QUANTUM TRANSPORT THEORY

by

GHI RYANG SHIN

A Dissertation Submitted to the Faculty of the

DEPARTMENT OF PHYSICS

In Partial Fulfillment of the Requirements
For the Degree of

DOCTOR OF PHILOSOPHY

In the Graduate College

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As members of the Final Examination Committee, we certify that we have read the dissertation prepared by SHIN, GHI RYANG

entitled QUANTUM TRANSPORT THEORY

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I hereby certify that I have read this dissertation prepared under my direction and recommend that it be accepted as fulfilling the dissertation requirement.

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SIGNED: Chi Ray Shi

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ABSTRACT

Within the framework of the quantum transport theory based on the Wigner transform of the density matrix I study first in non-relativistic and subsequently in relativistic formulation a number of applications. I also develop further the recently proposed relativistic theory: the classical limit is carefully derived and the integral equations of the relativistic Wigner function derived explicitly. I show how it is possible to obtain the Schwinger like particle production rate from relativistic quantum transport equations. Noteworthy numerical results address the shape of the relativistic Wigner function of a given quantum state. Other numerical studies are primarily oriented towards the time evolution of the Wigner function – I can presently solve only the nonrelativistic case in which there is no mixing between particle production and flow phenomena: I consider numerically the fate of the muon after muon catalyzed fusion.

CHAPTER 1

Wigner function[1]

Before I turn to the main topics of this thesis, I first present here the basic ideas regarding the classical and quantum phase space function (the Wigner distribution). I will survey here the time evolution equation for the Wigner function, which is a quantum transport equation. The integral form of a quantum transport equation will be also introduced.

1.1 Classical Phase Space

Let $\vec{r}$ be a position vector of a particle from a given origin [1]. The velocity of the particle is the time derivative,

$$\vec{v} = \frac{d\vec{r}}{dt}, \tag{1.1}$$

and the momentum of the particle is the product of velocity $\vec{v}$ with the particle mass m,

$$\vec{p} = m\vec{v}. \tag{1.2}$$

This particle evolves in time under the external potential V along the trajectory prescribed by Newton's equation of motion,

$$\frac{d\vec{p}}{dt} = \vec{F}, \tag{1.3}$$

where the force on the particle derived from the potential V is

$$\vec{F} = -\frac{dV}{d\vec{r}}. \tag{1.4}$$

To describe many body system or continuous media, it is convenient to generalize the Newtonian mechanics. Consider a system which consists of N identical particles of

[1]This chapter comprises introductory material as can be found e.g. in Ref.[8, 13]

mass m. We introduce generalized coordinates $(q_1, ..., q_{3N})$ and write down the Lagrangian L of the system,

$$L(q_i, \dot{q}_i) = T(q_i, \dot{q}_i) - V(q_i, \dot{q}_i), \tag{1.5}$$

where T and V are total kinetic and potential energy, respectively. $\dot{q}$ is the time derivative of coordinate q. In general, the potential V does not depend on time and velocity explicitly except for a dissipative system. The time evolution is then described by the set of Lagrange equations, obtainable from the least action principle,

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, \tag{1.6}$$

constrained by the initial and/or boundary conditions.

While the Lagrangian formulation can be applied to any system, there is yet another formulation to describe a physical system, namely, Hamiltonian formulation, which is especially useful for a system possessing symmetry. In this scheme, the basic variables are the generalized coordinates $q = (q_1, ..., q_{3N})$ and their momenta $p = (p_1, ..., p_{3N})$. The canonical momenta p are introduced by

$$p_i = \frac{\partial L}{\partial \dot{q}_i}. \tag{1.7}$$

The Hamiltonian H is the Legendre transform of L,

$$H(q,p,t) = \sum_i p_i \dot{q}_i - L(q, \dot{q}, t). \tag{1.8}$$

The system evolves in time according to the Hamilton equations of motion,

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \tag{1.9}$$

$$-\dot{p}_i = \frac{\partial H}{\partial q_i}, \tag{1.10}$$

which are $6N$ first order differential equations.

Consider now the $6N$ dimensional $(\vec{p}_i, \vec{q}_i)$-space which is made of $3N$ coordinates and $3N$ momenta. A system of N particles is completely described by a point in this space. The evolution of a system thus traces the trajectory prescribed by the set of Hamilton(or Lagrange) equations of motion. If there are conservation laws or constraints, the trajectory can be further restricted. For example, if the total energy of a system is conserved, the trajectory is on the $6N$ hyper–surface. This $6N$ dimensional $(\vec{p}_i, \vec{q}_i)$ space is the so-called phase space.

I recall the connection to statistical mechanics [2]. Consider a large number of classical system which have the same macroscopic properties. At any instance in time t, each can be represented as a point in the $6N$ phase space. Each region of the phase space can be assigned a certain weight according to the occupancy of discrete points found. In this way the distribution function $F(\vec{q}_i, \vec{p}_i, t)$ is introduced, which represents the (relative) weight at the phase point $(\vec{p}_i, \vec{q}_i)$ at time t. It is common to normalize the distribution function to 1,

$$\int d\Gamma\, F = 1, \tag{1.11}$$

where $d\Gamma = dq_1 ... dq_{3N} dp_1 ... dp_{3N}$ is a phase space volume element, so that the distribution F can be interpreted as a probability density to find the system at the point $(\vec{p}_i, \vec{q}_i)$ of the phase space at time t – note that by definition the function F also satisfies

$$F \geq 0. \tag{1.12}$$

The properties of a system can be obtained from the algebraic average;

$$\bar{B} = \int d\Gamma\, B(\vec{p}_i, \vec{q}_i) F(\vec{p}_i, \vec{q}_i, t), \tag{1.13}$$

where $B(\vec{p}_i, \vec{q}_i)$ is the microscopic function for the observable B.

There are several tacit postulates in this approach. First of all, we did not include here the internal variables to define the phase space. Secondly, we implicitly assumed that we can define (measure) the coordinate and momentum as precisely as we want. This is the major difference between the classical and quantum description and will be addressed just below. We also assumed the distribution function $F(\vec{q}_i, \vec{p}_i, t)$ comprises all physical informations the system can have, i.e. the state of a system is completely described given F at a certain time t.

The time evolution of a distribution function F can be described by

$$\frac{\partial F}{dt} = \{H, F\}_P, \tag{1.14}$$

where H is the microscopic Hamiltonian of the system under study. $\{H, F\}_P$ is the Poisson bracket,

$$\{H, F\}_P = \sum_{i=1}^{3N} \left(\frac{\partial H}{\partial q_i} \frac{\partial F}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial F}{\partial q_i} \right). \tag{1.15}$$

This is the well known Liouville equation and it follows from the Hamilton equation of motion. All of the classical transport equations can be derived from this equation. As an example, the transport equation of a one particle distribution function $f(\vec{p}, \vec{q}, t)$ can be obtained by integrating over allowed phase space all coordinates and momenta except the coordinates and momenta $(\vec{q}, \vec{p})$ of the particle considered. However, the resulting transport equation is not 'closed' in general. It is coupled to the higher order distributions, such as for example, 2 particle and/or 3 particle distribution functions and so on. Much of the effort in this field is devoted to finding suitable approximations which closes the system of transport equations.

1.2 Wigner Function

In quantum physics [3], we cannot measure a coordinate and its momentum simultaneously with an arbitrary precision because of the Heisenberg uncertainty principle,

$$\Delta x \Delta p \geq \hbar. \tag{1.16}$$

This causes a severe problem with the definition of the quantum distribution function. Nonetheless, it has been possible to define a function which has almost all the features expected of a phase space distribution function. Note first that the quantum expectation value of an observable $\hat{O}$ can be calculated by

$$< \hat{O} > = \mathrm{Tr} \hat{O} \hat{\rho}, \tag{1.17}$$

where Tr is the "trace" on any complete basis, $\hat{O}$ the quantum operator (not just number) of the observable and $\hat{\rho}$ the density matrix of the quantum system. The density matrix can be expressed in general as follows [4],

$$\hat{\rho} = \sum_i w_i |i\rangle\langle i|, \tag{1.18}$$

where the weighing factor has following properties:

1. $w_i \geq 0$,

2. $\sum_i w_i = 1$,

3. the set $\{|i>\}$ is a complete ortho-normal basis.

The corollary of the second property of the weighing factor is

$$\mathrm{Tr}\hat{\rho} \;=\; 1, \tag{1.19}$$

If all w_i but one of them are zero, the system is in a pure state. Otherwise, the system is in a mixed state.

Now I take the coordinate space $|\vec{x}>$ as the quantum basis so that the Eq.(1.17) becomes

$$\begin{aligned} <\hat{O}> &= \int d\vec{x} < \vec{x}|\hat{\rho}\hat{O}|\vec{x}> \\ &= \int d\vec{x} d\vec{p}\, W(\vec{x},\vec{p},t) O(\vec{x},\vec{p}), \end{aligned} \tag{1.20}$$

where

$$W(\vec{x},\vec{p}) = \int \frac{d\vec{y}}{(2\pi\hbar)^3} < \vec{x} + \frac{\vec{y}}{2}|\hat{\rho}|\vec{x} - \frac{\vec{y}}{2} > e^{-i\vec{p}\cdot\vec{y}/\hbar}, \tag{1.21}$$

$$O(\vec{x},\vec{p}) = \int d\vec{y} < \vec{x} + \frac{\vec{y}}{2}|\hat{O}|\vec{x} - \frac{\vec{y}}{2} > e^{-i\vec{p}\cdot\vec{y}/\hbar}. \tag{1.22}$$

This is so–called the Weyl transformation [5, 7]. In the derivation, I have used the relation

$$1 = \int d\vec{y} |\vec{y}>< \vec{y}|, \tag{1.23}$$

$$\delta(\vec{q}) = \frac{1}{(2\pi\hbar)^3} \int d\vec{p} e^{-i\vec{p}\cdot\vec{q}/\hbar}. \tag{1.24}$$

Note that not all quantum observables $\hat{O}$ can be transformed from an operator to a single scalar function by the Weyl transformation. Spin is a typical counter-example and its representation requires several functions. I shall address this issue further below.

Noticing close resemblance of the forms between the quantum expectation value Eq.(1.20) and the classical one Eq.(1.13), it is customary to call the *Wigner* function Eq.(1.21) also *the quantum distribution function.* I note that the function W was first introduced by Wigner [6] to study quantum corrections to classical statistical mechanics.

Ever since then, the Wigner function has found numerous applications in many fields of modern physics. I note further that the power of this approach to quantum mechanics is not limited to being only a calculational tool but it also provides fundamental insights into the relations between classical and quantum physics.

Let us consider some properties of the Wigner function of a pure state, i.e. we consider the Weyl transform of any quantum eigenstate $\psi_i(\vec{r}) = \langle \vec{x}|i\rangle$:

$$W_i(\vec{r},\vec{p}) \;=\; \int \frac{d\vec{s}}{(2\pi\hbar)^3}\, e^{-i\vec{p}\cdot\vec{s}/\hbar}\psi_i^*(\vec{r}-\vec{s}/2)\psi_i(\vec{r}+\vec{s}/2). \tag{1.25}$$

Note that with Eqs.(1.18,1.21) I have in general $W = \sum_i w_i W_i$. W_i has the following properties [8]:

1. $W_i(\vec{r},\vec{p},t)$ is a Hermitian, i.e. $W_i^\dagger(\vec{r},\vec{p},t) = W_i(\vec{r},\vec{p},t)$. Therefore, $W_i(\vec{r},\vec{p},t)$ is real.

2. The Wigner function satisfies the following relations

$$\int d\vec{p}\, W_i(\vec{r},\vec{p}) \;=\; |\psi_i(\vec{r})|^2, \tag{1.26}$$

$$\int d\vec{r}\, W_i(\vec{r},\vec{p}) \;=\; |\tilde{\psi}_i(\vec{p})|^2, \tag{1.27}$$

$$\int d\vec{r}\, d\vec{p}\, W_i(\vec{r},\vec{p}) \;=\; 1, \tag{1.28}$$

   where $\tilde{\psi}_i(\vec{p})$ is the wavefunction in momentum space.

3. The quantum expectation value of an observable $\hat{O}$ is

$$< i|\hat{O}|i >= \int d\vec{r}\, d\vec{p}\, W_i(\vec{r},\vec{p},t) O(\vec{r},\vec{p}), \tag{1.29}$$

   where $O(\vec{r},\vec{p})$ is the Weyl transform of the observable, self–adjoint operator $\hat{O}$.

4. For given wave functions $\psi_i(\vec{r})$ and $\psi_j(\vec{r})$, I have

$$|\int d\vec{r}\, \psi_i^*(\vec{r})\psi_j(\vec{r})|^2 \;=\; (2\pi\hbar)^3 \int d\vec{r} \int d\vec{p}\, W_i(\vec{r},\vec{p},t) W_j(\vec{r},\vec{p},t) \tag{1.30}$$

   For $i = j$ I obtain:

$$\int d\vec{r} \int d\vec{p}\, [W_i(\vec{r},\vec{p},t)]^2 \;=\; \frac{1}{(2\pi\hbar)^3}. \tag{1.31}$$

   Taking the wave functions ψ_i and ψ_j to be orthogonal to each other, I obtain

$$\int d\vec{r} \int d\vec{p}\, W_i(\vec{r},\vec{p},t) W_j(\vec{r},\vec{p},t) \;=\; 0. \tag{1.32}$$

Hence we see that in general $W_i(\vec{r},\vec{p})$ cannot be in everywhere positive, i.e. it must also assume negative values. For this reason one calls W_i a pseudo–probability density. However, there exists a simple way to make W_i positive definite by using the coarse graining (smearing) function [10, 11]:

$$G(\vec{r},\vec{p};\lambda_r,\lambda_p) = \frac{1}{(\pi\lambda_r\lambda_p)^3}e^{-\vec{p}^2/\lambda_p^2-\vec{r}^2/\lambda_r^2}. \tag{1.33}$$

The coarse grained distribution is

$$W_{s,i}(\vec{r},\vec{p},t;\lambda_r,\lambda_p) = \frac{1}{(\pi\lambda_r\lambda_p)^3}\int d\vec{r}'d\vec{p}'\; e^{-(\vec{p}-\vec{p}')^2/\lambda_p^2-(\vec{r}-\vec{r}')^2/\lambda_r^2}W_i(\vec{r}',\vec{p}',t). \tag{1.34}$$

I will return to discuss W_s below.

5. From the Cauchy–Schwartz inequality and the normalization condition of the wavefunction it further follows [9]:

$$\begin{aligned}|W_i(\vec{r},\vec{p},t)|^2 &\leq \int\frac{d\vec{s}}{(2\pi\hbar)^3}|\psi_i(\vec{r}+\vec{s}/2,t)|^2\int\frac{d\vec{s}}{(2\pi\hbar)^3}|\psi_i(\vec{r}-\vec{s}/2,t)|^2\\ &= \left(\frac{2}{2\pi\hbar}\right)^6.\end{aligned} \tag{1.35}$$

Therefore, I get

$$|W_i(\vec{r},\vec{p},t)| \leq \left(\frac{2}{2\pi\hbar}\right)^3. \tag{1.36}$$

This is another form of the uncertainty principle. Namely, since the probability to find the particle at point $(\vec{r},\vec{p})$ is less than $(\frac{2}{h})^3$, the phase volume needed to find the particle should be larger than $(\frac{h}{2})^3$.

Finally, I note that one can introduce a Wigner function in a gauge invariant way [12]. The idea is that a wave function can be written by using the translation operator as follows;

$$\psi(\vec{r}\pm\vec{s}/2) = e^{\pm\vec{s}\cdot\vec{\nabla}/2}\psi(\vec{r}), \tag{1.37}$$

and one then replaces the derivative $\vec{\nabla}$ by the covariant derivative $\vec{D}$ for a gauge transformation, i.e.

$$\vec{\nabla}\to\vec{D}=\vec{\nabla}-ie\vec{A}(\vec{r},t)/\hbar \tag{1.38}$$

for the electromagnetic fields. The final result for the Wigner function in electromagnetic fields is

$$W(\vec{r},\vec{p}) = \int\frac{d\vec{s}}{(2\pi\hbar)^3}\;\exp\left(-i\frac{\vec{s}}{\hbar}\cdot[\vec{p}+e\int_{-1/2}^{1/2}d\lambda\;\vec{A}(\vec{r}+\lambda\vec{s})]\right)\psi^*(\vec{r}-\vec{s}/2)\psi(\vec{r}+\vec{s}/2). \tag{1.39}$$

It is possible to generalize this definition further to allow for non-abelian gauge theory.

I note that there are many other definitions for a quantum distribution [14, 15] in addition to the one (Wigner) presented here. I will not consider these further. Also, I will mostly address pure state Wigner functions.

1.3 Quantum Transport Equation

The evolution equation of a Wigner function $W(\vec{r},\vec{p},t)$ follows directly from the Schrödinger equation,

$$i\hbar\frac{\partial}{\partial t}\psi(\vec{r},t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r})\right]\psi(\vec{r},t). \tag{1.40}$$

Differentiating the Wigner function Eq.(1.25) with respect to time gives

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}W(\vec{r},\vec{p},t) &= \int\frac{d^3s}{(2\pi\hbar)^3}\, e^{-i\vec{p}\cdot\vec{s}/\hbar}[-\frac{\hbar^2}{2m}\nabla^2\psi(\vec{r}+\vec{s}/2,t)\,\psi^*(\vec{r}-\vec{s}/2,t) \\ &+\frac{\hbar^2}{2m}\psi(\vec{r}+\vec{s}/2,t)\,\nabla^2\psi^*(\vec{r}-\vec{s}/2,t) \\ &+[V(\vec{r}+\vec{s}/2)-V(\vec{r}-\vec{s}/2)]\psi(\vec{r}+\vec{s}/2,t)\,\psi^*(\vec{r}-\vec{s}/2,t)]. \end{aligned} \tag{1.41}$$

First of all, I replace the derivative with respect to $\vec{r}$ by the derivative with respect to $\vec{s}$ and integrate by parts and then go from $\vec{s}$ back to $\vec{r}$–derivative to obtain

$$\begin{aligned} \left[\partial_t + \frac{1}{m}\vec{p}\cdot\nabla\right]W(\vec{r},\vec{p},t) &= \frac{1}{i\hbar}\int\frac{d^3s}{(2\pi\hbar)^3}\, e^{-i\vec{p}\cdot\vec{s}/\hbar}\,[V(\vec{r}+\vec{s}/2)-V(\vec{r}-\vec{s}/2)] \\ &\cdot\psi(\vec{r}+\vec{s}/2,t)\,\psi^*(\vec{r}-\vec{s}/2,t). \end{aligned} \tag{1.42}$$

Assuming the potential V can be expanded in Taylor series around point $\vec{r}$, I obtain finally,

$$\left[\partial_t + \frac{1}{m}\vec{p}\cdot\vec{\nabla}\right]W(\vec{r},\vec{p},t) = \sum_{k=0}^{\infty}\frac{(i\hbar)^{2k}}{2^{2k}(2k+1)!}(\vec{\nabla}'\cdot\vec{\partial})^{2k+1}V(\vec{r})W(\vec{r},\vec{p},t),$$

or, moving the classical force term $\vec{F} = -\vec{\nabla}V$ to the left hand side:

$$\left[\partial_t + \frac{1}{m}\vec{p}\cdot\vec{\nabla} + \vec{F}(\vec{r})\cdot\vec{\partial}\right]W(\vec{r},\vec{p},t) = \sum_{k=1}^{\infty}\frac{(-\hbar^2/4)^k}{(2k+1)!}(\vec{\nabla}'\cdot\vec{\partial})^{2k+1}V(\vec{r})W(\vec{r},\vec{p},t), \tag{1.43}$$

where $\vec{\nabla}'$ is the derivative with respect to coordinate $\vec{r}$ on the potential V very next to it and $\vec{\partial}$ the derivative with respect to momentum $\vec{p}$. This equation is the quantum transport equation for a one particle Wigner function.

In the classical limit $\hbar \to 0$, I have

$$\partial_t W(\vec{r},\vec{p},t) \;=\; -\left[\frac{1}{m}\vec{p}\cdot\vec{\nabla}+\vec{F}(\vec{r})\cdot\vec{\partial}\right] W(\vec{r},\vec{p},t)=\{H,W\}, \tag{1.44}$$

which is just a Liouville equation, as indicated. We will see that the classical limit ($\hbar \to 0$) of the *relativistic* quantum transport equation is actually a much more subtle matter, and I will study it in the context of the relativistic classical limit in more detail. Note further that in the classical limit the evolution of a Wigner function is determined by the solution of Hamilton's equations of motion, i.e.

$$W(\vec{r},\vec{p},t) \;=\; W(\vec{r}(\vec{r}_0,\vec{p}_0,t_0|t),\vec{p}(\vec{r}_0,\vec{p}_0,t_0|t)), \tag{1.45}$$

where $\vec{r}(\vec{r}_0,\vec{p}_0,t_0|t)$ and $\vec{p}(\vec{r}_0,\vec{p}_0,t_0|t)$ are the solution of Hamilton's equations Eq.(1.10) with initial condition $(\vec{r}_0,\vec{p}_0)$ at time t_0. This $(\vec{r}(t),\vec{p}(t))$ is the so-called Wigner trajectory. I will apply this method to study the muon after muon catalyzed $d-t$ fusion in the next chapter.

If the potential $V(\vec{r})$ is sufficiently smooth, the classical approximation and first few quantum corrections on the right hand side of Eq.(1.43) would suffice to effectively solve a given physical problem.

It is often useful to express the quantum transport equation in a integral equation [13]. To that end, I define the Green's function which satisfies the equation,

$$\left(\frac{\partial}{\partial t}+\frac{\vec{p}}{m}\cdot\vec{\nabla}\right)G(\vec{r},\vec{p},t) \;=\; \delta(\vec{r})\delta(t). \tag{1.46}$$

The integral form of Eq.(1.43) is then

$$W(\vec{r},\vec{p},t) \;=\; W_0(\vec{r},\vec{p},t)+\int d\vec{r}'dt'\; G(\vec{r}-\vec{r}',\vec{p},t-t')K(\vec{r}',\vec{p},t')W(\vec{r}',\vec{p},t'), \tag{1.47}$$

where $W_0(\vec{r},\vec{p},t)$ is the solution of a field free equation. The kernel K is defined by

$$K(\vec{r},\vec{p},t) \;=\; \sum_{k=0}^{\infty}\frac{(i\hbar)^{2k}}{2^{2k}(2k+1)!}(\vec{\nabla}'\cdot\vec{\partial})^{2k+1}V(\vec{r}). \tag{1.48}$$

The form of the Green's function is most easily obtained considering the integral equation Eq.(1.47) in the $(\vec{q},\vec{p},\omega)$ space: consider the Fourier transformations with respect to $\vec{r}$ and t

$$\tilde{f}(\vec{q},\vec{p},\omega) = \int d\vec{r}\, dt\, e^{-i\vec{q}\cdot\vec{r}/\hbar+i\omega t/\hbar}\, f(\vec{r},\vec{p},t). \tag{1.49}$$

The Green's function Eq.(1.46) becomes after this transformation,

$$\tilde{G}_{ret}(\vec{q},\vec{p},\omega) = \frac{i}{\omega + i\varepsilon - \vec{p}\cdot\vec{q}/m}, \tag{1.50}$$

where ε is an infinitesimal positive definite. Note that I explicitly incorporate the causality principle. I so obtain the integral equation in this representation,

$$\tilde{W}(\vec{q},\vec{p},\omega) = \tilde{W}_0(\vec{q},\vec{p},\omega) + \tilde{G}(\vec{q},\vec{p},\omega)\int \frac{d\vec{q}\,'d\omega'}{(2\pi\hbar)^4}\tilde{K}(\vec{q}-\vec{q}\,',\vec{p},\omega-\omega')\tilde{W}(\vec{q}\,',\vec{p},\omega'). \tag{1.51}$$

$\tilde{W}_0(\vec{q},\vec{p},\omega)$ is the solution of a homogeneous equation and the kernel $\tilde{K}(\vec{q},\vec{p},\omega)$ is the Fourier transform of $K(\vec{r},\vec{p},t)$,

$$\begin{aligned}\tilde{K}(\vec{q},\vec{p},\omega) &= \sum_{k=0}^{\infty} \frac{-i(\hbar)^{2k}}{2^{2k}(2k+1)!}(\vec{q}\cdot\vec{\partial})^{2k+1}\tilde{V}(\vec{q},\omega) \\ &= \frac{1}{i\hbar}\tilde{V}(\vec{q},\omega)\left(e^{+\frac{\hbar}{2}\vec{q}\cdot\vec{\partial}} - e^{-\frac{\hbar}{2}\vec{q}\cdot\vec{\partial}}\right).\end{aligned} \tag{1.52}$$

Since the exponential term on the right hand side is a translation operator, one obtains:

$$\begin{aligned}\tilde{W}(\vec{q},\vec{p},\omega) &= \tilde{W}_0(\vec{q},\vec{p},\omega) + \frac{1}{i\hbar}\tilde{G}(\vec{q},\vec{p},\omega)\int \frac{d\vec{q}\,'d\omega'}{(2\pi\hbar)^4}\,\tilde{V}(\vec{q}-\vec{q}\,',\omega-\omega') \\ &\cdot\left[\tilde{W}(\vec{q}\,',\vec{p}+\frac{\hbar}{2}(\vec{q}-\vec{q}\,'),\omega') - \tilde{W}(\vec{q}\,',\vec{p}-\frac{\hbar}{2}(\vec{q}-\vec{q}\,'),\omega')\right].\end{aligned} \tag{1.53}$$

If the potential $V(\vec{r},t)$ does not depend on time, the Fourier transform of the potential is

$$\tilde{V}(\vec{q},\omega) = \tilde{V}(\vec{q})(2\pi\hbar)\delta(\omega), \tag{1.54}$$

so that the integral equation simplifies to

$$\begin{aligned}\tilde{W}(\vec{q},\vec{p},\omega) &= \tilde{W}_0(\vec{q},\vec{p},\omega) + \frac{1}{i\hbar}\tilde{G}(\vec{q},\vec{p},\omega)\int \frac{d^3q\,'}{(2\pi\hbar)^3}\,\tilde{V}(\vec{q}-\vec{q}\,') \\ &\cdot\left[\tilde{W}(\vec{q}\,',\vec{p}+\frac{\hbar}{2}(\vec{q}-\vec{q}\,'),\omega) - \tilde{W}(\vec{q}\,',\vec{p}-\frac{\hbar}{2}(\vec{q}-\vec{q}\,'),\omega)\right].\end{aligned} \tag{1.55}$$

This integral form is often useful in quantum physics.

CHAPTER 2

Application of the Nonrelativistic Wigner Function - muon after dt-fusion[1]

In this chapter, I employ the theory introduced in Chapter 1 to the study of muons after muon catalyzed $d-t$ fusion. However, I shall begin with a brief review of muon catalyzed fusion in the first section. Next, I will compute and study the muon phase space distribution under the adiabatic approximation in Section 2.2 and obtain the energy spectrum of muons after fusion, as well as the initial sticking probability ω_s^0 in Section 2.3. I then will consider how the muon evolves and scatters elastically with background medium. I obtain the muon recapture and regeneration probability under this approximation in Section 2.4.

2.1 Muon Catalyzed Fusion[2]

Because of the large mass of a negative muon, $m_\mu \approx 207 m_e$, the Coulomb barrier between the hydrogen isotopes (d or t) of the muonic molecular ion $(dt\mu)^+$ decreases to the muonic Bohr radius, which is about 256fm. At this distance, hydrogen fusion ($(dt\mu)^+ \rightarrow \alpha + n + \mu$) can occur even at room temperature. This is so–called muon catalyzed fusion [17, 18, 19].

