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Dynamical systems and random perturbations

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DYNAMICAL SYSTEM AND ITS RANDOM PERTURBATIONS

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As members of the Final Examination Committee, we certify that we have read the dissertation prepared by Zheng Liu

entitled DYNAMICAL SYSTEM AND ITS RANDOM PERTURBATIONS

and recommend that it be accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy

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Final approval and acceptance of this dissertation is contingent upon the candidate's submission of the final copy of the dissertation to the Graduate College.

I hereby certify that I have read this dissertation prepared under my direction and recommend that it be accepted as fulfilling the dissertation requirement.

Dissertation Director Lai-Sang Young — Date 04-19-93

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SIGNED: ____________________

To my Mother and Father,

whose love and understanding transcend both time and distance.

To my wife, Haiyan, and my daughter, Natalie,

whose love, support and encouragement have made this dissertation possible.

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Abstract

This dissertation consists of two independent parts. In the first part we study the ergodic theory of surface endomorphisms. We consider non-uniformly expanding maps with generic singularities, and prove that the Pesin formula holds, which is to say that entropy is equal to the sum of the positive Lyapunov exponents if and only if the invariant probability measure in question is absolutely continuous with respect to Lebesgue measure. In the second part we study small random perturbations of the Feigenbaum map relative to the fixed point of Feigenbaum's renormalization operator for unimodal maps of the interval. We give a rigorous analysis of the changes in the geometry of the noisy attractor as noise level varies.

Introduction

I. General Discussion of the Thesis

This dissertation consists of two independent parts which we present as Chapters 1 and Chapter 2. Chapter 1 is about the ergodic theory of surface endomorphisms. Chapter 2 is about small random perturbations of certain dynamical systems.

Entropy, first introduced into ergodic theory by Kolmogorov in 1959, is used to measure the randomness of the disorder of a system. And Lyapunov exponents, first successfully used in studying the long-time behavior of a dynamical system by Oseledec in 1968, are used to measure the rate of divergence of nearby orbits. As a result, it is important to understand the relationship between these numbers in studying the complexity of a dynamical system. In 1967 Rohlin **[R2]** first solved this problem for expanding 1-dimensional maps. In 1977 Pesin **[P]** proved a formula (now called the Pesin entropy formula) for diffeomorphisms preserving smooth measures, i.e. entropy is equal to the sum of the positive exponents. Later, Ruelle **[R]** showed that for any invariant measure entropy is always bounded above by the sum of the positive exponents. The complete solution to this problem for C^2-diffeomorphisms was given by Ledrappier and Young **[LY]** in 1985, when they proved that Pesin's formula holds if and only if μ has absolutely continuous conditional measure on unstable manifolds.

In Chapter 1 of this dissertation we study differential maps of smooth surfaces which have singularities, and which may not be one-to-one. A dynamical system generated by a map f with singularities, has a much more complex behavior than one generated by a map without singularities, because near its singularities f may have a very complex local structure. It is also extremely hard to classify all singularities in general. (See, for example, [GG].) Therefore, it is very hard to estimate the distortion of f near its singularities. (The estimation of the distortion of a function f without singularities was the crucial step in the proof of Pesin's entropy formula.) In 1981 Ledrappier **[L]** proved that for a 1-dimensional map f, if both f and the invariant measure μ have certain nice properties near the singularities (f is non-flat, μ is non-degenerate, etc.), then the absolute continuity of μ with respect to Lebesgue measure is again a necessary and sufficient condition for Pesin's formula to hold. In this work we extend Ledrappier's 1-dimensional map result to higher dimensional maps. Under certain assumptions, we show that μ being smooth is still a necessary and sufficient condition for Pesin's formula to hold.

In Chapter 2, we turn to the problem of noise in certain dynamical systems. In the late 1970's, Feigenbaum **[F1]** and Coullet and Tresser **[CT]** independently observed that the limit of period-doubling bifurcations of unimodal maps on an interval I has certain universal scaling properties and they related these to certain properties of a particular fixed point for the period-doubling operator. The existence of such a fixed point, denoted by g, was proved by Lanford **[La]** and by Campanins and Epstein **[CE]**, using computer-assisted proofs. The qualitative properties of g and its attractor, Λ, were studied theoretically by many authors in 1980's. (See for example, **[G]**, **[M]**, **[R]**, **[VSK]**)

In iterating g, the approximation can be considered as random perturbations (or the error as noise) of g. For a given noise level ϵ, we have a Markov chain X_n^ϵ, $n \geq 0$, on the interval I, with the transition probability at $x \in I$ tending to the Dirac measure at $g(x)$ as $\epsilon \to 0$. If μ^ϵ is the invariant measure of the Markov chain $\{X_n^\epsilon\}$, then we call the support of μ^ϵ the *noisy attractor* for noise level ϵ and denote it by Λ_ϵ. In 1984 Vul, Sinai and Khanin **[VSK]** obtained some results on the statistical properties of μ^ϵ. We focus here on the *geometry* of Λ_ϵ. We prove for each fixed $\epsilon > 0$ that Λ_ϵ consists of 2^{n_ϵ} disjoint intervals cyclically permuted by $\{X_n^\epsilon\}$. As ϵ decreases, the number of components of Λ_ϵ increases. We will discuss the geometrical mechanism that leads to these changes in Λ_ϵ and give estimates on how n_ϵ scales with ϵ as ϵ changes.

II. Outline of the Thesis

Chapter 1. Entropy Formula of Non-Uniformly Expanding Maps with Singularities

In §1.0, we briefly discuss some background and state our main result in Chapter 1.

In §1.1, we first discuss some elementary properties of measure theory and some facts about the local structure of singularities for a certain class of maps. We then prove a local property of maps (Proposition 1.1.15), which, when combined with the unstable manifold theorem of Ruelle and Shub on the inverse limit $\bar{X}$ (Theorem 1.1.18), is important in constructing the partitions in the space considered.

In §1.2, we restate precisely the main theorem of Chapter 1 proved in this paper (Theorem 1.2.1) and its generalized version (Theorem 1.2.2).

In §1.3, we make a construction, based on the results discussed in §1.1, of 3 fundamental partitions in both X and its inverse limit $\bar{X}$. By considering the relevant properties of these partitions, we prove that one of them is actually a generator (Theorem 1.3.6). We also show how the map f behaves on the atoms of the generator (Theorem 1.3.12).

In §1.4, we carefully discuss the geometric difference between 1–dimensional and 2–dimensional maps. we then prove the main theorem of Chapter 1 under the assumption of ergodicity, using the methods of Ledrappier for 1–dimensional maps [L].

In §1.5, we prove stronger version of main theorem in which the assumption of ergodicity is dropped and generalize this result to dimensions higher than 2.

In the Appendix, we discuss in detail the local properties of two generic types of singularities, viz. fold and cusp.

Chapter 2. Random Perturbations of Feigenbaum Map

In §2.0, we introduce briefly some background about small random perturbations of a dynamical system, and state the main theorem of Chapter 2.

In §2.1, we discuss some well-known results for the unperturbed Feigenbaum map g, especially, a result proved by Vul, Sinai and Khanin in 1984 **[VSK]** (Theorem 2.1.5).

In §2.2, we study the relationship between the noisy attractor Λ_ϵ and the Feigenbaum attractor Λ. We show how the components of Λ_ϵ behave for small ϵ–noise (Theorem 2.2.7).

In §2.3, we focus on the geometric aspects of Λ_ϵ and prove that Λ_ϵ "explodes" at certain ϵ-noise levels as ϵ varies, i.e. the change in the geometry of Λ_ϵ is discontinuous even though the change in ϵ is continuous (Theorem 2.3.8).

In §2.4, we first estimate the distortion of the Feigenbaum map (Lemma 2.4.4), and, using this result, we then study the scaling of Λ_ϵ with respect to ϵ (Theorem 2.4.6).

Chapter 1

Entropy Formula of Non-Uniformly Expanding Maps with Singularities

§1.0. Introduction

Consider $f : (X, \mu) \longrightarrow (X, \mu)$ where X is a 2-dimensional Riemannian compact manifold, f is a smooth map and μ is an f-invariant Borel probability measure on X.

We make the following assumptions on f and μ:

1) f is *generic*, namely f belongs to a subset which is the countable intersection of open dense subsets in the space $C^\infty(X, X)$ with the Whitney C^∞ topology (see [**GG**], p.42 for details);

2) f is *asymptotically expanding*, i.e. for μ almost every $x \in X$, the Lyapunov exponents $\lambda_i(x) > 0,\ for\ i = 1, 2$;

3) μ is *non-degenerate* with respect to the set of critical points of f(the precise definition is given in the next section).

Let λ be the natural Lebesgue measure on X and write " $\mu \ll \lambda$" to mean that μ is absolutely continuous with respect to Lebesgue measure λ. We prove, in this paper, a generalization of a theorem of Ledrappier [**L**] for 1-dimensional piecewise monotone maps.

Main theorem. *Under the assumptions (1) - (3), we have that $\mu \ll \lambda$ if and only if Pesin's formula holds for f, i.e.*

$$h_\mu(f) = \int log|detDf_x|d\mu, \tag{1.0.1}$$

where $h_\mu(f)$ is the measure theoretic entropy.

Our proof follows closely to that given in [L], and it depends greatly on the local properties of f. For a given f, we consider the local structure of its singularities [GG], consider the local unstable manifold [RS], and then carefully construct good partitions on both X and its inverse limit space $\overline{X}$. We will discuss the relevant properties in detail in the next section.

The first paper which studied the connection between entropy and Lyapunov exponents was by Rohlin [R] in the 1960's. Then in the middle 1970's, Pesin [P] gave the following well-known formula under the assumptions that μ is equivalent to λ and f is a $C^{1+\alpha}$ diffeomorphism:

$$h_\mu(f) = \int \sum_{\lambda_i \geq 0} \lambda_i \cdot dim E_i d\mu, \tag{1.0.2}$$

where λ_i's are Lyapunov exponents and E_i's are the corresponding subspace in the tangent space.

Later, Ruelle[R] proved that when f is a differentiable map and μ is an arbitrary invariant Borel probability measure, the equality (1.0.2) is an inequality, i.e.

$$h_\mu(f) \leq \int \sum_{\lambda_i \geq 0} \lambda_i \cdot dim E_i d\mu. \tag{1.0.3}$$

The picture for C^2 diffeomorphisms was in some sense completed by Ledrappier and Young [LY] in 1985 when they proved that the equality of (1.0.2) holds if and only if μ has absolutely continuous conditional measures on unstable manifolds.

However, if f is a non-invertible transformation, and especially if f has critical points, one must take into consideration the properties of f and μ near the critical points of f. The first result of this type was proved by Ledrappier [Le] in 1981 for some maps of the closed interval. This paper is an attempt to generalize Ledrappier's result from one-dimensional to higher-dimensional compact manifolds.

§1.1. Preliminaries

In this section we present for the convenience of the reader some fundamental facts from measure theory and the theory of classification of generic maps. These will be useful in the proof of the main theorem.

A. Some Measure-Theoretic Background

Lemma 1.1.1. (Borel-Cantelli Lemma) *Let* $(X, \mathcal{B}, \mu)$ *be a probability space and* $E_i \subseteq X$ *be any sequence of measurable subsets of* X. *Set*

$$E = \{x;\ x \in E_i \text{ for infinitely many } i\}.$$

Then $\mu(E) = 0$ *provided* $\sum_{i=1}^{\infty} \mu(E_i) < +\infty$.

Lemma 1.1.2[L]. *Let* $I = [a, b]$ *be a closed interval in* $\mathbf{R}$ *and let* ν *be any Borel probability measure on* I. *Then for Lebesgue almost all* $x \in I$, *and for any* $0 < \alpha < 1$,

$$\limsup_n \frac{1}{n} \log \nu[x - \alpha^n, x + \alpha^n] < 0.$$

Proof. Set

$$A_k = \{x \in I; \qquad \nu[x - \alpha^k, x + \alpha^k] \leq 2k^2 \alpha^k\},$$

and cover the complementary $B_k = I \setminus A_k$ by intervals

$$C_i = [x_i - \alpha^k, x_i + \alpha^k],$$

where $x_i \in B_k$. That is, if we let λ be the Lebesgue measure, we must have

$$\nu[x_i - \alpha^k, x_i + \alpha^k] > 2k^2 \alpha^k = k^2 \lambda(C_i).$$

Moreover, we require that for any $x \in I$, x is at most in such two different intervals. Then we have the following estimation:

$$\lambda(B_k) = \lambda(I \setminus A_k) \leq \sum_i \lambda(C_i) \leq \frac{1}{k^2} \sum_i \nu(C_i) \leq \frac{2}{k^2}.$$

This implies that

$$\sum_{k=1}^{\infty} \lambda(B_k) \leq 2 \sum_{k=1}^{\infty} \frac{1}{k^2} < +\infty.$$

By the Borel-Cantelli Lemma, we know that for λ-a.e. $x \in I$, that x belongs only finitely many B_k. Hence, for λ-a.e. x, there is a sufficiently large N_x such that for any $n \geq N_x$, it follows that $x \notin B_n$, i.e,

$$\nu\{[x - \alpha^n, x + \alpha^n]\} \leq 2n^2\alpha^n, \quad \forall n \geq N_x, \quad \lambda - a.e. \tag{1.1.1}$$

Therefore,

$$\limsup_n \frac{1}{n} \log \nu[x - \alpha^n, x + \alpha^n] \leq \log \alpha < 0, \quad \lambda - a.e.$$

∎

We now assume that X is a Riemannian manifold of finite dimension and that μ is a non-atomic Borel probability measure on X. Since our purpose in this Chapter is to study those measures which are relatively smooth, we assume for any open set $O \subset X$ that $\mu(O) > 0$. Now for $x \in X$, let $B(x, \delta)$ be the open ball of radius $\delta > 0$ in X, centered at x i.e.

$$B(x, \delta) = \{y \in X; \quad d(x, y) < \delta\},$$

and let $\overline{B(x,\delta)}$ denote the closure of $B(x, \delta)$.

Define

$$\mu_x := \text{the normalized measure } \mu|_{\overline{B(x,\delta)}} \text{ on } \overline{B(x,\delta)}.$$

Then for every $x \in X$, the closed ball $\overline{B(x,\delta)}$ itself becomes a measurable space equipped with the probability measure μ_x for any $\delta > 0$.

We now define a map π_x: $\overline{B(x,\delta)} \longrightarrow I = [0, \delta]$ by letting

$$y \longmapsto d(x, y),$$

that is, π_x maps each sphere to its radius.

It is easy to check that π_x is continuous. Therefore, the measure μ_x on $\overline{B(x,\delta)}$ can be easily transfered to another Borel probability measure μ_x^* on I as follows:

Given any Borel subset $B \subset I$, define

$$\mu_x^*(B) = \mu_x(\pi_x^{-1}B). \tag{1.1.2}$$

Then by Kolmogorov extension theorem, there is a unique probability measure ν on I which is the extension of μ_x^* defined by (1.1.2), since the following three sufficient conditions are easily verified:

(i) $\mu_x^*(\phi) = \mu_x(\phi) = 0,$

(ii) $\mu_x^*(I) = \mu_x(\overline{B}(x,\delta)) = 1,$

(iii) $\mu_x^*(\bigcup\limits_{n=1}^{\infty} I_n) = \mu_x(\pi_x^{-1}\bigcup\limits_{n=1}^{\infty} I_n) = \mu_x(\bigcup\limits_{n=1}^{\infty} \pi_x^{-1}I_n) = \sum\limits_{n=1}^{\infty} \mu_x(\pi_x^{-1}I_n) = \sum\limits_{n=1}^{\infty} \mu_x^*(I_n),$
where the I_n are pairwise disjoint intervals in I.

Moreover, by Lemma 1.1.2, for λ-a.e $\xi \in [0,\delta]$ and for any $0 < \alpha < 1$, we have that $(\nu = \mu_x^*)$

$$\sum_{n=1}^{\infty} \mu_x^*[\xi - \alpha^n, \xi + \alpha^n] < \infty.$$

¿From this estimate, we obtain the following estimate for μ_x:

$$\sum_{n=1}^{\infty} \mu_x\{B(x;\xi - \alpha^n, \xi + \alpha^n)\} < \infty, \tag{1.1.3}$$

where

$$B(x;\xi - \alpha^n, \xi + \alpha^n) = \overline{B(x, \xi + \alpha^n)} \setminus B(x, \xi - \alpha^n).$$

For abbrevity, we introduce the following notation:

$\partial B_x^\delta = \partial B(x,\delta)$, the boundary of the ball $B(x,\delta)$,

$B_{\alpha^n}^{\delta,x} = \{y; dist(y, \partial B_x^\delta) \leq \alpha^n\} = B(x;\ \delta - \alpha^n, \delta + \alpha^n).$

We are now in aposition to prove the next proposition.

Proposition 1.1.3 . *Given any $x \in X$, there exists a $\delta_x > 0$ for which*

$$\sum_{n=1}^{\infty} \mu(B_{\alpha^n}^{\delta,x}) < \infty, \tag{1.1.4}$$

provided for λ-a.e $\delta \in (0, \delta_x)$ and any $0 < \alpha < 1$.

Proof. Note by the definition that we have $B_{\alpha^n}^{\delta,x} \subseteq B(x, \delta_x)$. when n is large enough. Hence, there is a constant independent of n such that

$$\mu(B_{\alpha^n}^{\delta,x}) \leq const.\mu_x(B_{\alpha^n}^{\delta,x}).$$

Therefore, (1.1.4) follows from (1.1.3). ∎

We now consider the dynamical systems (X, μ, f) and its natural extension $(\bar{X}, \bar{\mu}, \bar{f})$. That is, $\bar{X} \subseteq \prod_{i=0}^{\infty} X$ is the inverse limit space of X which can be specified as follows:

$$\bar{x} = (x_0, x_1, ...) \in \bar{X} \ \text{ iff } f(x_i) = x_{i-1}, \ i = 1, 2, ...,$$

and $\bar{f} : \bar{X} \longrightarrow \bar{X}$ is specifed as

$$\bar{f} : \bar{x} = (x_0, x_1, ...) \longmapsto \bar{f}\bar{x} = (f(x_0), x_0, x_1, ...).$$

It is obvious that $\bar{f}$ is invertible, i.e.

$$\bar{f}^{-1} : \bar{x} = (x_0, x_1, ...) \longmapsto (x_1, x_2, ...).$$

Moreover, if let π be the projection of $\bar{X}$ to X via $\pi(x_0, x_1, ...) = x_0$, and then the measure $\bar{\mu}$ on $\bar{X}$ can be specified by the following character,

$$\mu = \pi\bar{\mu}.$$

It is not hard to check the fact that if μ is ergodic, then so is $\bar{\mu}$.

More generally, we denote by π_i the projection of $\bar{X}$ to X by its i^{th} entry. Clearly, $\pi_0 = \pi$.

For convenience, we will frequently use the following notation. For any subset $C \subseteq X$ and $\bar{x} = (x_0, x_1, ...) \in \bar{X}$, let

$$\mathcal{N}_\epsilon(C) = \{x \in X; \quad dist(x, C) < \epsilon\},$$

$$\delta_c(\bar{x}) = dist(\pi\bar{x}, C) = dist(x_0, C).$$

Definition 1.1.4 . We say the measure μ is not too *concentrated* at C ($\subseteq X$) if there are constants $\alpha > 0$ and $\beta > 0$ such that

$$\mu(\mathcal{N}_\epsilon(C)) \leq \alpha\epsilon^\beta, \qquad \forall\, 0 < \epsilon < 1.$$

Remark. $\mu(C) = 0$.

