ and $g(M)$ are $\hat{Q}$-equivalent since $f(M) = (\alpha/\beta)g(M)$. Case two above is similar.

(2) Suppose M and N are contained in $\hat{R}N$ as stated. In case one above, let f be an isomorphism of $(\hat{R}N)^-$ with $\hat{Q}$. Then M and N determine the same $\hat{Q}$-equivalence class if and only if there is a unit $\alpha \in \hat{R}$ such that $f(N) = \alpha f(M)$, or $N = \alpha M$. Case two is similar. □

We will say that $\overline{M}$ and $\overline{N}$ are $\hat{Q}$-equivalent when M and N determine the same $\hat{Q}$-equivalence class in the above fashion. If M is an R-module with $rk(\hat{R}M) = 1$, then all torsion-free elements of M have equivalent height sequences because M is isotype in $\hat{R}M$. We denote this unique class of height sequences (an invariant of M) by $H(M)$.

Theorem 2.10. *Two sharp R-modules M and N are isomorphic if and only if they have equal Ulm invariants, $H(M) = H(N)$, and $\overline{M}$ is $\hat{Q}$-equivalent to $\overline{N}$.*

Proof: Necessity is clear by Lemma 2.9(1). To prove sufficiency, suppose the three invariants of M and N coincide. Clearly $U_{\hat{R}M} = U_{\hat{R}N}$, and the $\hat{R}$-modules are *WARF*-isomorphic since $H(M) = H(N)$. Thus, there is an isomorphism $\hat{R}M \to \hat{R}N$, and by replacing M by its isomorphic image we may assume M and

N are contained in $\hat{R}N$ with the quotients torsion-free. Since $\overline{M}$ is $\hat{Q}$-equivalent to $\overline{N}$, Lemma 2.4(2) implies $N = \alpha M$ for a unit $\alpha \in \hat{R}$, hence $N \cong M$. $\square$

In [8], I. Kaplansky has written about the problem of classification theorems in general: "How do we know when we have a satisfactory theorem? After all, a 'complete set of invariants' might turn out to be so complicated as to be virtually useless. We suggest that a tangible criterion be employed: the success of the alleged structure theorem in solving ... three test problems." Kaplansky's second test, the "square-root" problem, proves to be the most difficult of the three in our setting. We conclude this chapter with a corollary demonstrating Theorem 2.10 and settling the second test problem for a special case.

COROLLARY 2.11. *Let M be a reduced R-module of finite rank such that $rk(\hat{R}M) = 1$ and tM is totally projective. If $M \oplus M \cong N \oplus N$ for an R-module N, then $M \cong N$.*

PROOF: Let $T = tM$ and $S = tN$. Then $T \oplus T \cong S \oplus S$, so Ulm's theorem implies $T \cong S$. Clearly, N is reduced. Since $\hat{R}M \oplus \hat{R}M \cong \hat{R}(M \oplus M) \cong \hat{R}(N \oplus N) \cong \hat{R}N \oplus \hat{R}N$, $\hat{R}N$ is a rank 1 Warfield module. The following argument showing $H(M) = H(N)$ is due to Rotman [12]. Let $x \in M$ be torsion-free and denote the image of (x, x) in $N \oplus N$ by (a, b). Then for all $k \geq 0$, we have $|p^k x| = |(p^k a, p^k b)|$. If either a or b is torsion, the other is torsion-free and has height sequence equivalent to that of x. If neither is torsion, choose $m, n \geq 0$ and a unit $\alpha \in \hat{R}$ such that $p^m a = \alpha p^n b$ in $\hat{R}N$ (say $m \geq n$), and denote $y = p^m a$. Then for all k, $|p^k(p^m x)| = |(p^k y, \alpha^{-1} p^{k+m-n} y)| = |p^k y|$, so that $p^m x$ and y have the same height sequence and $H(M) = H(N)$. Next, note $\overline{N}$ is divisible or N splits according as $\overline{M}$ is divisible or M splits. Let ϕ be the extension of the given isomorphism to one of $\hat{R}M \oplus \hat{R}M$ with $\hat{R}N \oplus \hat{R}N$, and let $\overline{\phi}$ be the induced isomorphism $\hat{R}M/T \oplus \hat{R}M/T \to \hat{R}N/S \oplus \hat{R}N/S$. Identifying these quotients with $\hat{Q} \oplus \hat{Q}$ (if $\overline{M}$ is divisible) or $\hat{R} \oplus \hat{R}$ (if M splits), we can represent $\overline{\phi}$ by a 2×2 invertible matrix $[\alpha_{ij}]$ with entries in $\hat{Q}$. In particular, $\overline{N} \oplus \overline{N} = \overline{\phi}(\overline{M} \oplus \overline{M})$, hence $\overline{N} = \alpha_{11}\overline{M} + \alpha_{12}\overline{M}$. One of α_{11} and α_{12} is nonzero, say α_{11}. Since $\overline{M}$ and $\overline{N}$ have equal, finite dimension, we have $\overline{N} = \alpha_{11}\overline{M}$ and $\overline{M}$ is $\hat{Q}$-equivalent to $\overline{N}$. Theorem 2.10 now implies $M \cong N$. $\square$

CHAPTER 3
UNIFIED MODULES

3.1 Direct Sums

This section serves as the groundwork for more structure theorems that appear near the chapter's end. Our first goal here is the construction of a new class of R-modules from direct sums of the sharp modules in Chapter 2. The relative complexity of the proof of Corollary 2.11 suggests that a naive approach to direct sums of these modules could become overly complicated. With that in mind, we take a more sophisticated route that exploits the additive category $WALK$ and information about endomorphism rings.

LEMMA 3.1. *If M is a reduced R-module with torsion T and $rk(\hat{R}M) = 1$, then $End_R(M) \cong A \oplus Hom_R(M,T)$, where A is a pure R-subalgebra of $\hat{R}$ containing R.*

PROOF: Note $\overline{M}$ is contained in the $\hat{R}$-module $\hat{R}M/T$. Define $A = \{\alpha \in \hat{R} : \alpha\overline{M} \subseteq \overline{M}\}$, an R-subalgebra of $\hat{R}$ containing R. The purity of A in $\hat{R}$ follows from the purity of $\overline{M}$ in $\hat{R}M/T$. Since $M \subseteq \hat{R}M$, we can view M as a faithful left A-module to obtain $A \oplus Hom_R(M,T) \subseteq End_R(M)$. To see the reverse inclusion, extend any given element of $End_R(M)$ to an $\hat{R}$-endomorphism ϕ of $\hat{R}M$, and define $\overline{\phi}(x+T) = \phi(x)+T$ for all cosets $x+T$. Since $\hat{R}M/T \cong \hat{Q}$ or $\hat{R}$, $\overline{\phi}$ acts as multiplication by an element $\alpha \in \hat{Q}$. Let $x \in \hat{R}M$ be torsion-free and note $\phi(x) + T = \alpha(x + T)$, so that $p^i\alpha \in \hat{R}$ and $\phi(p^ix) = p^i\alpha x$ for some $i \geq 0$. Since ϕ cannot decrease heights, p^i divides $p^i\alpha$ in $\hat{R}$, hence $\alpha \in \hat{R}$. Now $\alpha\overline{M} = \overline{\phi}(\overline{M}) \subseteq \overline{M}$ and $(\overline{\phi} - \overline{\alpha})(\overline{M}) = 0$, so $\phi|_M = \alpha + (\phi|_M - \alpha) \in A \oplus Hom_R(M,T)$ as desired. $\square$

Like $WARF$, the category $WALK$ has been important to other accounts of mixed modules [4, 12, 17]. We obtain $WALK$ by equating homomorphisms which agree modulo torsion: the set of morphisms from an R-module A to B is the quotient $Hom_R(A,B)/Hom_R(A,tB)$, denoted $Hom_W(A,B)$. The following fact is proved in [4]: A and B are isomorphic in $WALK$ if and only if there exist torsion modules S and T such that $A \oplus S \cong B \oplus T$ in R-Mod. Thus, isomorphisms in R-Mod induce ones in $WALK$, and the latter category is of special benefit only when working with classes of mixed modules. We now state three preparatory results in succession; for the first, $End_W(A)$ denotes the ring $Hom_W(A,A)$.

LEMMA 3.2. *If M is a reduced R-module of finite rank and $rk(\hat{R}M) = 1$, then $End_W(M)$ is a local ring.*

PROOF: By the preceding lemma, we can identify $End_W(M)$ with the pure R-subalgebra A of $\hat{R}$. Choose a torsion-free element $x \in M$. Clearly, every R-independent subset $\{\alpha_1, \cdots, \alpha_n\}$ of A yields an R-independent subset $\{\alpha_1 x, \cdots, \alpha_n x\}$ of M. Since $rk(M)$ is finite, the rank of A as an R-module is finite. Thus, the Q-subalgebra QA of $\hat{Q}$ generated by A is a field since it is finitely generated as a module over Q. If $0 \neq \alpha \in A$, write $\alpha = \beta p^i$ ($i \geq 0$, β a unit of $\hat{R}$) and note $\beta^{-1} = p^i \alpha^{-1} \in QA \cap \hat{R} = A$ by purity. Hence β is a unit of A, and it follows that A is a discrete valuation ring with prime p. In particular, $End_W(M) = A$ is a local ring. $\square$

LEMMA 3.3. *Let M and N be reduced R-modules with $rk(\hat{R}M) = rk(\hat{R}N) = 1$. If M and N are $WALK$-isomorphic, then $H(M) = H(N)$ and $\overline{M}$ is $\hat{Q}$-equivalent to $\overline{N}$. The converse holds if M and N are sharp.*

PROOF: If $rk(\hat{R}M) = 1$ and T is a torsion module, we have $H(M) = H(M \oplus T)$ and can view $\overline{M} = \overline{M \oplus T}$. Thus, these two invariants coincide for $WALK$-

isomorphic modules M and N as stated. Conversely, if M and N are sharp R-modules, choose a large totally projective module T so that $M \oplus T$ and $N \oplus T$ have equal Ulm invariants. By theorem 2.10, $M \oplus T \cong N \oplus T$ if $H(M) = H(N)$ and $\overline{M}$ is $\hat{Q}$-equivalent to $\overline{N}$. $\square$

LEMMA 3.4. *Suppose $M = \bigoplus_{i \in I} M_i \cong \bigoplus_{k \in K} N_k$ in WALK, where the M_i and N_k are reduced R-modules with $rk(\hat{R}M_i) = rk(\hat{R}N_k) = 1$ for all i and k, and $rk(M_i)$ is finite for all i. Then there exists a bijection $\tau : I \to K$ such that M_i and $N_{\tau(i)}$ are WALK-isomorphic for all $i \in I$, and any direct summand of M is WALK-isomorphic to $\bigoplus_{j \in J} M_j$ for some $J \subseteq I$.*

A direct proof of Lemma 3.4 is beyond our scope. It will suffice to quote the following Azumaya-type theorem, and remark how it applies in our case.

