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The statistical analysis of fatigue data

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The University of Arizona, 1994
THE STATISTICAL ANALYSIS
OF
FATIGUE DATA

by
Chi-liu Shen

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THE UNIVERSITY OF ARIZONA
1994
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ABSTRACT

The overall objective of this study is to develop methods for providing a statistical summary of material fatigue stress-life (S-N) data for engineering design purposes. Specific goals are:

1. Development of an analytical model for characterizing fatigue strength. This model would include: (a) a description of the trend of the data (e.g., the median curve through the data), (b) a description of the scatter of the data (e.g., the standard deviation of N as a function of S), and (c) the statistical distribution of N given S or S given N.

2. Development of an algorithm for constructing a design curve from the data. The curve should be on the safe side of the data and should reflect uncertainties in the physical process as well as statistical uncertainty associated with small sample sizes.

3. Development of a statistical model that can be applied in a structural reliability analysis in which all design factors are treated as random variables.

Significant achievements are:

1. Demonstration, using representative fatigue data sets, that the bilinear model seems to provide a consistently adequate description of the trend of fatigue data.
2. Demonstration, using representative fatigue data sets, that the pure X error source model seems to provide a consistently adequate description of the uncertainties observed in heteroscedastic fatigue data. The pure X error source model is based on recognition of the uncertainties in local fatigue stress.

3. Development of a procedure for constructing a design curve using the tolerance limit concept developed by D.B. Owen. A more practical simplified or approximate Owen curve was shown to have a minimum loss of confidence level, relative to exact Owen theory, under fairly general conditions.

4. Recommendations for methods of developing a statistical model for reliability analysis. A comprehensive study of this issue was not pursued.
CHAPTER 1
INTRODUCTION

1.1 Goals of This Study

The general goal of this study is to develop formal methods for providing a statistical summary or synthesis of metal fatigue data for engineering design purposes. Specific goals are to:

1. Improve existing models or propose new models to better characterize fatigue strength.

2. Compare the performance of various models for characterizing fatigue strength.

3. Develop a strategy for constructing a design curve in the form of a stress-life curve or strain-life curve to characterize fatigue strength for use in a conventional factor of safety design approach.

4. Develop quality measure(s) of the design point or design curve. Assess the quality of the proposed design curve using the quality measure.

5. Compare the effect of different fatigue test strategies (i.e. test plans or experimental designs) on the quality of the proposed design curves.
6. Construct a statistical model based on fatigue test data to incorporate uncertainty of fatigue strength to be used in a comprehensive structural reliability analysis.

1.2 Fatigue Data: Preliminary Considerations

Fatigue failure in materials is due to cyclic loading. When a structural member experiences cyclic loading, a crack typically initiates at the point of stress concentration (See Fig. 1.1). The crack grows larger as cyclic loading continues until the component fractures. The number of cycles of loading at which a component experiences fatigue failure is called the fatigue life and is denoted as $N$. The fatigue life $N$, given stress, is a random variable and is generally observed to have a relatively large variance.

In a fatigue testing, the magnitude of load (stress/strain) is the controlled (or independent) variable and the life is the response (or dependent) variable. The load may be stress controlled (constant amplitude force applied to the component) or strain controlled (constant amplitude deformation of the component). The load ($S$ for stress, $\epsilon$ for strain) applied and corresponding fatigue life $N$ of each specimen in a total sample size $n$ is recorded as $(S_i, N_i)$ or $(\epsilon_i, N_i)$, $i = 1, n$. The stress $S$ may be recorded as the stress amplitude $S_a$ or stress range $S_R$ (Fig. 1.1). The recorded fatigue data $(S_i, N_i)$ or $(\epsilon_i, N_i)$ are then plotted on paper as illustrated in Fig. 1.2. By convention, the response variable $N$ is always plotted on the abscissa.

For mathematical convenience, fatigue data are often transformed before analysis. The log transformation is used most often. For example, the fatigue life $N$ and the strain $\epsilon$ are almost always plotted on a log scale; the stress $S$ may be plotted in linear or log scale. In this study, the raw fatigue data that are recorded are denoted as
Oscillatory load $S(t)$ applied over a long period of time

Crack initiates at point of stress concentration

Failure occurs at N cycles

Figure 1.1: Crack initiation and growth under cyclic loading.
The design curve represents fatigue strength for a conventional factor-of-safety design.
For example, for presentation of fatigue data on semi-log paper, $x$ is equivalent to $S$ and $y$ is equivalent to $\log N$ as shown in Fig. 1.2.

The fatigue data $(x_i, y_i)$ may be fitted by a curve using a statistical method, e.g. the least squares method. But for design purposes, a design curve $Y_D(X)$ targeting at the low percentile (i.e. the safe side) of the life distribution is required. A major goal of this study is develop rational procedures for constructing $Y_D(X)$.

### 1.3 The Importance of Fatigue in Engineering Design

Generally fatigue is considered to be the most important failure mode in mechanical design. It has been observed that fatigue accounts for $60 \sim 90\%$ of all observed service failures in mechanical and structural systems [60]. It's also been estimated that the annual cost of fatigue and fracture in the United States is 100 billion dollars [16]. Fatigue failures are frequently catastrophic; they often come without warning and may cause significant property damage and loss of life. In 1954, two British commercial jet airplanes disintegrated in calm air after approximately 1000 flights as a result of fatigue in the fuselage caused by cabin pressure cycling.

One of the reasons for the dominance of fatigue failure mode is that the fatigue process is inherently unpredictable as evidenced by the statistical scatter in laboratory data [80, 2], See Fig. 1.3. Scatter in cycles to failure frequently exceed one order of magnitude.
Figure 1.3: The S-N curve of 7075-T6 aluminum alloy.

See Ref. [2].
1.4 Methods of Fatigue Analysis

Three basic methods has been developed to model fatigue strength for engineering purposes [5, 24]. They are: (1) a stress based method, (2) a strain based approach, and (3) the fracture mechanics model [14].

1.4.1 The Stress Based or Stress-Life Approach

This method, which has been used for standard fatigue design for almost 100 years, is appropriate for high cycle fatigue (HCF). The term high cycle fatigue implies that the cyclic stress is elastic and that the fatigue life is greater than typically $10^8$ cycles. In a stress-life fatigue test of a specimen, structural system or subsystem, typically a constant amplitude stress range $S_i$ is chosen. The test article is subjected to this oscillatory stress until the article fails. The number of cycles to failure $N_i$ is then recorded. The test is repeated at different stress levels over a pre-determined range of $S$ for a sample size of $n$ identically prepared specimens. The data from this experiment are $(S_i, N_i), i = 1, n$. Fatigue data thus obtained are frequently plotted on and log-log or semi-log paper as shown in Fig. 1.3. A single curve that characterizes fatigue strength, as shown in Fig 1.2 is called "the S-N curve" or sometimes "the Wöhler curve". In Fig. 1.4, the horizontal part of the S-N curve is called the infinite-life region (i.e. $N = 10^6$ to $10^8$ cycles [23, page 16] or [5, page 2]). The stress below which the material has an "infinite" life is called the fatigue limit or endurance limit. Not all materials have a fatigue limit, but those having a body-centered cubic (BCC) crystal structure tend to exhibit this limit.
Figure 1.4: The S-N curve of ductile iron having $45^\circ$V notch.

See Ref. [2].
1.4.2 The Strain Based or Strain-life approach

The strain based method, an extension of the stress based approach to include cyclic plasticity and low cycle fatigue, was developed by Coffin [13], Manson [45] and Morrow [70]. The specimen undergoes a constant reverse strain range resulting in a cyclic stress-strain response as shown in Fig. 1.5. This figure demonstrates a cyclically hardening phenomena where the maximum stress in each cycle increases. Other materials may show a cyclically softening phenomena. In both cases, a cyclically stable condition will be achieved after about 20 ~ 40% of the fatigue life. Usually the stable cyclic response at half life (See Fig. 1.6) is used in the analysis. The total strain is divided into elastic and plastic components as shown in Fig 1.6. Given a test specimen, the number of cycles to failure, elastic strain, plastic strain and total strain values are recorded and plotted as shown in the example of Fig. 1.7.

This approach is suitable for high strain, low cycle fatigue(LCF), at points of stress concentration where plastic deformation plays a dominate role in the fatigue initiation process. With high strain, the cycle life is short (typically less than $10^4$ cycles). Under low strain conditions, most of the fatigue life is spent in the initiation stage, and this method is often used to provide an "initiation" life estimate.

1.4.3 Fracture Mechanics Approach

Developed in 1960s, the fracture mechanics model is suitable for describing fatigue crack propagation and is often used with non-destructive testing to determine the safe life of structures [14]. A component designed under a fracture mechanics approach will be such that if a crack does form, it will not grow to a critical size between specified inspection intervals.
Figure 1.5: The cyclic hardening in 2024-T4 aluminum alloy.
Figure 1.6: The stable stress-strain hysteresis loop.
Figure 1.7: The $\varepsilon$-$N$ curve of Cr-Mo-V steel.
Statistical analysis of crack growth data is not considered herein. This study will focus on S-N and ε-N data analysis.

1.5 Features of Fatigue Data

1. For carbon and low alloy steel, the S-N curve in log-log space is often characterized by a linear relationship and an endurance limit in the high cycle range as shown in Fig. 1.4.

2. For non-ferrous metals such as aluminum and titanium, the general character of the S-N data trend is similar to steels except that there does not seem to be a well defined endurance limit. However, the S-N curve is often nearly linear in log-log space except in the high cycle region.

3. Welded joint fatigue data tend to plot as a straight line in log-log space over a wide range of stresses. At low stress levels sometimes a bi- or tri-linear model is used as a design curve to describe endurance effects. Much welded joint data tend to be homoscedastic (constant scatter band of log life given stresses) except in high cycle range.

4. Fatigue data are generally heteroscedastic (i.e. the variance of log life given stress is not constant over the test range. Fig. 1.3, Fig. 1.8). The variance of log life given stress tends to increase at lower stress levels.

5. The coefficient of variation of fatigue life is relatively large, typically ranging from 0.3 to 1.5. Fig. 1.8 is a probability plot (i.e. empirical cumulative distribution function) which illustrates the large scatter in fatigue data.
Figure 1.8: Heteroscedastic nature of fatigue data of 7075-T6 aluminum alloy. Above is a probability plot on lognormal paper of cycles to failure data of a high performance 7075-T6 aluminum alloy. Wider spread of the data along the abscissa indicates larger variance. Heteroscedasticity is obvious across the different stress levels.[39]
6. The distribution of fatigue life given stress (or strain) level is not known. The Committee E-9 on Fatigue of the American Society for Testing and Materials suggested that the distribution of fatigue life given stress be modeled as lognormal [3]. In an unpublished study, Wirsching [76] has studied the distribution of several sets of fatigue data on a variety of materials and showed the life distribution is consistently better fit by a lognormal than the Weibull. The 3-parameter Weibull has also been suggested. In fact, the location parameter, as a function of $S$, could be used to define a design curve, $Y_D$.

7. In practice, specimens are often tested at many different stress levels with few replications at a given stress level.

8. In life testing, censored data (i.e. runouts) are quite common. Censored items are those specimens that have not failed at the time that the test is terminated. In the high cycle life region, a test may be terminated by plan at a pre-specified life simply because of practical constrains of time and cost.

The general features of fatigue data are summarized in Fig. 1.9. The figure shows fatigue data as heteroscedastic, nonlinear, having a large variance on life and having some censored items. Moreover the underlying distribution of fatigue life given stress (or strain) is unknown. All of these features complicate the statistical analysis of fatigue data.
Figure 1.9: Characteristics of fatigue data.
1.6 Contemporary Design and Analysis of Fatigue Experiments

In this section, various fatigue experimental designs and analysis methods used in engineering communities will be reviewed. Also included is a summary of standards for fatigue experiment design and data analysis used in different industrial countries.

1.6.1 Design and Analysis of Fatigue Experiments in Practice

A fatigue test can be categorized by the basic type of information one needs:

1. S-N test: used to construct a S-N (or ε-N) curve. The curve represents the median fatigue life \( N \) at given stress \( S \).

2. Life test: used to define the P-N curve, representing the estimate of the probability of fatigue failure \( P \) before \( N \) at a given stress \( S \).

3. Strength test: used to define the P-S curve which represents the estimate of the probability of failure as a function of \( S \) for a given life \( N \).

Note that the S-N curve and P-N (or P-S) curve can be combined to form a P-S-N surface on one plot composed of contours of different \( P \) values. See Fig. 1.10.

A fatigue test can also be categorized as a preliminary test or a design improvement test. Preliminary tests characterized by small sample sizes usually have no replications. Little or no statistical analysis is applied. Such data are used for problem recognition studies or for identification of gross fatigue effects (e.g. the shape of the S-N curve).
Figure 1.10: Relationship between P-S-N, S-N, P-N and P-S curves.
In planning fatigue tests, it is important to obtain random samples. Specimens should be randomly assigned to experiment runs. Experiment runs should be conducted in random order, e.g. not all of the replicates at one stress level should be run sequentially [40].

Different fatigue test plans and analysis procedures used in engineering communities are described in the following [3, 40, 42, 44, 30]:

1.6.1.1 Conventional S-N Test

The conventional S-N test is sometimes called a standard test or a classical Wöhler test. Constant amplitude stress cycles are applied to the specimens. These tests can be further categorized as:

1. Tests with no replicates. A single test specimen is selected at each stress level. This is done typically when only a small sample size is available. Four to six specimens are recommended if the shape or S-N curve is known. If the shape is unknown, six to 12 stress levels are suggested (Little [40, chapter 3]). Little also suggested that after gross fatigue effects are identified, the usefulness of a test with no replicates diminishes markedly. Nevertheless, fatigue tests in practice are often non-replicated.

2. Tests with replicate data. More than one specimen is tested at each stress level. Replicate tests are required in order to estimate the variability or the distribution of fatigue life. A sample size of at least four at each single stress level is suggested in order to estimate the variability of the data [3]. A sample size of 10 or more at each stress level is preferred to obtain some indication as to the shape of the life distribution. At least four or five different stress levels are
required to determine the P-S-N curves. To obtain an equal degree of precision throughout the range of the S-N curve, more specimens are needed in the long life range due to the greater variability in life. Two stress levels are used when the range of stress levels is small enough that the S-N curve can be considered as a straight line. Three or four stress levels are used when the S-N curve is smooth with gradual curvature. Five levels or more is not recommended due loss of replications when the sample size is limited.

1.6.1.2 Fatigue Strength Test

The fatigue strength test is also called the strength test, response test, or fatigue limit test. Constant amplitude cyclic stresses are applied. The goal of the test is to estimate the statistical distribution of strength given life.

1. Probit Method [43, 40]: To find the fatigue limit, i.e., the stress at which 50% will survive a specified cycle life $N_s$, tests are conducted at four or five different stress levels with a minimum sample size of five at each stress level. Larger sample sizes are necessary at stress levels having probability levels other than 50%. See Ref. [3] Table 1 for allocation of relative sample sizes at various stress levels. A total sample size of at least 50 is recommended. A test will be suspended at $N_s$ if the specimen is not failed.

2. Staircase (or up-and-down) Method [43, 3, 40]: A variation of the probit method which requires fewer specimens but takes more time because the test is conducted sequentially. It is useful only if the primary interest is the median fatigue strength at a given life $N_s$. The stress level of the next specimen is increased (or decreased) from the stress level of current specimen if the result
of the current specimen is not failed (or failed). At the analysis stage, only the results of less frequent events (failures or runouts) are used. Thus less than half of the samples contribute results. At least 30 specimens are recommended. Compared with the probit method, the staircase method estimates the median fatigue limit more efficiently, but it is not effective for estimating the response curve [40]. Use of the staircase method with a small sample size is possible if the type of fatigue strength distribution and its standard deviation is assumed to be known. See [42] for tables used for the staircase method with small samples.

3. Two Point Method [43, 40]: In this test, the staircase method is used to determine the most effective, closest to the median of the fatigue limit, of two neighboring stress levels. It then concentrates on the sequential test only at the two stress levels.

4. Modified Staircase Method [3]: For a faster result, specimens are divided into several groups. Each group undergoes its staircase method simultaneously and independently.

1.6.1.3 Increasing Amplitude Tests

To reduce test time, some tests sequentially increase the test stress level until the specimen fails.

1. Step Method [3]: A specimen is tested at one stress level to $N_s$. If it doesn’t fail in the first run, increase the stress level (about 5%) and repeat the run for the same $N_s$. Continue to increase the stress level until the specimen fails. The average stress value of the failed run and the last censored run are used as the fatigue strength of that specimen at $N_s$. The empirical cdf plot of the
fatigue strength at $N_s$ is then constructed from these average values. The step method should not be applied on materials that experience significant coaxing or under-stressing effects.

2. Prot Method [3]: Devised by Marcel Prot in France in 1945, this is a rapid method for estimating the fatigue limit. The applied stress level is initially set at about 60 to 70 per cent of its estimated fatigue limit, and then raised at a constant rate. All specimens are tested to failure. At least three different rates are used to establish and check the linear relationship between stress and the power of the loading rate, which is required in the Prot analysis. The fatigue limit of many alloy steels obtained this way are within a few per cent of those found by constant amplitude methods.

1.6.2 Standards in Different Industrial Countries

1.6.2.1 British Standard; B.S. 3518

1. By British standard (B.S. 3518:Part 5: 1966 [11]), the P-S-N curve is constructed from fatigue data tested at different stress levels with replicates of at least five at each stress level. The probability of failure for the i-th ordered life at a stress level is given by $\frac{i}{n+1}$, where $n$ is the number of samples at that stress level.

2. The fatigue strength at $N_s$ cycles is estimated from a total sample size of at least 50. The minimum sample size at each stress level is 5. Each stress level has a percentage of non failed specimens at $N_s$. An approximate probit analysis is conducted to regress the percentage against stress. The regressed model provides the percentage of failure before $N_s$ at any given stress.
3. The staircase method is useful only in estimating the mean stress level at a specific cycle life $N_s$. A variation of the probit method is used. This sequential test generally requires fewer test pieces than the probit, but the test time may be longer. In general at least 30 specimens are required.

4. To estimate the life at a fixed stress level, the British standard specifies analysis methods for: (1) censored data assuming that the underlying distribution is exponential, and (2) complete data assuming that the underlying distribution is lognormal. The goodness of fit test for normality (of log $N$) is based on sample moments of the distribution and requires at least a sample size of 25.

1.6.2.2 American Society for Testing and Materials ASTM E 739 (1980)

1. Assume that the fatigue life is lognormally distributed, and that the variance of log life is constant (homoscedastic) over the tested range [4].

2. The recommended sample size: $n = 6$ to 12 for preliminary and research and development tests, and $n = 12$ to 24 for design allowables and reliability tests.

3. Replication recommendations: percent replication of 17 to 33 for preliminary and exploratory, 33 to 50 for research and development testing for components and specimens, 50 to 75 for design allowable data and 75 to 88 for reliability data. Given a specified sample size $n$ and appropriate percent replication, the number of stress (or strain) levels $L$ may be determined by the relation: percent replication $= 100 \left(1 - \frac{L}{n}\right)$.

4. Method of analysis. A linear $S$-$N$ model and the least squares method is used for statistical analysis of linear or linearized fatigue data; no extrapolation is
allowed. Estimation of probabilities below 5% (i.e. quantiles below 5% of the life distribution given stress) should not be considered using this method.

1.6.2.3 Japan Society of Mechanical Engineers JSME S 002 (1981)

1. S-N testing: A total of 14 specimens are recommended, eight for "the slope part" (finite life region) at four stress levels (equally spaced in $S$ or log $S$) with two replications each, and six for the level part (fatigue limit region) using the staircase method. The finite life region data is assumed linear, and data are analyzed by the least squares method [29]. The fatigue limit is determined by taking the average of the stress levels in a staircase test.

2. Fatigue life test: A model of the distribution of life can be assumed as a lognormal, two-parameter Weibull or three-parameter Weibull. The mean, standard deviation, and percentile of the log life is determined. At least seven specimens are recommended for estimating the failure probability in the 10 ~ 90% range when the distribution type is not assumed. At least 25 specimens are suggested to check the distribution type using the $\chi^2$ test.

3. Fatigue strength test: The normal distribution is assumed for fatigue strength. The probit method, staircase method and two point method are adopted.

4. Statistic test: The $F$ test is used to test the linearity of the finite life region data of the S-N curve. For comparing two S-N curves, the $F$ test is used to test the ratio of the variances, while the $t$ test is used to test the intercept or slope. The $t$ test is also used to compare fatigue limits of two sources of specimens.
1.6.2.4 France; NF A 03-405 (1991)

1. By the French standard [36], the fatigue limit is obtained using the staircase method. A stress step about 5% of the estimated fatigue limit or 10 to 20 MPa is used. At least eight specimens are needed for estimating the mean. At least 15 are needed for estimating its error.

2. A graphical method using normal probability paper for \( \log N \) is applied using at least 10 specimens.

3. Linear S-N relations are analyzed with complete data. At least eight specimens are used to estimate the mean curve covering three decades of life. The test plan specifies five levels with a replicate of two at each of the top, median and bottom levels. For a reliability model to be used for design purposes, at least 25 specimens are needed to cover five stress levels with replicates of five at each stress level. The \( F \) test is used to check the linearity of the data. The minimum curve for design at \( p\% \) with 95% of confidence level is constructed using the one-sided \( t \)-distribution.

4. The \( P/S/N \) curve is described by a model of Bastenaire [71]

\[
N = A \frac{\exp[-(\frac{S-E}{B})^2]}{(S-E)}
\]  

(1.1)

A, B, C and E are model parameters, and E is the endurance limit.

1.7 Goals of Fatigue Data Analysis for the Design Engineer

Unlike the "build it and bust it" era of the past, modern design engineers rely on computer aided design to consider a variety of design possibilities prior to building of a
prototype. Such analytical design procedures depend upon reliable material property data and the analysis of these data [65, page 15]. To assist a designer in making decisions regarding the fatigue failure mode, it is customary to provide a statistical summary or synthesis of the data. The transformed test data of sample size n is 

\[(x_i, y_i), i = 1, n,\]

where \(x_i\) is the transformed stress or strain and \(y_i\) is the transformed fatigue life. A design point \((x_d, y_s)\) is a point on the design curve such that for a given service life \(y_s\), a proper design stress \(x_d\) is derived to ensure a certain pre-specified reliability (i.e., without fatigue failure) during service life \(y_s\).

Most commonly the statistical summary of the data is a design curve, constructed on the lower or safe side of the data as shown in Fig. 1.2. A simple method for defining the design fatigue strength might be to draw, by eye, a curve that follows the data and provides a little "white space" between the dots and the curve. A rigorous procedure following sound principals of mathematical statistics however can be extremely complicated but necessary for consistent design decisions. The goal of statistical fatigue data analysis for an design engineer is to construct a design curve which guarantees a pre-specified reliability with a pre-specified confidence level.

There is another issue. The design curve can provide the design point with pre-specified reliability and confidence only if the applied stress has a deterministic constant amplitude. In a probabilistic approach, a reliability model for fatigue strength is required when all design factors (e.g. stress, dimensions) are considered as random variables. After distribution information of these random variables is assessed, a probability of failure of a member is estimated using well known reliability analysis methods. In a fatigue strength reliability model, some or all of the model parameters should be considered to be random variables to quantify the uncertain nature of the
fatigue data. Here the general goal of fatigue data analysis is to translate the data into random variables of the model parameters.

Thus the general goals of statistical analysis of fatigue data as addressed herein are twofold: (1) define a design curve (or design points), and (2) construct a reliability model.

1.8 Why Statistical Analysis of Fatigue Data is Complicated

The features of typical fatigue data, as described in Sec. 1.5, combined with the goals of the fatigue data analysis mentioned above, suggest the following difficulties in providing a statistical analysis of fatigue data:

1. The S-N data will be, in general, nonlinear, even after a transformation, e.g. to log-log space.

2. The relatively large variance of fatigue life given stress suggests large uncertainties in estimates of the model parameters. Relatively larger sample sizes are desired to reduce uncertainty in the parameter estimates.

3. Sample sizes are often small (e.g. 10 ~ 50) due to the expense of testing. The fatigue test of one typical smooth laboratory specimen might cost 500 to 1,000 U.S. dollars.

4. To ensure high reliability, design curves targeting the low percentiles $y_\alpha$ of the fatigue life are required. The problem with estimating low probability points is that uncertainty in the estimates increases as the probability level $\alpha$ decreases [62].
5. Sometimes censored data (i.e. runouts) may exist. A specimen, particularly in the high cycle range might not fail within a reasonable time period.

6. The existence of an endurance limit presents mathematical difficulties. Fatigue life at the endurance limit is likely to be very long. Often failures do not occur within a practical time period. Thus many specimens may be censored at the fatigue limit. Censored data gives little information in statistical sense. Ultimately, mathematical modeling at the fatigue limit may be mostly engineering judgment without much sound physical evidence.

7. Data may be heteroscedastic (i.e. the variance of life is not constant across the stress range tested). Typically fatigue data as plotted in log-log space exhibit broader scatter at lower stress levels.

8. The underlying statistical distribution of fatigue life $N$ given stress $S$ is unknown.

9. It is common practice for the analyst to invert the independent and dependent variable in data analysis (i.e. doing regression of $x$ on $y$, assuming that life is the independent variable and load is the dependent variable). The quality of the estimates (e.g. the fitted curve, the design curve etc.) resulting from this inverse analysis is not well known.

In summary, difficulties in fatigue data analysis are due to the expense of testing and the long fatigue life required for components. If fatigue tests were inexpensive, large sample sizes could produce accurate estimates of: (a) the distribution of life given stress, and (b) a design curve in the low probability region. It is the author's
opinion that ultimately economic analysis could provide the most viable basis of determining a optimum experimental design and design curve.

1.9 Fatigue Design Curve

The stress-life (or strain-life) model with the estimated parameters represents the estimate of the median (or mean) life $N$ at given applied stress $S$/strain $\varepsilon$. Thus, using this model, a specimen would have roughly a 50% probability of failure before life $N$ at an applied stress/strain. However for design purposes, a curve representing a lower bound (safe side) of the data is desired to characterize fatigue strength. Methods of constructing such a curve are reviewed below. All the methods assume that $y_i$ (usually $y_i = \log N_i$) is normally distributed, homoscedastic and independent.

1. Design Curve By Eye. A simple way to construct a design curve is to draw, by eye, a curve that follows the data and provides a little “white space” between the data points and the curve. The design curve may not be too far removed from that derived by a formal statistical method. The drawback of this method is that it is subjective, and therefore the method lacks both consistency and a quantitative measure of its quality.

2. Lower 2-sigma or 3-sigma Design Curve. This design curve $Y_D(x)$ is derived by subtracting two or three times the sample standard deviation $s$ from the estimate of the mean curve $\hat{Y}$.

$$Y_D(x) = \hat{Y}(x) - Ks \quad (1.2)$$

where $K = 2$ or 3. This method fails to account for the statistical distribution of the estimated parameters. Therefore the error is larger for small sample sizes
Another version of this criterion is to select a lower probability level and compute the corresponding K factor. For example, choosing the lower 1% level as the design criteria implies that $K = 2.33$.

3. The ASME Boiler and Pressure Vessel Code: The $\varepsilon$-$N$ design curve is derived by applying simultaneously a safety factor of two on stress/strain and 20 on life relative to the mean curve. A lower bound envelope of the two curves is defined as the design curve. See [14, page 441] or [78].

4. One Dimensional Tolerance Limit. Use the factors $K_{p,\gamma,n}$ for the one-sided tolerance limit for a normal distribution (See [51] or [3, table 33]). The design point accounts for the uncertainty of estimated parameters and is derived as

$$Y_D(x) = \hat{Y}(x) - K_{p,\gamma,n}\hat{\sigma}$$

(1.3)

where $Y_D$ is the design point of transformed life $Y$, $1 - p$ is the lower probability level, $\gamma$ is the confidence level, and $\hat{Y}$ and $\hat{\sigma}$ are the sample mean and the sample standard deviation of $Y$ respectively. The design point guarantees that 100$\gamma$% of the time the probability of failure before life $Y_D$ is less than $1 - p$. The method is valid when samples are tested at a single stress level and when $Y$ has a normal distribution. The term "one dimensional" is used to imply that $Y$ is the only variable. However, this one dimensional tolerance factor is also used as an approximation to determine the tolerance limit in the two dimensional regression case in which both $X$ and $Y$ are variables. In the regression case, this method is incorrect because it accounts only partially for the uncertainty in the estimators.

5. Tolerance Limit in the Regression Case. Using Owen's tolerance factor [61]

$$K_{p,\gamma,\bar{x},d,x}$$

(where $1 - p$ is the lower probability level, $\gamma$ is the confidence level, $\bar{x}$
is a vector of dimension $n$ which depends upon the stress/strain level setting of each specimen in a test of size $n$, $x$ is the stress/strain level, and $d$ is the degree of the polynomial model $\hat{Y} = f(x)$; or more generally, $d + 1$ is the number of undetermined parameters of the linear model $\hat{Y} = f(x)$. The design curve $Y_D$ is defined as:

$$Y_D(x) = \hat{Y}(x) - K_{p,\gamma,d,x}x\hat{\sigma} \quad (1.4)$$

The Owen K-factor, which will be a function of the stress/strain level $x$, guarantees a confidence limit $\gamma$ at any given stress/strain level $x$. When all specimens are tested at the same stress/strain level $x$, Owen’s tolerance factor reduces to the one dimensional tolerance factors. See Appendix A for details of Owen’s theory.

6. The Simultaneous Tolerance Limit [50, page 123]: Owen’s tolerance limit guarantees the same confidence level at any specific stress level. The simultaneous tolerance limit guarantees that 100\% of the time every point $(x_d, y_d)$ on the design curve is such that $y_d < y_\alpha$ where $y_\alpha$ is the $\alpha = 1 - p$ quantile of $Y$ given load $x_d$. This statement is valid for any $x$ (stress or stain level). The simplest formulation of the simultaneous tolerance factor for the simple linear model is derived from the Bonferroni inequality (See [50, page 8]) and Working-Hotelling band (See [55, page 163,245]):

$$K_s = (2F_{2,n-2}^{\alpha/2})^{1/2} \left[ 1 + \frac{(x - \bar{x})^2}{\sum_i(x_i - \bar{x})^2} \right]^{1/2} + N(P) \left( \frac{n - 2}{\chi^2_{n-2}} \right)^{1/2} \quad (1.5)$$

Where $\chi^2_{n-2}$ is the $\alpha/2$ quantile of the $\chi^2_{n-2}$ distribution. See [38] for another calculation method for simultaneous tolerance factors.
1.10 The Scope of the Study

This research addresses the following:

1. Development of quality measure(s) of the design point or design curve. The basic measure considered herein is the confidence level associated with a design point corresponding to a specified lower probability level. The higher the confidence level, the better the quality of the design point.

2. Development and comparison of design points or design curves. An approximate Owen’s tolerance limit is developed for ease in calculation and incorporation into design equations. The mathematical and statistical justification of the approximate Owen’s tolerance limit is derived in detail. Comparison of the performance of the approximate Owen’s curve against that of the Owen’s curve is provided.

3. Comparison of different fatigue experimental designs. A test plan is characterized by the sample size and the stress level settings. Stress levels may be equally spaced and uniformly replicated across the test range or they may be weighted (i.e. non uniformly replicated) on one end or both ends of the test range.

4. Construction of a reliability model based on fatigue test data for reliability analysis. The S-N curve or design curve can be used to assess reliability only in cases where all design factors (e.g. the stress) are deterministic. When the fatigue model parameters are presented as random variables, incorporation of random design factors (e.g. the stress, dimensions) in a comprehensive reliability analysis is possible. Therefore, in a probabilistic approach to design, it is necessary to represent the fatigue model parameters as random variables in
order to reflect simultaneously the uncertainty in the inherent behavior of the material and the estimators. An elementary approach to translate the fatigue data into such parameters in a reliability model is presented.
CHAPTER 2
Statistical Models and Analysis to Characterize Fatigue Data

2.1 A General Guide to Notation

General rules for mathematical notation used in this article are listed below. When different rules are used in a specific section or paragraph, further explanation will be given.

1. Random variables are denoted in capital letters, e.g. $X$, $Y$. Note that a deterministic variable can be treated as a random variable with zero variance. The controlled variable stress/strain is denoted as a capital letter for general discussion because in some models they are considered as random.

2. Vectors are denoted in bold type lower case letters, e.g. $a$, $b$, $\eta$, $\lambda$.

3. Matrices are denoted in bold type upper case (capital) letters, e.g. $X$.

2.2 Models for Fatigue Data: General Considerations

The first step towards providing a statistical summary of fatigue data is defining the analysis model which describes:
1. The transformation of stress/strain and life. Transformations are frequently made before the raw data \((S,N)\) are analyzed in an attempt to linearize the data. Some transformations have free parameters to be determined during the analysis while others are fixed with no free parameters. An example of a fixed transformation is the widely used log transformation, \(X = \log S\), and \(Y = \log N\). An example of transformation with free parameters is the Box-Cox transformation \(Y = \frac{N^\lambda - 1}{\lambda}\), where \(\lambda\) is the free parameter to be determined in the analysis.

In general the transformation of stress/strain is denoted as:

\[
X = g(S; \eta) \tag{2.1}
\]

Where \(S\) is either stress or strain, and \(\eta\) is a vector of free parameters in the transformation function \(g\). For a fixed transformation, the number of free parameters is zero.

The transformation of life is denoted as:

\[
Y = h(N; \lambda) \tag{2.2}
\]

where \(N\) is fatigue life, and \(\lambda\) is a vector of free parameters in the transformation function \(h\). The transformed stress/strain and life are denoted as \(X\) and \(Y\) respectively in this article.