Ever since the muon catalyzed fusion was experimentally discovered [18], the number of fusions per muon has been of central importance to possible practical applications of MuCF(Muon Catalyzed Fusion). Since the Q-value of the fusion is 17.60 MeV and the optimistic minimal thermal energy cost to produce one muon is about 3 GeV [16], the process could be commercially beneficial if the number of fusions per muon is above 1000.

[1]Portions of this chapter have been published in [25]

[2]For further information, see the Ref.[20]

The achievable number of fusions per muon, Y, is obtained from the ratio of the fusion cycling rate λ_c to the muon loss rate λ_l,

$$Y = \frac{\lambda_c}{\lambda_l}. \tag{2.1}$$

A muon can be lost by natural decay(the mean life time of the muon is about $1/\lambda_0 = 2.2\mu s$) or by the probability per cycle ω_s of sticking to the fusion α-particle. Thus:

$$\lambda_l = \lambda_0 + \lambda_c \omega_s \tag{2.2}$$

and thus in general the number of fusions is limited by

$$Y = \frac{1}{\omega_s + \lambda_0/\lambda_c} < \frac{1}{\omega_s}. \tag{2.3}$$

The sticking probability per cycle comprises two effects: $\omega_s = \omega_s^0(1 - R)$. Here, ω_s^0 is the initial sticking probability and R incorporates all secondary effects, in particular muon regeneration and recapture.

Using Born-Oppenheimer three body wave function for the muonic molecular ion Jackson [19] found $\omega_s^0 \sim 1.2\,\%$. Full three-body non-adiabatic wave functions lead to a somewhat smaller value $\omega_s^0 = 0.89\%$. Incorporating the change of the muon fusion amplitude due to perturbative nuclear forces increase this value to $\omega_s^0 = 0.92\%$, which is today viewed as the best initial molecular value for the total initial sticking probability. The *dt* fusion sticking fraction results at LAMPF [21] were obtained by considering the cycling rate of muons and measuring the rate of muon loss, which is found to be somewhat greater than that generated alone by the muon decay. A somewhat different analysis of kinetics of the neutron emission and cycle dynamics was performed at the PSI [22], where the X-ray muo-atomic transitions from the MuCF cycle have also been measured [23]. Every experiment addressing *dt* fusion near liquid hydrogen density (LHD=4.25 10^{22}atoms/cc) has yielded sticking significantly below theoretical expectations, though the natural result should have been a greater than anticipated value (allowing for some residual and unidentified loss mechanisms). The very low sticking reported by LAMPF is about 0.35%, while PSI reports close to 0.45%; both results agree to within experimental errors. The conventional theoretical value including muon regeneration (inelastic collision striping) is at about 0.62%, which is about 1.5 times the experimental value. There is still no widely accepted explanation why the sticking in the $dt\mu$ fusion is smaller then theoretically predicted. This is at the heart of the interest in obtaining a comprehensive theoretical formulation of this problem in phase space. I wish in particular to study how the muon evolves under the influence of the fusion product, the α particle and the influence of the atoms of nearby matter.

I wish to explore the possibilities that free muons can be captured, and bound muons can be stripped from the fusion α particle due to the elastic or inelastic collision with nearby atoms. I note that since muons have substantially higher stopping power($\propto 1/m$) than the fusion product, especially muons found in front of the fusion α particle can be captured by the α particle. On the other hand, the initially bounded muon to the fusion α particle which has an initial velocity $v_{\alpha\mu} = 5.837\alpha c$ corresponding to the energy of 3.48 MeV, can be set free during the time the $(\alpha\mu)^+$ ion slows down in the matter. Evidently, the study of these processes requires the understanding of the transport processes of μ and $\alpha\mu$ in matter or/and in proximity of a moving α-particle.

With this in mind, I will put some efforts to understand in detail the evolution of the Wigner phase space distribution of the muon under the influence of nearby matter as well as the Coulomb field of the α-particle. I have not yet accounted for inelastic interactions with matter and the possible impact of external electro-magnetic fields. But I will show that even elastic interactions with surrounding matter impact significantly the fate of the muon, i.e. its probability to stick to the fusion product and thus establish the need for use of phase space in further studies of muon regeneration and sticking.

2.2 Muon Phase Space Distribution

In the adiabatic approximation the muon amplitude at the moment of the (dt) fusion is the hydrogen-like $(1s)$ state, i.e. $(H_e\mu)^+_{1s}$ state,

$$\Psi(\vec{r}, \vec{R} = 0) = \psi^{\mu}_{1s}(\vec{r}), \tag{2.4}$$

where $\vec{R}$ is the radius vector between d and t, $\vec{r}$ the radius vector from the center of mass of (dt) to the muon. The Wigner phase space distribution of the muon following fusion is then obtained according to Eq.(1.21),

$$f(\vec{r},\vec{p}) = (2\pi\hbar)^{-3} \int d\vec{x} \exp(-\frac{i\vec{p}\cdot\vec{x}}{\hbar})\psi^*(\vec{r}-\frac{\vec{x}}{2})\psi(\vec{r}+\frac{\vec{x}}{2}), \quad (2.5)$$

where $\vec{r}$ and $\vec{p}$ are the muon coordinate and momentum in a laboratory frame and by definition the fusion α-particle travels along the $\hat{z}$-axis. In this chapter I will employ muonic atomic units($\hbar = m_\mu = e = 1$) so that the atomic unit of velocity is $1a.u. = c/137$.

The integral in Eq.(2.5) cannot be evaluated in a closed analytical form. For further study, a suitable representation of the wave function is needed to carry out the integration. This representation must be valid for all sampled values of coordinate and momentum (not just one of these variables). I found this to be the case for a representation of the muon wave function given in terms of sufficiently long series of Gaussian germinals:

$$\begin{aligned} \psi_{1s}(\vec{r}) &= (\frac{Z^3}{\pi})^{1/2} \exp(-Zr) \\ &\simeq \sum_{i=1}^{N} a_i e^{-b_i r^2}, \end{aligned} \quad (2.6)$$

where $Z = 2$. The coefficients a_i and b_i follow in a straightforward fashion from the integral representation

$$e^{-2r} = \int_0^\infty dx \left(\frac{e^{-1/x}}{\sqrt{\pi}x^{\frac{3}{2}}}\right) e^{-xr^2} = \sum_i \Delta_i \frac{e^{-1/x_i}}{\sqrt{\pi}x_i^{\frac{3}{2}}} e^{-x_i r^2}. \quad (2.7)$$

The table 2.1 shows the coefficients which I use in this study. It seems that it would be

i	a_i	b_i
1	0.0269336	0.248628
2	0.1550654	0.552184
3	0.2913678	1.219208
4	0.3327314	2.842864
5	0.2824095	7.179696
6	0.2032226	19.660312
7	0.1363020	60.073376
8	0.0849006	218.792156
9	0.0477140	1016.070848
10	0.0242735	7107.102236

Table 2.1: Gaussian Expansion Coefficients of e^{-2r}.

easy to adjust the numerical coefficients arising from equation (2.6) to account for a more refined understanding of the muon fusion amplitude obtained in actual three body calculations [26]. I have verified that the Fourier transform of both the wave function and its approximation agree with each other.

Using this expansion, it is straightforward to obtain a simple expression for the Wigner function since all integrals can be carried out analytically [27]:

$$\begin{aligned} f(\vec{r},\vec{p}) &= \frac{1}{(2\pi)^{3/2}}\sum_{i=1}^{N}\frac{a_i^2}{b_i^{3/2}}\exp(-\frac{\vec{p}^2}{2b_i}-2b_i\vec{r}^2) \\ &+\frac{2}{\pi^{3/2}}\sum_{i>j}^{N}\frac{a_ia_j}{d_{ij}^{3/2}}\exp(-\frac{\vec{p}^2}{d_{ij}})\exp(-\frac{d_{ij}^2-c_{ij}^2}{d_{ij}}\vec{r}^2)\cos(\frac{2c_{ij}}{d_{ij}}\vec{p}\cdot\vec{r}) \end{aligned} \tag{2.8}$$

where $c_{ij} = b_i - b_j$ and $d_{ij} = b_i + b_j$.

2.3 Muon Energy Spectrum and Sticking

The initial energy spectrum in the α-rest frame is obtained using a projection operator onto a given energy of the muon in the phase space according to:

$$(\frac{dN}{dE})|_{\alpha-rest}(E,t=0) = \int f(\vec{r},\vec{p})P(\vec{r},\vec{p};E)d\vec{r}d\vec{p} \tag{2.9}$$

The quantum projection operator onto a given energy can be denoted by

$$\hat{P}(E) = \delta(\hat{H}-E), \tag{2.10}$$

where the Hamiltonian is:

$$\hat{H} = \frac{1}{2}\hat{\vec{v}}^2 - \frac{Z}{|\hat{\vec{r}}|}. \tag{2.11}$$

The Weyl transform Eq.(1.22) of this operator is

$$P(\vec{r},\vec{p};E) = \sum_n f_n(\vec{r},\vec{p})\delta(E_n-E), \tag{2.12}$$

where $|n>$ is the complete set of energy eigenstate and f_n is the Wigner function of a state $|n>$.

2.3.1 Approximate Sticking Calculations

In this subsection I shall explore how the muon spectrum Eq.(2.9) and the sticking probability depends on various physical assumptions I make in Eq.(2.9) regarding the phase space distributions f and the projector P.

Classical Projection

I consider first the intuitive classical projection with

$$P_{cl}(\vec{r},\vec{p};E) = \delta(\frac{1}{2}\vec{v}^2 - \frac{Z}{|\vec{r}|} - E). \quad (2.13)$$

In the α-rest frame we have:

$$(\frac{dN}{dE})|_{\alpha-rest}(E,t=0) = \int f(\vec{r},\vec{p})\delta(\frac{1}{2}(\vec{v}-\vec{v}_\alpha)^2 - \frac{Z}{r} - E)d\vec{r}d\vec{p} \quad (2.14)$$

where $\vec{v}_\alpha = 5.84\hat{z}$ is the velocity of fusion α-particle and $\vec{r}_\alpha = 0$ at t = 0. To integrate this equation I use the Monte Carlo Method. Details of the numerical method are given in APPENDIX A. Fig.(2.1) shows this energy spectrum. The initial sticking probability ω_s^0, the fraction of the negative energy the muon has in the fusion α-rest frame, is 2.1%. This value is much higher than 1.2% which would result if the projection was carried out in an exact quantum mechanical manner (in the adiabatic approximation).

Microcanonical Improvement

It is possible to improve this sticking result introducing the so-called micro-canonical approximation: I consider only the 6-dimensional hyper-phase space which satisfies the constraint to the classical energy $E_{lab}(p,q) = \frac{1}{2}\vec{v}^2 - \frac{2}{r} = -2$, corresponding to the muon binding energy of the fusing muon amplitude(in adiabatic approximation). I also re-normalizes this new distribution to unity. This approach, as it turns out, over-corrects the deficiency of the classical approximation of the quantum projection operator. Fig.(2.2) shows the energy spectrum in the α-rest frame for this new distribution. Now about 0.4% of all muons have negative energy relative to the α-particle. We thus see that the sticking fraction is highly sensitive to the proper treatment of the Wigner distribution.

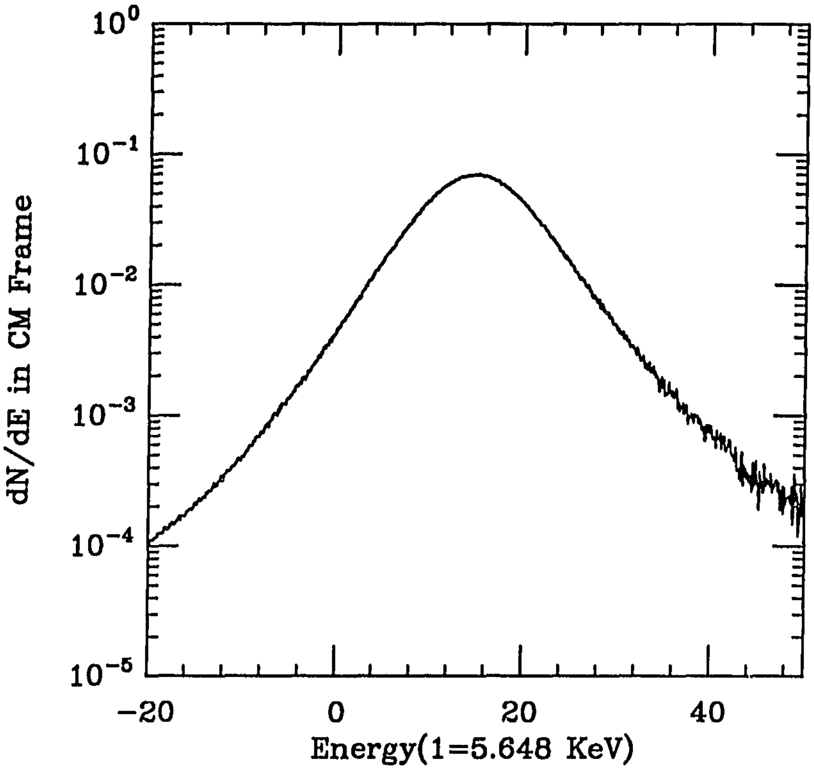

Figure 2.1: Muon energy spectrum in the α-rest frame

Smeared Wigner Function

Recall that the Wigner function defined by the Eq.(2.5) is not always positive definite as mentioned earlier and this is the major obstacle in the classical interpretation of its properties. As discussed above, the conventional resolution of this problem consist in the folding of the Wigner function with a Gaussian type smearing function, i.e., I consider:

$$f_s(\vec{r},\vec{p};\lambda) = \frac{1}{\pi^3}\int d\vec{r}_1 d\vec{p}_1 e^{-(\vec{p}-\vec{p}_1)^2/\lambda^2} e^{-\lambda^2(\vec{r}-\vec{r}_1)^2} f(\vec{r}_1,\vec{p}_1) \tag{2.15}$$

This smeared Wigner distribution is positive definite [10], as is required for proper classical interpretation of the initial muon phase-space distribution. However, this new distribution does depend on the smearing parameter λ. Thus the muon spectra which are calculated from this distribution will also depend on this parameter. In principle λ has some physical

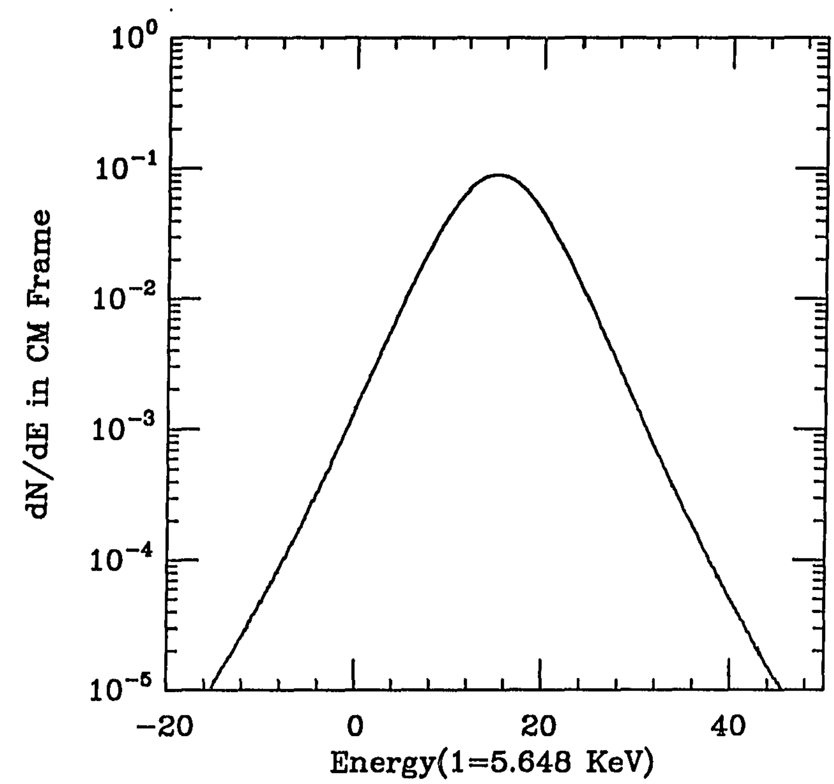

Figure 2.2: Muon energy spectrum in the α-rest frame in micro-canonical description

importance if the process of smearing is related to the measurement procedure.

It is straightforward to calculate the smeared Wigner function, Eq.(2.15), from the Wigner function I have employed, Eq.(2.7). We obtain the following expression in terms of the expansion coefficients a_i, b_i and the parameter λ:

$$
\begin{aligned}
f_s(\vec{r},\vec{p};\lambda) &= \frac{1}{\pi^{3/2}}\sum_i a_i^2\left(\frac{\lambda}{\lambda^2+d_{ij}}\right)^3 \exp\left(-\frac{\vec{p}^2}{\lambda^2+d_{ij}}\right)\exp\left(-\frac{\lambda^2 d_{ij}}{\lambda^2+d_{ij}}\vec{r}^2\right) \\
&+\frac{2}{\pi^{3/3}}\sum_{i>j} a_i a_j\left[\frac{\lambda^2}{(\lambda^2+d_{ij})^2-c_{ij}^2}\right]^{3/2}\exp\left[-\frac{\lambda^2+d_{ij}}{(\lambda^2+d_{ij})^2-c_{ij}^2}\vec{p}^2\right] \\
&\cdot\exp\left[-\frac{\lambda^4 d_{ij}+\lambda^2(d_{ij}^2-c_{ij}^2)}{(\lambda^2+d_{ij})^2-c_{ij}^2}\vec{r}^2\right]\cos\left[\frac{2\lambda^2 c_{ij}}{(\lambda^2+d_{ij})^2-c_{ij}^2}\vec{p}\cdot\vec{r}\right] \qquad (2.16)
\end{aligned}
$$

where $c_{ij} = b_i - b_j, d_{ij} = b_i + b_j$.

For the appropriate choice of λ, this distribution, which reflects a measurement of the position and momentum of the muon within the precision λ^{-1}, will yield the sticking probability of the muon. Note that when the muon attaches to the escaping α-particle, its position can be measured with a precision corresponding to the size of the muon orbit, which for the 1s state is 1/2 in natural units. It is thus natural to expect that the parameter $\lambda = 2$ represents the reasonable choice to describe the probability of muon sticking.[3] The initial sticking values are shown in Fig.(2.3) as function of the parameter λ. The square points show the initial sticking probability for the $(dd\mu \rightarrow He^3 + \mu + n)$ fusion and the + points for $(dt\mu \rightarrow \alpha + \mu + n)$ fusion. We see that taking the classical projection operator and a classical (that is smeared) phase space distribution I obtain reasonable sticking probabilities: $\omega_0 = 1.36\%$ for dt and 11.1% for dd fusion (for $\lambda = 2$) — recall that the full quantum sticking probability is 1.2% for the dt fusion and 15.5% for the dd fusion [28].

2.3.2 Exact Sticking Calculation

I now attempt a full calculation of sticking and muon spectrum within the complete quantum transport theory. The quantum operator of sticking can be represented by:

$$\hat{S} = \Theta(-\hat{H}_{rel}) \tag{2.17}$$

where the relative Hamiltonian with respect to the fusion α-particle is

$$\hat{H}_{rel} = \frac{(\hat{p} - \hat{p}_\alpha)^2}{2} + \frac{Z}{|\hat{r}_{rel}|}, \tag{2.18}$$

and Θ is a step function. I can express this projection operator by summing over appropriate eigenstates:

$$\hat{S} = \sum_{E_{nl}<0} {}_v| \, nl >< nl \, |_v \tag{2.19}$$

where ${}_v| \, nl >$ is the bound muon eigenstate of the fusion α-particle which has the velocity v. To simplify the situation, I consider the muon capture to the $(He\mu)^+_{1s}$ state, i.e., I choose

[3] It should be noted that in order to obtain the continuum spectrum of muons after fusion I must not choose $\lambda = 2$ since the muon is not localized to $\Delta_x < 1/2$ but practically speaking fully delocalized. Hence a small value of λ should be chosen for this purpose.

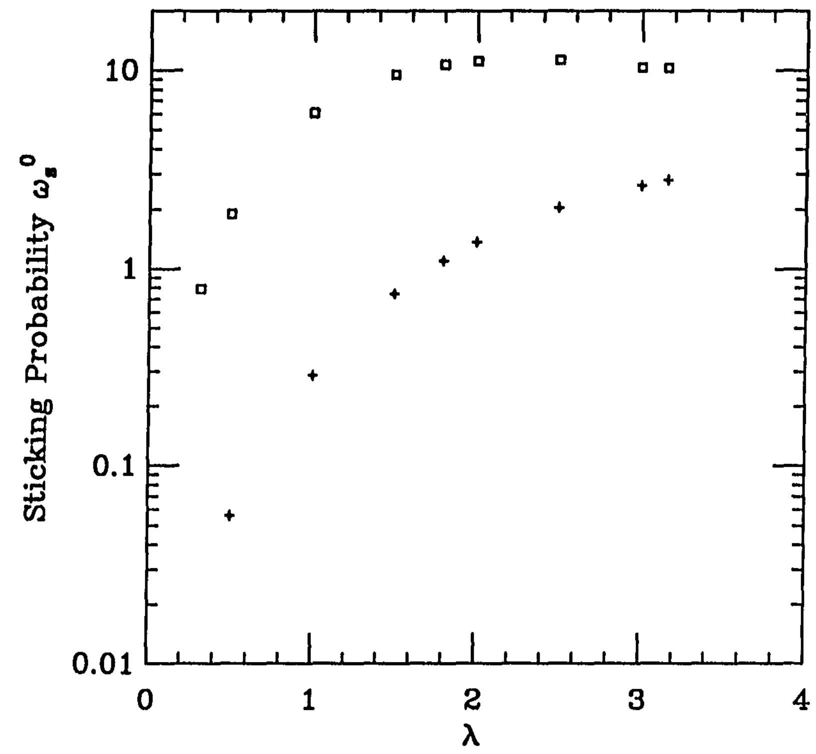

Figure 2.3: The sticking probability as function of the smearing parameter λ

for the sticking operator $\hat{S} = | v_\alpha, 1s >< v_\alpha, 1s |$.

The Weyl transformation of this operator is

$$\begin{aligned} S(\vec{r},\vec{p}) &= \int d\vec{x} < \vec{r} + \frac{\vec{x}}{2} \mid \hat{S} \mid \vec{r} - \frac{\vec{x}}{2} > \exp(-i\vec{p}\cdot\vec{x}) \\ &= \int d\vec{x}\, \exp(-i\vec{p}\cdot\vec{x}) \psi^*_{1s,v_\alpha}(\vec{r} - \frac{\vec{x}}{2}) \psi_{1s,v_\alpha}(\vec{r} + \frac{\vec{x}}{2}). \end{aligned} \tag{2.20}$$

Assuming that the momentum of the α-particle is well defined, the muon has a well defined translational velocity. Thus $\psi_v(\vec{r}) = \exp(im\vec{v}\cdot\vec{r})\psi(\vec{r})$, and we obtain:

$$S(\vec{r},\vec{p}) = \sum_{ij}^{N} a_i a_j (\frac{4\pi}{d_{ij}})^{3/2} \exp[-\frac{(\vec{p} - mv_\alpha)^2}{d_{ij}} - \frac{d_{ij}^2 - c_{ij}^2}{d_{ij}} \vec{r}^2 - \frac{2ic_{ij}}{d_ij} (\vec{p} - m\vec{v}_\alpha) \cdot \vec{r}] \tag{2.21}$$

Using the Gaussian germinal representation for the initial muon amplitude Eq.(2.8), the sticking probability is given by:

$$\omega_s^0 = (2\pi)^3 \sum_{ijkl} a_i a_j a_k a_l \left[\frac{1}{(d_{ij}+d_{kl})^2 - (c_{ij}-c_{kl})^2}\right]^{3/2}$$
$$\exp\left(-[\frac{1}{d_{ij}+d_{kl}} + \frac{1}{A_{ijkl}}(-\frac{c_{kl}}{d_{kl}} + \frac{d_{ij}}{d_{ij}+d_{kl}}(\frac{c_{ij}}{d_{ij}} + \frac{c_{kl}}{d_{kl}}))^2]m^2\vec{v}_\alpha{}^2\right) \quad (2.22)$$

where

$$A_{ijkl} = \frac{d_{ij}^2 - c_{ij}^2}{d_{ij}} + \frac{d_{kl}^2 - c_{kl}^2}{d_{kl}} + \frac{d_{ij}d_{kl}}{d_{ij}+d_{kl}}\left(\frac{c_{ij}}{d_{ij}} + \frac{c_{kl}}{d_{kl}}\right)^2 \quad (2.23)$$

and $d_{ij} = b_i + b_j$ and $c_{ij} = b_i - b_j$. Using the expansion coefficients in Table (2.1) for the adiabatic approximation, I compute $\omega_s^0 = 1.039\%$ which is the exact result under the assumption which I have made (adiabatic muon amplitude and only $1s$ sticking).

2.4 Time Evolution of the Phase Space Distribution

The spectrum of a muon seen in the laboratory frame of reference will undergo a time evolution which I will study here given the diverse limitations to the usefulness of the semi-classical treatment of the Wigner function I have explored in the last section.

2.4.1 Initial Energy Spectrum in Lab Frame

In the laboratory frame, at time $t = 0$ the energy spectrum is

$$(\frac{dN}{dE})\,|_{lab}\,(E, t=0) = \int f(\vec{r},\vec{p})\delta(\frac{1}{2}\vec{v}^2 - \frac{Z}{r} - E)d\vec{r}d\vec{p} \quad (2.24)$$

The result is shown in Fig.(2.4). This distribution shows that 89.2% of the muons have negative energy and 10.8% have positive energy and its maximum at about $E = -1.4$. The average energy is the same as the one given by the quantum mechanical expectation value, i.e.,

$$\int (\frac{dN}{dE})\,|_{lab}\, EdE = -2 \quad (2.25)$$

in (muonic) Hartree units of energy 1 = 5.648 KeV. Speaking in terms of quantum mechanical representation theory, this distribution is simply the energy representation of the

Helium 1s state. The correctness of the energy expectation value justifies in part the use of the classical limit of the Weyl transformation of the Hamilton operator Eq.(2.13). However, the fact that 11% of the muons have positive energy is an artifact of the energy representation of the wave function and it also demonstrates the shortcoming of interpreting the Wigner distribution as a classical probability. It shows again that one cannot with impunity proceed as if the quantum distribution was fully useful in classical calculations.