THEOREM [15]. *Let $\mathcal{A}$ be an additive category with kernels and infinite direct sums which satisfies a weak Grothendieck condition. If $M = \bigoplus_{i \in I} M_i = \bigoplus_{k \in K} N_k$ in $\mathcal{A}$, where $End_{\mathcal{A}}(M_i)$ is local and M_i is countably approximable for all i, then there is a bijection $\tau : I \to K$ such that M_i is $\mathcal{A}$-isomorphic to $N_{\tau(i)}$ for each $i \in I$, and any direct summand of M is $\mathcal{A}$-isomorphic to $\bigoplus_{j \in J} M_j$ for some $J \subseteq I$.*

It is routine to verify that *WALK* meets the given conditions for $\mathcal{A}$, and when M_i is as given in Lemma 3.4, it follows from Lemma 3.2 that $End_W(M_i)$ is local. Also, if M is any reduced R-module and $rk(\hat{R}M)$ is finite, then M is small in *WALK*, hence countably approximable. The argument when $rk(\hat{R}M) = 1$ is as follows: if $\phi : M \to P = \bigoplus_{i \in I} P_i$ is any representative of a *WALK*-homomorphism ϕ_W from M to P, choose a torsion-free element $x \in M$ and note that the projection of $\phi(x)$ onto each summand P_i must be zero for all i outside some finite $J \subseteq I$. If π denotes projection from P onto $\bigoplus_{j \in J} P_j$, then

$\phi - \pi\phi \in Hom_R(M, tP)$, so ϕ_W factors through a finite sum $\bigoplus_{j\in J} P_j$ in *WALK*. Thus, M is small.

With Lemma 3.4 now in place, we define a new invariant and a new class of modules. A structure theorem follows in the next section.

Definition 3.5. Let $M \cong \bigoplus_{i\in I} M_i$ in *WALK*, where the M_i are reduced R-modules of finite rank and $rk(\hat{R}M_i) = 1$ for all i. Let e denote an increasing sequence of ordinals and f a finite-dimensional subspace of $\hat{Q}$. Then $G_M(e, f)$ is the cardinality of the set of M_i for which $e \in H(M_i)$ and $\overline{M_i}$ is $\hat{Q}$-equivalent to f.

Definition 3.6. A *unified* R-module is a direct summand of a direct sum of sharp, finite-rank R-modules.

Lemmas 3.3 and 3.4 guarantee that the function G_M in the first definition is independent of the decomposition $M = \bigoplus_{i\in I} M_i$ chosen, and is invariant under isomorphism of M. In particular, this invariant is associated with any unified R-module M.

When M is unified, clearly $\hat{R}M$ is a Warfield module. We conclude this section by noting that all reduced Warfield $\hat{R}$-modules arise in this way.

PROPOSITION 3.7. *(1) A torsion module is unified if and only if it is totally projective.*

(2) If $\widetilde{M}$ is a reduced Warfield module over $\hat{R}$, then there is a unified R-module M such that $\widetilde{M} \cong \hat{R}M$.

PROOF: (1) Observe that a torsion module T is simply-presented over R if it is simply-presented over $\hat{R}$ (the same generators and relations serve to present T over both rings). If T is unified torsion, write $T \oplus N = \bigoplus_{i\in I} M_i$ as in Definition 3.6 and note $T \oplus \hat{R}N \cong \bigoplus_{i\in I} \hat{R}M_i$, so that T is a direct summand of a simply-presented $\hat{R}$-module. By a result in [4], T is simply-presented and hence totally projective. Conversely, if T is totally projective, then $R \oplus T$ is sharp and T is unified.

(2) We may assume $\widetilde{M}$ is nontorsion. By a result in [7], there is a totally projective module T so that $\widetilde{M} \oplus T = \bigoplus_{i \in I} \widetilde{M}_i$ is a direct sum of simply-presented, rank 1 $\hat{R}$-modules $\widetilde{M}_i$. For each i, choose a torsion-free element $x_i \in \widetilde{M}_i \cap \widetilde{M}$ and let M_i denote the purification of $R\langle x_i \rangle$ in $\widetilde{M}_i$. Let M be the purification of $\sum_{i \in I} R\langle x_i \rangle$ in $\widetilde{M}$. Noting $t(\widetilde{M} \oplus T) = tM \oplus T = \bigoplus_{i \in I} tM_i$, we have $M \oplus T = \bigoplus_{i \in I} M_i$. Since $\hat{R}M_i = \widetilde{M}_i$ for each i, M is unified. By construction $\hat{R}M \oplus T = \widetilde{M} \oplus T$ and $\hat{R}M \subseteq \widetilde{M}$, so $\hat{R}M = \widetilde{M}$ as desired. $\square$

3.2 A Structure Theorem

If $\{x_i : i \in I\}$ is a subset of a module M, a *subordinate* subset is of the form $\{p^{n_i}x_i : i \in I\}$, where $n_i \geq 0$ for all i. In the next two lemmas and theorem, we replace carefully-chosen full-rank submodules of our modules by height-isomorphic subordinates; under certain conditions, an isomorphism of the modules is induced.

LEMMA 3.8. *Suppose M and N are WALK-isomorphic, sharp R-modules. Then there are elements $x \in M$, $y \in N$ and a set $\{\alpha_i : i \in I\} \subseteq \hat{R}$ such that $\{x\} \cup \{\alpha_i x : i \in I\}$ and $\{y\} \cup \{\alpha_i y : i \in I\}$ are torsion-free bases for M and N, respectively.*

PROOF: By Theorem 2.10, we can choose a totally projective module T so that $M \oplus T \cong N \oplus T$. Let $\{z_i : i \in I\}$ be a torsion-free basis for M and fix a torsion-free element $z \in M$. For each $i \in I$, there exist nonzero $r_i \in R$ and $\beta_i \in \hat{R}$ such that $r_i z_i = \beta_i z$ in $\hat{R}M$. Choose $|\beta_{i_0}|_{\hat{R}}$ minimal in $\{|\beta_i|_{\hat{R}} : i \in I\}$ and set $x = \beta_{i_0} z$, $\alpha_i = \beta_i \beta_{i_0}^{-1}$ for all i. Then $\{x\} \cup \{\alpha_i x : i \in I\} = \{r_i z_i : i \in I\}$ is a torsion-free basis for M. By replacing x by a subordinate, we may assume there is an element $y' \in N$ with the same height-sequence as x. Since $\hat{R}\langle x \rangle$ and $\hat{R}\langle y' \rangle$ are height-isomorphic submodules of $\hat{R}M \oplus T$ and $\hat{R}N \oplus T$, there is an $\hat{R}$-isomorphism $\phi : \hat{R}M \oplus T \to \hat{R}N \oplus T$ with $\phi(x) = y'$. Now $\phi(M \oplus T) \cong N \oplus T$ in $\hat{R}N \oplus T$, so by Lemma 2.9(2) there exists a unit $\beta \in \hat{R}$ such that $N \oplus T = \beta\phi(M \oplus T)$. Since $\{y'\} \cup \{\alpha_i y' : i \in I\}$ is a torsion-free basis for $\phi(M \oplus T)$, $\{\beta y'\} \cup \{\alpha_i \beta y' : i \in I\}$ is one for $N \oplus T$. After replacing y' by a subordinate so that $\beta y' \in N$ and denoting $y = \beta y'$, we obtain the desired torsion-free basis $\{y\} \cup \{\alpha_i y : i \in I\}$ for N. □

The next lemma holds for modules over any discrete valuation ring R. In the proof of the subsequent theorem, we apply it to modules over $\hat{R}$. Recall that a torsion-free basis $\{x_i : i \in I\}$ for a module is a *decomposition basis* if $|\sum r_i x_i| = min\{|r_i x_i| : i \in I\}$ for all $(r_i)_{i \in I} \in \bigoplus_{|I|} R$. The lemma can be proved piecemeal from results in [7] and [18].