2. The trend of the data; e.g. the fitted curve as shown in Fig. 2.1. The general analytical form is

\[
\hat{y} = \hat{y}(x; a) \tag{2.3}
\]

Where \(\hat{y}\) is the transformed life, \(x\) is the transformed load (stress/strain); both are deterministic. \(a\) is a vector of parameters. However, statisticians often
Figure 2.1: Example of modeling stress-life relationship.
model the dependent variable as the sum of the trend of data and an error term:

\[ Y = y(x; a) + \epsilon \]  \hspace{1cm} (2.4)

The error term \( \epsilon \) is a random variable having \( E(\epsilon) = 0 \) which implies that \( \mu_Y(x) = \mu_{Y|X=x} = y(x; a) \). For most distribution types, an appropriate parameter to describe the trend is the mean, \( \mu_Y(x) \). Some distributions, e.g. Weibull, do not use the mean as the parameter. However their parameters can be transformed to a new set of parameters which does include the mean, Kececioglu (1991).

For convenience, the trend of data may also be denoted as \( \mu_Y(x) \):

\[ \mu_Y(x) = \mu_Y(x; a) \]  \hspace{1cm} (2.5)

For example, the trend of data may be modeled as quadratic with parameters \( a = (a_0, a_1, a_2) \)

\[ Y = a_0 + a_1 x + a_2 x^2 + \epsilon \]  \hspace{1cm} (2.6)

And,

\[ \mu_Y(x) = a_0 + a_1 x + a_2 x^2 \]  \hspace{1cm} (2.7)

3. The uncertainty or scatter in the life; a general analytical form of the standard deviation of transformed life \( Y \) given transformed load \( X = x \) is

\[ \sigma_Y(x) = \sigma_Y(x; b) \]  \hspace{1cm} (2.8)

Where \( b \) is a vector of parameters. For example, the standard deviation may be modeled as a linear function of stress/strain with parameter \( b = (b_0, b_1) \)

\[ \sigma_Y(x) = b_0 + b_1 x \]  \hspace{1cm} (2.9)
4. Some distribution types have three parameters (e.g. the three parameter Weibull). The third parameter may also modeled as a function of stress/strain

\[ \alpha_Y(x) = \alpha_Y(x; c) \]  

(2.10)

Where \( c \) is a vector of parameters.

5. The type of distribution of the transformed life \( Y \) given \( X = x \). Here only the type of distribution, e.g. normal or Weibull, is specified. Distribution parameters are defined as above. All together they define the distribution as the p.d.f. \( f_{Y|X=x}(y; \mu(x), \sigma(x), \alpha(x)) \), or simply \( f_{Y|X=x}(y; \theta(x)) \) where \( \theta(x) \) is a vector of \( \mu(x), \sigma(x), \alpha(x) \). It is customary to assume that the same distribution type applies for all \( S > 0 \). Historically the lognormal and the two and three parameter Weibull distributions have been most commonly used for \( N \). When \( N \) is assumed to be lognormal and \( Y = \log N \), then \( Y \) is normal. When \( N \) is assumed to be Weibull and \( Y = \log N \), then \( Y \) has a type 1 extreme value distribution (EVD) [53, page 36].

In summary, the total parameters describing the model is a vector \( \beta \)

\[ \beta = (\eta, \lambda, a, b, c) \]  

(2.11)

### 2.3 A Summary of Models Which Have Been Proposed

Although most of the models listed in this article were first developed for S-N curves, their use are general, and they could be used for e-N curves also. The models are strictly empirical.
2.3.1 Transformations

Transformations on $S$ and $N$ are used to make the data more amenable to analysis as well as to improve graphical presentation. In particular, transformed $S$-$N$ data is often required before utilizing the least squares method. Commonly used transformations are listed in Table 2.1. To be general, a non transformed variable is considered to be a transformed variable after a null transformation (Item 1 and 6 in Table 2.1). The $E$ in Items 3 and 4 denotes the endurance limit. The $\lambda$ in Items 8 to 12 denotes a free parameter. Some of these transformation are used in fatigue models as described in the next section.

<table>
<thead>
<tr>
<th></th>
<th>Analytical form</th>
<th>Remark</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$X = S$</td>
<td>null transform</td>
</tr>
<tr>
<td>2</td>
<td>$X = \log S$</td>
<td>log transform</td>
</tr>
<tr>
<td>3</td>
<td>$X = \log(S - E)$</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$X = \log(S/E)$</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$X = \begin{cases} S^n &amp; \eta \neq 0 \ \ln S &amp; \eta = 0 \end{cases}$</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>$Y = N$</td>
<td>null transform</td>
</tr>
<tr>
<td>7</td>
<td>$Y = \log N$</td>
<td>log transform</td>
</tr>
<tr>
<td>8</td>
<td>$Y = \log(N + \lambda)$</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>$Y = (\log N)^\lambda$</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>$Y = N^\lambda$</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>$Y = \begin{cases} \frac{N^\lambda - 1}{\lambda} &amp; \lambda \neq 0 \ \ln N &amp; \lambda = 0 \end{cases}$</td>
<td>Box-Cox transform type 1</td>
</tr>
<tr>
<td>12</td>
<td>$Y = \begin{cases} \frac{(N+\lambda_2)^{\lambda_1} - 1}{\ln(N+\lambda_2)} &amp; \lambda_1 \neq 0 \ \lambda_1 &amp; \lambda_1 = 0 \end{cases}$</td>
<td>Box-Cox transform type 2</td>
</tr>
</tbody>
</table>

Table 2.1: Commonly used transformations

Some of the important transformations are discussed below:

1. Variance stabilizing transformation [8, Page 231–238]. When data are heteroscedastic, a power transformation (Item 10 for example) on the dependent
variable, N, may be applied so that the transformed variable Y is nearly homoscedastic across the test range of the independent variable (stress S or X). The shape of the mean curve is changed after transformation.

2. Box-Cox transformation [9]. This is a transformation on the dependent variable N, with one or more undetermined parameters. If it can be assumed that the transformed dependent variable Y is homoscedastic and normally distributed, then the least squares method can be applied to the transformed data. Usually a simple linear model can be fitted to the transformed data. Two types of transformations have been proposed by Box and Cox, Items 11 and 12 in Table 2.1 having undetermined parameters λ's. But the concept of a Box-Cox transformation is general and not limited to these two forms. To determine λ, one may use the maximum likelihood method to solve for parameters β in Eq. 2.11 all at one time. But a more convenient method is to select values of λ and transform the data to a standardized variable as a function of the selected λ [55, page 149-150]. Then use the least squares method to find the error sum of squares of the standardized variable. The error is proportional to the negative of the maximum likelihood value of the non-transformed data. The λ that yields smallest value of the sum of squares of the error is considered to be the best estimate.

3. Transformation on the independent variable[10]. A transformation (Item 5) on the independent variable, S, may be helpful in simplifying the functional form of the mean curve, i.e. to linearize the mean curve, while leaving the distribution of the dependent variable (N or Y) intact. Note that the variance stabilizing transformation on N may change the distribution of the dependent variable N.
and the shape of the data trend at the same time. To solve for $\eta$, one may use
the maximum likelihood method to solve for parameters $\beta$ (which includes $\eta$)
in Eq. 2.11 all at one time.

### 2.3.2 Models to Describe The Trend of Data

The trend, or mean or median of the data is described by an analytical form as
if life were deterministic. The notation $N$, $Y$ or $\log N$ is used here instead of $\mu_Y$ or
$\mu_{\log N}$. Several proposed models for the trend of fatigue data are listed in Table 2.2.

<table>
<thead>
<tr>
<th>Reference</th>
<th>Analytical form</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Wöhler(1870)</td>
<td>$\log N = a - bS$</td>
</tr>
<tr>
<td>2 Basquin(1910)</td>
<td>$\log N = a - b \log S$</td>
</tr>
</tbody>
</table>
| 3 linear with endurance           | $\log N = a - b \log S$ for $S > E$
                                        | $\log N = \infty$ for $S < E$
| 4 three segment linear            | composed of three segment of straight lines |
| 5 Bilinear                        | $x = a_0 + a_1 \mu_Y + a_2 \sqrt{(\mu_Y - s_1)^2 + s_2}$ |
| 6 Strohmeyer(1914)                | $\log N = a - b \log (S - E)$ |
| 7 Palmgren(1924)                  | $\log (N + d) = a - b \log (S - E)$ |
| 8 Weibull(1949)                   | $\log (N + d) = a - b \log [\frac{(S - E) / (R - E)}]$ |
| 9 Bastenaire(1963)                | $N = A^{\exp[\frac{(S - E)}{B}]}$ |
| 10 Bastenaire(1963) log form      | $\log N = a - \log (S - E) + [(S - E)/b]^c$ |
| 11 ASME Boiler Code               | $S = BN^{-\frac{1}{2}} + S_e$ |
| 12 MIL-HDBK-5                     | $\log N = a - b \log (S_e - E)$ |
| 13 polynomial                     | $\log N = a_0 + \sum_{i=1}^{n} a_i (\log S)^i$ |
| 14 simple non-linear              | $\log N = a_0 + a_1 \log S + a_2 (\log S)^b$ |
| 15 variance stabilized non linear | $(\log N)^\lambda = a_0 + a_1 \log S + a_2 (\log S)^\xi$ |
| 16 general strain-life equation   | $A^*$ = $\frac{\sigma_f}{k} (2N_f)^b + c_f (2N_f)^c$ |

Table 2.2: Models to describe the trend of fatigue data
Modeling of S-N curves began in the nineteen century. In 1870, Wöhler introduced the linear model (Table 2.2, Item 1 ) for an S-N curve in log-linear space. In 1910, Basquin introduced the linear model in log-log space (Table 2.2, Item 2 ). This form is widely employed today. Both Wöhler’s model and Basquin’s model describe behavior only in the range where S is above the endurance limit. To model the endurance limit, one or more straight line segments can be used (Table 2.2, Item 3 ). For example, models for welded joint fatigue data may be composed of three line segments that intersect at two points (Table 2.2, Item 4 ). Another model that fits fatigue data well is the bilinear model (Table 2.2, Item 5 ) in which Y may be log N or log S and X may be log S or log N.

Strohmeyer (1914) and Palmgren (1924) modified Basquin’s model by introducing a shift in S or N, as shown in Table 2.2, Item 6 and Item 8 respectively. In these and the following S-N models, E represents the endurance limit and the R is the ultimate strength. Both models become a horizontal line at the endurance limit. The ASME Boiler Code model (Table 2.2, Item 12) has basically the same form as Strohmeyer model.

Weibull, Stussi and Bastenaire proposed an S-shaped model to describe the S-N curve as given in Table 2.2, Item 9, Item 7 and Item 10 respectively. Item 11 is equivalent to Item 10 if the a in Item 11 is replaced by the log A in Item 10. The S-shaped model could provide a general representation of the fatigue data over a range which includes plastic strain. However, in practice the sample size is often too small to obtain a good estimate of the parameters of the S-shaped model [71].

MIL-HDBK-5 [49, page 9-8811] models the trend of fatigue data using the form of Item 13 in Table 2.2, a form which is similar to the Strohmeyer model. This form accounts specifically for mean stress. Note that $S_{eq} = S_{max}(1 - R)^{A3}$, where $S_{max}$ is
the maximum stress in the stress cycle, \( R = \frac{S_{\text{min}}}{S_{\text{max}}} \), the stress ratio where \( S_{\text{min}} \) is the minimum stress, and \( A3 \) is a parameter.

Polynomial models are often used in industry, a general form of which is shown in Table 2.2, Item 14. A simple nonlinear model, Item 15, is commonly used by engineers. To stabilize the variance of life, Item 15 is modified by a second transformation on life to form Item 16.

The strain-life relation (Table 2.2, Item 17) was developed specifically for modeling the \( \epsilon-N \) curve when gross cycle plasticity is present. Geometrically speaking, the strain-life relation is bilinear in log-log space with curvature in the region of the intersection of the two lines. However the curvature can not be controlled as it is in the bilinear model of Item 5. In the general strain life model, \( \frac{\Delta \epsilon}{2} \) is the total strain amplitude and is the sum of elastic strain amplitude \( \frac{\Delta \epsilon}{2}^e \) and plastic strain amplitude \( \frac{\Delta \epsilon}{2}^p \). \( N_f \) is the cycles to failure, and \( E \) is the modulus of elasticity. The four unknown parameters to be fitted in the model are \( \sigma_f \) and \( b \), the fatigue strength coefficient and exponent respectively, and \( \epsilon_f \) and \( c \), the fatigue ductility coefficient and exponent respectively.

All the models listed above in Table 2.2 describe the median of life (or mean of log life) versus (log)stress. The variance of life versus stress is considered in the following.

### 2.3.3 Models for The Scatter of Life Given Stress

Typically the fatigue life distribution will be heteroscedastic. To deal with heteroscedastic data, three concepts have been proposed.

The first idea, proposed by Hinkle and Emptage\[26\], suggested the use of the Box-Cox transformation (See Item 11 in Table 2.1) on aluminum fatigue life \( N \) to stabilize
the variance of life and linearize the S-N curve at the same time. Sec. 2.3.1 describes details of this method.

The second approach was to model the homoscedasticity of life and fit the parameters by the maximum likelihood method. Nelson[54] has proposed a simple linear or quadratic model for the trend of the S-N curve and a simple linear model (Eq. 2.12) for the natural log of the standard deviation of log life.

\[ \sigma_Y = \exp(a + bx) \]  

(2.12)

where \( x = \log S \) and \( Y = \log N \). In general the standard deviation could be modeled as a polynomial function of the stress as

\[ \sigma_Y = b_0 + \sum_{i=1}^{d} b_i(x)^i \]  

(2.13)

where \( d \) is the degree of the polynomial.

The third idea is based on consideration of the constant scatter phenomena of data points in the \( X \), i.e. \( S \) or \( \log S \), direction. Described in the next section is the inverse model, a model for heteroscedasticity of fatigue data which belongs to this third approach. Other models based on the same concept are proposed in detail in Chapter 3.

2.3.4 The Inverse Model; Life \( N \) as the Independent Variable and \( S \) as the Dependent Variable

In this article, a model that incorrectly treats fatigue life as the independent variable and stress/strain as the dependent variable is called the “inverse model”; A model that correctly treats fatigue life as the dependent variable and stress/strain as the independent variable in analysis is called a “non-inverse model”.

Studies [6, 56, 69] of many fatigue data sets have shown that the variance of fatigue strength given life tends to be constant and relatively small (C.O.V. ≈ 10%) while the variance of log fatigue life given stress is a function of stress and relatively large. Some theoretical background for this observation can be found in [46, 68].

For non-inverse models, fatigue life $Y$ is often heteroscedastic. It is often difficult to model the standard deviation of $Y$ as a function of $X$ appropriately. Moreover, these models are generally difficult to fit numerically. If the life is placed in the ordinate, the trend of fatigue data in a non-inverse model tends to be vertical (a singular condition in mathematical sense) in the endurance limit region. This also causes numerical difficulties in fitting the data. The widely employed strain-life model (Item 17 in Table 2.2) expresses life as a function of strain and therefore makes the treatment of life as the independent variable much easier in the analysis. For these reasons, it is found that treating life $Y$ as the independent variable and stress/strain $X$ as the homoscedastic dependent variable produces a better fit in many cases. This becomes apparent from a visual examination of Fig. 2.2 [71, 57]. Some engineers model stress as the dependent variable in standard practice. And in the VAMAS (The Versailles Project on Advanced Materials and Standards) round-robin test [59, 58] for high cycle fatigue data analysis, six out of seven of the analyses (submitted by participants from Japan and Germany) used stress as the dependent variable.

2.3.5 Models for the Distribution of Life

Various statistical models have been used to model fatigue life given stress. The models that are most commonly used now are the lognormal and the two or three parameter Weibull. It has been found that Weibull and lognormal both perform well for a wide range of materials [28]. The three parameter Weibull has the promise of
$x = s$

$A(Y(X))$

- failed data
- censored data

$Y = \log N$

Figure 2.2: Typical fatigue data scatter in the direction of stress and life.
providing an improved fit by virtue of an extra parameter. However the engineering profession has for the most part adopted the lognormal as a model for statistical analysis of fatigue data.

It takes a relative large sample size (of about 35 or more of replicate data [41]) to distinguish the goodness of fit between Weibull and lognormal. Although such samples are often not available, some research [69] and an overwhelming body of evidence supports the lognormal as an acceptable distribution for \( N \), or equivalently, the normal as an appropriate distribution for \( \log N \). An unpublished study by Wirsching [81] has shown for a wide variety of fatigue data, the lognormal generally provides a better fit to the cycles to failure data than other two parameter families including the Weibull.

### 2.4 Statistical Analysis Methods

Given the type of transformation on stress/strain and life, the basic analytical form of \( \mu_{Y|X}, \sigma_{Y|X}, \alpha_{Y|X} \) and the distribution family for life, the whole set of model parameters are defined. The complete vector of parameters are

\[
\beta = (\eta, \lambda, a, b, c,)
\] (2.14)

Where \( \eta, \lambda, a, b, c \) are defined in Sec. 2.2. The fitting of the model involves translating the data \( (S_i, N_i) \) or \( (X_i, Y_i) \), \( i = 1, n \) into estimates of the parameters, denoted as \( \hat{\beta} \).

Three methods for estimating parameters are (1) least squares method (2) maximum likelihood method and (3) method of moments. These are reviewed in the following subsections.
2.4.1 Least Squares Method

The method of least squares may be the most widely used analysis procedure for fatigue data. Least squares analysis requires the following assumptions ([25, Pages 359-459] or [8, Pages 239-241]):

1. Independence for each $Y_i$. The least squares model assumes that the error terms are independent and identically distributed (i.i.d.). The correlation between nuisance variables and controlled variables should be avoided to guarantee the i.i.d. of $Y$. That is to say that the experiment should be randomized as described in Sec. 1.6.1. For example, the machines to test the specimens might be considered as a nuisance variable; different machines might have different biases. If all the specimens tested under 100MPa are tested on machine A, and all the specimens tested under 200MPa are tested on machine B, than the life $Y$ at the two stress levels will be correlated with the test machines and therefore no longer independent. Randomizing the test is essential.

2. The data or the transformed data, should be homoscedastic. In the case of fatigue data, variance of fatigue life, or a transformed life, is assumed to be constant over the stress range being tested. i.e., $\sigma_Y(x)$ is constant.

3. Fatigue life given stress is normally distributed. This assumption is required only when assessment of the confidence interval of estimates is required. However, the bootstrap method [18, 34, 17, 19, 37] can be used to assess the confidence interval without the normality assumption.

When the model for the mean of $Y$ is of the form in Eq. 2.15, it is called a linear model([1, Chapter 4] and [15, Chapter 5]). In Eq. 2.15, $Z_i(X)$ is any function
of $X$ without undetermined parameters. For example, $Z_i(X) = X^i$ will produce a polynomial model, e.g. Item 14 in Table 2.2.

$$Y = \sum_{i=0}^{k} a_i Z_i(X)$$ (2.15)

The least squares method provides analytical solutions for $a_i$, $i = 1, k$ and $\sigma$ in a linear model having a complete data set. However, Schmee and Hahn[66] proposed a iterative least squares method to estimate the parameters of linear model for a data set having runouts.

2.4.2 The Maximum Likelihood Method

Mutual independence of the $y_i$ is the only assumption required in applying the maximum likelihood method. The sampling distribution of the estimator $\hat{\beta}$ in general is unknown. However the asymptotic distribution of $\hat{\beta}$ is a normal multivariate [53, Page 313] given that the model is correct and given that the life distribution satisfies regular conditions, which are usually met in practical situations [74, 64]. This asymptotic distribution is the only convenient way to obtain an approximate confidence interval of the estimates. Maximum likelihood estimators (MLE) are generally biased, but they are always consistent. i.e. $E(\hat{\beta})$ approaches $\beta$ as the sample size $n \rightarrow \infty$. MLE have the invariance property; that is the MLE for any function of $\beta$ denoted as $f(\beta)$ is simply $f(\hat{\beta})$ provided that the inverse of the function $f(\beta)$ exists.

The asymptotic covariance of $\hat{\beta}$ is estimated by the inverse of Fisher information matrix [53, Pages 356–383] the terms of which are the negative second partial derivatives of the log-likelihood function. Large second partial derivative values imply large curvature of log-likelihood function (and therefore a sharp peak) at the maximum likelihood point. This suggests that the MLE has a narrow confidence interval and
small uncertainty [47, Page 73]. In practice, the local Fisher information matrix is used instead of the Fisher information matrix. The local Fisher information matrix is derived by substituting the $\beta$ in the Fisher information matrix by their estimates $\hat{\beta}$. In practice, the asymptotic distribution can be applied to small samples as a crude approximation.

Shenton and Bowman [67] provide the theory for higher order terms for greater accuracy and faster convergence of the asymptotic theory. Theoretically, this would be useful for the small sample size problem. However, a fairly large computational effort is required to evaluate the higher order terms [67]. And the number of significant digits is questionable after such massive computation. As they indicate [67, page 164], the problem is no longer a matter of correct algebraic manipulation but the interface of statistical theory and computer science. McCullagh [48] derived higher-order approximations based on the Edgeworth series.

The maximum likelihood method is flexible, applicable to any model including those having censored data. However an optimization routine is needed to find the maximum likelihood estimators. Numerical difficulties, such as convergence to a local maximum or divergence may occur, especially for a complex model or a model having many parameters.

### 2.4.3 The Method of Moments (MOM)

If the parameters of a distribution can be expressed explicitly in terms of the moments of the distribution, replacing the moments by sample moments will yield the estimates of the parameters. The MOM often yields estimators that are reasonably good. In general, moment estimators are asymptotically normally distributed and
consistent. However, their variance may be larger than those of estimators derived by other methods, such as the maximum likelihood method [25, Page 240].

The probability weighted moment method has a broader applicability than the method of moments [21]. Also the calculation is easier than that of the maximum likelihood method. Estimators of probability weighted moments have better small-sample properties than those of maximum likelihood and conventional method of moments [27, 35]. However, the MOM and the method of probability weighted moments can only be used to estimate the parameters of fatigue life distribution when the fatigue data is obtained at only one specific stress level.

2.5 Numerical Methods for Estimating Parameters

2.5.1 Introduction of the Program FEDADS

A FORTRAN program named fedads (Fatigue Experimental Design and Analysis, and Decision Simulation) was developed for this dissertation to study fatigue data analysis problems using simulation. The word “decision” refers to a design decision such as the determination of a design curve. The program is flexible and various combination of fatigue test plans, analysis methods, and design curves can be added. The code is in standard FORTRAN 77. Utilized are IMSL routines of random number generation, matrix operations, least squares parameter estimation for linear models, and the non-central $t$ distribution. Eleven subroutines from Numerical Recipes [75] are called. Some are modified for the optimization required in maximum likelihood parameter estimation.

The total program, excluding routines from IMSL and the Numerical Recipes, consists of 52 subroutines or functions and is more than 6000 lines. Ten input files
are used. Three of them serve as a data base, six of them control the program behavior as the user requires, and one stores the fatigue data. Inline documentation is implemented in both program source files and input files.

The program allows the user to input real fatigue data (into the file tested.dat) or generate fatigue data by simulation. Fatigue data are then analyzed under the analysis model chosen by the user. The method of parameter estimation is also chosen by the user. Statistics such as design points \((x_d, y_d)\) on selected stress levels or service life are calculated. In simulation studies, the procedure above is repeated for a specified number \(\text{NOsimu}\) (number of simulation) of times. The statistics of the estimators, e.g. mean of \(\text{NOsimu}\) design point, standard deviation of \(\text{NOsimu}\) design point etc. are also calculated.

To generate fatigue data by simulation, the user must first assume "nature's model" and then specify a test plan. Nature's model is specified through a "nature code" in a control input file named case.dat. The nature code is an index to items in a data base file named natrds.dat (for nature description) which store the definition of a collection of assumed nature's models. If the desired nature's model is not in the file natrds.dat, the user can define the desired model in the text file natrds.dat. To define a nature's model, the user needs to specify five functions and one distribution type using simple coding. This coding corresponds to the input arguments of the function routine funeval (for function evaluation) and rvsamp (for random variate sampling). Many functions or distributions have been defined in these routines. However the user can add more functions and distributions and re-compile.

An understanding of routine funeval and rvsamp is necessary before defining a new nature's model into the file natrds.dat. Specifying a test plan is achieved by specifying a "expdsn code" (for experimental design) in the file case.dat. The expdsn
code is an index to items in another data base file named expdds.dat (for experiment design description), which can also be expanded by the user for additional test plans. The definition of a test plan consists of the stress level settings for each specimen in the fatigue testing and the pre-specified censored time.

To specify the analysis model, which may be different from the assumed nature's model, the user needs to assign the "analsp code" (for analysis space) in the file case.dat. The analsp code is an index to items in the data base file analsp.dat which contain definitions of several analysis models. The file analsp.dat can also be expanded by adding more analysis models. Similar to natrds.dat, the definition of the analysis model consists of five functions and one distribution type. Appendix B lists currently available analysis models in fedads.

The method of parameter estimation (either least squares method or maximum likelihood) can be specified in case.dat. If the maximum likelihood method is chosen, the user may further specify in the file case.dat if he/she wants a global optimization method and some options in using the global optimization method. The analysis can also be conducted by the inverse model, i.e., treating life as independent variable in the analysis and the stress/strain as dependent variable, through setting the "IFxy code" in the file case.dat to be 1.

Statistics such as the design point and estimated mean and standard deviation of life can be examined at a finite number of levels (stress levels or life cycles). These levels can be specified in the file intlvl.dat (for interested levels). The type of design curve (Owens tolerance limit, simultaneous tolerance limit etc.) can also be specified in the file intlvl.dat.
2.5.2 Least Squares Method

A subroutine named \texttt{fndlse} (find least squares estimates) is developed to calculate the least squares estimates of the parameters. Currently the IMSL routine \texttt{RLINE} and \texttt{RCURV} is called within \texttt{fndlse} to conduct least squares analysis on polynomial linear models. The routine \texttt{RCURV} reparameterizes the model using orthogonal polynomials to reduce the loss of numerical accuracy. See Ref. [32] for the Forsythe algorithms and their modification by Shampine that are used in \texttt{RCURV}. When the Box-Cox transformation is applied, the least squares method is also used after the data is transformed, assuming that there is no runouts. The least squares method provides a direct and accurate calculation of the model parameters and is very fast. For example, 5000 simulations of least squares analysis of data with sample size 20 takes less than a minute CPU time on the CONVEX C240 computer.

2.5.3 Optimization Routines

The maximum likelihood method requires optimization routines to search for the model parameters that correspond to the maximum likelihood value. The parameters thus found are the maximum likelihood estimates (MLE) of the model parameters.

A subroutine named \texttt{fndmle} (find maximum likelihood estimates) calls several optimization routines from Numerical Recipes [75]. All optimization routines call a function named \texttt{func} which calculates the negative of logarithm of likelihood value for the given fatigue data set and the specified analysis model and model parameters. The optimization routine will search and find the parameters that maximize the likelihood value. Three optimization routines are used: \texttt{powell}, \texttt{amoeba} and \texttt{amebsa}.
When the model is simple having few parameters, routine `powell` works quite well. The reason to choose `powell` instead of other routines like `frprmn` or `dfpmin` is that `powell` does not need the first derivatives of the `func`. These can be quite complex. Another reason is that by comparison with the more advanced algorithm in `frprmn` or `dfpmin`, `powell` is somewhat slow, but not by much. Ease of use is considered to be more important than CPU time. The principal algorithm of `powell` seeks the minimum point in a quadratic approximation of a function to be minimized using a set of linearly independent and mutually conjugate directions [63]. The conjugate directions are found by series of searching the minimum point along some directions; therefore no derivative is needed.

In case `powell` does not converge, `fndmle` will call the routine `amoeba` to continue the search for the maximum likelihood estimates. The algorithm used by `amoeba` is called the downhill simplex method (Nelder-Mead method) [52]. For a function with \( n \) parameters, a simplex is a volume of dimension \( n \) defined by \( n + 1 \) points on its corners called vertices. The simplex algorithm find new vertices by contracting or extending current vertices according to the function values of current vertices until it contracts to points that yield the minimum function values. The method is very slow but extremely robust relative to the types of objective functions that it may encounter.

When the analysis model is complex, ill behaved or has many parameters, optimization routines may end up with a solution of a local maximum point instead of a global maximum point. A routine having a global optimization algorithm is necessary to deal with this problem [73]. The performance of global optimization algorithms are problem dependent and not general. And availability of global optimization routines is rather restricted. Numerical Recipes [75] offers a routine named `amebsa` based on
a simulated annealing global optimization algorithm [33]. In this algorithm, random jumps to higher function values are allowed occasionally. This provides some opportunities to search outside of local region to avoid being trapped at a local minimum. However, this method is extremely slow. The chance of finding the global maximum point is greater if more iterations are specified when calling the routine.

Troublesome numerical errors appear occasionally in using these unconstrained optimization routines on fatigue data. When the routine is searching for maximum likelihood parameter values, some combinations of parameters may yield a function value that causes numerical overflow. In other cases, a negative value for a standard deviation results. To overcome these problems, *powell* and *brent* from Numerical Recipes are modified to avoid overflow. The routine *funeval* provides special "positive guarantee functions" which ensure that the function values are positive in the test range of independent variable. This is done by expressing some functions in a different way and transforming the parameters passed from the optimization routine. For example, when the standard deviation of life given stress is modeled as a simple linear function of stress shown in Eq. 2.16, the standard deviation might be negative in some range of stress according the analysis model.

\[
\sigma = b_0 + b_1 x
\]

(2.16)

To avoid the negative value problem, re-express the function as Eq. 2.17.

\[
\sigma = (\sigma_2 - \sigma_1) \frac{x - x_1}{x_2 - x_1} + \sigma_1
\]

(2.17)

Where \( x_1 \) and \( x_2 \) are the minimum and maximum stress applied in the fatigue test. \( x_1 \) and \( x_2 \) are provided by the information from the test data and are not parameters in the model. The parameters in this model are \( \sigma_1 \) and \( \sigma_2 \) which represent the function value at \( x_1 \) and \( x_2 \) respectively. As long as \( \sigma_1 \) and \( \sigma_2 \) are positive, all function values
\( \sigma(x) \) for \( x_1 < x < x_2 \) are positive. The routine \texttt{funevl} provides such a function and it is coded as 17. The function coded 17 first transforms the parameters \( c_i \), which is passed from the optimization routine from real space \( (-\infty < c_i < \infty) \) to positive space \( (0 < c'_i < \infty) \) where \( c'_i \) is the transformed value of \( c_i \). Then treat \( c'_1 \) as \( \sigma_1 \) and \( c'_2 \) as \( \sigma_2 \). The rest of the calculations follow from Eq. 2.17. Some other functions like the quadratic and the bi-linear functions also have a positive guaranteed form in \texttt{funevl}. 
CHAPTER 3
Modeling Scatter in Fatigue Data Using Constant Stress Error Models

3.1 Background of the Constant Stress Error Models

In the traditional approach to fatigue data analysis, it is generally assumed that the fatigue stress $S$ (or $X$) is known and is deterministic. The error source, producing the observed scatter in fatigue life, is related to $N$ (or $Y$). This viewpoint was one of the central themes of Chapter 2. But a more general approach to modeling error sources has the promise of better characterizing the scatter of fatigue data. Featured is the concept of the constant stress error models.

Any model that considers stress as a homoscedastic random variable is categorized as a constant stress error model. A constant stress error model follows from (a) phenomenological considerations based on observations of fatigue data, and (b) physical considerations based on errors in the fatigue stress. From a phenomenological viewpoint, studies [6, 56, 69] have shown that for many fatigue data sets the variance of fatigue strength $S$ given life $N$ tends to be constant and relatively small (C.O.V. $\approx$ 10%) while the variance of the log of fatigue life given stress tends to increase as stress decreases. Moreover, modeling $S$ as an independent variable and $N$ as a dependent variable often fails to provide a good fit to heteroscedastic fatigue data as will be demonstrated in the following chapter. From physical considerations,
Shimokawa and Hamaguchi [69] explain that factors such as errors in setting the load level, variability in residual stresses, and the distribution of flaws in the test specimens may result in a constant variation of equivalent stress amplitude.

The phenomenological viewpoint will be discussed in some detail using a number of case studies in the following chapter. The physics of fatigue mechanism viewpoint is presented as follows. Start with the notion that the actual stress that produces a fatigue crack is a local or concentrated stress. The concentrated stress is a "local stress", not the "nominal stress" as calculated and recorded in fatigue test.

"Nominal stress" is the average stress, e.g. derived from dividing the applied axial force by the cross sectional area of the specimen. Fatigue test equipments today are able to control stress/strain loading with high precision. This means that the control of nominal stress/strain is very accurate. "Local stress" is the actual stress experienced by each point of the test material. The actual stresses are seldom uniform across the body of test specimens or engineering parts because of such factors as the geometric shape of the specimen, non-uniformly distributed residual stress, impurities and imperfections in crystal structure and heterogeneous microstructures, e.g. grain size and grain orientation.

Define local stress as the actual stress in the specimen that causes the fatigue crack to initiate and grow. Under a given load and shape geometry, the local stress at the critical point of fatigue depends on, by chance, the residual stress, impurities, microstructures, grain size and grain orientation at that point. Thus local stress at the critical point of different specimens of a random sample for a test, having the same type of material and shape geometry and nominal stress, can be considered as a random variable. Uncertainty in local stress results from the manufacture or specimen preparation process which affects the residual stress, impurities, microstructures,
grain size and grain orientation. Moreover material properties such as strength and hardness or modulus of elasticity may be non-uniformly distributed throughout the body of specimen as a result of unavoidable imperfections in the manufacturing process. These heterogeneous effects influence the fatigue strength of each local point of the specimen or mechanical part.