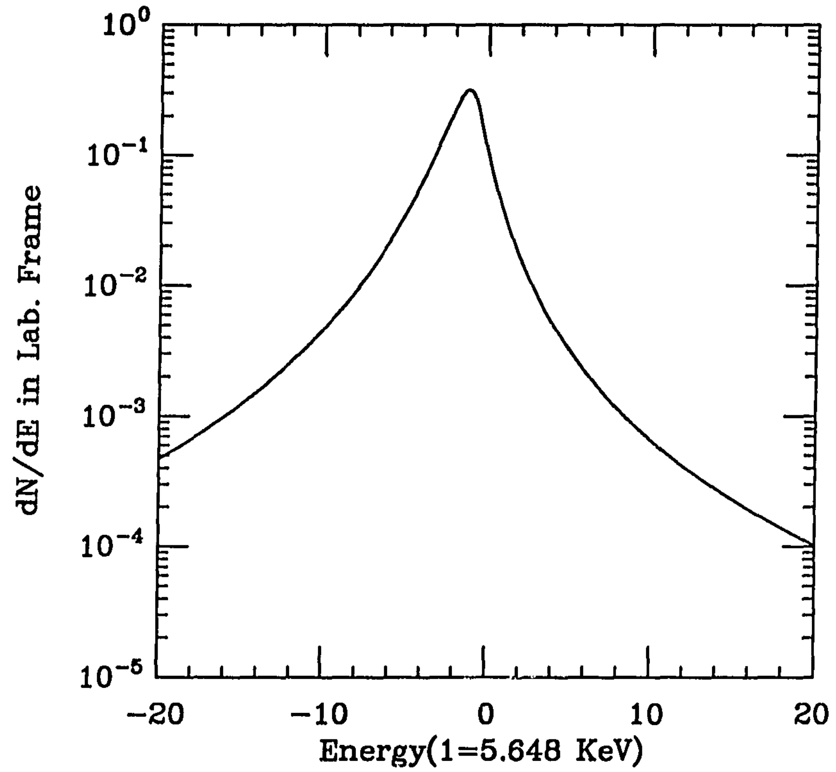

Figure 2.4: Muon energy spectrum in the Lab. frame

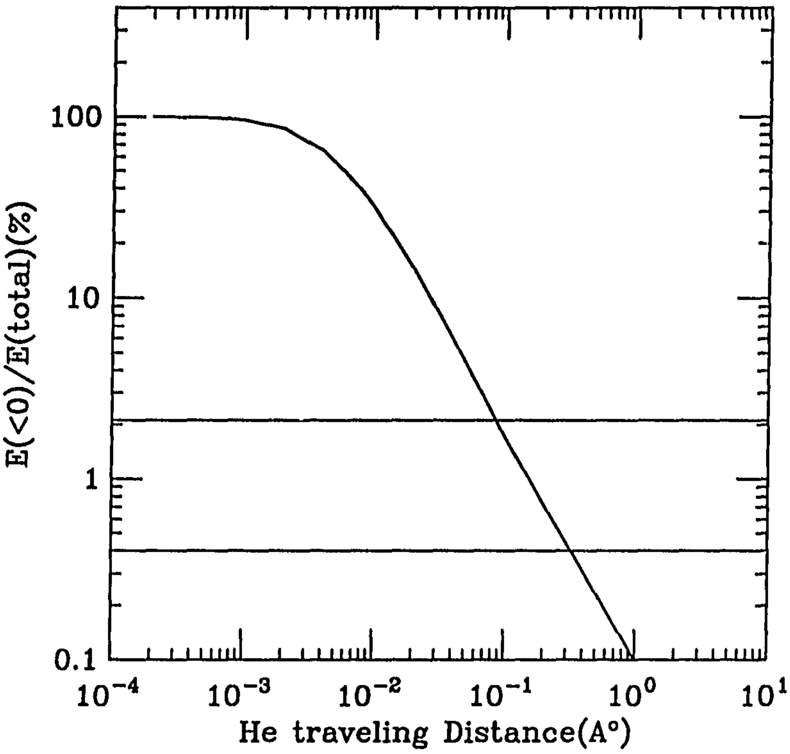

Figure 2.5: Fractions of muon which have the negative energy

2.4.2 Time Evolution of the Spectrum

I numerically evolve the muon phase space distribution along the classical trajectories in time, under the influence of the Coulomb field of the relatively rapidly moving α particle. The energy spectrum is:

$$\frac{dN}{dE}\Big|_{lab}(E,t) = \int f(\vec{r}(t),\vec{p}(t))\delta(\frac{1}{2}\vec{v}(t)^2 - \frac{2}{r(t)} - E)d^3r\, d^3p, \qquad (2.26)$$

where $\vec{r}(t)$ and $\vec{p}(t)$ are the Wigner trajectories. The initial position and momentum of the muon in the laboratory frame are determined using the Monte Carlo method of Appendix A. I then solve Newton's equation within the Coulomb field of moving α-particle working in the α-rest frame where the problem becomes exactly the Kepler problem, and is quite challenging in order to assure numerical precision.

I find that the negative energy fraction of the muon spectrum considered in the laboratory frame decreases quickly as is shown in Fig.(2.5) – recall that such negative energy fraction stays constant in the α-rest frame (sticking fraction). The important message contained in Fig.(2.5) is that after Helium traveled $0.1\AA$, the fraction of muons with negative energy in the laboratory frame becomes smaller than 1%. This is thus the approximate distance over which the quantum sticking is established. However, this is in terms of the scales of the problem at hand nearly a macroscopic distance and I should expect a rather significant interference of matter in direct neighborhood of the fusion event with the evolution of the muon amplitude.

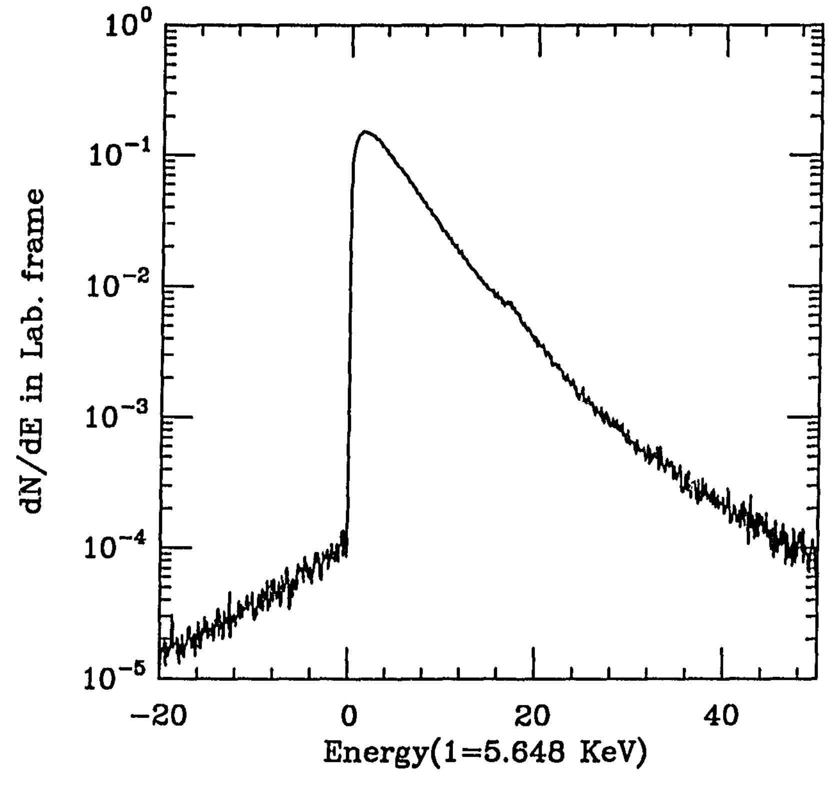

Figure 2.6: Muon energy spectrum in Lab. frame after $3\AA$

It can be expected given this result that the final form of the spectrum is established after a few Ångstrom propagation in matter. In Fig.(2.6) the energy spectrum is shown in the laboratory frame after the 3$\AA$ α traveling time evolved along the classical Wigner trajectory.

2.4.3 Impact of the Medium

In order to understand in full the impact of the background medium I would have to solve two coupled Boltzmann-like equations of motion of the quantum distributions, one for the muon and the other for the α–particle, including full scattering with surrounding matter. By full scattering I mean that the fusion α-particle and muon should scatter from the background and each other as well — recall that matter atoms are separated by $\AA$ distances, while μ and α are born initially at a fraction of this distance. The sensible approximation to make here is to assume that the α-particle does not interact with medium. Because of its much greater mass, its path will be affected only in unusually close collisions with other nuclei. The basic ingredients of the simplified picture to understand the muon's fate are thus as follows: A muon which initially is at the phase space point $(\vec{r},\vec{p})$ with weighing $f(\vec{r},\vec{p})$ or, respectively $f_s(\vec{r},\vec{p};\lambda)$, moves classically under the influence of the fusion α-particle Coulomb field up to a scattering point and after it scatters it will propagate to the next scattering event etc., until the prescribed time is up. As a first step in this complex study I consider the elastic scattering even though the inelastic scattering is of the same order of magnitude as that of elastic scattering – the natural expectation is that the elastic scattering increases the sticking fraction, while the inelastic scattering will decrease it.

The elastic scattering cross section of a muon on a hydrogen is given by the Mott formula [29]:

$$\frac{d\sigma}{d\cos\theta} = f(\cos\theta) = \frac{\eta^2}{2}\frac{(2+\eta^2v^2\sin^2\theta/2)^2}{(1+\eta^2v^2\sin^2\theta/2)^4} \tag{2.27}$$

in (πa_0^2) unit where η is the ratio of the muon mass m_μ to the electron mass m_e and v is in atomic unit (αc). The total cross section is given by integrating the angle over [-1,+1]

$$\sigma_T(v) = \frac{\eta^2(48+72\eta^2v^2+28\eta^4v^4)}{12(1+\eta^2v^2)^3} \tag{2.28}$$

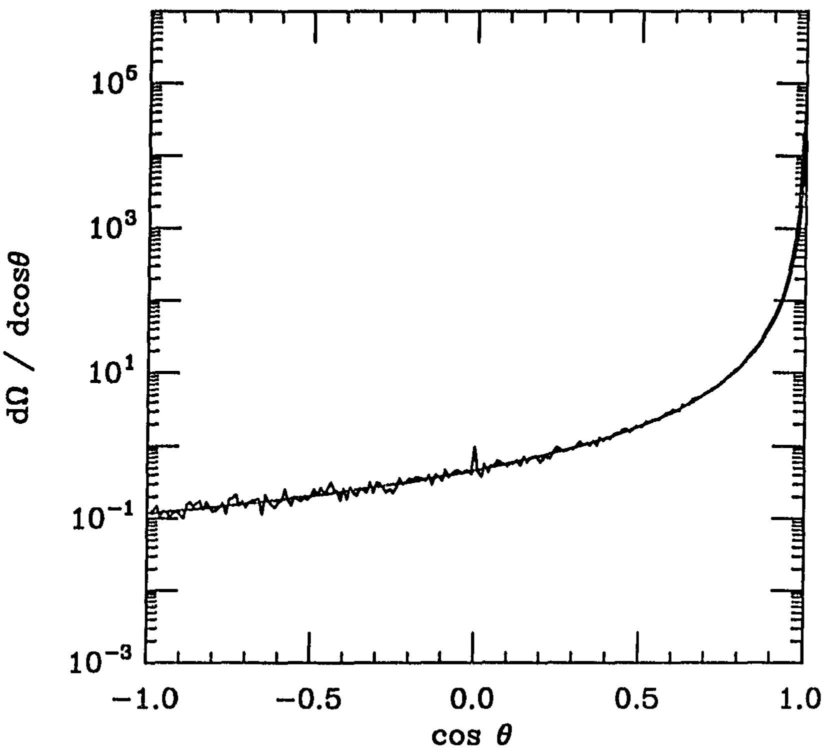

Figure 2.7: Differential cross section:solid line, Mott formula; dashed line, Monte Carlo sampled data

in (πa_0^2) unit. The scattering length will be chosen by the Monte Carlo method using the Poisson distribution [30, 31, 32, 33], i.e.

$$L = -L_0 \ln(r) \tag{2.29}$$

where $L_0 = 1/\rho\sigma_T(v)$ is a mean free path and r is a random number between 0 and 1. Once this scattering length is chosen, the muon moves up to this distance classically under the fusion α-particle Coulomb field. Note that the change of L coming from the change of the muon velocity due to the interaction with the α-particle during the transport has been ignored. At the scattering point, I can easily choose the azimuthal angle ϕ by

$$\phi = 2\pi r_1 \tag{2.30}$$

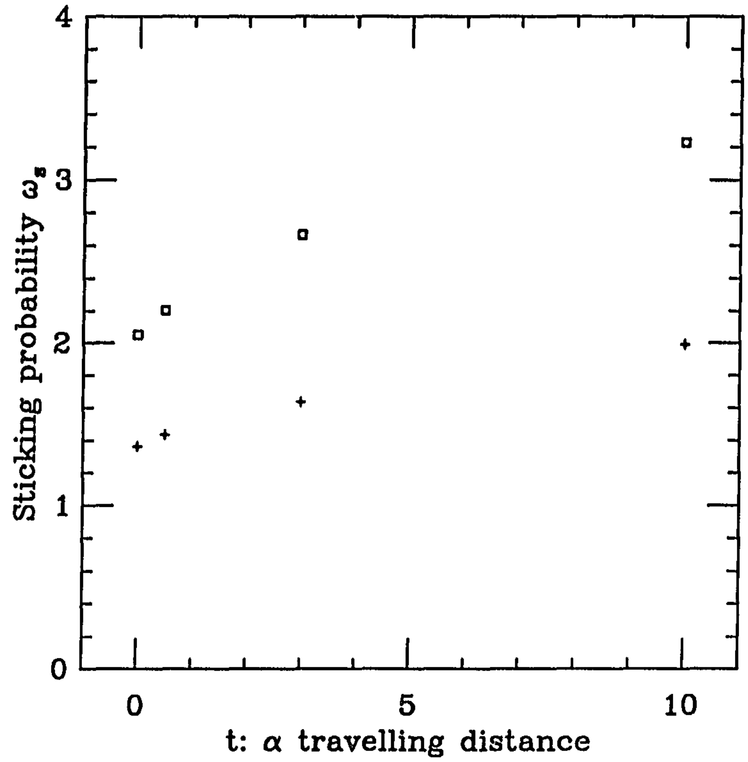

Figure 2.8: The evolution of the sticking probability

where $r_1 \in [0, 1]$ is another random number. It is rather difficult to choose the angle θ. Dividing Eq.(2.27) by the total cross section, I obtain

$$dr = \frac{f(t)}{\sigma_T} dt, \tag{2.31}$$

where $r = \sigma/\sigma_T$ is a random number between $[0, 1]$ and $t = \cos\theta$. The integral of this equation leads to:

$$r = \frac{1}{\sigma_T} \int_{-1}^{y} f(t) dt \tag{2.32}$$

Solution of this third order algebraic equation gives a value y for a given random number r. Now the angle θ is obtained by

$$\theta = \cos^{-1} y. \tag{2.33}$$

Fig.(2.7) shows the Monte Carlo sampling of the angle from Eq.(2.33) for a given v. It agrees well with the Mott formula.

I repeat this transport process until the fusion α–particle reaches the prescribed place.

This muon recapture calculation has been done for the hydrogen medium with density $\rho = 1.2$ LHD. The results are shown in Fig.(2.8). As shown the sticking probability is increased significantly by the interaction with the medium. This effect is seen for both smeared(+) and not smeared($\diamond$) phase space distributions, in both instances sticking rises by about 1%. Recall that by assuming only elastic scattering I effectively precluded the possibility of sticking becoming smaller. In this regard this work is clearly very preliminary, showing the need to perform a full phase space evolution calculation including inelastic interactions with the medium in which the muon catalyzed fusion occurs. In such a full treatment one would want to improve the muon amplitude to account for non-adiabatic effects, as well as relativistic and spin-structure corrections which so far have not been explored.

CHAPTER 3

Spin 1/2 Relativistic Wigner Function and Equation[1]

The Wigner transport formulation introduced in chapter 1 has been enlarged to allow for relativistic kinematics and particle production next to the matter flow processes – in this chapter I shall first survey the recent developments I will employ, with particular attention given to the methods developed in Refs. [48, 49].

3.1 Introduction

In Chapter 1, I introduced the one–particle Wigner function as the Weyl transform of the density matrix. While this Wigner function found applications in a variety of fields such as the chemical reactions [34], nuclear physics [35, 36], quantum optics [38] and solid state physics [37], this theory cannot describe the particle production process. Since the process of particle production is unavoidable in relativistic formulation and/or at sufficiently high energy, one has to develop a transport theory which has room for the process. Nearly 20 years ago Carruthers and Zachariasen [39] introduced a relativistic eight-dimensional Wigner function for spinless neutral particle fields:

$$F(p,x) \;=\; \int d^4y\, e^{ip\cdot y} < \Psi|\hat{\varphi}(x-y/2)\hat{\varphi}(x+y/2)|\Psi >, \qquad (3.1)$$

where $|\Psi>$ is the state vector. $\hat{\varphi}(x)$ is the 'second-quantized' Klein–Gordon field obeying the equation of motion

$$(\partial_\mu\partial^\mu + m^2)\hat{\varphi}(x) \;=\; \hat{\jmath}(x), \qquad (3.2)$$

where $\hat{\jmath}(x)$ is the source fixed in its form by the model under consideration. The equation of motion of this Wigner function can be obtained by applying $(\partial_\mu\partial^\mu + m^2)$ on Eq.(3.1)

[1]The subjects of this chapter have been extracted from the Ref.[48, 49]

and using the field equation. Since in general the source $\hat{j}(x)$ is a function of the field itself and/or other fields, the equation of motion cannot be closed unless one makes some kind of approximation (see also discussion in Chapter 1). This is one of the general properties of transport theory and is known as the BBGKY hierarchy problem — the dynamics of a two-point Wigner function is determined by a four– or higher–point Wigner function and that of a four-point Wigner function by six– or higher–point Wigner function, and so on. To break this hierarchy a suitable approximation, for example, MFA(mean field approximation), is made. It is important to remember that this transport theory inherits the intrinsic infinities from the relativistic field theory. Therefore one needs to renormalize the theory to produce meaningful quantities. This problem has been addressed by Cooper et. al. [40] for the scalar field.

This field theoretical approach has been extended to the Dirac field by Hakim [41] who was interested to study strongly interacting particles forming relativistic dense matter. While the scalar field Wigner function is in principle 2×2 matrix due to the particle and antiparticle sector, the spinor field Wigner function is 4×4 matrix coming from the spinor structure,

$$F(p,x) = \int \frac{d^4y}{(2\pi\hbar)^4} e^{-ip\cdot y} < \Psi|\hat{\bar{\psi}}(x+y/2) \otimes \hat{\psi}(x-y/2)|\Psi > . \tag{3.3}$$

This 4×4 matrix Wigner function has been decomposed into 16 functions on the basis of 16 linear independent matrices [42],

$$1_4, \gamma^\mu, \sigma^{\mu\nu}, \gamma^5, \gamma^\mu\gamma^5. \tag{3.4}$$

The dynamics of this Wigner function can be determined using the Dirac field equation.

While the formulations presented above are in general manifestly Lorentz covariant and can describe the particle production process, they miss one important ingredient, namely, a gauge covariance. It is well–known that the observable is gauge invariant and the physical process must be described by a gauge covariant theory. In particular, the gauge symmetry resides in the heart of modern physics, as for example in QED, QCD or standard model. Heinz [45] and Elze et al [46] were able to propose Wigner functions which had full gauge symmetry. To this end, they consider the Wigner operator [44],

$$\hat{W}(x,p) = \int \frac{d^4y}{(2\pi\hbar)^4} e^{-ip\cdot y} \hat{\bar{\psi}}(x+y/2) \otimes \hat{\psi}(x-y/2)$$

$$= \int \frac{d^4y}{(2\pi\hbar)^4} e^{-ip\cdot y} \hat{\bar{\psi}}(x) e^{+\frac{y}{2}\partial_x^\dagger} \otimes e^{-\frac{y}{2}\partial_x} \hat{\psi}(x), \tag{3.5}$$

where $\partial^\dagger$ operates on the function to the left of it and the relation $f(x \pm y) = e^{\pm y\cdot\partial_x} f(x)$. $\otimes$ is the tonsorial product in spinor space (4×4) as well as the internal quantum number such as the color. To make this function gauge covariant, it is only necessary to replace the derivative by the covariant derivative, i.e.

$$\partial_\mu \quad \rightarrow \quad D_\mu = \partial_\mu - igA_\mu, \tag{3.6}$$

where g is the coupling constant and A_μ is the gauge field. Thus, the gauge covariant Wigner operator is

$$\hat{W}(x,p) \;=\; \int d^4y \; \hat{\bar{\psi}}(x+y/2) U(x+y/2,x) \otimes U(x,x-y/2) \hat{\psi}(x-y/2), \tag{3.7}$$

with

$$U(a,b) \;=\; \exp\left[ig \int_b^a dx_\mu A^\mu\right], \tag{3.8}$$

where the path of the link operator U [47] must be straight line in order to interpret p as the physical 4–momentum.

Under the gauge transformation,

$$\psi(x) \;\rightarrow\; S(x)\psi(x), \quad S(x) = e^{i\theta_a(x)t_a}, \tag{3.9}$$

$$A^a_\mu \;\rightarrow\; A^a_\mu - \frac{1}{g}\partial_\mu\theta^a - f^a_{bc}\theta_b A_{c,\mu}, \tag{3.10}$$

the Wigner operator transforms covariantly, akin to the operator D^μ:

$$\hat{W}(x,p) \;\rightarrow\; S(x)\hat{W}(x,p)S^{-1}(x). \tag{3.11}$$

Here t_a is the generator of the gauge group and f_{abc} the structure constant of the gauge group. Of cause, the Wigner function is the quantum expectation of the Wigner operator in a given state $|\Psi>$. The dynamics of this function can be obtained by the field equation and it requires tedious operator ordering especially in the case of non–abelian gauge theory. This formulation is manifestly Lorentz covariant, and the dynamics described occur also off the mass–shell. Thus in order to calculate a physical observable one should project the results on the mass–shell, which is a (complex) constraint of the 8-dimensional dynamical motion.

3.2 BGR Functions and Equations

This projection requirement makes it difficult to extract the physical information from the transport functions, which are even more difficult to obtain. Consequently little progress was made regarding practical applications of the eight dimensional formulation. However, a different approach has been also recently proposed by Białynicki–Birula, Górnicki and Rafelski [48], who introduced the so–called Dirac–Heisenberg–Wigner (DHW) function, which is the Weyl transform of Dirac–Heisenberg density matrix. In many regards this formulation is similar to the conventional nonrelativistic Wigner theory. The DHW function for the matter field of the abelian gauge theory (QED), is introduced as follows,

$$W_{\alpha\beta}(\vec{r},\vec{p},t)=-\frac{1}{2}\int d^3s\; e^{-i\vec{p}\cdot\vec{s}} < \Psi | e^{-ie\int d\lambda\vec{s}\cdot\vec{A}(\vec{r}+\lambda\vec{s},t)} [\hat{\psi}_\alpha(\vec{r}+\vec{s}/2,t), \hat{\psi}^\dagger_\beta(\vec{r}-\vec{s}/2,t)] | \Psi >, \quad (3.12)$$

where $|\Psi>$ is a state vector and $\vec{A}$ is the gauge field. α and β are the spinor index. This DHW function has following properties:

1) since the DHW function is gauge invariant, I can fix the gauge in a most convenient way which is here the temporal gauge ($A_0 = 0$);

2) the DHW function is not manifestly Lorentz covariant because it has only one time t which is a laboratory time (this is a reason why it is called a single time formulation). However, it has full Poincare symmetry;

3) the field operators in Eq.(3.12) have been combined such that $W_{\alpha\beta}$ possesses the charge conjugation symmetry;

4) the transformation variable $\vec{p}$ is the physical kinetic momentum, a consequence of choosing the straight line integral in the phase factor which makes the function gauge invariant;

5) $W_{\alpha\beta}$ is a Hermitian by construction so that there are 16 linearly independent real functions defining the matrix.

One can decompose this 4×4 matrix **W** on the complete set of 4×4 Hermitian matrices,

$$\mathbf{W}(\vec{r},\vec{p},t) = \frac{1}{4}\left(f_0 + \sum_{i=1}^{3}\rho_i f_i + \vec{\sigma}\cdot\vec{g}_0 + \sum_{i=1}^{3}\rho_i\vec{\sigma}\cdot\vec{g}_i\right), \quad (3.13)$$

where the complete set of 4×4 Hermitian matrices is given in Appendix B. This decomposition allows a direct physical interpretation of the coefficient functions which are called 'BGR' functions. The physical meanings of the 16 component functions can be inferred from their momentum integrals:

$$\int \frac{d^3p}{(2\pi\hbar)^3} f_0(\vec{r},\vec{p},t) = \mathrm{Tr}[\bar{\psi}(\vec{r},t)\gamma^0\psi(\vec{r},t)], \quad (3.14)$$

$$\int \frac{d^3p}{(2\pi\hbar)^3} f_1(\vec{r},\vec{p},t) = \mathrm{Tr}[\bar{\psi}(\vec{r},t)i\gamma^0\gamma^5\psi(\vec{r},t)], \quad (3.15)$$

$$\int \frac{d^3p}{(2\pi\hbar)^3} f_2(\vec{r},\vec{p},t) = \mathrm{Tr}[\bar{\psi}(\vec{r},t)\gamma^5\psi(\vec{r},t)], \quad (3.16)$$

$$\int \frac{d^3p}{(2\pi\hbar)^3} f_3(\vec{r},\vec{p},t) = \mathrm{Tr}[\bar{\psi}(\vec{r},t)\psi(\vec{r},t)], \quad (3.17)$$

$$\int \frac{d^3p}{(2\pi\hbar)^3} \vec{g}_0(\vec{r},\vec{p},t) = -\mathrm{Tr}[\bar{\psi}(\vec{r},t)i\gamma^5\vec{\gamma}\psi(\vec{r},t)], \quad (3.18)$$

$$\int \frac{d^3p}{(2\pi\hbar)^3} \vec{g}_1(\vec{r},\vec{p},t) = \mathrm{Tr}[\bar{\psi}(\vec{r},t)\vec{\gamma}\psi(\vec{r},t)], \quad (3.19)$$

$$\int \frac{d^3p}{(2\pi\hbar)^3} \vec{g}_2(\vec{r},\vec{p},t) = -\mathrm{Tr}[\bar{\psi}(\vec{r},t)i\gamma^0\vec{\gamma}\psi(\vec{r},t)], \quad (3.20)$$

$$\int \frac{d^3p}{(2\pi\hbar)^3} g_3^k(\vec{r},\vec{p},t) = \mathrm{Tr}[\bar{\psi}(\vec{r},t)i\epsilon^{ijk}\gamma^{ij}\psi(\vec{r},t)], \quad (3.21)$$

where $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$, and $\gamma^{ij} = \gamma^i\gamma^j$. Tr stands here for the trace over the spinor space only. Thus $(f_0, \vec{g}_1)$ form the current four vector phase space distributions, f_3 is the mass density, $\vec{g}_0$ the spin density, $\vec{g}_3$ the magnetic moment density, etc. These interpretations can be further justified by the conservation laws I shall discuss below.