LEMMA 3.9. *Let A and B be Warfield modules with height-isomorphic decomposition bases $X = \{x_i : i \in I\}$ and $Y = \{y_i : i \in I\}$, respectively. Suppose A and B have equal Ulm invariants, $A/\langle X\rangle$ and $B/\langle Y\rangle$ are totally projective, and that $\langle X\rangle$ and $\langle Y\rangle$ are nice in A and B. Then there exist height-isomorphic subordinates X' of X and Y' of Y with these same properties, such that the relative Ulm invariants $U_{A,\langle X'\rangle}$ and $U_{B,\langle Y'\rangle}$ are equal.*

THEOREM 3.10. *Two unified R-modules M and N are isomorphic if and only if they have equal Ulm invariants and $G_M = G_N$.*

PROOF: By Proposition 3.7(1) and Ulm's theorem, we may assume M and N are nontorsion. Only sufficiency needs to be proved. Since M and N are unified and $G_M = G_N$, by Lemma 3.4 we can write $M \oplus S = (\bigoplus_{i\in I} M_i) \oplus T$ and $N \oplus S' = (\bigoplus_{i \in I} N_i) \oplus T'$, where S, S', T and T' are torsion modules and the M_i and N_i are sharp R-modules such that $H(M_i) = H(N_i)$ and $\overline{M_i}$ is $\hat{Q}$-equivalent to $\overline{N_i}$ for each $i \in I$. By Lemmas 3.3 and 3.8, for each $i \in I$ there exist

$x_i \in M_i$, $y_i \in N_i$ and a set $\{\alpha_{ij} : j \in J(i)\} \subseteq \hat{R}$ such that $X_i = \{x_i\} \cup \{\alpha_{ij}x_i : j \in J(i)\}$ and $Y_i = \{y_i\} \cup \{\alpha_{ij}y_i : j \in J(i)\}$ are torsion-free bases for M_i and N_i, respectively. By passing to subordinates, we may assume $X_i \subseteq M$, $Y_i \subseteq N$, and that x_i and y_i have identical height sequences for each $i \in I$. Note that M and N are the purifications of $R\langle\bigcup_{i\in I} X_i\rangle$ and $R\langle\bigcup_{i\in I} Y_i\rangle$ in $\hat{R}M$ and $\hat{R}N$, respectively. By construction, $X = \{x_i : i \in I\}$ and $Y = \{y_i : i \in I\}$ are height-isomorphic decomposition bases for $\hat{R}M$ and $\hat{R}N$, and satisfy the hypotheses of Lemma 3.9 over $\hat{R}$. Therefore, we may assume they have been replaced by height-isomorphic subordinates as stated in the lemma. Observe that our earlier statements regarding the torsion-free bases X_i and Y_i remain valid. By sending x_i to y_i for each $i \in I$, we induce an $\hat{R}$-isomorphism $\phi : \hat{R}M \to \hat{R}N$ such that $\phi(X_i) = Y_i$ for each $i \in I$. Since $\phi(M)$ is the purification of $R\langle\bigcup_{i\in I} \phi(X_i)\rangle$ in $\hat{R}N$, we have $\phi(M) = N$ and hence $M \cong N$. $\square$

Theorem 3.10 is appealing because it applies to a class of modules closed under the taking of direct sums and summands. We will soon prove that it subsumes the old structure theorem for Warfield R-modules. An example showing a natural limitation of the new theorem will be given in Chapter 5.

3.3 The Classes $\mathcal{M}(V)$

This section is devoted to special subclasses of the unified modules, and culminates with a refined version of Theorem 3.10.

Definition 3.11. Let V be a finite-dimensional Q-subspace of $\hat{Q}$. Then $\mathcal{M}(V)$ denotes the class of R-modules which are direct summands of modules of the form $\bigoplus_{i\in I} M_i$, where the M_i are sharp R-modules and $\overline{M_i}$ is $\hat{Q}$-equivalent to V for all $i \in I$.

The rank of each M_i in this definition is just the dimension of V, so $\mathcal{M}(V)$ consists of unified R-modules. We pause to observe what happens when V has dimension 1.

PROPOSITION 3.12. *$\mathcal{M}(Q)$ contains the reduced Warfield R-modules.*

PROOF: If T is a reduced, totally projective module, then $\overline{T \oplus R}$ is $\hat{Q}$-equivalent to Q, hence T is in $\mathcal{M}(Q)$. Assume M is a nontorsion, reduced Warfield R-module and write $M \oplus S = \bigoplus_{i \in I} M_i$ for a torsion module S, where the M_i are simply-presented R-modules of rank 1. Since all 1-dimensional Q-subspaces of $\hat{Q}$ are $\hat{Q}$-equivalent, $\overline{M_i}$ is $\hat{Q}$-equivalent to Q for all $i \in I$, hence M is in $\mathcal{M}(Q)$. $\square$

Recall that two Warfield modules are isomorphic if their Ulm and Warfield invariants coincide. After a lemma, we show that this is in fact true for all modules within each class $\mathcal{M}(V)$. The special notation used below is explained in Section 1.2.

LEMMA 3.13. *Let M be a reduced R-module such that $rk(\hat{R}M) = 1$. Then $W_M(e) = 1$ if $e \in H(M)$, and 0 otherwise.*

PROOF: Let $T = tM$ and write $End_R(M) = A \oplus Hom_R(M, T)$ as in Lemma 3.1. If $e \in H(M)$, we may assume e is the height sequence of an element $x \in M$. Map A/pA to $M(e)/M(e)^*$ by sending $\alpha + pA$ to $\alpha x + M(e)^*$ for all cosets $\alpha + pA$. Clearly, this defines an injective homomorphism of these R/pR-spaces. Given any nonzero coset $z + M(e)^*$ in $M(e)/M(e)^*$, write $p^i z = p^j \alpha x$ for some $i, j \geq 0$ and unit $\alpha \in \hat{R}$. Note that $\alpha \in A$. Also, $i \leq j$ since $z \in M(e)$, and $i \geq j$ since $z \notin M(e)^*$. Therefore $i = j$, and we have $z - \alpha x \in T \cap M(e) \subseteq M(e)^*$, so that the map is also surjective. Since R is pure and dense in A, we have $M(e)/M(e)^* \cong A/pA \cong R/pR$, so that $W_M(e) = dim(R/pR) = 1$. Finally, since all torsion-free elements of M have equivalent height sequences, if $e \notin H(M)$ then $M(e) = M(e)^*$ and $W_M(e) = 0$. $\square$

One consequence of this lemma is that M and $\hat{R}M$ have equal Warfield invariants when M is unified, a fact we will employ under different circumstances in Chapter 4. We conclude this chapter with a promised corollary to Theorem 3.10.

COROLLARY 3.14. *Two R-modules in $\mathcal{M}(V)$ are isomorphic if and only if they have the same Ulm and Warfield invariants.*

PROOF: Let M and N in $\mathcal{M}(V)$ have the same Ulm and Warfield invariants. By Lemma 3.4, we can write $M \cong \bigoplus_{i\in I} M_i$ and $N \cong \bigoplus_{i\in I} N_i$ in *WALK*, where all $\overline{M_i}$ and $\overline{N_i}$ are $\hat{Q}$-equivalent to V. Clearly, $G_M(e, V)$ counts the number of summands M_i such that $e \in H(M_i)$. Thus, by Lemma 3.13, $G_M(e, V) = \sum_{i\in I} W_{M_i}(e) = W_M(e)$. Similarly, $G_N(e, V) = W_N(e)$. Since $W_M = W_N$ by hypothesis, we have $G_M(e, V) = G_N(e, V)$ for all e. Also, $G_M(e, W) = G_N(e, W) = 0$ if W is not $\hat{Q}$-equivalent to V. Thus $G_M = G_N$, and Theorem 3.10 implies $M \cong N$. The converse is clear. $\square$

The well-known structure theorem for Warfield R-modules follows from this corollary and Proposition 3.12 by taking $V = Q$. But whereas the classic proof revolves around the existence of decomposition bases which generate nice submodules, when V has dimension 2 or greater the modules in $\mathcal{M}(V)$ will have no decomposition bases, and even their decomposition subsets need not generate nice submodules.

CHAPTER 4
ENDOMORPHISMS OVER INCOMPLETE RINGS

4.1 Introduction

In this chapter, we turn our attention toward the R-endomorphism algebras of some of the kinds of mixed R-modules we have encountered thus far. Broadly put, our goal is to determine the extent to which the algebra $End_R(M)$ reflects the structure of an R-module M from a given class. Traditionally, many results of this nature are manifested in *isomorphism theorems* of some type, which generally assert that two modules from a given class are isomorphic if their endomorphism algebras are isomorphic. In each case, there is something special about the structure of the underlying modules that makes such a theorem possible. Most results along these lines for the endomorphism algebras of mixed and torsion-free R-modules require R to be complete, a benefit we do not enjoy here. We shall, however, improve upon the one isomorphism theorem that was known to apply for mixed modules when R is incomplete (Theorem C below), as well as prove a weaker kind of isomorphism theorem for a significant class of mixed modules in the incomplete case. In certain places, the structure theory developed in the last two chapters will play an important role.

Our development commences in the next section with a study of the ideal $Hom_R(M, tM)$. It turns out that most of the information about M that can be found in $End_R(M)$ is carried by this ideal of homomorphisms into torsion. Consequently, most of the technical machinery will be in place when we conclude our study of $Hom_R(M, tM)$ in Section 2. The subsequent two sections are then devoted to the central issue of how the full algebra $End_R(M)$ determines the structure of M when R is incomplete. In Section 4, we will prove some of our strongest results for the special case where $rk(\hat{R}M) = 1$, including the fact that all algebra automorphisms of $End_R(M)$ are inner when M is sharp.

This section concludes with a list of relevant results from the theory of endomorphism algebras, some of which will serve as lemmas in what follows. An isomorphism $End_R(M) \to End_R(N)$ is *induced* if it is of the form $\sigma \mapsto \phi\sigma\phi^{-1}$ for an isomorphism $\phi : M \to N$.