A few figures from Stulen[72] may give us an idea how serious the variation of local stress/strength might be in some materials: Body-centered cubic iron has a modulus of elasticity about 132.4 GPa along one set of axes and a modulus of 284.13 GPa along another set. Face-centered cubic copper has a modulus that varies from 66.89 GPa to 190.34 GPa. In these materials, local stress reacted by a single grain will depend on the crystalographic orientation of the grain with respect to the principal stress direction and on the adjacent grains in the surrounding region.

In general, the location of crack initiation will be determined by the combination of local stress and local strength at each point, i.e. a point having a relatively low local strength and relatively high local stress is likely to be the location of crack initiation. The crack might not be initiated at the exact point of maximum stress concentration as predicted using the theory of solid mechanics. However, in actual fatigue tests, most crack initiation sites are found in a region, not an exact point, of the stress concentration.

Different materials and manufacturing processes might cause different degrees of variability in local stress. For cases where the variability of local stress is significant, a special model might be needed.
3.2 Modeling Local Stress/Strength in General

3.2.1 Error Sources in the Fatigue Model

As discussed in Sec. 2.2, the regression model is of the general form

\[ Y = y(x; a) + \epsilon \]  

(3.1)

where \( a \) is a vector of the model parameters and the error term \( \epsilon \) is a resulting error in \( Y \) from all error sources. \( \epsilon \) is called "the net error term" in this article. Usually the net error term \( \epsilon \) is assumed to be normally distributed as \( N(0, \sigma^2) \) for ease of analysis, regardless of how many or what sources have contributed to \( \epsilon \). The assumption of \( \epsilon \sim N(0, \sigma^2) \) however is not valid for most fatigue data.

Consider the separation of the source of error due to the variation of local stress from the net error term \( \epsilon \). First, assume that the local stress \( x_L \) can be controlled. The regression model is

\[ Y = y_L(x_L; a) + \epsilon_{y_L} \]  

(3.2)

Where \( x_L \) is the local stress (the actual fatigue stress) at the point of crack initiation and \( \epsilon_{y_L} \) is the variation of \( Y \) due to sources other than local stress \( x_L \). \( y_L(x_L; a) \) is the mean curve of \( Y \) when \( x_L \) is controlled, i.e. \( E(Y|x_L) = y_L(x_L; a) \). This implies that \( E(\epsilon_{y_L}) = 0 \).

Now consider the possibility that the local stress \( x_L \) can not be controlled. Assume that \( E(x_L) = x \) where \( x \) is the nominal stress controlled in the experiment. The relationship between nominal and local stress can be expressed as

\[ x_L = x + \epsilon_{x_L} \]  

(3.3)
where the error term $\epsilon_{x_L}$ represents the randomness of local stress $x_L$. It is assumed that $E(\epsilon_{x_L}) = 0$. Substituting Eq. 3.3 into Eq. 3.2,

$$Y = y_L(x + \epsilon_{x_L}; a) + \epsilon_{y_L}$$  \hspace{1cm} (3.4)

Furthermore assume that $\epsilon_{x_L} \sim N(0, \sigma_{x_L}^2)$ and $\epsilon_{y_L} \sim N(0, \sigma_{y_L}^2)$. If $X = \log S$, then the local stress $s_L$ is lognormally distributed. Eq. 3.4 is called the “mixed error source model”. When $\epsilon_{x_L}$ is absent, the model is called the “pure $Y$ error source model” and expressed as

$$Y = y_L(x; a) + \epsilon_{y_L}$$  \hspace{1cm} (3.5)

which is equivalent to Eq. 3.1. When $\epsilon_{y_L}$ is absent, the model is called the “pure $X$ error source model” and expressed as

$$Y = y_L(x; a)$$  \hspace{1cm} (3.6)

$$= y_L(x + \epsilon_{x_L}; a)$$  \hspace{1cm} (3.7)

Note that all the error sources in Eq. 3.4 to Eq. 3.7 could be lumped into the net error term to form Eq. 3.1. The net error term $\epsilon$ in Eq. 3.1 might not be normally distributed when $\epsilon_{x_L}$ exists.

### 3.2.2 Constant Stress Error Only

Fig. 3.1 illustrates how the distribution of $Y$ is affected by the variability of $x_L$ in the pure $X$ error source model where $\epsilon_{y_L}$ is absent. The $X$ error source $\epsilon_{x_L}$ has an assumed normal distribution. In Fig. 3.1, two specimens are tested at the same nominal stress $x_1 = x_2$. Another two are tested at nominal stress $x_3 = x_4$. The local stresses of magnitude $x_{L1} \cdots x_{L4}$ are induced in the four specimens respectively. Local stress is determined by nature; it value can not be controlled or measured. The values
Figure 3.1: The pure X error source model.
of $y_1 \cdots y_4$ are then determined by the function $y_L(x_L; a)$. These points $(x_{L1}, y_i)$ are represented by the squares in Fig. 3.1. The engineer doesn't know the value of local stress $x_{L1}$. He records $(x_1, y_1) \cdots (x_4, y_4)$ as represented by dots (or circles if censored) in Fig. 3.1.

Regarding censored data, note that $y_{L2} = y_L(x_{L2})$ has a life longer than the censoring time (suspended life) $y_c$ set by experimenter. Therefore the failure life $y_{L2}$ will not be realized. Instead, a censored life of $y_2 = y_c$ is observed and recorded for specimen 2. Denote $x_c$ as the local stress corresponding to the censoring time $y_c$ through the relation $y_c = y_L(x_c)$. Thus all specimens which have a local stress less then $x_c$ will result in censored data.

Note that in the region of $x_3$, $y_L(x_L; a)$ is nearly a straight line and therefore the distribution of $Y$ is approximately normal. However, in the region of $x_1$ the slope of $y_L(x_L; a)$ is changing. In this region the distribution of $Y$ is not normal but skewed to the right because of the curvature of $y_L(x_L; a)$. These features are observed in many fatigue data sets.

3.2.3 Model for Errors Due to All Sources

Consider the distribution of $Y$ when both $\epsilon_{xL}$ and $\epsilon_{yL}$ exist. Denote the error in $Y$ that is due to the local stress error source $\epsilon_{xL}$ as $\epsilon_{yx}$. The net error term will be

$$\epsilon = \epsilon_{yx} + \epsilon_{yL}$$

(3.8)

The variance of $\epsilon_{yL}$ is assumed to be constant. But the variance of $\epsilon_{yx}$ becomes larger as $x$ approaches the fatigue limit region as shown in Fig. 3.1. In the fatigue limit region where the slope of $y_L(x_L; a)$ approaches to zero, the enlargement of the
variance of $\varepsilon_{yx}$ is so great that the effect of local stress cannot be ignored even when the variance of $\varepsilon_{xz}$ is small compared to that of $\varepsilon_{yl}$.

The distribution of $\varepsilon_{yx}$ can be related to the distribution of the error source $\varepsilon_{zl}$. Denote the probability density function of the distribution of $\varepsilon_{yx}$ as $f_{YX}(yX)$ and that of $\varepsilon_{zl}$ as $f_{XL}(xL)$. Then,

$$f_{YX}(yX) = f_{XL}(xL)$$

(3.9)

$$f_{YX}(yX) = \left| \frac{dx}{dy} \right| f_{XL}(xL)$$

(3.10)

where $\left| \frac{dx}{dy} \right|$ in Eq. 3.10 is the Jacobian relating the transformed distribution $f_{YX}(yX)$ to the distribution $f_{XL}(xL)$. In the upper part of the S-N curve where the curve is nearly a straight line, the Jacobian $\left| \frac{dx}{dy} \right|$ is nearly constant and the distribution of $\varepsilon_{yx}$ is the same as that of $f_{XL}(xL)$ except for the scale of $\left| \frac{dx}{dy} \right|$. Assuming that $\varepsilon_{yx}$ and $\varepsilon_{yl}$ are independent, the resulting net error term $\epsilon$ in the upper part of S-N curve will still be normally distributed according to the reproductivity property of the normal distribution.

### 3.3 Proposed Models for Local Stress/Strength

Due to the complexity of the mixed error source model, only the pure X error source model and the pure Y error source are proposed. The mixed error source model with small $\sigma_{yl}$ can be approximated by a pure X error source model. However, the mixed error source model with small $\sigma_{zl}$ may not be approximated by the pure Y error source model because of the dominating influence of local stress at the fatigue limit region. However, if the fatigue data do not extend into the high cycle region, then a pure Y error source model might be adequate.
The pure $Y$ error source model is equivalent to the regression model with a homoscedastic normally distributed net error term. Its analysis can be accomplished using the least squares method.

In attempt to describe the constant stress error phenomena, four models are discussed as follows: (a) the constant $\sigma_z$, basic model, (b) the inverse model, (c) the back-projection model, and (d) the pure $X$ error source model. Terminology regarding the models is summarized below.

1. Non-inverse model. Model the (transformed) stress/strain as the independent variable and the (transformed) life as the dependent variable.

2. Inverse model. Model the (transformed) stress/strain as the dependent variable and the (transformed) life as the independent variable. The scatter of $X$ given $Y$ is assumed to be homoscedastic and normal.

3. Pure $X$ error source model. This is a non-inverse model having a distribution of (transformed) life $Y$ given (transformed) stress/strain $X$ that is derived from Eq. 3.10.

4. Pure $Y$ error source model. This is a non-inverse model having a homoscedastic normally distributed net error term $\epsilon$.

5. Basic model. This is a non-inverse model having a normally distributed net error term $\epsilon$. It may be homoscedastic or heteroscedastic. In the heteroscedastic case, the standard deviation of the net error term could be modeled for example as a linear or quadratic function of stress/strain.
3.3.1 The constant $\sigma_x$ basic model

The variance of $X$, where $X = S$ or $X = \log S$, is assumed to be constant. It is also assumed that the net error term $\epsilon$ is \textit{normally distributed but heteroscedastic}. The heteroscedasticity is derived from a constant scatter $\sigma_x$. The derivation of this model is illustrated in Fig 3.2. In this figure, the trend of the data is denoted as $y(x)$ or its inverse function $x(y)$. The $\mu - \sigma$ curve, denoted as $x(y) - \sigma_x$ or its inverse function $y(x + \sigma_x)$, is constructed by subtracting $\sigma_x$ from $x(y)$. Because $\sigma_x$ is constant, all points on $x(y) - \sigma_x$ have the same vertical distance from $x(y)$. Therefore the construction of $x(y) - \sigma_x$ is achieved by shifting the $x(y)$ down a distance of $\sigma_x$ to get $x(y) - \sigma_x$. This is equivalent to shifting the $y(x)$ in the decreasing direction of $x$ to get $y(x + \sigma_x)$. Thus, $\sigma_y(x) = y(x) - y(x - \sigma_x)$ is the model for the scatter of transformed life $Y$.

This model assumes that the net error term $\epsilon$ is heteroscedastic but normally distributed. The heteroscedasticity is defined by

$$\sigma_y(x) = y(x) - y(x - \sigma_x)$$

where $\sigma_y(x)$ is the standard deviation of the net error term $\epsilon$. It is a basic model and therefore named the constant $\sigma_x$ basic model

3.3.2 The Inverse Model

In the current practice of fatigue data analysis, the inverse model is frequently used. The inverse model assumes that $X$ is the dependent variable and $Y$ the independent variable. The error of the dependent variable $X$ is assumed to be homoscedastic and normally distributed. The least squares method can be used for the inverse model if
Figure 3.2: The constant $\sigma_x$ basic model.
there is no censored data. The model is easy to use, but not correct from a statistical point of view as will be discussed in Section 3.4.

Another issue related to the inverse model is how to treat censored data. Censored data are right censored when life is considered as the dependent variable. But when stress is treated as the dependent variable, it is not obvious that the censored data should be treated as right censored or left censored. Because stress is always placed on the ordinate in fatigue data analysis, right censored can be thought of as “top” censored and left censored as “bottom” censored. This censored data issue is clarified by the back-projection model discussed as follows.

### 3.3.3 The Back-Projection Model

Shimokawa and Hamaguchi [69] employed a concept that they called “equivalent stress amplitude”. Equivalent stress amplitude is identical to the local stress concept as described earlier. In their work, the homoscedasticity of fatigue data is examined using a scheme that is called the back-projection method in this article. This method uses the fatigue life data to back-project local stress values.

The principal of the back-projection method is described as follows. In the pure X error source model, the relationship between failure life $y_L$ and its corresponding local stress $x_L$ is deterministic and is described by the function $y_L(x_L; a)$ as in Eq. 3.6. Simply by using the inverse function of $y_L(x_L; a)$ and the failure life $y_L$ the correct estimate of $x_L$ can be obtained provided that $y_L(x_L; a)$ is correct. This operation is illustrated in Fig. 3.1. Given the failure data points at stress levels $x_3$ and $x_4$, start from each data point $(x_3, y_3), (x_4, y_4)$ and go vertically to the curve $y_L(x_L)$. Then go horizontally to the ordinate. The intersected values on the ordinate, $x_{L3}$ and $x_{L4}$, are the corresponding local stresses.
For censored data, e.g., \((x_2, y_2)\), this operation will back-project to the \(x_c\) value on the ordinate. This means that the true local stress \(x_{L2}\) cannot be estimated because the failure life \(y_{L2}\) is not realized. However the true local stress \(y_{L2}\) must be some value smaller than \(x_c\). Under the censoring time planned at \(y_c\), all local stresses above \(x_c\) will have their failure life realized; and all local stresses below \(x_c\) will result in censored data. From this point of view, censored data, after being back-projected to local stress space, should be treated as left censored or "bottom" censored as shown in Fig. 3.1. Shimokawa and Hamaguchi use this method and a best fit S-N curve (i.e. the \(y_L(x_L)\) curve) to "recover" the local stress of fatigue data at each stress level and test the hypothesis \(H_0: the\ local\ stress\ distribution\ is\ homoscedastic\). Their results support the hypothesis.

### 3.3.4 Relationship Between the Inverse Model and the Back-Projection Model

It is shown in this section that the back-projection and the inverse model are essentially the same in terms of estimating the model parameters.

To help understand how to use the back-projection method in estimating the parameters of a model, and the difference between inverse model and back-projection model, consider Fig. 3.3. The parameter vector \(a\) in \(y_L(x_L; a)\) and \(\sigma_{x_L}\) are undetermined. In the maximum likelihood method, different parameter values are selected and the likelihood value corresponding to each set of parameters is calculated. An optimization routine is employed to search for the set of parameters that yield the maximum likelihood value.
Figure 3.3: Comparison of inverse model and back-projection model.
Consider the inverse model. Given the independent variable \( y \), the mean value of the dependent variable, i.e. \( E(X) = x_L(y) \), is determined by the inverse function of \( y_L(x_L; a) \) using a graphical method of Fig. 3.3. Project vertically from each data point, e.g. \((x_1, y_1)\) until the projection line intersects the \( y_L(x_L) \) curve. The ordinate value, e.g. \( x_{L1} \), is the mean value of dependent variable \( x \) given \( y_1 \). Assuming that \( x \) is normally distributed as \( N(x_{L1}, \sigma^2_{x_L}) \), the likelihood value of the failure data point \((x_1, y_1)\) is,

\[
f_{X_1}(x_1) = \frac{1}{\sigma_{x_L}} \phi\left(\frac{x_1 - x_{L1}}{\sigma_{x_L}}\right)
\]

(3.12)

where \( \phi \) is the standard normal probability density function distribution. This equation is valid when there are no free parameters in Eq. 2.1, e.g. \( X = S \) or \( X = \log S \).

However when \( X \) is not a null transformation of \( S \), the Jacobian \( \left| \frac{dx}{ds} \right| \) should be considered. The likelihood value should then be

\[
f_{S_1}(s_1) = \left| \frac{ds}{dx} \right| f_{X_1}(x_1)
\]

(3.13)

where \( s_1 \) is the untransformed stress corresponding to \( x_1 \). If there are no free parameters in Eq. 2.1, \( \eta \) is said to be null. For example, in the case \( X = \log S \), the Jacobian can be ignored in estimating the parameters of the model because the Jacobian is the same for all values of the model parameters. When \( \eta \) is null, the likelihood value with or without the Jacobian differ only by a constant ratio. Therefore the location of the maximum of the likelihood function is independent of the Jacobian.

In the back-projection model, \( x_{L1} \) is treated as an realization of local stress and is a dependent variable given nominal stress \( x_1 \). The mean value of local stress \( x_{L1} \) is the nominal stress \( x_1 \). Here the \( x_L \) (instead of \( x \) in the inverse model) is the normally distributed random variable. The likelihood for this realization under the assumed
model parameters is

$$f_{x_{L_1}}(x_{L_1}) = \frac{1}{\sigma_{x_L}} \phi\left(\frac{x_{L_1} - x_1}{\sigma_{x_L}}\right)$$

(3.14)

Again, when \(X\) is not a null transformation of \(S\), the Jacobian \(|\frac{dx}{ds}|\) should be considered and Eq. 3.15 should be applied.

$$f_{s_{L_1}}(s_{L_1}) = \left| \frac{ds}{ds} \right| f_{x_{L_1}}(x_{L_1})$$

(3.15)

where \(x_{L_1} = g(s_{L_1}; \eta)\) as in Eq. 2.1. Because the probability density function of the normal distribution is symmetric, Eq. 3.12 and Eq. 3.14 will result in the same likelihood value if the Jacobian is not needed. When the Jacobian is needed, the likelihood value will differ only slightly because Eq. 3.13 uses nominal stress \(x\) while Eq. 3.15 uses local stress \(x_L\). However from the experience of the author, the difference is small and can be ignored. Part of the reason is that the expected value of local stress \(x_L\) is the nominal stress \(x\). Therefore the overall value of the Jacobian in both models should be very close for large sample sizes.

For censored data \((x_2, y_2)\) in the back-projection model, the random variable \(x_L\) has a mean value of \(x_2\) but is censored at \(x_c\). From the previous discussion it is known that \(x_L\) should be treated as left (or "bottom") censored. The likelihood value is the probability of the unrealized portion represented by the shaded area in Fig. 3.3 and expressed mathematically as

$$F_{x_{L_2}}(x_c) = \Phi\left(\frac{x_c - x_2}{\sigma_{x_L}}\right)$$

(3.16)

where \(\Phi\) is the cumulative density function of standard normal distribution. Note that the Jacobian is not needed for censored data because the probability represented by the shaded area will not be changed under transformation.

For censored data \((x_2, y_2)\) in the inverse model, the random variable \(x\) has a mean value of \(x_c\) but is censored at \(x_2\). Consider treating the \(x_2\) as right (or "top") censored
first. The likelihood value is the probability of the shaded area and expressed as

\[
1 - F_{X_L}(x_c) = 1 - \Phi\left(\frac{x_2 - x_c}{\sigma_{x_L}}\right)
\]

\[
= \Phi\left(\frac{x_c - x_2}{\sigma_{x_L}}\right)
\]  
(3.17)

But this is the same as Eq. 3.16. Treating \(x_2\) as left ("bottom") censored in the inverse model will result in a likelihood value equal to the unshaded area, which is different from that of shaded area. The back-projection method provides a reasonable view of how censored data should be treated. The treatment of censored data in the inverse model should be consistent with that of the back-projection model. Therefore it is suggested that the censored data in the inverse model should be treated as right (or top) censored.

3.3.5 The Pure X Error Source Model

The pure X error source model correctly views transformed life \(Y\) as the dependent variable. The data are \((s_i, n_i)\) or \((x_i, y_i)\) where \(s_i\) is deterministic nominal stress/strain and \(n_i\) is fatigue life given \(s_i\) (or \(x_i\) if there are no free parameters in the transformation of \(s_i\)). The correct way of estimation using the maximum likelihood method is to express the distribution of \(n_i\) given \(s_i\) and then calculate the likelihood value. Because the transformations on \(s_i\) in fatigue data analysis are either \(X = S\) or \(X = \log S\), the discussion below concentrates on cases where there are no free parameters in the transformation on \(s_i\). The source of randomness of \(Y\) is assumed to be local stress \(x_L\). By examining Fig. 3.1 and referring to Eq. 3.10 and 3.14, the likelihood function of the failure data \((x_1, y_1)\) for example is

\[
f_{y_1}(y_1) = \left|\frac{dx}{dy}\right| \frac{1}{\sigma_y} \phi\left(\frac{x_{L1} - x_1}{\sigma_{x_L}}\right)
\]  
(3.18)
where $|\frac{dy}{dx}|$ is the absolute value of the inverse of the derivative of Eq. 2.5. When the transformation on life has free parameter(s), the Jacobian $|\frac{dy}{dx}|$, where $\frac{dx}{dn}$ is the derivative of Eq. 2.2, should be included. And the likelihood function becomes,

$$f_{N_1}(n_1) = \left| \frac{dy}{dn} \right| f_{Y_1}(y_1)$$  \hspace{1cm} (3.19)

where $n_1$ is the untransformed fatigue life corresponding to $y_1$, i.e. $y_1 = h(n_1; \lambda)$ as in Eq. 2.2.

The likelihood value for censored data $(x_2, y_2)$ for example, is simply the shaded area and is expressed in the same form as Eq. 3.16.

The likelihood value for a data set is then

$$L(N_1, \eta) = \prod_{i=1}^{n_f} f_{N_1}(n_i) \prod_{j=1}^{n_c} [1 - F_{N_j}(n_j)]$$  \hspace{1cm} (3.20)

where $n_f$ and $n_c$ are number of failure data and censored data respectively

$$f_{N_1}(n_i) = \left| \frac{dy}{dn} \right| \left| \frac{dx}{dy} \right| \frac{1}{\sigma_{x_L}} \phi \left( \frac{x_{L_i} - x_i}{\sigma_{x_L}} \right)$$  \hspace{1cm} (3.21)

The term $[1 - F_{N_j}(n_j)]$ is the equivalent to the shaded area in Fig. 3.1 and is

$$1 - F_{N_j}(n_j) = 1 - F_{Y_j}(y_j) = F_{X_L}(x_c) = \Phi \left( \frac{x_c - x_j}{\sigma_{x_L}} \right)$$  \hspace{1cm} (3.22)

### 3.3.6 Summary of the Constant Stress Error Source Models

The inverse model treats nominal stress as the dependent variable while the back-projection model treats local stress as dependent variable. We have shown that they will give practically the same result in estimating the model parameters provided that the censored data is treated as right(or top) censored in the inverse model. Their slight differences relate to the Jacobian as defined in Eq. 3.13 and Eq. 3.15. The pure X error source model treats life as the dependent variable but assumes that the
variability comes from the homoscedastic local stress. When both the transformation on stress and life have no free parameters, e.g. $X = \log S$ and $Y = \log N$, the only difference in the likelihood value expression between the pure X error source model and the inverse(or the back-projection) model is the Jacobian $|\frac{dy}{dx}|$. How this Jacobian makes a difference will be examined.

3.4 The Inverse Model Applied in Simple Linear Regression: Theoretical Considerations

In this section the validity of using the inverse model is examined when (1) the trend of data is modeled as a simple linear function, and (2) the least squares method is used in estimation of the parameters.

The pure Y error source model for the simple linear case is

$$Y = a_0 + a_1 x + \epsilon_{yl}$$ \hspace{1cm} (3.23)

where $\epsilon_{yl} \sim N(0, \sigma_{yl}^2)$. The mixed error source model for the simple linear case is

$$Y = a_0 + a_1(x_L) + \epsilon_{yl}$$ \hspace{1cm} (3.24)

However the value of local stress $x_L$ is not known. Using the relationship $x_L = x + \epsilon_{xL}$,

$$Y = a_0 + a_1(x + \epsilon_{xL}) + \epsilon_{yl}$$

$$= a_0 + a_1x + \epsilon$$ \hspace{1cm} (3.25)

where $\epsilon = a_1 \epsilon_{xL} + \epsilon_{yl}$. The error $\epsilon$ is normally distributed as $N(0, \sigma^2)$ if both $\epsilon_{xL}$ and $\epsilon_{yl}$ are normally distributed and independent. The pure X error source model for a simple linear trend can also be expressed in Eq. 3.25 except that $\epsilon = a_1 \epsilon_{xL}$. Note that Eq. 3.23 and Eq. 3.25 are of the same form. Therefore, for simple linear trend data,
Eq. 3.25 is a general form for either a pure $X$, or a pure $Y$ or a mixed error source model. Eq. 3.25 for an experiment of sample size $n$ may be expressed as

$$y_i = a_0 + a_1 x_i + e_i \quad i = 1, \ldots n$$  \hspace{1cm} (3.26)

The corresponding matrix form is

$$Y = Xa + \epsilon$$  \hspace{1cm} (3.27)

where

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$  \hspace{1cm} (3.28)

$$X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}$$  \hspace{1cm} (3.29)

$$a = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$  \hspace{1cm} (3.30)

$$\epsilon = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix}$$  \hspace{1cm} (3.31)

Using least squares analysis on Eq. 3.25 will result in unbiased estimates of the parameters $a_0$, $a_1$ and $\sigma^2$ if $y$ is treated as the dependent variable. The unbiased estimator of $a_0$ and $a_1$ in matrix form is

$$\hat{a} = (X'X)^{-1} X'Y$$  \hspace{1cm} (3.32)
where $X'$ is the transpose of $X$.

In the inverse model, $y$ is treated as independent variable. Eq. 3.25 is reorganized into the inverse model as follows

$$x = \left( \frac{-a_0}{a_1} \right) + \left( \frac{1}{a_1} \right) Y + \left( -\frac{\epsilon}{a_1} \right)$$

(3.33)

Denoting $a_{x0} = \left( -\frac{a_0}{a_1} \right)$, $a_{x1} = \left( \frac{1}{a_1} \right)$ and $\epsilon_x = \left( -\frac{\epsilon}{a_1} \right)$, Eq. 3.33 becomes

$$x = a_{x0} + a_{x1} Y + \epsilon_x$$

(3.34)

In this equation, $x$ is kept lower case and $Y$ is kept upper case to emphasize the fact that the $x$ is a deterministic controllable variable and $Y$ is a random variable. To use Eq. 3.34 as an analysis model, the matrix form would be

$$X = Ya_x + \epsilon_x$$

(3.35)

where

$$X = \begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{bmatrix}$$

(3.36)

$$Y = \begin{bmatrix}
  1 & y_1 \\
  1 & y_2 \\
  \vdots & \vdots \\
  1 & y_n
\end{bmatrix}$$

(3.37)

$$a = \begin{bmatrix}
  a_{x0} \\
  a_{x1}
\end{bmatrix}$$

(3.38)
The least squares solution for the model parameter $a_x = (a_{x0}, a_{x1})$ is

$$\hat{a}_x = (Y'Y)^{-1}Y'X$$  \hspace{1cm} (3.40)

Expanding $X$ in this equation using Eq. 3.35, the expected value of the estimator is

$$E[\hat{a}_x] = E[(Y'Y)^{-1}Y'X]$$
$$= E[(Y'Y)^{-1}Y'Y a_x + (Y'Y)^{-1}Y'\varepsilon_x]$$
$$= a_x + E[(Y'Y)^{-1}Y'\varepsilon_x]$$  \hspace{1cm} (3.41)

Thus the bias in $\hat{a}_x$ is $E[(Y'Y)^{-1}Y'\varepsilon_x]$.

If the independent variable matrix $Y$ were deterministic, the estimator would be unbiased because the $(Y'Y)^{-1}Y'$ can be taken out of the expectation operator. And $E[\varepsilon_x] = 0$. Thus,

$$E[(Y'Y)^{-1}Y'\varepsilon_x] = (Y'Y)^{-1}Y'E[\varepsilon_x]$$
$$= 0$$

In the non-inverse model, the independent variable matrix $X$ in Eq. 3.32 is deterministic, and therefore the estimator is unbiased. But that is not the case for inverse model.

The conclusion on modeling simple linear trend data is that regardless of whether nature's model is pure $X$, pure $Y$ or mixed error source, the unbiased estimator is always attainable by treating life as the dependent variable in simple linear regression. Treating life as the independent variable will result in a biased estimator of the model parameters.
3.5 Comparison of the Inverse Model and the Pure X Error Source Model in Simple Linear Regression

3.5.1 A Description of the Example

The inverse model and the pure X error source model applied in simple linear regression are compared using simulation. The non inverse model is used as a reference. The inverse and non inverse models use the least squares method to estimate the parameters, while the pure X error source model uses the maximum likelihood method. Statistical parameters of the UK Department of Energy E-curve [22] are used in this study. However results of this study are general to all simple linear regression cases.

The reason that the results of this study are general to all simple linear regression cases is discussed as follows. In regression, coordinates can be shifted or scaled without affecting the resulting regression line. Use of different scales, for example, occurs when the data is expressed in English and SI units. Regression lines based on different units are physically the same except they are numerically expressed in different units. Knowing that shifting and scaling will not affect the results of regression, all simple linear models can be shifted and scaled to a model with zero intercept and slope of −1.

The simple linear model with zero intercept and slope of −1 is called a standardized simple linear model in this article. The negative slope is chosen because all fatigue curves have negative slope. However a positive slope, or any value, could be chosen as the slope of the standardized simple linear model. All simple linear models can be shifted and scaled as a standardized simple linear model. The only difference between these standardized simple linear models is their standard deviation. This means that
all simple linear models can be characterized by only one value, their standard deviation in the standardized coordinate. This value is called the *standardized standard deviation* and is denoted as $\sigma_s$. Thus to explore different simple linear models, one needs only to consider different $\sigma_s$.

The UK Department of Energy E-curve has a intercept of 12.5172 and a slope of $-3$ with a standard deviation of $\sigma = 0.251$ where $\sigma$ = standard deviation of $\log_{10} N$. Units of stress for this example are MPa. Its standardized simple linear model has a standardized standard deviation of $0.251/| -3| = 0.0837$, i.e. the E-curve is characterized by $\sigma_s = 0.0837$. In this study, the standard deviation of the UK Department of Energy E-curve will be doubled and halved to examine the bias in different simple linear models. Standard deviations of $\sigma = 0.502$ and 0.125 in a $\log_{10}$ cycle life scale are considered.

Five types of test plans are examined. Stress level settings in all five test plans are equally spaced from 1.85 to 3.17 in $\log_{10}$ stress space. There may be different sample sizes at each of the five stress levels. In Table 3.1, the stress level setting $[1 \ 1 \ 1 \ 1 \ 1]$ means a sample size of five with one specimen tested at each of the five equally spaced stress levels. A stress level setting of $[3 \ 3 \ 3 \ 3 \ 3]$ means that a sample size of fifteen with three specimens tested at each of the five equally spaced stress levels is considered. $[2 \ 2 \ 3 \ 3 \ 10]$ means two specimens are tested at the lowest stress level, another two at the second lowest stress level, three specimens tested at each of the third and the fourth stress levels, and ten specimens tested at the highest stress level.
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12.535
12.530
12.498
12.512
12.555
12.510
12.514
12.515
12.566
12.666
12.120
12.141
12.159
12.716
13.114
13.318
13.406
13.512
13.321

standard
error

a

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-3.001
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-3.001
-2.998
-2.999
-3.002
-3.003
-3.002
-2.996
-2.999
-3.003
-3.006
-3.004
-2.992
-2.997
-3.001
-3.002
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Sao
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0.214
0.151
0.171
0.137
0.620
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0.303
0.344
0.275
1.240
0.862
0.607
0.689
0.551
0.309
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0.151
0.172
0.137
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standardized
bias

Zao
0.41
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0.55
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1.35
-2.10
-0.82
0.50
1.38
1.26
-2.18
-0.92
0.64
1.58
1.31
-2.14
-0.62
0.55
1.44
1.58
-2.09
-0.82
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-1.91
-0.61
8.65
11.38
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Zal
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-1.30
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0.97
-0.47
-1.43
-1.55
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0.83
-0.46
-1.38
-1.49
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0.76
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-1.40
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0.16
-8.68
-11.49
-21.24
-23.9"
-25.60
-11.01
-33.68
-52.99
-50.29
-52.15
-34.11
-65.77
-103.4
-101.9
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relative
bias

R.B·a

R.B·a

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-0.01 ill
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0.02
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-0.06
-0.03
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0.11
0.08
-0.12

0.07~

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-0.08 0
-0.03 0
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-0.17
-0.06
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0.05
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0.14 0
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11.81 0
12.02~

Table 3.1: Estimated bias for parameters in simple linear regression using inverse,
non inverse or pure X error source model


3.5.2 Simulation Results

Forty five cases are examined using simulation. 5000 simulations are performed for each case. For each case, the sample mean and the sample standard deviation (i.e. standard error) of the estimators \( \hat{a}_0 \) and \( \hat{a}_1 \) for parameters \( a_0 \) and \( a_1 \) in Eq. 3.25 are computed and listed in Table 3.1. The bias of the estimators are measured in two values: the standardized bias and the relative bias. The standardized bias of \( \hat{a}_0 \) for example is defined as \( z_{\hat{a}_0} = (\hat{a}_0 - a_0)/s_{\hat{a}_0} \) where \( \hat{a}_0 \) is the sample mean of the 5000 estimates of \( a_0 \), and \( a_0 \) is the true value of the parameter; \( s_{\hat{a}_0} \) is the estimator of the standard deviation of \( \hat{a}_0 \). Note that \( s_{\hat{a}_0} = s_{\hat{a}_0}/(\sqrt{5000}) \), where 5000 is the number of the simulation and \( s_{\hat{a}_0} \) is the standard error of the 5000 estimates \( \hat{a}_0 \). The relative bias of \( \hat{a}_0 \) for example is defined as \( R.B.\hat{a}_0 = \frac{\hat{a}_0 - a_0}{a_0} \).