Definition 1.1.5.[L] We say the measure μ is *non-degenerate* at C ($\subseteq X$) if

$$\liminf_n \frac{1}{n} \log \delta_c(\bar{f}^{-n}\bar{x}) = 0, \qquad \bar{\mu} - a.e.\ \bar{x} \in \bar{X}.$$

Definition 1.1.6. Define the set of critical points of f as the following set

$$C_f = \{x \in X; \quad \det Df_x = 0\ \}.$$

Note if $C_f \neq \emptyset$ for a dynamical system (X, f, μ), then its dynamical behavior is much more complicated than when $C_f = \emptyset$. Basically this is because f loses the property of local diffeomorphism near critical points. It is also hard to estimate the distortion near the critical set. We have to carefully discuss the local structure of f near C_f.

Remark. If $C = C_f$ in the definitions 1.4 and 1.5, then the version 1.4 is natural, and is commonly used.(See for instance [**KS**]). However, the following argument shows that definition 1.5 is a weaker version.

Proposition 1.1.7. *If the measure μ is not too concentrated at a subset $C \subseteq X$, then μ must be non-degenerate at C.*

Proof. Let

$$\bar{B} = \{\bar{x} \in \bar{X}; \qquad \liminf_{n} \frac{1}{n} \log \delta_c(\bar{f}^{-n}\bar{x}) < 0 \},$$

and for $\epsilon > 0$, let

$$\bar{B}_\epsilon = \{\bar{x} \in \bar{X}; \qquad \liminf_{n} \frac{1}{n} \log \delta_c(\bar{f}^{-n}\bar{x}) < -\epsilon \}.$$

It is clear that

$$\bar{B} = \cup_{m=k}^{\infty} \bar{B}_{1/m}, \qquad \forall k \geq 1.$$

Therefore, to show $\bar{\mu}(\bar{B}) = 0$, it suffices to show that for any $\epsilon > 0$,

$$\bar{\mu}(\bar{B}_\epsilon) = 0.$$

Note if $\bar{x} \in \bar{B}_\epsilon$, that there exists a sequence $\{n_i\}$ with $n_i \to \infty$ as $i \to \infty$ such that

$$\delta_c(\bar{f}^{-n_i}\bar{x}) < e^{-n_i\varepsilon}.$$

Then, we let $\bar{A}_k^\epsilon \in \bar{X}$ defined as follows:

$$\bar{A}_k^\epsilon = \{\bar{x} \in \bar{X}; \qquad \delta_c(\bar{f}^{-k}\bar{x}) < e^{-k\varepsilon} \}.$$

We now claim that

$$\bar{B}_\epsilon = \cap_{n \geq 1} \cup_{k \geq n} \bar{A}_k^\epsilon. \tag{1.1.5}$$

In fact, if $\bar{x} \in \bar{B}_\epsilon$, then there is a sequence $\{n_i \to +\infty\}$ such that $\delta_c(\bar{f}^{-n_i}\bar{x}) < e^{n_i\epsilon}$, i.e.

$$\bar{x} \in \bar{A}_{n_i}^\epsilon \subseteq \cup_{k=i}^{\infty} \bar{A}_k^\epsilon,$$

for all $i \geq 1$, $n_i \geq i$. Therefore,

$$\bar{x} \in \cap_{i=1}^{\infty} \cup_{k \geq i} \bar{A}_k^\epsilon.$$

We have thus proved that

$$\bar{B}_\epsilon \subseteq \cap_{n\geq 1} \cup_{k\geq n} \bar{A}_k^\epsilon.$$

On the other hand, however, if $\bar{x} \in \cap_{n\geq 1}\cup_{k\geq n}\bar{A}_k^\epsilon$, then for any j, $\bar{x} \in \bigcup_{k\geq j} \bar{A}_k^\epsilon$. Hence, we can pick an increasing sequence $n_j \to +\infty$ such that $\bar{x} \in \bar{A}_{n_j}^\epsilon$, i.e. $\delta_c(\bar{f}^{-n_j}) < e^{-n_j\epsilon}$.

This implies that

$$\liminf_n \frac{1}{n} \log(\bar{f}^{-n}\bar{x}) < -\epsilon,$$

and hence $\bar{x} \in \bar{B}_\epsilon$. Thus

$$\cap_{n\geq 1} \cup_{k\geq n} \bar{A}_k^\epsilon \subseteq \bar{B}_\epsilon,$$

so (1.1.5) is true.

Now for any n, we have that

$$\bar{B}_\epsilon \subseteq \cup_{k\geq n}(\bar{A}_k^\epsilon).$$

It is obvious by the definition of $\bar{A}_k^\epsilon$ that

$$\pi\bar{f}^{-k}(\bar{A}_k^\epsilon) \subseteq \mathcal{N}_{e^{-k\epsilon}}(C).$$

Then by the invariance of f and μ being not too concentrated at C, we obtain as $n \to \infty$ that

$$\begin{aligned}
\bar{\mu}(\bar{B}_\epsilon) &\leq \bar{\mu}(\cup_{k\geq n}\bar{A}_k^\epsilon) \\
&\leq \sum_{k\geq n} \bar{\mu}(\bar{A}_k^\epsilon) \\
&= \sum_{k\geq n} \bar{\mu}(\pi^{-1}\pi\bar{f}^{-k}(\bar{A}_k^\epsilon)) \\
&\leq \sum_{k\geq n} \bar{\mu}(\pi^{-1}(\mathcal{N}_{e^{-k\epsilon}}(C))) \\
&= \sum_{k\geq n} \mu(\mathcal{N}_{e^{-k\epsilon}}(C)) \\
&\leq \sum_{k\geq n} \alpha e^{-k\epsilon\beta} \longrightarrow 0.
\end{aligned}$$

Therefore,

$$\bar{\mu}(\bar{B}_\epsilon) = 0.$$

Since $\epsilon > 0$ can be chosen arbitrarily small, this implies for $\bar{\mu}$-a.e. $\bar{x}$,

$$\liminf_n \frac{1}{n} \log \delta_C(\bar{f}^{-n}\bar{x}) = 0.$$

We have thus proved that μ is non-degenerate at C. ∎

Corallary 1.1.8. *For any measurable set $C \subset X$ and $0 < \alpha < 1$, let*

$$C_{\alpha^n} = \{x;\ d(x, C) < \alpha^n\ \}.$$

If $\sum_{n=0}^{\infty} \mu(C_{\alpha^n}) < \infty$, then μ is non-degenerate at C.

Proof. For any n, we have by Prop. 1.7,

$$\bar{\mu}(\bar{B}_\epsilon) \le \sum_{k \ge n} \mu(C_{\alpha^k}).$$

Hence, $\bar{\mu}(\bar{B}_\epsilon) = 0$ for any $\epsilon > 0$. ∎

The Corallary above with Prop. 1.3 gives the following proposition:

Proposition 1.1.9. *Given $x \in X$ and a small $\delta_x > 0$. Then μ is non-degenerate at $\partial B_\delta(x)$ for λ-a.e. $\delta \in (0, \delta_x)$.*

We next define the *Rohlin decomposition* of the measure μ with respect to a measurable partition ξ.

For a measurable partition ξ of X, we define the modulo space X/ξ to be a measurable space having the atoms of ξ as points and an associated measure μ_ξ on X/ξ defined by

$$\mu_\xi(K) = \mu(P^{-1}K),$$

where P is the projection from $X \to X/\xi$ by $x \mapsto \xi(x)$.

By a decomposition of μ with respect to ξ, we mean a collection of measures $\mu_{\xi(x)}$ for any $x \in X$, satisfying

(1) $\mu_{\xi(x)}$ is a Borel probability measure in each atom $\xi(x) \in \xi$;

(2) For any measurable set $A \subseteq X$, the set $A \cap \xi(x)$ is measurable in the space $\xi(x)$, for μ_ξ-a.e. x, and the function $\mu_{\xi(x)}(A \cap \xi(x))$ is μ_ξ measurable satisfying the following equality:

$$\mu(A) = \int_{X/\xi} \mu_{\xi(x)}(A \cap \xi(x))\mu_\xi(dx).$$

B. Maps with Some Generic Properties

In this section we examine the generic properties of maps that are related to fold and cusp singularities. (See **[GG]** for instance.) We will restrict our investigation here to 2-dimensional Riemannian manifolds.

Definition 1.1.10.[GG] Call a smooth map $f : X \to X$ *finite* if for any $x \in X$, the estimate $dim_R \mathcal{R}_f(x) < \infty$ holds, where $\mathcal{R}_f(x)$ is the local ring of germs of f at x.

Proposition 1.1.11. *If X is compact and $f : X \to X$ finite, then there is an integer $N = N(f) \in \mathcal{Z}^+$ such that for any $x \in X$,*

$$Card\{f^{-1}x\} < N.$$

Proof. By the property of finite maps (See **[GG]**, Prop. 2.4, p.168), for $x \in X$, there is a neighborhood $\mathcal{U}_x \subset X$ such that for any $y \in \mathcal{U}_x$,

$$Card\{f^{-1}y\} \leq Card\{f^{-1}x\} < \infty.$$

Since finitely many open sets $\mathcal{U}_x$'s cover X, the result follows. ∎

Proposotion 1.1.12.([**GG**], Theorem 2.6, p.169) *The finite maps $f : X \to X$ are a residual subset of $C^\infty(X, X)$.*

Definition 1.1.13. Let $f : X \to X$ be a smooth map.

(1) We say a critical point $p \in X$ of f is a *fold* if one can choose a system of coordinates (x, y) near p and (u, v) near $q = f(p)$ such that f is of the form:

$$(x, y) \longmapsto (x, y^2) \tag{1.1.6}$$

near p.

(2) We say a critical point $p \in X$ of f is a simple *cusp* if one can choose a system of coordinates (x, y) near p and (u, v) near $q = f(p)$ such that f is of the form

$$(x, y) \longmapsto (x, xy + y^3) \tag{1.1.7}$$

near p.

For more details on *fold* and *cusp*, see the Appendix.

Proposition 1.1.14. (Whitney classification, **[GG]**, p.145) *There is a residual subset in $C^\infty(X, X)$ such that if f belongs to this set, its critical points are either folds or simple cusps.*

We now make the following clear statement on a *generic* property for maps by Propositions 1.10 and 1.12.

Property – A: We say $f \in C^\infty(X, X)$ satisfies the *property-A* if f is finite and its singularities are either folds or cusps.

Remark. It is obvious that the subset $\mathcal{H} = \{f \in C^\infty;\ f \text{ is of } property-A\ \}$ is a residual subset of $C^\infty(X, X)$. From now on we assume that the map under consideration has property-A.

We end this part by proving a local property of f which is useful when we construct partitions on X.

Proposition 1.1.15. *Let X be a 2-dimensional compact Riemannian manifold, $f : x \to X$ be smooth and μ be an f-invariant Borel probability measure. For any $x \in X$ and a δ_x small enough, there is a collection of small neighborhoods*

$\{\mathcal{N}_\delta(x)\}_{\delta\in[0,\delta_x]}$ *of* x *such that for* λ*-a.e.* $\delta \in [0, \delta_x]$,

$$\mu\{\cup_{-\infty}^{+\infty} f^n(\partial\mathcal{N}_\delta(x))\} = 0.$$

Proof. Let $\mathcal{N}_\delta(x)$ be the δ-ball centered at x, i.e.

$$\mathcal{N}_\delta(x) = \{y \in X;\ \ d(x,y) < \delta\ \}.$$

Then clearly, if $\delta_1 \neq \delta_2$,

$$\partial\mathcal{N}_{\delta_1}(x) \cap \partial\mathcal{N}_{\delta_2}(x) = \emptyset.$$

This implies that for λ-a.e. $\delta \in [0, \delta_x]$,

$$\mu(\partial\mathcal{N}_\delta(x)) = 0.$$

Hence by the invariance, for λ-a.e. δ, we have

$$\mu(\cup_{n=0}^{+\infty} f^{-n}(\partial\mathcal{N}_\delta(x))) = 0.$$

We next show that

$$\mu(\cup_{n\geq 0} f^n(\partial\mathcal{N}_\delta(x))) = 0, \quad \lambda - a.e.\ \delta.$$

For $i \geq 1$, define

$$\Delta_i = \{\delta \in [0, \delta_x];\ \ \mu(f^i(\partial\mathcal{N}_\delta(x))) > 0\ \}.$$

Then it suffices to show that for all $i \geq 1$,

$$\lambda(\Delta_i) = 0.$$

Since, if this equality holds, then $\Delta_\infty := \bigcap_{i\geq 1} \Delta_i^c$ must have the full measure, where Δ_i^c is the complement of Δ_i. Also it is clear that for any $\delta \in \Delta_\infty$,

$$\mu(\cup_{n\geq 0} f^n(\partial\mathcal{N}_\delta(x))) = 0.$$

Now let $\Im$ be the collection of Borel subset of X. We define an equivalence relation, "$\sim$", on $\Im$ as follows:

$$A \sim B \iff \mu(A \Delta B) = 0,$$

where "Δ" denotes the symmetric difference of two sets. Let $[A]$ represent the equivalence class of A.

We now consider Δ_1, and for brevity, let

$$A_\delta = f(\partial \mathcal{N}_\delta(x)).$$

Let $\mathcal{F}_1$ be the collection of those A_δ with $\delta \in \Delta_1$ such that $[A_{\delta_1}] \neq [A_{\delta_2}]$ for $A_{\delta_1} \neq A_{\delta_2} \in \mathcal{F}_1$, i.e. for each equivalent class, we specify a representative as follows:

$$\mathcal{F}_1 = \{A_{\delta_i};\ \delta_i \in \Delta_1, \text{ and } [A_{\delta_i}] \neq [A_{\delta_j}] \text{ if } \delta_i \neq \delta_j\ \}.$$

Suppose that $\lambda(\Delta_1) > 0$. This implies that Δ_1 is an uncountable subset.

Now, if $\mathcal{F}_1$ were countable, then one could choose an infinite sequence $\{\delta_i\} \subseteq \Delta_1$ with all δ_i's different in such a way that for the associated sequence $\{A_{\delta_i}\}$, one would have $[A_{\delta_1}] = [A_{\delta_j}]$, for $j \geq 1$. But then it would follow by induction that for any $n \geq 1$,

$$\mu(A_{\delta_1} \cap A_{\delta_2} \cdots \cap A_{\delta_n}) = \mu(A_{\delta_1}) > 0.$$

Hence,

$$A_{\delta_1} \cap A_{\delta_2} \cdots \cap A_{\delta_n} \neq \emptyset.$$

This implies for any $n \geq 1$ that there is a $y \in X$ such that $Card\{f^{-1}y\} \geq n$, contradicting the hypothesis that f is a finite map.

Now since $\mathcal{F}_1$ is uncountable, we can construct from $\mathcal{F}_1$ a second collection $\mathcal{F}_2$ of subsets as follows:

$$\mathcal{F}_2 = \{A_{\delta_1\delta_2} = A_{\delta_1} \cap A_{\delta_2};\ A_{\delta_1} \neq A_{\delta_2} \in \mathcal{F}_1,\ \mu(A_{\delta_1} \cap A_{\delta_2}) > 0,$$

$$\text{and if } (\delta_1, \delta_2) \neq (\delta_3, \delta_4),\ [A_{\delta_1} \cap A_{\delta_2}] \neq [A_{\delta_3} \cap A_{\delta_4}]\ \}.$$

But $\mathcal{F}_2$ is also uncountable, since otherwise we we could order the A_δ's used in constructing $\mathcal{F}_2$ by $\mathcal{F}_2' = \{A_{\delta_1}, \cdots, A_{\delta_n}, \cdots\}$. Then let

$$\mathcal{F}_1' = \mathcal{F}_1 - \mathcal{F}_2',$$

for which it follows that $\mathcal{F}_1'$ is uncountable. There are now two cases:

(1) Uncountably many A_δ in $\mathcal{F}_1'$ are such that for any two of them, $\mu(A_{\delta_1} \cap A_{\delta_2}) = 0$. This implies that there are uncountably many subsets in X with positive measure which are pairwise disjoint. This contradicts the hypothesis that X is a probability space.

(2) Uncountably many indices (δ_i, δ_j) are such that $A_{\delta_i}, A_{\delta_j} \in \mathcal{F}_1'$ and $A_{\delta_1\delta_2} \in \mathcal{F}_2$ with $[A_{\delta_1\delta_2}] = [A_{\delta_i\delta_j}]$ for any such an index (δ_i, δ_j). This implies that one can choose an infinite sequence of different indices $\{(\delta_{n_i}, \delta_{m_i})\}$ such that for some fixed $A_{\delta_1\delta_2} \in \mathcal{F}_2$,

$$[A_{\delta_{n_i}\delta_{m_i}}] = [A_{\delta_1\delta_2}], \quad i \geq 1.$$

Reasoning as above,

$$\mu(A_{\delta_{n_1}\delta_{m_1}} \cap \cdots \cap A_{\delta_{n_k}\delta_{m_k}}) = \mu(A_{\delta_1\delta_2}) > 0.$$

This again implies for any $N \in \mathcal{Z}^+$ that there is a element in X such that the number of its pre-images is greater than N, which contradicts the finiteness of f. Therefore $\mathcal{F}_2$ must be uncountable.

Inductively, for $n > 2$, we can construct $\mathcal{F}_n$ based on $\mathcal{F}_{n-1}$ with $\mathcal{F}_n$ uncountable. Note that an element in $\mathcal{F}_n$ has the form

$$A_{\delta_1} \cap A_{\delta_2} \cap \cdots \cap A_{\delta_{2^n}}.$$

Again, by the same reasoning, this contradicts the finiteness of f. So we have proved that $\lambda(\Delta_1) = 0$.

For $i > 1$, we need only set $g = f^i$, and the rest is exactly the same as above. This finishes the proof. ∎

C. Smooth Dynamical Systems

We assume in this section that X is a smooth Riemannian manifold of finite dimension (not necessarily with dim = 2).

We first present the following well-known result in ergodic theory due to Oseledec [**O**].

Theorem 1.1.16. (multiplicative ergodic theorem) *Let $f : X \to X$ be a measurable map, μ be an f-invariant Borel probability measure, and $A : X \to M(m, \mathcal{R})$ be a measure map such that*

$$\int_X \log^+ \|A(x)\| d\mu(x) < +\infty,$$

where $\log^+ a = \max\{\log a, 0\}$ and $M(m, \mathcal{R})$ is the set of all real $m \times m$ matrices. Then for $x \in X$, there is an integer $s(x) \in [1, m]$ and a measurable filtration

$$\{0\} = L_0(x) \subseteq L_1(x) \subseteq \cdots \subseteq L_{s(x)}(x) = \mathcal{R}$$

where $\{L_i)x)\}_{1 \leq i \leq s(x)}$ are linear subspaces of $\mathcal{R}^m$ such that $L_i(x) \neq L_j(x), i \neq j$, $A(x)L_i(x) = L_i(fx)$ and that for all $0 \neq v \in L_i(x) \setminus L_{i-1}(x)$, $1 \leq i \leq s(x)$, the following limit exists

$$\lim_{n \to \infty} \frac{1}{n} \log \|A(n.x)v\| = \lambda_i(x),$$

where $A(n, x) = A(f^{n-1}x)A(f^{n-2}) \cdots A(x)$, and

$$-\infty \leq \lambda_1(x) < \lambda_2(x) < \cdots < \lambda_{s(x)}(x) < +\infty.$$

Remark. For our purpose in this paper, let $A(x) = Df_x$. Then the associated numbers $\{\lambda_i(x)\}_{1 \leq i \leq s(x)}$ are the well-known *Lyapunov exponents* of f at x. The number $k_i(x) = dimL_i(x) - dimL_{i-1}(x)$ is called the *multiplicity* of the exponent λ_i. Both exponent and its multiplicity are invariant along an orbit.

Definition 1.1.17. We say a smooth map $f : x \to X$ is a *non-uniformly expanding map* if $\lambda_i(x) > 0$ for μ-a.e. $x \in X$, $1 \le i \le s(x)$.