The time evolution of this DHW function, thus the relativistic quantum transport equation, can be obtained by differentiating Eq.(3.12) with respect to time and using Dirac field equations,

$$i\partial_t\psi_\mu = [\vec{\alpha}\cdot(-i\vec{\nabla} - e\vec{A}) + \beta m]_{\mu\nu}\psi_\nu(\vec{r},t), \quad (3.22)$$

$$-i\partial_t\psi_\mu^\dagger = \psi_\nu^\dagger(\vec{r},t)[\vec{\alpha}\cdot(i\nabla - e\vec{A}) + \beta m]_{\nu\mu}. \quad (3.23)$$

Keeping track of arguments $(\vec{r} - \vec{y}/2)$ and $(\vec{r} + \vec{y}/2)$ of the field operators carefully, one obtains the time evolution,

$$D_t\mathrm{W} = -\frac{c}{2}\vec{D}\cdot\{\rho_1\vec{\sigma}, \mathrm{W}\} - \frac{ic}{\hbar}[\rho_1\vec{\sigma}\cdot\vec{P} + \rho_3 mc, \mathrm{W}], \quad (3.24)$$

where the integro-differential operators are

$$D_t = \partial_t + e\int_{-1/2}^{1/2} d\lambda \vec{E}(\vec{r} + i\hbar\lambda\vec{\partial}_p, t)\cdot\vec{\partial}_p, \quad (3.25)$$

$$\vec{D} = \vec{\nabla} + \frac{e}{c}\int_{-1/2}^{1/2} d\lambda \vec{B}(\vec{r} + i\hbar\lambda\vec{\partial}_p, t) \times \vec{\partial}_p, \tag{3.26}$$

$$\vec{P} = \vec{p} - \frac{ie\hbar}{c}\int_{-1/2}^{1/2} d\lambda\lambda \vec{B}(\vec{r} + i\hbar\lambda\vec{\partial}_p, t) \times \vec{\partial}_p. \tag{3.27}$$

Since the formulation is constructed on the temporal gauge, the electric and magnetic fields are given by

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t}, \tag{3.28}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}. \tag{3.29}$$

One further assumption was made to obtain these equations: The expectation value of the products of Dirac field operators with gauge field strength was replaced by the product of corresponding expectation values,

$$< \Psi|\vec{E} \exp\left(-ie\int d\lambda\vec{s}\cdot\vec{A}(\vec{r} + \lambda\vec{s}, t)\right) [\psi(\vec{r} + \vec{s}/2, t), \psi(\vec{r} - \vec{s}/2, t)]|\Psi > \rightarrow$$
$$< \Psi|\vec{E}|\Psi > < \Psi|\exp\left(-ie\int d\lambda\vec{s}\cdot\vec{A}(\vec{r} + \lambda\vec{s}, t)\right) [\psi(\vec{r} + \vec{s}/2, t), \psi(\vec{r} - \vec{s}/2, t)]|\Psi >, \tag{3.30}$$

$$< \Psi|\vec{B} \exp\left(-ie\int d\lambda\vec{s}\cdot\vec{A}(\vec{r} + \lambda\vec{s}, t)\right) [\psi(\vec{r} + \vec{s}/2, t), \psi(\vec{r} - \vec{s}/2, t)]|\Psi > \rightarrow$$
$$< \Psi|\vec{B}|\Psi > < \Psi|\exp\left(-ie\int d\lambda\vec{s}\cdot\vec{A}(\vec{r} + \lambda\vec{s}, t)\right) [\psi(\vec{r} + \vec{s}/2, t), \psi(\vec{r} - \vec{s}/2, t)]|\Psi > . \tag{3.31}$$

If one does not make this approximation, the equation of motion cannot be closed since the gauge field strength will introduce the Dirac field as a source. This is similar to the BBGKY hierarchy as mentioned before. In this approximation one neglects the fluctuation in the number of the photons, while all fluctuations of the matter field are retained. Consequently, this approach is particularly suitable to the study of the matter field in the presence of strong gauge fields.

After substituting the expansion of DHW function W, Eq.(3.13), into the evolution equation, one obtains BGR equations,

$$D_t f_0 + c\vec{D}\cdot\vec{g}_1 = 0, \tag{3.32}$$

$$D_t f_1 + c\vec{D}\cdot\vec{g}_0 = -\frac{2mc^2}{\hbar} f_2, \tag{3.33}$$

$$D_t f_2 + \frac{2c}{\hbar}\vec{P}\cdot\vec{g}_3 = +\frac{2mc^2}{\hbar} f_1, \tag{3.34}$$

$$D_t f_3 - \frac{2c}{\hbar}\vec{P}\cdot\vec{g}_2 = 0, \tag{3.35}$$

$$D_t\vec{g}_0 + c\vec{D}f_1 - \frac{2c}{\hbar}\vec{P}\times\vec{g}_1 = 0, \tag{3.36}$$
$$D_t\vec{g}_1 + c\vec{D}f_0 - \frac{2c}{\hbar}\vec{P}\times\vec{g}_0 = -\frac{2mc^2}{\hbar}\vec{g}_2, \tag{3.37}$$
$$D_t\vec{g}_2 + c\vec{D}\times\vec{g}_3 + \frac{2c}{\hbar}\vec{P}f_3 = +\frac{2mc^2}{\hbar}\vec{g}_1, \tag{3.38}$$
$$D_t\vec{g}_3 - c\vec{D}\times\vec{g}_2 - \frac{2c}{\hbar}\vec{P}f_2 = 0. \tag{3.39}$$

To close the set of equations, Maxwell equations must be added,

$$\partial_t\vec{B} = -\vec{\nabla}\times\vec{E}, \tag{3.40}$$
$$\vec{\nabla}\cdot\vec{B} = 0, \tag{3.41}$$
$$\partial_t\epsilon_0\vec{E} = \vec{\nabla}\times\mu_0^{-1}\vec{B} - \vec{j}_t, \tag{3.42}$$
$$\vec{\nabla}\cdot\epsilon_0\vec{E} = \rho_t, \tag{3.43}$$

where charge and current density including a back reaction are

$$\rho_t(\vec{r},t) = e\int d\vec{p}\, f_0(\vec{r},\vec{p},t) + \rho_{ext}(\vec{r},t), \tag{3.44}$$
$$\vec{j}_t(\vec{r},t) = e\int d\vec{p}\, \vec{g}_1(\vec{r},\vec{p},t) + \vec{j}_{ext}(\vec{r},t), \tag{3.45}$$

and where ρ_{ext} and $\vec{j}_{ext}$ is the external charge and current density. The total charge, energy, momentum and angular momentum for the closed system are given by, respectively,

$$Q = e\int d\Gamma\, f_0(\vec{r},\vec{p},t), \tag{3.46}$$
$$E = \int d\Gamma\, [c\vec{p}\cdot\vec{g}_1(\vec{r},\vec{p},t) + mc^2 f_3(\vec{r},\vec{p},t)] + \frac{1}{2}\int d^3r\, [\epsilon_0\vec{E}^2(\vec{r},t) + \mu_0^{-1}\vec{B}^2(\vec{r},t)], \tag{3.47}$$
$$\vec{P} = \int d\Gamma\, \vec{p}f_0(\vec{r},\vec{p},t) + \int d^3r\, [\epsilon_0\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)], \tag{3.48}$$
$$\vec{M} = \int d\Gamma\, [\vec{r}\times\vec{p}f_0(\vec{r},\vec{p},t) + \frac{\hbar}{2}\vec{g}_0(\vec{r},\vec{p},t)] + \int d^3r\, \vec{r}\times[\epsilon_0\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)], \tag{3.49}$$

where $d\Gamma$ is a phase space volume element, $d\Gamma = d^3r d^3p/(2\pi\hbar)^3$. ϵ_0 and μ_0 is the electric permittivity and magnetic permeability. It is straightforward to prove that those quantities are constants of motion. Note that equations (3.46-3.49) give further motivation for the interpretation of the distributions $f_0, f_3, \vec{g}_0, \vec{g}_3$ presented above.

3.3 Integral Equations

It is useful to represent the transport equation in an integral form. Especially, the perturbative solution can be obtained in this equations systematically. To this end, one rewrites the BGR equation in the following symbolic form [49],

$$i\hbar\partial_t \mathbf{W} = (\mathbf{L} + \mathbf{N})\mathbf{W}, \tag{3.50}$$

where the column vector $\mathbf{W}$ is composed of f_i's and $\vec{g}_i$'s as follows,

$$\mathbf{W} = \begin{pmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \end{pmatrix}, \tag{3.51}$$

and each column vector is again

$$W_i = \begin{pmatrix} f_i \\ \vec{g}_i \end{pmatrix}. \tag{3.52}$$

The field free 16×16 matrix $\mathbf{L}$ is given by

$$\mathbf{L} = -i \begin{pmatrix} 0 & \mathcal{R} & 0 & 0 \\ \mathcal{R} & 0 & 2m \cdot 1_4 & 0 \\ 0 & -2m \cdot 1_4 & 0 & {}^*\mathcal{R} \\ 0 & 0 & -{}^*\mathcal{R} & 0 \end{pmatrix}, \tag{3.53}$$

where 1_4 is a 4×4 unit matrix and

$$\mathcal{R} = \begin{pmatrix} 0 & \nabla_1 & \nabla_2 & \nabla_3 \\ \nabla_1 & 0 & 2p_3 & -2p_2 \\ \nabla_2 & -2p_3 & 0 & 2p_1 \\ \nabla_3 & 2p_2 & -2p_1 & 0 \end{pmatrix}, {}^*\mathcal{R} = \begin{pmatrix} 0 & 2p_1 & 2p_2 & 2p_3 \\ 2p_1 & 0 & -\nabla_3 & \nabla_2 \\ 2p_2 & \nabla_3 & 0 & -\nabla_1 \\ 2p_3 & -\nabla_2 & \nabla_1 & 0 \end{pmatrix}. \tag{3.54}$$

The field part, denoted by $\mathbf{N}$, is

$$\mathbf{N} = -i \begin{pmatrix} \mathcal{E} & \mathcal{B} & 0 & 0 \\ \mathcal{B} & \mathcal{E} & 0 & 0 \\ 0 & 0 & \mathcal{E} & {}^*\mathcal{B} \\ 0 & 0 & -{}^*\mathcal{B} & \mathcal{E} \end{pmatrix}, \tag{3.55}$$

where the electric field part is a diagonal matrix,

$$\mathcal{E} = \begin{pmatrix} \mathcal{E}^0 & 0 & 0 & 0 \\ 0 & \mathcal{E}^0 & 0 & 0 \\ 0 & 0 & \mathcal{E}^0 & 0 \\ 0 & 0 & 0 & \mathcal{E}^0 \end{pmatrix}, \tag{3.56}$$

and magnetic field parts are

$$\mathcal{B} = \begin{pmatrix} 0 & \mathcal{B}_1^0 & \mathcal{B}_2^0 & \mathcal{B}_3^0 \\ \mathcal{B}_1^0 & 0 & \mathcal{B}_3^1 & -\mathcal{B}_2^1 \\ \mathcal{B}_2^0 & -\mathcal{B}_3^1 & 0 & \mathcal{B}_1^1 \\ \mathcal{B}_3^0 & \mathcal{B}_2^1 & -\mathcal{B}_1^1 & 0 \end{pmatrix}, {}^*\mathcal{B} = \begin{pmatrix} 0 & \mathcal{B}_1^1 & \mathcal{B}_2^1 & \mathcal{B}_3^1 \\ \mathcal{B}_1^1 & 0 & -\mathcal{B}_3^0 & \mathcal{B}_2^0 \\ \mathcal{B}_2^1 & \mathcal{B}_3^0 & 0 & -\mathcal{B}_1^0 \\ \mathcal{B}_3^1 & -\mathcal{B}_2^0 & \mathcal{B}_1^0 & 0 \end{pmatrix}. \tag{3.57}$$

The integro–differential field operators are

$$\mathcal{E}^0 = e\int_{-1/2}^{1/2} d\lambda \vec{E}(\vec{r} + i\hbar\lambda\vec{\partial}_p, t)\cdot\vec{\partial}_p, \tag{3.58}$$

$$\mathcal{B}^0 = \frac{e}{c}\int_{-1/2}^{1/2} d\lambda \vec{B}(\vec{r} + i\hbar\lambda\vec{\partial}_p, t)\times\vec{\partial}_p, \tag{3.59}$$

$$\mathcal{B}^1 = -\frac{ie\hbar}{c}\int_{-1/2}^{1/2} d\lambda\lambda \vec{B}(\vec{r} + i\hbar\lambda\vec{\partial}_p, t)\times\vec{\partial}_p, \tag{3.60}$$

where for later convenience all $\hbar$ have been put in explicitly. One introduces a retarded Green's function which satisfies the causality,

$$(i\hbar\partial_t - \mathrm{L})G_{ret}(\vec{r}, \vec{p}, t) = \delta(t)\delta(\vec{r}). \tag{3.61}$$

Note that the Green's function is 16×16 matrix. Then the BGR equations can be written in integral form as follows,

$$\mathbf{W}(\vec{r},\vec{p},t) = \mathbf{W}^0(\vec{r},\vec{p},t) + \int dt'd^3r'G_{ret}(\vec{r}-\vec{r}',\vec{p},t-t')\mathrm{N}(\vec{r}',\vec{p},t')\mathbf{W}(\vec{r}',\vec{p},t'), \tag{3.62}$$

where $\mathbf{W}^0$ is the field free solution. It is easy to see that this is the solution of BGR equation Eq.(3.50) by multiplying $(i\hbar\partial_t - \mathrm{L})$ from the left. While this integral equation is equivalent to the set of BGR equations, Eq.(3.32 - 3.37), except for the explicit implementation of the boundary condition, it has little advantage from the calculational point of view since the Green's function has spacial derivatives in it. To make further advance one introduces a Fourier transform akin to the case of non-relativistic Wigner formulation in integral equation form.

For any function A the Fourier transformation is

$$A(\vec{r},t) = \int \frac{d\omega}{2\pi\hbar} \int \frac{d^3q}{(2\pi\hbar)^3} e^{-i\omega t + i\vec{q}\cdot\vec{r}} A(\vec{q},\omega). \tag{3.63}$$

Thus the integral form of BGR equations after Fourier transformation is

$$\tilde{\mathbf{W}}(\vec{q},\vec{p},\omega) = \tilde{\mathbf{W}}^0(\vec{q},\vec{p},\omega) + \tilde{G}(\vec{q},\vec{p},\omega) \int d\Gamma' \tilde{N}(\vec{q}-\vec{q}\,',\vec{p},\omega-\omega')\tilde{\mathbf{W}}(\vec{q}\,',\vec{p},\omega') \tag{3.64}$$

where

$$d\Gamma' = d\vec{q}\,'d\omega'/(2\pi\hbar)^4. \tag{3.65}$$

Since the spacial derivative $\vec{\nabla}$ is replaced by $i\vec{q}$ in $\tilde{L}$, the Green's function is the solution of the algebraic equation,

$$(\omega - \tilde{L})\tilde{G}(\vec{q},\vec{p},\omega) = 1, \tag{3.66}$$

where the $i\epsilon$ prescription was implicitly assumed to incorporate the boundary condition, the causality, i.e. $\omega = \omega + i\epsilon$, with an infinitesimal positive number ϵ. Since the Green's function, 16×16 matrix, is too big to obtain its inverse matrix directly, one needs to partition the matrix $(\omega - \tilde{L})$ as follows,

$$\omega - \tilde{L} = A = \begin{pmatrix} A_{11}(8\times 8) & A_{12}(8\times 8) \\ A_{21}(8\times 8) & A_{22}(8\times 8) \end{pmatrix}. \tag{3.67}$$

and use the relations for the partitioned matrix,

$$AB = BA = 1, \tag{3.68}$$

so that

$$B_{11} = \left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1}, \tag{3.69}$$

$$B_{12} = -B_{11}A_{12}A_{22}^{-1}, \tag{3.70}$$

$$B_{21} = -A_{22}^{-1}A_{21}B_{11}, \tag{3.71}$$

$$B_{22} = A_{22}^{-1} - B_{21}A_{12}A_{22}^{-1}, \tag{3.72}$$

where A_{22} and $A_{11} - A_{12}A_{22}^{-1}A_{21}$ are not singular. The matrix B is of course the Green's function we seek. I have done this calculation explicitly by using *Mathematica* and the set of integral equations is given explicitly in Appendix D.

These integro–differential equations are equivalent to the original BGR equations. The advantages of this formulation are that one can obtain the iterative (also perturbative) solutions for the given field and in particular one can perform the normalization of the theory. Further this formulation also allows an intuitive physical interpretation. Namely, one studies the pole structures of the integral equations which have four poles, two of them are acoustic poles and two are pair production poles. For further information, see the reference [49].

CHAPTER 4

Classical Limit of Quantum Transport Theory[1]

Perhaps the most profound reason to develop the Wigner phase space theory is to scrutinize the close resemblance between the classical and quantum distributions, which can improve the understanding of the classical interpretation of quantum physics. Therefore I here first turn to study the classical limit of the relativistic matter field. I first uncover the difficulties that occur when the classical limit ($\hbar \rightarrow 0$) of relativistic quantum phase space is studied. I resolve these folding the Wigner function with a Gaussian, non-covariant coarse graining function. I then expand the smeared Wigner function using $\hbar$ as the expansion parameter.

The classical limit will be achieved by taking the limit of $\hbar \rightarrow 0$ [51, 52].

I will in that way obtain a set of differential equations between the BGR functions. In order to give them appropriate physical meaning suitable manipulations have to be performed and definitions introduced in order to arrive at the distribution functions we are familiar in the classical limit. Namely, in relativistic quantum transport theory we have to deal with probabilities which naturally encompass the particle and antiparticle duality inherent in the relativistic formulation. It was already noted in reference [48] that in a particular treatment of the theory the relativistic Vlasov equation, which describes the classic flow of charged relativistic particles subject to the electro-magnetic field, emerges naturally from the single time formulation of the relativistic quantum Wigner function. I will show here that in the classical limit Vlasov equations for both the particle and antiparticle fluids emerge. I will then turn to study other, in classical limit non-vanishing distributions, and relate these to the motion of the spin and magnetic moment distributions.

[1]Portions of this chapter have been published in [50]

4.1 Nonrelativistic Theory

To begin our study of the problem of the $\hbar$–expansion in the limit of $\hbar \to 0$, let us take a simple example, namely, the Wigner function of a plane wave in 1–dimension,

$$\psi(x) = e^{+ipx/\hbar}. \tag{4.1}$$

The nonrelativistic Wigner function is:

$$\begin{aligned} W^{(1)}(x,p,t) &= \int \frac{ds}{2\pi} e^{-iq(x/\hbar - s/2)} e^{+iq(x/\hbar + s/2)} e^{-ips} \\ &= \delta(p-q), \end{aligned} \tag{4.2}$$

where I have made all $\hbar$ explicit. This Wigner function corresponds to sharp momentum and completely delocalized configuration space occupancy. The $\hbar \to 0$ problem [34] occurs first when I consider a wavefunction consisting of two plane waves, i.e.,

$$\psi(x) = e^{+ip_1x/\hbar} + e^{+ip_2x/\hbar}, \tag{4.3}$$

(here I took the weighing factor 1 for each plane wave). The corresponding Wigner function is:

$$W^{(2)}(x,p,t) = \delta(p-p_1) + \delta(p-p_2) + 2\cos[\frac{(p_1-p_2)x}{\hbar}]\,\delta(p-\frac{p_1+p_2}{2}). \tag{4.4}$$

The last interference term between two plane waves is singular in the limit of $\hbar \to 0$ unless $x = 0$. To remove this rapid oscillation I coarse grain, folding Eq.(4.4) with a Gaussian, normalized distribution function,

$$G_\lambda(x) = \frac{1}{\lambda\sqrt{\pi}} e^{-x^2/\lambda^2}. \tag{4.5}$$

After folding I find that the last term vanishes for small $\hbar$ at finite λ_r,

$$\int dx' \frac{1}{\lambda_r\sqrt{\pi}} e^{-(x-x')^2/\lambda_r^2} \cos[\frac{p_1-p_2}{\hbar}x'] = e^{-\frac{\lambda_r^2}{4}(\frac{p_1-p_2}{\hbar})^2} \cos[\frac{p_1-p_2}{\hbar}x] \to 0. \tag{4.6}$$

Here the last relation arises from the fact that the Gaussian term suppress the highly oscillating cosine term in the limit $\hbar \to 0$. Hence the coarse grained Wigner function for two plane waves is

$$\tilde{W}^{(2)}(x,p,t) = \delta(p-p_1) + \delta(p-p_2). \tag{4.7}$$

In general, let us consider the wave function of the form;

$$\psi(x) = \int dq\, \phi(q) e^{iqx/\hbar}, \tag{4.8}$$

where the coefficient function is

$$\phi(q) \;=\; \sum_i c_i\delta(q_i) + \varphi(q). \tag{4.9}$$

The function φ should be non-singular and could be a function of $\hbar$. The Wigner function is then

$$\begin{aligned} W(x,p,t) \;=\; & \sum_i c_i^2\delta(p-q_i) + 2\sum_{i>j} c_i c_j \delta(p-\frac{q_i+q_j}{2})\cos\frac{q_i-q_j}{\hbar}x \\ & +\frac{1}{\pi}\sum_i c_i\varphi(2p-q_i)e^{-2ipx/\hbar} + \frac{1}{\pi}\sum_i c_i\varphi^*(2p-q_i)e^{2ipx/\hbar} \\ & +\frac{1}{\pi}\int dq\,\varphi(q)\varphi^*(2p-q)e^{2i(p-q)x/\hbar}. \end{aligned} \tag{4.10}$$

The rapid oscillations in the limit $\hbar \to 0$ will disappear for a smooth function $\varphi(q)$ after I fold the Wigner function with the coarse graining function Eq.(4.5) as I have seen in the case of two plane waves. We thus can conclude that this coarse graining procedure throws away the quantum interference, which is a prominent signature of quantum physics. What this teaches us is that a smooth classical limit $\hbar \to 0$ of quantum distributions requires coarse graining of the Wigner function.

After performing the coarse graining, I expand the Wigner function around $\hbar = 0$ as follows,

$$W_s(x,p,t) \;=\; W_s^c(x,p,t) + \hbar W_s^{(1)}(x,p,t) + \hbar^2 W_s^{(2)}(x,p,t) + \ldots. \tag{4.11}$$

The transport equation Eq.(1.43) becomes in the limit $\hbar \to 0$,

$$\left(\partial_t + \frac{1}{m}\vec{p}\cdot\vec{\nabla} + \vec{F}(\vec{r})\cdot\vec{\partial}\right) W_s^c(\vec{r},\vec{p},t) \;=\; 0, \tag{4.12}$$

which is just the Vlasov equation. In fact, the coarse grained Wigner function is positive definite so that I can interpret W_s^c as a probability to find the particle at $(\vec{r},\vec{p})$ at time t.

4.2 The Classical Limit of Relativistic Wigner Function

Now I apply the idea of coarse graining to the relativistic formulation. This case needs more care since the theory has particle and antiparticle sector mixed together. To begin with, I rewrite the DHW function making $\hbar$ explicit,

$$W_{\alpha\beta}(\vec{r},\vec{p},t) = -\frac{1}{2}\int d^3s \, \exp(-i\vec{p}\cdot\vec{s}) < \Psi| \exp\left(-\frac{ie}{c}\int_{-1/2}^{1/2} dl \, \vec{s}\cdot\vec{A}(\vec{r}+l\hbar\vec{s})\right)$$
$$[\psi_\alpha(\vec{r}+\hbar\vec{s}/2,t),\psi^\dagger_\beta(\vec{r}-\hbar\vec{s}/2,t)]|\Psi>, \quad (4.13)$$

where the indices α and β go from 0 to 3. The coordinate $\vec{r}$ and momentum $\vec{p}$ are the (classical) phase space variables and hence do not contain $\hbar$. The vector potential $\vec{A}$, chosen in the Heisenberg–Pauli(temporal) gauge, is in principle composed of two contributions. One is the external sources which do not undergo the dynamic evolution here discussed and are assumed to be smooth even in (microscopic) world. The other is the so–called *back reaction* terms, arising from the induced matter field source current, which include the highly oscillating part (*Zitterbewegung*) of frequency $\nu \sim 2mc^2/\hbar$ with a mean distance $\lambda_e \sim \hbar/mc$ (for QED, $\nu \approx 2\ 10^{21}$ and $\lambda_e \approx 380$ fm). I assume that this quantum fluctuation will be averaged out by coarse graining procedure. In this case (to be verified) the vector potential is a well–behaved function for $\hbar \to 0$. The rapid oscillation due to the quantum interference, originating from the field operators acting at two different places, should also be averaged out after I fold the relativistic Wigner function with the coarse graining function, as was the case in the nonrelativistic case. Given these considerations, I can work assuming that the coarse grained Wigner function has a physically well–defined limit $\hbar \to 0$ and I can try to find a consistent systematic $\hbar$–expansion.

To obtain the classical limit of the BGR transport theory, I introduce the Gaussian type coarse graining(folding) function G, which does not affect the gauge invariance symmetry:

$$G(\vec{r}-\vec{r}',\vec{p}-\vec{p}') = (\frac{1}{\pi\lambda_r\lambda_p})^3 e^{-(\vec{p}-\vec{p}')^2/\lambda_p^2-(\vec{r}-\vec{r}')^2/\lambda_r^2}, \quad (4.14)$$

with the new phase space cell dimension

$$\lambda_r\lambda_p \gg \hbar. \quad (4.15)$$

Note that I normalized the folding function G, so it becomes a delta function in the limit of $\lambda_p, \lambda_r \to 0$, i.e.,

$$G(\vec{r}-\vec{r}',\vec{p}-\vec{p}') \to \delta(\vec{r}-\vec{r}')\delta(\vec{p}-\vec{p}'). \quad (4.16)$$

Clearly, the folding procedure does not alter anything in that limit. However, I note that the coarse graining is meaningful only when the phase space volume $(\lambda_r \lambda_p)^3$ is considerably larger than the elementary quantum phase space volume $(2\pi\hbar)^3$.

The coarse grained BGR functions are, by definition:

$$F_s(\vec{r},\vec{p},t) = \int d\vec{r}'d\vec{p}'G(\vec{r}-\vec{r}',\vec{p}-\vec{p}')F(\vec{r}',\vec{p}',t), \tag{4.17}$$

where $F = f_i$ or $\vec{g}_i$, $i = 0,1,2,3$ are all 16 BGR functions. As discussed, I suppose that the coarse graining procedure allows us to expand the smeared BGR functions around $\hbar = 0$. However, I have other problems: the first one is that the coarse grained BGR functions are in general not Lorentz invariant any more since the folding function itself is not, and the other is the possibility that even the Gaussian tail of the coarse graining function could introduce the artificial correlation between two phase space points far away which can impair some other physical properties. On the other hand, one could argue that the coarse graining with proper choice of λ_r, λ_p could embody the experimental procedure measurement [69].

From now on in this chapter, unless otherwise stated all BGR functions are assumed to have been coarse grained. Our objective first is to see what form will take the differential equations for the BGR functions. I expand using $\hbar$ as an expansion parameter all the 16 (coarse grained) BGR functions, which by assumption are analytic around $\hbar = 0$ and hence:

$$f_i = f_i^c + \hbar f_i^{(1)} + \hbar^2 f_i^{(2)} + ..., \tag{4.18}$$

$$\vec{g}_i = \vec{g}_i^c + \hbar \vec{g}_i^{(1)} + \hbar^2 \vec{g}_i^{(2)} + ..., \tag{4.19}$$

where $i = 0,1,2,3$. The effect of the differential operators can also be obtained in an $\hbar$-expansion, noting what they do to smeared functions — I obtain:

$$\begin{aligned} D_t &= D_t^c - \frac{e\lambda_r^2}{2}[\frac{4\pi}{\epsilon_0}\vec{\nabla}\rho(\vec{r}) + \frac{1}{c}\partial_t(\vec{\nabla}\times\vec{B}(\vec{r}))]\cdot\vec{\partial}_p \\ &+\frac{e\lambda_r^2}{2}\vec{\nabla}\cdot[(\vec{\partial}\cdot\vec{\nabla})\vec{E}(\vec{r}) - \frac{1}{c}\vec{\partial}_p\times\partial_t\vec{B}(\vec{r})] - \frac{e\hbar^2}{24}(\vec{\nabla}'\cdot\vec{\partial})^2\vec{E}\cdot\vec{\partial} + ..., \end{aligned} \tag{4.20}$$

$$\begin{aligned} \vec{D} &= \vec{D}^c - \frac{e\lambda_r^2}{2c}\frac{\mu_0\epsilon_0}{c}\partial_t(\vec{\nabla}\times\vec{E}(\vec{r})) - \frac{2\pi e\mu_0\lambda_r^2}{c^2}(\vec{\nabla}\times\vec{J}(\vec{r}))\times\vec{\partial}_p \\ &+\frac{e\lambda_r^2}{2c}\vec{\nabla}\cdot\vec{\nabla}(\vec{B}(\vec{r})\times\vec{\partial}_p) - \frac{e\hbar^2}{24c}(\vec{\nabla}'\cdot\vec{\partial})^2\vec{B}\times\vec{\partial} + ..., \end{aligned} \tag{4.21}$$

$$\vec{P} = \vec{p} + \frac{\lambda_p^2}{2}\vec{\partial}_p - \frac{e\hbar^2\lambda_r^2}{48c}\frac{\mu_0\epsilon_0}{c}\partial_t\left(\vec{\nabla}\times(\vec{\partial}\cdot\vec{\nabla}')\vec{E}(\vec{r})\right)$$
$$-\frac{\pi e\mu_0\hbar^2\lambda_r^2}{12c^2}\left(\vec{\nabla}\times(\vec{\partial}\cdot\vec{\nabla}')\vec{J}(\vec{r})\right)\times\vec{\partial}_p$$
$$+\frac{e\hbar^2\lambda_r^2}{48c}\vec{\nabla}\cdot\vec{\nabla}\left((\vec{\partial}\cdot\vec{\nabla}')\vec{B}(\vec{r})\times\vec{\partial}_p\right) + \frac{e\hbar^2}{12c}(\vec{\nabla}'\cdot\vec{\partial})\vec{B}\times\vec{\partial} + ..., \tag{4.22}$$

where $\vec{\nabla}$ is the spacial derivative and the prime on it indicates that $\vec{\nabla}$ operates only on the electromagnetic field very next to it. The semiclassical portions of the operator are:

$$D_t^c = \partial_t + e\vec{E}\cdot\vec{\partial}, \tag{4.23}$$
$$\vec{D}^c = \vec{\nabla} + \frac{e}{c}\vec{B}\times\vec{\partial}, \tag{4.24}$$

where $\vec{\partial}$ is the momentum derivative.