The standardized bias tests if the bias is significant in a statistical sense or just a statistical error in sampling. By the central limit theorem, a sample mean of a sample size of 5000 is approximated by the normal distribution \( N(\mu_{\hat{a}_0}, \sigma_{\hat{a}_0}) \). If the true value \( a_0 \) is outside of the confidence interval \( (\hat{a}_0 - 1.96s_{\hat{a}_0}, \hat{a}_0 + 1.96s_{\hat{a}_0}) \), then reject the null hypothesis \( H_0: \hat{a}_0 \) is not biased with confidence level of 95%.

Replacing the unknown \( \sigma_{\hat{a}_0} \) in the confidence interval expression by the known \( s_{\hat{a}_0} \), the hypothesis can be rejected with confidence level of 95% if the standardized bias is outside of the interval \((-1.96, 1.96)\). If the result of this test indicates that there is significant bias, then the relative bias gives an estimate of the degree of bias.

By examining the standardized bias in table 3.1, both the non inverse model and pure X error source model have unbiased estimators for \( a_0 \) and \( a_1 \). The inverse model shows clear evidence of bias. Consider the relative bias of the inverse model. Comparing three different true standard deviations for each stress level setting, it is found that the relative bias increases by a factor of four when the standard deviation...
is doubled. This applies to both $\hat{a}_0$ and $\hat{a}_1$. For the E-curve, the relative bias for $\hat{a}_0$ is $1\% \sim 2\%$ and that for $\hat{a}_1$ is $2\% \sim 3\%$. When the standard deviation is doubled, the relative bias for $\hat{a}_0$ is increased to $5\% \sim 8\%$, and that for $\hat{a}_1$ is increased to $8\% \sim 12\%$.

Considering the relative bias in cases with stress level settings $[1\ 1\ 1\ 1\ 1]$ , $[2\ 2\ 2\ 2\ 2]$ and $[4\ 4\ 4\ 4\ 4]$, it is surprising that the relative bias is increased as the sample size increases. Comparing the relative bias in cases having a stress level setting of size 20, $[4\ 4\ 4\ 4\ 4]$, $[2\ 2\ 3\ 3\ 10]$ and $[10\ 3\ 3\ 2\ 2]$, it is seen that while bias of $\hat{a}_0$ decreases as the weighting is changed from a high stress level weighting to a low stress level weighting, the bias of $\hat{a}_1$ varies with the more irregular pattern.

The conclusion is that for the simple linear model, regardless of whether the error term is pure X error source or pure Y error source or mixed error source, statistical analysis using the non inverse model produces unbiased estimates of the parameters.

For a nonlinear trend of data and with the assumption of a constant stress error, is the inverse model a reasonable approximation? The inverse model is easy to apply using the least squares method, while the pure X error source model requires the maximum likelihood method and frequently some programming. In simulation on simple linear cases, it was found that although the only difference in the calculation between the inverse model and pure X error source model is the Jacobian, the difference in resulting parameter estimates can be very significant. Parameter estimates of the inverse model are biased and that of pure X error source is not. In the next chapter, differences between the inverse model and the pure X error source model will be examined by analysis of five fatigue data sets using different nonlinear models.
CHAPTER 4
Comparison of Models for Fatigue Life Data

4.1 Preliminary Remarks

In this chapter, the relative performance of models to characterize fatigue data are studied. There are two classes of models. They are: (a) models to describe the mean, median, or "trend" of the data, and (b) models to describe the scatter of the data. There are many possible models for both, and the number of feasible combinations can be quite large. In this study, only a limited number of models were studied. The models studied were those considered from subjective judgment to be the most viable.

A fundamental assumption which is made is that the error sources are normally distributed in the transformed space. Thus if a log transformation is made on stress and/or life, the implication is that the error source has a lognormal distribution in the original space.

In order to distinguish which of several competing models provides the best fit, a large sample size is desired. A large sample size increases the power of test, i.e. the probability to reject an inappropriate model. Five large fatigue data sets were chosen by the author. These data sets have different characteristics and in whole were considered to be representative of the range of possibilities. One exception is that there is no data set considered that has a simple linear trend and homoscedastic scatter. Such data might result from a test in which an initial crack is present and
fatigue is principally a matter of crack propagation. This is the case for welded joint fatigue data, or more generally in fabricated sections where flaws invariably exist. For such data, the simple linear model would be appropriate, as discussed in Chapter 3. No matter what the error source, the traditional regression model will fit as long as the source of error is homoscedastic and normally distributed. The theme of this chapter and a major contribution of this dissertation is a comparison of the performance of nonlinear models for fatigue data.

There are three methods which are used herein to test the goodness-of-fit of a model for the mean and for the scatter given a set of data. The test methods are: (a) visual examination of the mean curve and the $\pm 3\sigma$ envelope, (b) visual examination of the standardized residual plot, and (c) the likelihood ratio test. In total, these tests are a blend of subjective judgment and quantitative measure.

For a convenient comparison of the results of these tests, all figures and tables that described these tests are placed together at the end of this chapter.

4.2 Data Sets Used for Comparison

Five large sample size fatigue data sets (Table 4.1) are used to study the goodness-of-fit of the models for the mean and scatter. These data is provided in Appendix D. They are plotted in Fig. 4.1 to Fig. 4.6. The Ti64 data consists of pooled fatigue data tested at different temperatures: 75°F, 300°F, 500°F, 650°F and 750°F. It is assumed that the fatigue strength of this material does not vary within the temperature range. Those data tested at 75°F or 300°F (lower temperature) are used to form a subset denoted as Ti64-300. Ti64-300 results are compared with Ti64. Note
Table 4.1: Data sets used in this study

<table>
<thead>
<tr>
<th>abbreviation</th>
<th>material type</th>
<th>sample size</th>
<th>no. of censor data</th>
<th>data source</th>
</tr>
</thead>
<tbody>
<tr>
<td>2024-T4</td>
<td>2024-T4 Aluminum Alloy</td>
<td>252</td>
<td>0</td>
<td>T. Shimokawa and Y. Hamaguchi[69]</td>
</tr>
<tr>
<td>In 718</td>
<td>Inconel 718</td>
<td>246</td>
<td>4</td>
<td>G.E.A.E.[12]</td>
</tr>
<tr>
<td>AAW</td>
<td>annealed aluminum wire</td>
<td>200</td>
<td>0</td>
<td>Freudenthal[20]</td>
</tr>
<tr>
<td>Ti64</td>
<td>titanium(Ti-6Al-4V)</td>
<td>96</td>
<td>25</td>
<td>G.E.A.E.[12]</td>
</tr>
<tr>
<td>Ti64-300</td>
<td>titanium(Ti-6Al-4V)</td>
<td>48</td>
<td>12</td>
<td>G.E.A.E.[12]</td>
</tr>
<tr>
<td>wire</td>
<td>steel wire</td>
<td>75</td>
<td>10</td>
<td>private source</td>
</tr>
</tbody>
</table>

that the AAW (annealed aluminum wire) data set appears to be homoscedastic on life while the rest of data sets appear to be heteroscedastic.

Consider the different characteristics of the fatigue data in Figs. 4.1 through 4.6. While all have an endurance region, the data sets of Fig. 4.4 and 4.5 have many censored data. The data sets of Fig. 4.2 and 4.6 have some censored data. In the low cycle range, some data seem to have a linear trend i.e. Fig. 4.1, 4.2 and 4.6; at first glance, a model having two straight line segments might be appropriate. In Fig. 4.1, 4.3 and 4.4, there are many data in the endurance region.

As a whole, the five data sets chosen are considered to have all of the characteristic which might be encountered in most fatigue test data.

A special thanks is made to Mr. Gerald T. Cashman of General Electric Aircraft Engines for providing the titanium Ti64 and Inconel 718 data, and to Mr. Ken Bauer in AlliedSignal Aircraft Engines for preparing the aluminum 2024-T4 data from the work of T. Shimokawa and Y. Hamaguchi [69].
4.3 Models Under Consideration

4.3.1 Transformation of the Data

Transformations considered in this study are:

1. Transformation of stress/strain $S$.
   
   (a) Null transform ($X = S$)
   
   (b) Base 10 log transform ($X = \log S$)

   Except when mentioned otherwise, all transformations on stress/strain are log transforms. In Table 4.4 to Table 4.9, "X=S" is marked in the column "model for scatter" when the transformation of stress/strain is null.


   (a) Base 10 log transform ($Y = \log N$).

   (b) Box-Cox transformation of Type 1; Item 11 in Table 2.1.

4.3.2 Models for the Mean (the Trend of the Data)

Models for the mean considered in this study were selected from the list in Table 2.2. They are listed in Table 4.2. The MIL-HDBK-5 (MIL) and general strain-life equation ($e-N$) were selected for their broad application in modeling the $S-N$ curve and the strain-life curves respectively. Also selected was the five parameter bilinear model (bi) for its flexibility in modeling the mean curve using two straight line segments. Each segment has an arbitrary slope, and there is arbitrary curvature at the intersection. Because the fatigue limit may be reasonably modeled as a horizontal
Table 4.2: Models for mean considered in this study

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Name</th>
<th>Formula</th>
<th>Number of Parameters</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>bi</td>
<td>bilinear</td>
<td>$x = a_1 + a_2(\mu \nu + a_3\sqrt{(\mu \nu - a_3)^2 + a_4})$</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>l-bi</td>
<td>leveling-off</td>
<td>$x = a_1 + a_2(\mu \nu - \sqrt{(\mu \nu - a_3)^2 + a_4})$</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>e-N</td>
<td>general</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>strain-life</td>
<td>$\epsilon = a_1 N^{a_2} + a_3 N^{a_4}$</td>
<td>4</td>
<td>$Y = \log N$</td>
</tr>
<tr>
<td></td>
<td>equation</td>
<td></td>
<td></td>
<td>$X = \log \epsilon$</td>
</tr>
<tr>
<td>MIL</td>
<td>MIL-HDBK-5</td>
<td>$Y = a_1 + a_2 \log(X - a_3)$</td>
<td>3</td>
<td>$Y = \log N$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$X = S$</td>
</tr>
</tbody>
</table>
| Box    | Box-Cox          | $\left\{ \begin{array}{l} \frac{N^2}{N} - 1 \\
|        |                  | \frac{\ln N}{N} \quad \lambda = 0 \\
|        |                  | \mu \nu = a_0 + a_1 x \\
|        |                  | \sigma \nu = b_0 \\
|        |                  | \end{array} \right.$                                                  | 1                    | +2 +1=4                  |

line, the bilinear model can be modified, by letting $a_5 = -1$, to force the asymptotic slope, as $N \to \infty$, to be zero. This leads to the four parameter leveling-off bilinear model (l-bi), where “leveling-off” refers to the asymptotic zero slope. In the case of the Box-Cox transformation (Box) on life, only the simple linear model is used as a model for the mean.

### 4.3.3 Models for Scatter

Models for scatter are listed in Table 4.3. In the case of the Box-Cox transformation on life, only the homoscedastic scatter model is used. As described in Chapter 3, modeling of the scatter of fatigue data may be categorized as follows:

1. Basic model. Model the net error term $\epsilon$ in Eq. 3.1 as *normally distributed* and homoscedastic or heteroscedastic regardless of the source of error. This is the most general model. Errors from all sources are lumped into the net error term. The underlying structure of the error source is ignored. Models in this category
Table 4.3: Models for scatter considered in this study

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Name</th>
<th>Formula</th>
<th>Number of Parameters</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>qua</td>
<td>quadratic</td>
<td>$\sigma_Y(x) = b_0 + b_1 x + b_2 x^2$</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>lin</td>
<td>linear</td>
<td>$\sigma_Y(x) = b_0 + b_1 x$</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>homo</td>
<td>homoscedastic</td>
<td>$\sigma_Y(x) = b_0$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>exp1</td>
<td>type 1 explode</td>
<td>$\sigma_Y(x) = b_0 (x - \log(b_1))^{b_3}$</td>
<td>3</td>
<td>see Chapter 3 and Fig. 3.2</td>
</tr>
<tr>
<td>exp2</td>
<td>type 2 explode</td>
<td>$\sigma_Y(x) = b_0 (\frac{1}{x+1} + b_2)$</td>
<td>3</td>
<td>see Chapter 3</td>
</tr>
<tr>
<td>2Sy</td>
<td>constant $\sigma_x$, non inverse model</td>
<td>$\sigma_Y(x) = y(x) - y(x - \sigma_x)$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>pure X</td>
<td>pure X error source model</td>
<td>$\sigma_{x_2}(x) = b_0$</td>
<td>1</td>
<td>see Chapter 3</td>
</tr>
<tr>
<td>(y,x)</td>
<td>inverse model</td>
<td>$\sigma_X = b_0$</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

are: (a) homoscedastic (homo), (b) linearly heteroscedastic (lin), (c) quadratically heteroscedastic (qua), (d) "explode" heteroscedastic in which scatter increases dramatically, i.e. explodes, at the fatigue limit, and (e) the constant $\sigma_x$ basic model (2Sy). The “2Sy” is an abbreviation of “CSx2Sy” which in turn is an abbreviation of “constant standard deviation of X to standard deviation of Y”.

2. Inverse model, denoted as $(y,x)$. Stress/strain is treated as the dependent variable. The scatter of stress/strain is assumed to be homoscedastic and normally distributed. This model is statistically incorrect, but it is widely used in engineering practice because it seems to give reasonable results.

3. Pure X error source model (pure X). It is assumed that the major error source is homoscedastic and normally distributed local stress. The distribution of the net error term $\epsilon$ is derived from the distribution of the local stress through the Jacobian (See Eq. 3.10). Stress/strain is considered as the independent variable.

The combinations of mean and scatter models used in considering each data set are listed in the Table 4.4 to Table 4.9. The default transformation for stress is base 10 log. If the transformation of stress is null, the model will be marked “X=S” in the
column "model for scatter". In the case of "X = S", the pure X error source model has normally distributed local stress $S_L$ while in case of "X = log S" the model has a normally distributed $X_L$ and lognormally distributed local stress $S_L$.

4.4 Comparison Using the Mean Curve and the $\pm 3\sigma$ Envelope

4.4.1 A description of the Mean Curve and $\pm 3\sigma$ Envelope

Visual examination of the mean curve and the $\pm 3\sigma$ envelope provides a subjective method of assessing the goodness-of-fit of a model. The fitted mean curve is

$$\hat{\mu}_Y(x) = \mu_Y(x; \hat{\alpha})$$

(4.1)

where $\hat{\alpha}$ is the vector of parameter estimates. A $\pm 3\sigma$ envelope can be constructed by

$$\hat{\mu}_Y(x) \pm 3\hat{\sigma}_Y(x)$$

(4.2)

where $\hat{\sigma}_Y(x)$ is the estimate of $\sigma_Y(x)$, the model for scatter of fatigue data

$$\hat{\sigma}_Y(x) = \sigma_Y(x; \hat{\beta})$$

(4.3)

and $\hat{\beta}$ is a vector of parameter estimates.

For the inverse model and the pure X error source model where uncertainty is defined by $\sigma_X$, the $\pm 3\sigma$ envelope is constructed as

$$\hat{\mu}_Y^{-1}(y) \pm 3\hat{\sigma}_X$$

(4.4)

where $\hat{\mu}_Y^{-1}(y)$ is the inverse function of $\hat{\mu}_Y(x)$.

The fitted mean curve and the $\pm 3\sigma$ envelope provides a visual examination of the goodness-of-fit. The mean curve should follow the trend of data and the envelope
should cover most of the data points. For ease of comparing results, all figures and tables are placed at the end of this chapter.

In a visual examination of the mean curve and ±3σ envelope, first look to see if the envelope has a reasonable shape. In the basic model (defined in Sec. 4.3.3), if scatter is modeled as heteroscedastic, the fitted ±3σ envelope sometimes appears to be inappropriate, e.g. the concave envelope as in Fig. 4.7. If this concave envelope does not reasonably follow the data in the test range, it will be considered as a poor fit and will be denoted as "concave" in this article. "Concave" envelopes will be marked as "c" in column "±3σ env. test" in Tables 4.4 to 4.9 ("±3σ env. test" is an abbreviation for the "±3σ envelope test"). The column "±3σ env. test" marks those models having a poor fit as established subjectively by visual examination.

Some envelopes explode at the fatigue limit region to accommodate the scatter in that region, e.g. Fig. 4.8. However if the explosion goes too fast such that the envelope turn away from the data points in the test range of stress/strain, classify this model as "explode" and mark an "e" in the column "±3σ env. test". For example, in Fig. 4.8 the -3σ curve at -2.42 log strain has a log life of 3, far removed from the data points at that stress level. This kind of explosion does not reasonably follow the data and will be considered as a poor fit.

Another situation is an explosion of the -3σ curve which doubles back, e.g. Fig. 4.9.

An irregular envelope will result when both the model for the mean and the model for scatter have significant curvature and each dominate at different stress/strain levels. For example, in Fig. 4.10 the mean curve has a kink at stress level 2.57 which also causes a kink for the ±3σ envelope. Below that level the mean curve becomes a straight line while the scatter grows exponentially. The domination of the curvature
of the scatter can be seen at stress level 2.52. The two effects cause two kinks for the ±3σ envelope. This situation is considered as a poor fit and is called "irregular". Irregular envelopes will be marked as "i" in the column "±3σ env. test". Envelopes that do not appear to fit the data well, but do not belong to the categories above are simply called "bad fit" and are marked as "b".

4.4.2 Aluminum 2024-T4 Data Set

Models that provide a poor fit for the 2024-T4 data set are identified in the column "±3σ env. test" of Table 4.4. Those models that are not marked as having a poor fit are, the inverse model, the pure X error source model and the Box-Cox model. Further comparisons for these unmarked models are:

1. Consider the null transformation \( X = S \) and the leveling-off bilinear for the mean. The inverse model and the pure X error source model are compared in Fig. 4.11. In the test stress range, the \(-3\sigma\) curve for both models are similar while the pure X error source model has a more conservative mean curve and \(+3\sigma\) curve than those of the inverse model. Below the test stress range, all curves of the pure X error source are more conservative than those of the inverse model. At stress level 2.47 the difference in life is one order of magnitude. Above the test stress range, the inverse model is slightly more conservative than the pure X error source model. For example, at stress 2.75, the \(-3\sigma\) curve for the inverse model is 0.025 (in log life) less than that of pure X error source model. This means that the life been predicted by the \(-3\sigma\) of inverse model is \(10^{0.025} = 1.06\) times of that of pure X error source model. Note that the upper part of the envelope is slightly curved. This is because the leveling-off bilinear is fitted in \(X = S\) space and plotted in \(X = \log S\) space A straight line in \(X = S\) space is
curved in $X = \log S$ space. As can be seen in the Fig. 4.11 the S-shape envelope fits the data quite well.

2. For the bilinear model, the pure X error source model is the same as the inverse model in the upper line segment region (Fig. 4.12) and more conservative in the lower line segment region. The $-3\sigma$ curve of the pure X error source model predicts life 78% less than that of the inverse model at a stress level of 2.45.

3. For $X = \log S$ and the leveling-off bilinear, the pure X error source model is approximately the same as the inverse model in the test range but is conservative at lower stress levels (Fig. 4.13). Because the slope at the right asymptotically approaches zero, the conservatism is severe. For example at log stress = 2.5, the pure X error source model has a mean curve of $10^{6.8} = 6.31 \times 10^6$ cycles, while the inverse model has a mean curve approaching infinity.

4. Using the MIL-HDBK-5 model for the mean, the pure X error source is more conservative at the fatigue limit. The $-3\sigma$ curve is about $10^{0.25} - 1 = 78\%$ lower in life than the inverse model at stress level 270 MPa. The two models are essentially identical in the upper part of the test region.

5. Comparing the Box-Cox model with the leveling-off bilinear with $X = S$ and the pure X error source model, the Box-Cox is more conservative as it has a relative large standard deviation (Fig. 4.14). This suggests that the Box-Cox model fails to fit the data better than the leveling-off bilinear. The envelope of the Box-Cox covers that of the leveling-off bilinear model for most of the range.
4.4.3 Inconel 718 Data Set

Models which provide a poor fit for the Inconel 718 data set are so marked in the column “±3σ env. test” of Table 4.5. An example of a poor fit is shown in Fig. 4.15. Models for scatter which survived the visual test are the inverse model, the pure X error source model and the Box-Cox model. The leveling-off bilinear and the bilinear model for the mean perform the same in terms of fitting. This is because of the observed fatigue limit in the data. Because models having fewer parameters are preferred, the leveling-off bilinear model will be discussed for this data set. Further comparisons among these survivors are:

1. For the leveling-off bilinear model (Fig. 4.16), the −3σ curve of the pure X error source model is more conservative than that of the inverse model in the region below the “kink”. i.e. there is about 12% less in life at the lowest tested strain level. The difference grows exponentially below the lowest test strain level because the slope at the right become zero asymptotically. The inverse model is more conservative at the upper part of the tested region, about 7% less in life.

2. For the mean modeled by the generalized strain-life equation, the −3σ curve of pure X error source model is more conservative than that of the inverse model. There is about 26% less in life at the lowest tested strain level as shown in Fig. 4.17. The inverse model is more conservative in the upper part of test region, about 12% less in life at the highest strain level tested.

3. The −3σ curve of the Box-Cox model is more conservative than that of the leveling-off bilinear with the pure X error source model in region below the log strain level of -1.83 as shown in Fig. 4.18. The curvature of the Box-Cox model
does not accommodate the data very well at the top of the test range. The leveling-off bilinear model appears to fit the data well.

4. Generally, the pure X error source model is more conservative in the lower part while inverse model is a little bit conservative in the upper part.

4.4.4 AAW Data Set

Models which provide a poor fit for the AAW data set are identified in the column “±3σ env. test” of Table 4.6. As a contrast to the other data sets studied herein, the inverse model, the pure X error source model and the Box-Cox model provide a poor fit to the homoscedastic AAW data. The use of the bilinear mean in combination with the inverse and the pure X error source models is illustrated in Fig. 4.19. Box-Cox model results are shown in Fig. 4.20. Fitting homoscedastic life data is not a feature of these models. The homoscedastic, linear, quadratic or even the explode scatter model fit the data well. An example of general strain-life equation with homoscedastic scatter or quadratic scatter is shown in Fig. 4.21.

4.4.5 Ti-64 Titanium Data Set

Models which provide a poor fit for the Ti64 data set are identified in the column “±3σ env. test” of Table 4.7. Models that generally fit well are the inverse model, the pure X error source model and the Box-Cox model. As the leveling-off bilinear model and the bilinear model are essentially the same in terms of fitting for this data set, only the leveling-off bilinear model will be discussed because the data suggests that there is a fatigue limit. Further comparisons are:
1. For the leveling-off bilinear model shown in Fig. 4.22, $-3\sigma$ of the pure $X$ error model is much more conservative at lower strain levels; the inverse model is more conservative in the upper region.

2. For the leveling-off bilinear and a null transformation $X = S$ (Fig. 4.23), the $-3\sigma$ curve of the pure $X$ error model is more conservative below the strain level being tested than that of the inverse model. In the region above the test range, the inverse model is more conservative. Note that the scatter of log strain of the pure $X$ error source model in Fig. 4.23 is not constant. However the scatter of strain is constant.

3. For the general strain-life model, the $-3\sigma$ curve of the pure $X$ error source model is also conservative in the lower strain region relative to that of the inverse model.

4.4.6 Ti64-300 Titanium Data Set

Models which provide a poor fit for the Ti64-300 data set are identified in the column "±3σ env. test" of Table 4.8. Models that generally fit well are the inverse model, the pure $X$ error source model and the Box-Cox model. As the leveling-off bilinear model and the bilinear model are essentially the same for this data set, only the leveling-off bilinear model will be discussed because the data suggests that there is a fatigue limit. Further comparisons are:

1. For the leveling-off bilinear case shown in Fig. 4.24, the inverse model becomes horizontal below -2.2 log strain, while the pure $X$ error source model still has a non-zero slope. The difference in life predicted by the inverse model and the pure $X$ error source is infinite in this region. The inverse model is slightly more
conservative in life than that of the pure X error source model in the higher strain region.

2. For the general strain-life model (Fig. 4.25), the $-3\sigma$ curve of the pure X error source model is more conservative than that of the inverse model at the lower strain level. The pure X error source model and the inverse model are very close at higher stress levels.

3. For the pure X error source model (Fig. 4.26), the general strain-life and the leveling-off bilinear curves are close. The $-3\sigma$ curve of the general strain-life model is more conservative in life at higher strain levels.

4. For most of the strain region, Fig. 4.27 illustrates that the Box-Cox model is more conservative than the leveling-off bilinear with pure X error source model.

### 4.4.7 Steel Wire Data Set

Models which provide a poor fit for the wire data set are identified in the column "±3\sigma env. test" of Table 4.9. Models that generally fit well are the inverse model, the pure X error source model and the Box-Cox model. As the leveling-off bilinear model and the bilinear model are essentially the same in terms of fitting for this data set, only the leveling-off bilinear model will be discussed as there appears to be a fatigue limit. Further comparisons are:

1. As shown in Fig. 4.28, the pure X error source model is more conservative than the inverse model.

2. For the general strain-life model (Fig. 4.29), the pure X error source model is more conservative than that of the inverse model at lower stress levels. The
inverse model is slightly conservative in the higher stress region, about 7% less in life relative to the pure X error source.

3. For the MIL-HDBK-5 model, the pure X error source is still more conservative relative to the inverse model at lower stress levels. The figure for this case is not shown.

4.4.8 Conclusions

From the six data sets analyzed, general conclusions based on visual examination of fitted mean curve and $\pm 3\sigma$ envelope are:

1. For heteroscedastic data, the inverse model, the pure X error source model and the Box-Cox model seem to provide a consistently reasonable fit. For homoscedastic data, the basic models seem to work well.

2. The Box-Cox model seemed to provide more conservative results (wider $\pm 3\sigma$ envelope) in most of the test stress/strain region. The plots are always curved in log-log space. The model is not suitable for data having a straight line portion in this space.

3. While the inverse model and the pure X error source model are based on the assumption that the homoscedastic local stress is the major source of error, the Box-Cox model recognizes no fatigue mechanism. The Box-Cox model relies purely on its mathematical flexibility to deal with heteroscedastic data. The inverse model and the pure X error source model rely on recognition of a fatigue mechanism.
4. Among all cases studied, the pure X error source model is consistently more conservative than the inverse model in the lower test stress region, i.e. the high cycle region. At the fatigue limit, the difference in life between the two models is large because the slope is close to zero.

5. The inverse model is always slightly more conservative than the pure X error source in upper part of the test region. In this region however, the difference in life between the two models is small due to the steep slope. General speaking, the pure X error source is more conservative than the inverse model.

6. In the cases studied in which a fatigue limit appears to be present, the four parameter leveling-off bilinear model seems to fit as well as the five parameter model.

4.5 Comparison Using the Standardized Residual Plot

4.5.1 Introduction to the Standardized Residual Plot

A standardized residual plot is helpful in examining the goodness-of-fit of a model. When a valid model is used, the standardized residual plot should be homoscedastic. A heteroscedastic standardized residual plot implies that the model fits poorly. In this study, the definition of the standardized residual depends upon the scatter model. For the basic model, a residual $e_i$ for data point $(x_i, y_i)$ is defined as

$$e_i = y_i - \hat{\mu}_y(x_i) \quad (4.5)$$

The corresponding standardized residual $z_i$ is defined as

$$z_i = \frac{e_i}{\hat{\sigma}_y(x_i)} \quad (4.6)$$
The standardized residual plot is constructed by plotting \((x_i, z_i)\) across all \(x_i\).

For the inverse model, the definitions and the construction of the standardized residual plot are the same as the basic model except that \(x_i\) and \(y_i\) are interchanged.

For the pure X error source model, the homoscedastic and normally distributed error source is assumed for local stress \(x_L\). The local stress \(x_{Li}\) for a data point \((x_i, y_i)\) is derived using the back-projection method:

\[
\hat{x}_{Li} = \hat{\mu}_Y^{-1}(y_i; \hat{\alpha})
\]  

where \(\hat{\mu}_Y^{-1}\) is the inverse function of \(\hat{\mu}_Y\). By the invariance property (Sec. 2.4.2), when \(\hat{\alpha}\) is the maximum likelihood estimator of \(\alpha\), \(\hat{x}_{Li}\) will also be the maximum likelihood estimator of \(x_{Li}\). The local stress \(x_{Li}\) is assumed to be normally distributed with the mean equal to \(x_i\), nominal stress, and standard deviation of \(\sigma_{x_L}\). The residual of local stress is defined as

\[
e_i = \hat{x}_{Li} - x_i
\]  

(4.8)

The standardized residual of local stress is

\[
z_i = \frac{e_i}{\hat{\sigma}_{x_L}}
\]  

(4.9)

where \(\hat{\sigma}_{x_L}\) is the estimator of \(\sigma_{x_L}\). The standardized residual plot for the pure X error source model is constructed by plotting \((x_i, z_i)\).

Results of visual examination of the standard residual plot for each data set are listed in column "\(z_i\) test" of Tables 4.4 to Table 4.9. "\(z_i\) test" is short for "standardized residual test". Models which provide a good fit should have a standardized residual plot showing homoscedastic scatter and a mean curve which is essentially a horizontal straight line. When the standardized residual is heteroscedastic, a "\(t\)" (for "hetero") is listed in the column "\(z_i\) test" of the table. When the mean curve of the standardized
residual is not a straight line, a "n" (for "nonlinear") is listed. When censored data points show an unreasonable pattern, a "c" (for "censor") is listed.

4.5.2 Aluminum 2024-T4 Data Set

1. In all cases the inverse model seems to have an unusual residual plot, an example of which is shown in Fig. 4.30: Data tested at one stress level results in the residuals spread along a curve. This is because the inverse model treats life as the independent variable.

2. The bilinear with a null transformation "X=S" has identical residual plots for both quadratic scatter and the pure X error source. As shown in Fig. 4.31, the residuals appear to be homoscedastic except for those at the lowest stress levels.

3. In Table 4.4, all models listed in the rows below bilinear and "X=S" with "exp2" scatter have heteroscedastic or nonlinear residuals (Fig. 4.32). All models below leveling-off bilinear with quadratic scatter have severe heteroscedasticity (Fig. 4.33).

4.5.3 Inconel 718 Data Set

1. The inverse model in all cases has severe heteroscedastic residuals as illustrated in Fig. 4.34. Standardized residuals $z_i$ at log life greater than 5 are between -1 and 1. See also Table 4.5.

2. The Box-Cox and all models with the pure X error source model have homoscedastic residuals (e.g. Fig. 4.35). The residuals $z_i$ of these models are all within a $\pm 3$ band.
3. All models with quadratic or "exp" type scatter have reasonably homoscedastic residuals. In comparison with the pure X error source model, the only obvious difference is that some standardized residuals exceed three as shown in Fig. 4.36.

4. All homoscedastic scatter models have heteroscedastic residuals with some exceeding five (Fig. 4.37).

4.5.4 AAW Data Set

1. The inverse model in all cases has heteroscedastic residuals (Fig.4.38). See also Table 4.6.

2. All basic models have homoscedastic residuals for all scatter models considered (Fig. 4.39). The poorest performing case in this category is the MIL-HDBK-5 model having a homoscedastic scatter model. But even its residual plot, shown in Fig. 4.40, indicates close to homoscedastic conditions.

3. The Box-Cox model and all pure X error source models have heteroscedastic residuals (Fig. 4.41).

4.5.5 Ti-64 Titanium Data Set

1. Most of the inverse models have heteroscedastic residuals. All inverse models seem to have an unreasonable spread of censored residuals. As illustrated in Fig. 4.42, censored residuals at a log life of 6 vary from $z = -4.5$ to $z = 1$ with only one failure residual at $z = -3.9$. A reasonable censored data spread should be such, for example, that for one censored data point at the 90 percentile (or $z = 1.28$) there should be nine failure data points in the region $z < 1.28$. 
In other words when censoring is set at 90 percentile, the ratio of number of censored and failure data should be 1:9. A model having an unreasonable spread of censored data is identified as a "c" in Table 4.7.

2. The Box-Cox model and all pure X error source models or quadratic scatter models have homoscedastic residuals and a reasonable spread of censored residuals (Fig. 4.43).

3. All "explode" type scatter models have homoscedastic residuals. But the censored residual spread is curved (Fig. 4.44).

4. All homoscedastic scatter or linear scatter models have heteroscedastic residuals.

4.5.6 Ti64-300 Titanium Data Set

The residual plots for the Ti64-300 data set are basically the same as that of the Ti64 data. See Table 4.8.

4.5.7 Steel Wire Data Set

1. Most inverse models have heteroscedastic and nonlinear residuals. See Fig. 4.45 and Table 4.9.

2. The leveling-off bilinear with the pure X error source model has nearly homoscedastic residuals as shown in Fig. 4.46.

3. In Table 4.9 all the models listed below the leveling-off bilinear with pure X error source have heteroscedastic residuals.
4.5.8 Conclusion

1. Most cases of the inverse model seem to have heteroscedastic standardized residuals. Censored residuals seem to have an unreasonable spread for the inverse model.

2. Censored residuals spread reasonably for most of the basic models except the "explode" models for scatter.

3. General speaking, models having larger maximum likelihood values seem to have more reasonable residual plots. However, this conclusion does not apply to the inverse model because the maximum likelihood value of the inverse model cannot be compared with that of non-inverse models. The explanation is given in next section.