A. THEOREM [8]. *If M and N are torsion R-modules, then every isomorphism $End_R(M) \to End_R(N)$ is induced by an isomorphism $M \to N$.*

B. THEOREM [10, 11]. *Suppose M is a reduced R-module and M/tM is divisible. If Φ is an isomorphism of $End_R(M)$ with $End_R(N)$ or of $Hom_R(M, tM)$ with $Hom_R(N, tN)$, then there is an isomorphism $\phi : tM \to tN$ such that $\Phi(\sigma)$ and $\phi\sigma\phi^{-1}$ agree on tN for all σ in $End_R(M)$ or $Hom_R(M, tM)$.*

C. THEOREM [9]. *Suppose M and N are reduced R-modules of rank 1 and tM is totally projective. Then every isomorphism $End_R(M) \to End_R(N)$ is induced by an isomorphism $M \to N$.*

D. THEOREM [11]. *In $\hat{R}$-Mod, suppose A is a reduced, mixed module and every full-rank submodule of A contains a nice submodule B such that A/B is totally projective. Then every isomorphism $End_{\hat{R}}(A) \to End_{\hat{R}}(C)$ is induced by an isomorphism $A \to C$.*

E. COROLLARY. *If M is a module as given in A, C or D above, then every algebra automorphism of $End_R(M)$ is inner, i.e., is of the form $\sigma \mapsto \phi\sigma\phi^{-1}$ for a unit $\phi \in End_R(M)$.*

4.2 Homomorphisms into Torsion

Our goal in this section is to show that under appropriate conditions, $\hat{R}M$ is determined up to isomorphism by $Hom_R(M, tM)$. Besides preparing the way for our subsequent study of $End_R(M)$, this result will yield the kind of intrinsic characterization of sharp R-modules that eluded us in Chapter 2.

Several conventions will be in effect. Recall that if M is a reduced R-module with torsion T and M/T is divisible, then $M^\bullet = T^\bullet$ and every endomorphism of M has a unique extension to an $\hat{R}$-endomorphism of $T^\bullet$. Consequently, $End_R(M)$ is naturally embedded in $End_{\hat{R}}(T^\bullet)$ as an R-subalgebra, as is $End_R(N)$ for all R-modules $N \subseteq T^\bullet$ with $T^\bullet/N$ torsion-free. Since $T^\bullet/T$ is divisible, the restriction homomorphism $\sigma \mapsto \sigma|_T$ from $End_{\hat{R}}(T^\bullet)$ to $End_R(T)$ is bijective, hence $End_{\hat{R}}(T^\bullet)$ is naturally identified with $End_R(T)$. Thus, we regard $Hom_R(N,T) \subseteq End_R(N) \subseteq End_{\hat{R}}(T^\bullet) = End_R(T)$ for the kinds of modules N stated above.

Lemmas 4.1 through 4.5 take place in $\hat{R}$-*Mod*. Our goal in this sequence of lemmas is to show that certain $\hat{R}$-submodules of $T^\bullet$ are completely determined by their homomorphisms into T. Our results will then be brought to bear on $\hat{R}M$ for appropriate mixed R-modules M. The proof of Lemma 4.1 can be found in [11].

LEMMA 4.1. *Let T be an unbounded torsion module contained in a totally projective module $\widetilde{T}$ with $\ell(\widetilde{T}) < \ell(T) + \omega$, and let σ be the least ordinal such that $p^\sigma T$ is bounded. Then $rk(p^\sigma(T^\bullet)) \geq 2^{\aleph_0}$ unless σ has cofinality exceeding ω, in which case $p^\sigma(T^\bullet)$ is torsion.*

In the next few lemmas, we use the fact that finite subsets of $\hat{R}$-modules generate nice submodules [3].

LEMMA 4.2. *In $\hat{R}$-Mod, suppose A is a reduced module containing a finite torsion-free basis X with $A/\langle X\rangle$ totally projective. If $T = tA$, then there is a totally projective module $\widetilde{T}$ containing T such that $\ell(\widetilde{T}) < \ell(T) + \omega$.*

PROOF: In $\hat{R}$-*Mod*, suppose B is a reduced module with $T = tB$, and $x \in B$ is torsion-free. We claim that $\ell(t(B/\langle x\rangle)) < \ell(T)+\omega$. It suffices to prove that the supremum of heights of nonzero socle elements of $B/\langle x\rangle$ is less than $\ell(T)+\omega$. If there exists a nonzero socle element $z_0 + \langle x\rangle \in B/\langle x\rangle$ with $|z_0 + \langle x\rangle| \geq \ell(T)$, we may assume $|z_0 + \langle x\rangle| = |z_0|$ since $\langle x\rangle$ is nice in B. In this case we claim $\ell(t(B/\langle x\rangle)) \leq |pz_0|$, which suffices since $pz_0 \neq 0$ and $\ell(B) \leq \ell(T)+\omega$. For the sake of contradiction, assume $z + \langle x\rangle$ is a nonzero socle element of $B/\langle x\rangle$ with $|z+\langle x\rangle| = |z| \geq |pz_0|$. Then $|pz| > |z| \geq |pz_0|$ and $pz, pz_0 \in \langle x\rangle$, so that $pz = p\alpha(pz_0)$ for some $\alpha \in \hat{R}$. Thus $0 \neq z - \alpha pz_0 \in T$ and $|z - \alpha pz_0| > \ell(T)$, a contradiction. This proves our initial claim. To prove the lemma, let A and X be as stated and write $X = \{x_1, \cdots, x_n\}$, $T_k = t(A/\langle x_1, \cdots, x_k\rangle)$ for $1 \leq k \leq n$. Denote $T_0 = T$, $\widetilde{T} = T_n$, and note the latter is totally projective. There are natural inclusions $T_{k-1} \subseteq T_k$ for $1 \leq k \leq n$, and by the fact proved above there are finite ordinals m_k such that $\ell(T_k) = \ell(T_{k-1}) + m_k$ for all such k. Thus $\ell(\widetilde{T}) = \ell(T) + m_1 + \cdots + m_n < \ell(T)+\omega$ as desired. □

LEMMA 4.3. *In $\hat{R}$-Mod, let A be a reduced module with torsion T. Assume that A/T is divisible, and A contains a finite basis X with $A/\langle X\rangle$ totally projective. If μ is an ordinal with $\mu+\omega \leq \ell(T^\bullet)$, then $rk(p^\mu(T^\bullet)) \geq 2^{\aleph_0}$.*

PROOF: Clearly, T is unbounded. By Lemma 4.2, T meets the hypotheses of Lemma 4.1. Let σ be the least ordinal such that $p^\sigma T$ is bounded, and assume $\mu+\omega \leq \ell(T^\bullet)$. Then $p^\mu(T^\bullet)$ cannot be bounded, since otherwise $\mu+\omega > \ell(T^\bullet)$. By Lemma 4.1, $rk(p^\mu(T^\bullet)) \geq 2^{\aleph_0}$ unless $\mu \geq \sigma$ and $p^\mu(T^\bullet)$ is torsion. But if this occurred, since T is isotype in $T^\bullet$ we would have $p^\mu(T^\bullet) = p^\mu T$, contradicting the unboundedness of $p^\mu(T^\bullet)$. □

LEMMA 4.4. *In $\hat{R}$-Mod, let A be a reduced module with torsion T such that A/T is divisible and assume $A/\langle X\rangle$ is totally projective for a finite torsion-free basis*

X. If $\langle X\rangle = \langle x\rangle \oplus \langle Y\rangle$ for $x \neq 0$, then there exists an ordinal μ with $\mu+\omega \leq \ell(T^\bullet)$ such that:

(1) Given $z \in p^\mu(T^\bullet)$, there exists $\alpha \in End_{\hat{R}}(T^\bullet)$ with $\alpha(x) = z$ and $\alpha(Y) = 0$.

(2) For α as in (1), if $y \in T^\bullet$ satisfies $|p^k\alpha(y)|_{T^\bullet} < \mu$ for all $k \geq 0$, then there exists $\beta \in End_{\hat{R}}(T^\bullet)$ with $\beta(y) = \alpha(y)$ and $\beta(X) = 0$.

PROOF: (1) By the proof of Lemma 4.2, $\ell(A/\langle X\rangle) < \ell(T) + \omega$. Put $T' = t(A/\langle Y\rangle)$. Since there are natural inclusions $T \subseteq T' \subseteq A/\langle X\rangle$ and all modules are reduced, $\ell(T) \leq \ell(T') < \ell(T) + \omega$. We claim there exists an ordinal μ with $\mu + \omega \leq \ell(T^\bullet)$ such that $|p^kx + \langle Y\rangle| \leq \mu + k$ for all $k \geq 0$ (heights without subscripts are taken in $A/\langle Y\rangle$ throughout). First assume $\ell(T^\bullet) > \ell(T)$, so that $\ell(T^\bullet) = \ell(T) + \omega$. If $|p^k + \langle Y\rangle| > \ell(T')$ for some k, let k_0 be the least such k and write $|p^{k_0}x + \langle Y\rangle| = \ell(T') + m$. Then clearly $|p^kx + \langle Y\rangle| \leq \ell(T') + m + k$ for all $k \geq 0$. Put $\mu = \ell(T') + m$ and note $\mu + \omega \leq \ell(T) + \omega = \ell(T^\bullet)$ by the inequality given above. For the other case, we have $\ell(T) = \ell(T^\bullet)$ and Lemma 4.1 implies $\sigma \leq \ell(T) = \ell(T^\bullet) < \sigma + \omega$, where σ is an ordinal of cofinality greater than ω. Note $\ell(A) = \ell(T)$ since $T \subseteq A \subseteq T^\bullet$, and $\ell(A/\langle Y\rangle) \leq \ell(A)$ since $\langle Y\rangle$ is nice in A. Since $x + \langle Y\rangle$ is torsion-free in $A/\langle Y\rangle$, $\{|p^kx + \langle Y\rangle|\}_{0\leq k<\omega}$ is an increasing sequence of ordinals strictly less than $\ell(A/\langle Y\rangle)$. Let μ be the supremum of this sequence and note $\mu + \omega < \sigma$ by the cofinality of σ. This yields the desired μ, and proves the claim. Now suppose $z \in p^\mu(T^\bullet)$, and note $|p^kx + y|_A \leq |p^kx + \langle Y\rangle| \leq \mu + k$ for all $k \geq 0$ and $y \in \langle Y\rangle$. Hence, mapping $x \mapsto z$ and $Y \mapsto 0$ induces a homomorphisim $\langle X\rangle \to T^\bullet$ which does not decrease heights relative to A and $T^\bullet$. Since $\langle X\rangle$ is nice and $A/\langle X\rangle$ is totally projective, we get a homomorphism $A \to T^\bullet$ which extends to the desired $\alpha \in End_{\hat{R}}(T^\bullet)$.