Next we state a theorem due to Ruelle and Shub **[RS]** which is useful in the construction of good partitions on X as well as on the inverse limit $\bar{X}$.

Theorem 1.1.18.(unstable manifold theorem on $\overline{X}$) *Let X be a smooth Riemannian manifold of finite dimension, $f : X \to X$ be a C^2 endomorphism, and μ be an f-invariant Borel probability measure. Let $\overline{X}$ be the inverse limit of X, $\overline{f} : \overline{X} \to \overline{X}$ be the extension of f, and $\overline{\mu}$ be the extension of μ which is $\overline{f}$-invariant. Then*

(i) There is a Borel set $\bar{\Gamma} \subseteq \bar{X}$ such that $\bar{f}(\bar{\Gamma}) \subseteq \bar{\Gamma}$ and $\bar{\mu}(\bar{\Gamma}) = 1$. Moreover, if $\bar{x} = \{x_n\} \in \bar{\Gamma}$, then there is an $s \in [0, m]$, where $m = dimX$, $\mu^{(1)} > ... > \mu^{(s)}$, and nested subspaces of T_{x_0},

$$\{0\} \subseteq \bar{V}_{\bar{x}}^{(0)} \subseteq \bar{V}_{\bar{x}}^{(1)} \subseteq ... \subseteq \bar{V}_{\bar{x}}^{(s)} \subseteq T_{x_0}$$

such that if $\{u_n\}_{n\ge 0}$ satisfies $u_n \in T_{x_n}$ with $Df_{x_{n+1}} u_{n+1} = u_n$ and

$$\lim_{n\to+\infty} \frac{1}{n} \log \|u_n\| < +\infty,$$

then $u_0 \in \bar{V}_{\bar{x}}^{(s)}$. Conversely, for every $u_0 \in \bar{V}_{\bar{x}}^{(s)}$, there is a unique sequence $\{u_n\}_{n\ge 0}$ such that

$$\lim_{n\to+\infty} \frac{1}{n} \log \|u_n\| = -\mu^{(r)},$$

provided $u_0 \in \bar{V}_{\bar{x}}^{(r)} \setminus \bar{V}_{\bar{x}}^{(r-1)}$, $r = 1, 2, ..., s$.

(ii). Let θ, ν, γ be $\bar{f}$-invariant Borel functions on $\bar{\Gamma}$ with $\theta > 0$, $\nu > 0$, $\gamma \in [0, s]$ and

$$\mu^{(\gamma+1)} < \nu < \mu^{(\gamma)}.$$

Then there are Borel functions $\bar{\beta} > \bar{\alpha} > 0$ and $\bar{\gamma} > 0$ on $\bar{\Gamma}$ with the following properties:

(a) If $\bar{x} = (x_n) \in \bar{\Gamma}$, the set

$$\bar{W}^u_{\bar{x}} = \{\bar{y} = (y_n) \in \bar{X};\quad d(x_0, y_0) \leq \bar{\alpha}(\bar{x}) \text{ and for all } n \geq 0,\ d(x_n, y_n) \leq \bar{\beta}(\bar{x})e^{-n\nu(\bar{x})}\}$$

is contained in $\bar{\Gamma}$; the map π restricted to $\bar{W}^u_{\bar{x}}$ is injective $(1-1)$ and $\pi\bar{W}^u_{\bar{x}}$ is a submanifold of the ball $B_{x_0} = \{x \in X;\ d(x_0, x) \leq \bar{\alpha}(\bar{x})\ \}$. Moreover, for each $\bar{y} \in \bar{W}^u_{\bar{x}}$, we have

$$T_{y_0}\pi\bar{W}^u_{\bar{x}} = \bar{V}^{(\gamma)}_{\bar{y}}.$$

(b) If $\bar{y} = (y_n)$, $\bar{z} = (z_n) \in \bar{W}^u_{\bar{x}}$, then

$$d(y_n, z_n) \leq \bar{\gamma}(\bar{x})d(y_0, z_0)e^{-n\nu(\bar{x})}.$$

(c) If $\bar{x} \in \bar{\Gamma}$, then $\bar{\alpha}(\bar{f}^{-n}\bar{x})$ and $\bar{\beta}(\bar{f}^{-n}\bar{x})$ decrease less fast with n than the exponential $e^{-n\theta}$.

Remark. Comparing Theorem 1.14 to Theorem 1.16, if we let $A(x) = Df_x$, then $-\mu^{(i)}(\bar{x}) = \lambda_i(x)$ for $\bar{x} \in \overline{\Gamma}$ with $\pi\bar{x} = x$.

Based on this theorem, we can now define a local unstable manifold at x for μ-a.e. x as follows. Let $\Gamma = \pi\overline{\Gamma}$, then $\mu(\Gamma) = 1$ and for $x \in \Gamma$, let $\overline{x} \in \overline{\Gamma}$ be such that $\pi\overline{x} = x$. Moreover for such a $\overline{x}$, we assume that the following relationship holds:

$$\mu^{(1)} > \cdots > \mu^{(r)} > 0 > \mu^{(r+1)} > \cdots > \mu^{(s)}, \quad \text{for some } r.$$

We then fix a positive measurable function $\nu(\overline{y}) > 0$ on $\overline{\Gamma}$ such that

$$0 < \nu(\overline{x}) < \mu^{(r)},$$

and define a local unstable manifold $W^u_{x,loc}$ at x by

$$W^u_{x,loc} = \pi\overline{W}^u_{\overline{x}}$$

where $\overline{W}^u_{\overline{x}}$ is defined by Theorem 1.16, (ii)-(a). It is clear that $W^u_{x,loc}$ is not unique, it depends greatly on the choice of $\overline{x} \in \overline{X}$.

Theorom 1.1.19[SR]. (unstable manifold theorem on X) *For μ-a.e. $x \in X$, there is a $\overline{x} = \{x_n\} \in \overline{\Gamma}$ (defined by Theorem 1.16) with $\pi\overline{x} = x$, and a submanifold $W^u_{x,loc}$ passing through x such that for any $y \in W^u_{x,loc}$, there is a $\overline{y} = \{y_n\} \in \overline{X}$ with $\pi\overline{y} = y$, and for any $n \geq 1$,*

$$d(x,y) < \overline{\alpha}(\overline{x}), \quad \textit{and} \quad d(x_n, y_n) < \overline{\beta}(\overline{x})e^{-n\nu(\overline{x})},$$

where the positive measurable functions $\overline{\alpha}(\overline{x})$ and $\overline{\beta}(\overline{x})$ are given by Theorem 1.16. That is,

$$W^u_{x,loc(\overline{x})} = \{y \in X; \;\; \exists \overline{y} \in \overline{X} \text{ with} \pi\overline{y} = y \;\; \text{and} \;\; d(x,y) < \overline{\alpha}(\overline{x}), \; d(x_n,y_n) < \overline{\beta}(\overline{x})e^{-n\nu(\overline{x})}\}.$$

Remark. If we consider expanding maps, then $W^u_{x,loc}$ must contain an open ball centered at x.

§1.2. Precise Statement of the Main Theorem of Chapter 1

We now restate our main theorem precisely as Theorem 1.2.1. In higher dimensions, if the singularities of a map have certain nice properties, for instance, the singularities are locally of polynomial type, then our proof works also. This will be stated as Theorem 1.2.2.

Theorem 1.2.1. *Suppose that X is a 2-dimensional compact Riemannian manifold without boundary, $f : x \to X$ is a smooth endomorphism, and μ is an f-invariant Borel probability measure. Assume that*

(1) f is non-uniformly expanding;

(2) f has the **Property-A**;

(3) μ is non-degenerate at the critical set C.

Then $\mu \ll \lambda$ (= Lebesgue measure) if and only if Pesin's formula holds, i.e.

$$h_\mu(f) = \int_X \log |detDf_x| d\mu(x).$$

Theorem 1.2.2 *Suppose that X is a compact Riemannian manifold of dimension higher than 2 without boundary, and $f : X \to X$ is a smooth non-uniformly expanding map, and μ is an f-invariant Borel probability measure. Assume that*

(1) There is a finite measurable partition of X, $\xi = \{\xi_1, \cdots, \xi_k\}$, such that the interior of ξ_i is non-empty for each i and $f|_{int\xi_i}$ is one-to-one to its image;

(2) μ is non-degenerate at the boundary $\partial\xi = \cup\partial\xi_i$.

Then $\mu \ll \lambda$ if and only if Pesin's formula holds, i.e.

$$h_\mu(f) = \int_X \log |detDf_x| d\mu(x).$$

Remark. (i) To simplify the proof, we will always assume that the invariant measure μ is ergodic. However this condition is not necessary since by the ergodic decomposition theorem, μ can be expressed as the convex combination of ergodic measures. We will discuss this condition more in detail in §5.

(ii) The proof of Theorem 2.2 will be explained in §5 under some weaker assumption on f.

§1.3. 3 Useful Partitions

A. An Elementary Partition ξ on X

For the rest of the discussions in this Chapter, we assume that all the hypothesis of Theorem 1.2.1 are satisfied. We especially emphasize that $dimX = 2$ and $\lambda_1(x) \geq \lambda_2(x) > 0$ for μ-a.e. $x \in X$. Then by Prop. 1.13, we will construct ous first elementary finite measurable partition of X based on the local structure of singularities as follows.

Proposition 1.3.1. *There is a finite measurable partition $\xi = \{\xi_1, \xi_2, ...\xi_n\}$ of X such that for each i,*

$$f_i = f|_{\xi_i^0} : \xi_i^0 \longmapsto f(\xi_i^0), \qquad i = 1, 2, ...n,$$

is a diffeomorphism, where the interior of ξ_i, denoted by ξ_i^0, is not empty.

Proof. Given $x \in X$, choose a small open neighborhood $\mathcal{N}_x$ of x in such a way that:

(a) If x is a regular point of f, then choose $\mathcal{N}_x$ so that $f|_{\mathcal{N}_x^0}$ is a diffeomorphism from $\mathcal{N}_x^0$ to its image;

(b) If x is a critical point of f, then choose $\mathcal{N}_x$ so that under the change of coordinate, the local structure of f in $\mathcal{N}_x$ has the form either (1.1.6), as a fold, or (1.1.7), as a cusp. (Cf. the definition 1.1.2)

Moreover by Prop. 1.9 and 1.13, we may assume that every $\mathcal{N}_x$ is chosen adequatedly so that μ is non-degenerate at $\partial\mathcal{N}_x$.

Since X is compact, there are finitely many such open sets, $\mathcal{N}_x$, denoted by $\mathcal{N}_1, ..., \mathcal{N}_m$, which cover X. Then by the properties (A.1) and (A.3) in the Appendix, if $\mathcal{N}_i$ is the neighborhood of critical point, then it can be split into at most seven subsets such that the interior of each subset is non-empty and f is a diffeomorphism when restricted to this interior. Also if $\mathcal{N}_j$ is the neighborhood of a regular point, we do not split it. Therefore, from this finite measurable covering of X, it is easy to construct a finite partition $\xi = \{\xi_1, \xi_2, ..., \xi_n\}$ and clearly it satisfies the required condition. This finishes the proof. ∎

Clearly if we connect this partition to the inverse limit space $\bar{X}$ of X, it turns out that there is finite $\bar{\mu}$-measurable partition $\bar{\xi}$ on $\bar{X}$ induced by ξ defined as follows:

$$\bar{\xi} = \{\pi^{-1}\xi_1, \pi^{-1}\xi_2, ..., \pi^{-1}\xi_n\} = \{\bar{\xi}_1, \bar{\xi}_2, ..., \bar{\xi}_n\}.$$

Remark. If $\mathcal{N}_x$ is a neighborhood of cusp, the curves which splite $\mathcal{N}_x$ into a few components as Prop. 3.1 may not satisfy Propositions 1.3 and 1.13. However by the same method used in the construction of the partition, we can always choose nice curves which satisfies Proposition 3.1. (For details please see the Appendix.) Therefore, μ is non-degenerate not only at $\partial\mathcal{N}_x$ and on the critical set, but also at those curves which split $\mathcal{N}_x$. So techniquely, we make the following assumption on the partition ξ.

H-1. Let $\partial\xi_i$ be the boundary of the set ξ_i, and let $\partial\xi$ denote $\bigcup_{i=1}^{n} \partial\xi_i$, then

$$\mu(\bigcup_{-\infty}^{+\infty} f^n(\partial\xi)) = 0.$$

H-2. μ is non-degenerate at $\partial\xi$.

By $H-2$, we let $\overline{\Sigma} \subseteq \overline{X}$ be the set

$$\overline{\Sigma} = \{\overline{x} \in \overline{X};\ \liminf_n \frac{1}{n} \log \delta_{\partial\xi}(\bar{f}^{-n}\overline{x}) = 0\ \}.$$

It is clear that $\overline{\mu}(\overline{\Sigma}) = 1$.

B. An Elementary Partition $\bar{P}$ on $\bar{X}$

In the view of the elementary partition ξ on X in the previous section, we construct a fundamental partition $\bar{P}$ on $\bar{X}$ as follows:

Given $\bar{x} = (x_0, x_1, \cdots) \in \bar{X}$, the atom $\bar{P}(\bar{x})$ in $\bar{P}$, which contains $\bar{x}$, is defined as follows:

$$\bar{P}(\bar{x}) = \{\bar{y} = (y_n);\ \ \xi(x_n) = \xi(y_n),\ n = 0, 1, 2, ...\}.$$

Theorem 1.3.2. *The partition $\bar{P}$ defined above has the following properties:*
(i) $\bar{P} = \bigvee_{n=0}^{\infty} \bar{f}^n \bar{\xi}$;
(ii) $\pi \bar{P}(\bar{x})$ *is open in* $\xi(x_0)$ *for* $\bar{\mu} -$ *a.e.* $\bar{x} \in \bar{X}$, *where* $\bar{x} = (x_0, x_1, ...)$.

To prove Theorem 1.3.2, we need to discuss some local properties of x and $\bar{x}$. As before, let $\partial\xi$ denote the boundary of partition ξ, i.e. $\partial\xi = \bigcup_{k=1}^{n} \partial\xi_i$ where $\xi = \{\xi_1, ...\xi_n\}$.

Definition 1.3.3. Say a point $\bar{x} = (x_n) \in \bar{X}$ *typical* if $\bar{x} \in \bar{\Gamma} \cap \bar{\Sigma}$ and for every n, $x_n \notin \partial\xi$.

If we let $\bar{T}$ denote the set of typical points in $\bar{X}$, it is easy to check:

Proposition 1.3.4. *$\bar{T}$ has full $\bar{\mu}$-measure, i.e. $\bar{\mu}(\bar{T}) = 1$.*

Proof. Assumption H-1 gives the following fact

$$\bar{\mu}(\bigcup_{n=0}^{+\infty} \bar{f}^{-n}(\pi^{-1}(\partial\xi))) = 0.$$

Note that $\bar{T}^c \subseteq \bigcup_{n=0}^{+\infty} \bar{f}^{-n}(\bar{\pi}^{-1}(\partial\xi))$, this implies that $\bar{T}$ has full measure. ∎

For the rest of this Chapter, we will follow closely what Ledrappier did in his paper[L]. By generalizing his methods, we can get some similar propositions for X, f and μ specified in this part, and the proofs we adopted here have the same flavor as those of Ledrappier's.

Lemma 1.3.5. *If $\bar{x} = (x_n) \in \bar{X}$ is typical, then there is a $\zeta_{\bar{x}} > 0$ such that for every $y_0 \in B(x_0, \zeta_{\bar{x}})$, there exists a unique $\bar{y} = (y_0, y_1, ...) \in \bar{X}$ satisfying*

$$d(x_n, y_n) < \frac{1}{2} d(x_n, \partial\xi).$$

Remark. This proposition means that for $\overline{x} = \{x_n\}$ typical, there is a collection of open sets $\mathcal{O}(x_n)$, $n \geq 0$, in X such that for any $y \in \mathcal{O}(x_0)$, there is a $\overline{y} \in \overline{X}$ typical satisfying $\pi\overline{y} = y$ and $y_n \in \mathcal{O}(x_n)$.

Proof. By the assumption that the measure μ is non-degenerate on the boundary $\partial\xi$ of the partition ξ, then for a given typical point $\bar{x} = (x_n) \in \bar{X}$, we have

$$\liminf_{n\to+\infty} \frac{1}{n} \log \delta_{\partial\xi}(x_n) = 0.$$

Thus for any $\epsilon > 0$, there is an integer $N_1 > 0$ such that for $n \geq N_1$, we have

$$\delta_{\partial\xi}(x_n) > e^{-n\epsilon}, \qquad \text{for } n \geq N_1.$$

But on the other hand, by Theorem 1.16, we can fix a $\lambda > 0$ with

$$0 < \lambda < \lambda_2(x) \leq \lambda_1(x)$$

such that if $d(x_0, y_0) < \bar{\alpha}(\bar{x})$, we have a unique $\bar{y} \in \bar{W}^u_{\bar{x}}$ satisfying

$$d(x_n, y_n) < \bar{\beta}(\bar{x})e^{-n\lambda}, \qquad n > 0.$$

Note that since $\epsilon > 0$ is arbitrary, we can choose $0 < \epsilon < \lambda$ and hence, there exists an $N_2 > 0$ such that for $n > N_2$, we have

$$e^{-n\epsilon} > 2\bar{\beta}(\bar{x})e^{-n\lambda},$$

that is,

$$e^{n(\lambda-\epsilon)} > 2\bar{\beta}(\bar{x}).$$

Let $N = max\{N_1, N_2\}$, then for $n \geq N$, we obtain

$$2d(x_n, y_n) < \delta_{\partial\xi}(x_n) = d(x_n, \partial\xi),$$

where $\bar{y} \in \bar{W}^u_{\bar{x}}$ is uniquely determined by the given initial point $y_0 \in X$ with $d(x_0, y_0) < \bar{\alpha}(\bar{x})$.

Set

$$B_k = \{\bar{y} = (y_n) \in \bar{X};\ \ d(y_i, x_i) < \min\{\bar{\alpha}(\bar{x}),\ \frac{1}{2}\delta_{\partial\xi}(x_i),\ i = 0, 1, ..., k\}.$$

Then obviously, $\pi(B_N \bigcap \bar{W}^{\lambda}_{\bar{x}})$ contains an open neighborhood of x_0. This implies that we can choose a $\zeta_{\bar{x}} > 0$ such that

$$B(x_0, \zeta_{\bar{x}}) \subseteq \pi(B_N \bigcap \bar{W}^u_{\bar{x}}).$$

Therefore, it is clear that given any $y_0 \in B(x_0, \zeta_{\bar{X}})$, there is a unique $\bar{y} \in \bar{X}$ *with* $\pi\bar{y} = y_0$ and for all $n \geq 0$,

$$d(x_n, y_n) < \frac{1}{2}d(x_n, \partial\xi).$$

We finish the proof. ∎

<u>Proof of Theorem 1.3.2</u>:

(i) It follows easily by the induction, since from the definition of "$\bigvee$" we have that

$$\begin{aligned}(\bar{\xi} \bigvee \bar{f}\bar{\xi})(\bar{x}) &= \{\bar{y} = (y_n);\ \xi(x_0) = \xi(y_0),\ \xi(\pi\bar{f}^{-1}\bar{x}) = \xi(\pi\bar{f}^{-1}\bar{y})\} \\ &= \{\bar{y} = (y_n);\ \xi(x_0) = \xi(y_0),\ \xi(y_1) = \xi(x_1)\}.\end{aligned}$$

and in general, we have for any n that,

$$(\bar{\xi} \bigvee \bar{f}\bar{\xi} \bigvee ... \bigvee \bar{f}^n\bar{\xi})(\bar{x}) = \{\bar{y} = (y_n);\ \xi(x_i) = \xi(y_i),\ i = 0, 1, ..., n\}.$$

(ii) By the hypothesis that μ is non-degenerate on $\partial\xi$, that is, for $\bar{\mu}-a.e.\ \bar{x} \in X$, there is an $\epsilon_0 > 0$ such that

$$d(x_n, \partial\xi) \geq e^{-n\epsilon_0},\ \ \forall n > 0.$$

But on the other hand by Lemma 1.3.5, for any $y_0 \in B(x_0, \zeta_{\bar{x}})$, there is a unique $\bar{y} = (y_0, y_1, ...) \in \bar{X}$ such that

$$\bar{y} \in \bar{P}(\bar{x}).$$

Since $d(x_n, y_n) < \frac{1}{2}d(x_n, \partial\xi)$, this implies that $\xi(x_n) = \xi(y_n)$ *for* $n \geq 0$. Hence, $\pi\bar{P}(\bar{x})$ contains an open ball centered at x_0, denoted as $B(x_0)$, and it is obvious that

$$B(x_0) \subseteq \pi\bar{P}(\bar{x}) \subseteq \xi(x_0).$$

∎

Remark. It is clear that $\overline{P}$ may not be a generator. Since if the partition ξ satisfies that $f^{-1}\partial\xi \subseteq \partial\xi$, then it is easy to verify $(\bigvee_{n\geq 0} \bar{f}^{-n}\bar{P})(\bar{x}) = \bar{P}(\bar{x})$. Hence if $\bar{x}$ is typical, then the Theorem 1.3.2 implies that $\pi\overline{P}(\bar{x})$ contains an open set $B(x_0)$, and for any $y_0 \in B(x_0)$, there is a $\bar{y} \in \overline{X}$ such that $\bar{y} \in \overline{P}(\bar{x})$.