4.6 Comparison Using the Likelihood Ratio Test (LRT)

4.6.1 Likelihood-ratio Test in General Application

The likelihood ratio test (LRT) for single and for multiple random variables is described in [47] and [7] respectively. To apply LRT on regression models, one needs to understand the LRT in a general setting. Suppose $Y$ is a random variable having a density function $f(y; \beta)$ where $\beta$ is the vector of parameters. Assume that $\beta$ is unknown; but $\beta$ is known to be in a set $\Omega$. We wish to test the hypothesis $H_0$: $\beta \in \Omega_1$ vs. $H_1$: $\beta \notin \Omega_1$, where $\Omega_1 \subset \Omega$. If $\Omega$ is the universal set and $\Omega_2 = \Omega_1$, i.e. $\Omega_1 \cap \Omega_2 = \emptyset$ and $\Omega_1 \cup \Omega_2 = \Omega$, the above hypothesis is equivalent to $H_0$: $\beta \in \Omega_1$ vs. $H_1$: $\beta \in \Omega_2$. 
The likelihood-ratio is defined as

\[ \Lambda = \max_{\beta \in \Omega_1} \frac{\prod_{i=1}^{n} f(y_i, \beta)}{\max_{\beta \in \Omega} \prod_{i=1}^{n} f(y_i, \beta)} \]  

(4.10)

where \( y_i, i = 1, n \) is the sample of size \( n \), \( \beta \) is vector of the model parameters, and \( \prod_{i=1}^{n} f(y_i, \beta) \) is the likelihood value of the data set \( y \) under specific model parameters \( \beta \). \( \max_{\beta \in \Omega_1} \prod_{i=1}^{n} f(y_i, \beta) \) is the maximum likelihood value of the data set \( y \) under model \( \beta \in \Omega_1 \). This is the largest likelihood value found by adjusting \( \beta \) within the space of \( \Omega_1 \).

From maximum likelihood theory, \( -2 \ln \Lambda \sim \chi^2_{\nu} \) (i.e. has a chi-square distribution) as \( n \to \infty \). \( \nu \) is the degrees of freedom of the \( \chi^2 \) (chi-square) distribution and is defined as

\[ \nu = \text{dimension of } \Omega - \text{dimension of } \Omega_1 \]  

(4.11)

\[ = p - p_1 \]  

(4.12)

where dimension is defined as the number of independent parameters required to define the space \( \Omega \) or \( \Omega_1 \) and is denoted as \( p \) and \( p_1 \) respectively in Table 4.4 to Table 4.9.

Denote the likelihood values as

\[ L_1^* = \max_{\beta \in \Omega_1} \prod_{i=1}^{n} f(y_i, \beta) \]  

(4.13)

\[ L^* = \max_{\beta \in \Omega} \prod_{i=1}^{n} f(y_i, \beta) \]  

(4.14)

Then,

\[ -2 \ln \Lambda = -2 \ln \left( \frac{L_1^*}{L^*} \right) \]  

(4.15)

\[ = -2 \ln L_1^* + 2 \ln L^* \]  

(4.16)

\[ = 2 (\ln L^* - \ln L_1^*) \]  

(4.17)
For simplicity, $2(\ln L^* - \ln L_1^*)$ is denoted herein as $\Delta 2\ln L$. Then,

$$\Delta 2\ln L \sim \chi^2_\nu$$  \hspace{1cm} (4.18)

The likelihood-ratio test (LRT) rejects the hypothesis $H_0: \beta \in \Omega_1$ at significance level of $\alpha$ if

$$\Delta 2\ln L > \chi^2_{\alpha,\nu}$$  \hspace{1cm} (4.19)

where $\chi^2_{\alpha,\nu}$ is the $100(1 - \alpha)$ percentile of $\chi^2_\nu$. Because $\Delta 2\ln L$ is used in the LRT, the maximum likelihood values listed in the tables are two times the maximum likelihood values, denoted as $2\ln L$.

The LRT is used to compare the goodness-of-fit of any two models. If model A is rejected by model B in an LRT, then model B is said to be significantly better than model A in fitting the specific data set. If model A is not rejected by model B, then there is no significant difference between the two models in fitting the specific data. Application of the LRT on the reference fatigue data is described in the following.

### 4.6.2 Comparison of Models Using the LRT

The models presented in Table 4.4 to Table 4.9 are sorted by their maximum likelihood value ($2\ln L$). The inverse model treats stress/strain as the dependent variable, and likelihood values are calculated using the density function of stress/strain. The non-inverse models, including the pure $X$ error source model, treat life as a dependent variable and use the density function of life to calculate the likelihood value. Thus maximum likelihood values of the inverse model cannot be compared with those of the non-inverse models. For this reason, the inverse model is not considered in this LRT analysis. Maximum likelihood values for all models (except the inverse model)
in each data set are plotted in Fig. 4.47 to Fig. 4.52. Various models for the mean are categorized on the abscissa.

The maximum likelihood value is denoted as $L^*$. For each data set, the model having the largest $L^*$ is considered to be the "best fit model". Other models are then compared with the best fit model using the LRT and are called the "model to be tested". If the model to be tested is significantly worse than the best fit model under the LRT, a "!" is marked in the column "LRT" in Tables 4.4 through 4.9. Otherwise a "-" is marked to show that the model fits the data equally well. In other words, those models which are marked with "!" are rejected by the LRT. Those been marked as "-", along with the best fit model, are survivors of the LRT.

In these tables, $p_1$ is the number of free (or undetermined) parameters of the model to be tested $\Omega_1$, i.e. the total number of parameters in Eq. 2.11. For example, Item 6 in Table 4.4 is a model with a log transform on both stress $S$ and life $N$, a five parameter bilinear model for the mean curve, and one parameter for the assumed constant scatter of local stress. Because the log transformation has no free parameters, the total number of parameters for this model is six and is so listed in column "$p_1$" in the table. The number of free parameters for the various models for the mean and scatter can be found in Tables 4.2 and 4.3 respectively.

Consider Item 11 in Table 4.7 as another example. The model applies a one parameter Box-Cox transformation on life and a simple linear function (two parameters) as a model for the mean and homoscedastic scatter (one parameter); the total number of free parameters $p_1$ for this model is four.

The $p$ in these tables is the number of free parameters of $\Omega$. And $\Omega$ is the union of both the model to be tested and the best fit model. Two type of situations may
occur: (1) one model is a subset of the other, and (2) none of the models is a subset of the other.

When one model is a subset of the other, \( p \) equals the number of parameters of the larger set. For example, the only difference between the model of Item 5 in Table 4.5 and the best fit model is that one uses the bilinear as model for the mean while the other uses the leveling-off bilinear. The leveling-off bilinear is a subset of the bilinear and is derived from the bilinear by setting the parameter \( a_5 \) in Table 4.2 to \(-1\). The best fit model is a subset of the model in Item 5. Thus \( p \) equals the \( p_1 \) of the larger set model, i.e. the model of Item 5, and is equal to 6. Similarly, model of Item 8 in Table 4.6 is a subset of the best fit model in that table, and the model of Item 7 in Table 4.8 is a subset of the best fit model in that table.

When none of the models is a subset of the other, an "index" parameter is needed to integrate two models into a union. For example, consider Item 7 in Table 4.7 and the best fit model (i.e., Item 5). Each is not a subset of the other. For simplicity the model of Item 7 is denoted as \( \Omega_1(\beta_1, \beta_2, \ldots, \beta_5) \), where \((\beta_1, \beta_2, \ldots, \beta_5)\) are the five free parameters of the model. The best fit model is denoted as \( \Omega_2(\beta_1, \beta_2, \ldots, \beta_8) \) where \((\beta_1, \beta_2, \ldots, \beta_8)\) are the eight free parameters of the model. The two models can be integrated into the union model \( \Omega \) by

\[
\Omega = \rho \Omega_1(\beta_1, \beta_2, \ldots, \beta_5) + (1 - \rho) \Omega_2(\beta_1, \beta_2, \ldots, \beta_8)
\]

(4.20)

where \( \rho \) is the index parameter having a value of either 0 or 1. When \( \rho = 1 \), the model of Item 7 is specified. When \( \rho = 0 \), the best fit model is specified. The two models have totally different models for the mean (one uses a bilinear while the other uses the general strain-life equation) and for the scatter (quadratic versus pure X error source). However they can share the same parameters \( \beta_1, \beta_2, \ldots, \beta_5 \) in Eq. 4.20 without conflict because the index parameter \( \rho \) has already separated \( \beta_1, \beta_2, \ldots, \beta_5 \)
of the two models into different parameter spaces. For example $\beta_1 = 3.5$ when $\rho = 1$

is not the same with $\beta_1 = 3.5$ when $\rho = 0$. The former $\beta_1$ is the first parameter of
the general strain-life equation while the latter is the first parameter of the bilinear. Therefore a total of $8 + 1 = 9$ parameters (one for the “index” parameter $\rho$ and eight for $\beta_i$) is necessary and sufficient to define the union space of the two models. This
value is listed in column “p”. We conclude that when none of the models are subsets
of the others, $p$ equals to one plus the larger of the number of parameters of each
model.

The degrees of freedom $\nu$ is equal to $p - p_1$ and is also listed in Tables 4.4 through
4.9. Chi-square($\chi^2_{\alpha,\nu}$) values for different $\nu$ and a level of significance $\alpha = 0.05$
are listed in Table 4.10. In this study, none of the $\nu$ will exceed seven, which corresponds
to $\chi^2_{\alpha,\nu}$ value of 14.07. Thus all models to be tested having a $\Delta 2 \ln L$ value greater
than 14.07 will be rejected (see Eq. 4.19). This is the reason why the $p_1$, $p$ and $\nu$

in Tables 4.4 to 4.9 are listed only for cases with $\Delta 2 \ln L$ less than 14.07. All models to
be tested having a $\Delta 2 \ln L$ greater than 14.07 are directly marked with “!” without
the need for computing of $p_1$, $p$ and $\nu$.

4.6.3 Comparison of Models for the Mean

Which of the models for the mean provides the best fit of the fatigue data? This

is a complex question because the model for mean and the model for scatter have a
synergistic relationship relative to the goodness-of-fit. However a rough comparison
using a result of the LRT is made as follows. Various models for the mean that
appear as survivors of the LRT for each of the six data sets are listed in Table 4.11.
For example, both survivors of the LRT in the 2024-T4 data set use the bilinear as
the model for the mean. One of the LRT survivors of the In718 data set uses the
bilinear while the other uses the leveling-off bilinear as a model for mean. The total number of appearances of each model for the mean in the LRT survivors among six data sets are listed at lower part of the table. For example, the bilinear model for the mean appears in 11 out of the 27 (a ratio of 41%) LRT survivors. Because Ti64-300 is a subset of Ti64, it may be ignored in this comparison because the smaller data set has more uncertainty in a statistical test. When Ti64-300 results are ignored, the total times that the bilinear model appears as a LRT survivor is nine out of 19 (47%). From this table, clearly the bilinear, the leveling-off bilinear and the general strain-life models appear to perform better than the MIL-HDBK-5 and the Box-Cox model. Note that the Box-Cox model in this dissertation is not only a model for the mean but a complete model that consists of the Box-Cox transformation on life with a simple linear model for the mean with homoscedastic scatter. However the Box-Cox model is listed in Table 4.11 for comparison purposes.

4.6.4 Comparison of Models for the Scatter

A similar comparison to determine which model for the scatter best fits the fatigue data is performed as follows. Models for scatter that are among the survivors of the LRT for each of the six data sets are listed in Table 4.12. For example, one of the survivors in the 2024-T4 data set has a quadratic scatter model while the other uses a pure X error source. The total number of appearances of each model for scatter in the LRT survivors among the six data sets are listed at lower part of Table 4.12. For example, the quadratic model for scatter appears in eight out of the 26 (31%) LRT survivors. As mentioned earlier, the AAW data set appears to be approximately homoscedastic and the pure X error source model and the Box-Cox model was not intended for modeling homoscedastic data. If we focus on comparisons
of heteroscedastic models and exclude the AAW data set as shown in Table 4.12, the number of appearances of the quadratic model drops to six out of 20 (30%). The pure X error source model has the same number but an improved ratio, from 11 out of 26 (42%) to 11 out of 20 (55%). If the Ti64-300 data set is also excluded, the appearance of the quadratic model is reduced to three out of 12 (25%) and that of pure X error source is reduced to eight out of 12 (67%). But the ratio for the pure X error source is increased, and that for the quadratic is decreased slightly. Note that the explode type 2 scatter is excluded at the final stage. Clearly the winner in this competition is the pure X error source model.

Another rough method of comparing models for scatter is to conduct the LRT between models for scatter given the same model for mean. In this study only the pure X error source model is compared with other models for the mean. Results are listed in Table 4.13. For example, the first item in this table lists all models using MIL-HDBK-5 as the mean curve. The MIL model which uses the pure X error source as model of scatter has a $2\ln L_1$ value of -6102.15. The MIL model which uses quadratic scatter is compared with that using the pure X error source model. The $\Delta 2\ln L$ is listed in the column “qua”. The $\Delta 2\ln L$ is defined as

$$\Delta 2\ln L = 2\ln L_2 - 2\ln L_1$$

(4.21)

where $2\ln L_2$ is the $2\ln L$ value of $\Omega_2$, in this example the $2\ln L$ of quadratic scatter. The value of $\Delta 2\ln L$ is used to test if $\Omega_1$ or $\Omega_2$ provides a better fit. Note that the two tests have different degrees of freedom $\nu$ in the LRT.

Consider the Ti64 data set using the leveling-off bilinear and “X=S” for example. The $\Delta 2\ln L$ in column “qua” is -4.36 which means that the pure X error source model has a larger $2\ln L$ value than that of the quadratic model. To check if the quadratic model is significantly worse a LRT against $H_0: \beta \in \Omega_2$ is performed. Here
$p_1$ is the number of parameters for $\Omega_2$, $4 + 3 = 7$, where four is for the leveling-off bilinear and three is for the quadratic. Thus $p = 7 + 1 = 8$, where seven is the larger of number of parameters between $\Omega_1$ and $\Omega_2$ and one is the "index" parameter to integrate the two models. $\nu$ is therefore equal to one. From Table 4.10, a difference of 3.84 or greater will reject $\Omega_2$. In this example of the Ti64 data set, $-4.36$ is large enough to reject the quadratic model in preference to the pure X error source model. A "**" is then marked in the column "qua". Those marked as "N.A." (not available) means the model is not studied.

Consider the 2024-T4 data set using the bilinear as another example. $\Delta 2 \ln L = 6.57$, which means that the quadratic has a larger $2 \ln L$ value than that of the pure X error source model. To test if the pure X error source is significantly worse, a LRT against $H_0: \beta \in \Omega_1$ is performed. Here the $p_1$ is number of parameters for $\Omega_1$, $5 + 1 = 6$, where five is for the bilinear and one is for the homoscedastic pure X error source model. The model using the quadratic has $5 + 3 = 8$ parameters, three for the quadratic. Thus $p = 8 + 1 = 9$ where eight is the larger of number of parameters between $\Omega_1$ and $\Omega_2$ and one is the index parameter. $\nu$ is therefore equal to $9 - 6 = 3$. From Table 4.10, a difference of 7.81 or higher is required to reject $\Omega_1$. In this example of the 2024-T4 data set, 6.57 is not large enough to reject the pure X error source model in preference of the quadratic model and a "-" is marked in column "qua".

Each model for the scatter in $\Omega_2$ is compared against the pure X error source model in the same way. If no scatter model in $\Omega_2$ rejects the pure X error source in the same row then a "**" is marked in the column "model for mean". Otherwise a "!" is marked. Given the same model for the mean, this marks indicates whether the pure X error source model is acceptable as a model for scatter in comparison with
the others. Upon examining these marks, 14 out of 23 (61%) cases accept the pure X error source model as a model for scatter. If the AAW data set is excluded for reasons mentioned earlier, 14 out of 20 (70%) cases accept the pure X error source. If the Ti64-300 data set is further excluded, 11 out of 16 (69%) cases accept the pure X error source model.

4.7 Combination of All Three Tests

Models that pass all three tests (±3σ envelope test, standard residual test and LRT) are marked with a "*" in the column "pass all test" in Tables 4.4 to Table 4.9.

4.7.1 Comparison of Models for the Mean

Table 4.14 lists the (number of) appearance of various models for the mean among survivors of all three tests. The bilinear model for the mean appears in all data sets. When the Ti64-300 data set is excluded, the leveling-off bilinear, general strain-life, MIL-HDBK-5 and Box-Cox model appears three, two, zero and one out of five data sets respectively. If the number of appearance is counted and the Ti64-300 data set is excluded, the bilinear model appears eight out of 17 (47%) while the leveling-off bilinear and the general strain-life appears four out of 17 (24%) cases.

Note that the bilinear model and the leveling-off bilinear model are identical in terms of fitting the wire data set. The bilinear model for the mean is not listed in Table 4.9 to reduce the size of the table. It is included in Table 4.14.
4.7.2 Comparison of Models for the Scatter

For homoscedastic data (e.g. the AAW data set), the models that pass all three tests are the quadratic, the explode type 2, and the homoscedastic model for the scatter. Because models having fewer parameters are preferred, the homoscedastic model is recommended.

For the five heteroscedastic data sets (e.g. excluding the AAW data set), models for the scatter that appear in the survivors of all three tests are listed in Table 4.15. This table is derived from the column “pass all test” from all heteroscedastic data sets. The Pure X error source model appeared in all data sets while Box-Cox appeared in only two data sets. If Ti64-300 data set is excluded, both the Box-Cox and the quadratic appeared only in one data set. If the number of appearances is counted, the pure X error source appears in 11 out of 14 (79%) while the Box-Cox appears only in two out of 14 (14%) and the quadratic appears in one out of 14 (7%). Based on the data sets examined, the pure X error source model outperforms all other models for scatter.

4.8 Conclusion

The following conclusions are based on a limited but representative group of five data sets.

For modeling the mean curve of fatigue data, the bilinear, or the leveling-off bilinear, and the general life equation seem to perform far better than MIL-HDBK-5 model.

For modeling the scatter of heteroscedastic fatigue data, the pure X error source outperforms the other models considered.
The inverse model fails to have a sound statistical basis. Its residual plot appears to be unreasonable, especially when there are many censored data. But it does recognize the homoscedastic feature of stress/strain which appears in many fatigue data sets. And experience has shown that design curves derived from the inverse model seem to be reasonable.

Modeling heteroscedasticity by a linear, quadratic or explode type model usually results in an irregular or an exploding ±3σ envelope. These models rely purely on their mathematical flexibility (by having more parameters) in dealing with heteroscedasticity and are not based on the recognition of fatigue mechanism.

For data which consists of a curved and linear region at the same time, a typical characteristic of fatigue data, the Box-Cox model does not fit well. The Box-Cox model produces an envelope that is curved in the whole test stress range in the log-log space. The Box-Cox model also relies on its mathematical flexibility (but with fewer parameters) to deal with heteroscedasticity.

The pure X error source model appears to be the most reasonable model for scatter of heteroscedastic fatigue data. It appeared most frequently as a survivor of the combination test. Its residual plot seems to be more reasonable than that of the inverse model. It usually does not have an irregular or exploding envelope such as the quadratic or the explode type models do. Most importantly, it is based on the assumption of homoscedastic local stress/strain (an assumption regarding the fatigue mechanism) and a rigorous statistical development. Results of the tests suggest that behind the awkward appearance of heteroscedastic fatigue data, there may lies a simple homoscedastic random source.
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<th>(p)</th>
<th>(\nu)</th>
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(a). Symbol in column "model for mean" and "model for scatter" follows Table 4.2 and 4.3.
(b). In column "LRT",
!: the model is rejected by the best fit model which is the first model under the double horizontal line.
-: the model is statistically as good as the best fit model.
(c). In column "±3\(\sigma\) env. test",
c: the model fits poorly with a concave envelope.
e: the model fits poorly with a "explode" type envelope.
i: the model fits poorly with a irregular envelope.
b: the model fits poorly with an envelope not belonging to the category above.
(d). In column "\(z_i\) test" (standard residual test),
c: standardized residuals for censored data have unreasonable spread.
t: standardized residuals have a heteroscedastic spread.
(e). In column "pass all tests",
*: means the model passes all three tests.

Table 4.4: Multiple tests of the models on the aluminum 2024-T4 data set.
### Table 4.5: Multiple tests of the models on the Inconel 718 data set.

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<th>(p_1)</th>
<th>(p)</th>
<th>(\nu)</th>
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<th>(\pm 3\sigma) env. test</th>
<th>(z_i) test</th>
<th>pass all tests</th>
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</tbody>
</table>

(a). Symbol in column “model for mean” and “model for scatter” follows Table 4.2 and 4.3.
(b). In column “LRT”,
*!: the model is rejected by the best fit model which is the first model under the double horizontal line.
*: the model is statistically as good as the best fit model.
(c). In column “\(\pm 3\sigma\) env. test”,
c: the model fits poorly with a concave envelope.
e: the model fits poorly with a "explode" type envelope.
i: the model fits poorly with an irregular envelope.
b: the model fits poorly with an envelope not belonging to the category above.
(d). In column “\(z_i\) test” (standard residual test),
c: standardized residuals for censored data have unreasonable spread.
t: standardized residuals have a heteroscedastic spread.
(e). In column “pass all tests”,
*: means the model passes all three tests.
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(a). Symbol in column "model for mean" and "model for scatter" follows Table 4.2 and 4.3.
(b). In column "LRT",
!: the model is rejected by the best fit model which is the first model under the double horizontal line.
:-: the model is statistically as good as the best fit model.
(c). In column "$\pm 3\sigma$ env. test",
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b: the model fits poorly with an envelope not belonging to the category above.
(d). In column "$z_i$ test" (standard residual test),
c: standardized residuals for censored data have unreasonable spread.
t: standardized residuals have a heteroscedastic spread.
(e). In column "pass all tests",
*: means the model passes all three tests.

Table 4.6: Multiple tests of the models on the AAW data set.
Table 4.7: Multiple tests of the models on the Ti64 data set.
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<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11 MIL exp2 (X=S)</td>
<td>-812.35</td>
<td>-7.66</td>
<td>3+3=6</td>
<td>9</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12 e-N exp2</td>
<td>-812.42</td>
<td>-7.73</td>
<td>4+3=7</td>
<td>9</td>
<td>2</td>
<td>!</td>
<td>c</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>13 Box</td>
<td>-812.66</td>
<td>-7.97</td>
<td>1+2+1=4</td>
<td>9</td>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>14 l-bi exp2</td>
<td>-813.77</td>
<td>-9.08</td>
<td>4+3=7</td>
<td>9</td>
<td>2</td>
<td>!</td>
<td>e</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15 bi exp2</td>
<td>-814.67</td>
<td>-9.98</td>
<td>5+3=8</td>
<td>9</td>
<td>1</td>
<td>!</td>
<td>e,i</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>16 MIL pure X (X=S)</td>
<td>-826.95</td>
<td>-22.26</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>17 MIL exp1 (X=S)</td>
<td>-826.96</td>
<td>-22.27</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>18 MIL homo (X=S)</td>
<td>-827.58</td>
<td>-22.89</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(a). Symbol in column “model for mean” and “model for scatter” follows Table 4.2 and 4.3.
(b). In column “LRT”,
   !: the model is rejected by the best fit model which is the first model under the double horizontal line.
   -: the model is statistically as good as the best fit model.
(c). In column “$\pm 3\sigma$ env. test”,
c: the model fits poorly with a concave envelope.
e: the model fits poorly with a “explode” type envelope.
i: the model fits poorly with an irregular envelope.
b: the model fits poorly with an envelope not belonging to the category above.
(d). In column “$z_i$ test” (standard residual test),
c: standardized residuals for censored data have unreasonable spread.
t: standardized residuals have a heteroscedastic spread.
(e). In column “pass all tests”,
*: means the model passes all three tests.

Table 4.8: Multiple tests of the models on the Ti64-300 data set.
<table>
<thead>
<tr>
<th>model for mean</th>
<th>model for scatter</th>
<th>$2 \ln L$</th>
<th>$\Delta 2 \ln L$</th>
<th>$p_1$</th>
<th>$\nu$</th>
<th>LRT</th>
<th>±3σ env. test</th>
<th>$z_i$ test</th>
<th>pass all tests</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 bi</td>
<td>(y,x)</td>
<td>-1318.18</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>t.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 l-bi</td>
<td>(y,x)</td>
<td>-1319.27</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>t.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 e-N</td>
<td>(y,x)</td>
<td>-1370.26</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>t, n</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4 MIL</td>
<td>(y,x) (X=S)</td>
<td>-1372.74</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>t, n</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5 l-bi</td>
<td>pure X</td>
<td>-1509.30</td>
<td>4+1=5</td>
<td>6</td>
<td>1</td>
<td>!</td>
<td>e t.</td>
<td></td>
<td>*</td>
</tr>
<tr>
<td>6 l-bi</td>
<td>2Sy=1</td>
<td>-1513.60</td>
<td>-4.30</td>
<td>4+1=5</td>
<td>6</td>
<td>1</td>
<td>e t.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7 l-bi</td>
<td>2Sy=2</td>
<td>-1515.68</td>
<td>-6.38</td>
<td>4+1=5</td>
<td>6</td>
<td>1</td>
<td>e t.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8 l-bi</td>
<td>exp2</td>
<td>-1519.09</td>
<td>-9.79</td>
<td>4+3=7</td>
<td>8</td>
<td>1</td>
<td>e t.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9 l-bi</td>
<td>2Sy=3</td>
<td>-1521.88</td>
<td>-12.58</td>
<td>4+1=5</td>
<td>6</td>
<td>1</td>
<td>e t.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10 l-bi</td>
<td>qua</td>
<td>-1529.33</td>
<td>-20.03</td>
<td>4+3=7</td>
<td>8</td>
<td>1</td>
<td>i t.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11 e-N</td>
<td>qua</td>
<td>-1533.16</td>
<td>-23.86</td>
<td></td>
<td></td>
<td>!</td>
<td>c t.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12 MIL</td>
<td>exp2 (X=S)</td>
<td>-1538.01</td>
<td>-28.71</td>
<td></td>
<td></td>
<td>!</td>
<td>e t.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>14 e-N</td>
<td>pure X</td>
<td>-1543.73</td>
<td>-34.43</td>
<td></td>
<td></td>
<td>!</td>
<td>t.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15 MIL</td>
<td>qua (X=S)</td>
<td>-1545.43</td>
<td>-36.13</td>
<td></td>
<td></td>
<td>!</td>
<td>c. t</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16 MIL</td>
<td>pure X (X=S)</td>
<td>-1549.52</td>
<td>-40.22</td>
<td></td>
<td></td>
<td>!</td>
<td>t.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>17 MIL</td>
<td>exp1 (X=S)</td>
<td>-1563.77</td>
<td>-54.47</td>
<td></td>
<td></td>
<td>!</td>
<td>b t.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>18 l-bi</td>
<td>lin</td>
<td>-1561.32</td>
<td>-72.02</td>
<td></td>
<td></td>
<td>!</td>
<td>b t.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>19 l-bi</td>
<td>homo</td>
<td>-1654.23</td>
<td>-144.93</td>
<td></td>
<td></td>
<td>!</td>
<td>b t.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20 MIL</td>
<td>homo (X=S)</td>
<td>-1657.31</td>
<td>-148.01</td>
<td></td>
<td></td>
<td>!</td>
<td>b t.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(a). Symbol in column “model for mean” and “model for scatter” follows Table 4.2 and 4.3.
(b). In column “LRT”,
!: the model is rejected by the best fit model which is the first model under the double horizontal line.
:: the model is statistically as good as the best fit model.
(c). In column “±3σ env. test”,
c: the model fits poorly with a concave envelope.
e: the model fits poorly with a “explode” type envelope.
i: the model fits poorly with an irregular envelope.
b: the model fits poorly with an envelope not belonging to the category above.
(d). In column “$z_i$ test” (standard residual test),
c: standardized residuals for censored data have unreasonable spread.
t: standardized residuals have a heteroscedastic spread.
(e). In column “pass all tests”,
*: means the model passes all three tests.

Table 4.9: Multiple tests of the models on the wire data set.

<table>
<thead>
<tr>
<th>degrees of freedom, $\nu$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi^2_{0.05,\nu}$</td>
<td>3.84</td>
<td>5.99</td>
<td>7.81</td>
<td>9.49</td>
<td>11.07</td>
<td>12.59</td>
<td>14.07</td>
</tr>
</tbody>
</table>

Table 4.10: 95 percentile of the chi-square variate $\chi^2_{\nu}$
The numbers represent the appearances of a model for the mean among the survivors of the LRT for a specific data set.

Table 4.11: Comparison of the models for the mean among survivors of the LRT

<table>
<thead>
<tr>
<th>model for mean</th>
<th>bi</th>
<th>l-bi</th>
<th>e-N</th>
<th>MIL</th>
<th>(Box)</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>data set</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2024-T4</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>In718</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>AAW</td>
<td>3</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td>6</td>
</tr>
<tr>
<td>Ti64</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td></td>
<td></td>
<td>7</td>
</tr>
<tr>
<td>Ti64-300</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td></td>
<td>8</td>
</tr>
<tr>
<td>wire</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>all data set</td>
<td>11</td>
<td>5</td>
<td>7</td>
<td>2</td>
<td>2</td>
<td>27</td>
</tr>
<tr>
<td>exclude</td>
<td>9</td>
<td>4</td>
<td>5</td>
<td>0</td>
<td>1</td>
<td>19</td>
</tr>
<tr>
<td>Ti64-300</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The numbers represent the appearances of a model for the scatter among the survivors of the LRT for a specific data set.

Table 4.12: Comparison of the models for the scatter among survivors of the LRT

<table>
<thead>
<tr>
<th>model for scatter</th>
<th>qua</th>
<th>pure X</th>
<th>homo</th>
<th>exp2</th>
<th>(Box)</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>data set</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2024-T4</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>In718</td>
<td>2</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>AAW</td>
<td>2</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td>6</td>
</tr>
<tr>
<td>Ti64</td>
<td>2</td>
<td>4</td>
<td></td>
<td></td>
<td>1</td>
<td>7</td>
</tr>
<tr>
<td>Ti64-300</td>
<td>3</td>
<td>3</td>
<td></td>
<td></td>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>wire</td>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>all data set</td>
<td>8</td>
<td>11</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>26</td>
</tr>
<tr>
<td>exclude AAW</td>
<td>6</td>
<td>11</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>20</td>
</tr>
<tr>
<td>exclude Ti64-300</td>
<td>3</td>
<td>8</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>12</td>
</tr>
</tbody>
</table>

The numbers represent the appearances of a model for the scatter among the survivors of the LRT for a specific data set.

Table 4.12: Comparison of the models for the scatter among survivors of the LRT
(a). $\Omega_1$: hypothesis that the pure X error source model is the underlying model for the scatter.

(b). $\Omega_2$: hypothesis that one of the six models in column "$\Omega_2$" is the underlying model for the scatter.

(c). Column "$\Omega_1$", The number in this column is the likelihood value $2 \ln L$ for pure X error source model along with the corresponding model for the mean.

(d). Column "$\Omega_2$", The numbers are the $\Delta 2 \ln L$ in the LRT.

* : this symbol means that there is no significant difference between the pure X error source model and a model in column "$\Omega_2$", as the result of the LRT.

*: this symbol means that the pure X error source model is significantly better than the other model in the LRT.

!: this symbol means that the other model in the LRT is significantly better than the pure X error source model.

N.A.: this means the LRT is not performed for this pair of models.

(e). In the column "model for mean", *

* : this symbol means that on the same row, none of the models in the column "$\Omega_2$" tested is significantly better than the pure X error source model.

!: this symbol means that on the same row, at least one of the models in the column "$\Omega_2$" tested is significantly better than the pure X error source model.

Table 4.13: Comparison of the pure X error source model with other models for scatter.
"Yes" means a model for the mean has appeared at least once among the survivors of the combination test for a specific data set. The number in the parentheses is the frequency of such appearance.

**Table 4.14:** Comparison of models for the mean among survivors of all three tests

<table>
<thead>
<tr>
<th>model for mean</th>
<th>bi</th>
<th>l-bi</th>
<th>e-N</th>
<th>MIL</th>
<th>(Box)</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>data set</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2024-T4</td>
<td>Yes(2)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(2)</td>
</tr>
<tr>
<td>In718</td>
<td>Yes(1)</td>
<td>Yes(1)</td>
<td></td>
<td></td>
<td></td>
<td>(2)</td>
</tr>
<tr>
<td>AAW</td>
<td>Yes(3)</td>
<td>Yes(3)</td>
<td></td>
<td></td>
<td></td>
<td>(6)</td>
</tr>
<tr>
<td>Ti64</td>
<td>Yes(1)</td>
<td>Yes(2)</td>
<td>Yes(1)</td>
<td></td>
<td>Yes(1)</td>
<td>(5)</td>
</tr>
<tr>
<td>Ti64-300</td>
<td>Yes(1)</td>
<td>Yes(1)</td>
<td>Yes(1)</td>
<td></td>
<td>Yes(1)</td>
<td>(4)</td>
</tr>
<tr>
<td>wire</td>
<td>Yes(1)</td>
<td>Yes(1)</td>
<td></td>
<td></td>
<td></td>
<td>(2)</td>
</tr>
<tr>
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<td>4/6</td>
<td>3/6</td>
<td>0/6</td>
<td>2/6</td>
<td></td>
</tr>
<tr>
<td>exclude</td>
<td>5/5</td>
<td>3/5</td>
<td>2/5</td>
<td>0/5</td>
<td>1/5</td>
<td></td>
</tr>
<tr>
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<td>0</td>
<td>2</td>
<td>(21)</td>
</tr>
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<td>4</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>(17)</td>
</tr>
</tbody>
</table>

"Yes" means a model for the scatter has appeared at least once among the survivors of the combination test for a specific data set. The number in the parentheses is the frequency of such appearance.