(2) Let α be as above, and assume $y \in T^\bullet$ satisfies $|p^k\alpha(y)|_{T^\bullet} < \mu$ for all $k \geq 0$. Then since $\alpha(\langle X\rangle) \subseteq p^\mu(T^\bullet)$ and $\langle X\rangle$ is nice, $|p^ky+\langle X\rangle| \leq |p^k\alpha(y)|_{T^\bullet}$ for all $k \geq 0$ (here, unmarked heights are taken in $T^\bullet/\langle X\rangle$). Let $\widetilde{A} = \langle y, X\rangle_* \subseteq$

$T^\bullet$ and observe $A/\langle X\rangle \cong (\langle y\rangle \oplus A)/\langle y, X\rangle \subseteq \widetilde{A}/\langle y, X\rangle$, with $\widetilde{A}/(\langle y\rangle \oplus A) \cong \langle y\rangle_*/\langle y, T\rangle$ countably-generated. Thus, $\widetilde{A}/\langle y, X\rangle$ is totally projective since it is isomorphic to a countably-generated extension of $A/\langle X\rangle$. Note $\langle y, X\rangle$ is nice in $\widetilde{A}$. Since $|p^k y + w|_{\widetilde{A}} \leq |p^k y + \langle X\rangle| \leq |p^k \alpha(y)|_{T^\bullet}$ for all $w \in \langle X\rangle$ and all $k \geq 0$, the homomorphism $\langle y, X\rangle \to T^\bullet$ induced by $y \mapsto \alpha(y)$ and $X \mapsto 0$ does not decrease heights relative to $\widetilde{A}$ and $T^\bullet$, hence extends to a homomorphism $\widetilde{A} \to T^\bullet$. This last homomorphism then extends to $\beta \in End_{\hat{R}}(T^\bullet)$ with $\beta(y) = \alpha(y)$ and $\beta(X) = 0$. $\square$

LEMMA 4.5. *In $\hat{R}$-Mod, let A be a reduced, nontorsion module with torsion T such that A/T is divisible, and assume $A/\langle X\rangle$ is totally projective for a finite torsion-free basis X. If B is a submodule of $T^\bullet$ with $T^\bullet/B$ torsion-free and $Hom_{\hat{R}}(A,T) = Hom_{\hat{R}}(B,T)$, then $A = B$.*

PROOF: We first show $B \subseteq A$. Let X be as stated, and suppose $y \notin A$. Put $C = \langle y, X\rangle_* \subseteq T^\bullet$, and note $\langle y, X\rangle = \langle y\rangle \oplus \langle X\rangle$. Then $A/\langle X\rangle$ is naturally embedded in $C/\langle y, X\rangle$ with $(C/\langle y, X\rangle)/(A/\langle X\rangle) \cong \langle y\rangle_*/\langle y, T\rangle$ countably-generated, so that $C/\langle y, X\rangle$ is totally projective. By Lemma 4.4, there is an ordinal μ with $\mu + \omega \leq \ell(T^\bullet)$ such that given $z \in p^\mu(T^\bullet)$, there exists $\alpha \in End_{\hat{R}}(T^\bullet)$ with $\alpha(y) = z$ and $\alpha(X) = 0$. We can choose a torsion-free $z \in p^\mu(T^\bullet)$ by Lemma 4.3, and if $\alpha \in End_{\hat{R}}(T^\bullet)$ is as just stated then $\alpha \in Hom_{\hat{R}}(A,T) = Hom_{\hat{R}}(B,T)$ and $\alpha(y) \notin T$, so that $y \notin B$. This proves $B \subseteq A$. Now suppose $B \neq A$, and note $B \cap \langle X\rangle$ is a full-rank submodule of B. Since $rk(B) < rk(A)$, we can write $\langle X\rangle = \langle Z\rangle \oplus \langle w\rangle$, where $B \cap \langle X\rangle \subseteq \langle Z\rangle$ and $w \neq 0$. By Lemmas 4.3 and 4.4, there exists a torsion-free element $z \in T^\bullet$ together with $\alpha \in End_{\hat{R}}(T^\bullet)$ such that $\alpha(w) = z$ and $\alpha(Z) = 0$. Now $\alpha \in Hom_{\hat{R}}(B,T)$ and $\alpha(A) \nsubseteq T$, a contradiction. Thus, $A = B$. $\square$

We now possess the tools needed to prove the main theorems of this chapter. Before turning our attention to $End_R(M)$ for mixed R-modules M, we prove two results about $Hom_R(M, tM)$. Recall that when M is reduced with torsion T and M/T is divisible, then $Hom_R(M,T)$ is embedded in $End_{\hat{R}}(T^{\bullet}) = End_R(T)$. In fact, if $\alpha \in End_{\hat{R}}(T^{\bullet})$, then $\alpha(M) \subseteq T$ if and only if $\alpha(\hat{R}M) \subseteq T$, so that $Hom_{\hat{R}}(\hat{R}M, T) = Hom_R(M,T)$.

PROPOSITION 4.6. *Let M be a reduced, nontorsion R-module with totally projective torsion T, and assume M/T is divisible. The following are equivalent.*

(1) M is sharp.

(2) $Hom_R(M,T) = Hom_R(N,T)$ for a rank 1 submodule N with M/N torsion-free.

(3) $Hom_R(M,T) = Hom_R(N,T)$ for all nontorsion submodules N with M/N torsion-free.

PROOF: Clearly, (3) implies (2). As in (2), assume $Hom_R(M,T) = Hom_R(N,T)$ for a rank 1 submodule N with M/N torsion-free, and note $\hat{R}N$ is a rank 1 Warfield $\hat{R}$-module. We have $Hom_{\hat{R}}(\hat{R}N,T) = Hom_{\hat{R}}(\hat{R}M,T)$ by what was noted above, and by setting $A = \hat{R}N$ and $B = \hat{R}M$ in Lemma 4.5 we obtain $\hat{R}N = \hat{R}M$, so that (1) holds. To see that (1) implies (3), let $N \subseteq M$ be a nontorsion submodule with M/N torsion-free. Let $\sigma \in Hom_R(N,T)$, and choose a torsion-free element $x \in N$. If $y \in M$, then $\alpha y = \beta x$ for some $\alpha, \beta \in \hat{R}$, hence $\alpha\sigma(y) = \beta\sigma(x) \in T$ and $\sigma(M) \subseteq T$. Thus, $Hom_R(N,T) \subseteq Hom_R(M,T)$. The reverse inclusion is clear, so (3) holds. $\square$

When M is mixed, $Hom_R(M, tM)$ is an R-algebra without identity. All isomorphisms $Hom_R(M, tM) \to Hom_R(N, tN)$ are assumed to be algebra isomorphisms.

PROPOSITION 4.7. *Suppose M is a reduced, nontorsion R-module such that M/tM is divisible and $\hat{R}M/\hat{R}\langle X\rangle$ is totally projective for a finite basis X of $\hat{R}M$. If N is a reduced R-module and $Hom_R(M,tM) \cong Hom_R(N,tN)$, then $\hat{R}M \cong \hat{R}N$.*

PROOF: Let $\Phi : Hom_R(M,tM) \to Hom_R(N,tN)$ be the given isomorphism and denote $T = tN$. By Theorem B, there is an isomorphism $\phi : tM \to T$ such that $\Phi(\sigma)$ and $\phi\sigma\phi^{-1}$ agree on T for all $\sigma \in Hom_R(M,tM)$. Recall from Lemma 2.4 that N/T is divisible if $\hat{R}N/T$ is, hence N/T is divisible if $Hom_{\hat{R}}(\hat{R}N/T,T) = 0$. Let $\pi : \hat{R}N \to \hat{R}N/T$ be the natural map and observe that $\Phi^{-1}(\lambda\pi)$ agrees with $\phi^{-1}(\lambda\pi)\phi$ on tM for all $\lambda \in Hom_{\hat{R}}(\hat{R}N/T,T)$. Now $\big(\phi^{-1}(\lambda\pi)\phi\big)(tM) = 0$ and M/tM is divisible, hence $\Phi^{-1}(\lambda\pi) = 0$ and we have $\lambda = 0$ for all $\lambda \in Hom_{\hat{R}}(\hat{R}N/T,T)$, as desired. Let $\phi^\bullet$ be the extension of ϕ to an isomorphism $(tM)^\bullet \to T^\bullet$, and denote $M' = \phi^\bullet(M)$. Then $(tM)^\bullet/M \cong T^\bullet/M'$ via the map $x + M \to \phi^\bullet(x) + M'$, so M' and N are contained in $T^\bullet$ with $T^\bullet/M'$ and $T^\bullet/N$ torsion-free. The map Φ' given by $\Phi'(\sigma) = \Phi\big((\phi^\bullet)^{-1}\sigma\phi^\bullet\big)$ for all $\sigma \in Hom_R(M',T)$ is easily checked to be an isomorphism $Hom_R(M',T) \to Hom_R(N,T)$. In fact, since $\Phi'(\sigma)$ agrees with $\phi(\phi^{-1}\sigma\phi)\phi^{-1} = \sigma$ on T and N/T is divisible, we have $\Phi'(\sigma) = \sigma$ for all $\sigma \in Hom_R(M',T)$. Thus $Hom_{\hat{R}}(\hat{R}M',T) = Hom_{\hat{R}}(\hat{R}N,T)$ and Lemma 4.5 implies $\hat{R}M' = \hat{R}N$, so that $\hat{R}M \cong \hat{R}N$ as desired. □

4.3. Isomorphism of Endomorphism Algebras

Our purpose here is to establish some new kinds of isomorphism theorems for the endomorphism algebras of mixed R-modules. Our main results state that under certain conditions, $End_R(M)$ determines the structure of $\hat{R}M$, and sometimes even that of M itself. Theorem D is a paradigm for the case where R is complete, and in keeping we focus here on R-modules M for which $\hat{R}M$ contains a basis X with $\hat{R}M/\hat{R}\langle X\rangle$ totally projective.