C. A Generator $\bar{\eta}$ on $\bar{X}$

To construct a generator on $\bar{X}$, we consider the measurable functions $\bar{\alpha}(\bar{x})$ and $\bar{\beta}(\bar{x})$ which are given by Theorem 1.16, and two fixed constants $\alpha_0 > 0$, and $\beta_0 > 0$ such that the measurable set $\bar{A} = \{\bar{\alpha} \geq \alpha_0,\ \bar{\beta} \leq \beta_0\}$ has positive measure, i.e.

$$\bar{\mu}(\bar{A}) > 0.$$

Moreover, fix an $x_0 \in X \setminus \partial\xi$ and an $\epsilon > 0$ with $\epsilon < \alpha_0/3$, and set

$$\bar{A}_\epsilon = \bar{A} \bigcap \pi^{-1}(B(x_0, \epsilon)).$$

It is clear that $\bar{A}_\epsilon \subseteq \bar{X}$ is measurable and we may choose such an x_0 and a small ϵ, such that each component of $f^{-1}(B(x_0, \epsilon))$ is contained inside exactly one atom of the partition ξ, and

$$\bar{\mu}(\bar{A}_{\frac{\epsilon}{2}}) > 0.$$

This can be done because of the compactness of X and the finiteness of f with $\mu(\partial\xi) = 0$.

Following Ledrappier [L], we now construct a measurable partition $\bar{\eta}$ on$\bar{X}$ as follows and then prove that $\bar{\eta}$ is a generator with some interesting properties.

Let

$$\bar{\eta} = \bigvee_{n=0}^{\infty} \bar{f}^n(\{\bar{A}_{\frac{\epsilon}{2}}, \bar{X} \setminus \bar{A}_{\frac{\epsilon}{2}}\} \bigvee \bar{\xi}).$$

Theorem 1.3.6. *$\bar{\eta}$ has the following properties:*

(i) *$\bar{\eta}$ increases, i.e. $\bar{f}^{-1}\eta > \bar{\eta}$;*

(ii) *$\bar{\eta}(\bar{x}) \subseteq \bar{P}(\bar{x})$;*

(iii) *$\bar{\eta}$ is a generator.*

Proof. (i) It is obvious by the definition.

(ii) Since $\bar{\eta} > \bigvee_{n=0}^{\infty} \bar{f}^n \bar{\xi} = \bar{P}$, the result follows immediately.

(iii) Given $\bar{x} = (x_n)$ and $\bar{y} = (y_n) \in \bigvee_{n=0}^{\infty} \bar{f}^{-n}\bar{\eta}(\bar{x})$, i.e. $\bar{f}^n\bar{x}$ and $\bar{f}^n\bar{y}$ belong to the same element of $\bar{\eta}$, $\forall n \geq 0$. Note that also

$$\bar{\eta} > \{\bar{A}_{\frac{\epsilon}{2}}, \bar{X} \setminus \bar{A}_{\frac{\epsilon}{2}}\} \bigvee \bar{P}.$$

Then we can choose an increasing subsequence n_i such that

$$\bar{f}^{n_i}\bar{x},\ \bar{f}^{n_i}\bar{y}\ \in\ \bar{A}_{\frac{\epsilon}{2}}. \tag{$*$}$$

This is because that $\bar{\mu}(\bar{A}_{\frac{\epsilon}{2}}) > 0$ and $\bar{\mu}$ is ergodic. We now have the following estimate:

$$\begin{aligned} d(\pi\bar{x}, \pi\bar{y}) &= d(\pi\bar{f}^{-n_i}(\bar{f}^{n_i}\bar{x}), \pi\bar{f}^{-n_i}(\bar{f}^{n_i}\bar{y})) \\ &\leq 2\beta_0 e^{-n_i\lambda}, \end{aligned}$$

where $\bar{f}^{n_i}\bar{x}$ and $\bar{f}^{n_i}\bar{y}$ belong to the same unstable manifold by (*).

Therefore, $x_0 = \pi\bar{x} = \pi\bar{y} = y_0$. Note that π is injective when restricted to the local unstable manifold, we have proved the result by applying Theorem 1.16 that

$$\bar{x} = \bar{y}.$$

It follows that $\bar{\eta}$ is a generator. ∎

Proposition 1.3.7. *For $\bar{\mu}$ – a.e. $\bar{x} \in \bar{X}$, $\pi\bar{\eta}(\bar{x}) \subseteq \pi\bar{P}(\bar{x})$ is open.*

Proof. By an appropriate choice of $\epsilon > 0$, we may assume that

$$\bar{E} = \bar{A}\bigcap \pi^{-1}(\partial B(x_0,\epsilon)) \neq \emptyset.$$

Define $\omega(\bar{y})$ as the follwing distance function

$$\omega(\bar{y}) = \operatorname{dist}(\pi\bar{y}, \partial B(x_0,\epsilon)).$$

Note that for $0 < \tau < 0$,

$$\bar{\mu}(\{\omega(\bar{f}^{-n}\bar{y}) < \tau^n\}) \leq \bar{\mu}\{\bar{y};\ \ d(\pi\bar{y}, \partial B(x_0,\epsilon)) < \tau^n\}.$$

Similarly, we can choose an ϵ so that μ satisfies the condition of Prop. 1.9 on $\partial B(x_0,\epsilon)$. This yields

$$\sum_{n=1}^{\infty} \bar{\mu}(\{\omega(\bar{f}^{-n}\bar{y}) < \tau^n\}) \leq \sum \bar{\mu}\{\bar{y};\ d(\pi\bar{y}, \partial B(x_0,\epsilon)) < \tau^n\} < +\infty.$$

Thus in the view of Borel-cantelli Lemma, it follows that for any $\chi > 0$, the following function

$$\psi(\bar{x}) := \inf\{\omega(\bar{f}^{-n}\bar{x})e^{n\chi},\ \ n \geq 0\}, \quad \bar{\mu} - a.e.$$

is positive.

For $\bar{x} = (x_n) \in \bar{X}$, we now consider those $y_0 \in X$ such that

$$d(x_0, y_0) < \min\{\bar{\alpha}(\bar{x}),\ \zeta_{\bar{x}},\ \psi(\bar{x})\backslash 2\beta(\bar{x})\}.$$

It is clear that there exists a unique $\bar{y}(y_0, y_1, ...) \in \bar{X}$ with $\bar{y} \in \bar{P}(\bar{x})$ such that

$$\begin{aligned} d(\pi\bar{f}^{-n}\bar{x}, \pi\bar{f}^{-n}\bar{y}) &\leq \bar{\beta}(\bar{x})e^{-n\chi}d(x_0,y_0) \\ &\leq \bar{\beta}(\bar{x})e^{-n\chi}\psi(\bar{x})\backslash 2\bar{\beta}(\bar{x}) < \omega(\bar{f}^{-n}\bar{x}). \end{aligned}$$

This implies that $\bar{f}^{-n}\bar{x}$ and $\bar{f}^{-n}\bar{y}$ are either in $\bar{A}_\epsilon$ or in $\bar{X} \setminus \bar{A}_\epsilon$, for all $n \geq 0$. But $d(x_0, y_0) < \zeta_{\bar{x}}$, this implies that $\bar{x}$ and $\bar{y}$ must be in the same atom of $\bar{\eta}$. Therefore, there exists an open neighborhood of x_0, denoted as $\mathcal{N}(x_0)$, such that

$$\mathcal{N}(x_0) \subseteq \pi\bar{\eta}(\bar{x}) \subseteq \bar{\pi}\bar{P}(\bar{x}).$$

Proposition 1.3.8. *For the generator $\bar{\eta}$, we have*

$$h_\mu(f) = H(\bar{f}^{-1}\bar{\eta} \mid \bar{\eta}).$$

Proof. See [**L**].

Up to now, we have constructed three relevant partitions, ξ, $\bar{P}$, and $\bar{\eta}$, and discussed some interesting properties which are useful in the proof of the main theorem of this Chapter.

D. An Important Estimate on $\overline{\eta}$

Let $C = C_f$ be the critical set of f.

Proposition 1.3.9. *If $x \in X$ is not a critical point, then*

$$D^{-1} \leq \frac{|detDf_x|}{|d(x, C)|} \leq D,$$

where D is a constant independent of the choice of x.

Remark. This inequality means that for maps satisfying the generic *property-A*, their critical points satisfy the condition very similar to that of "non-flat" condition for maps defined in an interval.

Proof. By Whitney's classification theorem (Prop. 1.14), the only critical points are folds and cusps. For $\epsilon > 0$ small enough, let $\mathcal{N}_\epsilon(C)$ denote the open ϵ-eighborhood of C.

It is obvious that for x in the complement of $\mathcal{N}_\epsilon(C)$, there is a constant D_1 independent of x such that

$$D_1^{-1} \leq \frac{|\det Df_x|}{d(x, C_f)} \leq D_1.$$

We now suppose that $x \in \mathcal{N}_\epsilon(C)$.

Case 1. Assume that x is near a fold. Clearly, there is a constant D_2 such that

$$\frac{|\det Df_x|}{d(x,C)} \leq \frac{D_2 d(x,C)}{d(x,C)} = D_2.$$

Moreover, we assume that locally near the fold, we have already had the local coordinate such that f has the form $f:(u,v) \longmapsto (u,v^2)$. Then it is easy to check that $d(x,C) = |v|$, ($v = 0$ corresponding critical point), and $\det Df_x = 2v$, hence, let $D_2 > 2$, we obtain,

$$D_2^{-1} \leq \frac{|\det Df_x|}{d(x,c)} \leq D_2.$$

Case 2. Assume that x is near a cusp. Also we assume that near this cusp we have already had $f:(u,v) \longmapsto (u, uv+v^3)$. Clearly, there is a D_3 such that

$$\frac{|\det Df_x|}{d(x,c)} < D_3.$$

Note that locally, the critical set is a 1-dimensional manifold: $u + 3v^2 = 0$, and hence for fixed $x_0 = (u_0, v_0) \notin C$, $y_0 = (-3v_0^2, v_0) \in C$, we have

$$d(x_0, C) \leq d(x_0, y_0) = |u_0 + 3v_0^2|.$$

and

$$\det Df_{x_0} = u_0 + 3v_0^2$$

. Therefore if we let $D_3 > 1$, we have obtained

$$D_3^{-1} < \frac{|\det Df_x|}{d(x,c)} < D_3.$$

Let $D = \max\{D_1, D_2, D_3\}$, then for $x \in X \setminus C$, we obtain

$$D^{-1} < \frac{|\det Df_x|}{d(x,C)} < D.$$

It is clear that D is independent of $x \in X \setminus C$. ∎

Proposition 1.3.10. *There is a constant $A > 0$ independent of all typical points $\bar{x} \in \bar{T} \subseteq \bar{X}$ such that for $\bar{y} \in \bar{X}$ determined by Lemma 1.3.5 with $d(x_0, y_0) < \zeta_{\bar{x}}$, we have*

$$A^{-1} < \frac{|detDf_{x_n}|}{|detDf_{y_n}|} < A, \qquad \forall\, n \geq 0.$$

Proof. Since $\bar{x}$ is typical, it implies by Pro.3.4 that

$$\det Df_{y_n} \neq 0, \qquad \det Df_{x_n} \neq 0, \qquad \forall n \geq 0.$$

Now for each $n \geq 0$, let $x_n' \in \partial\xi$ and $y_n' \in \partial\xi$ be such that

$$\begin{aligned}\frac{d(x_n, \partial\xi)}{d(y_n, \partial\xi)} &= \frac{d(x_n, x_n')}{d(y_n, y_n')} \\ &\geq \frac{d(x_n, x_n')}{d(x_n, y_n) + d(x_n, \partial\xi)} \\ &\geq \frac{d(x_n, x_n')}{d(x_n, x_n')/2 + d(x_n, x_n')} > \frac{1}{2}.\end{aligned}$$

But on the other hand, geometrically we have,

$$d(y_n, y_n') \geq \frac{1}{2} d(x_n, x_n').$$

Thus we obtain

$$\frac{1}{2} \leq \frac{d(x_n, \partial\xi)}{d(y_n, \partial\xi)} \leq 2.$$

Note that

$$\frac{|\det Df_{x_n}|}{|\det Df_{y_n}|} = \left\{ \frac{|\det Df_{x_n}|}{d(x_n, \partial\xi)} \Big\backslash \frac{|\det Df_{y_n}|}{d(y_n, \partial\xi)} \right\} \cdot \frac{d(x_n, \partial\xi)}{d(y_n, \partial\xi)}.$$

By setting $A = 2D^2$, we finish the proof by combining the above estimate with Prop.3.5 that

$$A^{-1} \leq \frac{|\det Df_{x_n}|}{|\det Df_{y_n}|} < A, \qquad \forall\, n \geq 0.$$

∎

Now for any $\bar{x} = (x_n) \in \bar{X}$ be typical, let $\bar{y} = (y_n) \in \bar{X}$ satisfy Prop. 3.10 , and we set

$$\Delta(\bar{x}, \bar{y}) = \prod_{n=0}^{\infty} \frac{|\det Df_{x_n}|}{|\det Df_{y_n}|}.$$

Proposition 1.3.11. *There are measurable functions $\nu_1(\bar{x})$ and $\nu_2(\bar{x})$ on $\bar{X}$ such that*

$$0 < \nu_1(\bar{x}) \leq \Delta(\bar{x}, \bar{y}) \leq \nu_2(\bar{x}), \qquad \bar{\mu} - a.e.\ \bar{x}.$$

Proof. Let $B > 0$ be the constant such that for any x and y in X, the following estimate holds

$$|\det Df_x - \det Df_y| \leq B \cdot d(x, y).$$

Then for $\bar{x} = (x_n)$ typical, and $\bar{y} = (y_n)$ in $\bar{X}$ with $d(x_0, y_0) < \zeta_{\bar{x}}$, by applying the unstable manifold theorem we obtain

$$|\det Df_{x_n} - \det Df_{y_n}| \leq B \cdot d(x_n, y_n) \leq B \cdot \beta(\bar{x}) e^{-n\lambda}, \qquad \forall n \geq 0.$$

It follows that

$$\begin{aligned} |\frac{\det Df_{x_n}}{\det Df_{y_n}} - 1| &\leq \frac{B\beta(\bar{X}) e^{-n\lambda}}{|\det Df_{y_n}|} \\ &\leq \frac{AB\beta(\bar{x}) e^{-n\lambda}}{|\det Df_{x_n}|} \leq \frac{AB\beta(\bar{x}) e^{-n\lambda}}{\frac{1}{D} \cdot d(x_n, \partial\xi)} \\ &\leq const.\beta(\bar{x}) e^{-n\lambda} \cdot e^{n\epsilon_0} \\ &= const.\beta(\bar{x}) e^{-n(\lambda - \epsilon_0)}. \end{aligned}$$

Hence it is easy to choose the measurable functions $\nu_2(\bar{x}) \geq \nu_1(\bar{x}) > 0$ based on the above estimate that

$$0 < \nu_1(\bar{x}) \leq \Delta(\bar{x}, \bar{y}) \leq \nu_2(\bar{x}).$$

∎

Theorem 1.3.12. *For $\bar{\mu}-a.e.\ \bar{x} \in \bar{X}$, we have*

$$0 < \int_{\bar{\eta}(\bar{x})} \Delta(\bar{x},\bar{y})d\bar{y} < \infty,$$

where $d\bar{y}$ denotes the natural Lebesgue measure on each element of $\bar{\eta}$.

Proof. The integral having positive value follows directly from the previous Proposition 3.11, since $\bar{\eta}(\bar{x})$ contains an open neighborhood of $\bar{x}$, which has positive Lebesgue measure. And on $\bar{\eta}(\bar{x})$, we have proved that $\Delta(\bar{x},\bar{y}) > \nu_1(\bar{x}) > 0$. Hence, we only need to verify the integral bounded above.

Note that $\bar{\eta} > \{\bar{A}_{\frac{\epsilon}{2}},\ \bar{X} \setminus \bar{A}_{\frac{\epsilon}{2}}\} \bigvee \bar{P}$, hence if $\bar{\eta}(\bar{x}) \subseteq \bar{A}_{\frac{\epsilon}{2}}$, then by the proof of Prop.2.7, we have

$$\Delta(\bar{x},\bar{y}) \leq \nu_2(\bar{x}) \leq const.$$

The result follows immediately since $\beta(\bar{x}) \leq \beta_0 = constant.$

On the other hand, if $\bar{\eta}(\bar{x}) \subseteq \bar{X} \setminus \bar{A}_{\frac{\epsilon}{2}}$, then by the ergodicity, there is an integer k such that $\bar{f}^{-k}\bar{x} \in \bar{A}_{\frac{\epsilon}{2}}$. Note that

$$\bar{\eta}(\bar{x}) = \bar{f}^k([\bar{f}^{-k}\bar{\eta}](\bar{f}^{-k}\bar{x})),$$

it follows that

$$\int_{\bar{\eta}(\bar{x})} \Delta(\bar{x},\bar{y})d\bar{y} = \int_{\bar{f}^k([\bar{f}^{-k}\eta](\bar{f}^{-k}\bar{x}))} \Delta(\bar{f}^{-k}\bar{x},\bar{f}^{-k}\bar{y}) \prod_{i=1}^{k} \frac{|\det Df_{x_i}|}{|\det Df_{y_i}} d\bar{y}$$

$$= \prod_{i=1}^{k} |\det Df_{\pi \bar{f}^{-i}\bar{x}}| \int_{[\bar{f}^{-k}\eta](\bar{f}^{-k}\bar{x})} \Delta(\bar{f}^{-k}\bar{x},\bar{y})d\bar{y} < +\infty.$$

Thus the boundness of the integral has been proved. ∎

§1.4. The Proof of the Main Theorem of Chapter 1

A. Prove: "Pesin's formula holds $\Longrightarrow \mu \ll \lambda$".

Lemma 1.4.1.(following **[L]**) *The Rohlin decomposition of the measure $\bar{\mu}$ on $\bar{X}$ with respect to the partition $\bar{\eta}$ is given by $q(\bar{x},\cdot)$ on $\bar{\eta}(\bar{x})$, for $\bar{x} \in \bar{X}$, for which $q(\bar{x},\cdot)$ is specified as follows: for any measurable subset $\bar{B} \subseteq \bar{X}$,*

$$q(\bar{x},\bar{B}) = \frac{\int_{\bar{B}\cap\bar{\eta}(\bar{x})} \Delta(\bar{x},\bar{y})d\bar{y}}{\int_{\bar{\eta}(\bar{x})} \Delta(\bar{x},\bar{y})d\bar{y}}.$$

Proof. It is obvious, by Prop. 1.3.12, that $q(\bar{x},\cdot)$ is a probability measure. This can be checked easily that

(i) $q(\bar{x},\eta(\bar{x})) = 1$;

(ii) $q(\bar{x}_1,\cdot) = q(\bar{x}_2,\cdot)$, if $\bar{\eta}(\bar{x}_1) = \bar{\eta}(\bar{x}_2)$.