**Table 4.15:** Comparison of models for the scatter among survivors of three tests

<table>
<thead>
<tr>
<th>model for scatter</th>
<th>qua</th>
<th>pure X</th>
<th>homo</th>
<th>exp2</th>
<th>(Box)</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>data set</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2024-T4</td>
<td>Yes(1)</td>
<td>Yes(1)</td>
<td></td>
<td></td>
<td></td>
<td>(2)</td>
</tr>
<tr>
<td>In718</td>
<td>Yes(2)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(2)</td>
</tr>
<tr>
<td>AAW</td>
<td>Yes(2)</td>
<td>Yes(2)</td>
<td>Yes(2)</td>
<td></td>
<td></td>
<td>(6)</td>
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<td>Ti64</td>
<td>Yes(4)</td>
<td></td>
<td></td>
<td></td>
<td>Yes(1)</td>
<td>(5)</td>
</tr>
<tr>
<td>Ti64-300</td>
<td>Yes(3)</td>
<td></td>
<td></td>
<td></td>
<td>Yes(1)</td>
<td>(4)</td>
</tr>
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<td>Yes(1)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(1)</td>
</tr>
<tr>
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<td>5/6</td>
<td>1/6</td>
<td>1/6</td>
<td>2/6</td>
<td></td>
</tr>
<tr>
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<td>5/5</td>
<td>0/5</td>
<td>0/5</td>
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<td>0/4</td>
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<td>1/4</td>
<td></td>
</tr>
<tr>
<td>exclude Ti64-300</td>
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<td>11</td>
<td>2</td>
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<td>2</td>
<td>20</td>
</tr>
<tr>
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<td>11</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>14</td>
</tr>
<tr>
<td>all data set</td>
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<td>8</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>10</td>
</tr>
</tbody>
</table>

"Yes" means a model for the scatter has appeared at least once among the survivors of the combination test for a specific data set. The number in the parentheses is the frequency of such appearance.

**Table 4.15:** Comparison of models for the scatter among survivors of three tests
Figure 4.1: 2024-T4 fatigue data.

Figure 4.2: In718 fatigue data.
Figure 4.3: AAW fatigue data.

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mean curve and ±3σ envelope plot

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CHAPTER 5
S-N Design Curve Using the Approximate Owen Tolerance Factor

5.1 Introduction

5.1.1 Preliminary Comments

The goal of this chapter is to develop a method for constructing a design curve to characterize fatigue strength to be used in a conventional factor of safety design approach. General considerations of the design curve were discussed in Sec. 1.9. In brief, the design curve should fall to the left, or the safe side of the data, i.e., the low end of the distribution of life given stress/strain. It should be a lower bound to majority of the data. A rational criterion is to define the design curve targeting the low $1 - p$ quantile, denoted as $y_{1-p}$, of the distribution of life at a given stress/strain. All $y_{1-p}$ in a specific range of stress/strain constitutes a $y_{1-p}$ curve in a S-N plot. Given no other random design factors, $p$ would be the probability of survival before life $y_{1-p}$ and $1 - p$ is the probability of failure. However many other factors will affect the probability of failure and in general the $p$ value alone doesn't imply "probability of survival". In this context the $p$ value is chosen, not by probability of failure considerations, but by engineering judgment. The choice of $p$ value is therefore "arbitrary", but values of 0.95 or 0.99 are often used.
In general, a design point at a specific stress/strain level $x_h$ is constructed using

$$Y_D(x_h) = \hat{\mu}(x_h) - K(x_h)\hat{\sigma}$$  \hspace{1cm} (5.1)$$

where $\hat{\mu}(x_h)$ and $\hat{\sigma}$ are the estimated mean and standard deviation of (transformed) life $Y$ at (transformed) stress/strain level $x_h$. $Y_D(x_h)$ is a random variable because $\hat{\mu}(x_h)$ and $\hat{\sigma}$ are random variables. $K$ is a factor. Various choices of $K$ are discussed in Sec. 1.9. One choice is Owen's tolerance factor $K(x_h)$ [61] which guarantees that 100$\gamma$% of the time, the design point $Y_D(x_h)$ will lie at left (conservative) side of $y_{1-p}(x_h)$. $\gamma$ is called the confidence level. For example, using Owen's tolerance $K(x_h)$ with $\gamma = 0.95$ in constructing the design point $Y_D(x_h)$, 95% of the time $Y_D$ defines a design point that has a probability of survival greater than $p$. Again, note that the "probability of survival" applies only if scatter in the data were the only random design factor. The design curves constructed using Owen's tolerance factor $K(x_h)$ are called "Owen's curve" in this article.

Owen's tolerance factor accounts for the uncertainty of the estimators $\hat{\mu}(x_h)$ and $\hat{\sigma}$ and therefore is recommended particularly for small sample size situations, as is the case in many fatigue tests.

From practical engineering considerations, it is desirable that a design curve would have the same shape as, i.e. be parallel with at all $x_h$, the median curve through the data. Note that if the $K$ in Eq. 5.1 were constant, not a function of $x_h$, then $Y_D(x_h)$ would in fact be parallel with $\hat{\mu}(x_h)$. A constant $K$, called "approximate Owen's tolerance factor", is developed and its effect on the confidence level $\gamma$ of the design curve is examined. In this study, the amount of loss in confidence level $\gamma$ when using the approximate (i.e., "incorrect") Owen curve will be shown. It is the author's hope that this assessment of error will help engineers to determine if they wish to take the advantage of the simpler approximate Owen curve.
Owen's theory is valid for linear regression models. Therefore the design curve discussed in this chapter applies only to the linear regression model where the following conditions are satisfied.

- The model for the mean is linear in terms of the undetermined coefficients $a$ defined in Eq 2.5.

- The scatter of $Y$ is homoscedastic and normally distributed. Note that there should be no undetermined parameters in the transformation of life, i.e. there should be no $\lambda$ as in Eq. 2.2.

- All $y_i$, $i = 1, n$ are mutually independent. $n$ is the sample size.

Relative to the error source models considered herein, the above conditions imply the following:

- The design curve in this chapter does not in general apply to the pure X error source model.

- When the model for the mean is simple linear, the pure X error source model is the linear regression model. However in this case, the pure X error source is equivalent to the pure Y error source. Furthermore, both are also equivalent to the mixed error source model.

- The pure Y error source model with the linear model for the mean is also a linear regression model.

For the pure X error source model, another method to construct a design curve is necessary. One such method is proposed in Sec. 5.8.1 of this chapter.
5.1.2 Introduction of the Owen Tolerance Factor

The purpose of the Owen tolerance factor is to provide an estimate of a quantile \((1 - p)\) having a given confidence level \(\gamma\).

Owen's tolerance factor may be applied for samples of normal univariates or for the linear regression model. An example of normal variate samples is when all specimens are tested at one stress/strain level. The response transformed life \(Y\) is an univariate. An example of a linear regression model is when specimens are tested at various stress/strain levels and the model for the mean is linear. Transformed life \(Y(z_h)\) is a random variable.

For samples of normal univariates, the design point is defined as

\[
Y_D = \hat{\mu} - K\hat{\sigma}
\]

where \(\hat{\mu}\) and \(\hat{\sigma}\) are the sample mean and sample standard deviation respectively. The derivation of \(K\) can be found in Appendix A. Values of \(K\) can be found in tables of one-sided tolerance limits (See [51] or [3, table 33]). \(Y_D\) is an estimate of \(y_{1-p}\). At least 100\(p\)% of population of \(Y\) will lie above \(Y_D\) with confidence \(\gamma\).

For the linear regression cases, the design curve is defined in Eq. 5.1. Values of \(K(z_h)\) are not tabulated and must be computed. The derivation of \(K(z_h)\) is presented in Appendix A.

The Owen tolerance factor is based on the non-central \(t\) distribution having two parameters: the non-centrality \(\delta\) and the number of degrees of freedom \(f\). For practical reasons, in this study, the Owen tolerance factor will be treated as a function of \(a, f, p\) and \(\gamma\) where \(a\) equals to \(1/n\) if the sample is from a normal variate; \(a\) is defined later by Eq. 5.9 for the case when the linear regression model is applied. \(f\) is the degrees of freedom and is equal to \(n - 1\) if the sample is from a normal variate, and
equal to \( n - m \) for the linear regression model having \( m \) unknown parameters. For a polynomial regression model of degree \( d \), \( m = d + 1 \). Details of the Owen tolerance factor is presented in Appendix A.

5.2 Concept of the Approximate Owen's Curve

5.2.1 Review of Various Design Curves

Using an example, a visual comparison of design curves constructed using different methods is made in this subsection. All of these curves are estimates of the \( \alpha = 1 - p = 0.01 \) quantile of life given stress/strain. The assumed underlying model (nature's choice) is based on aluminum 7075-T6 presented in MIL-HDBK-5D Fig.3.7.4.1.8(c). This homoscedastic model is described by

\[
Y = 8.932 - 0.04803x + 0.00010746x^2 + \epsilon
\]  

(5.3)

where \( Y = \log_{10} N \), \( x = \) stress in MPa and \( \epsilon \) is normally distributed with zero mean and \( \sigma = 0.2051 \) in \( \log_{10} N \) space. This implies a COV of \( N \) of 50%. The assumed analysis model is chosen as the same as nature's model except that the parameters are undetermined.

A sample of size \( n = 10 \) is obtained from nature's model. The least squares method is used to fit the data. After estimates of the mean curve \( \hat{Y}(x) \) and the standard deviation \( \hat{\sigma} \) are determined, various design curves may be constructed. The design curves under comparison are: (1) the \( K = 2.33 \) design curve (Eq. 5.4), (2) Owen's one dimensional tolerance design curve (Eq. 1.3) which is a S-N curve constructed by using \( K \) in Eq. 5.2 instead of the correct \( K \) of Eq. 1.4, (3) Owen's tolerance design curve for the regression model (Eq. 1.4), and (4) the simultaneous tolerance
design curve (Eq. 1.5). Except for the $K = 2.33$ design curve, all design curves are constructed with confidence level of $\gamma = 0.95$. The $K = 2.33$ design curve is

$$Y_D(x) = \hat{Y}(x) - 2.33\hat{\sigma}$$

(5.4)

where $-2.33$ is the $\alpha = 0.01$ quantile corresponding to the standard normal distribution.

These curves are plotted in Fig. 5.1 and 5.2. In this example, most data points are at the right of the true 50% curve. Thus the least squares curve falls to the right of the true mean. Because the expected value of $K = 2.33$ curve is the true 1% curve, approximately 50% of time this curve is unconservative. As seen in Fig. 5.1, the $K = 2.33$ design curve is unconservative in this case as it results in a higher design stress than that of true 1% curve for a given service life. However, Owen's curve guarantees that at any specific stress level $x$, $100\gamma\%$ of time the design point at stress level $x$ will be to the left of the true $1 - p$ quantile $y_{1-p}$.

The simultaneous design curve guarantees that at least $100p\%$ of time all points on the curve are simultaneously below the true $y_{1-p}$ curve. In this example, both Owen's curve and the simultaneous curve are conservative even though the fitted mean curve falls to the right of true mean. Experience has shown that the simultaneous curve is usually grossly conservative and not appropriate for most design situations.

As shown in Fig. 5.2, Owen's one dimensional tolerance design curve (using $K$ as in Eq. 5.2 instead of $K(x_A)$ in Eq. 5.1) is less conservative than Owen's curve. This is because the one dimensional tolerance limit did not consider uncertainty in estimating the regression parameters. The one dimensional curve is not mathematically correct even though its use is fairly common. There is no theoretical means to estimate the loss of confidence level $\gamma$ when the one dimensional design curve is applied in the
regression model. The benefit of using this form is the ease of determining the one-dimensional tolerance factors $K$ using existing tables[51]. On the other hand, to find Owen's tolerance factor for the regression model, a computer software to calculate variates associated with the non-central $t$ distribution is required.

### 5.2.2 Approximate Owen's Curve

Owen's tolerance factor, denoted simply as $K$ in this chapter, is a function of stress/strain $x$ in the regression model. Visual examination of curves constructed from Owen's $K$ suggest that the variation of $K(x)$ within the test stress range is generally relatively small for sample sizes greater than six. Therefore an average value of $K(x)$ having an acceptably small error within the stress range is suggested. The average value of $K(x)$ is called the "approximate Owen's $K" in this chapter and is defined as

$$\bar{K} = \frac{1}{x_b - x_a} \int_{x_a}^{x_b} K(x) \, dx$$  \hspace{1cm} (5.5)

If the average value within the test stress range is desired, then use $x_a$ and $x_b$ as the lowest and highest stress/strain levels in the test plan. An approximate Owen's curve, plotted in Fig. 5.2, is compared with the exact Owen curve. Visually their difference is small. The question addressed herein is the significance of the difference in a statistic sense.

Because $\bar{K}$ is the average value within a given stress/strain range, $\bar{K}$ must be larger than $K(x)$ somewhere in that range and smaller than $K(x)$ in the rest of that range. Compared with the Owen's curve, the approximate Owen's curve is more conservative somewhere in the range and less conservative in the rest of the range. If the $p$ value is kept the same ($p = 0.99$ in this example), the confidence level $\gamma$ of the approximate Owen's curve, denoted as $\gamma(x)$, will vary from point to point. At
a stress level $x$ where $\bar{K}$ is larger than $K(x)$, the approximate Owen’s curve has a $\gamma(x) > 0.95$ and vice versa.

To determine if an approximate Owen’s curve is an acceptable approximation of Owen’s curve in statistical sense, a measure of the quality of the approximation is adopted using $\Delta \gamma = \gamma_{\text{min}} - \gamma$, where $\gamma = 0.95$ in this study; $\gamma_{\text{min}}$ is the smallest $\gamma(x)$ value in the range where $K(x)$ is being approximated. If an approximate Owen’s curve has a $|\Delta \gamma| < 0.03$, all design points on approximate Owen’s curve have at least a confidence level of $\gamma(x) > \gamma - 0.03 = 0.92$. This will be arbitrarily considered herein as an acceptable value compared with the original goal of $\gamma = 0.95$. On the other hand, we hope that the $\gamma_{\text{max}}$, the largest $\gamma(x)$ value in the range, is not too conservative. Because the maximum possible value of $\gamma$ is 100%, the value of $\gamma_{\text{max}}$ is not appropriate in measuring the over conservatism.

An alternative method for measuring the over conservatism is to compare $\Delta K_{\text{max}}$ and $\Delta K_{\text{min}}$, where $\Delta K_{\text{max}} = \bar{K} - K_{\text{max}}$, and $\Delta K_{\text{min}} = \bar{K} - K_{\text{min}}$. $K_{\text{max}}$ and $K_{\text{min}}$ are the maximum and minimum value of $K(x)$ in the range being considered. $K_{\text{max}}$ will yield the $\gamma_{\text{min}}$ while $K_{\text{min}}$ will yield the $\gamma_{\text{max}}$. $\Delta K_{\text{max}}$ can be thought of as a measure of non-conservatism of the least conservative point on the curve, while $\Delta K_{\text{min}}$ can be thought of as a measure of over conservatism of the most conservative point on the curve. If the absolute value of $\Delta K_{\text{min}}$ is not much larger than the absolute value of $\Delta K_{\text{max}}$, the amount of over conservatism at the most conservative point is not large in comparison to the amount of non-conservatism at the least conservative point on the curve. In addition, if the absolute value of $\Delta \gamma$ (which corresponds to $K_{\text{max}}$ at the least conservative point on the curve) is not large, the approximate Owen’s curve is considered to be acceptable.
In order to examine the quality of the approximate Owen's curve, further study of Owen's $K$ is necessary. Among the factors $a, f, p, \gamma$ in determining $K$, the $a$ value is the key to generalizing the results of this study. The $a$ value is examined in detail in the following sections.

5.3 A Study of the $a$ Value

5.3.1 Definition of $a$ Value

The quantity denoted as $a$ in Owen's work [61] is described in this section.

Suppose that a fatigue test has been specified. Denote the stress level settings for test of the $n$ specimens as the vector $x$

$$
\begin{align*}
    x &= \begin{pmatrix}
        x_1 \\
        \vdots \\
        x_n
    \end{pmatrix}
\end{align*}
$$

$x_i$ is the stress level setting for the $i$th specimen. A model is chosen for data analysis. As described in Sec. 2.2, a complete analysis model consists of: (1) transformations on stress/strain and life, (2) a model for the mean and (3) a model for the scatter and (4) a model for the distribution of life given stress/strain. Owen's theory assumes a linear model. Conditions for the choice of linear models are: (1) any transformation without undetermined parameters, e.g. log transformation, is allowed, (2) the model for the mean should be linear in terms of the undetermined parameters, (3) the model for scatter should be homoscedastic, and (4) $Y$ should have a normal distribution.

Given the stress level setting $x$ and the analysis model, the analysis matrix, denoted as $X$, can be constructed. For example, assume that the analysis model is
quadratic

\[ y_i = a_0 + a_1 x_i + a_2 x_i^2 + \epsilon_i \]  
(5.7)

Then the element in the \( i \)-th row and the \( j \)-th column of the analysis matrix \( X \) are the "coefficients" of the \( a_j \) terms.

\[
X = \begin{pmatrix}
1 & x_1 & x_1^2 \\
1 & x_2 & x_2^2 \\
\vdots \\
1 & x_n & x_n^2
\end{pmatrix}
\]  
(5.8)

The value of \( a \) at stress/strain level \( x_h \) is then defined as

\[ a(x_h) = x_h^T (X^T X)^{-1} x_h \]  
(5.9)

where \( X^T \) is the transpose of the analysis matrix. \( x_h^T \) is equal to the analysis matrix having only one stress level \( x_h \) which may be any stress level of interest.

\[ x_h^T = \begin{pmatrix} 1 & x_h & x_h^2 \end{pmatrix} \]  
(5.10)

The \( a \) value is common in linear regression [55]. For example, the fitted mean curve \( \hat{Y}(x) \) in linear regression is a random variable. The fitted mean value at stress level \( x_h \), denoted as \( \hat{Y}(x_h) \), has a variance of

\[ \sigma^2 \{ \hat{Y}(x_h) \} = \sigma^2 a(x_h) \]  
(5.11)

where \( a(x_h) \) is defined as above. It is seen that \( a(x_h) \) represents the uncertainty involved in estimating the mean curve; a larger \( a \) implies larger uncertainty. Note that uncertainty is a function of position \( x_h \) relative to the test stress range. As shown later, \( a(x_h) \) becomes relatively large as \( x_h \) moves away from test stress range.
To illustrate the functional form of \( a(x_h) \), consider a simple case. Assume a simple linear model for the mean. The analysis matrix \( X \) will be an \( n \times 2 \) matrix. The inverse matrix \((X^TX)^{-1}\) in Eq. 5.9 will be a \( 2 \times 2 \) matrix,

\[
(X^TX)^{-1} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}
\]

(5.12)

where the \( a_{ij} \) are constants that are computed from the stress level settings in the test plan. Then \( a(x_h) \) will be a polynomial function of degree two

\[
a(x_h) = \begin{pmatrix} 1 & x_h \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 \\ x_h \end{pmatrix} = a_{11} + (a_{12} + a_{21})x_h + a_{22}x_h^2
\]

(5.13)

Because the inverse matrix \((X^TX)^{-1}\) contains only constant values calculated from stress level settings, only the matrix \( x_h \) affects the degree of the polynomial. Multiplication of \( x_h^T \) and \( x_h \) is involved in the calculation of \( a(x_h) \), and the resulting degree of the polynomial is twice the highest order in \( x_h \). In this example the analysis model is simple linear (polynomial of degree one) and \( a(x_h) \) is a polynomial function of degree two. If the analysis model were quadratic, then \( a(x_h) \) would be a polynomial function of degree four and so on.

### 5.3.2 Invariance of \( a \) Under a Linear Transformation of Stress

Eq. 5.9 shows that \( a \) is function of the stress level setting \( x \) in the fatigue test plan and that \( x_h \) is the stress level of interest. It is a practical impossibility to derive a general relationship between \( x \), \( x_h \) and \( a \) because the number of possible \( x \) is infinite. One way to simplify the relationship is to consider normalized stress levels, \( u \). The
normalized stress level setting \( u \) is defined as follows:

\[
\mathbf{u} = \begin{pmatrix}
    u_1 \\
    \vdots \\
    u_n 
\end{pmatrix}
\]  

(5.14)

The element \( u_i \) in \( u \) is derived through a linear transformation of \( x_i \) defined as

\[
u_i = \frac{x_i - x_{\text{min}}}{x_{\text{max}} - x_{\text{min}}}
\]

(5.15)

where \( x_{\text{min}} \) and \( x_{\text{max}} \) represent the lowest and highest stress levels in the stress level setting of a fatigue test plan. Note that all \( u_i \) values will fall in the range of \([0, 1]\).

The linear transformation in Eq. 5.15 is called a normalizing transformation. The normalized stress level of \( x_h \) is denoted as \( u_h \).

It would greatly simplify the study of \( a \) if \( x, x_h \) and their corresponding normalized values \( u, u_h \) produced exactly the same value of \( a \). Because no mathematical theory is available to prove this, a numerical investigation is performed in which the calculation of \( a \) both ways is compared. First calculate \( a \) (denoted as \( a_x \) here) using \( x \) and \( x_h \).

\[
a_x = x_h^T (X^T X)^{-1} x_h
\]

(5.16)

Then calculate \( a \) (denoted as \( a_u \)) using \( u \) and \( u_h \).

\[
a_u = u_h^T (U^T U)^{-1} u_h
\]

(5.17)

\( U \) is the analysis matrix defined as:

\[
U = \begin{pmatrix}
    1 & u_1 & u_1^2 \\
    1 & u_2 & u_2^2 \\
    \vdots \\
    1 & u_n & u_n^2
\end{pmatrix}
\]

(5.18)
The relative error of $a_u$ (denoted as $R.E._a$) is defined as

$$R.E._a = \frac{a_u - a_x}{a_x}$$

(5.19)

If $a$ is invariant to the normalized transformation of $x$ (Eq. 5.15), then $R.E._a$ should be zero in all cases. $R.E._a$ has been calculated for the following cases using double precision in a FORTRAN program.

- Analysis model: simple linear, quadratic and cubic.
- Normalized stress level settings $u$: 23 different settings as shown in Table 5.1.
- $x_{\text{max}}$ (Eq. 5.15): values for $x_{\text{max}}$ considered are 1,000,000, 10,000, 100, and 10.
- $\bar{x}_{\text{min}}$: values considered are .999, .99, .9, .7, .5, .2, .05, 0, -.05, -.2, -.5, -.7, -.9, -.99, -.999.
- $u_h$: values of $u_h$ that are considered are 57 equally spaced values in the range $[-0.2,1.2]$, i.e. a 20% of extrapolation at both the higher end and the lower end of the stress range $[x_{\text{min}}, x_{\text{max}}]$ is considered. Note that the range $[x_{\text{min}}, x_{\text{max}}]$ of $x_h$ corresponds to the range $[0,1]$ of $u_h$.

Double precision is used throughout the calculations of $R.E._a$. The calculation procedure is demonstrated by the following example:

1. Choose a normalized stress level setting $u$ from Table 5.1. As an example, consider Item 5: $u_1 = 0$ and $u_2 = 1$.

2. $x_{\text{max}} = 100$ and $\bar{x}_{\text{min}}_{\text{max}} = 0.7$ is chosen for this example. Then the stress level setting $x$ is determined using information from (1) and (2), i.e. two specimens are tested at highest level 100 and one tested at lowest level, $0.7 \times 100 = 70$. 
23 cases have been studied. The sample size and normalized stress level settings for each case are listed. The number of replicates at each stress level is shown in parentheses. All normalized stress levels are in the range of \([0, 1]\). The stress levels in each case are equally spaced but not necessarily uniformly distributed. Stresses in cases 1, 2, 3, 4, 21, 22, and 23 are uniformly distributed. Those in cases 5~12 are linearly weighted at one side. Those in cases 13~20 are quadratically weighted at both ends.
(3) As an example, $u_h$ is chosen to be $-0.1$. From Eq. 5.15, $x_h = (-0.1) \times (100 - 70) + 70 = 67$.

(4) A quadratic model is chosen as an example. At this point, both the analysis matrix $X$ and $U$ are determined; both the value of $a_x(x_h)$ and $a_u(u_h)$ can be calculated.

(5) $R.E.a$ is computed. 57 different $u_h$ values (from $-0.2$ to $1.2$) are considered. Their corresponding $R.E.a$ is calculated using the conditions defined in (1), (2) and (4). The maximum value of 57 $R.E.a$ is denoted as $\max R.E.a$. Given the conditions defined from (2) and (4), there are 23 $\max R.E.a$ values corresponding to 23 normalized stress levels settings from Table 5.1. To conserve space, only the range of these 23 maximum $R.E.a$ values in Table 5.2 are listed for selected cases. Consider the first item in Table 5.2 for example. The column “$\max R.E.a$” shows the range of 23 $R.E.a$ values, for $x_{\max} = 1,000,000$ and $\bar{x}_{\max} = 0.9$ and a simple linear model. Double precision in FORTRAN has 15 significant digits. A $\max R.E.a$ value close to $1E-15$ suggests that the only difference between the value of $a_x$ and $a_u$ is numerical truncation rather than a difference in nature.

Note that cases having larger $\max R.E.a$ values also have relatively large $\bar{x}_{\min}$ values. A large $\bar{x}_{\min}$ implies that all $x$ are “close”, i.e., their normalized values are all close to 1. Adding or subtracting close values will likely lose some numerical precision. The calculation of $a$ involves a calculation of a polynomial in $x$. And if the analysis model is simple linear, then a polynomial of degree two will be involved in the calculation of $a$. If the analysis model is quadratic, then a polynomial of degree four will be involved. Consider $\bar{x}_{\min} = 0.9$ for example. The worst case in this regard is Case 23 of Table 5.2 which has 7 equally spaced stress level from 0.9 to 1:0.900, 0.917, 0.933, 0.950, 0.967, 0.983 and 1.000. Thus neighboring $x_i$ have a relative difference
Table 5.2: Comparison of $a$ values calculated by $x$ and by $u$.

The $R.E.a$ is the relative error between $a$ values calculated by $x$ and by $u$. The results suggest that all relative errors are due to numerical truncation. The conclusion, based on mathematical intuition, is that the $a$ value is invariant under transformation (Eq. 5.15) of $x$. 

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of about \( \frac{z_{max} - z_{min}}{6} = 1 - 0.9 = 0.017 \). This implies a loss of 1 to 2 significant digits in adding or subtracting the neighboring \( x \), e.g. \( 0.983 - 0.967 = 0.016 \), a loss of one significant digit from three to two. Because a polynomial of degree two is involved in calculating the \( a \) under a simple linear model, adding or subtracting \( x^2 \) and \( x \) terms doubles the truncation error. This results in a loss of two to four significant digits in this example.

Because double precision carries 15 significant digits, all numerical \( R.E.a \) values inherently have a minimum value (minimum error) of \( 1 \times 10^{-15} \). From Table 5.2, a \( R.E.a = 1 \times 10^{-13} \) to \( 1 \times 10^{-12} \) implies a loss of 2 to 3 significant digits. When a quadratic model is used, the loss of significant digits is doubled due to the \( x^4 \) terms. Table 5.2 shows a lost of 6 significant digits for all \( \frac{z_{min}}{z_{max}} = 0.9 \) cases. By the same reasoning, the table shows a lose of 9 significant digits when the cubic model is used.

Table 5.2 suggests, on the basis of mathematical intuition, that \( a_x \) and \( a_u \) are the same, i.e. \( a \) is invariant under the normalizing transformation (Eq. 5.15) of \( x \). This is important property regarding the calculation of \( a \), because all test stress ranges can now be transformed to the normalized stress range of \([0,1]\). What is important is the relative distribution of the \( x_i \), i.e. the \( u \). For example, five specimens tested at five equally spaced stress ranging from 20 MPa to 100 MPa will have the same value of \( a \) at 60 MPa as that at 45 MPa when five specimens are tested at five equally spaced stresses that range from 40 MPa to 50 MPa. A general relationship between stress level settings \( u \) and \( a \) can now be studied.
5.4 $a_f$: The Normalized $a$

In the previous section, it was shown that the $a$ value depends only on the relative distribution of the stress level setting of an experiment, i.e. the $u$. In this section a normalized $a$ value is introduced. As will be shown, the normalized $a$ depends only on the "shape" of relative distribution of stress level settings and not upon the sample size $n$. This is important to generalize the results of this study.

The normalized $a$, denoted as $a_f$, is defined as

$$a_f = na$$

(5.20)

where $n$ is the sample size of the experiment. As shown below, $a_f$ is invariant under replication of the experiment.

5.4.1 Invariance of $a_f$ Under Replication

Let the stress level settings and the analysis matrix of an experiment of sample size $n_1$ be $x_1$ and $X_1$ respectively. Another experiment, with its stress level settings denoted as $x$, composed of $r$ replicates of previous experiment $x_1$, will have a sample size of $r \times n$ and an analysis matrix $X$ as

$$X = \begin{pmatrix}
X_1 \\
X_2 \\
\vdots \\
X_r
\end{pmatrix}$$

(5.21)

where $X_1 = X_2 = \cdots = X_r$ is $r$ replicates of $X_1$. Note that each row in an analysis matrix that represents one specimen. $X_1$ for example has $n_1$ rows, and therefore $X$
has \( r \times n \) rows. Further development provides,

\[
X^TX = \begin{pmatrix} X_1^T & X_2^T & \cdots & X_r^T \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_r \end{pmatrix}
\]

\[
= X_1^TX_1 + X_2^TX_2 + \cdots + X_r^TX_r
\]

\[
= r(X_1^TX_1)
\]

And,

\[
(X^TX)^{-1} = \left[ r(X_1^TX_1) \right]^{-1}
\]

\[
= \frac{1}{r}(X_1^TX_1)^{-1}
\]

The \( a \) value corresponding to experiments \( x_1 \) and \( x \) are respectively,

\[
a_1(x_h) = x_h^T(X_1^TX_1)^{-1}x_h
\]

\[
a(x_h) = x_h^T(X^TX)^{-1}x_h
\]

Clearly the two differ by a ratio of \( r \). The normalized \( a \) of experiment \( x_1 \) is

\[
a_f(x_h) = n_1a_1(x_h)
\]

\[
= n_1x_h^T(X_1^TX_1)^{-1}x_h
\]

Using Eq. 5.26, the normalized \( a \) of experiment \( x \) is

\[
a_f(x_h) = n_1ra(x_h)
\]

\[
= n_1rx_h^T(X^TX)^{-1}x_h
\]

\[
= n_1rx_h^T\left[ \frac{1}{r}(X_1^TX_1)^{-1} \right]x_h
\]

\[
= n_1x_h^T(X_1^TX_1)^{-1}x_h
\]

\[
= a_f(x_h)
\]
Thus normalized $a$ of experiment $x$ is equal to normalized $a$ of experiment $x_1$. The normalized value $a_f(x_h)$, while maintaining the same shape as $a(x_h)$, is independent of the sample size and number of replications. Thus $a_f$ depends only on the "shape" of distribution of stress level settings.

5.4.2 $a_f(u_h)$ for Various Stress Level Settings

As mentioned in Sec. 5.3.1, $a(u_h)$ and therefore $a_f(u_h)$ are polynomial functions of degree $2d$, where $d$ is the degree of the polynomial model for the mean. In this section $a_f(u_h)$ for various stress level settings is examined. A total of 94 cases have been considered; their stress level distributions are listed in Table 5.3. Because $a$ has proved to be invariant under a linear transformation of stress, all 94 cases are expressed in normalized stress $u$. Except when specified otherwise, all stress levels in these cases are equally spaced between 0 and 1. The same notation for stress level settings as in Table 3.1 is used. Consider case 10 in Table 5.3 for example. The notation $[4 \ 5 \ 6]$ means four specimens are tested at the lowest stress level, five specimens tested at the middle stress level and six specimens tested at the highest stress level. Categories of stress level settings considered are: (1) equally weighted at each stress level: e.g. Cases 1, 9, 28, 44, 61, 62, 79, 80, 94; (2) weighted more to one side of the stress range: e.g. Case 3 having higher stress weighted as twice the lower stress, while Case 4 weights higher stresses as three times that of lower stress; (3) Concave: Weighted more at both ends, e.g. Cases 19 to 22 and 32 to 35 etc; (4) Convex: Weighted more at middle than the two ends, e.g. Case 23 to 27 and 55 to 58 etc.; (5) singly peaked: Only one stress level is weighted more while the rest are equally weighted, e.g. Cases 63 to 78, 81 to 85 and 87 to 92 etc. Some cases may apply to several categories, e.g. Case 2 is weighted at one side and singly peaked at
the same time. The severity of weighting can be quantified using the ratio of number of specimens at the heavily weighted stress level to that at least weighted stress level, e.g. Case 63 is singly peaked in a ratio of $5 \div 4 = 1.25$ while Case 66 is singly peaked in a ratio of 2.

Typical plots of $a_f(u_h)$ are shown in Fig. 5.3 to 5.7. It was observed that all 94 cases have an $a_f(u_h)$ plot very close to those in these figures. Polynomials of degree one, two and three as models for the mean are examined in each of the 94 cases.