Inserting these expressions into BGR equations Eqs.(3.32-3.39), I obtain the following relations among BGR functions which arises in $\hbar^{-1}$ order[2],

$$f_1^c = \frac{1}{mc}\vec{p}\cdot\vec{g}_3^c, \tag{4.25}$$
$$f_2^c = 0, \tag{4.26}$$
$$\vec{g}_1^c = \frac{1}{mc}\vec{p}f_3^c, \tag{4.27}$$
$$\vec{g}_2^c = \frac{1}{mc}\vec{p}\times\vec{g}_0^c. \tag{4.28}$$

where I dropped those terms, which are proportional to λ_r or λ_p, since I assume that λ_r is some value characteristic of an inter–atomic distance, Å, still very small in the macroscopic world and λ_p also small enough even if $\lambda_r\lambda_p \gg \hbar$. I implicitly assume this approximation in each order of $\hbar$. Eq.(4.26) tells us that f_2 (pseudo-scalar density distribution) is a genuine quantum function, which vanishes exactly for $\hbar \to 0$. f_3^c, $\vec{g}_0^c$ and $\vec{g}_3^c$ fix the values of f_1^c, f_2^c, $\vec{g}_1^c$ and $\vec{g}_2^c$. In addition I note that f_0^c is in general non-vanishing, independent function to this order in $\hbar$.

Eq.(4.27) suggests that (f_0, f_3) are charge and mass distributions of particles and holes rather than particles and antiparticles because the current density is proportional to the mass density but not to the charge density. This will be clear when I discuss Vlasov equations below.

[2]This indicates that the expansion before coarse graining is non-analytical in $\hbar$

In next order in $\hbar$, I find that the classical limit equations of motion *are driven by quantum corrections* as follows;

$$\begin{aligned}
D_t^c f_1^c + c\vec{D}^c \cdot \vec{g}_0^c &= -2mc^2 f_2^{(1)} , & (4.29)\\
D_t^c f_3^c &= 2c\vec{p} \cdot \vec{g}_2^{(1)} , & (4.30)\\
D_t^c \vec{g}_0^c + c\vec{D}^c f_1^c &= 2c\vec{p} \times \vec{g}_1^{(1)} , & (4.31)\\
D_t^c \vec{g}_1^c + c\vec{D}^c f_0^c &= 2c\vec{p} \times \vec{g}_0^{(1)} - 2mc^2 \vec{g}_2^{(1)} , & (4.32)\\
D_t^c \vec{g}_2^c + c\vec{D}^c \times \vec{g}_3^c &= -2c\vec{p} f_3^{(1)} + 2mc^2 \vec{g}_1^{(1)} , & (4.33)\\
D_t^c \vec{g}_3^c - c\vec{D}^c \times \vec{g}_2^c &= 2c\vec{p} f_2^{(1)} , & (4.34)
\end{aligned}$$

while there also arise two constraints between the functions, one for the (c)-order and one for the (1)-order in $\hbar$ expansion:

$$\begin{aligned}
0 &= D_t^c f_0^c + c\vec{D}^c \cdot \vec{g}_1^c, & (4.35)\\
0 &= 2c\vec{p} \cdot \vec{g}_3^{(1)} - 2mc^2 f_1^{(1)} , & (4.36)
\end{aligned}$$

I shall next show that the dynamical equations in the classical limit really arise using these equations in order $\hbar$. It is therefore incorrect to naively neglect all the quantum corrections in order to obtain the classical limit.

4.3 Vlasov Equation

Multiplying Eq.(4.32) by $\vec{p}$ and substituting that into Eq.(4.30) to eliminate $\vec{g}_2^{(1)}$, I obtain,

$$0 = D_t^c f_3^c + \frac{1}{mc}\vec{p} \cdot D_t^c \vec{g}_1^c + \frac{1}{m}\vec{p} \cdot \vec{D}^c f_0^c \ . \qquad (4.37)$$

I use the constraint Eq.(4.27) in Eq.(4.35) and Eq.(4.37) to obtain the symmetric expressions,

$$\begin{aligned}
0 &= \partial_t f_0^c + \vec{v} \cdot \vec{\nabla} f_3'^c + e\vec{E} \cdot \vec{\partial}_p f_0^c + \frac{e}{c}\vec{v} \times \vec{B} \cdot \vec{\partial}_p f_3'^c , & (4.38)\\
0 &= \partial_t f_3'^c + \vec{v} \cdot \vec{\nabla} f_0^c + e\vec{E} \cdot \vec{\partial}_p f_3'^c + \frac{e}{c}\vec{v} \times \vec{B} \cdot \vec{\partial}_p f_0^c , & (4.39)
\end{aligned}$$

where I put

$$f_3'^c = \frac{E_p}{mc^2} f_3^c, \qquad (4.40)$$

and the energy and velocity are,

$$E_p = \sqrt{\vec{p}^2c^2 + m^2c^4}, \tag{4.41}$$
$$\vec{v} = \vec{p}c^2/E_p. \tag{4.42}$$

I now introduce particle and antiparticle phase space distributions such that the number of particles and antiparticles in vacuum state are equal to each other:

$$f^c_\pm(\vec{p}) = \pm\frac{1}{2}[f^c_0(\pm\vec{p}) \pm f_3'^c(\pm\vec{p})] = \pm\frac{1}{2}[f^c_0(\pm\vec{p}) \pm \frac{E_p}{mc^2}f^c_3(\pm\vec{p})], \tag{4.43}$$

and I then obtain,

$$\partial_t f^c_\pm + \vec{v}\cdot\vec{\nabla}f^c_\pm \pm e(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B})\cdot\vec{\partial}_p f^c_\pm = 0, \tag{4.44}$$

which is the relativistic particle and antiparticle Vlasov equation as can be seen noting the signs of the EM–charge.

Integrating Eq.(4.44) over $\vec{p}$ results in the continuity equations for particle and antiparticle,

$$0 = \partial_t\rho^c_\pm + \vec{\nabla}\cdot\vec{j}^c_\pm\ , \tag{4.45}$$

where

$$\rho^c_\pm = \int f^c_\pm d\vec{p}, \tag{4.46}$$
$$\vec{j}^c_\pm = \int \frac{\vec{p}c^2}{E_p}f^c_\pm d\vec{p}. \tag{4.47}$$

I thus find that the number of particle and antiparticle is conserved separately in the classical limit. This remark clearly cannot be true in general, and thus I can already conclude that the particle (pair) production is a non-perturbative process in $\hbar$.

I note that this derivation extends the preliminary result presented in Ref.[48] in which it is valid also when $f^c_- \neq 0$. In Ref.[48] it was implicitly assumed that only particles could be present, while the current development shows that in a relativistic transport theory classical particle and antiparticle flow occurs at equal footing. Proper perception of this point is a necessary requirement to adequately describe the process of pair production and its relation to flow.

4.4 BMT Equations in Phase Space

I can in the same way treat the vector function $\vec{g}_i$: I combine Eqs.(4.29,4.34) to eliminate $f_2^{(1)}$,

$$D_t^c \vec{g}_3^c - c\vec{D}^c \times \vec{g}_2^c + \frac{\vec{p}}{m}[D_t^c f_1^c + c\vec{D}^c \cdot \vec{g}_0^c] = 0. \tag{4.48}$$

Using the constraints Eqs.(4.25,4.28) and manipulating the vector relations one obtains

$$D_t^c \vec{g}_3^c + \frac{1}{(mc)^2}\vec{p} D_t^c(\vec{p}\cdot\vec{g}_3^c) - \frac{1}{m}\vec{D}^c \times (\vec{p}\times\vec{g}_0^c) + \frac{1}{m}\vec{p}(\vec{D}^c\cdot\vec{g}_0^c) = 0. \tag{4.49}$$

On the other hand, I take vector product of Eq.(4.33) with $\vec{p}$ and substitute that into Eq.(4.31) to obtain

$$D_t^c \vec{g}_0^c - \frac{1}{(mc)^2}\vec{p}\times D_t^c(\vec{p}\times\vec{g}_0^c) - \frac{1}{m}\vec{p}\times(\vec{D}^c\times\vec{g}_3^c) + \frac{1}{m}\vec{D}^c(\vec{p}\cdot\vec{g}_3^c) = 0. \tag{4.50}$$

After I manipulate these two equations Eqs.(4.49,4.50), they finally become

$$\begin{aligned} 0 &= \partial_t \vec{g}_0^c - \frac{c^2}{E_p^2}\vec{p}(\vec{p}\cdot\partial_t\vec{g}_0^c) + e(\vec{E}\cdot\vec{\partial})\vec{g}_0^c \\ &\quad -\frac{ec^2}{E_p^2}\{(\vec{p}\cdot\vec{g}_0^c)\vec{E} - (\vec{p}\cdot\vec{E})\vec{g}_0^c + \vec{p}[\vec{p}\cdot(\vec{E}\cdot\vec{\partial})\vec{g}_0^c]\} \\ &\quad +\frac{1}{m}\frac{mc^2}{E_p}(\vec{p}\cdot\vec{\nabla})\vec{g}_3'^c + \frac{e}{mc}\frac{mc^2}{E_p}\vec{B}\times\vec{g}_3'^c + \frac{e}{mc}\frac{mc^2}{E_p}[\vec{p}\cdot\vec{B}\times\vec{\partial}]\vec{g}_3'^c, \end{aligned} \tag{4.51}$$

$$\begin{aligned} 0 &= \partial_t \vec{g}_3'^c + \frac{1}{m^2c^2}\vec{p}(\vec{p}\cdot\partial_t\vec{g}_3'^c) + e(\vec{E}\cdot\vec{\partial})\vec{g}_3'^c + \frac{ec^2}{E_p^2}(\vec{E}\cdot\vec{p})\vec{g}_3'^c \\ &\quad +\frac{e}{m^2c^2}\{(\vec{E}\cdot\vec{g}_3'^c)\vec{p} + \vec{p}[\vec{p}\cdot(\vec{E}\cdot\vec{\partial})\vec{g}_3'^c] + \frac{c^2}{E_p^2}(\vec{E}\cdot\vec{p})(\vec{p}\cdot\vec{g}_3'^c)\vec{p}\} \\ &\quad +\frac{1}{m}\frac{mc^2}{E_p}(\vec{p}\cdot\vec{\nabla})\vec{g}_0^c + \frac{e}{mc}\frac{mc^2}{E_p}\vec{B}\times\vec{g}_0^c + \frac{e}{mc}\frac{mc^2}{E_p}[\vec{p}\cdot\vec{B}\times\vec{\partial}]\vec{g}_0^c, \end{aligned} \tag{4.52}$$

where I have set

$$\vec{g}_3'^c = \frac{mc^2}{E_p}\vec{g}_3^c. \tag{4.53}$$

These are phase space version of Bargmann-Michel-Telegdi spin equation of motion [53, 54]. The great complexity of these equations arises very probably as consequence of the fact that the BGR equations are not manifestly Lorentz covariant, even though they contain the full Poincare symmetry group implicitly in their structure. To see the detailed meaning of equations Eq.(4.51, 4.52), I study equations Eqs.(4.51, 4.52) in the particle rest frame so that the momentum vanishes in the external electromagnetic field,

$$\partial_t \vec{g}_+ + \frac{e}{mc}\vec{B}\times\vec{g}_+ = 0, \tag{4.54}$$

$$\partial_t \vec{g}_- - \frac{e}{mc}\vec{B}\times\vec{g}_- = 0, \tag{4.55}$$

where the particle and antiparticle spin density are

$$\vec{g}_{\pm} = \pm\frac{1}{2}[\vec{g}_3'^{c}(\pm\vec{p}) \pm \vec{g}_0^{c}(\pm\vec{p})], \quad (4.56)$$

These are just the classical equations of motion of spin with g–factor 2.

In conclusion of this chapter, I recall that the $\hbar$–expansioon around $\hbar = 0$ has been justified by considering the coarse grained Wigner function. I have obtained explicitly the classical dynamical equations of the gauge invariant and relativistic transport theory. These are the Vlasov equations and a candidate for a phase space version of Bargmann-Michel-Telegdi spin equation for particle and antiparticle fluids respectively. I further have shown that in order to obtain a closed set of dynamical equations in the classical limit, it is necessary to consider the first order $\hbar$–corrections of the Wigner functions. In many ways the classical limit presented here constitutes the construction of a consistent relativistic classical limit of the Dirac field.

CHAPTER 5

Explicit Forms of BGR Function

It is interesting in developing understanding of the relativistic quantum transport theory to see next some explicit forms of single time relativistic Wigner functions or BGR functions for certain given quantum states. In section 1, I obtain the BGR functions for the $J^\pi = \frac{1}{2}^+$ state and show that the BGR functions are indeed the appropriate generalization of a nonrelativistic Wigner function. I also present the phase space version of the Landau orbit in a constant magnetic field in section 2.

5.1 Hydrogen-like 1s state and Cavity state[1]

I obtain the explicit form of the BGR functions for the $J^\pi = \frac{1}{2}^+$ Coulomb-like state and cavity relativistic state of the MIT bag model. Our objectives of this section is to show that 4×4 matrix DHW-function is the appropriate generalization of a nonrelativistic density matrix/Wigner function. As a incidental but interesting issue I also study the sharing of the total angular momentum of a quantum state between the spin and rotational degrees of freedom, arising from the spin-orbit coupling. Such a investigation could not rigorously have been done before. The relativistic spin 1/2 particle in a given quantum wave function ψ_α can be described by the 16 gauge invariant Wigner functions which are derived from the definition of DHW function [48],

$$W_{\alpha\beta}(\vec{r},\vec{p},t) = \int d\vec{x} e^{-i\vec{p}\cdot\vec{x}} \psi_\alpha(\vec{r}+\vec{x}/2,t)\psi^\dagger_\beta(\vec{r}-\vec{x}/2,t) \exp\{-ie\int_{-1/2}^{+1/2} d\lambda \vec{x}\cdot\vec{A}(\vec{r}+\lambda\vec{x},t)\}, \quad (5.1)$$

where α and β are the spinor indices. The gauge term is an oscillatory phase in the case of a Coulomb potential which adds to each wave function a factor $\exp(\pm itV(\vec{r} \pm \lambda\vec{x}/2))$,

[1]Portions of this section have been published in [55]

and which disappears for the cavity state case – I will not discuss it further here.

For a localized stationary spin 1/2 particle in the $J^\pi = 1/2^+$ state the four component Dirac wave function is given by [56]

$$\Psi_s(\vec{r}) = \begin{pmatrix} G(r) \\ -i\vec{\sigma}\cdot\hat{r}F(r) \end{pmatrix}\chi_s, \tag{5.2}$$

where χ_s is the 2 component spin function and $\vec{\sigma}$ is the Pauli spin matrices.

Using the relation between the DHW functions and BGR functions in Appendix C, BGR functions can be written in the form:

$$\begin{aligned}
f_0 &= A + [(\vec{r})^2 + \tfrac{\hbar^2}{4}(\vec{\partial}_p)^2 - \hbar\vec{s}\cdot(\vec{r}\times\vec{\partial}_p)]D, & (5.3)\\
\vec{g}_0 &= \vec{s}A + [\hbar\vec{r}\times\vec{\partial}_p + 2\vec{r}(\vec{r}\cdot\vec{s}) + \tfrac{\hbar^2}{2}\vec{\partial}_p(\vec{\partial}_p\cdot\vec{s}) - \vec{s}(\vec{r}^2 + \tfrac{\hbar^2}{4}\vec{\partial}_p^2)]D, & (5.4)\\
f_1 &= [\hbar\vec{s}\cdot\vec{\partial}_p]B - [2\vec{s}\cdot\vec{r}]C, & (5.5)\\
\vec{g}_1 &= [\hbar\vec{\partial}_p - 2\vec{s}\times\vec{r}]B - [2\vec{r} + \hbar\vec{s}\times\vec{\partial}_p]C, & (5.6)\\
f_2 &= -[2\vec{s}\cdot\vec{r}]B - [\hbar\vec{s}\cdot\vec{\partial}_p]C, & (5.7)\\
\vec{g}_2 &= -[2\vec{r} + \hbar\vec{s}\times\vec{\partial}_p]B - [\hbar\vec{\partial}_p - 2\vec{s}\times\vec{r}]C, & (5.8)\\
f_3 &= A - [(\vec{r})^2 + \tfrac{\hbar^2}{4}(\vec{\partial}_p)^2 - \hbar\vec{s}\cdot(\vec{r}\times\vec{\partial}_p)]D, & (5.9)\\
\vec{g}_3 &= \vec{s}A - [\hbar\vec{r}\times\vec{\partial}_p + 2\vec{r}(\vec{r}\cdot\vec{s}) + \tfrac{\hbar^2}{2}\vec{\partial}_p(\vec{\partial}_p\cdot\vec{s}) - \vec{s}(\vec{r}^2 + \tfrac{\hbar^2}{4}\vec{\partial}_p^2)]D, & (5.10)
\end{aligned}$$

where I introduced the spin direction vector, $\vec{s} = \chi_s^\dagger\vec{\sigma}\chi_s$, which becomes $\vec{s} = (0, 0, \pm 1)$ for $\chi_s = \chi_\pm$. The following auxiliary functions have been used:

$$\begin{aligned}
A(\vec{r},\vec{p}) &= \int d\vec{x}\, e^{-i\vec{p}\cdot\vec{x}/\hbar} G(\vec{r}+\vec{x}/2)G(\vec{r}-\vec{x}/2), & (5.11)\\
B(\vec{r},\vec{p}) &= \mathcal{R}\mathrm{e}[\int d\vec{x}\, e^{-i\vec{p}\cdot\vec{x}/\hbar} G(\vec{r}+\vec{x}/2)\frac{F(\vec{r}-\vec{x}/2)}{|\vec{r}-\vec{x}/2|}], & (5.12)\\
C(\vec{r},\vec{p}) &= \mathcal{I}\mathrm{m}[\int d\vec{x}\, e^{-i\vec{p}\cdot\vec{x}/\hbar} G(\vec{r}+\vec{x}/2)\frac{F(\vec{r}-\vec{x}/2)}{|\vec{r}-\vec{x}/2|}], & (5.13)\\
D(\vec{r},\vec{p}) &= \int d\vec{x}\, e^{-i\vec{p}\cdot\vec{x}/\hbar} \frac{F(\vec{r}+\vec{x}/2)}{|\vec{r}+\vec{x}/2|}\frac{F(\vec{r}-\vec{x}/2)}{|\vec{r}-\vec{x}/2|}. & (5.14)
\end{aligned}$$

The total energy and charge contained in the state are the generators of the symmetry transformations and are given by Eqs.(3.46) and (3.47), respectively,

$$E = \int d\Gamma(c\vec{p}\cdot\vec{g}_1 + mc^2 f_3), \tag{5.15}$$

$$Q = \int d\Gamma f_0, \tag{5.16}$$

where the phase space volume $d\Gamma = d\vec{r}d\vec{p}/(2\pi)^3$. One easily finds that the charge Q and energy E have the same values as the spin-unpolarized quantities, namely, $E = E^+ = E^-$ and $Q = Q^+ = Q^-$ where $\pm$ represent the spin up and down, respectively.

Similarly, the total angular momentum is given by Eq.(3.49),

$$\begin{aligned} \vec{J} &= \int d\Gamma(\vec{r}\times\vec{p})f_0 + \int d\Gamma\frac{\hbar}{2}\vec{g}_0 \\ &= \vec{L} + \vec{S}, \end{aligned} \tag{5.17}$$

where the rotational degree of freedom is

$$\begin{aligned} \vec{L} &= \int d\Gamma(\vec{r}\times\vec{p})f_0(\vec{r},\vec{p},t) \\ &= \vec{s}\frac{\hbar}{2}\frac{4}{3}\int d\vec{r}F(r)F(r) \end{aligned} \tag{5.18}$$

and the spin is

$$\begin{aligned} \vec{S} &= \int d\Gamma(\frac{\hbar}{2})\vec{g}_0(\vec{r},\vec{p},t) \\ &= \vec{s}\frac{\hbar}{2}\int d\vec{r}\{G(r)G(r) - \frac{1}{3}F(r)F(r)\}. \end{aligned} \tag{5.19}$$

In results, the total angular momentum is as expected,

$$\begin{aligned} \vec{J} &= \vec{s}\frac{\hbar}{2}\int d\vec{r}\{G(r)G(r) + F(r)F(r)\} \\ &= \vec{s}\frac{\hbar}{2}Q \end{aligned} \tag{5.20}$$

where the last factor is unity for a normalized state with $Q = 1$. To see the ratio between the rotational degree of freedom and spin, I first consider the Hydrogen-like $1s$-atomic state [43] in which the wave function is given by,

$$G(r) = \frac{(2mZ\alpha)^{3/2}}{\sqrt{4\pi}}\left(\frac{1+\gamma}{2\Gamma(1+2\gamma)}\right)^{1/2}(2mZ\alpha r)^{\gamma-1}e^{-mZ\alpha r}, \tag{5.21}$$

$$F(r) = -G(r)\frac{1-\gamma}{Z\alpha}, \tag{5.22}$$

where α is the fine structure constant, Z the charge number and $\gamma = \sqrt{1 - Z^2\alpha^2}$. Substituting Eqs.(5.21) and (5.22) into Eqs.(5.18) and (5.19), I obtain

$$\vec{L} = \vec{s}\frac{\hbar}{2}\frac{2-2\gamma}{3} \tag{5.23}$$

and

$$\vec{S} = \vec{s}\frac{\hbar}{2}\frac{1+2\gamma}{3} , \tag{5.24}$$

In the nonrelativistic limit $\gamma \rightarrow 1 - Z^2\alpha^2/2$, I find that practically all angular momentum is in the spin density of the state for small Z. For $Z\alpha \rightarrow 1$, $\gamma \rightarrow 0$ and I find that 2/3 of the total angular momentum is in the rotational degrees of freedom. Figure (5.1) shows the ratio between rotational degree of freedom and spin as function of charge number Z. This

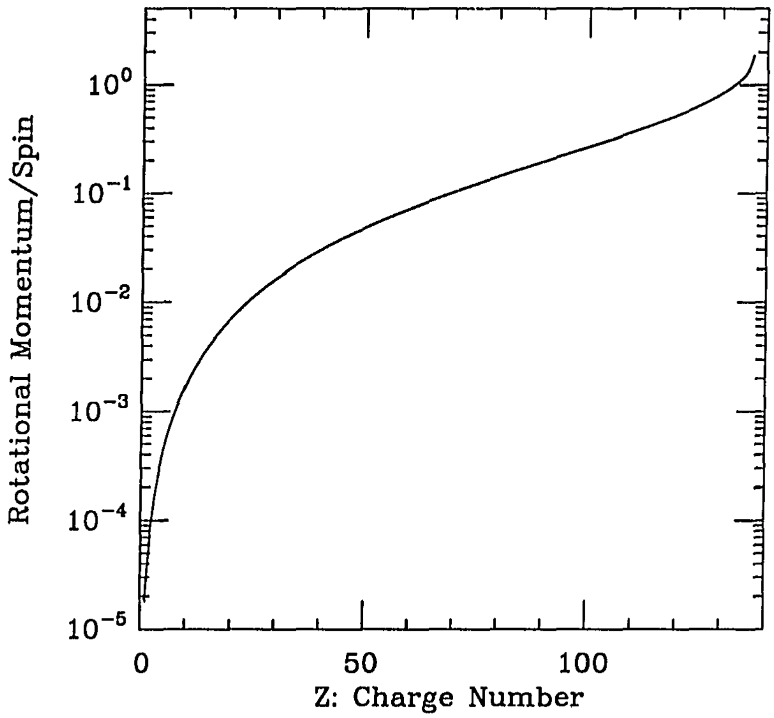

Figure 5.1: The Ratio of Rotational degree of freedom and Spin

figure tells us that all the angular momentum resides more in rotational degree of freedom than in the spin from $Z = 133$.

As another example of $J^\pi = \frac{1}{2}^+$, I take the bag type cavity states [57],

$$G(r) = \frac{N}{\sqrt{4\pi}} j_0(\frac{\omega r}{R}), \quad (5.25)$$

$$F(r) = -\frac{N}{\sqrt{4\pi}} j_1(\frac{\omega r}{R}), \quad (5.26)$$

where j_0 and j_1 are the Bessel functions, ω the eigen–energy and the normalization constant

$$N = \sqrt{\frac{\omega^3}{2R^3(\omega - 1)\sin^2\omega}}. \quad (5.27)$$

With a cavity of radius R=1 I obtain:

$$\begin{aligned} \vec{L} &= \vec{s}\frac{\hbar}{2}\{\frac{4}{3} - \frac{1}{3}\frac{\omega - \sin\omega\cos\omega}{(\omega-1)\sin^2\omega}\} \\ &\simeq \vec{s}\frac{\hbar}{2}0.35\,, \end{aligned} \quad (5.28)$$

$$\begin{aligned} \vec{S} &= \vec{s}\frac{\hbar}{2}\{\frac{1}{3}\frac{\omega - \sin\omega\cos\omega}{(\omega-1)\sin^2\omega} - \frac{1}{3}\} \\ &\simeq \vec{s}\frac{\hbar}{2}0.65\,, \end{aligned} \quad (5.29)$$

where the eigen-energy ω is 2.04 in natural units for $J^\pi = 1/2^+$, $\kappa = -1$ state. About a third of the spin is transferred in this case to the rotational degree of freedom. This interesting property of the relativistic cavity state was noted in the context of hadronic structure studies, as it reconciled the ratio of the weak interaction axial vector to vector coupling constants g_A/g_V with experiment [58]. Along this line I note the magnitude of the magnetic moment of a (confined) fermion

$$\vec{\mu} = \frac{1}{2}\int d\vec{r}(\vec{r} \times \vec{j}) \quad (5.30)$$

which can be written as,

$$\vec{\mu} = \frac{1}{2}\int d\Gamma\,(\vec{r} \times \vec{g}_1)\,, \quad (5.31)$$

and which yields $\mu_z = 0.2R$ for $\vec{s} = (0, 0, \pm 1)$ as in Ref.[58].