If we assume that the Rohlin decomposition for $\bar{\mu}$ with respect to $\bar{\eta}$ is given by $p(\bar{x},\cdot)$ on $\bar{\eta}(\bar{x})$, then by taking an integer n, we have the following estimate by Prop. 1.3.8 that

$$nh_\mu(f) = H(\bar{f}^{-n}\bar{\eta} \mid \bar{\eta}) = -\int \log p(\bar{x},[\bar{f}^{-n}\bar{\eta}](\bar{x}))\bar{\mu}(d\bar{x}). \tag{1.4.1}$$

But on the other hand, we will have the following estimate by Prop.2.10 that

$$q(\bar{x},[\bar{f}^{-n}\bar{\eta}](\bar{x})) = \frac{\int_{[\bar{f}^{-n}\bar{\eta}](\bar{x})} \Delta(\bar{x},\bar{y})d\bar{y}}{\int_{\bar{\eta}(\bar{x})} \Delta(\bar{x},\bar{y})d\bar{y}}, \tag{1.4.2}$$

and

$$q(\bar{x},[\bar{f}^{n}\bar{\eta}](\bar{x})) = \frac{\int_{\bar{f}^{-n}(\bar{\eta}(\bar{f}^n\bar{x}))} \Delta(\bar{x},\bar{y})d\bar{y}}{\int_{\bar{\eta}(\bar{x})} \Delta(\bar{x},\bar{y})d\bar{y}}. \tag{1.4.3}$$

Set $k(\bar{y}) = \int_{\bar{\eta}(\bar{y})} \Delta(\bar{y},\bar{y}')d\bar{y}'$, we obtain the following equality easily that

$$q(\bar{x},[\bar{f}^{n}\bar{\eta}](\bar{x})) = \frac{k(\bar{f}^n\bar{x})}{|\det Df^n_{\pi\bar{x}}|k(\bar{x})}. \tag{1.4.4}$$

It follows that

$$-\int \log q(\bar{x}, [\bar{f}^n\bar{\eta}](\bar{x}))d\bar{\mu} = \int \log |\det Df^n_{\pi\bar{x}}|d\bar{\mu}$$
$$= \int \log |\det Df^n_x|d\mu = n\int \log |\det Df_x|d\mu. \qquad (1.4.5)$$

By comparing (1.4.1) and (1.4.5), we obtain the relationshi between q and p as follows:

$$\int \log \frac{q(\bar{x}, [\bar{f}^{-n}\bar{\eta}](\bar{x}))}{p(\bar{x}, [\bar{f}^{-n}\bar{\eta}](\bar{x}))}\bar{\mu}(d\bar{x}) = 0. \qquad (1.4.6)$$

We now define a measure $\bar{\nu}$ on the $\bar{\sigma}$-algebra generated by measurable subsets of $\bar{f}^{-n}\bar{\eta}$ by

$$\bar{\nu}(\bar{B}) = \int \frac{q(\bar{x}, \bar{B})}{p(\bar{x}, \bar{B})}\bar{m}u(d\bar{x}).$$

Then it is clear that (1.4.6) implies

$$\int \log \frac{d\bar{\nu}}{d\bar{\mu}}d\bar{\mu} = 0.$$

And this is possible only if $\bar{\mu} = \bar{\nu}$ by the concavity of log Therefore, $q(\bar{x}, \bar{B}) = p(\bar{x}, \bar{B})$ for any $\bar{B}$ being $\bar{f}^{-n}\bar{\eta}$ measurable. Note that $\bar{f}^{-n}\bar{\eta}$ generates, by letting $n \to +\infty$, we obtain the result that

$$q(\bar{x}, \cdot) = p(\bar{x}, \cdot).$$

∎

We are now back to the proof of the main theorem. Suppose $A \subseteq X$ is measurable with Lebesgue measure zero. Clearly $\pi^{-1}(A)$ is of $q(\bar{x}, \cdot)$ measure zero for $\bar{\mu} - a.e.$ $\bar{x}$ by the lemma 1.3.2. Therefore, by averaging in $\bar{x}$, we conclude that $\pi^{-1}(A)$ is of $\bar{\mu}$-measure zero. It follows that $\mu(A) = 0$. We have proved that

$$\mu \ll \lambda$$

provided that Pesin's formula holds.

B. Prove: "$\mu \ll \lambda \Longrightarrow$ Pesin's formula holds".

Note that Ruelle's inequality gives the estimate,

$$h_\mu(f) \leq \int \log |\mathrm{det} Df_x| d\mu, \tag{1.4.7}$$

we need only estimate the entropy bounded from below, i.e.

$$h_\mu(f) \geq \int \log |\mathrm{det} Df_x| d\mu. \tag{1.4.9}$$

Lemma 1.4.2. *Assume that ρ is the density of μ, i.e. $d\mu = \rho d\lambda$, then for $\mu - a.e.$ $x \in X$, we have*

$$\rho(x) = \sum_{y \in f^{-1}x} \frac{\rho(y)}{|detDf_y|}.$$

Proof. Let $C_0 = C$ be the set of critical points of f, and for $n = \pm 1, \pm 2, \cdots$, set

$$C_n = f^n(C_0).$$

Set

$$C_\infty = \bigcup_{-\infty}^{+\infty} C_n.$$

It is clear by $\mathbf{H-1}$ that $\mu(C_\infty) = 0$.

Now for any $x \in X \setminus C_\infty$, let

$$f^{-1}x = \{y_1,\ y_2, \cdots, y_m\}$$

where m is some fixed integer less than or equal to some fixed integer N by the finiteness of f.

We notice that $\mathrm{det} Df_{y_i} \neq 0$ for $i = 1, 2..., m$, and there is a sufficiently large K such that for all $n \geq K$,

$$f:\ B(y_i, 1/n) \longmapsto f(B(y_i, 1/n)$$

is a diffeomorphism for $i = 1, 2, ..., m$. By fixed a large n_0, we may assume that

$$f^{-1}(B(x, 1/n_0)) = \bigcup_{i=1}^{m} B(y_i),$$

where $B(y_i)$ is an open neighborhood of y_i such that $B(y_i) \bigcap B(y_j) = \emptyset$ for $i \neq j$. Furthermore, by changing the variables, we have following equality:

$$\begin{aligned}\int_{B(x,1/n_0)} \rho(x) d\lambda(x) &= \sum_{i=1}^{m} \int_{B(y_i)} \rho(y) d\lambda(y) \\ &= \int_{B(x,1/n_0)} \sum_{i=1}^{m} \frac{\rho(f^{-1}x)}{|\det Df_{f^{-1}x}|} d\lambda(x) \\ &= \int_{B(x,1/n_0)} \sum_{y \in f^{-1}x} \frac{\rho(y)}{|\det Df_y|} d\lambda(x),\end{aligned}$$

in which the first equality holds by the f-invariance of μ. Since n_0 can be chosen arbitrarily large, the neighborhood $B(x, 1/n_0)$ of x can be arbitrarily small. Hence we obtain

$$\rho(x) = \sum_{y \in f^{-1}x} \frac{\rho(y)}{|\det Df_y|}, \qquad \mu - a.e\ x.$$

∎

Lemma 1.4.3. *Let Σ denote the partition by point, then the Rohlin decomposition for $\mu = \rho\lambda$ with respect to $f^{-n}\Sigma$ is given by, (with $\frac{0}{0} = 0$),*

$$\begin{aligned}E_\mu(B \mid f^{-n}\Sigma)(x) &= E_{\rho\lambda}(B \mid f^{-n}\Sigma)(x) \\ &= \sum_{y \in f^{-n}(f^n x)} \frac{\rho(y) 1_B(y)}{|detDf_y^n|} \Big\backslash \sum_{y \in f^{-n}(f^n x)} \frac{\rho(y)}{|detDf_y^n|}, \qquad \text{for } \mu - a.e\ x,\end{aligned}$$

where $1_B(y)$ is the characteristic function.

Proof. Assume that

$$f^{-n}(f^n x) = \{y_1, \cdots, y_k\}.$$

For simplifying, we will only prove the Lemma for $n = 1$.

Obviously, the point in the modulo space $X\backslash f^{-1}\Sigma$ has the form: $f^{-1}x$, $\forall x \in X$. A set $K \subseteq X\backslash f^{-1}\Sigma$ is measurable if there is a measurable set $A \subseteq X$ such that

$$K = \bigcup_{x \in A} \{f^{-1}x\}.$$

It is easy to check that the associated probability measure $\mu_{f^{-1}\Sigma}$ on $X\backslash f^{-1}\Sigma$ can be defined by

$$\mu_{f^{-1}\Sigma}(K) = \mu(f^{-1}A), \quad \text{where } K = \bigcup_{x \in A} \{f^{-1}x\}. \tag{1.4.9}$$

We now consider any sufficiently small open ball $B \subseteq X$ such that $f|_B$ is a diffeomorphism. Then clearly for $x \in X$, there is at most one $y = y_x \in f^{-1}x$ such that $y \in B$. In the view of Rohlin decomposition, we then have the following estimate:

$$\int_{X\backslash f^{-1}\Sigma} E_{\rho\lambda}(B \bigcap f^{-1}\Sigma(x) | f^{-1}\Sigma)(x) d\mu_{f^{-1}\Sigma}$$

$$= \int_{X\backslash f^{-1}\Sigma} \frac{\rho(y_x)1_B}{|\det Df_{y_x}|} \Big\backslash \sum_{y \in f^{-1}fx} \frac{\rho(y)}{|\det Df_y|} d\mu_{f^{-1}\Sigma}. \tag{1.4.10}$$

By (1.4.9), it is easy to check that

$$d\mu_{f^{-1}\Sigma}(x) = \sum_{y \in f^{-1}fx} \frac{\rho(y)}{|\det Df_y|} d\lambda(x).$$

It follows that (1.4.10) becomes

$$\int_X \frac{\rho(y_x)1_B}{|\det Df_{y_x}|} d\lambda(x) = \int_B |\det Df_x^{-1}| d\mu(x) = \mu(f^{-1}B).$$

Note that μ is f-invariant, we obtain

$$\mu(B) = \int_{X\backslash f^{-1}\Sigma} E_{\rho\lambda}(B \bigcap f^{-1}\Sigma(x) \mid f^{-1}\Sigma)(x) d\mu_{f^{-1}\Sigma}.$$

This finishes the proof. ∎

Lemma 1.4.4. **[P]**

$h_\mu(f) = \sup\{H(\xi \mid f^{-1}\xi);\ \xi \text{ is a measurable partition of X and } f^{-1}\xi < \xi\}.$

Proof. See **[P]**, " Entropy and generators in engodic theory", on p.57. ∎

We now estimate $h_\mu(f)$ from the below.

If we let $I(\cdot \mid \cdot)$ denote the conditional information, then by its definition we have the following estimate

$$\begin{aligned} I(\Sigma \mid f^{-1}\Sigma)(x) &= -\sum_{y\in X} 1_{\{y\}}(x)\log E_\mu(1_{\{y\}} \mid f^{-1}\Sigma) \\ &= -\log E_\mu(1_{\{x\}} \mid f^{-1}\Sigma) \\ &= -\log\left\{\frac{\rho(x)}{|\det Df_x|} \Big\backslash \sum_{y\in f^{-1}fx} \frac{\rho(y)}{|\det Df_y|}\right\}. \end{aligned}$$

The last equality is from Lemma 4.3. Furthermore, it is easy to verify, by Lemma 1.4.2, that

$$\rho(fx) = \sum_{y\in f^{-1}fx} \frac{\rho(y)}{|\det Df_y|}.$$

Thus, we obtain

$$I(\Sigma \mid f^{-1}\Sigma)(x) = \log|\det Df_x| + \log\rho(fx) - \log\rho(x).$$

By taking the ergodic average to the above formula, we obtain

$$H(\Sigma \mid f^{-1}\Sigma) = \int \log|\det Df_x| d\mu.$$

By Lemma 1.4.4, we have obtained the estimate (3.8), i.e.

$$h_\mu(f) \geq \int \log|\det Df_x| d\mu.$$

The theorem is proved.

§1.5. **Extension of the Proof in** §1.4

A. **Ergodic Decomposition**

We show that the hypothesis of ergodicity in the proof of Theorem 1.2.1 is not necessary.

Let $M(X, f)$ be the set of f-invariant Borel probability measures on X, and $E(X, f) \subset M(X, f)$ be the set of ergodic measures. Note that $M(X, f) \neq \emptyset$ and $E(X, f) \neq \emptyset$ since X is compact.

Proposition 1.5.1[W]. *(1) For $\mu \in M(X, f)$, there is a unique Borel probability measure τ on the compact metrizable space $M(X, f)$ such that $\tau(E(X, f)) = 1$ and*

$$\mu = \int_{E(X,f)} m d\tau(m).$$

(2) $h_\mu(f) = \int_{E(X,f)} h_m(f) d\tau(m)$.

Using this proposition, we assume that the f-invariant measure μ considered in this Chapter has the ergodic decomposition given by (1).

We note that if "$\mu \ll \lambda$", then Pesin's entropy formula holds regardless of whether μ is ergodic or not. This fact is clear if we check carefully the proof of the sufficiency of Pesin's formula in §4. So the ergodicity is only used when we prove the necessity of Pesin's formula. Now we show that ergodicity is not necessary.

We assume that Pesin's formula holds. Then,

$$\begin{aligned} h_\mu(f) &= \int \log |detDf_x| d\mu(x) \\ &= \int_X \log |detDf_x| dm(x) \int_{E(X,f)} d\tau(m) \\ &= \int_{E(X,f)} \{ \int_X \log |detDf_x| dm(x) \} d\tau(m). \end{aligned}$$

On the other hand, (2) above gives

$$h_\mu(f) = \int_{E(X,f)} h_m(f) d\tau(m).$$

Therefore,

$$\int_{E(X,f)} h_m(f)d\tau(m) = \int_{E(X,f)} \{\int_X \log|detDf_x|dm(x)\}d\tau(m).$$

By Ruelle's inequality, we have

$$h_m(f) \leq \int_X \log|detDf_x|dm(x).$$

This implies that

$$h_m(f) = \int_X \log|detDf_x|dm(x), \quad \tau - a.e.m.$$

Thus "$m \ll \lambda,\ \tau - a.e.m$", which implies that "$\mu \ll \lambda$".

B. Dimensions Higher than 2

We know that in the case of dimension = 2, the property that singularities of a smooth map are either folds or cusps is generic. Then one can divide a neighborhood of a single singularity into finitely many disjoint pieces such that f, restricted to the interior of each piece, is one-to-one, which leads to some nice partitions on both X and $\bar{X}$.

However singularities in higher dimensional case are so complicated that there does not exist a similar generic property for maps with singularities as in the case of "dim X = 2". Therefore, we can only prove a similar result in higher dimension for those maps satisfying the following hypothesis (such as the polynomial-type singularities):

For $x \in C_f$, there are a neighborhood $\mathcal{N}_x \subset X$ and finitely many codimension 1 submanifolds such that

(1) $\mathcal{N}_x$ can be split into finitely many disjoint domains by these submanifolds,

(2) There is no singularities in the interior of each domain, and f, restricted to the interior of each domain, is one-to-one.

Then we can easily construct some nice partitions as we have done in §1.3. Moreover all the methods used in §1.4 can be carried out. Therefore Theorem 1.2.2 can be proved in a similar way as Theorem 1.2.1.

Appendix. Local Properties of 2-Generic Maps

By the remark of Prop. 1.14, for a generic map of a 2-manifold, the critic points are only of the two types:

(1) fold: $(x, y) \longmapsto (x, y^2)$,

(2) cusp: $(x, y) \longmapsto (x, xy + y^3)$.

Then locally the sets of singularities of a fold and a cusp are $\{y = 0\}$ and $\{x + 3r^2 = 0\}$, respectively. We will discuss their local properties as follows.

(1) Fold. Given two points (x, y) and $(\bar{x}, \bar{y})$ in the neighborhood of the origin, it is obvious that $f(x, y) = f(\bar{x}, \bar{y})$ iff $x = \bar{x}$, and $y^2 = \bar{y}^2$, i.e. the points having the same image must lie on a same vertical line . Hence we need only cut the disk B_δ defined by

$$B_\delta = \{(x, y);\ \ d((x, y), (0, 0)) < \delta\} \qquad (A-1)$$

into two pieces along the x-axis. Then it is easy to check that when f is restricted to either the upper half or the lower half of the disk, it is a diffeomorphism.

(ii) Cusp If (x, y) and $(\bar{x}, \bar{y})$ are in a neighborhood of cusp, then they have the same image under the map f iff $x = \bar{x}$ and $xy + y^3 = \bar{x}\bar{y} + \bar{y}^3$. It follows that $\bar{y}^2 + y\bar{y} + y^2 + x = 0$. Hence we have

$$\bar{y} = \frac{-y \pm \sqrt{-3y^2 - 4x}}{2} \qquad (A-3)$$

This implies if (x, y) lies in the right half disk ($x \geq 0$), the mapping f is one-to-one. So we need only consider the case that $x < 0$.

Proposition A.1. *Let C_ε be the critical set near a cusp, i.e.*

$$C_\varepsilon = \{(x, y);\ \ x + 3y^2 = 0,\ x^2 + y^2 < \varepsilon^2\}$$

Then $f^{-1}(f(C_3))$ consists of two smooth curves:

$$C_\varepsilon = \{(x, y);\ \ x + 3y^2 = 0,\ x^2 + y^2 < \varepsilon^2\},$$

and

$$C_\varepsilon' = \{(x,y);\ \ 4x + 3y^2 = 0,\ x^2 + y^2 < \varepsilon^2\}.$$

(See the figure *Figure 1.1*).

Proof. It is easy to check that for a fixed small $x_0 < 0$, if (x_0, y_0) on C_3 and $(x_0, \bar{y})$ on C_ε', then $f(x_0, y) = f(x_0, \bar{y})$ iff $\bar{y} = (-y \pm \sqrt{-3y^2 - 4x_0})/2$. Note that $x_0 + 3y^2 = 0$, then $-3y^2 - 4x_0 = -3y^2 + 12y^{=}9y^2$, i.e.

$$\bar{y} = \frac{-y \pm \sqrt{9y^2}}{2} = \frac{-y \pm 3y}{2} = \begin{cases} -2y \\ y \end{cases}$$

Hence $\bar{y} = -2y$. It follows that

$$4x_0 + 3(-2y)^2 = 4x_0 + 12y^2 = 4(x_0 + 3y^2) = 0.$$

Moreover, since $\bar{y}$ is the only solution which is not a critical point, we have proved that C_ε and C_ε' are the only sets such that

$$f(C_\varepsilon) = f(C_\varepsilon').$$

∎

Set

$$C_\varepsilon'' = \{(x,y);\ \ x + y^2 = 0,\ x^2 + y^2 < \varepsilon^2\}.$$

Proposition A.2. *Given any $-\varepsilon < x_0 < 0$, the vertical line: $x = x_0$ intersects C_ε'' and the x-axis at the following three points:*

$$(x_0, 0),\ (x_0, \sqrt{-x_0}),\ (x_0, -\sqrt{-x_0}).$$

And

$$f(x_0, 0) = f(x_0, -\sqrt{-x_0}) = f(x_0, \sqrt{-x_0}).$$

Proof. Obvious. ∎

Proposition A.3. *The curves C_ε, $C_\varepsilon^{'}$, $C_\varepsilon^{''}$ and the negative x-axis divide B_ε into seven subsets, $P_1, P_2, ..., P_7$. And moreover $f|_{P_i^0} : P_i^0 \to f(P_i^0)$ are diffeomorphisms for $i = 1, 2, ..., 7$, where K^0 denote the interior of a set K.*

(Please see the figure *Figure 1.2*).