It is worth noting that even in the case where $rk(\hat{R}M) = 1$, we cannot conclude that $End_R(M)$ faithfully reflects the structure of M. Let T be an unbounded, totally projective module and x a torsion-free element of $T^\bullet$. Choose an element $\alpha \in \hat{R}$ transcendental over R, and let $M = (R\langle x, \alpha x\rangle)_*$, $N = (R\langle x, \alpha^2 x\rangle)_*$ in $T^\bullet$. We will see in Chapter 5 that $End_R(M) = End_R(N)$, but M and N are nonisomorphic since they are sharp and $Q \oplus \alpha Q$ is not $\hat{Q}$-equivalent to $Q \oplus \alpha^2 Q$.

Our first result indicates how $End_R(M)$ determines $Hom_R(M, tM)$ for mixed R-modules M, thus providing a crucial link with results in the previous section. The technical Lemma 4.4 will be put to full use for the first time. To set up notation, assume T is a torsion module and M is an R-submodule of $T^\bullet$ with $T^\bullet/M$ torsion-free. We define M^* to be the maximal $End_{\hat{R}}(T^\bullet)$-module contained in M, i.e., $M^* = \{x \in M : \alpha(x) \in M \text{ for all } \alpha \in End_{\hat{R}}(T^\bullet)\}$. Clearly, M^* is a pure $\hat{R}$-submodule of $T^\bullet$ containing T.

LEMMA 4.8. *Assume M is a reduced, nontorsion R-module with torsion T, M/T is divisible, and $\hat{R}M/\hat{R}\langle X\rangle$ is totally projective for a finite basis X of $\hat{R}M$. If N is an R-submodule of $T^\bullet$ with $T^\bullet/N$ torsion-free and $End_R(M) = End_R(N)$, then $Hom_R(M, T) = Hom_R(N, T)$.*

PROOF: We first show $Hom_R(M, M^*) = Hom_R(N, N^*)$. Let $\alpha \in Hom_R(M, M^*)$. For all $\beta \in End(T^\bullet)$ we have $\beta\alpha(M) \subseteq M^*$, hence $\beta\alpha \in End_R(M) = End_R(N)$ and $\alpha(N) \subseteq N^*$. Thus $Hom_R(M, M^*) \subseteq Hom_R(N, N^*)$, and by symmetry we also have the reverse inclusion. We conclude the proof by showing $M^* = T$ and then $N^* = T$. Let X be the given basis for $\hat{R}M$. If $M^* \neq T$, choose a nonzero element $w \in M^* \cap \hat{R}\langle X\rangle$ and write $\hat{R}\langle X\rangle = \hat{R}\langle x\rangle \oplus \hat{R}\langle Y\rangle$, where $w \in \hat{R}\langle x\rangle$. By Lemma 4.4, there is an ordinal μ with $\mu + \omega \leq \ell(T^\bullet)$ such that for all $z \in p^\mu(T^\bullet)$, there exists $\alpha \in End_{\hat{R}}(T^\bullet)$ with $\alpha(x) = z$. Thus

$p^\mu(T^\bullet) \subseteq M^*$ by definition of M^*. But $p^\mu(T^\bullet)$ has infinite rank by Lemma 4.3, contrary to our assumption on the rank of $\hat{R}M$. Hence, $M^* = T$ and we have $Hom_R(M,T) = Hom_R(N,N^*)$. Now suppose $N^* \neq T$ and choose a torsion-free element $y \in N^*$. Set $A = (\hat{R}\langle y, X\rangle)_* \subseteq T^\bullet$ and note as in the proof of Lemma 4.5 that A satisfies the hypotheses of Lemma 4.4. Applying part (1) of the latter to the basis $\hat{R}\langle y\rangle \oplus \hat{R}\langle X\rangle$ for A, we obtain μ with $\mu + \omega \leq \ell(T^\bullet)$ such that given $z \in p^\mu(T^\bullet)$, there exists $\alpha \in End_{\hat{R}}(T^\bullet)$ with $\alpha(y) = z$. Thus, $p^\mu(T^\bullet) \subseteq N^*$. Let $0 \neq x \in X$. Lemma 4.3 guarantees the existence of a torsion-free element $z \in p^\mu(T^\bullet)$, and by Lemma 4.4(1) there exists $\alpha \in End_{\hat{R}}(T^\bullet)$ with $\alpha(x) = z$. We claim $\alpha(N) \subseteq N^*$. Let $w \in N$ and note by purity that $\alpha(w) \in N^*$ if $|p^k\alpha(w)| \geq \mu$ for some $k \geq 0$. Thus, assume $|p^k\alpha(w)| < \mu$ for all k. By Lemma 4.4(2), there exists $\beta \in End_{\hat{R}}(T^\bullet)$ with $\beta(w) = \alpha(w)$ and $\beta(X) = 0$, so that $\beta \in Hom_R(M,T) = Hom_R(N,N^*)$ and $\alpha(w) \in N^*$ as desired. Now we have $\alpha \in Hom_R(N,N^*) = Hom_R(M,T)$ and $\alpha(x) = z \notin T$, a contradiction. Hence, $N^* = T$. $\square$

We can now prove our main result by piecing together two lemmas. Some substantial corollaries will follow. In the next section, we will give a slightly improved version of the theorem for the case where $\hat{R}M$ has rank 1.

THEOREM 4.9. *Suppose M is a reduced R-module such that M/tM is divisible and $\hat{R}M/\hat{R}\langle X\rangle$ is totally projective for a finite basis X. If $End_R(M) \cong End_R(N)$, then $\hat{R}M \cong \hat{R}N$.*

PROOF: Let $\Phi : End_R(M) \to End_R(N)$ be the given isomorphism and denote $T = tN$. By Theorem B, there exists an isomorphism $\phi : tM \to T$ such that $\Phi(\sigma)$ and $\phi\sigma\phi^{-1}$ agree on T for all $\sigma \in End_R(M)$. If $N = P \oplus N'$ with $P \cong Q$, let $\pi : N \to P$ be projection and note that $\Phi^{-1}(\pi)(M)$ is a direct summand of M isomorphic to Q. This contradiction shows that N must be reduced. The argument used in the proof of Proposition 4.7 shows that N/T is divisible. Let $M' = \phi^\bullet(M) \subseteq T^\bullet$, and note $T^\bullet/M'$ is torsion-free. As in the proof of the proposition, the isomorphism $End_R(M') \to End_R(N)$ given by $\sigma \mapsto \Phi(\phi^{-1}\sigma\phi)$ is the identity map, and we have $End_R(M') = End_R(N)$. Thus $Hom_R(M', T) = Hom_R(N, T)$ by Lemma 4.8, so that $Hom_{\hat{R}}(\hat{R}M', T) = Hom_{\hat{R}}(\hat{R}N, T)$. Lemma 4.5 implies $\hat{R}M' = \hat{R}N$, hence $\hat{R}M \cong \hat{R}N$. □

In this theorem, no assumptions are made about the second module N. We obtain the following corollaries by making further assumptions about both M and N. First, recall the class $\mathcal{M}(V)$ of R-modules associated to each finite-dimensional Q-subspace V of $\hat{Q}$ (Definition 3.11 and Corollary 3.14).

COROLLARY 4.10. *Let M and N be R-modules in $\mathcal{M}(V)$. Assume M has finite rank and M/tM is divisible. If $End_R(M) \cong End_R(N)$, then $M \cong N$.*

PROOF: $\hat{R}M$ is a Warfield module of finite rank, hence Theorem 4.9 implies $\hat{R}M \cong \hat{R}N$. By Corollary 3.14, we have $M \cong N$ if M and N have the same Ulm and Warfield invariants. As in Chapter 3, there is a torsion module S so that $M \oplus S = M_1 \oplus \cdots \oplus M_n$ is a direct sum of sharp R-modules. Note that $\hat{R}M \oplus S = \hat{R}M_1 \oplus \cdots \oplus \hat{R}M_n$. By Lemma 3.13, $W_{\hat{R}M_i} = W_{M_i}$ for all i and we have $W_{\hat{R}M} = W_{\hat{R}M \oplus S} = \sum_{1 \le i \le n} W_{\hat{R}M_i} = \sum_{1 \le i \le n} W_{M_i} = W_{M \oplus S} = W_M$, hence M and $\hat{R}M$ have the same Warfield invariants. The same argument shows $W_{\hat{R}N} = W_N$, so that $W_M = W_N$ as desired. □

We bring the section to a close with a special case of this corollary.