With these results I have conclusively demonstrated that the Wigner density matrix formulation presented for the strong field QED is the appropriate generalization of the Wigner density matrix concept of nonrelativistic quantum mechanics. I also have presented in this section a consistent discussion of spin-polarized relativistic $J^\pi = \frac{1}{2}^+$ states with emphasis on angular momentum sharing between rotational and spin degree of freedom.

5.2 BGR functions of Dirac Particle in a Constant Magnetic Field - Landau orbit

One other problem which I can also solve exactly is that of a Dirac particle in a constant magnetic field. Even though I do not have immediate applications of the resulting rich structures of the phase space distributions, the importance of Landau orbits in many areas of physics justifies this effort and certainly provides for additional experiences with the relativistic phase space theory.

I can begin from known wave functions which represent the so-called Landau orbits. Consider a spin 1/2 particle in a constant magnetic field which is in z-direction. To solve the Dirac equation in a constant magnetic field, I choose a gauge[2] such that

$$A^0 = A^1 = A^3 = 0, \tag{5.32}$$

$$A^2 = Bx, \tag{5.33}$$

where B is the magnetic field strength. The solutions [59] of Dirac equation in this gauge are for the spin-up

$$\Psi_{\uparrow,n}(\vec{r}) = Ne^{-iE_n t/\hbar + i(p_y y + p_z z)/\hbar} \begin{pmatrix} \varphi_n(\xi) \\ 0 \\ \frac{p_z}{E_n+mc^2}\varphi_n(\xi) \\ \frac{i\sqrt{eB\hbar 2(n+1)}}{E_n+mc^2}\varphi_{n+1}(\xi) \end{pmatrix} \tag{5.34}$$

where the energy E_n, harmonic oscillator wave function φ_n and normalization constant N are given

$$E_n^2 = m^2c^4 + p_z^2c^2 + eB\hbar c 2(n+1), \tag{5.35}$$

$$\varphi_n(\xi) = (\sqrt{\pi}n!2^n)^{-1/2}e^{-\xi^2/2}H_n(\xi), \tag{5.36}$$

$$\xi = \sqrt{\frac{eB}{\hbar}}(x + \frac{p_y c}{eB}), \tag{5.37}$$

$$N = \sqrt{\frac{E_n + mc^2}{2E_n}}, \tag{5.38}$$

[2]This is not the most symmetric gauge choice to define the Landau orbits, but it allows us to calculate analytically BGR functions for all quantum numbers; I will return to the symmetric gauge choice below.

and for the spin-down state

$$\Psi_{\downarrow,n}(\vec{r}) = Ne^{-iE_n t/\hbar + i(p_y y + p_z z)/\hbar} \begin{pmatrix} 0 \\ \varphi_n(\xi) \\ -\frac{i\sqrt{eB\hbar 2n}}{E_n + mc^2}\varphi_{n-1}(\xi) \\ -\frac{p_z}{E_n + mc^2}\varphi_n(\xi) \end{pmatrix}, \tag{5.39}$$

where

$$E_n^2 = m^2c^4 + p_z^2c^2 + eB\hbar c2n. \tag{5.40}$$

The wave functions are characterized by five independent quantum numbers; two linear momenta p_y and p_z, the integer n which quantizes the orbit radius and radial energy, the signature ϵ of the total energy, and the spin η.

For the gauge which I have chosen, the linear path integration of the phase factor of Eq.(5.1) can be done easily,

$$\int_{-1/2}^{1/2} d\lambda \vec{s}\cdot\vec{A}(\vec{r} + \lambda\vec{s}, t) = Bxs_y, \tag{5.41}$$

so that the Wigner functions are

$$W_{\alpha\beta}(\vec{r}, \vec{q}, t) = \int d\vec{s} e^{-i(\vec{q} - e\vec{A}(\vec{r}))\cdot\vec{s}/\hbar} \Psi_\alpha(\vec{r} + \vec{s}/2, t)\Psi_\beta^\dagger(\vec{r} - \vec{s}/2, t). \tag{5.42}$$

The Wigner function are for the state of energy E_n,

$$\begin{aligned} W_{\alpha\beta,n}(\vec{r}, \vec{q}, t) &= \int ds_y e^{-i(q_y - eBx - p_y)s_y/\hbar} \int ds_z e^{-i(q_z - p_z)s_z/\hbar} \\ &\quad \cdot N^2 c_\alpha c_\beta^* \int ds_x e^{-iq_x s_x/\hbar}\varphi_n(\xi_+)\varphi_{n'}^*(\xi_-) \qquad (5.43) \\ &= (2\pi\hbar)^2 N^2 c_\alpha c_\beta^* \delta(q_y - eBx - p_y)\delta(q_z - p_z) \\ &\quad \cdot\sqrt{\frac{\hbar}{eB}} \int ds'_x e^{-i\sqrt{\frac{\hbar}{eB}}q'_x s'_x/\hbar}\varphi_n(x' + s'_x/2)\varphi_{n'}(x' - s'_x/2) \qquad (5.44) \\ &= N^2 c_\alpha c_\beta^* \mathcal{L}_{p_y,p_z,nn'}(\vec{r}, \vec{q}), \qquad (5.45) \end{aligned}$$

where

$$x' = \sqrt{\frac{eB}{\hbar}}(x + \frac{p_y c}{eB}), \tag{5.46}$$

$$s' = \sqrt{\frac{eB}{\hbar}}s, \tag{5.47}$$

$$E_n^2 = m^2c^4 + p_z^2c^2 + eB\hbar c(2n + 1 + \eta), \tag{5.48}$$

and c_α, c_β are the coefficients in front of harmonic oscillator wave function φ except the plane wave and normalization constant in the wave function Ψ. The phase space form of the Landau orbit is

$$\begin{aligned}\mathcal{L}_{p_y,p_z,nn'} &= (2\pi\hbar)^2\delta(q_y - eBx - p_y)\delta(q_z - p_z) \\ &\cdot\sqrt{\frac{4\pi\hbar}{eB}}\sum_{lL} < lL|nn' > (-i)^l\varphi_l(\sqrt{\frac{2}{eB\hbar}}q_x)\varphi_L(\sqrt{2}x')\end{aligned} \quad (5.49)$$

To derive the last form, I used the Talmi transformation [60, 61]. The multiplication of two harmonic oscillator wave functions can be represented by two harmonic oscillators of CM and relative coordinates respectively,

$$\varphi_n(q_1)\varphi_{n'}(q_2) = \sum_{lL} < lL|nn' > \varphi_l(q)\varphi_L(Q), \quad (5.50)$$

where

$$q = (q_1 - q_2)/\sqrt{2}, \quad (5.51)$$

$$Q = (q_1 + q_2)/\sqrt{2}. \quad (5.52)$$

The Talmi transformation coefficient is given by

$$< lL|nn' > = (\frac{l!L!}{n!n'!})^{1/2}2^{-(n+n')/2}\sum_\gamma(-1)^\gamma\begin{pmatrix} n \\ l-\gamma\end{pmatrix}\begin{pmatrix} n' \\ \gamma\end{pmatrix}, \quad (5.53)$$

which has to satisfy the relation $n + n' = l + L$ to conserve the energy. I also used the Fourier transform of the harmonic oscillator wave function [62, 63]

$$\int e^{-ipq}\varphi_n(q)dq = (2\pi)^{1/2}(-i)^n\varphi_n(p). \quad (5.54)$$

Let us calculate explicitly the DHW functions and BGR functions of the ground ($n = 0$), spin-down state as an example. The H.O. wave function is given by for this state

$$\varphi_0(x) = \frac{1}{\pi^{1/4}}e^{-x^2/2}, \quad (5.55)$$

and the Talmi transformation coefficient is

$$< 00|00 > = 1. \quad (5.56)$$

I can see that the third component of the wave function Eq.(5.39) is zero because the integer n must satisfy $n \geq 0$. Therefore, I find the DHW functions as follows,

$$W^{\downarrow}_{11,0} = \frac{E_0 + mc^2}{2E_0}\mathcal{L}_{p_y,p_z,00}, \quad (5.57)$$

$$W^{\downarrow}_{13,0} = \frac{-p_z}{2E_0}\mathcal{L}_{p_y,p_z,00}, \tag{5.58}$$

$$W^{\downarrow}_{31,0} = W^{\downarrow}_{13,0}, \tag{5.59}$$

$$W^{\downarrow}_{33,0} = \frac{E_0 + mc^2}{2E_0}\left(\frac{p_z}{E_0 + mc^2}\right)^2\mathcal{L}_{p_y,p_z,00}, \tag{5.60}$$

and all others are zero. The Landau orbit has:

$$\mathcal{L}_{p_y,p_z,00} = (2\pi\hbar)^{5/2}\sqrt{\frac{2}{eB}}\delta(q_y - eBx - p_y)\delta(q_z - p_z) \cdot\frac{1}{\sqrt{\pi}}\exp\{-\frac{1}{eB\hbar}q_x^2 - \frac{eB}{\hbar}(x + \frac{p_y c}{eB})^2\}. \tag{5.61}$$

Note that $\vec{p}$ is quantum number and $\vec{q}$ the physical momenta here. This function shows that the (y, z) plane represents the classical Landau orbit and the (x, q_x)-plane has Gaussian distribution at the center $(x = -\frac{p_y c}{eB}, q_x = 0)$. Figure (5.2) shows the (x, q_x)-plane phase distribution in arbitrary scale with $B = 1, p_y = 1$ in $\hbar = c = e = 1$ unit. Notice that the energy can be positive or negative,

$$E_0 = \epsilon\sqrt{m^2c^4 + p_z^2c^2},\ \epsilon = \pm 1. \tag{5.62}$$

Using the relations between DHW and BGR functions in Appendix C, I obtain the BGR functions of the ground, spin-down state. They are

$$f^{\downarrow}_{0,0} = \mathcal{L}_{p_y,p_z,00}(\vec{r}, \vec{q}), \tag{5.63}$$

$$f^{\downarrow}_{1,0} = -\frac{p_z}{E_0}\mathcal{L}_{p_y,p_z,00}(\vec{r}, \vec{q}), \tag{5.64}$$

$$f^{\downarrow}_{2,0} = 0, \tag{5.65}$$

$$f^{\downarrow}_{3,0} = \frac{mc^2}{E_0}\mathcal{L}_{p_y,p_z,00}(\vec{r}, \vec{q}), \tag{5.66}$$

$$\vec{g}^{\downarrow}_{0,0} = -f^{\downarrow}_0\hat{k}, \tag{5.67}$$

$$\vec{g}^{\downarrow}_{1,0} = -f^{\downarrow}_1\hat{k}, \tag{5.68}$$

$$\vec{g}^{\downarrow}_{2,0} = 0, \tag{5.69}$$

$$\vec{g}^{\downarrow}_{3,0} = -f^{\downarrow}_3\hat{k}, \tag{5.70}$$

where the subscript 0 of BGR functions denotes the energy quantum number n. These BGR functions are one of the stationary solutions of BGR equations under a constant magnetic field. For this set of solutions, the charge density has a Gaussian distribution as seen in Fig.(5.2). The mass density has the Lorentz factor as I expected and the current density, $\vec{g}_1$, is just the multiplication of the charge density by the velocity. The spin density has the correct direction. While these functions properly describe what I expect, it is rather surprising to find that $\vec{g}_3$ has the Lorentz factor and f_1, which is believed to be related with *Zitterbewegung*, is the velocity times the charge density. Finally, note that the particle distribution, f_+, is not zero while f_-, the antiparticle distribution, vanishes.

A more symmetric gauge choice is[3],

$$A_x = -\frac{1}{2}By, \tag{5.71}$$

$$A_y = +\frac{1}{2}Bx, \tag{5.72}$$

$$A_z = 0. \tag{5.73}$$

I denote by n for the energy quantum number, ϵ the signature of energy eigenvalue, η the spin, p_z the z-component of momenta, and l the angular quantum number of orbit with respect to the origin. The following are the known relativistic Landau wave functions of good angular momentum:

$$\Psi_n^{\uparrow} = G^{-1}\begin{pmatrix} (mc^2+E_n)\varphi_n \\ 0 \\ cp_z\varphi_n \\ \sqrt{2e\hbar cB(n+1)}\varphi_{n+1} \end{pmatrix}, \tag{5.74}$$

where

$$E_n^2 = c^2p_z^2 + m^2c^4 + 2e\hbar cB(n+1), \tag{5.75}$$

$$\varphi_{n,l} = (-i)^n(2^{n+l+1}\pi l!n!)^{-1/2}\lambda^{-1}\exp[\frac{x^2+y^2}{4\lambda^2} + ip_zz/\hbar] \cdot\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)^n\left(\frac{x-iy}{\lambda}\right)^l\exp[-(x^2+y^2)/2\lambda^2], \tag{5.76}$$

$$G = \sqrt{2E(E+mc^2)}, \tag{5.77}$$

$$\lambda^2 = \hbar c/eB, \tag{5.78}$$

and

$$\Psi_n^{\downarrow} = G^{-1}\begin{pmatrix} 0 \\ (mc^2+E_n)\varphi_n \\ \sqrt{2e\hbar cBn}\varphi_{n-1} \\ -cp_z\varphi_n \end{pmatrix}, \tag{5.79}$$

[3]This is right choice for each individual Landau orbit of fixed angular momentum, but I can calculate explicitly BGR functions only for $n = l = 0$ state.

where

$$E_n^2 = c^2p_z^2 + m^2c^4 + 2e\hbar cBn, \tag{5.80}$$

For $l = n = 0$, spin-down state, I obtain the Wigner functions:

$$W_{\alpha\beta}^{\downarrow} = \frac{c_\alpha c_\beta^*}{2E_0(E_0 + mc^2)}\mathcal{L}_{00}(\vec{r}, \vec{q}), \tag{5.81}$$

$$\mathcal{L}_{00}(\vec{r}, \vec{q}) = 8\pi\hbar\delta(q_z - p_z)\exp\{-\frac{x^2+y^2}{2\lambda^2} - 4\lambda^2[(\frac{q_x}{\hbar} + \frac{eB}{2c}y)^2 + (\frac{q_y}{\hbar} - \frac{eB}{2c}x)^2]\}, \tag{5.82}$$

so that

$$W_{11}^{\downarrow} = \frac{E_0 + mc^2}{2E_0}\mathcal{L}_{00}, \tag{5.83}$$

$$W_{13}^{\downarrow} = \frac{-cp_z}{2E_0}\mathcal{L}_{00}, \tag{5.84}$$

$$W_{31}^{\downarrow} = W_{13}^{\downarrow}, \tag{5.85}$$

$$W_{33}^{\downarrow} = \frac{c^2p_z^2}{2E_0(E_0 + mc^2)}\mathcal{L}_{00}. \tag{5.86}$$

The BGR functions are then

$$f_0^{\downarrow} = \mathcal{L}_{00}, \tag{5.87}$$

$$f_1^{\downarrow} = -\frac{p_z}{E_0}\mathcal{L}_{00}, \tag{5.88}$$

$$f_2^{\downarrow} = 0, \tag{5.89}$$

$$f_3^{\downarrow} = \frac{mc^2}{E_0}\mathcal{L}_{00}, \tag{5.90}$$

$$\vec{g}_0^{\downarrow} = -f_0^{\downarrow}\hat{k}, \tag{5.91}$$

$$\vec{g}_1^{\downarrow} = -f_1^{\downarrow}\hat{k}, \tag{5.92}$$

$$\vec{g}_2^{\downarrow} = 0, \tag{5.93}$$

$$\vec{g}_3^{\downarrow} = -f_3^{\downarrow}\hat{k}. \tag{5.94}$$

I have substituted these solutions into the BGR equations to check if they are consistent. Keeping in mind that I explicitly used the charge "e" as the absolute value of an electronic charge, I have verified that these solutions satisfy the BGR equations.

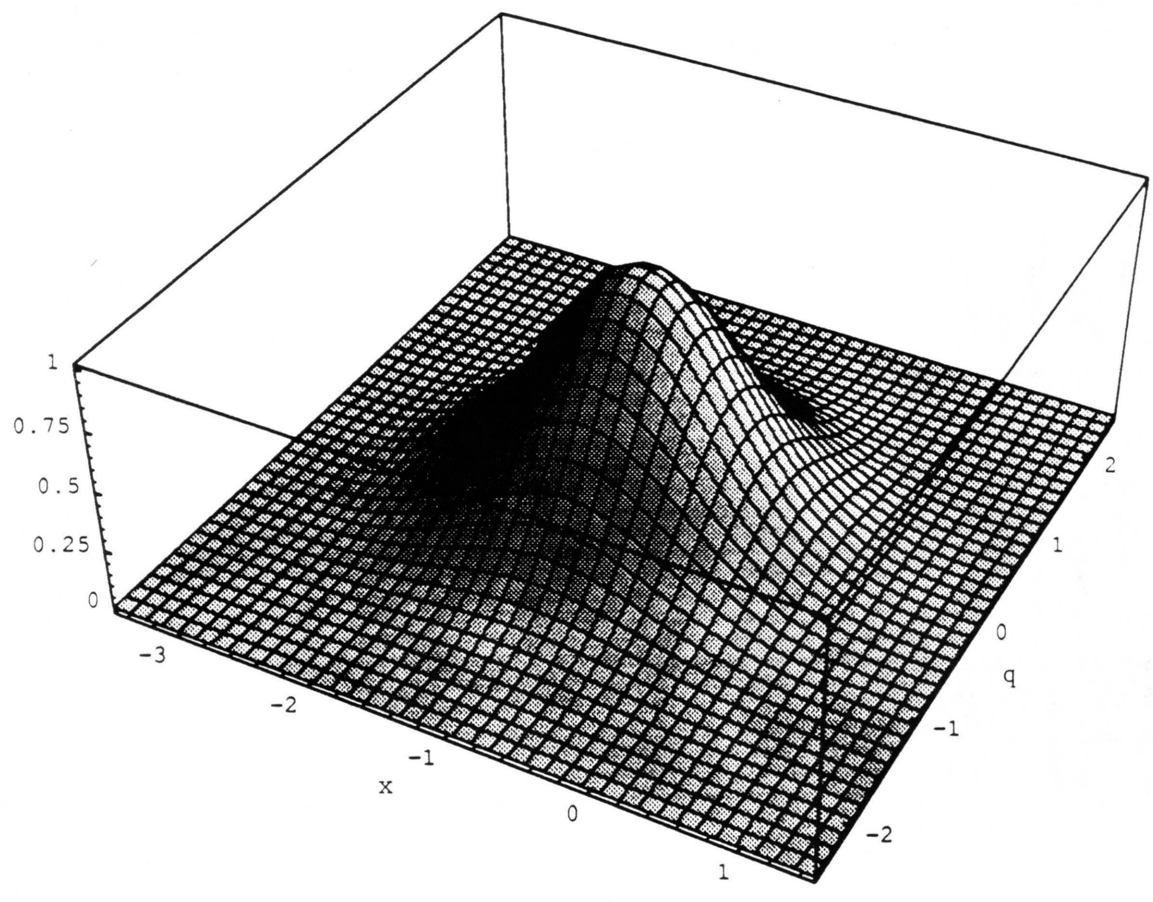

Figure 5.2: (x, q_x)-plane phase distribution of Phase Space Landau Orbit: $\mathcal{L}_{00}$

CHAPTER 6

Solutions of BGR Equations

In general solutions of the BGR equations are very difficult to obtain, as I have here 16 coupled differential equations of first order in 7 dimensional space, with often highly oscillatory behavior, in particular when particle production occurs. Therefore I can only hope to solve the BGR equations for some specific given symmetric cases with appropriate initial and/or boundary conditions, a task which I will begin to study in this chapter. In section 1, I consider two infinite conductive parallel plates on which the field lines end and recover the Schwinger particle production rate.

6.1 The Finite Time and Size Effects on Particle Production

Even though there are many investigations [65, 66, 67, 68, 69, 70] of the particle production from an infinite or a finite system in a constant/slowly varying electric field following on Schwinger's work, it is still interesting to see how the effect of finite size and finite time would impact our study. The finite time here means the time during which the external field was turned on. Better understanding here can lead to better particle production models in realistic systems. To that end, consider a system which consists of two infinite parallel plates with a separation distance $2L$ between them. The system is in a prescribed vacuum state until at $t = 0$ when I turn on the electric field parallel to z–axis. I turn the field off at $t = T$ while I keep the magnetic field zero at all times. While the system evolves under the external field, pairs of particles can be produced.

First notice that the system depends only on the following phase space variables: $(z, p_z, p_\perp, t)$ where $p_\perp = \sqrt{p_x^2 + p_y^2}$ because of the symmetry here introduced. Vector BGR functions, $\vec{g}_1$ and $\vec{g}_2$, thus can be written in the form:

$$\vec{g}_i = g_{iz}\hat{z} + \hat{p}_\perp g_{i\perp}, \tag{6.1}$$

while the pseudo-vectors, $\vec{g}_0$ and $\vec{g}_3$, can be written

$$\vec{g}_i = \hat{z} \times \hat{p}_\perp g_i. \tag{6.2}$$

The partial derivative of momentum is in the cylindrical coordinate system,

$$\partial_p = \hat{z}\partial_{p_z} + \hat{p}_\perp \partial_{p_\perp} + \hat{\phi}\frac{1}{p_\perp}\partial_\phi. \tag{6.3}$$

The integral–differential operators for this system are

$$D_t = \partial_t + e\int d\lambda E_z(z + i\lambda\partial_{p_z})\partial_{p_z} \equiv \bar{D}_t, \tag{6.4}$$

$$\mathbf{D} = \hat{z}\nabla_z, \tag{6.5}$$

$$\mathbf{P} = \hat{z}p_z + \mathbf{p}_\perp. \tag{6.6}$$

Substituting these Eq.(6.1-6.6) into the set of BGR equations, I obtain:

$$f_1 = f_2 = 0, \tag{6.7}$$

and there remain 8 differential equations

$$\partial_t f_0(z,p_{z0},t) + \nabla_z g_{1z}(z,p_{z0},t) = 0, \tag{6.8}$$

$$\partial_t f_3(z,p_{z0},t) - 2p_z g_{2z}(z,p_{z0},t) = +2p_\perp g_{2\perp}(z,p_{z0},t), \tag{6.9}$$

$$\partial_t g_0(z,p_{z0},t) - 2p_z g_{1\perp}(z,p_{z0},t) = -2p_\perp g_{1z}(z,p_{z0},t), \tag{6.10}$$

$$\partial_t g_{1z}(z,p_{z0},t) + \nabla_z f_0(z,p_{z0},t) = +2p_\perp g_0(z,p_{z0},t) - 2g_{2z}(z,p_{z0},t), \tag{6.11}$$

$$\partial_t g_{1\perp}(z,p_{z0},t) + 2p_z g_0(z,p_{z0},t) = -2g_{2\perp}(z,p_{z0},t), \tag{6.12}$$

$$\partial_t g_{2z}(z,p_{z0},t) + 2p_z f_3(z,p_{z0},t) = +2g_{1z}(z,p_{z0},t), \tag{6.13}$$

$$\partial_t g_{2\perp}(z,p_{z0},t) - \nabla_z g_3(z,p_{z0},t) = -2p_\perp f_3(z,p_{z0},t) + 2g_{1\perp}(z,p_{z0},t), \tag{6.14}$$

$$\partial_t g_3(z,p_{z0},t) - \nabla_z g_{2\perp}(z,p_{z0},t) = 0, \tag{6.15}$$

where I use the natural unit $\hbar = c = 1$ and in addition choose the fermion mass as the dimension of energy and inverse length, hence $m = 1$ so that the equations are dimensionless.

To obtain this set of equations, I used the method of characteristics to separate the classical motion out, i.e. I rewrote the BGR functions in the form:

$$W(\vec{p},t) = \int \frac{d^3p_0}{(2\pi)^3} W(\vec{p}_0,t)\delta(\vec{p} - \vec{p}(\vec{p}_0|t)), \tag{6.16}$$

where $\vec{p}(\vec{p}_0|t)$ is the solution of the classical equation of motion in the field,

$$\frac{d\vec{p}}{dt} = e\vec{E}(t), \tag{6.17}$$

so that

$$\vec{p}(\vec{p}_0|t) = \vec{p}_0 + e\int dt\, \vec{E}(t), \tag{6.18}$$

with the initial condition $\vec{p}(\vec{p}_0|t=0) = \vec{p}_0$.

I take the approximate vacuum BGR functions between two infinite plates separated by a distance $2L$ in the form (see Appendix E),

$$f_3 = -2\frac{2\pi}{2L}\sum_n \frac{m}{E_n}A_n, \tag{6.19}$$

$$g_{1\perp} = -2\frac{2\pi}{2L}\sum_n \frac{p_\perp}{E_n}A_n, \tag{6.20}$$

$$g_{1z} = -2\frac{2\pi}{2L}\sum_n \frac{k_n}{E_n}A_n, \tag{6.21}$$

where

$$A_n = \frac{1}{\pi}\frac{\sin 2(p_z - k_n)(L-|z|)}{p_z - k_n}\Theta(L^2 - z^2), \tag{6.22}$$

$$k_n = \frac{2\pi n}{2L}, \tag{6.23}$$

$$E_n = \sqrt{m^2 + p_\perp^2 + k_n^2}, \tag{6.24}$$

with all other functions vanishing. Considering these initial BGR functions, I can make following observations:

1. The function $g_{2\perp}$ and g_3 vanish always.
2. The function g_0 and $g_{1\perp}$ are proportional to the function g_{2z} and f_3, respectively, i.e.,

$$g_0 = -p_\perp g_{2z}, \tag{6.25}$$

$$g_{1\perp} = +p_\perp f_3. \tag{6.26}$$

With these observations, the number of equations reduces from 8 to 4 so that the following set of equations need to be solved considering initial/boundary conditions,

$$\partial_t f_0 + \nabla_z g_{1z} = 0, \tag{6.27}$$

$$\partial_t g_{1z} + \nabla_z f_0 = -2m_\perp^2 g_{2z}, \tag{6.28}$$
$$\partial_t f_3 - 2p_z g_{2z} = 0, \tag{6.29}$$
$$\partial_t g_{2z} + 2p_z f_3 = +2g_{1z}, \tag{6.30}$$

where

$$m_\perp = \sqrt{m^2 + p_\perp^2}. \tag{6.31}$$

To see the Schwinger particle production factor, I assume first that the initial conditions between $-L$ to $+L$ are the same as those of field free vacuum of infinite system,

$$f_3 = -\frac{2}{E_p}, \tag{6.32}$$
$$\vec{g}_1 = -\frac{2\vec{p}}{E_p}. \tag{6.33}$$

I apply the Neumann boundary condition such that the normal gradients of the functions are zero at boundaries. I so obtain $f_0 = 0$ since there is no spacial dependence. I then introduce the 3 remaining independent functions as follows,

$$f_3 = -\frac{2}{E_p}\left[a_0 - \frac{p_z}{m_\perp}a_1\right], \tag{6.34}$$
$$g_{1z} = -\frac{2}{E_p}\left[p_z a_0 + m_\perp a_1\right], \tag{6.35}$$
$$g_{2z} = -\frac{2}{m_\perp}a_2. \tag{6.36}$$

Substituting these into the Eq.(6.28)-(6.30), I obtain the set of equations,

$$\partial_t a_0 = \frac{\mathcal{E} m_\perp}{E_p^2}a_1, \tag{6.37}$$
$$\partial_t a_1 = -\frac{\mathcal{E} m_\perp}{E_p^2}a_0 - 2E_p a_2, \tag{6.38}$$
$$\partial_t a_2 = 2E_p a_1, \tag{6.39}$$

where $\mathcal{E} = eE$. These three equations are identical to those of Eq.(66)-(68) in reference [48] and give, as was recently shown [71] exactly the analytical form of the Schwinger particle production rate arising at large times,

$$\frac{1-a_0}{2} = e^{-\pi m_\perp^2/\mathcal{E}}. \tag{6.40}$$

Returning now to the finite system, I first define new functions which are a spacial average of BGR functions,

$$\bar{f}_3 = \frac{1}{2L}\int_{-L}^{L} dz f_3, \tag{6.41}$$

and so on. Since I take the Dirichlet boundary condition, such that all BGR functions vanish at $z = \pm L$, I have,

$$\partial_t \bar{g}_{1z} = -2m_\perp^2 \bar{g}_{2z}, \tag{6.42}$$

$$\partial_t \bar{f}_3 - 2p_z \bar{g}_{2z} = 0, \tag{6.43}$$

$$\partial_t \bar{g}_{2z} + 2p_z \bar{f}_3 = +2\bar{g}_{1z}, \tag{6.44}$$

Substituting Eqs.(6.34)-(6.36) into these equations, I obtain the same set of equations Eqs.(6.37)-(6.39) which, when solved, recover the Schwinger particle production rate. This conclusion is valid only if the back reaction is completely ignored and the boundary conditions I imposed are satisfied and the initial conditions good enough. However, there is one important difference between a finite system and infinite system, namely, the mass m for a finite system should be understood as the lowest eigen-energy which depends on the size of a system,

$$m = m(D). \tag{6.45}$$

The eigen-spectrum of a Dirac particle between two plates at distance D is (see Appendix F),

$$\tan kD = -\frac{k}{m_0}, \tag{6.46}$$

where $k = \sqrt{E^2 - m_0^2}$. I explicitly introduced here the subscript 0 to denote the rest mass associated with an infinite system. Figure (6.1) shows the lowest eigen-energy as function of distance D. Thus, the particle production rate is

$$\frac{1-\bar{a}_0}{2} = \eta e^{-\pi(m_0^2+p_\perp^2)/\mathcal{E}}, \tag{6.47}$$

where the suppressing factor is

$$\eta = e^{-\pi\alpha(\alpha+2m_0)/\mathcal{E}}. \tag{6.48}$$

The function α is defined by

$$E_{min} = m_0 + \alpha. \tag{6.49}$$

The figure (6.1) shows strong suppressing at small distance. The Table (6.1) shows the suppressing factor for $e|\vec{E}| = 1$ as function of the separation distance D.