Proof. If $x_0 \geq 0$, it is trivial to check the one-to-one property. So we need only check that $x_0 < 0$. Also by symmetry, we need only to check $y > 0$.

(i) If $y > 0$ is outside P_i, $i = 2, ..., 7$, then the points (x, y) satisfies $4x+3y^2 > 0$. By (A-3),

$$\Delta = -4x - 3y^2 < 0,$$

Hence, (A-3) has no solution other than (x, y). Therefore

$$f|_{P_1^0} : P_1^0 \to f(P_1^0) \quad \text{is a diffeomorphism.}$$

(ii) If $(x_0, y) \in P_2$, then (x_0, y_0) is at C_ε, i.e. $x_0 + 3y_0^2 = 0$. Clearly, $(x_0, 2y_0)$ is at $C_\varepsilon^{'}$ and $(x_0, \sqrt{3}y_0)$ is at $C_\varepsilon^{''}$. Hence,

$$\sqrt{3}y_0 < y < 2y_0, \quad x_0 + 3y_0^2 = 0.$$

We now consider a different point $(x_0, \bar{y})$ from (x_0, y) such that it has same image as (x_0, y), then by (A-3),

$$\bar{y} = \frac{-y \pm \sqrt{-4x_0 - 3y^2}}{2} = \bar{y}_- \textit{ and } \bar{y}_+$$

and clearly $\bar{y}_- \notin P_2$. If $\bar{y}_+ < 0$, then we are done since $0 > \bar{y}_+ \notin P_2$. So we assume that $\bar{y}_+ > 0$. It follows that $\sqrt{-4x_0 - 3y^2} > y > 0$, and hence $x_0 + y^2 < 0$ It is a contradiction since we assume that $y \in P_2$ which implies $x_0 + y^2 > 0$.

(iii) $(x_0, y) \in P_3$, i.e. $0 < y_0 < y < \sqrt{3}y_0$. Then as $x_0 + 3y_0^2 = 0$

$$\bar{y}_+ = \frac{-y + \sqrt{-4x_0 - 3y^2}}{2} = \frac{-y + \sqrt{-4(-3y_0)^2 - 3y^2}}{2}$$

$$= \frac{-y + \sqrt{12y_0^2 - 3y^2}}{2} < \frac{-y_0 + \sqrt{12y_0^2 - 3y_0^2}}{2} = y_0.$$

Hence, $\bar{y}_+ \notin P_3$ and $f|_{P_3^0}$ is diffeomorphic.

(iv) $(x_0, y_0) \in P_4$ i.e. $0 < y < y_0$, with $x_0 + 3y_0^2 = 0$.

$$\bar{y}_+ = \frac{-y + \sqrt{-4x_0 - 3y^2}}{2} = \frac{-y + \sqrt{12y_0^2 - 3y^2}}{2}$$

$$> \frac{-y_0 + \sqrt{12y_0^2 - 3y_0^2}}{2} = y_0$$

Hence, $(x_0, \bar{y}_\pm) \in P_4$, i.e. $f|_{P_4^0}$ is diffeomorphic.

By symmetry, we can check P_5, P_6, and P_7 in the same way. Hence we have finished the proof that

$$f|_{P_i^0} : P_i^0 \to f(P_i^0), \; i = 1, 2, ..., 7$$

is a diffeomorphism.]qed

Chapter 2

Random Perturbations of Feigenbaum Map

§2.0. Introduction

Let g be the *Feigenbaum map.* That is, g is a fixed point of the period doubling operator

$$T : f(x) \longmapsto \alpha f(f(\alpha^{-1}x)), \quad x \in [-1,1], \tag{2.0.1}$$

defined on a certain class of functions on $[-1,1]$. A more precise description of g will be given in the next section.

For each $\epsilon > 0$, we consider the Markov chain X_n^ϵ, $n = 0, 1, \cdots$, defined as follows:

$$X_{n+1}^\epsilon = g(X_n^\epsilon) + \xi_{n+1}^\epsilon, \tag{2.0.2}$$

where $\{\xi_n^\epsilon\}$ are i.i.d. random variables whose distributions have densities positive on $[-\epsilon, \epsilon]$ and 0 elsewhere.

We say that a probability measure μ^ϵ on I is an *invariant measure* of the Markov chain X_n^ϵ if for every Borel set $\Gamma \subseteq I$,

$$\int P^\epsilon(x,\Gamma)\mu^\epsilon(dx) = \mu^\epsilon(\Gamma), \tag{2.0.3}$$

where $P^\epsilon(x,\Gamma)$ is the transition probability of the chain X_n^ϵ, and it is specified by the density of the random variable ξ^ϵ (see **[Ki]** for instance). It is well known that μ^ϵ exists, and that if

$$\mu^{\epsilon_i} \longrightarrow \mu \ \ \textit{weakly as} \ \ \epsilon_i \longrightarrow 0, \tag{2.0.4}$$

then μ is a g-invariant probability measure. This means that for every Borel set $\Gamma \subseteq I$,

$$\mu(g^{-1}(\Gamma)) = \mu(\Gamma). \tag{2.0.5}$$

Some authors have considered the limit behavior of the measure μ^{ϵ}'s as $\epsilon \to 0$ for various kinds of transformations. For example Kifer **[Ki]** and Young **[Y]** studied the case where f has a hyperbolic attractor. Katok and Kifer **[KK]** and Benedicks and Young **[BY]** studied the one-parameter family $f_\lambda(x) = \lambda x(1-x)$. There are various other results, mostly numerical in nature (see **[CNR]**,**[SWN]** and **[Ka]** for example).

In **[VSK]**, Vul, Sinai and Khanin studied the Feigenbaum map g. Among other things they considered random perturbations of g and obtained some results on the statistical properties of the invariant measure μ^{ϵ}. This paper is also about perturbations of g. We focus however on the *geometry of the support of* μ^{ϵ}.

More precisely, we consider the perturbed Feigenbaum system X_n^{ϵ}, $n = 0, 1, \cdots$ defined at the beginning of the introduction. For each $\epsilon > 0$, we let Λ_ϵ denote the support of the invariant measure μ^{ϵ} and call it the *noisy attractor* of X_n^{ϵ}. We will prove some rigorous results about how the geometry of Λ_ϵ changes as ϵ varies. We will show in particular that as ϵ increases, our noisy attractor does not change continuously. Instead, it "explodes" at certain values of ϵ as adjacent components of Λ_ϵ merge together. Then borrowing some estimates from **[VSK]**, we will discuss also how the number of components of Λ_ϵ scales with ϵ as ϵ changes. More precisely, we prove the following result in this part:

Main Theorem. *There exists a strictly decreasing sequence* $\epsilon_n \to 0$ *such that for every* ϵ *with* $\epsilon_n \geq \epsilon > \epsilon_{n+1}$, *the noisy attractor* Λ_ϵ *consists of exactly* 2^n *disjoint intervals, the union of which contains the Feigenbaum attractor* Λ. *These intervals are cyclically permuted by* X_n^{ϵ}. *Moreover,* ϵ_n *goes to zero exponentially fast as* $n \to \infty$.

For numerical results along similar lines, see for instance **[CNR]**, **[Ka]**, **[SWM]** and **[CFH]**.

§2.1. Properties of the Map g

Let M denote the space of continuously differentiable even maps ϕ of the interval $[-1,1]$ into itself such that

M1. $\phi(0) = 1$,

M2. $x\phi'(x) < 0$ *for* $x \neq 0$.

M2 says that ϕ is strictly increasing on $[-1,0)$ and strictly decreasing on $(0,1]$.

M1 says that the unique critical point 0 is mapped to 1. We want to consider those ϕ's which maps 1 slightly - but not too far - to the left of 0. It may then be possible to find nonoverlapping intervals I_0 about 0 and I_1 near 1 which are exchanged by ϕ. Technically, let $a = -\phi(1) = -\phi^2(0)$ and $b = \phi(a)$. We suppress from the notation the dependence of a and b on ϕ. Define $D(T)$ to be the set of all ϕ's in M such that:

D1. $a > 0$,

D2. $b > a$,

D3. $\phi(b) \leq a$.

The two intervals $I_0 = [-a,a]$ and $I_1 = [b,1]$ are then nonoverlapping and ϕ maps I_0 into I_1 and vice versa. If $\phi \in D(T)$, then $\phi \circ \phi|_{I_0}$ has a single critical point, which is minimum. By making the change of variables $x \mapsto -ax$, we replace I_0 by $[-1,1]$ and the minimum by the maximum, i.e. if we define the period doubling operator T by

$$T\phi(x) = -\frac{1}{a}\phi \circ \phi(-ax) \qquad for\ x \in [-1,1],$$

then $T\phi$ is again in M. Thus T defines a map of $D(T)$ into M. (See e.g. **[La]**.)

M. Feigenbaum **[F1]** and Coullet and Tresser **[CT]** independently observed certain universal scaling properties in the limit of period doubling bifurcations of unimodal maps. They proposed to explain their observation in terms of a fixed point with certain properties for T. The existence of this fixed point was later proved rigorously by Lanford **[La]** and Campanins and Epstein **[CE]** using computer-assisted proofs. They showed that T has a fixed point g which is analytic with $g(1) = -\alpha^{-1}$, $\alpha = 2.5029\cdots$.

A proof that is not computer-assisted has recently been obtained by Sullivan [S1].

Definition 2.1.1. We call this fixed point g of T the *Feigenbaum map.*

It is easy to check that, for any $n \geq 1$, g satisfies the functional equation:

$$g(x) = (-1)^n \alpha^n g^{2^n}(\alpha^{-n}x). \tag{2.1.1}$$

In order to understand the perturbed system X_n^ϵ, $n = 0, 1, \cdots,$, we will need to know some properties of g.

(0) g is an even function;

(1) g has a unique unstable periodic orbit of period 2^n for each $n \geq 0$;

(2) g has no other periodic orbits;

Moreover, if we let

$$\Delta_0^{(n)} = [-\alpha^{-n}, \alpha^{-n}], \quad \textit{for each } n \geq 0,$$

and

$$\Delta_i^{(n)} = g^i(\Delta_0^{(n)}), \quad (i = 1, 2, \ldots, 2^n - 1),$$

then,

(3) $\Delta_i^{(n)} \cap \Delta_j^{(n)} = \emptyset$, *for* $i \neq j$;

(4) $g^{2^n}(\Delta_0^{(n)}) \subseteq \Delta_0^{(n)}$;

(5) Each $\Delta_k^{(n-1)}$ contains exactly two subintervals of the next generation, namely $\Delta_k^{(n)}$ and $\Delta_{k+2^{n-1}}^{(n)}$, $0 \leq k \leq 2^{(n-1)} - 1$;

(6) Each point of the unstable orbit of period 2^n lies in some $\Delta_k^{(n)}$, strictly between $\Delta_k^{(n+1)}$ and $\Delta_{k+2^n}^{(n+1)}$.

(7) let $|\Delta|$ denote the length of an interval Δ, then

$$|\Delta_0^{(n)}| = \max_{0 \leq i < 2^n} |\Delta_i^{(n)}|.$$

For a proof of these properties, we refer the reader to, for example, **[VSK]**.

If we let d_n denote the distance between the orbit of period 2^n and the origin, then it is easy to obtain the following result.

Proposition 2.1.2. *There is a constant $C_0 > \alpha^{-1}$, such that*

$$d_n = C_0 \alpha^{-(n-1)} .$$

Proof. Let C_0 be the distance between the origin and the fixed point of g. Since $T(g) = g$, we have for $n \geq 1$,

$$d_n : \alpha^{-n} = d_{n-1} : \alpha^{-(n-1)}$$

and the proposition then follows. ∎

Definition 2.1.3. The closed set $\bigcap_{n\geq 1} \bigcup_{k=0}^{2^n-1} \Delta_k^{(n)} = \Lambda$ is called the *Feigenbaum attractor.*

Proposition 2.1.4. *The map g has a unique invariant ergodic probability measure μ on I. This measure is supported on Λ, and $g|_\Lambda$ is minimal.*

Proof. Cf. **[VSK]** or **[M]**. ∎

We end this section by quoting a theorem proved in **[VSK]** which provides us with some quantitative estimates on g.

Theorem 2.1.5. [VSK] *There are positive numbers λ and σ such that*

(i)
$$\lim_{n\to\infty} \frac{1}{n} log[\frac{1}{2^n}(\sum_{k=1}^{2^n-1} |\Delta_k^{(n)}|)] = log\sigma,$$

and

(ii)
$$\lim_{n\to\infty} \frac{1}{n} log[1 + \sum_{k=1}^{2^n-1} \prod_{i=k}^{2^n-1} (g'(x_i))^2] = 2(log\lambda - log\sigma)$$

where

$$x_i = g^i(x_0),\ \ x_0 \in \Delta_0^{(n)}.$$

Numerically, one has $\lambda = 6.6\ldots$ *and* $\sigma = 0.29\ldots$.

§2.2. The Noisy Attractor Λ_ϵ

Throughout this section, we assume that $\epsilon > 0$ is fixed so that we will suppress from the notation the dependence on ϵ where there is no confusion.

As we assumed before that the i.i.d. random variables ξ_n^ϵ have densities, it follows easily that the transition probability $P(x, \cdot)$ must be absolutely continuous (denoted by "$\ll$") with respect to the Lebesgue measure λ, i.e. for any $x \in I$,

$$P(x, \cdot) \ll \lambda. \tag{2.2.1}$$

Let μ^ϵ be an invariant measure of the Markov chain X_n^ϵ. It follows from (2.2.1) that

$$\mu^\epsilon \ll \lambda.$$

Let "*supp* μ^ϵ" denote the support of μ^ϵ.

Lemma 2.2.1. *For any* $n \geq 0$,

$$X_n^\epsilon(supp\ \mu^\epsilon) = supp\ \mu^\epsilon. \tag{2.2.2}$$

Proof. This follows from the definition of invariant measure. ∎

Definition 2.2.2.[B] Say a set $A \subseteq I$ is *closed* under the Markov chain X_n^ϵ, if for all $x \in A$,

$$P(x, A) = 1.$$

Lemma 2.2.3. *Let A be a set that is closed under X_n^ϵ, and let $\mathcal{U}_\epsilon(\Lambda)$ denote the ϵ-neighborhood of the Feigenbaum attractor Λ. Then*

$$\mathcal{U}_{\frac{\epsilon}{2}}(\Lambda) \subseteq A, \quad \lambda - a.e.$$

Proof. Let $z \in \Lambda$ be an arbitrary point. We will show that $\mathcal{U}_{\epsilon/2}(z) \subseteq A$ up to a set of Lebesgue measure 0. Let $y \in \Lambda$ be such that $gy = z$. Since Λ is an attractor and $g|\Lambda$ is topologically transitive, for every $x_0 \in A$ and every $\delta > 0$, there exist an n such that $P^n(x_0, \mathcal{U}_\delta(y)) > 0$. If δ is small enough so that $|g(y \pm \delta) - z| < \epsilon/2$, then it will follow that the distribution of $P^{n+1}(x_0, \cdot)$ has a positive density on $\mathcal{U}_{\epsilon/2}(z)$. Since A is closed under X_n^ϵ, we have $P(x, A) = 1,\ \forall\, x \in A$. So $P^{n+1}(x_0, A) = 1$, which implies that $\mathcal{U}_{\epsilon/2} \subseteq A,\ \lambda - a.e.$. ∎

Proposition 2.2.4. *For $\epsilon > 0$, μ^ϵ is the unique invariant measure for the Markov chain X_n^ϵ.*

Proof. Suppose both A_1 and A_2 are closed under X_n^ϵ, then according to lemma 2.3, one has

$$\mathcal{U}_{\frac{\epsilon}{2}}(\Lambda) \subseteq A_1 \cap A_2, \quad \lambda - a.e.$$

Obviously, $\lambda(\mathcal{U}_{\frac{\epsilon}{2}}(\Lambda)) > 0$, so there cannot be two disjoint closed sets. By a standard fact about Markov chains (see for instance [B], Theorem 7.16, pp.136), the invariant measure μ^ϵ is unique. ∎

Definition 2.2.5. Let Λ_ϵ denote the support of the unique invariant measure μ^ϵ with respect to the Markov chain X_n^ϵ. Call Λ_ϵ the *noisy attractor* of X_n^ϵ.

The following lemma is sometimes useful:

Lemma 2.2.6. *(i) If K is a nonempty compact set closed under X_n^ϵ, then*

$$\Lambda_\epsilon \subseteq K.$$

(ii) Λ_ϵ is the topological closure of $\cup_{n \geq 0} X_n^\epsilon(\Lambda)$.

Proof. (i) Since for all $n \geq 0$, X_n^ϵ is well defined on K, so there exists an invariant measure ν^ϵ with support in K. By the uniqueness of the invariant measure for g on I(Prop. 2.4), we conclude that

$$\nu^\epsilon \equiv \mu^\epsilon.$$

It hence follows that

$$\Lambda_\epsilon \subseteq K.$$

(ii) Let C be the set in question. It is easy to verify that it is closed under X_n^ϵ. By (i), $\Lambda_\epsilon \subset C$. To prove the other inclusion, we know from Lemma 2.2.3 that Λ_ϵ must contain the topological closure of $\cup_{n\geq 0} X_n^\epsilon(\mathcal{U}_{\frac{\epsilon}{2}}(\Lambda))$, which obviously contains C. ∎

Because of the nature of our transition probabilities, it is easy to see that Λ_ϵ consists of finitely many closed disjoint intervals, each has length at least 2ϵ. Moreover, these intervals must be cyclically permuted by X_n^ϵ. This is because each interval has positive μ^ϵ-measure and Λ_ϵ is considered to be the smallest invariant set of X_n^ϵ in I in the sense of the Lemma 2.6. More precisely, we prove the following result.

Theorem 2.2.7. *For fixed $\epsilon > 0$, there exist an integer n_ϵ and 2^{n_ϵ} pairwise disjoint closed intervals I_i^ϵ, $i = 0, 1, \cdots, 2^{n_\epsilon} - 1$, satisfying the following properties:*

(1) $\Lambda_\epsilon = \cup_{i=0}^{2^{n_\epsilon}-1} I_i^\epsilon$;

(2) $0 \in I_0^\epsilon$; $I_k^\epsilon = X_k^\epsilon(I_0^\epsilon)$, *for* $k = 1, 2, \cdots, 2^{n_\epsilon} - 1$, *and* $I_0^\epsilon = X_{2^{n_\epsilon}}^\epsilon(I_0^\epsilon)$;

(3) $\Delta_k^{(n_\epsilon)} \subseteq I_k^\epsilon$, *for* $k = 1, 2, \cdots, 2^{n_\epsilon} - 1$. *(This need not be true for $k = 0$.)*

Proof. Since Λ_ϵ is closed under X_n^ϵ, we know by Lemma 2.2.3 that $\mathcal{U}_{\frac{\epsilon}{2}}(\Lambda) \subseteq \Lambda_\epsilon$. We let I_0^ϵ be the component of Λ_ϵ containing 0, and let $I_i^\epsilon = X_i^\epsilon(I_0^\epsilon)$, for $i \geq 1$. Since Λ_ϵ consists of finitely many closed intervals, there must be an integer $l \geq 1$ such that $I_0^\epsilon, \cdots, I_{l-1}^\epsilon$ are mutually disjoint, $X_l^\epsilon(I_0^\epsilon) = I_0^\epsilon$, and

$$\Lambda_\epsilon = \cup_{i=0}^{l-1} I_i^\epsilon.$$

Furthermore it is clear that 0 is in the interior of Λ_ϵ, and that for m sufficiently large, $\Delta_0^{(m)} \subseteq I_0^\epsilon$.