For a polynomial model of degree one, $a_f(u_h)$ is a quadratic function (Fig. 5.3 and 5.4), and is symmetric when the stress levels are "symmetrically weighted" and asymmetric when weighted to one side of the stress range or the other. When the stress levels are asymmetric, the side weighted less has a larger $a_f$ value. This is reasonable because $a$ represents the uncertainty in estimating the mean curve. The side having fewer specimens should have a larger variance in estimating the mean. As shown in Fig. 5.4, Case 1 has the smallest $a$, and Case 28 has the largest. The explanation is that the simple linear model will provide a perfect fit, with no uncertainty, if there are only two data points. The distribution of stress level settings in Case 1 concentrates at two levels and best describes the two point situation. The next good analogy to the two point situation is Case 20 where more weight is placed on the two ends, while the worst analogy to the two point situation is Case 28 having the most uniformly distributed stress level settings. Note that $a$ equals to $\frac{2f}{n}$, and therefore $a$ values become smaller than those on the plot as the sample size increases. The observed difference of $a_f(u_h)$ between the various cases is not very large. For a polynomial of degree one, the largest $a_f(u_h)$ in the $u_h$ range of $[-0.2, 1.2]$ among the 94 cases is about 6.
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Table 5.3 (continued) Stress level settings of cases studied.
For a polynomial of degree two, the \( a_f \) value suddenly becomes very large at normalized stresses lower than 0 and higher than 1. This means that the uncertainty in estimating the mean curve becomes very large for stresses out of the test range. The conclusion is that extrapolation of the approximate Owen curve outside of the test stress range is likely to produce a significant loss of confidence when the quadratic model is used. However the \( a_f \) values within the test stress range did not exceed 10 in any of the 94 cases studied.

Fig. 5.7 shows \( a_f(u_h) \) for the polynomial model of degree three. The \( a_f(u_h) \) value "explodes" beyond \( u_h \) range of \([0, 1]\). Again extrapolation using the cubic model is not recommended. As in the quadratic model, the \( a_f \) value within \( u_h \) the range of \([0, 1]\) did not exceed 10 in any of the 94 cases.

For these limited cases studied, it is noted that the range of \( a_f(u_h) \) is never below 1 or exceed 10, within the test stress range, \( u_h \in [0, 1] \).

### 5.5 Approximate Owen’s \( K \) Using an Average \( a_f \)

#### 5.5.1 Relationship between \( a_f \) and \( K \)

The functional relationship between Owen’s \( K \) and \( a_f \) given \( p, \gamma, d \) and \( n \) is examined. Cases that are considered are: (1) degree of polynomial model for mean, \( d = 1, 2, 3 \). (2) probability level \( p = 0.85 \) to 0.99. (3) confidence level \( \gamma = 0.75 \) to 0.95. (4) sample sizes \( n = 5 \) to 50. (5) \( a_f = 1 \) to 10, using the experience of studying \( a_f \) values from the previous section. It is thought that these ranges cover most situations encountered in practical experiments.

Figures 5.8 to 5.10 demonstrate the functional relationship between Owen’s \( K \) and \( a_f \) for various \( d \) and \( n \) given \( p = 0.99 \) and \( \gamma = 0.95 \). Clearly the relationship is nearly
linear in the range considered. Similar functional relationships are observed for other cases that have been studied but not shown in this article.

What this linear relationship suggests is that instead of calculating the average value of $K$ as in Eq. 5.5, one may obtain a reasonable approximation by just calculating the average value of $a_f$ and using this average $a_f$ to calculate $K$. To demonstrate this mathematically, first assume that the linear relation between $K$ and $a_f$ is

$$K(x) = c_0 + c_1 a_f(x) \quad (5.36)$$

where $c_0$ and $c_1$ are parameters. The average of $K$ will then be

$$\bar{K} = \frac{1}{x_b - x_a} \int_{x_a}^{x_b} K(x) \, dx \quad (5.37)$$

$$= \frac{1}{x_b - x_a} \int_{x_a}^{x_b} c_0 + c_1 a_f(x) \, dx \quad (5.38)$$

$$= c_0 + c_1 \frac{1}{x_b - x_a} \int_{x_a}^{x_b} a_f(x) \, dx \quad (5.39)$$

$$= c_0 + c_1 \bar{a_f} \quad (5.40)$$

Equation 5.40 simplifies the calculations of $K$. We need only compute one $K$ value, $\bar{K} = K(p, \gamma, \bar{a_f}, f)$. Because the functional relationship between $K$ and $a_f$ is not "exactly" linear, the average $K$ derived in this way is denoted as $K_{\bar{a_f}}$ in order to distinguish it from $\bar{K}$.

### 5.5.2 Error When $\bar{a_f}$ is Used

When $\bar{a_f}$ is used in calculating the approximate Owen's $K$, denoted as $K_{\bar{a_f}}$, an error is introduced from this approximation. This section will examine the error in the $a_f$ value when $\bar{a_f}$ is used, and the errors in $K$ and $\gamma$ when $K_{\bar{a_f}}$ is used. All different stress level settings listed in Table 5.3 are examined for $p = 0.99$ and $\gamma = 0.95$. Polynomials of degree one, two and three as models for the mean are considered. In
each case, four normalized stress ranges are considered: $u_h = [-0.2 : 1.2], [-0.1 : 1.1], [0 : 1]$ and $[0.1 : 0.9]$. Average values of $a_f(u_h)$, i.e. $\bar{a}_f$, for each of the four ranges are calculated. Consider the polynomial of degree one and Case 4 in Table 5.3 for example. $a_f(u_h)$ is given in Fig. 5.3. Its average value $\bar{a}_f$ in the $u_h$ range of $[0 : 1]$ is 1.7822. When $\bar{a}_f$ is used across the range $[0 : 1]$, the most conservative point is at $u_h = 0.75$ where $a_f(u_h) = 1$, but $\bar{a}_f = 1.7822$ is used. Larger $a_f$ values produce larger $K$ resulting in a more conservative design points. The least conservative point in this range is at $u_h = 0$ where $a_f(u_h) = 4$, but $\bar{a}_f = 1.7822$ is used. A smaller $a_f$ implies smaller $K$ and a design point having a confidence level $\gamma < 0.95$. For an $u_h = [-0.2 : 1.2]$, $\bar{a}_f = 2.2107$. The most conservative point is still at $u_h = 0.75$. But the least conservative point is $u_h = -0.2$.

The error in $a_f$ when $\bar{a}_f$ is used is defined as

$$\Delta a_f = \bar{a}_f - a_f(u_h)$$

(5.41)

$\Delta a_f$ at the least conservative point (one of the end points of the test stress range) for each case and range are plotted in Fig. 5.11 to 5.13 for polynomials of degree one, two and three respectively. $\Delta a_f$ values at the most conservative point (somewhere in the middle of the stress range) are plotted in Fig. 5.17 to 5.19. For the least conservative point within the range $[0 : 1]$, $|\Delta a_f|$ was found to be less than three for polynomials of degree one, less than seven for polynomials of degree two or three. $|\Delta a_f|$ of the most conservative point is always smaller than that of least conservative point. That means the conservatism is not as severe as non-conservatism. Therefore the conservative point should not be the main concern. The following discussion will focus on examination of error at the least conservative point only.
When $\bar{a}_f$ is used instead of $a_f(u_h)$, the tolerance factor will be $K_{\bar{a}_f}$ instead of $K(u_h)$. The error in $K$ is defined as

$$\Delta K = K_{\bar{a}_f} - K(u_h)$$

(5.42)

Values of $\Delta K$ are plotted in Fig. 5.20 to 5.22 for the various degrees of polynomial. For $u_h$ in the range of $[0 : 1]$, all $|\Delta K|$ are smaller than 0.6.

When $K_{\bar{a}_f}$ is used instead of $K(u_h)$ at the least conservative point, the design point has a smaller $\gamma$. The error in $\gamma$ is defined in the example as

$$\Delta \gamma = \gamma - 0.95$$

(5.43)

where $\gamma$ is the confidence level of the design curve at the least conservative point. Values of $|\Delta \gamma|$ are plotted in Fig. 5.23 to 5.25. For a polynomial of degree one, $|\Delta \gamma|$ are small even for $u_h$ range of $[-0.2 : 1.2]$. For polynomials of degree two and three, it is clear that extrapolation should be avoided. It seems to be a contradiction that, for an $u_h$ range of $[0.1 : 0.9]$, the $|\Delta \gamma|$ from a polynomial of degree three has a smaller value (not more than 0.01) than that of degree two (not more than 0.02) and degree one. For an $u_h$ range of $[0 : 1]$ and a polynomial of degree two or three, some cases (i.e. Case 45, 63, 67, 71, 75 and 87 to 92) have a $|\Delta \gamma|$ greater than 0.03. Roughly speaking, larger sample sizes will result in larger $|\Delta \gamma|$. And larger numbers of different stress levels will also produce a larger $|\Delta \gamma|$ in polynomials of degree two and three.

### 5.5.3 Estimate of Error in $\gamma$ for the General Case

Ninety four different stress level settings have been examined in this study. While this set does not exhaust the list of possible stress level settings, it does cover a large range of possibilities. Given a stress level setting that differs from the list, an
interpolation method may be employed. For example, if a stress level setting [3 4] is applied, one may consider it as weighted at one side by a ratio of $4 \div 3 = 1.33$ which is between Case 1 (ratio=1) and Case 2 (ratio=1.5). Therefore $\Delta \gamma$ for the new case that is somewhere between that of Case 1 and Case 2 can be estimated by interpolation.

If interpolation of the cases of Table 5.3 is not possible, an easy method (avoiding calculating Owen's $K$) to estimate the error is desired. To achieve this, the functional relationship between $K$, $a_f$ and $\gamma$ is examined. Fig. 5.26 shows this relationship for a polynomial of degree one and $n = 10$. Polynomials of degree one, two and three along with $n = 5$ to $n = 50$ are examined, and they all show a similar relationship as that of Fig. 5.26. Because a linear relationship is easier to work with, a function of $\Delta \gamma$ is used on the abscissa. As there is only concern about the least conservative point, we need only the linear relation for $\gamma < 0.95$ side. $|\Delta \gamma|^{0.7}$ is found to be linear with $K$ for $a_f = 1$ to $a_f = 10$. We conclude that $K$ is nearly a linear function of $a_f$ and $|\Delta \gamma|^{0.7}$ in the region where $\Delta \gamma < 0$. This means that $K$ can be approximated using a Taylor expansion of order one. Denote $|\Delta \gamma|^{0.7}$ as $|\Delta \gamma|^c$ and we have

$$K \approx K_o + K_{a_f} \Delta a_f + K_{\gamma |\Delta \gamma|^c}$$ \hspace{1cm} (5.44)

where $K_o$ is a constant, and $K_{a_f}$ and $K_{\gamma |\Delta \gamma|^c}$ are partial derivatives of $K$ with $a_f$ and $|\Delta \gamma|^c$ respectively. When $\bar{a}_f$ is used instead of $a_f(u_h)$, $K$ is approximated as,

$$K_{\bar{a}_f} \approx K_o + K_{a_f} \Delta a_f$$ \hspace{1cm} (5.45)

where $\Delta a_f = \bar{a}_f - a_f(u_h)$. On the other hand if we have used the correct $a_f(u_h)$ but have specified a different $\gamma = 0.95 + \Delta \gamma$, the $K$ value we will use is

$$K \approx K_o + K_{\gamma |\Delta \gamma|^c}$$ \hspace{1cm} (5.46)
Equating the last two, an expression for $\Delta \gamma$ can be derived

$$K_0 + K_a \Delta a_f = K_0 + K_{a_{\gamma}} |\Delta \gamma|^c$$  \hspace{1cm} (5.47)

$$\Delta \gamma = - \left| \frac{K_{a_f}}{K_{\gamma}} \Delta a_f \right|^{\frac{1}{c}}$$  \hspace{1cm} (5.48)

This form gives an estimate of the error in $\gamma$ when an error in $a_f$ is introduced by using $\bar{a}_f$ instead of $a_f(u_h)$.

Tables 5.4 to 5.6 list $\frac{K_{a_f}}{K_{a_{\gamma}}}$ values for various $n, a_f$ and degree of the polynomial for the mean. The use of these tables is demonstrated by the following example.

Consider Case 91 in Table 5.3 for example. The $a_f$ at the least conservative point for $u_h$ range of $[-0.1: 1.1]$ is 5.0037 and $\bar{a}_f = 2.3098$; therefore $\Delta a_f = -2.6939$.

Case 91 has a sample size of 12. In Table 5.4 for $a_f = 5.0037$ and $n = 12$, we get $\frac{K_{a_f}}{K_{a_{\gamma}}} = 0.026806$. The error in $\gamma$ at the least conservative point is calculated as

$$\Delta \gamma = - \left| \frac{K_{a_f}}{K_{a_{\gamma}}} \Delta a_f \right|^{\frac{1}{c}} = -0.026806 \times -2.6939^{\frac{1}{c}} = -0.0234$$  \hspace{1cm} (5.49)

This value compares well with the $\Delta \gamma = -0.022$ of Fig. 5.23. The consequence of this number is that the confidence level $\gamma$ of the lower $1 - p = 0.01$ point is degraded by the approximation from $\gamma = 0.95$ to $\gamma = 0.95 - \Delta \gamma = 0.95 - 0.0234 = 0.9236$.

5.6 An Improved Algorithm for the Calculation of $K$

Calculation of Owen’s $K$ for the regression model requires calculation of the non-central $t$ distribution which is not available in most statistical computer packages. Calculation of the non-central $t$ distribution is slow as it involves complex procedures such as repeated integration by parts. Attempts have been made to find the approximation of $K$ using much easier algebra. One method called “approximation
### Table 5.4: $K_{sl}$ for a polynomial model of degree one given $p = 0.99$ and $\gamma = 0.95.$

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### Table 5.5: $K_{sl}$ for a polynomial model of degree two given $p = 0.99$ and $\gamma = 0.95.$

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Table 5.6: $K_{a_f}$ for a polynomial model of degree three given $p = 0.99$ and $\gamma = 0.95$.

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<td>0.02084</td>
<td>0.01964</td>
<td>0.01852</td>
<td>0.01695</td>
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<td>0.03210</td>
<td>0.02951</td>
<td>0.02659</td>
<td>0.02457</td>
<td>0.02267</td>
<td>0.02142</td>
<td>0.02013</td>
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<td>0.03159</td>
<td>0.02900</td>
<td>0.02639</td>
<td>0.02497</td>
<td>0.02338</td>
<td>0.02196</td>
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<td>0.03857</td>
<td>0.03444</td>
<td>0.03121</td>
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<td>0.02641</td>
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<td>0.03707</td>
<td>0.03341</td>
<td>0.03027</td>
<td>0.02813</td>
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<td>0.02928</td>
<td>0.02713</td>
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<td>0.04279</td>
<td>0.03823</td>
<td>0.03463</td>
<td>0.03171</td>
<td>0.02928</td>
<td>0.02723</td>
<td>0.02438</td>
<td>0.02211</td>
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</table>

D”, denoted as $K_D$ here, in Owen’s paper [61] has been improved in this study. To examine the accuracy of approximation D, the ratio $K/K_D$ is plotted as in Fig. 5.27 where $p = 0.99$ and $\gamma = 0.95$. As seen in the figure, the error is about 4% for sample sizes smaller than 10. Curves having different $a$ in Fig. 5.27 have a scatter of about 1% vertically. Fig. 5.28 is basically the same parameter plot as Fig. 5.27 but uses $a_f$ instead of $a$. The scatter between curves of different $a_f$ is smaller than that of various $a$. This suggests that if a function could be found which fits the curve in Fig. 5.28, then this function can be used to correct approximation D and improve its accuracy. For $\gamma = 0.95$ the function is found empirically to be

$$K/K_D \approx 0.9968238 + \frac{0.1596499}{f_{0.6}} - 2.635553 \exp(-f) \quad (5.50)$$

Multiply the $K_D$ value from approximation D by the expression above to get an improved approximation of $K$ value denoted as $K_a$. The accuracy is improved greatly as shown in Fig. 5.29. This correction is implemented in a FORTRAN function code appdk and is listed in Appendix C. It can be used with $\gamma = 0.95, 0.90, 0.85, 0.80$ and $p = 0.99$ down to $p = 0.85$. The argument $K_p$ and $K_r$ of this FORTRAN function are
$p$ quantile and $\gamma$ quantile of standard normal distribution. For $p = 0.99$, $K_p = 2.32635$. For $\gamma = 0.95$, $K_r = 1.64485$.

5.7 An Example of Constructing the Approximate Owen's Curve

A short example is presented here to demonstrate the construction of the approximate Owen's curve with $p = 0.99$ and $\gamma = 0.95$. Assume the model as selected by nature is the same as defined in Eq. 5.3. An analysis model of quadratic mean and homoscedastic scatter was chosen. The plan is to test two specimens each at five equally spaced stress levels: 70, 100, 130, 160 and 190 MPa for a total sample size, $n = 10$. There will be two replicates at each stress level. A random sample of data is given as Table 5.7. The following illustrates the steps in the analysis to determine approximate Owen $K$ in the test range. Note that $K$ can be determined prior to the collection of the data.

<table>
<thead>
<tr>
<th>no.</th>
<th>$y = \log_{10} N$</th>
<th>$x = S$ (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6.326469</td>
<td>70.0</td>
</tr>
<tr>
<td>2</td>
<td>6.126969</td>
<td>70.0</td>
</tr>
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<td>3</td>
<td>5.775955</td>
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<td>9</td>
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<td>190.0</td>
</tr>
<tr>
<td>10</td>
<td>3.873905</td>
<td>190.0</td>
</tr>
</tbody>
</table>

Table 5.7: Data of sample size $n = 10$
1. Use least squares analysis to estimate the parameters of the mean

\[ \hat{\mu}(x) = a_0 + a_1 x + a_2 x^2 \]  

(5.51)

where \( x = S \) (null transformation), and \( \hat{\mu}(x) \) is the estimate of the mean value of \( \log N \) given \( S \). The least squares estimates are \( \hat{a}_0 = 9.1722, \hat{a}_1 = -0.04837, \hat{a}_2 = 1.05651 \times 10^{-4} \) and \( \hat{\sigma} = 0.1834 \). The mean curve is shown in Fig. 5.30.

2. Determine \( \bar{a}_f \). Due to the invariance of \( a_f \) under replication, this test plan is identical to Case 44 (five equally spaced, uniformly weighted stress levels) in Table 5.3. From Fig. 5.15, \( \bar{a}_f = 2.344 \) for the normalized stress range \([0:1]\). At this point, one may either use Fig. 5.9 to determine \( K = 4.5 \) or, to provide a little more accuracy, use the following calculation procedures.

3. Determine \( \bar{a} \). Here \( n = 10 \) so that

\[ \bar{a} = \frac{\bar{a}_f}{n} = \frac{2.344}{10} = 0.2344 \]  

(5.52)

4. Determine the number of degrees of freedom \( f = n - d - 1 = 10 - 2 - 1 = 7 \), where \( d \) is the degree of the polynomial model for the mean.

5. Determination of \( K_{\bar{a}_f} \). This can be done by using the FORTRAN code appdk in Appendix C. Or one could follow the same procedure in appdk using a calculator: (1) input arguments \( K_p = 2.32635, K_r = 1.64485 \) which are the \( p = 0.99, \gamma = 0.95 \) quantiles of standard normal distribution respectively; (2) input \( a = \bar{a} = 0.2344 \) and \( f = 7 \); (3) because \( f = 7 > 2 \), follow segment [1.2] in appdk to find coefficients \( c_{11}, c_{20}, \) and \( c_{22} \); (4) follow the approximate D method in segment [1.3] of appdk to find \( K_D \); (5) because \( \gamma = 0.95 \) is desired, follow segment [2.1] to correct the \( K_D \) by a ratio. The final result, \( K_{\bar{a}_f} = 4.49 \).
6. Then the approximate Owen curve is shown on Fig. 5.30, and is expressed as

\[ Y_D(x) = \hat{\mu}(x) - 4.49\hat{\sigma} \]  

(5.53)

5.8 Proposed Design Curve for the Pure X Error Source Models

5.8.1 General Concept

As demonstrated in Chapter 4, the bilinear model for the mean, and the pure X error source model for scatter perform well for heteroscedastic fatigue data. The implication is that the fatigue model is nonlinear. But Owen’s theory does not apply to nonlinear models. Yet it may be possible that the approximate Owen curve can be used for the nonlinear model with little loss of accuracy.

The pure X error source model is based on an assumption that the scatter of local stress is homoscedastic. Denote the standard deviation of local stress as \( \sigma_{xL} \), and the \( 1 - p \) quantile of (transformed) life as \( y_{1-p}(x) \) or its inverse function \( y^{-1}_{1-p}(y) \). Then

\[ y^{-1}_{1-p}(y) = y^{-1}(y) - K_p \sigma_{xL} \]  

(5.54)

where \( y^{-1}(y) \) is the inverse function of the function relationship \( y(x_L) \), and \( -K_p \) is the \( 1 - p \) quantile of a standard normal variate. Compare Eq. 5.54 with the \( 1 - p \) quantile of life for the linear model having homoscedastic normally distributed life

\[ y_{1-p}(x) = \mu_y(x) - K_p \sigma_y \]  

(5.55)

The two have a similar form. In estimating \( y_{1-p}(x) \) for the linear model, \( \mu_y(x) \) is replaced by \( \hat{\mu}(x) \) and \( \sigma_y \) replaced by \( \hat{\sigma} \). Then \( K_p \) is replaced by Owen’s tolerance factor \( K(x) \) to accommodate the uncertainty from the estimates \( \hat{\mu}(x) \) and \( \hat{\sigma} \). The
derivation of Owen's tolerance factor (See Appendix A) is based on the assumption that $\hat{\mu}(x)$ is normally distributed, $\hat{\sigma}$ is related to the chi-square distribution, and $\hat{\mu}(x)$, $\hat{\sigma}$ are independent. If the estimates of $y^{-1}(y)$ and $\sigma_{xL}$ in the pure $X$ error source model possess similar properties, then the estimate of $y_{1-p}^{-1}(y)$ may be constructed as

$$y_{1-p}^{-1}(y) = y^{-1}(y) - K\hat{\sigma}_{xL}$$

(5.56)

While Owen's theory has not been developed for the nonlinear case, intuition of the author suggests that the Owen tolerance factor may be used as a reasonable approximation of the tolerance factor for nonlinear cases. To verify this approximation, simulation is required. Unfortunately the numerical process as described in Chapter 4 is slow. Operator intervention is needed for the initial guess of parameters for the optimization routine to ensure that a global maximum is reached. Verification using simulation may be impractical at this stage.

In summary, it is suggested herein that one may adopt the approximate Owen factor as described above as a first approximation of a tolerance factor for the case of a nonlinear model for the mean and the pure $X$ error source model for scatter.

5.8.2 Example

The titanium data Ti64-300 in Chapter 4 is used as an example of constructing an approximate tolerance curve when the pure $X$ error source model is used.

1. Consider the leveling-off bilinear as model for the mean and the pure $X$ error source as model for the scatter. The leveling-off bilinear model is

$$x = y^{-1}(y) = a_1 + a_2(y - \sqrt{(y - a_3)^2 + a_4})$$

(5.57)

2. Sample size is $n = 48$ including 36 failed specimens and 12 censored data.
3. Determine $\bar{a}_f$.

(a) The shape of distribution of stress level settings is near uniform.

(b) Uniform stress level settings in the 94 cases have $\bar{a}_f = 1.8$ for the simple linear model and $\bar{a}_f = 2.5$ for the quadratic model.

(c) The bilinear model consists of two line segments and a curvature. It is assumed that $\bar{a}_f = 2.5$ for the quadratic model should provide a reasonable estimate of $\bar{a}_f$ for the bilinear model.

4. Determination of $\bar{a}$.

(a) Determine an equivalent sample size $n$ when censored data exist. The 12 censored data are spread evenly from low quantile to high quantile. It is assumed (using intuition only) that the information contained in the 12 evenly distributed censored data is equivalent to $12/2 = 6$ failure data. The equivalent sample size $n = 36 + 6 = 42$ is used to compensate the loss of information from censored data.

(b) $\bar{a} = \bar{a}_f/n = 2.5/42 = 0.06$.

5. Determination of $f$. $f = n$—number of parameters in the model for the mean $= 42 - 4 = 38$. Note that $n = 42$ is used as same reason above.

6. Determination of $K_{\bar{a}f}$ using FORTRAN function appdk.

(a) input $K_p = 2.326$, $K_\gamma = 1.645$, $a = 0.06$ and $f = 38$.

(b) $K_{\bar{a}f} = 2.993 \times 1.0148 = 3.04$.

7. Maximum likelihood estimates of the parameters are $a_1 = -1.484$, $a_2 = -0.1671$, $a_3 = 4.719$, $a_4 = 1.885$, $\sigma_{\varepsilon_i} = 0.02866$. 
8. The proposed design curve is

\[ x_D = \hat{y}^{-1}(y) - K_{\hat{a}}\hat{\sigma}_x \]

\[ = -1.484 - 0.1671(y - \sqrt{(y - 4.719)^2 + 1.885}) - 3.04 \times 0.02866 \]

\[ = -1.57 - 0.1671(y - \sqrt{(y - 4.719)^2 + 1.885}) \quad (5.58) \]

The curve is shown in Fig. 5.31.
Figure 5.1: Comparison of design curves (i).

Figure 5.2: Comparison of design curves (ii).
Figure 5.3: $a_f(u_h)$ plot with the simple linear model for cases 1,3 and 4.

Figure 5.4: $a_f(u_h)$ plot with the simple linear model for cases 1,9,20 and 28.
Figure 5.5: $a_f(u_h)$ plot with the quadratic model for cases 9, 20 and 28.

Figure 5.6: $a_f(u_h)$ plot with the quadratic model for cases 28, 45 and 46.
Figure 5.7: $a_f(u_h)$ plot with the cubic model for cases 28, 35, 45 and 62.

Figure 5.8: Owen's $K$ as function of $a_f$; simple linear model; $p = 0.99$, $\gamma = 0.95$. 
under polynomial model of degree two

![Graph showing Owen's K as function of \(a_f\); quadratic model; \(p = 0.99, \gamma = 0.95\).](image)

Figure 5.9: Owen's K as function of \(a_f\); quadratic model; \(p = 0.99, \gamma = 0.95\).

under polynomial model of degree three

![Graph showing Owen's K as function of \(a_f\); cubic model; \(p = 0.99, \gamma = 0.95\).](image)

Figure 5.10: Owen's K as function of \(a_f\); cubic model; \(p = 0.99, \gamma = 0.95\).
Figure 5.11: $\Delta a_f$ at the least conservative point for various cases with the simple linear model.
Figure 5.12: $\Delta a_f$ at the least conservative point for various cases with the quadratic model.
Figure 5.13: $\Delta a_f$ at the least conservative point for various cases with the cubic model.
Figure 5.14: $\bar{a}_f$ at the least conservative point for various cases with the simple linear model.
Figure 5.15: $\bar{a}_f$ at the least conservative point for various cases with the quadratic model.
Figure 5.16: $\bar{a}_f$ at the least conservative point for various cases with the cubic model.
Figure 5.17: $\Delta a_f$ at the most conservative point for various cases with the simple linear model.
Figure 5.18: $\Delta a_f$ at the most conservative point for various cases with the quadratic model.
Figure 5.19: $\Delta a_f$ at the most conservative point for various cases with the cubic model.
Figure 5.20: $\Delta K$ at the least conservative point for various cases with the simple linear model.
Figure 5.21: $\Delta K$ at the least conservative point for various cases with the quadratic model.
polynomial of degree three

normalized stress range [0.1:0.9] —
normalized stress range [0.0:1.0] —
normalized stress range [-0.1:1.1] —
normalized stress range [-0.2:1.2] —

Figure 5.22: $\Delta K$ at the least conservative point for various cases with the cubic model.
Figure 5.23: $\Delta \gamma$ at the least conservative point for various cases with the simple linear model.
Figure 5.24: $\Delta \gamma$ at the least conservative point for various cases with the quadratic model.
Figure 5.25: $\Delta \gamma$ at the least conservative point for various cases with the cubic model.
Figure 5.26: Functional relationship of Owen’s $K$, $\Delta \gamma$ and $a_f$ for the simple linear model having sample size $n = 10$.

Figure 5.27: Ratio of Owen’s $K$ and $K_D$ by the “approximation D” method for various $a$ and degrees of freedom $f$. 
Figure 5.28: Ratio of Owen’s $K$ and $K_D$ by the “approximation D” method for various $a_f$ and degrees of freedom $f$.

Figure 5.29: Ratio of Owen’s $K$ and $K_a$ by the improved “approximation D” method for various $a_f$ and degrees of freedom $f$. 
Figure 5.30: Example of the approximate Owen's curve.

Figure 5.31: Example of the approximate Owen's curve for the pure X error source model.
CHAPTER 6
Summary and Recommendations

A strategy for fatigue data analysis which addressed several research questions was pursued. Given a random sample of fatigue data: (1) What is the appropriate analysis model for the mean and scatter as well as the distribution of (transformed) life \( Y \) given (transformed) stress \( X \)? (2) What is the appropriate design curve to be used when employing a traditional factor of safety approach? (3) How will the test plan, i.e. specification of stress level settings, affect the quality of design curve? (4) How can the uncertainty of estimated parameters be incorporated into a reliability model in order to perform reliability based design? These complex and closely related questions are addressed in this dissertation. This chapter provides: (1) a summary of the results of this study, (2) some recommendations of strategies for fatigue data analysis based the results of study, and (3) gaps in this study and a recommendation of future research topics.
6.1 Summary of Results

6.1.1 Models for Scatter of Life Given Stress/Strain of Fatigue Data

The question regarding the appropriate model for the mean, the scatter and, the distribution of (transformed) life \( Y \) given (transformed) stress \( X \) is addressed in Chapters 3 and 4. In Chapter 3, the concept of a constant stress error is introduced. Various models for constant stress error are discussed theoretically: the inverse model, the back-projection model, the pure \( X \) error source model and the constant \( \sigma_x \), and the non-inverse model. These models are compared using simulation for the simple linear cases. The back-projection model provides an insight on how censored data should be treated. The pure \( X \) error source model corrects shortcomings in the inverse model and the back-projection model by adding Jacobian terms in the log likelihood value. The pure \( X \) error source model offers an explanation on why some fatigue data sets tend to be homoscedastic and normally distributed in the upper (or low cycle) region of the S-N curve and heteroscedastic, skewed to the right, in the high cycle region. It was also concluded that the existence of an \( X \) error source is more dominant than the existence of a \( Y \) error source regarding their influence on the resulting distribution of life given stress.

Simulation results of various simple linear models on complete failure data show that the parameters estimated using the pure \( X \) error source model are unbiased while those using the inverse model are biased. Furthermore theoretical development shows that for a simple linear model for the mean, a normally distributed \( X \) (stress/strain) error source will result in a normally distributed \( Y \) error source. A basic model may
be used, and unbiased parameter estimates can be obtained using the least squares method.

6.1.2 Comparison of Models for Fatigue Life Data

In Chapter 4, various combinations of models for the mean and scatter are examined using five representative sets of real data having relatively large sample sizes. Three methods are used to compare the goodness fit of various combinations of models: (1) the mean curve and \( \pm 3\sigma \) envelope plot; (2) the standardized residual plot; and (3) the likelihood ratio test. Methods (1) and (2) are based on subjective judgment using visual examination and method (3) is based on a quantitative measure of the log likelihood values. Conclusions from these tests are:

1. The mean curve and \( \pm 3\sigma \) envelope plots show that for homoscedastic data, all of the basic models for scatter seem to work well. However for heteroscedastic data, only the inverse model, the pure X error source model and the Box-Cox model seem to provide reasonable fits. The Box-Cox model relies on mathematical flexibility in fitting the heteroscedasticity, while the inverse model and the pure X error source model rely on recognition of physics of the fatigue process. The pure X error source seems to be consistently more conservative than the inverse model in the region close to fatigue limit, and always a little bit less conservative than inverse model in low cycle region.

2. Regarding models for the mean, the bilinear model and the leveling-off bilinear seem to perform better than the general strain-life equation model. The MIL-HDBK-5 and Box-Cox models did not seem to perform as well.
3. The standardized residual plot amplifies the fundamental error of the inverse model. Standardized residuals of the inverse model are heteroscedastic, and the spread of censored residuals appears to be unreasonable. Moreover the inverse model does not seem to be robust relative to the design of the experiment, i.e. the stress level settings. The "explode" type of model for the scatter also has an unreasonable spread of censored residuals.

4. Considering the survivors of the likelihood ratio test on the data sets considered, the best candidates for the mean are the bilinear model, the leveling-off bilinear model and the general strain-life equation. For modeling heteroscedastic scatter, the clear winner is the pure X error source model followed by the quadratic model for scatter.

5. In summary, considering combinations of all three tests on modeling heteroscedastic fatigue data, the winners are the bilinear, leveling-off bilinear and general strain-life equation as models for the mean and the pure X error source model for scatter.

6.1.3 Approximate Owen's Design Curve

Construction of a design curve is considered in Chapter 5. Because of widespread acceptance in the design community of the tolerance limit concept, the Owen curve, based on a tolerance limit, is considered to be the approach of choice. Extensive examination of Owen's curve for typical test plans suggests that an average Owen's tolerance factor $K$ may be used without significant loss in the confidence level $\gamma$. 
6.2 Recommendations

The following recommendations for the models are based on limited data sets. These data were arbitrarily chosen but they are considered to be representative as they have large sample sizes and have the general appearance of most published fatigue data.

6.2.1 Recommendation of Models for Mean

The bilinear model for the mean seemed generally to provide the best fit relative to the other models considered. In some cases however, the leveling-off bilinear model fit as well as the bilinear. These two models are purely empirical. Also the empirical general strain-life equation model fits some data quite well. For large sample sizes, it would be desirable if possible to fit all three models and compare their performance using tests as described in Chapter 4.