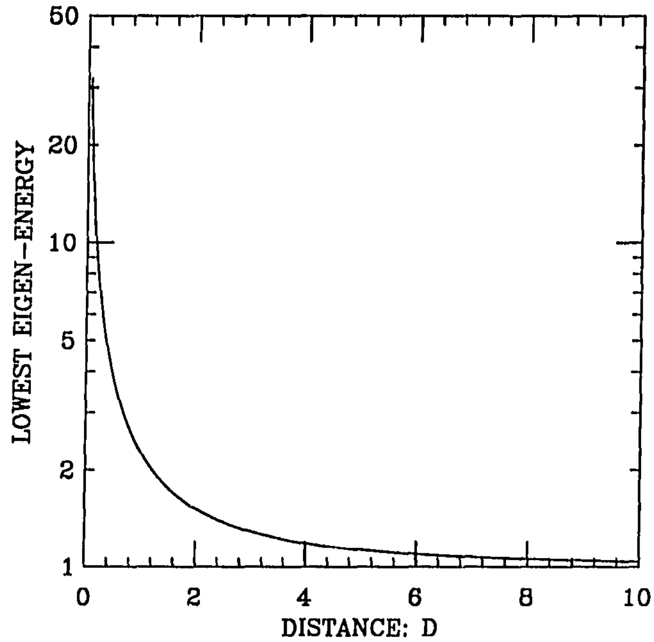

Figure 6.1: Lowest Energy Spectrum of Dirac Particle

D	E_{min}	η
0.2	8.5024	$5.4\ 10^{-98}$
0.5	3.8069	$3.9\ 10^{-19}$
1.0	2.2618	$2.4\ 10^{-6}$
2.0	1.5198	$1.6\ 10^{-2}$
3.0	1.2923	$1.22\ 10^{-1}$
5.0	1.1321	$4.23\ 10^{-1}$
10.0	1.0402	$7.73\ 10^{-1}$

Table 6.1: Particle Production Suppressing Factor.

CHAPTER 7

Conclusions and Outlook

I introduced and summarized essential elements of the nonrelativistic Wigner function and its properties and time evolution in Chapter 1.

In Chapter 2, I considered muon catalyzed fusion problems and in particular studied the sticking of muons to the fusion α particle. The nonrelativistic Wigner formulation has been applied to the study of a muon after (dt) fusion. The Wigner phase space distribution of a muon has been obtained explicitly under the adiabatic approximation to the molecular state. The energy spectrum of the muon for the various approximations were studied and the initial sticking probability ω_s^0, introduced here as the conditional probability when the muon has a negative relative energy with respect to the fusion α-particle, was calculated. A semi-classical treatment of the quantum distribution gives us twice as much sticking than the expected value. This shows that I cannot treat the Wigner distribution as a classical phase space distribution. Restricting the muon Wigner function to the binding energy (micro-canonical distribution) gives about half of the expected result. Only when I fold the Wigner function with coarse graining function did I obtain about 1.1% for the smearing parameter $\lambda = 2$. I also found the probability by a full calculation to be 1.04% which is in good agreement with the result from other theory, but with the added insight of having a full Wigner phase space distribution for the muon amplitude. Finally, I evolved the muon phase space distribution along the Wigner trajectory including elastic collisions with matter and found an increase of sticking probability as time goes on. However, this calculation did not include the inelastic collisions which are very important. I hope to return to study these issues in the near future.

I then turned to study a relativistic Wigner function and introduced the single time formulation, developed by Białynicki-Birula, Górnicki and Rafelski(BGR), which is equivalent to the nonrelativistic Wigner function in Chapter 3. This relativistic phase space formulation is especially good for the study of vacuum and can describe the particle production as well as the flow of particles.

To deepen the understanding of the relation between quantum and classical transport theory, I studied how to extract classical contents from nonrelativistic and relativistic phase space formulation in Chapter 4. I found that the classical relativistic flow can be separated from the perturbative quantum fluctuations and non-perturbative particle production phenomena by taking $\hbar \to 0$ after I fold the quantum distribution with the coarse graining function such as is the Gaussian function. In that way, I consistently derived the relativistic Vlasov equations for particles and antiparticles. Applying the same method, the phase space version of Bargmann–Michel–Telegdi spin equation of motion was also determined.

I then proceeded to obtain the relativistic Wigner functions of Coulombic state and MIT cavity $J^{\pi} = \frac{1}{2}^{+}$ state in Chapter 5. The initial sticking probability addressed in Chapter 2 can be refined using the Coulombic relativistic distribution given here. The phase space distribution of the MIT bag state is derived and applied to understand the sharing of the total angular momentum between the degree of freedom and spin which are 35% and 65%, respectively. I also obtained the Wigner function of Landau orbits which could be of importance e.g. in QCD models of vacuum structure.

In the final chapter, I considered the relativistic transport equations for the case of finite system with constant electric field and recovered the Schwinger particle production rate. I am currently investigating different boundary conditions to obtain the surface effects on the particle production.

With this experience I can now take advantages of the wealth of information provided by the quantum phase space theory to study such problems as are the vacuum structure and/or particle production in strong fields. In particular given the groundwork here laid I hope to investigate

1. muon sticking with relativistic dynamics and inelastic interaction with matter,
2. vacuum structure, viz. non-trivial time independent solutions of the BGR equations, and
3. particle production in homogeneous fields with finite space-time extent.

APPENDIX A

Monte Carlo Sampling Method

I briefly describe here the Monte Carlo method I employ in order to evaluate Eq.(2.14) and choose the initial coordinate and momentum of muon in laboratory frame. The details of this approach are as follows: The phase space distribution is a function of $r, p, \cos\theta$ only, where θ is the angle between $\vec{r}$ and $\vec{p}$. I choose $(r, p, \cos\theta)$ using the proper bin size to optimize the normalization and computer running time. It is enough for the hydrogen-like $(1s)$ state to choose r from 0 to r_{max}= 4.2 and p from 0 to p_{max}= 12 and $\cos\theta$ from -1 to +1. The r_{max} and p_{max} are given by the corresponding wave function and Wigner function, i.e., $\psi_{1s}(r > r_{max}) \approx 0$ and $\phi_{1s}(p > p_{max}) \approx 0$ and $f(r > r_{max}\ or\ p > p_{max}, cos\theta) \approx 0$.

Another 3 angles $(\phi, \cos\theta^*, \phi^*)$ are chosen from random number such that $\phi, \phi^* \in [0, 2\pi]$ and $cos\theta^* \in [-1, +1]$. The angle ϕ is the azimuthal angle of $\vec{r}$ with respect to $\vec{p}$ so that $\vec{r}$ has (r, θ, ϕ) with respect to $\vec{p}$. The $(-\theta^*, -\phi^*)$ are the rotation angles of $\vec{p}$. The rotation $R_y(-\theta^*)$ is followed by the rotation $R_z(-\phi^*)$. So $\vec{p}$ has the coordinate (p, θ^*, ϕ^*) with respect to this new coordinate system which is the laboratory frame. In result, $\vec{r}$ is given by in laboratory frame,

$$\vec{r}_{lab} = R_z(-\phi^*)R_y(-\theta^*)\vec{r}(r, \theta, \phi). \tag{A.1}$$

These $\vec{r}$ and $\vec{p}$ are used as an initial state to get Wigner trajectory.

To integrate Eq.(2.14) and related integrals, I cut the phase space into bins denoted by $(r, p, \cos\theta)$, each of which has volume $(dV_{\vec{r}}dV_{\vec{p}})$ at $(r, p, \cos\theta, \phi, \cos\theta^*, \phi^*)$ or $(\vec{r}, \vec{p})$ with the weighing factor $W(\vec{r}, \vec{p}) = f(\vec{r}, \vec{p})dV_{\vec{r}}dV_{\vec{p}}$ where I include the bin volume into the weighing factor for later convenience. In order to find a more efficient way to integrate, I introduce new variables such that each bin has approximately same weighing factor,

$$W(\vec{r}, \vec{p}) = f(r, p, cos\theta)8\pi^2 r^2 p^2 \frac{dr}{d\alpha}\frac{dp}{d\beta}\frac{dcos\theta}{d\eta}d\alpha d\beta d\eta. \tag{A.2}$$

I found that the following choice works well,

$$\frac{dr}{d\alpha} = \frac{\alpha}{5} + 0.1, \qquad \alpha \in (0,6), \tag{A.3}$$

$$\frac{dp}{d\beta} = \frac{\beta}{20} + 0.1, \qquad \beta \in (0,20), \tag{A.4}$$

$$\frac{dcos\theta}{d\eta} = \frac{\eta}{10} + \frac{1}{30}, \qquad \eta \in [-3,+3]. \tag{A.5}$$

APPENDIX B

16 Hermitian Matrix

16 linear independent 4×4 Hermitian matrices can be obtained by the combination of $(1_2, \vec{\sigma})$ and $(1_2, \vec{\rho})$ where 1_2 is a 2×2 unit matrix and $\vec{\sigma}$ and $\vec{\rho}$ are the Pauli matrices. I present the explicit forms here;

$$1_4 = 1_2 \otimes 1_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \rho_1 = \rho_1 \otimes 1_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\rho_2 = \rho_2 \otimes 1_2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ +i & 0 & 0 & 0 \\ 0 & +i & 0 & 0 \end{pmatrix}, \quad \rho_3 = \rho_3 \otimes 1_2 = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\sigma_1 = 1_2 \otimes \sigma_1 = \begin{pmatrix} 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & +1 & 0 \end{pmatrix}, \quad \sigma_2 = 1_2 \otimes \sigma_2 = \begin{pmatrix} 0 & -i & 0 & 0 \\ +i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & +i & 0 \end{pmatrix},$$

$$\sigma_3 = 1_2 \otimes \sigma_3 = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \rho_1 \otimes \sigma_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & +i & 0 \end{pmatrix},$$

$$\rho_1 \otimes \sigma_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & +i & 0 \\ 0 & -i & 0 & 0 \\ +i & 0 & 0 & 0 \end{pmatrix}, \quad \rho_1 \otimes \sigma_3 = \begin{pmatrix} 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \\ +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$\rho_2 \otimes \sigma_1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & +i & 0 & 0 \\ +i & 0 & 0 & 0 \end{pmatrix}, \quad \rho_2 \otimes \sigma_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & +1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\rho_2 \otimes \sigma_3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & +i \\ +i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad \rho_1 \otimes \sigma_1 = \begin{pmatrix} 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$$\rho_3 \otimes \sigma_2 = \begin{pmatrix} 0 & -i & 0 & 0 \\ +i & 0 & 0 & 0 \\ 0 & 0 & 0 & +i \\ 0 & 0 & +i & 0 \end{pmatrix}, \quad \rho_3 \otimes \sigma_3 = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}$$

APPENDIX C

Relations between Wigner and BGR functions

I obtain explicit relations between 16 Wigner functions and 16 BGR functions by solving 16 functions,

$$\mathrm{W} = \frac{1}{4}\left(f_0 + \sum_{i=1}^{3} \rho_i f_i + \sigma \cdot \mathrm{g}_0 + \sum_{i=1}^{3} \rho_i \sigma \cdot \mathrm{g}_i\right), \tag{C.1}$$

where the matrix $(\rho, \sigma, \rho\sigma)$ are given in Appendix B. In results, BGR functions are

$$f_0 = W_{00} + W_{11} + W_{22} + W_{33}, \tag{C.2}$$

$$f_1 = W_{02} + W_{20} + W_{13} + W_{31}, \tag{C.3}$$

$$f_2 = i(W_{02} - W_{20} + W_{13} - W_{31}), \tag{C.4}$$

$$f_3 = W_{00} + W_{11} - W_{22} - W_{33}, \tag{C.5}$$

$$\begin{aligned} g_{01} &= W_{01} + W_{10} + W_{23} + W_{32}, \\ g_{02} &= i(W_{01} - W_{10} + W_{23} - W_{32}), \\ g_{03} &= W_{00} + W_{22} - W_{11} - W_{33}, \end{aligned} \tag{C.6}$$

$$\begin{aligned} g_{11} &= W_{03} + W_{12} + W_{21} + W_{30}, \\ g_{12} &= i(W_{21} - W_{12} + W_{03} - W_{30}), \\ g_{13} &= W_{02} - W_{13} + W_{20} - W_{31}, \end{aligned} \tag{C.7}$$

$$\begin{aligned} g_{21} &= i(W_{03} - W_{30} + W_{12} - W_{21}), \\ g_{22} &= W_{12} + W_{21} - W_{03} - W_{30}, \\ g_{23} &= i(W_{02} - W_{20} + W_{31} - W_{13}), \end{aligned} \tag{C.8}$$

$$
\begin{aligned}
g_{31} &= W_{01} + W_{10} - W_{23} - W_{32}, \\
g_{32} &= i(W_{01} - W_{10} + W_{32} - W_{23}), \\
g_{33} &= W_{00} - W_{11} - W_{22} + W_{33},
\end{aligned}
\tag{C.9}
$$

APPENDIX D

Integral Equation in Momentum and Frequency space

I introduce the following notations to make equations short

$$\tilde{A}\circ\tilde{B} = \int d\Gamma'\tilde{A}(\vec{q}-\vec{q}',\vec{p},\omega-\omega')\tilde{B}(\vec{q}',\vec{p},\omega'), \tag{D.1}$$

$$\tilde{\vec{A}}\odot\tilde{\vec{B}} = \int d\Gamma'\tilde{\vec{A}}(\vec{q}-\vec{q}',\vec{p},\omega-\omega')\cdot\tilde{\vec{B}}(\vec{q}',\vec{p},\omega'), \tag{D.2}$$

$$\tilde{\vec{A}}\otimes\tilde{\vec{B}} = \int d\Gamma'\tilde{\vec{A}}(\vec{q}-\vec{q}',\vec{p},\omega-\omega')\times\tilde{\vec{B}}(\vec{q}',\vec{p},\omega'). \tag{D.3}$$

Using the method of partition and the *Mathematica*, I have obtained the integral equations for each function:

$$\begin{aligned}
\tilde{f}_0 = {} & \tilde{f}_0^0 - \frac{i\hbar}{\Delta\Omega}\{[-4(\vec{p}\cdot\vec{q})\vec{p}+\omega^2\vec{q}]\cdot[\tilde{\mathcal{E}}\circ\tilde{\vec{g}}_1+\tilde{\vec{\mathcal{B}}}^0\circ\tilde{f}_0-2\tilde{\vec{\mathcal{B}}}^1\otimes\tilde{\vec{g}}_0] \\
& +\omega[-4m^2-4\vec{p}^2+\omega^2][\tilde{\mathcal{E}}\circ\tilde{f}_0+\tilde{\vec{\mathcal{B}}}^0\odot\tilde{\vec{g}}_1]-4m[\vec{p}\cdot\vec{q}][\tilde{\mathcal{E}}\circ\tilde{f}_3-2\tilde{\vec{\mathcal{B}}}^1\odot\tilde{\vec{g}}_2] \\
& -[2im\omega\vec{q}]\cdot[\tilde{\mathcal{E}}\circ\tilde{\vec{g}}_2+\tilde{\vec{\mathcal{B}}}^0\otimes\tilde{\vec{g}}_3+2\tilde{\vec{\mathcal{B}}}^1\circ\tilde{f}_3] \\
& -[2i\omega(\vec{p}\times\vec{q})]\cdot[\tilde{\mathcal{E}}\circ\tilde{\vec{g}}_0+\tilde{\vec{\mathcal{B}}}^0\circ\tilde{f}_1-2\tilde{\vec{\mathcal{B}}}^1\otimes\tilde{\vec{g}}_1]\},
\end{aligned} \tag{D.4}$$

$$\begin{aligned}
\tilde{f}_1 = {} & \tilde{f}_1^0 - \frac{i\hbar}{\Delta\Omega}\{[-2i\omega(\vec{p}\times\vec{q})]\cdot[\tilde{\mathcal{E}}\circ\tilde{\vec{g}}_1+\tilde{\vec{\mathcal{B}}}^0\circ\tilde{f}_0-2\tilde{\vec{\mathcal{B}}}^1\otimes\tilde{\vec{g}}_0] \\
& -[4m\omega\vec{p}]\cdot[\tilde{\mathcal{E}}\circ\tilde{\vec{g}}_3-\tilde{\vec{\mathcal{B}}}^0\otimes\tilde{\vec{g}}_2-2\tilde{\vec{\mathcal{B}}}^1\circ\tilde{f}_2]-[2im\omega^2][\tilde{\mathcal{E}}\circ\tilde{f}_2+2\tilde{\vec{\mathcal{B}}}^1\odot\tilde{\vec{g}}_3] \\
& +[-4(\vec{p}\cdot\vec{q})\vec{p}+\omega^2\vec{q}]\cdot[\tilde{\mathcal{E}}\circ\tilde{\vec{g}}_0+\tilde{\vec{\mathcal{B}}}^0\circ\tilde{f}_1-2\tilde{\vec{\mathcal{B}}}^1\otimes\tilde{\vec{g}}_1] \\
& +[\omega(-4\vec{p}^2+\omega^2)][\tilde{\mathcal{E}}\circ\tilde{f}_1+\tilde{\vec{\mathcal{B}}}^0\odot\tilde{\vec{g}}_0]\},
\end{aligned} \tag{D.5}$$

$$\begin{aligned}
\tilde{f}_2 = {} & \tilde{f}_2^0 - \frac{i\hbar}{\Delta\Omega}\{2i[(\vec{p}\cdot\vec{q})\vec{q}-\omega^2\vec{p}]\cdot[\tilde{\mathcal{E}}\circ\tilde{\vec{g}}_3-\tilde{\vec{\mathcal{B}}}^0\otimes\tilde{\vec{g}}_2-2\tilde{\vec{\mathcal{B}}}^1\circ\tilde{f}_2] \\
& +[2i\omega(\vec{p}\times\vec{q})]\cdot[\tilde{\mathcal{E}}\circ\tilde{\vec{g}}_2+\tilde{\vec{\mathcal{B}}}^0\otimes\tilde{\vec{g}}_3+2\tilde{\vec{\mathcal{B}}}^1\circ\tilde{f}_3]+[\omega(-\vec{q}^2+\omega^2)][\tilde{\mathcal{E}}\circ\tilde{f}_2+2\tilde{\vec{\mathcal{B}}}^1\odot\tilde{\vec{g}}_3] \\
& +[2im\omega\vec{q}]\cdot[\tilde{\mathcal{E}}\circ\tilde{\vec{g}}_0+\tilde{\vec{\mathcal{B}}}^0\circ\tilde{f}_1-2\tilde{\vec{\mathcal{B}}}^1\otimes\tilde{\vec{g}}_1]+[2im\omega^2][\tilde{\mathcal{E}}\circ\tilde{f}_1+\tilde{\vec{\mathcal{B}}}^0\odot\tilde{\vec{g}}_0]\},
\end{aligned} \tag{D.6}$$

$$\begin{aligned}
\tilde{f}_3 = {} & \tilde{f}_3^0 - \frac{i\hbar}{\Delta\Omega}\{-[4m\omega\vec{p}]\cdot[\tilde{\mathcal{E}}\circ\tilde{\vec{g}}_1+\tilde{\vec{\mathcal{B}}}^0\circ\tilde{f}_0-2\tilde{\vec{\mathcal{B}}}^1\otimes\tilde{\vec{g}}_0] \\
& -[4m(\vec{p}\cdot\vec{q})][\tilde{\mathcal{E}}\circ\tilde{f}_0+\tilde{\vec{\mathcal{B}}}^0\odot\tilde{\vec{g}}_1]+[\omega(-4m^2-\vec{q}^2+\omega^2)][\tilde{\mathcal{E}}\circ\tilde{f}_3-2\tilde{\vec{\mathcal{B}}}^1\odot\tilde{\vec{g}}_2] \\
& +[2i\omega(\vec{p}\times\vec{q})]\cdot[\tilde{\mathcal{E}}\circ\tilde{\vec{g}}_3-\tilde{\vec{\mathcal{B}}}^0\otimes\tilde{\vec{g}}_2-2\tilde{\vec{\mathcal{B}}}^1\circ\tilde{f}_2]
\end{aligned}$$

$$
\begin{aligned}
& -2i[(\vec{p}\cdot\vec{q})\vec{q}-\omega^2\vec{p}]\cdot[\tilde{\mathcal{E}}\circ\tilde{\vec{g}}_2+\tilde{\vec{\mathcal{B}}}^0\otimes\tilde{\vec{g}}_3+2\tilde{\vec{\mathcal{B}}}^1\circ\tilde{\vec{f}}_3]\}, && \text{(D.7)}\\
\tilde{\vec{g}}_0 &= \tilde{\vec{g}}_0{}^0-\frac{i\hbar}{\Delta\Omega}\{-2i[(\vec{p}\cdot\vec{q})\vec{q}-\omega^2\vec{p}]\times[\tilde{\mathcal{E}}\circ\tilde{\vec{g}}_1+\tilde{\vec{\mathcal{B}}}^0\circ\tilde{\vec{f}}_0-2\tilde{\vec{\mathcal{B}}}^1\otimes\tilde{\vec{g}}_0]\\
&+[2i\omega(\vec{p}\times\vec{q})][\tilde{\mathcal{E}}\circ\tilde{\vec{f}}_0+\tilde{\vec{\mathcal{B}}}^0\odot\tilde{\vec{g}}_1]-[4m(\vec{p}\cdot\vec{q})][\tilde{\mathcal{E}}\circ\tilde{\vec{g}}_3-\tilde{\vec{\mathcal{B}}}^0\otimes\tilde{\vec{g}}_2-2\tilde{\vec{\mathcal{B}}}^1\circ\tilde{\vec{f}}_2]\\
&+[4m\omega\vec{p}]\times[\tilde{\mathcal{E}}\circ\tilde{\vec{g}}_2+\tilde{\vec{\mathcal{B}}}^0\otimes\tilde{\vec{g}}_3+2\tilde{\vec{\mathcal{B}}}^1\circ\tilde{\vec{f}}_3]-[2im\omega\vec{q}][\tilde{\mathcal{E}}\circ\tilde{\vec{f}}_2+2\tilde{\vec{\mathcal{B}}}^1\odot\tilde{\vec{g}}_3]\\
&+\omega[\omega^2-4m^2-\vec{q}^2-4\vec{p}\vec{p}\cdot+\vec{q}\vec{q}\cdot][\tilde{\mathcal{E}}\circ\tilde{\vec{g}}_0+\tilde{\vec{\mathcal{B}}}^0\circ\tilde{\vec{f}}_1-2\tilde{\vec{\mathcal{B}}}^1\otimes\tilde{\vec{g}}_1]\\
&+[-4(\vec{p}\cdot\vec{q})\vec{p}+\omega^2\vec{q}][\tilde{\mathcal{E}}\circ\tilde{\vec{f}}_1+\tilde{\vec{\mathcal{B}}}^0\odot\tilde{\vec{g}}_0]\}, && \text{(D.8)}\\
\tilde{\vec{g}}_1 &= \tilde{\vec{g}}_1{}^0-\frac{i\hbar}{\Delta\Omega}\{\omega[\omega^2-\vec{q}^2-4\vec{p}\vec{p}\cdot+\vec{q}\vec{q}\cdot][\tilde{\mathcal{E}}\circ\tilde{\vec{g}}_1+\tilde{\vec{\mathcal{B}}}^0\circ\tilde{\vec{f}}_0-2\tilde{\vec{\mathcal{B}}}^1\otimes\tilde{\vec{g}}_0]\\
&+[-4(\vec{p}\cdot\vec{q})\vec{p}+\omega^2\vec{q}][\tilde{\mathcal{E}}\circ\tilde{\vec{f}}_0+\tilde{\vec{\mathcal{B}}}^0\odot\tilde{\vec{g}}_1]-[4m\omega\vec{p}][\tilde{\mathcal{E}}\circ\tilde{\vec{f}}_3-2\tilde{\vec{\mathcal{B}}}^1\odot\tilde{\vec{g}}_2]\\
&-[2im\omega^2][\tilde{\mathcal{E}}\circ\tilde{\vec{g}}_2+\tilde{\vec{\mathcal{B}}}^0\otimes\tilde{\vec{g}}_3+2\tilde{\vec{\mathcal{B}}}^1\circ\tilde{\vec{f}}_3]\\
&-2i[(\vec{p}\cdot\vec{q})\vec{q}-\omega^2\vec{p}]\times[\tilde{\mathcal{E}}\circ\tilde{\vec{g}}_0+\tilde{\vec{\mathcal{B}}}^0\circ\tilde{\vec{f}}_1-2\tilde{\vec{\mathcal{B}}}^1\otimes\tilde{\vec{g}}_1]\\
&-[2im\omega\vec{q}]\times[\tilde{\mathcal{E}}\circ\tilde{\vec{g}}_3-\tilde{\vec{\mathcal{B}}}^0\otimes\tilde{\vec{g}}_2-2\tilde{\vec{\mathcal{B}}}^1\circ\tilde{\vec{f}}_2]+[2i\omega(\vec{p}\times\vec{q})][\tilde{\mathcal{E}}\circ\tilde{\vec{f}}_1+\tilde{\vec{\mathcal{B}}}^0\odot\tilde{\vec{g}}_0]\}, && \text{(D.9)}\\
\tilde{\vec{g}}_2 &= \tilde{\vec{g}}_2{}^0-\frac{i\hbar}{\Delta\Omega}\{\omega[\omega^2-4\vec{p}^2+4\vec{p}\vec{p}\cdot-\vec{q}\vec{q}\cdot][\tilde{\mathcal{E}}\circ\tilde{\vec{g}}_2+\tilde{\vec{\mathcal{B}}}^0\otimes\tilde{\vec{g}}_3+2\tilde{\vec{\mathcal{B}}}^1\circ\tilde{\vec{f}}_3]\\
&+[-4(\vec{p}\cdot\vec{q})\vec{p}+\omega^2\vec{q}]\times[\tilde{\mathcal{E}}\circ\tilde{\vec{g}}_3-\tilde{\vec{\mathcal{B}}}^0\otimes\tilde{\vec{g}}_2-2\tilde{\vec{\mathcal{B}}}^1\circ\tilde{\vec{f}}_2]\\
&+2i[(\vec{p}\cdot\vec{q})\vec{q}-\omega^2\vec{p}][\tilde{\mathcal{E}}\circ\tilde{\vec{f}}_3-2\tilde{\vec{\mathcal{B}}}^1\odot\tilde{\vec{g}}_2]\\
&+[2im\omega^2][\tilde{\mathcal{E}}\circ\tilde{\vec{g}}_1+\tilde{\vec{\mathcal{B}}}^0\circ\tilde{\vec{f}}_0-2\tilde{\vec{\mathcal{B}}}^1\otimes\tilde{\vec{g}}_0]+[2im\omega\vec{q}][\tilde{\mathcal{E}}\circ\tilde{\vec{f}}_0+\tilde{\vec{\mathcal{B}}}^0\odot\tilde{\vec{g}}_1]\\
&-[2i\omega(\vec{p}\times\vec{q})][\tilde{\mathcal{E}}\circ\tilde{\vec{f}}_2+2\tilde{\vec{\mathcal{B}}}^1\odot\tilde{\vec{g}}_3]-[4m\omega\vec{p}]\times[\tilde{\mathcal{E}}\circ\tilde{\vec{g}}_0+\tilde{\vec{\mathcal{B}}}^0\circ\tilde{\vec{f}}_1-2\tilde{\vec{\mathcal{B}}}^1\otimes\tilde{\vec{g}}_1]\} && \text{(D.10)}\\
\tilde{\vec{g}}_3 &= \tilde{\vec{g}}_3{}^0-\frac{i\hbar}{\Delta\Omega}\{[-2im\omega\vec{q}]\times[\tilde{\mathcal{E}}\circ\tilde{\vec{g}}_1+\tilde{\vec{\mathcal{B}}}^0\circ\tilde{\vec{f}}_0-2\tilde{\vec{\mathcal{B}}}^1\otimes\tilde{\vec{g}}_0]\\
&-[2i\omega(\vec{p}\times\vec{q})][\tilde{\mathcal{E}}\circ\tilde{\vec{f}}_3-2\tilde{\vec{\mathcal{B}}}^1\odot\tilde{\vec{g}}_2]\\
&+[\omega(\omega^2-4m^2-4\vec{p}^2+4\vec{p}\vec{p}\cdot-\vec{q}\vec{q}\cdot)][\tilde{\mathcal{E}}\circ\tilde{\vec{g}}_3-\tilde{\vec{\mathcal{B}}}^0\otimes\tilde{\vec{g}}_2-2\tilde{\vec{\mathcal{B}}}^1\circ\tilde{\vec{f}}_2]\\
&+[4(\vec{p}\cdot\vec{q})\vec{p}-\omega^2\vec{q}]\times[\tilde{\mathcal{E}}\circ\tilde{\vec{g}}_2+\tilde{\vec{\mathcal{B}}}^0\otimes\tilde{\vec{g}}_3+2\tilde{\vec{\mathcal{B}}}^1\circ\tilde{\vec{f}}_3]\\
&-2i[(\vec{p}\cdot\vec{q})\vec{q}-\omega^2\vec{p}][\tilde{\mathcal{E}}\circ\tilde{\vec{f}}_2+2\tilde{\vec{\mathcal{B}}}^1\odot\tilde{\vec{g}}_3]\\
&-[4m(\vec{p}\cdot\vec{q})][\tilde{\mathcal{E}}\circ\tilde{\vec{g}}_0+\tilde{\vec{\mathcal{B}}}^0\circ\tilde{\vec{f}}_1-2\tilde{\vec{\mathcal{B}}}^1\otimes\tilde{\vec{g}}_1]-[4m\omega\vec{p}][\tilde{\mathcal{E}}\circ\tilde{\vec{f}}_1+\tilde{\vec{\mathcal{B}}}^0\odot\tilde{\vec{g}}_0]\}, && \text{(D.11)}
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta\Omega &= \omega^4-\omega^2(4m^2+4\vec{p}^2+\vec{q}^2)+4(\vec{p}\cdot\vec{q})^2, && \text{(D.12)}\\
\tilde{f}_i &= \tilde{f}_i(\vec{q},\vec{p},\omega), && \text{(D.13)}\\
\tilde{\vec{g}}_i &= \tilde{\vec{g}}_i(\vec{q},\vec{p},\omega), && \text{(D.14)}
\end{aligned}
$$