Let n_ϵ be the smallest integer such that either $\alpha^{-n_\epsilon} \in I_0^\epsilon$ or $-\alpha^{-n_\epsilon} \in I_0^\epsilon$. Recall that $\Delta_0^{(n_\epsilon)} = [-\alpha^{-n_\epsilon}, \alpha^{-n_\epsilon}]$, and note that for $i < 2^{n_\epsilon}$, since g^i maps $[-\alpha^{-n_\epsilon}, 0]$ or $[0, \alpha^{-n_\epsilon}]$ onto $\Delta_i^{(n_\epsilon)}$, it follows from our definition of n_ϵ that $\Delta_i^{(n_\epsilon)}$ is contained in $X_i^\epsilon(I_0^\epsilon)$, which is equal to some I_k^ϵ.

<u>*Claim.*</u> For $j = 1, 2, \cdots, 2^{n_\epsilon} - 1$,

$$\Delta_j^{(n_\epsilon)} \cap I_0^\epsilon = \emptyset.$$

Suppose this was not the case. Then by the connectedness of I_0^ϵ and our observation about that every $\Delta_i^{(n_\epsilon)}$ is contained in some I_k^ϵ, I_0^ϵ must contain one of the two $\Delta^{(n_\epsilon)}$-intervals adjacent to $\Delta_0^{(n_\epsilon)}$. If $\Delta_{2^{n_\epsilon}-1}^{(n_\epsilon)} \subset I_0^\epsilon$, then $[-\alpha^{-(n_\epsilon-1)}, 0] \subset I_0^\epsilon$. If I_0^ϵ contains the $\Delta_j^{(n_\epsilon)}$, for some j other than 0 and $2^{n_\epsilon} - 1$, on the other side, then $[0, \alpha^{-(n_\epsilon-1)}] \subset I_0^\epsilon$. Both scenarios contradict our choice of Λ_ϵ.

(Please see the figure *Figure 2.1*).

Next we claim that for $i < j < 2^{n_\epsilon}$, $\Delta_i^{(n_\epsilon)}$ and $\Delta_j^{(n_\epsilon)}$ must lie in different components of Λ_ϵ. If they were in the same I_k^ϵ, then we would have $g^{l-k}(\Delta_i^{(n_\epsilon)}), g^{l-k}(\Delta_j^{(n_\epsilon)})$ $\subseteq I_0^\epsilon$, which contradicts the claim above.

We have argued that each component of Λ_ϵ contains exactly one $\Delta_i^{(n_\epsilon)}$, which proves our assertion with $2^{n_\epsilon} = l$. ∎

§2.3. Geometry of the Noisy Attractor Λ_ϵ as ϵ Varies: a Qualitative Analysis

We begin this section with an easy proposition.

Proposition 2.3.1.(Monotonicity) *If* $\epsilon_2 > \epsilon_1 > 0$, *then*

$$\Lambda_{\epsilon_1} \subseteq \Lambda_{\epsilon_2}.$$

Proof. This follows from Lemma 2.2.6(ii). ∎

Proposition 2..3.1 together with Theorem 2.2.7 in the previous section show that there is an integer-valued function $\epsilon \mapsto n_\epsilon$, with n_ϵ increasing as ϵ decreases, such that Λ_ϵ is the disjoint union of exactly 2^{n_ϵ} intervals. In this section, we study the geometrical mechanism leading to the "jumps" in n_ϵ as ϵ decreases. We will show that for every $n \in \mathbf{Z}^+$, there is an interval with positive Lebesgue measure such that for every ϵ in this interval, $n_\epsilon = n$.

For all the discussion below, we assume that the integer n considered is an even integer unless otherwise stated. The analysis for n odd is the same.

We now fix n and consider the image of $\Delta_0^{(n)} = [-\alpha^{-n}, \alpha^{-n}]$ under the random process $X_m^\epsilon, m = 1, 2, \cdots$, for ϵ small. Note that g folds $\Delta_0^{(n)}$, and $g^i|_{g\Delta_0^{(n)}}$ is a homeomorphism for $i = 1, \cdots, 2^n - 1$. We set

$$\Delta_{0,\epsilon}^{(n)} = [A_0^\epsilon, B_0^\epsilon] = [0, \alpha^{-n}]$$

and for $k > 0$,

$$\Delta_{k,\epsilon}^{(n)} = [A_k^\epsilon, B_k^\epsilon] = X_{k\cdot 2^n}^\epsilon[0, \alpha^{-n}].$$

Also set

$$\Delta_\epsilon^{(n)} = \bigcup_{k\geq 0} \Delta_{k,\epsilon}^{(n)}.$$

The following fact is simple but useful.

Lemma 2.3.2. *For fixed n and ϵ,*

$$\Delta_{k,\epsilon}^{(n)} \subseteq \Delta_{k+1,\epsilon}^{(n)}, \quad for\ k \geq 0.$$

Proof. Since $g^{2^n}[0, \alpha^{-n}] = [-\alpha^{-(n+1)}, \alpha^{-n}]$, it is clear that $\Delta_{0,\epsilon}^{(n)} \subseteq \Delta_{1,\epsilon}^{(n)}$, and hence $X_{2^n}^{\epsilon}\Delta_{0,\epsilon}^{(n)} \subseteq X_{2^n}^{\epsilon}\Delta_{1,\epsilon}^{(n)}$. The rest follows by induction. ∎

Recall that one of the fundamental properties of g is that for each $n \geq 1$ and $i = 0, 1, \cdots, 2^n - 1$, $\Delta_i^{(n)}$ contains exactly one point from the 2^n-periodic orbit, denoted by $\{x_i^{(n)};\ i = 0, 1, \cdots, 2^n - 1\}$. We refer this orbit as $Orb(x_0^{(n)})$, and assume that these points are labeled in such a way that

$$x_0^{(n)} = g^{2^n}(x_0^{(n)}) \in \Delta_0^{(n)},$$

and for $i = 1, 2, \cdots, 2^n - 1$,

$$x_i^{(n)} = g^i(x_0^{(n)}) \in \Delta_i^{(n)}.$$

Let $-x_0^{(n)}$ be the point symmetric to $x_0^{(n)}$ by the origin.

For n even, the geometry near the origin 0 is as follows:

(Please see the figure *Figure 2.2*).

Note that for any $x_0 \in [-1, 1]$, $X_i^{\epsilon}\{x_0\}$ is an interval of length $\geq 2\epsilon$. In the analysis to follow, we will be speaking of the left end point and the right end point of $X_i^{\epsilon}\{x_0\}$.

Lemma 2.3.3. *For $\epsilon > 0$ small enough,*

$$\Delta_{\epsilon}^{(n)} = [A_2^{\epsilon}, B_2^{\epsilon}],$$

and $\pm x_0^{(n-1)} \notin \Delta_{\epsilon}^{(n)}$.

Proof. We only need to prove that for sufficiently small ϵ, $X_{2^n}^{\epsilon}[A_2^{\epsilon}, B_2^{\epsilon}] = [A_2^{\epsilon}, B_2^{\epsilon}]$ and $\pm x_0^{(n-1)} \notin [A_2^{\epsilon}, B_2^{\epsilon}]$.

Observe that for fixed $k \geq 1$,

$$\lim_{\epsilon \to 0} \Delta_{k,\epsilon}^{(n)} = [-\alpha^{-(n+1)}, \alpha^{-n}].$$

We can therefore choose $\epsilon > 0$ small enough that the following hold:

(1) $X_k^\epsilon(\Delta_0^{(n)}) \cap \{0\} = \emptyset$, *for* $0 < k < 2 \cdot 2^n$, $k \neq 2^n$;

(2) $|A_1^\epsilon| \leq B_1^\epsilon$, *and* $|A_2^\epsilon| \leq B_2^\epsilon$.

Since $X_k^\epsilon(\Delta_0^{(n)}) \cap \{0\} = \emptyset$ for $0 < k < 2^n$, it follows from the geometry of g that B_1^ϵ is the right end point of $X_{2^n}^\epsilon\{0\}$, and A_1^ϵ is the left end point of $X_{2^n}^\epsilon\{\alpha^{-n}\}$.

(Please refer to the figure *Figure 2.3*).

Furthermore, since $|A_1^\epsilon| \leq |B_1^\epsilon|$, the same reasoning tells us that $B_2^\epsilon = B_1^\epsilon$, and A_2^ϵ is the left end point of $X_{2^n}^\epsilon\{B_1^\epsilon\}$. From this it follows that $\Delta_{k,\epsilon}^{(n)} = \Delta_{2,\epsilon}^{(n)}$ $\forall k \geq 2$. One could easily arrange to have $\pm x_0^{(n-1)} \notin \Delta_{2,\epsilon}^{(n)}$ for ϵ small. ∎

For $\delta > 0$ small, we define the δ-neighborhood of $\pm x_0^{(n-1)}$ as follows,

$$V_n(\delta) = (x_0^{(n-1)} - \delta, x_0^{(n-1)} + \delta) \cup (-x_0^{(n-1)} - \delta, -x_0^{(n-1)} + \delta).$$

We now fix an $\tau_n > 0$ such that for $\epsilon < \tau_n$, Lemma 2.3.3 holds.

Lemma 2.3.4. *There is a $\delta_n > 0$ depending on τ_n such that for all $\epsilon \geq \tau_n$, if $V_n(\delta_n) \cap [A_k^\epsilon, B_k^\epsilon] \neq \emptyset$ for some $k \geq 1$, then*

$$x_0^{(n-1)} \in [A_{k+1}^\epsilon, B_{k+1}^\epsilon].$$

Proof. Let $\delta_n > 0$ be such that for $y \in V_n(\delta_n)$,

$$|g^{2^n}(y) - g^{2^n}(\pm x_0^{(n-1)})| = |g^{2^n}(y) - x_0^{(n-1)}| < \tau_n.$$

Our claim follows since

$$[A_{k+1}^\epsilon, B_{k+1}^\epsilon] \supset \mathcal{U}_\epsilon(g^{2^n}[A_k^\epsilon, B_k^\epsilon]).$$

Lemma 2.3.6. *For $\epsilon > \epsilon_n$, Λ_ϵ consists of at most 2^{n-1} disjoint components.*

Proof. Let $\epsilon > \epsilon_n$ be arbitrarily close to ϵ_n. We may assume that

$$B_2^\epsilon < |x_0^{(n-1)}| \text{ and } A_2^\epsilon > x_0^{(n-1)}$$

because these inequalities hold for ϵ_n. It is clear that $|A_2^\epsilon| > B_2^\epsilon$, otherwise $[A_2^\epsilon, B_2^\epsilon]$ will be invariant under $X_{k\cdot 2^n}^\epsilon$, and hence $x_0^{(n-1)} \notin [A_2^\epsilon, B_2^\epsilon] = \Delta_\epsilon^{(n)}$, which contradicts the choice of $\epsilon > \epsilon_n$. Moreover, our choice of ϵ implies that there is a k_0 such that

$$A_{k_0}^\epsilon > x_0^{(n-1)}, \text{ but } A_{k_0+1}^\epsilon \leq x_0^{(n-1)}.$$

Again let I_0^ϵ be the component of Λ_ϵ containing 0. We first show that

$$x_0^{(n-1)} \in I_{j\cdot 2^n}^\epsilon, \text{ for some } j.$$

<u>Case 1</u>. $|A_1^\epsilon| \geq B_1^\epsilon$.

It is easy to verify inductively that for $j = 1, \cdots, k_0$, A_j^ϵ is reached by an ϵ-pseudo orbit of length $j \cdot 2^n$ starting at α^{-n}. An argument similar to that in Lemma 2.3.4 then tells us that $x_0^{(n-1)}$ can be reached by an ϵ-pseudo orbit of length $(k_0+1)\cdot 2^n$. Since $g^{2^n}(0) = \alpha^{-n}$, we obtain that

$$x_0^{(n-1)} \in X_{(k_0+2)\cdot 2^n}^\epsilon(\{0\}) \subseteq I_{(k_0+2)\cdot 2^n}^\epsilon.$$

<u>Case 2</u>. $|A_1^\epsilon| < B_1^\epsilon$.

In this case, A_2^ϵ is reached by an ϵ-pseudo orbit starting at B_1^ϵ, which in turn is reached by an ϵ-pseudo orbit starting at 0. The rest is the same as in the Case 1.

Having shown that $x_0^{(n-1)} \in I_{j\cdot 2^n}^\epsilon$ for some j, we note that $g^{2^{n-1}}(x_0^{(n-1)}) = x_0^{(n-1)}$, which implies that

$$I_{j\cdot 2^n}^\epsilon \cap I_{j\cdot 2^n + 2^{n-1}} \neq \emptyset.$$

This proves that the number of components of Λ_ϵ must be $\leq 2^{n-1}$. ∎

Lemma 2.3.7. $\epsilon_n > \epsilon_{n+1}$.

Proof. Clearly, Lemmas 2.3.5 and 2.3.6 imply that $\epsilon_n \geq \epsilon_{n+1}$, and hence $X^{\epsilon_{n+1}}_{2^{n+1}}\{0\} \subseteq X^{\epsilon_n}_{2^{n+1}}\{0\}$. We next argue that there is a $j \geq 1$ such that

$$X^{\epsilon_{n+1}}_{j \cdot 2^{n+1}}\{0\} \neq X^{\epsilon_n}_{j \cdot 2^{n+1}}\{0\},$$

which is possible only if $\epsilon_n > \epsilon_{n+1}$.

By Lemma 2.3.5, $x_0^{(n)} \notin \Delta^{(n)}_{\epsilon_{n+1}}$. This implies that $X^{\epsilon_{n+1}}_{2^{n+1}}\{0\}$ is contained in the interval bounded by $\pm x_0^{(n-1)}$.

On the other hand, arguing as in the last lemma, we see that $B_2^{\epsilon_n}$ can be reached by an ϵ_n-pseudo orbit of length $2 \cdot 2^n = 2^{n+1}$ starting at 0 (case 2 in Lemma 3.6), or $A_3^{\epsilon_n}$ is reached by a pseudo orbit of length $4 \cdot 2^n = 2 \cdot 2^{n+1}$ starting at 0 (case 1 in Lemma 3.6). Since $|A_3^{\epsilon_n}| \geq |A_2^{\epsilon_n}| \geq B_2^{\epsilon_n} > \alpha^{-n} > |x_0^{(n-1)}|$, we have proved our claim. ∎

We are now ready to summarize the above propositions as the following result which forms the first half of the main theorem.

Theorem 2.3.8. *There exists a strictly decreasing sequence $\{\epsilon_n\}$, with $\epsilon_n \to 0$ as $n \to +\infty$, such that for any $\epsilon \in (\epsilon_{n+1}, \epsilon_n]$, the noisy attractor Λ_ϵ consists of exactly 2^n disjoint intervals. Moreover, these intervals are cyclically permuted by the process X^ϵ.*

§2.4. Scaling of Λ_ϵ with respect to ϵ

In practice it is useful to have some estimates on the the rate of convergence of ϵ_n. We first introduce the following notations.

Let i_n be the smallest integer such that $|A^\epsilon_{i_n}| \geq \alpha^{-n}$. From our previous discussion we know that i_n is either 1 or 2. Let

$$D_n = |A^{\epsilon_n}_{i_n}| - \alpha^{-(n+1)}.$$

(Please see the figure *Figure 2.4*).)

For $i = 0, 1, \cdots, 2^n - 1$, we let $Q^{(n)}_i$ denote the gap inside $\Delta^{(n)}_i$ between $\Delta^{n+1)}_i$ and $\Delta^{n+1)}_{i+2^n}$. (Note that the picture looks slightly different for $i = 0$.)

Lemma 2.4.1. *We have the following estimate:*

$$D_n = \epsilon_n \cdot (1 + \sum_{k=1}^{2^n-1} \prod_{j=k}^{2^n-1} |g'(y_j)|), \tag{2.4.1}$$

where y_j and $y_{j+2^{n-1}} \in Q^{(n-1)}_j$, for $j = 1, \cdots, 2^{n-1} - 1$, $y_{2^{n-1}} \in Q^{(n-1)}_0$.

Proof. We first consider the case $i_n = 1$.

¿From the last section, we know that $\{X^{\epsilon_n}_i(\Delta^{(n)}_0),\ \ i = 0, 1, \cdots, 2^n - 1\}$ are pairwise disjoint intervals with $X^{\epsilon_n}_i(\Delta^{(n)}_0) \supset g^i(\Delta^{(n)}_0)$. We want to estimate how much bigger $X^{\epsilon_n}_i(\Delta^{(n)}_0)$ is than $g^i(\Delta^{(n)}_0)$.

It is clear from the geometry of g that $g^{2^n}(\alpha^{-n}) = -\alpha^{-(n+1)}$ and that of $g^i(\alpha^{-n})$ is one of the end points of $g^i(\Delta^{(n)}_0)$. It is also clear that there exists an ϵ_n-pseudo orbit $u^{\epsilon_n}_0, \cdots, u^{\epsilon_n}_{2^n}$ such that $u^{\epsilon_n}_0 = \alpha^{-n}$, $u^{\epsilon_n}_{2^n} = A^{\epsilon_n}_1$, and $u^{\epsilon_n}_i$ is one of the end point of $X^{\epsilon_n}_i(\Delta^{(n)}_0)$ for $0 \leq i \leq 2^n$. Moreover, it is easy to argue that the interval between $u^{\epsilon_n}_i$ and $g^i(\alpha^{-n})$ lies in the gap $Q^{(n-1)}_j$ for $i = j$ or $j + 2^n - 1$ with $Q^{(n-1)}_0 = Q^{(n-1)}_{2^n-1}$.

We estimate $|u^{\epsilon_n}_i - g^i(\alpha^{-n})|$ as follows. For $i = 1$, $|u^{\epsilon_n}_1 - g^i(\alpha^{-n})| = \epsilon_n$. We assume that

$$|u^{\epsilon_n}_i - g^i(\alpha^{-n})| = \epsilon_n \cdot (1 + \sum_{k=1}^{i-1} \prod_{j=k}^{i-1} |g'(y_j)|)$$

where $y_j,\ j \le i$, are as in the statement of the lemma. Since

$$|u_{i+1}^{\epsilon_n} - g^{i+1}(\alpha^{-n})| = |g'(y_{i+1})||u_i^{\epsilon_n} - g^i(\alpha^{-n})| + \epsilon_n$$

where $y_{i+1} \in Q_i^{(n-1)}$, it follows that

$$|u_{i+1}^{\epsilon_n} - g^{i+1}(\alpha^{-n})| = \epsilon_n \cdot (1 + \sum_{k=1}^{i} \prod_{j=k}^{i} |g'(y_j)|).$$

The assertion in this lemma is the $2^{n^{th}}$-step of this induction.

Next we consider the case $i_n = 2$. Note that $B_1^{\epsilon_n}$ lies between α^{-n} and $-x_0^{(n-1)}$. Hence it is clear that $g^i(B_1^{\epsilon_n})$ must lie in the same gap as $g^i(\alpha^{-n})$. Reasoning as in the first case, there exists an ϵ_n-pseudo orbit $v_0^{\epsilon_n}, \cdots, v_{2^n}^{\epsilon_n}$ such that $v_0^{\epsilon_n} = B_1^{\epsilon_n}$ and $v_{2^n}^{\epsilon_n} = A_2^{\epsilon_n}$, and $v_i^{\epsilon_n}$ is one of the end point of $X_{i+2^n}^{\epsilon_n}(\Delta_0^{(n)})$ for $0 \le i \le 2^n$. Also the interval between $v_i^{\epsilon_n}$ and $g^i(B_1^{\epsilon_n})$ lies in the same gap as $g^i(\alpha^{-n})$ does. So the rest is the same as in the case $i_n = 1$. ∎

We will need the following important fact about the Feigenbaum attractor Λ.