COROLLARY 4.11. *Let M and N be Warfield R-modules. Assume M is reduced, has finite rank, and M/tM is divisible. If $End_R(M) \cong End_R(N)$, then $M \cong N$.*

PROOF: By Proposition 3.12, M and N are contained in $\mathcal{M}(Q)$. Corollary 4.10 now implies $M \cong N$. □

4.4. Endomorphisms of Sharp Modules

In Chapter 2, we saw that rank 1 R-modules with totally projective torsion submodules are automatically sharp, and that sharp R-modules enjoy many of the properties of the former kind. The main result of this section strengthens the connection: the corollary to Theorem C is extended to encompass the entire class of sharp R-modules. Since isomorphisms $End_R(M) \to End_R(N)$ need not be induced by isomorphisms $M \to N$ in this case, it is surprising that $End_R(M)$ possesses only inner automorphisms.

First, we give an improved version of Theorem 4.9 for our setting that does not require M/tM to be divisible.

COROLLARY 4.12. *If M is a sharp R-module that is not torsion-free and $End_R(M) \cong End_R(N)$, then $\hat{R}M \cong \hat{R}N$.*

PROOF: Let π_L denote the projection of M onto any direct summand L of M. If M/tM is divisible, then $\hat{R}M \cong \hat{R}N$ by Theorem 4.9. Thus, by Lemma 2.4(2), we may assume $M = A \oplus T$, where $\hat{R}A \cong \hat{R}$ and $T \neq 0$ is torsion. Let $\langle t \rangle$ be a nonzero direct summand of T, and let ρ_A, ρ_T and $\rho_{\langle t \rangle}$ denote the respective images of π_A, π_T and $\pi_{\langle t \rangle}$ under the given isomorphism. Writing $B = \rho_A(N)$, $S = \rho_T(N)$ and $\langle s \rangle = \rho_{\langle t \rangle}(N)$, we have $End_R(S) \cong End_R(T)$ and $End_R(B) \cong End_R(A)$, so that $S \cong T$ (by Theorem A) and B is torsion-free and reduced (reduced since B cannot be isomorphic to Q or Q/R and $End_R(B)$ contains no nontrivial idempotents, and torsion-free since $End_R(B)$ is torsion-free). We have $\hat{R}B/p(\hat{R}B) \cong B/pB$ and $A/pA \cong R/pR = R(p)$ since B and R are pure and dense in $\hat{R}B$ and A, respectively. Observe $Hom_R(A, \langle t \rangle) \cong Hom_R(B, \langle s \rangle)$ via the isomorphism $\pi_{\langle t \rangle} End_R(M) \pi_A \cong \rho_{\langle t \rangle} End_R(N) \rho_A$, hence $R(p) \cong Hom_R(A/pA, R(p)) \cong Hom_R(A, \langle t \rangle)[p] \cong Hom_R(B, \langle s \rangle)[p] \cong Hom_R(B/pB, R(p))$. If $rk(\hat{R}B) > 1$, then $B/pB \cong R(p) \oplus R(p) \oplus C$ (some

C) and the last module above could not be cyclic. This contradiction shows $\hat{R}B \cong \hat{R}$, hence $\hat{R}N \cong \hat{R} \oplus T \cong \hat{R}M$. $\square$

THEOREM 4.13. *If M is a sharp R-module, then every algebra automorphism of $End_R(M)$ is inner.*

PROOF: Let Φ be an automorphism of $End_R(M)$. If $\phi \in End_{\hat{R}}(\hat{R}M) = Hom_{\hat{R}}(\hat{R}M, T) \oplus \hat{R}$, write $\phi = \sigma + \alpha$ with $\sigma(M) \subseteq T$ and $\alpha \in \hat{R}$. Define $\hat{\Phi} : End_{\hat{R}}(\hat{R}M) \to End_{\hat{R}}(\hat{R}M)$ by $\hat{\Phi}(\phi) = \Phi(\sigma) + \alpha$. The main fact needed to prove that $\hat{\Phi}$ is an $\hat{R}$-algebra isomorphism is that $\alpha\Phi(\sigma) = \Phi(\alpha\sigma)$ for all $\alpha \in \hat{R}$ and $\sigma \in Hom_R(M,T)$. To see that this holds, let D be the submodule of $\hat{R}M$ generated by elements of the form $\alpha\Phi(\sigma)(m) - \Phi(\alpha\sigma)(m)$, where $\alpha \in \hat{R}$, $\sigma \in Hom_R(M,T)$, and $m \in \hat{R}M$. Given $k \geq 0$ and a generator of D as above, write $\alpha = p^k\alpha_0 + r$ ($\alpha_0 \in \hat{R}, r \in R$) and note $\alpha\Phi(\sigma)(m) - \Phi(\alpha\sigma)(m) = p^k(\alpha_0\Phi(\sigma)(m) - \Phi(\alpha_0\sigma)(m)) + r\Phi(\sigma)(m) - \Phi(r\sigma)(m) \in p^kD$. Thus, D is divisible. Since $\hat{R}M$ is reduced, $D = 0$ and the needed fact is proved. A similar argument shows that $\hat{\Phi}$ extends Φ. By the corollary to Theorem D, there is an automorphism λ of $\hat{R}M$ such that $\hat{\Phi}(\phi) = \lambda\phi\lambda^{-1}$ for all $\phi \in End_{\hat{R}}(\hat{R}M)$. Since M and $\lambda(M)$ are isomorphic sharp modules contained in $\hat{R}M$, Lemma 2.9(2) implies $M = \alpha\lambda(M)$ for a unit $\alpha \in \hat{R}$. Now $\psi = \alpha\lambda|_M$ is an automorphism of M, and for all $\phi \in End_R(M)$ we have $\psi\phi\psi^{-1} = (\alpha\lambda)\phi(\alpha\lambda)^{-1}|_M = \lambda\phi\lambda^{-1}|_M = \hat{\Phi}(\phi)|_M = \Phi(\phi)$ because α commutes with endomorphisms. Thus, ψ induces Φ. $\square$

Our proof of this theorem makes heavy use of the fact that $\hat{R}M$ has rank 1. In the last chapter, we will see that $End_R(M)$ can have outer automorphisms when $\hat{R}M$ has greater rank.

CHAPTER 5
CONSTRUCTIONS

For convenience, our constructions will be carried out in the setting $R = \mathbb{Z}_{(p)}$, where p is a prime number. The countability of R will prove advantageous here. We refer to our R-modules now as abelian groups G, and take $\hat{R}G$ as a module over the p-adic integers $\hat{R}$. In this brief chapter, we construct three special kinds of abelian groups whose properties illuminate results from Chapters 2-4 in a different way. All three constructions make use of special elements of $\hat{R}$; recall that $\alpha \in \hat{R}$ is *transcendental* over R if the set $\{\alpha^i : 0 \leq i < \omega\}$ is R-linearly independent, and that the *transcendence degree* of $\hat{R}$ over R is at least λ if there is a subset of $\hat{R}$ of cardinality λ consisting of transcendental, R-independent elements. In [6], it is proved that the transcendence degree of $\hat{R}$ over R is exactly $2^{\aleph_0}$ when the latter is countable.

Only our first construction requires additional preparation. In [12, 13], a reduced R-module M with $\ell(M) \leq \omega$ is called *taut* if $\ell(M) = \ell(M \bigotimes_R \hat{R})$, otherwise *slack*. These early accounts give structure theorems for limited classes of taut modules, but left even the existence of slack modules an open matter. The sole examples of slack modules came later in [14], with considerable effort expended in their construction. Although the theory of taut modules has not prospered beyond the confines of these few papers, we include a result demonstrating the ease with which we are now able to construct the once-elusive variety of slack modules.

PROPOSITION 5.1. *If T is an unbounded, totally projective group and $2 \leq \lambda \leq 2^{\aleph_0}$, then there exist $2^{\aleph_0}$ nonisomorphic, slack abelian groups G of rank λ with $tG = T$ and G/T divisible.*

PROOF: Let $R = \mathbb{Z}_{(p)}$ and write $T \cong R(p^{n_i}) \oplus S_i$ for $0 \leq i < \omega$, where $n_0 < n_1 < n_2 < \cdots$. Choose a torsion-free element $x \in T^{\bullet}$ with $|x| \geq \omega$ by Lemma 4.1. Let e_0 denote the sequence $\{0, 1, 2, \cdots\}$. If $e = \{\sigma_0, \sigma_1, \sigma_2, \cdots\}$ is any increasing sequence of finite ordinals not equivalent to e_0 (in the ense of height sequences) and such that $\sigma_k \in \{n_i - 1 : 0 \leq i < \omega\}$ whenever $\sigma_{k+1} > \sigma_k + 1$, results in [3] guarantee the existence of a rank 1 group G with $tG = T$, G/T divisible, and $e \in H(G)$. Thus G embeds purely in $T^{\bullet}$, and latter contains an element x_e with height sequence equivalent to e. By choosing a set $\{e_i : i \in 2^{\aleph_0}\}$ consisting of mutually inequivalent sequences satisfying the above conditions and denoting $x_i = x_{e_i}$, we obtain $2^{\aleph_0}$ nonisomorphic modules $(\hat{R}\langle x, x_i\rangle)_* \subseteq T^{\bullet}$ of length $\omega 2$. Since $\hat{R}$ has transcendence degree $2^{\aleph_0}$ over R, given λ with $2 \leq \lambda \leq 2^{\aleph_0}$ we can choose $\{\alpha_j : j \in J\} \subseteq \hat{R}$ such that J has cardinality $\lambda - 1$ and the set $\{1\} \cup \{\alpha_j : j \in J\}$ is R-linearly independent. For each $i \in 2^{\aleph_0}$, define $G_i = (R\langle x_i, \alpha_j x_i + x : j \in J\rangle)_* \subseteq T^{\bullet}$. Then all torsion-free elements of G_i have finite height, so $\ell(G_i) = \omega$. Clearly $rk(G_i) = \lambda$ for all i, and $\hat{R}G_i = (\hat{R}\langle x, x_i\rangle)_*$. Let D_i be the maximal divisible submodule of $G_i \bigotimes_R \hat{R}$. Proposition 2.1(1) implies $\hat{R}G_i \oplus D_i \cong G_i \bigotimes_R \hat{R}$, so that $\ell(G_i \bigotimes_R \hat{R}) = \ell(\hat{R}G_i) = \omega 2 > \ell(G_i)$ and each G_i is slack. Since the $\hat{R}G_i$ are mutually nonisomorphic, the same holds for the collection of $2^{\aleph_0}$ slack groups G_i. $\square$

As a by-product, this construction shows that M and $\hat{R}M$ can have unequal Warfield invariants. If e denotes the height sequence of x above, then $W_{\hat{R}G_i}(e) = 1$ since $(\hat{R}G_i)(e)/((\hat{R}G_i)(e))^* \cong \hat{R}\langle x\rangle/\hat{R}\langle px\rangle \cong \hat{R}/p\hat{R}$, while $W_{G_i}(e) = 0$ since $G_i(e) = 0$. In particular, this shows the necessity of taking $rk(\hat{R}M) = 1$ in Lemma 3.13.