and the field operators are given by

$$\tilde{\mathcal{E}} = e\int_{-1/2}^{1/2} d\lambda e^{-\lambda\vec{q}\cdot\vec{\delta}_p} \tilde{\vec{E}}(\vec{q},\omega)\cdot\vec{\delta}_p, \tag{D.15}$$

$$\tilde{\vec{\mathcal{B}}}^0 = \frac{e}{c}\int_{-1/2}^{1/2} d\lambda e^{-\lambda\vec{q}\cdot\vec{\delta}_p} \tilde{\vec{B}}(\vec{q},\omega)\times\vec{\delta}_p, \tag{D.16}$$

$$\tilde{\vec{\mathcal{B}}}^1 = -\frac{ie\hbar}{c}\int_{-1/2}^{1/2} d\lambda\lambda e^{-\lambda\vec{q}\cdot\vec{\delta}_p} \tilde{\vec{B}}(\vec{q},\omega)\times\vec{\delta}_p. \tag{D.17}$$

APPENDIX E

Initial Conditions: BGR functions between two parallel plates

I obtain here the vacuum BGR function between two infinite parallel plates at distance $2L$. To get these functions, I need to obtain the Dirac field operator $\hat{\psi}(x)$ between two plates. This operator can be infered from the free field operator in an infinite space, namely, it can be expanded on the complete set of plane waves as follows,

$$\hat{\psi}(x) = \frac{1}{2L}\sum_n \int \frac{d\vec{k}_\perp}{(2\pi)^{1/2}}\sqrt{\frac{m}{k_0}}\sum_{i=1,2}\left[\hat{b}_i(k)u^{(i)}(k)e^{-ikx} + \hat{d}_i^\dagger(k)v^{(i)}(k)e^{+ikx}\right], \quad (E.1)$$

where the z–space is between $z = -L$ and L. I assume that the field obeys the periodic condition so that

$$k_z = k_n = \frac{2\pi n}{2L}, \quad (E.2)$$

and n is an integer. I explicitly find that the operator here constructed satisfies the Dirac equation and obeys the confinement condition, $\bar{\psi}\psi|_{z=\pm L} = 0$. As appropriate for the DHW function, t stands for laboratory time.

The creation and destruction operator obey the anticommutation relation,

$$\{\hat{b}_i(k), \hat{b}_j^\dagger(k')\} = \delta_{ij}\frac{2L}{2\pi}\delta_{nn'}\delta^2(\vec{k}_\perp - \vec{k}'_\perp), \quad (E.3)$$

$$\{\hat{d}_i(k), \hat{d}_j^\dagger(k')\} = \delta_{ij}\frac{2L}{2\pi}\delta_{nn'}\delta^2(\vec{k}_\perp - \vec{k}'_\perp), \quad (E.4)$$

where I used $\delta(k_n - k_{n'}) = \frac{2L}{2\pi}\delta_{nn'}$. We can see easily that the field becomes the free field operator in the limit $L \to \infty$.

The DHW functions $W_{\alpha\beta}$, in this case where $A_\mu = 0$, (In fact, this is a wrong assumption. If I include the back reaction, the average $\bar{A}_\mu$ is zero but not A_μ, and A_μ

could be related to a noise function) are,

$$
\begin{aligned}
W_{\alpha\beta} &= -\frac{1}{8L^2}\int d^3s e^{-i\vec{p}\cdot\vec{s}}\sum_n\int\frac{d\vec{k}_\perp}{(2\pi)^{1/2}}\sqrt{\frac{m}{k_0}}\sum_{n'}\int\frac{d\vec{k}'_\perp}{(2\pi)^{1/2}}\sqrt{\frac{m}{k'_0}} \\
&\quad \Theta(L^2-(z-s_z/2)^2)\Theta(L^2-(z+s_z/2)^2) \\
&\quad <0|[\sum_{i=1,2}(\hat{b}_i(k)u^{(i)}(k)e^{-ik_0t+i\vec{k}\cdot(\vec{x}+\vec{s}/2)}+\hat{d}_i^\dagger(k)v^{(i)}(k)e^{+ik_0t-i\vec{k}\cdot(\vec{x}+\vec{s}/2)}), \\
&\quad \sum_{j=1,2}(\hat{b}_j^\dagger(k')u^{(j)\dagger}(k')e^{+ik'_0t+i\vec{k}'\cdot(\vec{x}-\vec{s}/2)}+\hat{d}_j(k')v^{(j)\dagger}(k')e^{-ik'_0t+i\vec{k}'\cdot(\vec{x}-\vec{s}/2)})]|0>, \qquad \text{(E.5)}
\end{aligned}
$$

where I explicitly put in the step function $\Theta(x)$ to include the cut at $z=\pm L$. Considering that the creation and destruction operator applies to the vacuum, the only non-vanishing terms are

$$
\begin{aligned}
W_{\alpha\beta} &= -\frac{1}{8L^2}\int d^3s e^{-i\vec{p}\cdot\vec{s}}\sum_n\int\frac{d\vec{k}_\perp}{(2\pi)^{1/2}}\sqrt{\frac{m}{k_0}}\sum_{n'}\int\frac{d\vec{k}'_\perp}{(2\pi)^{1/2}}\sqrt{\frac{m}{k'_0}} \\
&\quad \Theta(L^2-(z-s_z/2)^2)\Theta(L^2-(z+s_z/2)^2) \\
&\quad <0|[\hat{b}_1(k),\hat{b}_1^\dagger(k')]u_\alpha^{(1)}(k)u_\beta^{(1)\dagger}(k')e^{+i(\vec{k}+\vec{k}')\cdot\vec{s}/2-i(k-k')x} \\
&\quad +[\hat{b}_2(k),\hat{b}_2^\dagger(k')]u_\alpha^{(2)}(k)u_\beta^{(2)\dagger}(k')e^{+i(\vec{k}+\vec{k}')\cdot\vec{s}/2-i(k-k')x} \\
&\quad +[\hat{d}_1^\dagger(k),\hat{d}_1(k')]v_\alpha^{(1)}(k)v_\beta^{(1)\dagger}(k')e^{-i(\vec{k}+\vec{k}')\cdot\vec{s}/2+i(k-k')x} \\
&\quad +[\hat{d}_2^\dagger(k),\hat{d}_2(k')]v_\alpha^{(2)}(k)v_\beta^{(2)\dagger}(k')e^{-i(\vec{k}+\vec{k}')\cdot\vec{s}/2+i(k-k')x}|0> . \qquad \text{(E.6)}
\end{aligned}
$$

Using the anticommutation relation Eq.(E.3,E.4), I obtain

$$
\begin{aligned}
W_{\alpha\beta} &= -\frac{1}{2}\frac{2\pi}{2L}\int d^3s e^{-i\vec{p}\cdot\vec{s}}\sum_n\int\frac{d\vec{k}_\perp}{(2\pi)^3}\frac{m}{k_0}\Theta(L^2-(z-s_z/2)^2)\Theta(L^2-(z+s_z/2)^2) \\
&\quad \{u_\alpha^{(1)}(k)u_\beta^{(1)\dagger}(k)e^{+i\vec{k}\cdot\vec{s}}+u_\alpha^{(2)}(k)u_\beta^{(2)\dagger}(k)e^{+i\vec{k}\cdot\vec{s}} \\
&\quad -v_\alpha^{(1)}(k)v_\beta^{(1)\dagger}(k)e^{-i\vec{k}\cdot\vec{s}}-v_\alpha^{(2)}(k)v_\beta^{(2)\dagger}(k)e^{-i\vec{k}\cdot\vec{s}}\}. \qquad \text{(E.7)}
\end{aligned}
$$

I use here the relation

$$
\Theta(L^2-(z-s_z/2)^2)\Theta(L^2-(z+s_z/2)^2)=\Theta(L^2-z^2)\Theta(L-|z|-s_z/2)\Theta(L-|z|+s_z/2), \qquad \text{(E.8)}
$$

and integrate over $\vec{s}_\perp$ and then $\vec{k}_\perp$ to find

$$
\begin{aligned}
W_{\alpha\beta} &= -\frac{1}{2}\frac{2\pi}{2L}\sum_n\frac{m}{k_0}\int_{-2L+2|z|}^{+2L-2|z|}\frac{ds_z}{2\pi}e^{-ip_zs_z}\Theta(L^2-z^2) \\
&\quad \{u_\alpha^{(1)}(\vec{p}_\perp,k_n)u_\beta^{(1)\dagger}(\vec{p}_\perp,k_n)e^{+ik_ns_z}+u_\alpha^{(2)}(\vec{p}_\perp,k_n)u_\beta^{(2)\dagger}(\vec{p}_\perp,k_n)e^{+ik_ns_z} \\
&- \quad v_\alpha^{(1)}(-\vec{p}_\perp,k_n)v_\beta^{(1)\dagger}(-\vec{p}_\perp,k_n)e^{-ik_ns_z}-v_\alpha^{(2)}(-\vec{p}_\perp,k_n)v_\beta^{(2)\dagger}(-\vec{p}_\perp,k_n)e^{-ik_ns_z}\} \qquad \text{(E.9)}
\end{aligned}
$$

so that the result reads

$$
\begin{aligned}
W_{\alpha\beta} &= -\frac{1}{2}\frac{2\pi}{2L}\sum_n \frac{m}{E}\{u_\alpha^{(1)}(\vec{p}_\perp, k_n)u_\beta^{(1)\dagger}(\vec{p}_\perp, k_n)A_n + u_\alpha^{(2)}(\vec{p}_\perp, k_n)u_\beta^{(2)\dagger}(\vec{p}_\perp, k_n)A_n \\
&\quad -v_\alpha^{(1)}(-\vec{p}_\perp, k_n)v_\beta^{(1)\dagger}(-\vec{p}_\perp, k_n)B_n - v_\alpha^{(2)}(-\vec{p}_\perp, k_n)v_\beta^{(2)\dagger}(-\vec{p}_\perp, k_n)B_n\}, \quad \text{(E.10)}
\end{aligned}
$$

where I introduced

$$
A_n = \frac{1}{\pi}\frac{\sin 2(p_z - k_n)(L - |z|)}{p_z - k_n}\Theta(L^2 - z^2), \quad \text{(E.11)}
$$

$$
B_n = \frac{1}{\pi}\frac{\sin 2(p_z + k_n)(L - |z|)}{p_z + k_n}\Theta(L^2 - z^2), \quad \text{(E.12)}
$$

$$
E = \sqrt{m^2 + p_\perp^2 + k_n^2}. \quad \text{(E.13)}
$$

The spinors for the free field are

$$
u^{(1)}(\vec{p}_\perp, k_n) = \sqrt{\frac{E+m}{2m}}\begin{pmatrix} 1 \\ 0 \\ \frac{k_n}{E+m} \\ \frac{p_+}{E+m} \end{pmatrix}, \quad u^{(2)}(\vec{p}_\perp, k_n) = \sqrt{\frac{E+m}{2m}}\begin{pmatrix} 0 \\ 1 \\ \frac{p_-}{E+m} \\ -\frac{k_n}{E+m} \end{pmatrix}, \quad \text{(E.14)}
$$

$$
v^{(1)}(-\vec{p}_\perp, k_n) = \sqrt{\frac{E+m}{2m}}\begin{pmatrix} +\frac{k_n}{E+m} \\ -\frac{p_+}{E+m} \\ 1 \\ 0 \end{pmatrix}, \quad v^{(2)}(-\vec{p}_\perp, k_n) = \sqrt{\frac{E+m}{2m}}\begin{pmatrix} -\frac{p_-}{E+m} \\ -\frac{k_n}{E+m} \\ 0 \\ 1 \end{pmatrix} \text{(E.15)}
$$

Noting that $A_n = B_{-n}$ and $k_n = -k_{-n}$, I can explicitly calculate the Wigner functions:

$$
W_{00} = -\frac{1}{2}\frac{2\pi}{2L}\sum_n \frac{m}{E_n}A_n, \quad \text{(E.16)}
$$

$$
W_{11} = -\frac{1}{2}\frac{2\pi}{2L}\sum_n \frac{m}{E_n}A_n, \quad \text{(E.17)}
$$

$$
W_{22} = +\frac{1}{2}\frac{2\pi}{2L}\sum_n \frac{m}{E_n}A_n, \quad \text{(E.18)}
$$

$$
W_{33} = +\frac{1}{2}\frac{2\pi}{2L}\sum_n \frac{m}{E_n}A_n, \quad \text{(E.19)}
$$

$$
W_{01} = 0, \quad \text{(E.20)}
$$

$$
W_{10} = 0, \quad \text{(E.21)}
$$

$$
W_{02} = -\frac{1}{2}\frac{2\pi}{2L}\sum_n \frac{k_n}{E_n}A_n, \quad \text{(E.22)}
$$

$$W_{20} = -\frac{1}{2}\frac{2\pi}{2L}\sum_n \frac{k_n}{E_n}A_n, \tag{E.23}$$
$$W_{03} = -\frac{1}{2}\frac{2\pi}{2L}\sum_n \frac{p_-}{E_n}A_n, \tag{E.24}$$
$$W_{30} = -\frac{1}{2}\frac{2\pi}{2L}\sum_n \frac{p_+}{E_n}A_n, \tag{E.25}$$
$$W_{12} = -\frac{1}{2}\frac{2\pi}{2L}\sum_n \frac{p_+}{E_n}A_n, \tag{E.26}$$
$$W_{21} = -\frac{1}{2}\frac{2\pi}{2L}\sum_n \frac{p_-}{E_n}A_n, \tag{E.27}$$
$$W_{13} = +\frac{1}{2}\frac{2\pi}{2L}\sum_n \frac{k_n}{E_n}A_n, \tag{E.28}$$
$$W_{31} = +\frac{1}{2}\frac{2\pi}{2L}\sum_n \frac{k_n}{E_n}A_n, \tag{E.29}$$
$$W_{23} = 0, \tag{E.30}$$
$$W_{32} = 0, \tag{E.31}$$

so that the BGR functions are

$$f_0 = w_{00} + w_{11} + w_{22} + w_{33} = 0, \tag{E.32}$$
$$f_1 = w_{02} + w_{20} + w_{13} + w_{31} = 0, \tag{E.33}$$
$$f_2 = i(w_{02} - w_{20} + w_{13} - w_{31}) = 0, \tag{E.34}$$
$$\begin{aligned} f_3 &= w_{00} + w_{11} - w_{22} - w_{33} \\ &= -2\frac{2\pi}{2L}\sum_n \frac{m}{E_n}A_n, \end{aligned} \tag{E.35}$$
$$g_{01} = w_{01} + w_{10} + w_{23} + w_{32} = 0, \tag{E.36}$$
$$g_{02} = i(w_{01} - w_{10} + w_{23} - w_{32}) = 0, \tag{E.37}$$
$$g_{03} = w_{00} - w_{11} + w_{22} - w_{33} = 0, \tag{E.38}$$
$$\begin{aligned} g_{11} &= w_{03} + w_{30} + w_{12} + w_{21} \\ &= -2\frac{2\pi}{2L}\sum_n \frac{p_x}{E_n}A_n, \end{aligned} \tag{E.39}$$
$$\begin{aligned} g_{12} &= i(w_{03} - w_{30} + w_{21} - w_{12}) \\ &= -2\frac{2\pi}{2L}\sum_n \frac{p_y}{E_n}A_n, \end{aligned} \tag{E.40}$$
$$\begin{aligned} g_{13} &= w_{02} - w_{13} + w_{20} - w_{31} \\ &= -2\frac{2\pi}{2L}\sum_n \frac{k_n}{E_n}A_n, \end{aligned} \tag{E.41}$$
$$g_{21} = i(w_{03} - w_{30} + w_{12} - w_{21}) = 0, \tag{E.42}$$

$$
\begin{aligned}
g_{22} &= w_{12} - w_{03} + w_{21} - w_{30} = 0, && \text{(E.43)}\\
g_{23} &= i(w_{02} + w_{31} - w_{20} - w_{13}) = 0, && \text{(E.44)}\\
g_{31} &= w_{01} + w_{10} - w_{23} - w_{32} = 0, && \text{(E.45)}\\
g_{32} &= i(w_{01} - w_{10} - w_{23} + w_{32}) = 0, && \text{(E.46)}\\
g_{33} &= w_{00} - w_{11} - w_{22} + w_{33} = 0. && \text{(E.47)}
\end{aligned}
$$

Thus, I finally obtain the initial BGR functions as follows,

$$
\begin{aligned}
f_3 &= -2\frac{2\pi}{2L}\sum_n \frac{m}{E_n} A_n, && \text{(E.48)}\\
g_{1\perp} &= -2\frac{2\pi}{2L}\sum_n \frac{p_\perp}{E_n} A_n, && \text{(E.49)}\\
g_{1z} &= -2\frac{2\pi}{2L}\sum_n \frac{k_n}{E_n} A_n. && \text{(E.50)}
\end{aligned}
$$

APPENDIX F

Dirac Solutions between Two Plates

I solve here the Dirac equation in order to obtain the energy spectrum of fermions between two infinite parallel plates at $z = \pm L/2$. The potential depends on the z–coordinate only so it is enough to set $p_x = p_y = 0$ for our purpose, especially for the ground state. In general, I can assume the plane wave solution for the (x, y)–plane and follow the procedure I introduce here. The naive Dirac equation is

$$(\alpha_z \hat{p}_z + \beta m)\Psi = (E + V)\Psi. \tag{F.1}$$

However, this equation suffers from an unphysical solution in the case of the confinement of Dirac particle where the potential goes to infinity at $z = \pm L/2$. The reason is that the total energy of the equation is always positive since the potential goes to infinity. To eliminate this problem, I put the confinement potential into the mass term as was done in the Bogoliubov bag model, i.e.,

$$[\alpha_z \hat{p}_z + \beta(m + V)]\Psi = E\Psi. \tag{F.2}$$

I construct Ψ in terms of two–component spinors

$$\Psi(z) = \begin{pmatrix} \varphi(z) \\ \chi(z) \end{pmatrix}. \tag{F.3}$$

I then have the equations of two-component spinors in Dirac representation,

$$\sigma_z \hat{p}_z \chi_{I,III} + (m + V_0)\varphi_{I,III} = E\varphi_{I,III}, \tag{F.4}$$

$$\sigma_z \hat{p}_z \varphi_{I,III} - (m + V_0)\chi_{I,III} = E\chi_{I,III}. \tag{F.5}$$

¿From the second equation, I obtain

$$\chi_{I,III} = \frac{1}{m + E + V_0}\sigma_z \hat{p}_z \varphi_{I,III}, \tag{F.6}$$

and put this into the first equation to get

$$\hat{p}_z^2\varphi_{I,III} + [(m+V_0)^2 - E^2]\varphi_{I,III} = 0. \tag{F.7}$$

Note that I ignored the term $\frac{\delta(z\pm L/2)}{(m+V+E)}\hat{p}_z\varphi$ at boundaries. This δ–function potential introduces the discontinuity of the first derivative wave function at the boundaries. The solution of this equation which does not depend on the spinor structure is

$$\varphi_{I,III} = Ae^{\kappa z} + Be^{-\kappa z}, \tag{F.8}$$

where

$$\kappa = \sqrt{(m+V_0)^2 - E^2}. \tag{F.9}$$

Using the boundary conditions at $z = \pm\infty$ and choosing the positive helicity state $\begin{pmatrix}1\\0\end{pmatrix}$, I have

$$\varphi_I(z) = A\begin{pmatrix}1\\0\end{pmatrix}e^{\kappa z}, \tag{F.10}$$

$$\varphi_{III}(z) = B\begin{pmatrix}1\\0\end{pmatrix}e^{-\kappa z}, \tag{F.11}$$

so that the wave functions in the region I and III are

$$\psi_I(z) = Ae^{\kappa z}\begin{pmatrix}1\\0\\\frac{-i\kappa}{m+V_0+E}\\0\end{pmatrix}, \tag{F.12}$$

$$\psi_{III}(z) = Be^{-\kappa z}\begin{pmatrix}1\\0\\\frac{+i\kappa}{m+V_0+E}\\0\end{pmatrix}. \tag{F.13}$$

On the other hand, in region II I have the equations

$$\sigma_z\hat{p}_z\chi_{II} + m\varphi_{II} = E\varphi_{II}, \tag{F.14}$$

$$\sigma_z\hat{p}_z\varphi_{II} - m\chi_{II} = E\chi_{II}. \tag{F.15}$$

It is not difficult to prove that no solution exists if $|E| < m$. Therefore, I find the general solution,

$$\varphi_{II} = Ce^{+ikz} + De^{-ikz}, \tag{F.16}$$

where

$$k = \sqrt{E^2 - m^2}. \tag{F.17}$$

The complete wave function of the positive helicity state is

$$\psi_{II}(z) = \begin{pmatrix} Ce^{+ikz} + De^{-ikz} \\ 0 \\ \frac{k}{m+E}(Ce^{+ikz} - De^{-ikz}) \\ 0 \end{pmatrix}. \tag{F.18}$$

To fix the integral constants, I use the matching condition at $z = -L/2$

$$Ae^{-\kappa L/2} = Ce^{-ikL/2} + De^{ikL/2}, \tag{F.19}$$

$$-i\frac{m+E}{m+V_0+E}\frac{\kappa}{k}Ae^{-\kappa L/2} = Ce^{-ikL/2} - De^{ikL/2}, \tag{F.20}$$

And I also have at $z = L/2$,

$$Be^{-\kappa L/2} = Ce^{+ikL/2} + De^{-ikL/2}, \tag{F.21}$$

$$i\frac{m+E}{m+V_0+E}\frac{\kappa}{k}Be^{-\kappa L/2} = Ce^{+ikL/2} - De^{-ikL/2}, \tag{F.22}$$

I obtain equations for C and D by eliminating the constants A and B, To have non-trivial solutions for C and D, the determinant of the coefficient matrix must be zero, which gives us the energy quantization equation,

$$\tan kL = -\frac{k}{m}, \tag{F.23}$$

in the limit of $V_0 \to \infty$. I require the normalization of wave function to be 1

$$\int_{-L/2}^{+L/2} \psi_{II}^\dagger(z)\psi_{II}(z) = 1. \tag{F.24}$$

In this way, I can get explicit expressions of C and D,

$$C = \frac{1}{2}\sqrt{\frac{\beta^2}{1+\beta^2}(L + \frac{1}{m}\cos^2 kL)^{-1}} \tag{F.25}$$

$$D = C\frac{1+\beta^2}{1-\beta^2}\cos kL = \pm sign(E)\, C, \tag{F.26}$$

where

$$\beta = \frac{m+E}{k}, \tag{F.27}$$

and the plus sign is for the 1^{st} and 3^{th} ... eigen–energy of Eq.(F.23) while the minus sign is for the 2^{nd} and 4^{th} ... energy. I can also obtain the negative helicity state in the same way.

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