6.2.2 Recommendation of a Model for Scatter

The work presented in Chapter 4 strongly suggests that for scatter the pure X error source model seems to fit heteroscedastic data better than the other models considered. Although the inverse model and the pure X error source model have a similar mean and $\pm 3\sigma$ envelope for most data sets, they differ significantly when there are many censored data. Even in the cases where both models have a similar envelope, minor differences in the fatigue limit region produces large differences in the design life due to the near zero slope in that region. General speaking, the pure X error source model is more conservative than the inverse model because it seems to be consistently more conservative in fatigue limit region.
For homoscedastic data, clearly the best model for scatter is a constant standard deviation of N. As mentioned in Chapter 3, it is the engineer’s judgment to decide if variance of local stress is large enough to consider using the pure X error source as model for scatter. When the variance of local stress is small, scatter can be approximated by the pure Y error source model, and the homoscedastic model for scatter will fit the data. When it is uncertain whether the variance of local stress is large or small, one needs to make a practical judgment. The engineer should choose the pure X error source model for heteroscedastic data and the homoscedastic model for constant variance life data as a model for scatter.

When no censored data exist, the least squares method can be applied on the homoscedastic model for scatter as well as the inverse model. All other models for scatter must use the maximum likelihood method.

6.2.3 Recommendation of a Design Curve

The following recommendations for a design curve apply only to the linear model. And there should be no censored data.

Because it accounts for the uncertainty of the estimators, Owen’s curve is mathematically correct. Owen’s curve may be replaced by an approximate Owen’s curve, a design curve derived using an average value of Owen’s tolerance factor $K$ in a specified stress region. The loss in confidence level $\gamma$ in this approximation was shown to be “small” for typical fatigue test plans. Loss of confidence $\gamma$ for a specific test plan can be evaluated using Tables 5.4 to 5.6 before the test is performed because the calculation depends only on $a_f(x_h)$ and the mean $\bar{a}_f$, which is functions of the stress level settings only. Such an exercise is useful in checking how well a test plan
supports the use of the approximate Owen's curve. In fact, an engineer may use these tables to compare several test plans.

6.3 Limitations of the Current Research and Recommendations for Future Studies

Some issues that were left unresolved by this study, but which could be addressed in a continuation of the theme of this research are:

1. Comparison of test plans for non-linear models. Chapter 5 examines test plans under a linear model using $\Delta \gamma$ as measure of their quality. But what would be optimal test plans for those nonlinear models of Chapter 4? Because a theoretical approach may be very difficult for nonlinear models, simulation may be necessary. However, experience has shown that current numerical methods in dealing with global optimization are slow. Except for a very simple model, a trial and error is frequently required to fit just one set of data. This is not practical for thousands of simulations.

2. The tolerance factor for the nonlinear model. Owen's $K$ was developed for the linear model only. Can the approximate Owen's $K$ be employed for a nonlinear model as well? What is the loss in confidence $\gamma$ when the approximate Owen's $K$ is used for a nonlinear model? Again it is likely that simulation must be used. And again, the efficiency of the numerical method needs to be improved in order to study this issue.

3. The loss in $\gamma$ when the approximate Owen's $K$ is used and there are censored data. The design curve as presented in Chapter 5 considers only complete
data. For censored data, there is no theoretical tool to estimate the loss in $\gamma$. Simulation must be used. Again, numerical methods need to be improved in order to study the nonlinear model.

4. A reliability model where the uncertainty of estimates is incorporated. This topic is considered briefly in the next section.

5. Measures of quality of the design curve other than $\Delta \gamma$. For example, a measure based on cost analysis may be useful. The benefit of a measure based on cost is that many different and important factors can be lumped into one quantity and compared. Therefore the influence from many factors can be assessed. The downside is that some factors are controversial and difficult to quantify, e.g. the value of a human life.

Recommended topics for future consideration are:

1. In Chapter 5, the approximate of Owen's $K$ was defined as the average of $K$ in a specified region. Consider a polynomial model of degree one for example. Visual examination of $a_f$ suggests that, for various stress level settings $a_f$ may be approximated by one quadratic curve. And $a_f$ for a polynomial model of degree two may be approximated by a single polynomial curve of degree four, etc. In fact we may only need one $a_f$ curve as an approximation of $a_f$ for all various stress level settings. A study of the derivation of such a curve and the error in $\gamma$ when this curve is used as a replacement of $a_f$ would be useful.

2. The mixed error source model. When both the $X$ and the $Y$ error sources coexist, what is the impact on the distribution of resulting life given stress/strain? To analyze this model, e.g. to calculate the likelihood value for this model, a
line integration in the space of the two random variables (X error source variate and Y error source variate) is required. This will further degrade efficiency of the numerical optimization procedure.

3. Improvement of the numerical method for optimization. Many issues listed above hinge on the availability of an efficient simulation approach in analyzing the nonlinear model. The major problem is the inefficiency of the optimization routine used for maximum likelihood estimators. A strategy of finding a good initial guess of the parameters may be the key to any improvements.

6.4 Future Studies: Construction of a Reliability Model Based on Fatigue Test Data

Development of a reliability model was not a major goal of this study. However some initial considerations on this issue are presented in the following.

Statistical uncertainty of estimates of model parameters from fatigue data analysis may be incorporated into the reliability model by an overall probability consideration or through the confidence level. A discussion of these approaches is presented below.

6.4.1 Reliability Model that Incorporates the Uncertainty of Estimates into Overall Probability; Stress is Deterministic

A reliability model is a function in which some parameters are considered as random variables. The probability of failure is calculated by an analytical or numerical method or by simulation. If the underlying model for the mean is linear and the error
is normally distributed and homoscedastic (pure $Y$ error source), then the underlying reliability model for the fatigue strength is

$$Y = X\beta + \varepsilon$$
$$= X\beta + \sigma Z$$
$$= \mu + \sigma Z$$  \hspace{1cm} (6.1)

Where $Z$ is a standard normal variate and $\sigma$ is the standard deviation of the error $\varepsilon$. Note that $X\beta$ has been replaced by $\mu$ for simplicity and clarity.

The reliability of an item within the service life $y_s$ can be evaluated first by constructing the limit state function,

$$G = Y - y_s$$  \hspace{1cm} (6.2)

$G$ is defined such that $G < 0$ is the event of failure of item before the intended service life $y_s$. The probability of failure is,

$$P_f = P(G < 0)$$
$$= P(Y - y_s \leq 0)$$  \hspace{1cm} (6.3)

But $\mu$ and $\sigma$ are not known so that $P_f$ can not be evaluated immediately. To construct a reliability model from estimated $\mu$ and $\sigma$, i.e. $\hat{\mu}$ and $\hat{\sigma}$, the relation between $\mu$, $\sigma$ and $\hat{\mu}$, $\hat{\sigma}$ can be employed,

$$\hat{\mu} \sim N(\mu, a\sigma^2)$$  \hspace{1cm} (6.4)

$$\frac{(n - m)\hat{\sigma}^2}{\sigma^2} \sim \chi^2_{n-m}$$  \hspace{1cm} (6.5)

where $a$ is the same as defined in Chapter 5, and $\chi^2_{n-m}$ is the chi-square variate with $n - m$ degrees of freedom. $m$ is the number of parameters in the model for the mean.
Note that \( \hat{\mu} \) and \( \hat{\sigma} \) are independent, when Eq. 6.1 is valid (See appendix A for the proof). Thus,

\[
\hat{\sigma} = \sigma \sqrt{\frac{X^2_{n-m}}{n-m}} \tag{6.6}
\]

\[
\sigma = \hat{\sigma} \sqrt{\frac{n-m}{X^2_{n-m}}} \tag{6.7}
\]

\[
\hat{\mu} = \mu + \sqrt{a} \sigma Z_1 \tag{6.8}
\]

\[
\mu = \hat{\mu} - \sqrt{a} \sigma Z_1 \tag{6.9}
\]

Where \( X^2_{n-m} \sim \chi^2_{n-m} \), and \( Z_1 \) is a standard normal variate. Substituting \( \sigma \) of Eq. 6.7 into Eq. 6.9,

\[
\mu = \hat{\mu} - \hat{\sigma} Z_1 \sqrt{\frac{n-m}{X^2_{n-m}}} \tag{6.10}
\]

Substituting Eq. 6.10 and 6.7 into Eq. 6.1,

\[
Y = \hat{\mu} - \hat{\sigma} Z_1 \sqrt{\frac{n-m}{X^2_{n-m}}} + \hat{\sigma} Z \sqrt{\frac{n-m}{X^2_{n-m}}} \tag{6.11}
\]

Note that \( Z \) defines the uncertainty of a future observation. \( Z_1 \) and \( X^2_{n-m} \) define the randomness of the estimates. Therefore \( Z \) is independent of \( Z_1 \) and \( X^2_{n-m} \). Also note that \( Z_1 \) defines the randomness of \( \hat{\mu} \), and \( X^2_{n-m} \) defines the randomness of \( \hat{\sigma} \). Therefore \( Z_1 \) and \( X^2_{n-m} \) are independent.

Given \( \hat{\mu} \), \( \hat{\sigma} \) and a specific \( X^2_{n-m} \) value \( x^2_{n-m} \), the sum of \( -\hat{\sigma} Z_1 \sqrt{\frac{n-m}{x^2_{n-m}}} \) and \( \hat{\sigma} Z \sqrt{\frac{n-m}{x^2_{n-m}}} \) is a normal variate (by the reproductive property of normal distribution), with mean = 0 because \( E(Z_1) = E(Z) = 0 \). The variance equals to

\[
a \frac{n-m}{x^2_{n-m}} \hat{\sigma}^2 + \frac{n-m}{x^2_{n-m}} \hat{\sigma}^2 = (a + 1) \frac{n-m}{x^2_{n-m}} \hat{\sigma}^2 \tag{6.12}
\]

Thus,

\[
Y = \hat{\mu} + \hat{\sigma} Z_2 \sqrt{a + 1} \sqrt{\frac{n-m}{X^2_{n-m}}} \tag{6.13}
\]
The random variable \( X^2_{n-m} \) and \( Z_2 \) (a standard normal variate) are independent. Therefore \( Z_2 \sqrt{\frac{n-m}{X^2_{n-m}}} \) forms a student \( t \) variate \( T_{n-m} \) with \((n - m)\) degrees of freedom [25, page 217]. Simplifying Eq. 6.13

\[
Y = \mu + \sigma T_{n-m} \sqrt{a + 1}
\]

(6.14)

Comparing Eq. 6.14 and 6.1, the random variable \( Z \) in the reliability model becomes \( T_{n-m} \sqrt{a + 1} \) while \( \mu, \sigma \) are replaced by \( \hat{\mu}, \hat{\sigma} \). The \( T_{n-m} \sqrt{a + 1} \) contains all the randomness from \( Z \), \( \hat{\mu} \) and \( \hat{\sigma} \). Note however that \( a \) is function of stress.

This model has same form as the prediction interval [25, page 373] which is used to predict the next observation. From this viewpoint, the model, where uncertainty of the estimates lump into the overall probability, is only useful in predicting the distribution of the next outcome. In other words, this model relates the distribution of the next outcome \( Y \) to the estimates \( \hat{\mu} \) and \( \hat{\sigma} \). However the Owen's tolerance factor described in the next section, lumps uncertainty of estimates into the confidence level \( \gamma \). It is this separation of uncertainty of estimates from the overall probability that makes estimation of the \( 1 - p \) quantile of \( Y \) (by Eq. 1.4) to be possible. What Owen's tolerance factor addresses is not the next outcome but the unknown distribution. Each \( 1 - p \) quantile of the distribution can be estimated with certain confidence level \( \gamma \). The reliability model based on Owen's concept is derived in the next section.
6.4.2 A Reliability Model that Incorporates the Uncertainty of Estimates into the Confidence Level; Stress is Deterministic

Owen’s estimate of any quantile \( 1 - p \) using estimates \( \hat{\mu} \) and \( \hat{\sigma} \) with a confidence level of \( \gamma \) is

\[
\hat{y}_{1-p} = \hat{\mu} - K_{p,\gamma,a,f} \hat{\sigma}
\]  

(6.15)

where \( K \) is Owen’s tolerance factor. The confidence level \( \gamma \) is arbitrary, and \( \gamma = 95\% \) is often used. For a given test plan and analysis model, the \( a \) and \( f \) are determined and \( K \) may be thought of as a function of \( p \) only, i.e., \( K(p) \). Its inverse function will be denoted as \( p(K) \). Note that \( p \) is defined between 0 and 1. Therefore \( p(K) \) can be viewed as a CDF of the random variable \( K \). A reliability model may be constructed using Eq. 6.15 treating \( K_{p,\gamma,a,f} \) as a random variable with CDF defined as \( p(K) \):

\[
\hat{y}_{1-p} = \hat{\mu} - K \hat{\sigma}
\]  

(6.16)

The CDF, \( p(K) \) can be calculated numerically and be input to a reliability analysis program. In the Eq. 6.16, only the \( K \) is treated as random variable. This equation may be compared with Eq. 6.1 and 6.14. One major difference is that the \( K \) has one additional independent variable \( \gamma \) when compared with \( T_{n-m} \) in Eq. 6.14.

Both Eq. 6.14 and 6.16 are for cases where stress/strain \( x \) is deterministic during the service life. However a reliability model must be capable of incorporating additional random variables such as stress/strain. This makes the formulation of the reliability model more difficult, as will be seen in next subsection.
6.4.3 Reliability Model with Stress as a Random Variable

First assume that the stress/strain during the service life is known, i.e. the parameters of random variable $X$ are known. A reliability model at the $\gamma$ confidence level may be considered as,

$$\hat{y}_{1-p} = \hat{\mu}(X) - K(X)\hat{\sigma}$$

where both $\hat{\mu}$ and $K$ are functions of the known random variable $X$ (stress/strain). As shown in Chapter 5, $K(x)$ may be approximated by an average $K$ in the stress/strain range considered. Thus $K(X)$ may be replaced by $K_{\bar{a}}$ in Eq. 6.17. And $\hat{\mu}(X)$ may be treated as follows. Consider the polynomial model of degree one as an example

$$\hat{\mu}(X) = \hat{a}_0 + \hat{a}_1 X$$

A reliability model may be directly constructed as

$$\hat{y}_{1-p} = \hat{a}_0 + \hat{a}_1 X - K_{\bar{a}}\hat{\sigma}$$

where $\hat{a}_0$, $\hat{a}_1$ and $\hat{\sigma}$ are estimates treated as deterministic values, and $X$, $K_{\bar{a}}$ are random variables.

There are two questions regarding the correctness of Eq. 6.19. First, Owen's tolerance factor considers a tolerance limit at a single stress/strain level at a time. To consider a range of stress/strain over which the random variable $X$ spans at one time, one should use the simultaneous tolerance factor. However the simultaneous tolerance factor is derived using the Bonferroni inequality (See [50, page 8]) which leads to a very conservative and seemingly inappropriate result as discussed in Chapter 5. The simultaneous tolerance limit may not be appropriate unless the variance of $X$ is very large. Is the range of random variable $X$ small enough that $K_{\bar{a}}$ may be employed
with small error? Does the confidence level $\gamma$ and distribution of $X$ interact with each other?

Moreover, note that the randomness of $X$ is unknown and must be estimated. Estimates of $X$ include the sample mean $\mu_X$ and sample deviation $\sigma_X$. Should Owen's tolerance factor also be applied on these estimates thereby introducing another confidence level $\gamma_X$? How is this $\gamma_X$ related to $\gamma$? Is there interaction or dependence between them? How $\gamma_X$ and $\gamma$ be combined to form a single resulting confidence $\gamma_r$ that relates to the overall probability? Clearly there are several mathematical issues that need to be addressed to develop a reliability model having a sound mathematical basis.
Appendix A

Owen's Tolerance Factor: Derivation and Applications

A summary of the theoretical basis of the tolerance limit concept developed by D. B. Owen [61] for a normal univariate or a linear model is presented.

A.1 Definition of the Non-Central t Distribution

The non-central \( t \) random variable with degrees of freedom, \( f \) and non-centrality parameter, \( \delta \) is defined as

\[
T_f(\delta) = \frac{X_1 + \delta}{\sqrt{X_2/f}}
\]

where

- \( X_1 \) is a standard normal variate, i.e. \( X_1 \sim N(0,1) \).
- \( X_2 \) is a chi-square variate with degrees of freedom, \( f \), i.e. \( X_2 \sim \chi_f^2 \).
- \( X_1 \) and \( X_2 \) are independent.
- \( \delta \) is constant.

Note that \( X_1 + \delta \sim N(\delta,1) \) is a non-central normal variate.
A.2 Application of the Non-Central $t$ to Tolerance Limits

Consider a random variable $Y$, normally distributed: $Y \sim N(\mu, \sigma^2)$. Note that $Y$ may be a univariate of a response (i.e. a dependent variable) at a specific stress/strain in a regression model. Denote the estimators of $\mu$ and $\sigma$ as $\hat{\mu}$ and $\hat{\sigma}$ respectively. Note that $\hat{\mu}$ and $\hat{\sigma}$ are random variables. The estimator of the $1 - p$ quantile $Y_{1-p}$ is then defined as

$$\hat{Y}_{1-p} = \hat{\mu} - K\hat{\sigma}$$  \hspace{1cm} (A.2)

where $K$ is so chosen that $\hat{Y}_{1-p}$, also a random variable, will lie on the left of the true quantile $Y_{1-p}$ with probability $\gamma$. $\gamma$ is called the confidence level. The event that $\hat{Y}_{1-p}$ lies on the left of the true quantile $Y_{1-p}$ is called event A. The probability of event A should be at least $\gamma$. The goal of analysis is, given $p$ and $\gamma$, to determine $K$.

Event A can be expressed mathematically as

$$\frac{\hat{Y}_{1-p} - \mu}{\sigma} \leq -K_p$$  \hspace{1cm} (A.3)

where $-K_p$ is the $1 - p$ quantile of standard normal variate. $K_p$ is the $p$ quantile of standard normal variate. This inequality can be written as,

$$\frac{\mu - \hat{Y}_{1-p}}{\sigma} \geq -K_p$$  \hspace{1cm} (A.4)

Substituting $\hat{Y}_{1-p}$ using Eq. A.2,

$$\frac{\mu - \hat{\mu}}{\sigma} + \frac{K\hat{\sigma}}{\sigma} \geq -K_p$$  \hspace{1cm} (A.5)

$\hat{\mu}$ is normally distributed as $\hat{\mu} \sim N(\mu, a\sigma^2)$, where $a$ is defined in Eq. 5.9. For the special case when $Y$ is univariate, $a = 1/n$. The quantity $\frac{\hat{\mu} - \mu}{\sqrt{a}\sigma}$ or $\frac{\hat{\mu} - \mu}{\sqrt{a}\sigma}$ will be a standard normal. It is related to $X_1$ in Eq. A.1. Divide both sides of Eq. A.5 by $\sqrt{a}$,

$$\frac{\mu - \hat{\mu}}{\sqrt{a}\sigma} + \frac{K\hat{\sigma}}{\sqrt{a}\sigma} \geq -\frac{K_p}{\sqrt{a}}$$  \hspace{1cm} (A.6)
Because \( f \frac{\delta^2}{\sigma^2} \) has a \( \chi^2 \) distribution, i.e. \( \delta^2 \sim \sigma^2 \chi^2_f / f \), we may use \( \delta / \sigma \) to form the quantity \( \sqrt{\chi^2_f / f} \) in Eq. A.1. First interchange two terms on both sides of the inequality of Eq. A.6 to form

\[
\frac{\mu - \hat{\mu}}{\sqrt{a} \sigma} - \frac{K_p}{\sqrt{a}} \geq - \frac{K \hat{\delta}}{\sqrt{a}} \tag{A.7}
\]

Divide both sides by \( \frac{\hat{\delta}}{\sigma} \)

\[
\frac{(\mu - \hat{\mu}) - \frac{K_p}{\sqrt{a}}}{(\hat{\delta} / \sigma)} \geq - \frac{K}{\sqrt{a}} \tag{A.8}
\]

Denote \( - \frac{K_p}{\sqrt{a}} \) on the left hand side as \( \delta \). The left hand side is a non-central \( t \) variate if \( \hat{\mu} \) and \( \hat{\delta} \) are independent. The proof of independence between \( \hat{\mu} \) and \( \hat{\delta} \) is given in Sec. A.5. Denote the right hand side as \( t_o = - \frac{K}{\sqrt{a}} \). The probability of even \( A \) should be at least \( \gamma \). This can be described mathematically as

\[
Pr\{T_f(\delta) \geq t_o\} = \gamma \tag{A.9}
\]

where \( T_f(\delta) \) and \( t_o \) are the left hand and right hand side of the last inequality. \( t_o \) is the \( \gamma \) quantile of the variate \( T_f(\delta) \). The estimate of the \( (1 - p) \) quantile of \( Y \) is related to a non-central \( T \) variate with \( f \) degrees of freedom and non-centrality parameter \( \delta \).

In the next sections, Owen's tolerance factor \( K \) will be derived to satisfy the goal in Eq. A.9 for various cases.

### A.3 Tolerance Factor \( K \) for the Univariate Case

When specimens are tested at one stress/strain level only, the (transformed) fatigue life \( Y \) is a univariate. The sample size is \( n \). The sample mean \( \hat{\mu} \) and sample variance \( \hat{\sigma} \) are distributed as

\[
\hat{\mu} \sim N(\mu, \sigma \sqrt{n}) \tag{A.10}
\]

\[
\hat{\sigma}^2 \sim \frac{\sigma^2}{n - 1} \chi^2_{n-1} \tag{A.11}
\]
Therefore, \( a = 1/n \) and \( f = n - 1 \) in the case when \( Y \) is an univariate. Thus

\[
\delta = -\sqrt{n}K_p \tag{A.12}
\]
\[
t_o = -\sqrt{n}K \tag{A.13}
\]

Owen's tolerance factor \( K \) is determined by the following procedure. First calculate \( \delta \), then find the quantile \( t_o \) of non-central \( t \) distribution with degrees of freedom \( n - 1 \) and non-centrality \( \delta \). \( t_o \) is then used to calculate \( K = -t_o/\sqrt{n} \).

In the case where \( Y \) is a univariate, \( K \) depends only on \( p, \gamma \) and \( n \). Tables for \( K \) are available ([51] or [3, table 33]). Two typical combinations of \( p \) and \( \gamma \) are: (1) \( \gamma = 95\%;\ p = 99\%; \) (2) \( \gamma = 95\%;\ p = 90\% \). \( K \) values for Case(1) are called "A-values" and \( K \) values for Case(2) are called "B-values" in MIL-HDBK-5[49].

### A.4 Tolerance Factor \( K \) for the Regression Case

When specimens are tested under various stress/strain levels, a linear regression model which relates the independent variable \( X \) and dependent variable \( Y \) is applied. The estimated mean of \( Y \) at given stress/strain level \( x_h \) is denoted as \( \hat{\mu}_h \); and the estimated variance is denoted as \( \hat{\sigma}^2 \). They are distributed as

\[
\hat{\mu}_h \sim N(\mu_h, a(x_h)\sigma^2) \tag{A.14}
\]
\[
\hat{\sigma}^2 \sim \frac{\sigma^2}{n - m} \chi^2_{n-m} \tag{A.15}
\]

where \( \mu_h \) is the true mean of \( Y \) at \( x_h \), and \( m \) is the number of parameters of the model for the mean. For example, consider a polynomial of degree one as a model of the mean. The number of parameters is two, \( f = n - 2 \). To find the Owen's \( K \), first calculate \( \delta = -K_p/\sqrt{a(x_h)} \), then find the \( \gamma \) quantile \( t_o \) of non-central \( t \) with non-centrality \( \gamma \) and degrees of freedom \( f \). Then \( K = -t_o/\sqrt{a(x_h)} \).
In the case of the regression model, \( K \) is function of \( p, \gamma, f \) and \( a(x_h) \) which is difficult to tabulate. A computer program, such as the one listed in Appendix C, is needed to determine \( K \).

A.5 Proof of Independence Between the Sample Mean and Sample Variance

To show that the left hand side of inequality A.8 is a non-central \( t \) variate, one has to prove that \( (\hat{\mu} / \hat{\sigma}) \) and \( (\hat{\sigma} / \sigma) \) are independent. Because the only random variables that appear in these quantities are \( \hat{\mu} \) and \( \hat{\sigma} \), it is sufficient to prove that \( \hat{\mu} \) and \( \hat{\sigma} \) are independent. This will be proven for cases where \( Y \) is the response variable of the regression model. Because the case where \( Y \) is univariate is a special case of the regression model where all tests are conducted at one stress/strain level, a separate proof would be unnecessary.

A theorem in linear model analysis is used: Let \( Y \sim N_n(\mu, \sigma^2I), \sigma^2 > 0, \) and \( A \) be a \( m \times n \) matrix, \( C \) be a \( n \times n \) symmetric matrix. \( N_n(\mu, \sigma^2I) \) denotes an \( n \)-dimensional independent normal distribution. If \( AC = 0 \), then \( AY \) and \( Y'CY \) are independent.

From linear regression, \( \hat{\sigma}^2 \) is expressed as

\[
\hat{\sigma}^2 = \frac{SS_E}{n - m} = \frac{1}{n - m} Y'(I - H)Y
\]  

(A.16)

where \( m \) is the number of parameters in the model for the mean, \( I \) is an unit matrix, and \( H = X(X'X)^{-1}X' \). \( X \) is the analysis matrix defined in Chapter 5. \( Y \) is a \( n \times 1 \) matrix that consists of \( n \) outcomes \( y_i, i = 1, n \) and is distributed as \( N_n(\mu, \sigma^2I) \). \( \mu \) is a \( n \times 1 \) matrix that consists of \( \mu \) corresponding to true mean value of \( y_i \) in \( Y \).

The estimator of the mean of \( Y \) is

\[
\hat{\mu} = x_h^T \hat{a}
\]  

(A.17)
\[ x_h^T (X'X)^{-1} X'Y \] (A.18)

where \( \hat{\alpha} \) is the estimated parameters as defined in Eq. 3.32, and \( x_h^T \) is defined in Eq. 5.10. Let \( A = x_h^T (X'X)^{-1} X' \) and \( C = (I - H) \). If it can be proven that \( AC = 0 \), then \( \hat{\mu} \) and \( \hat{\sigma}^2 \) are independent. Note that the constant \( \frac{1}{n-m} \) in Eq. A.16 can be ignored in this proof. By direct substitution,

\[
AC = x_h^T (X'X)^{-1} X'(I - H) \quad (A.19)
= x_h^T (X'X)^{-1} X' - x_h^T (X'X)^{-1} X'H \quad (A.20)
= x_h^T (X'X)^{-1} X' - x_h^T (X'X)^{-1} X'X(X'X)^{-1} X' \quad (A.21)
= x_h^T (X'X)^{-1} X' - x_h^T (X'X)^{-1} X' \quad (A.22)
= 0 \quad (A.23)
\]

Thus \( \hat{\mu} \) and \( \hat{\sigma}^2 \) are independent.
Appendix B
Analysis Models Provided in Program FEDADS

The program FEDADS provides a flexible way of specifying an analysis model. As mentioned in Sec. 2.2, an analysis model consists of (a) the type of transformation applied on stress/strain or life (Eq. 2.1 and 2.2), (b) the model for the mean (Eq. 2.5), (c) the model for the scatter (Eq. 2.8), and (d) the model for the distribution type of $Y$ given $X$. The program is organized in such a way that all transformations and models for the mean or scatter are defined by specifying a function type contained in the function $\text{FUNevl}$. By this organization, the user may define the analysis model using different combinations of functions applied on the transformation (on stress/strain and life), the model for the mean and the model for scatter. However, not all functions in $\text{FUNevl}$ are suitable to define a transformation and not all are suitable to define scatter. For example, only functions that are coded in such a way that all function values in the range of test stress are positive may be used as model for scatter. This type of function is called "positive guaranteed".

The distribution types allowed in FEDADS are normal, lognormal, Weibull, EVD, exponential and uniform. However only the normal distribution was used in this study and is therefore well tested.

The current applicable transformations and models for mean and models for scatter in program FEDADS are listed in the tables below.
Table B.1: Available transformation in program FEDADS

<table>
<thead>
<tr>
<th>name</th>
<th>formula</th>
<th>number of parameter</th>
<th>remark</th>
</tr>
</thead>
<tbody>
<tr>
<td>log</td>
<td>$Y = \log N$ or $X = \log S$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>ln</td>
<td>$Y = \ln N$ or $X = \ln S$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>Box-Cox type 1</td>
<td>$Y = \frac{N^{\lambda-1}}{\lambda}$ for $\lambda \neq 0$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$Y = \ln N$ for $\lambda = 0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Box-Cox type 2</td>
<td>$Y = \frac{(N+\lambda_1)^{\lambda_1}-1}{\lambda_1}$ for $\lambda_1 \neq 0$</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$Y = \ln(N + \lambda_2)$ for $\lambda_1 = 0$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table B.2: Available models for the mean in program FEDADS

<table>
<thead>
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<th>name</th>
<th>formula</th>
<th>number of parameter</th>
<th>remark</th>
</tr>
</thead>
<tbody>
<tr>
<td>linear</td>
<td>$\mu_Y = a_0 + a_1 x$</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>quadratic</td>
<td>$\mu_Y = a_0 + a_1 x + a_2 x^2$</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>exponential</td>
<td>$\mu_Y = \exp x$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>log</td>
<td>$\mu_Y = \log x$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>ln</td>
<td>$\mu_Y = \ln x$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>bilinear</td>
<td>$x = a_1 + a_2(\mu_Y + a_3\sqrt{\mu_Y - a_3})^2 + a_4)$</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>leveling-off</td>
<td>$x = a_1 + a_2(\mu_Y - \sqrt{\mu_Y - a_3})^2 + a_4)$</td>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>
| general               | $\epsilon = a_1N^a_2 + a_3N^a_4$            | 4                   | $Y = \log N$  
| strain-life           |                                              |                     | $X = \log \epsilon$ |
| equation              |                                              |                     | $X = \epsilon$ |
| MIL-HDBK-5            | $Y = a_1 + a_2 \log(X - a_3)$                | 3                   | $Y = \log N$  
<p>|                       |                                              |                     | $X = S$            |</p>
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<td>$\sigma_Y(x) = y(x) - y(x - \sigma_x)$</td>
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Table B.3: Available models for the scatter in program FEDADS
Appendix C

Improved Approximation of Owen's $K$

The following is a FORTRAN function to calculate Owen's tolerance factor with improved precision. The numerical algorithm is based on the approximate D method in Owen's work[61]. The approximate D method is applied first, then a correction factor $\frac{K}{K_D}$ is calculated and multiplied with the result of the approximate D method. In this FORTRAN function, the improvement is performed in cases $\gamma = 0.95, 0.90, 0.85, 0.80$. The $p$ may be continuous from $p = 0.99$ down to $p = 0.85$. $K_p$ and $K_r$ are the $p$ and $\gamma$ quantile of standard normal variate respectively.

```fortran
real function appdk(kp, kr, f, a)
c R (R, R, R, R)
c == improved approximation of Owen's K. At first the
c == approximation D(see below) is applied, then Shen's
c == improvement is applied for f >= 3.
c
c [1] approximate Owen's K by Approximation D
c [Owen, Technometrics, 1968 p. 469]
c == Note: n in the Owen's paper(p.469) has replaced by 1/a here.
real kp, kr, f, a, c11, c20, c22
c [1.1]
```
if( f.le.2.) then
  c === in case of f =< 2, use Johnson and Welch’s approximation
  c === to avoid numerical difficulty
  c11=1
  c20=1.
  c22=1./(2.*f)
c [1.2]
else
  c11=1.+(3./(4.*(f-1.042))
  c20=f/(f-2.)
  c22=c20-c11**2
endif
c [1.3]
appdk=c11*kp+kr*sqrt(c22*kp**2+c20*a)
c
c [2] [ improve ]
if( f.ge.3) then

if( abs(Kr- 1.64485). lt. 0.0001 ) then
  c [2.1] [ r=.95 ]
    ratio= 0.9968238 +0.1596499/(f**.6) -2.635553*exp(-f)
elseif( abs(Kr- 1.28155). lt. 0.0001 ) then
  c [2.2] [ r=.90 ]
    ratio= 1.003493 -6.016034/(f**3) +1.098676*exp(-f)
elseif( abs(Kr- 1.03643). lt. 0.0001 ) then
c [2.3] [ r=.85 ]
   ratio= 1.000637 -0.7211758/(f**1.5) -1.485648*exp(-f)
elseif( abs(Kr- 0.84162). lt. 0.0001 ) then
  c [2.4] [ r=.80 ]
  ratio= 1.000748 -0.6366483/(f**1.25) -1.554196*exp(-f)
else
  c [2.5] [ no improvement for other cases]
  ratio= 1.
endif
appdk= appdk *ratio

endif
return
end
Appendix D

Fatigue Data Used in This Research

The complete fatigue data sets used in this study is listed in this appendix. The Titanium and Inconel 718 are strain controlled. The others are stress controlled. Except for complete data sets, censored data are indicated as "1" and failure data are indicated as "0" in column "c". For Titanium data, testing temperatures are indicated in column "t" in Fahrenheit.

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Table D.1: The annealed aluminum wire (AAW) fatigue data.
Table D.2: Steel wire fatigue data.
Table D.3: Titanium fatigue data.

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Table D.4: 2024-T4 aluminum alloy fatigue data.
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Table D.5 (continued) Inconel 718 fatigue data.

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