The relevance of our next construction is clear in light of Corollary 2.11, Lemma 3.4 and, especially, Theorem 3.10.

Example 5.2. There are *WALK*-nonisomorphic, sharp abelian groups G and H of countable rank such that $G \oplus G \cong H \oplus H$.

As before, we commence by taking $R = \mathbb{Z}_{(p)}$, $T = \bigoplus_{1 \leq i < \omega} R(p^i)$, and $x \in T^\bullet$ torsion-free. Choose $\alpha \in \hat{R}$ transcendental over R with $|\alpha|_{\hat{R}} > 0$, and let $\beta = 1 + \alpha^2$. Members of the polynomial ring $R[\alpha]$ will be denoted $f(\alpha)$. Let G and H be the purifications in $T^\bullet$ of the groups

$$R\langle \beta^{-n} f(\alpha)x : 0 \leq n < \omega, deg(f) \leq 2n \rangle$$

and

$$R\langle \beta^{-n} f(\alpha)x : 0 \leq n < \omega, deg(f) \leq 2n + 1 \rangle,$$

respectively. Since $\alpha^2 - 1 \neq 0$, the matrix $\begin{bmatrix} \alpha & 1 \\ 1 & \alpha \end{bmatrix}$ induces an automorphism Φ of $T^\bullet \oplus T^\bullet$ in the obvious way. Clearly $\Phi(G \oplus G) = H \oplus H$ by the way G and H were constructed, hence $G \oplus G \cong H \oplus H$.

Since $rk(\hat{R}G) = rk(\hat{R}H) = 1$, Lemma 3.3 implies G and H are *WALK*-nonisomorphic if $\overline{G}$ and $\overline{H}$ are not $\hat{Q}$-equivalent. It follows from the transcendence of α over R that $\overline{G}$ and $\overline{H}$ can be represented as fixed subspaces of $\hat{Q}$ by the bases

$$\{\beta^{-n}\alpha^i : 0 \leq n < \omega, 0 \leq i \leq min\{1, n\}\}$$

and

$$\{\beta^{-n}\alpha^i : 0 \leq n < \omega, 0 \leq i \leq 1\},$$

respectively. Note both $\overline{G}$ and $\overline{H}$ contain 1. If $\overline{H} = \gamma\overline{G}$ for some $\gamma \in \hat{Q}$, then $\gamma = \sum_{i,n} q_{in}\beta^{-n}\alpha^i \in \overline{H}$ for elements $q_{in} \in Q$, and $1 = \gamma \sum_{i,n} q'_{in}\beta^{-n}\alpha^i = \gamma\delta$ for elements $q'_{in} \in Q$. Let $N = max\{n : q_{in} \neq 0 \text{ for some } i\}$ and $N' = max\{n : q'_{in} \neq 0 \text{ for some } i\}$. Then $\beta^N\gamma$, $\beta^{N'}\delta \in Q[\alpha]$, and $\beta^{N+N'} = (\beta^N\gamma)(\beta^{N'}\delta)$. Since β is irreducible in $Q[\alpha]$, it follows from the last equality that $\gamma = 1$.

Thus $\overline{H} = \overline{G}$, so that $\alpha \in \overline{G}$. Writing $\alpha = \sum_{i,n} q_{in}\beta^{-n}\alpha^i$ for elements $q_{in} \in Q$ and taking N as above, we have $\beta^N\alpha = \sum_{i,n} q_{in}\beta^{N-n}\alpha^i$ in $Q[\alpha]$. Now $deg(\beta^{N-n}\alpha^i) \leq 2(N-n)+i \leq 2(N-n)+2n = 2N$ when $q_{in} \neq 0$, so the sum has degree at most $2N$, contradicting $deg(\beta^N\alpha) = 2N+1$. Thus, $\overline{G}$ and $\overline{H}$ cannot be $\hat{Q}$-equivalent.

If M is a reduced R-module with M/tM divisible, the proof of Theorem 29 in [8] can be adapted to show that the center of $End_R(M)$ is contained in $\hat{R}$. We complete our work with an example reflecting upon Theorem 4.13.

Example 5.3. There is a reduced abelian group G with tG totally projective and $rk(\hat{R}G) = 2$ such that $End_R(G)$ possesses an outer automorphism.

There is much leeway in our construction. Let $R = \mathbb{Z}_{(p)}$, and choose a unit $\alpha \in \hat{R}$ transcendental over R. Let G_0 be any reduced, rank 1 group with $tG_0 = T$ totally projective and G_0/T divisible. Viewing $G_0 \subseteq T^\bullet$, define $G_1 = G_0 + \alpha G_0$ and $G_2 = G_0 + \alpha^2 G_0$. Before defining G, we prove two facts: (1) $Hom_R(G_i, G_j) = Hom_R(G_i, T)$ if $i \neq j$, and (2) $End_R(G_i) = End_R(G_0)$ for $i = 1, 2$.

To prove (1), choose $x \in G_0$ torsion-free and let $\phi \in Hom_R(G_1, G_2)$. There exist r, $s \in R$ and $n \geq 0$ such that $p^n\phi(x) = rx + s\alpha^2 x$. Thus $p^n\phi(\alpha x) = r\alpha x + s\alpha^3 x \in G_2$, so that one of the four sets $\{r\alpha, s\alpha^3\}$, $\{r\alpha, s\alpha^3, 1\}$, $\{r\alpha, s\alpha^3, \alpha^2\}$ and $\{r\alpha, s\alpha^3, 1, \alpha^2\}$ is R-linearly dependent. Since α is transcendental over R, this implies $rs = 0$. Repeating this argument with $r = 0$ or $s = 0$ yields $r = s = 0$, so that $p^n\phi(x) = 0$ and $\phi(G_1) \subseteq T$. A similar argument shows $Hom_R(G_2, G_1) = Hom_R(G_2, T)$.

The inclusion $End_R(G_0) \subseteq End_R(G_1)$ is clear in (2). If $\phi \in End_R(G_1)$, write $p^n\alpha\phi(x) = r\alpha x + s\alpha^2 x \in G_1$ similar to before and note that $s = 0$ in this case. Thus $p^n\phi(x) = rx$, and it follows that $\phi(G_0) \subseteq G_0$ since $G_0/\langle x\rangle$ is torsion and G_0 is pure in $T^\bullet$. Similar arguments show $End_R(G_2) = End_R(G_0)$.

Define $G = G_1 \oplus G_2 \subseteq T^\bullet \oplus T^\bullet$, and let λ denote the automorphism $(a,b) \mapsto (b,a)$ of $T^\bullet \oplus T^\bullet$. Let Φ be the inner automorphism of $End_{\hat{R}}(T^\bullet \oplus T^\bullet)$ induced by λ. Relative to the decomposition $G = G_1 \oplus G_2$, we have

$$End_R(G) = \begin{bmatrix} End_R(G_1) & Hom_R(G_2, G_1) \\ Hom_R(G_1, G_2) & End_R(G_2) \end{bmatrix}$$

$$= \begin{bmatrix} End_R(G_2) & Hom_R(G_2, T) \\ Hom_R(G_1, T) & End_R(G_1) \end{bmatrix}$$

by what was proved above. Given $\phi \in End_R(G)$ and $(a,b) \in G_1 \oplus G_2$, decompose $\phi = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$ as in the last matrix and observe $\lambda\phi\lambda(a,b) = (\phi_{21}(b) + \phi_{22}(a), \phi_{11}(b) + \phi_{12}(a)) \in G_1 \oplus G_2$. Thus $\lambda End_R(G)\lambda \subseteq End_R(G)$, and since $\lambda^2 = 1$ we see that Φ restricts to an automorphism of $End_R(G)$.

Finally, suppose Φ is induced by an automorphism ϕ of G. Then for all $\sigma \in End_R(G)$, we have $\lambda\sigma\lambda = \phi\sigma\phi^{-1}$, so that $\lambda\phi$ is contained in the center of $End_R(G)$. Thus $\lambda\phi = \alpha \in \hat{R}$, or $\lambda = \alpha^{-1}\phi$. Choosing $x \in M_0$ torsion-free and decomposing ϕ as above, we obtain $x = \alpha^{-1}\phi_{21}(x) \in T$. This impossibility shows that the restriction of Φ to $End_R(G)$ must be an outer automorphism, and our construction is complete.

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