Lemma 2.4.2.[G],[S] *The attractor Λ of g has bounded geometry, i.e. there is a constant $c > 0$ independent of n such that for all k, $0 \le k \le 2^n - 1$,*

$$c^{-1} < \frac{|\Delta_k^{(n+1)}|}{|\Delta_k^{(n)}|} < c, \quad c^{-1} < \frac{|\Delta_{k+2^n}^{(n+1)}|}{|\Delta_k^{(n)}|} < c \ \ and \ \ c^{-1} < \frac{|Q_k^{(n)}|}{|\Delta_k^{(n)}|} < c. \tag{2.4.2}$$

Using the bounded geometry of Λ, we obtain the following estimate, which is also used in **[VSK]**.

Lemma 2.4.3. *There exists a universal constant R such that*

$$\sup_n \max_{x_i \in \Delta_i^{(n)}} \sum_{i=1}^{2^n-1} \frac{|\Delta_i^{(n)}|}{|g'(x_i)|} = R < +\infty. \tag{2.4.3}$$

Proof. Note that for any $i = 1, \dots, 2^n - 1$,

$$\Delta_i^{(n)} = \Delta_i^{(n+1)} \cup Q_i^{(n)} \cup \Delta_{i+2^n}^{(n+1)}.$$

So

$$\max_{x\in\Delta_i^{(n)}} \frac{|\Delta_i^{(n)}|}{|g'(x)|} = \max_{x\in\Delta_i^{(n)}} \frac{|\Delta_i^{(n+1)}| + |Q_i^{(n)}| + |\Delta_{i+2^n}^{(n+1)}|}{|g'(x)|}$$

$$\geq \max_{x\in\Delta_i^{(n+1)}} \frac{|\Delta_i^{(n+1)}|}{|g'(x)|} + \max_{x\in\Delta_i^{(n)}} \frac{|Q_i^{(n)}|}{|g'(x)|} + \max_{x\in\Delta_{i+2^n}^{(n+1)}} \frac{|\Delta_{i+2^n}^{(n+1)}|}{|g'(x)|}.$$

By lemma 2.4.2, we have

$$\max_{x\in\Delta_i^{(n)}} \frac{|Q_i^{(n)}|}{|g'(x)|} \geq c^{-1} \max_{x\in\Delta_i^{(n)}} \frac{|\Delta_i^{(n)}|}{|g'(x)|}.$$

This gives

$$(1 - c^{-1}) \cdot \max_{x\in\Delta_i^{(n)}} \frac{|\Delta_i^{(n)}|}{|g'(x)|} \geq \max_{x\in\Delta_i^{(n+1)}} \frac{|\Delta_i^{(n+1)}|}{|g'(x)|} + \max_{x\in\Delta_{i+2^n}^{(n+1)}} \frac{|\Delta_{i+2^n}^{(n+1)}|}{|g'(x)|}$$

Summing over i, we obtain

$$(1 - c^{-1}) \cdot \sum_{i=1}^{2^n-1} \max_{x\in\Delta_i^{(n)}} \frac{|\Delta_i^{(n)}|}{|g'(x)|}$$

$$\geq \sum_{i=1}^{2^{n+1}-1} \max_{x\in\Delta_i^{(n+1)}} \frac{|\Delta_i^{(n+1)}|}{|g'(x)|} - \max_{x\in\Delta_{2^n}^{(n+1)}} \frac{|\Delta_{2^n}^{(n+1)}|}{|g'(x)|}. \tag{2.4.4}$$

Moreover, by the fundamental properties of g discussed in the first section,

$$|\Delta_{2^n}^{(n+1)}| \leq |\Delta_0^{(n+1)}| = 2\alpha^{-(n+1)},$$

and

$$\min_{x\in\Delta_{2^n}^{(n+1)}} |g'(x)| \geq |g'(\alpha^{-(n+1)})| = const.\alpha^{-(n+1)}.$$

Therefore,

$$\max_{x\in\Delta_{2^n}^{(n+1)}} \frac{|\Delta_{2^n}^{(n+1)}|}{|g'(x)|} \leq \frac{2\alpha^{-(n+1)}}{const.\alpha^{-(n+1)}} = const. = k_0,$$

where $k_0 = k_0(g)$ is independent of n. (Set $k_0 \geq 1$). Let

$$l = 1 - c^{-1}.$$

Then (2.4.4) is reduced to

$$l \cdot \sum_{i=1}^{2^n-1} \max_{x\in\Delta_i^{(n)}} \frac{|\Delta_i^{(n)}|}{|g'(x)|} + k_0 \geq \sum_{i=1}^{2^{n+1}-1} \max_{x\in\Delta_i^{(n+1)}} \frac{|\Delta_i^{(n+1)}|}{|g'(x)|}$$

for all $n \geq 1$. Inductively we obtain

$$\sum_{i=1}^{2^n-1} \max_{x\in\Delta_i^{(n)}} \frac{|\Delta_i^{(n)}|}{|g'(x)|} \leq k_0(1 + l + \ldots + l^{n-2}) + l^{n-1} \leq \frac{k_0}{1-l} = ck_0 = R < \infty.$$

Lemma 2.4.3 is proved. ∎

Lemma 2.4.4.(Distortion lemma) *Given $n \geq 1$ and any two sequences $\{x_i\}_{i=1}^{2^n-1}$ and $\{y_i\}_{i=1}^{2^n-1}$ such that $x_i \in \Delta_i^{(n)}$ and $y_i \in \Delta_i^{(n)}$, $i = 1, 2, \ldots, 2^n - 1$, we have*

$$D^{-1} \leq \frac{\prod_{i=1}^{2^n-1} |g'(x_i)|}{\prod_{i=1}^{2^n-1} |g'(y_i)|} \leq D \tag{2.4.5}$$

for some D independent of n.

Proof. By the mean value theorem, we have

$$\sum_{i=1}^{2^n-1} |\log|g'(x_i)| - \log|g'(y_i)||$$

$$= \sum_{i=1}^{2^n-1} \frac{|g''(\xi_i)|}{|g'(\xi_i)|}|x_i - y_i|$$

$$\leq M_2 \sum_{i=1}^{2^n-1} \frac{|\Delta_i^{(n)}|}{|g'(\xi_i)|} \leq M_2 R$$

where $M_2 = \max|g''(x)|$ and $\xi_i \in |\Delta_i^{(n)}|$, and the last inequality is from lemma 2.4.3 .

By taking $D = \exp(M_2 R)$, we finish the proof. ∎

Corallary 2.4.5. *There is a constant $C = C(g) > 0$ independent of n such that for the sequence $\{y_i\}_{i=1}^{2^n-1}$ obtained in the Lemma 2.4.1,*

$$\frac{\prod_{i=k}^{2^n-1} |g'(y_i)|}{\prod_{i=k}^{2^n-1} |g'(z_i)|} \leq C, \quad 1 < k < 2^n, \tag{2.4.6}$$

where $z_i = g^i(\alpha^{-n})$.

Proof. We note that for $i \neq 2^{n-1}$, y_i, $y_{i+2^{n-1}} \in Q_i^{(n-1)} \subset \Delta_i^{(n-1)}$. So for $2^{n-1} < k < 2^n - 1$, (2.4.6) is obtained by (2.4.5) with $C = D$.

For $k < 2^{n-1}$, we have

$$\frac{\prod_{i=k}^{2^n-1} |g'(y_i)|}{\prod_{i=k}^{2^n-1} |g'(z_i)|} = \frac{\prod_{i=k}^{2^{n-1}} |g'(y_i)|}{\prod_{i=k}^{2^{n-1}} |g'(z_i)|} \cdot \frac{\prod_{i=2^{n-1}+1}^{2^n-1} |g'(y_i)|}{\prod_{i=2^{n-1}+1}^{2^n-1} |g'(z_i)|} \cdot \frac{|g'(y_{2^{n-1}})|}{|g'(z_{2^{n-1}})|}.$$

Note that $0 < |y_{2^{n-1}}| < |z_{2^{n-1}}|$. Thus

$$\frac{|g'(y_{2^{n-1}})|}{|g'(z_{2^{n-1}})|} < 1.$$

Hence we can obtain the estimate (4.6) by choosing $C = D^2$. ∎

We now prove the second half of the main theorem.

Theorem 2.4.6 *There exist constants $\alpha_1 (> \alpha)$, E and E_1 such that*

$$E_1 \alpha_1^{-n} \leq \epsilon_n \leq E\alpha^{-n}.$$

Proof. It is obvious by Lemma 2.3.3 that

$$\epsilon_n \leq E\alpha^{-n}, \qquad \text{for some constant } E.$$

So we only need to estimate a lower bound for ϵ_n.

Let $A^{\epsilon}_{i_n}$ be as defined at the beginning of §2.4. Then

$$D_n = ||A^{\epsilon_n}_{i_n}| - \alpha^{-(n+1)}| \geq \alpha^{-n} - \alpha^{-(n+1)} = const \cdot \alpha^{-n}.$$

By Lemma 4.1,

$$D_n = \epsilon_n \cdot (1 + \sum_{k=1}^{2^n-1} \prod_{j=k}^{2^n-1} |g'(y_j)|) \geq const \cdot \alpha^{-n}.$$

Hence,

$$\epsilon_n > const. \frac{\alpha^{-n}}{1 + \sum_{k=1}^{2^n-1} \prod_{j=k}^{2^n-1} |g'(y_j)|}.$$

Now by Corollary 2.4.5, we have

$$1 + \sum_{k=1}^{2^n-1} \prod_{j=k}^{2^n-1} |g'(y_j)| \leq C(1 + \sum_{k=1}^{2^n-1} \prod_{j=k}^{2^n-1} |g'(z_j)|),$$

where $z_j = g^j(\alpha^{-n})$. Moreover by Holder inequality we have

$$1 + \sum_{k=1}^{2^n-1} \prod_{j=k}^{2^n-1} |g'(z_j)| \leq 2^{\frac{n}{2}} \cdot (1 + \sum_{k=1}^{2^n-1} \prod_{j=k}^{2^n-1} (g'(z_j))^2)^{\frac{1}{2}},$$

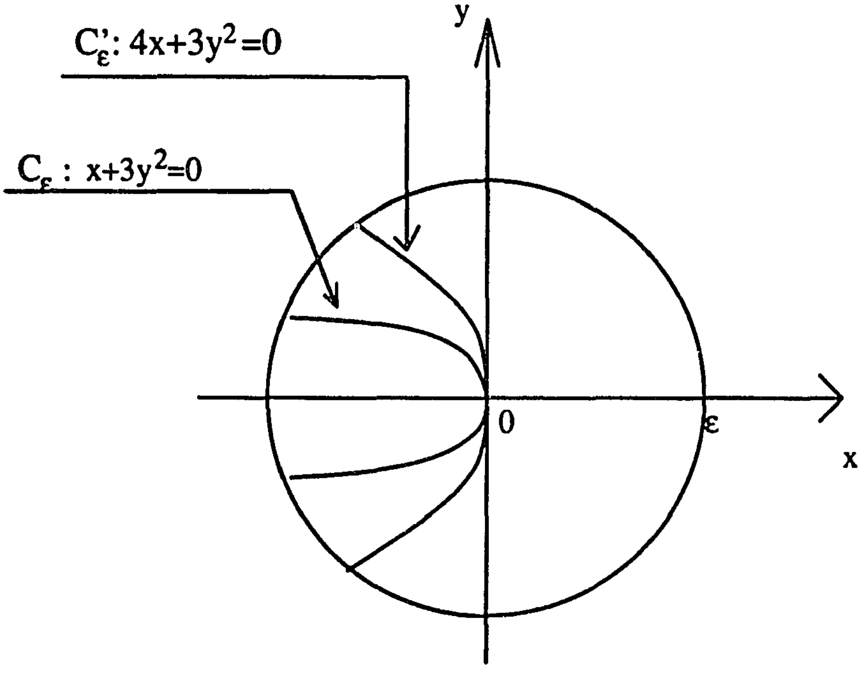
where we let $a_i = 1,\ i = 0, 1, \cdots, 2^n - 1,\ \ b_0 = 1$, and $b_j = \prod_{j=k}^{2^n-1} |g'(z_j)|,\ j = 1, \cdots, 2^n - 1$.

We now use an estimate from [VSK] (stated as Theorem 1.5 in the first section) to obtain

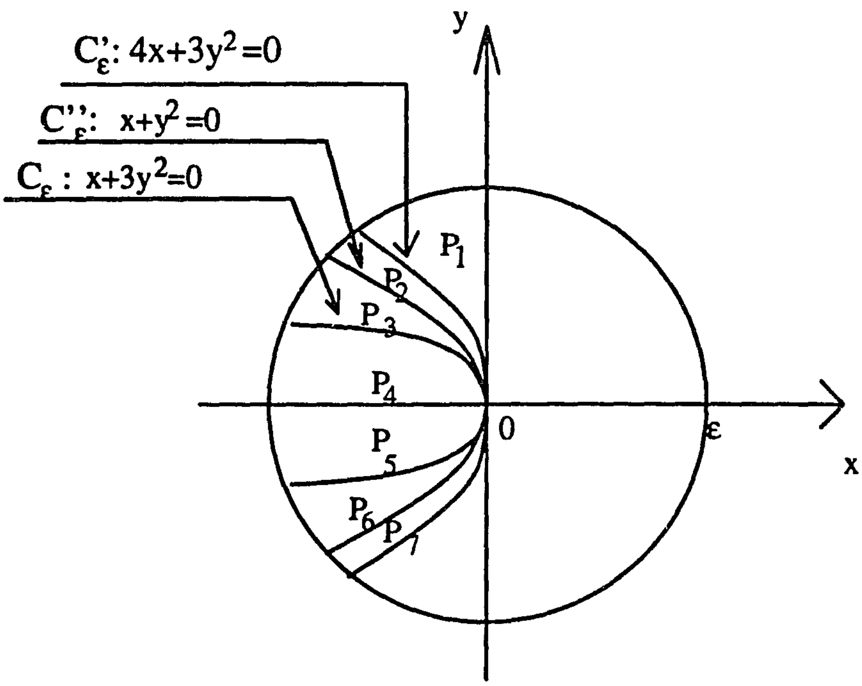
$$\overline{\lim}_{n\to\infty} \frac{1}{n}(1 + \sum_{k=1}^{2^n-1} \prod_{j=k}^{2^n-1} |g'(y_j)| \leq \sqrt{2}\lambda\sigma^{-1},$$

and so α_1 can be chosen to slightly bigger than $\sqrt{2}\lambda\sigma^{-1}\alpha$. We finish the proof of Theorem 2.4.6. ∎

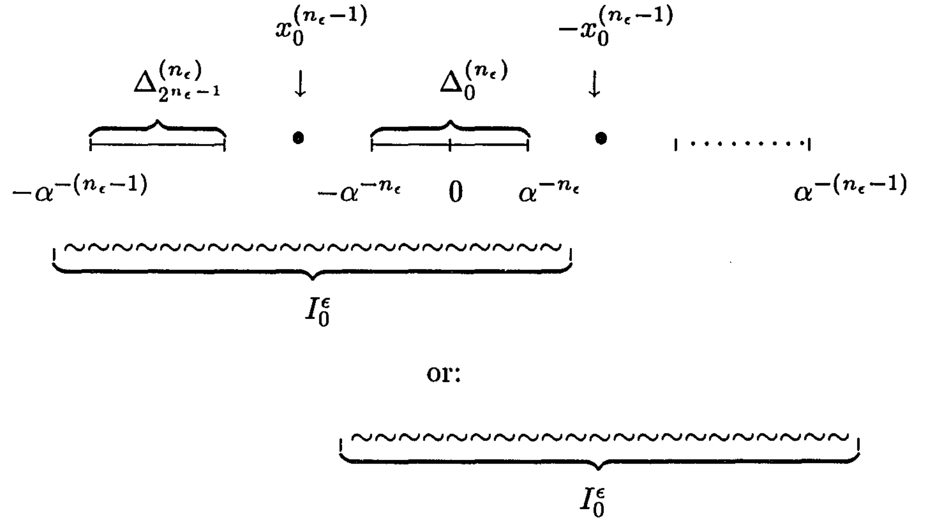
We remark that $\alpha = 2.50\cdots$, and $\sqrt{2}\lambda\sigma^{-1}\alpha$ is numerically estimated to be $\sim 80.55\cdots$.



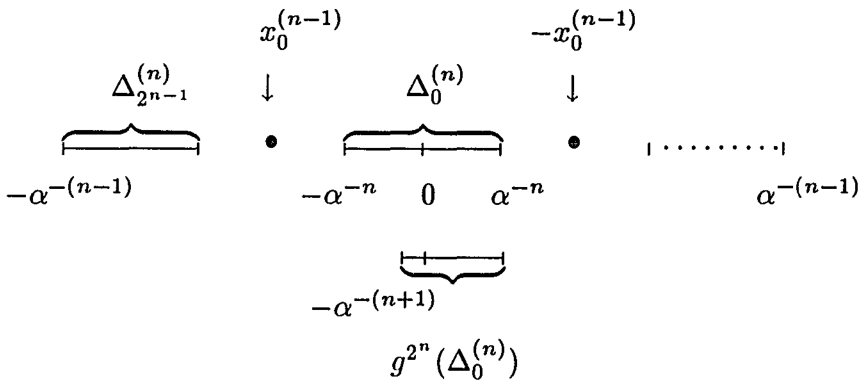
(Figure 1.1) Smooth curves generated by $f^{-1}(f(C_3))$



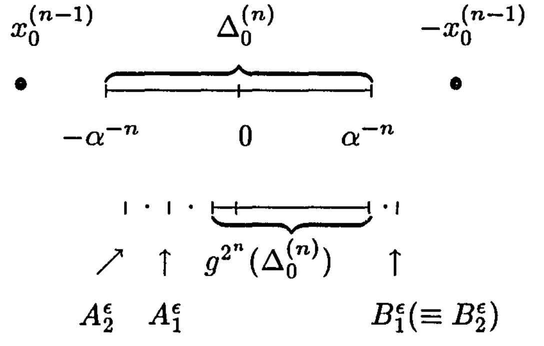
(Figure 1.2) The geometry near a cusp



(Figure 2.1) The relative position of the interval I_0^ϵ

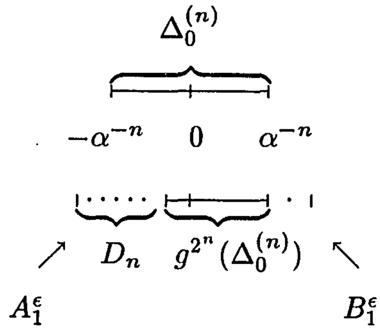


(Figure 2.2) The geometry of g near the origin

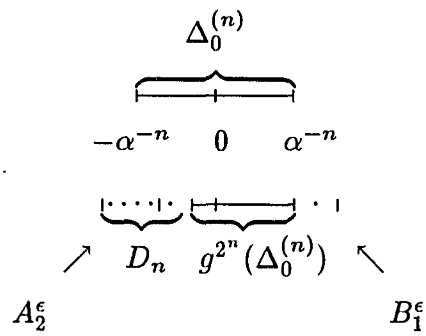


(Figure 2.3) The relative positions of A_i^ϵ and B_i^ϵ, $i = 1, 2$

Case 1: $i_n = 1$.



Case 2: $i_n = 2$.



(Figure 2.4) The length of D_n

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