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# Integrable curve dynamics

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# INTEGRABLE CURVE DYNAMICS

by

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As members of the Final Examination Committee, we certify that we have read the dissertation prepared by Annalisa Maria Calini

entitled Integrable Curve Dynamics

and recommend that it be accepted as fulfilling the dissertation requirement for the Degree of Ph.D.

Nicholas Ercolani — Date 10/10/94

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Date

Date

Final approval and acceptance of this dissertation is contingent upon the candidate's submission of the final copy of the dissertation to the Graduate College.

I hereby certify that I have read this dissertation prepared under my direction and recommend that it be accepted as fulfilling the dissertation requirement.

Dissertation Director Nicholas Ercolani — Date 10/10/94

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SIGNED: Amalia Soli

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# PREFACE

Che lo studio dei nodi abbia un aspetto liberatorio è una teoria che non ho mai visto scritta nemmeno in una nota a pié di pagina. Eppure, forse che non si parte da un nodo e, maneggiandolo con precauzione, osservandolo come il meccanismo di una cassaforte, non si studiano le mosse necessarie per ridurlo ad un semplice, inutile pezzo di corda?

Il nodo ha una storia varia e una dualità complessa, ma al giorno d'oggi dopo secoli di catene, le attuali rese piu' forti e sottili da vincoli mentali, morali e sociali, il nodo richiama immagini non amate. Si è perduto il senso del lavoro umano: tessuti fatti di nodi, lavori nei campi, l'animale guidato dal contadino, l'asino costretto a portare il basto, l'acqua issata a fatica dai pozzi, le vele di grandi navi piene di carichi e di esploratori.

Ci sono libri che esaltano i nodi, simboli di progresso, di unione. Eppure, diciamoci la verità, questi nodi noi li vogliamo sciogliere. Tutte queste teorie matematiche alla fin fine ruotano attorno ad un'idea fissa: capire il nodo e disfarlo, comprendere quando due nodi sono uguali e usare la stessa tecnica per vanificarne lo scopo, compilare un catalogo di tutti i nodi del mondo al fine di distruggerli sistematicamente.

E allora via le catene, via le vele, via le navi alla deriva, liberi asini e buoi a calpestare i campi, liberi gli uomini dalle vesti, liberi i capelli delle donne; cadono gli impiccati col culo per terra e urlano "Mamma, sono ancora vivo!", volano i rocciatori nel nulla trasformandosi in aquile, piove l'acqua nei pozzi ritornando alla terra.

Si sciogliessero anche i nodi del cuore e quelli, piú subdoli, nella testa della gente, avremmo raggiunto una grandissima scoperta.

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# ABSTRACT

The Heisenberg Model of the integrable evolution of a continuous spin chain can be used to describe an integrable dynamics of curves in $R^3$.

The role of orthonormal frames of the curve is explored. In this framework a second Poisson structure for the Heisenberg Model is derived and the relation between the Heisenberg Model and the cubic Non-Linear Schrödinger Equation is explained.

The Frenet frame of a curve is shown to be a Legendrian curve in the space of orthonormal frames with respect to a natural contact structure. As a consequence, generic singularities of the solution of the Heisenberg Model and topological invariants of the curve are computed.

The family of multi-phase solutions of the Heisenberg Model and the corresponding curves are constructed with techniques of algebraic geometry. The relation with the Non-Linear Schrödinger Equation is explained also in this context.

A formula for the Bäcklund tranformation for the Heisenberg Model is derived and applied to construct orbits homoclinic to planar circles. As a result singular knots are obtained.

# Chapter 1 Introduction

The story of this thesis begins in Italy, in my country (just a coincidence!) at the turn of the century [Ric91]. It begins with another thesis by Luigi Da Rios, a student of Tullio Levi Civita. In his work he studied the motion of an isolated vortex filament in an indefinite domain filled with an incompressible inviscid fluid. A region of vorticity in a fluid or a gaseous medium is a region where the velocity field possesses a rotational component. When the vorticity is non-zero only in the interior of a thin tube, (skilled smokers are able to produce a variety of examples), two ideas come to mind to start understanding the motion of the filament:

- The shape of the filament can be approximated by a curve in space, assuming that its thickness vanishes and that its internal structure is irrelevant. Therefore its motion can be described only in terms of the position vector $\vec{\gamma}$ of an abstract curve.

- When the filament is curved, nearby segments create a potential field which causes the vortex to move. To simplify the model even more, one remembers that at each point every curve is approximated by a circular arc, and assumes the potential to be local. In this limit, the velocity field of $\vec{\gamma}$ at one point will depend only on the value of its curvature at that point.

This effort of simplification creates a remarkable equation which, in its more modern formulation, has the following form [Ham65, Bet65],

$$\frac{d\vec{\gamma}}{dt} = k\vec{b}. \tag{1.1}$$

Here $k$ is the curvature of $\vec{\gamma}$, and $\vec{b}$ is its binormal vector; both are functions of time and of the arclength parameter. Its form reflects directly the main feature of the physical approximation: regions of greater curvature move with a larger speed. Thinking once more of smoke rings, smaller rings travel faster through the air.

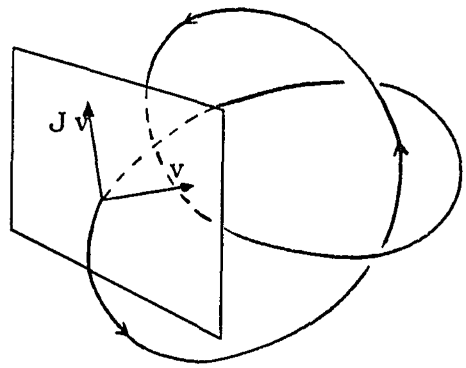


Figure 1.1, The complex structure of the space of knots

Marsden and Weinstein [MW83] showed that the Filament Equation (1.1) has a natural hamiltonian formulation which can be derived from the Euler equations for a perfect fluid.

If $C$ is a "point" in the phase space $M$ of equation (1.1), that is an arclength parametrized curve, a tangent vector to $M$ at $C$ is a smooth choice of normal vectors along the curve (Fig. 1.1). Thinking of the normal plane at a point along $\vec{\gamma}$ as the plane of complex numbers, we can endow it with a notion of "multiplication by $i$" and induce a skew-symmetric operation $J$ on tangent vectors in the following natural way ($\times$ is the cross product in $R^3$)

$$J\vec{u}(s) = \frac{d\vec{\gamma}}{ds}(s) \times \vec{u}(s), \qquad \vec{u} \in T_C M. \tag{1.2}$$

The Marsden-Weinstein Poisson bracket on $C^\infty(M)$ is defined by

$$\{f, g\} = \int_C df \cdot J dg \qquad f, g \in C^\infty(M) \tag{1.3}$$

($df, dg$ are the gradients of $f, g$ and $\cdot$ is the scalar product in $R^3$). A Poisson bracket associated with a skew-symmetric operator $J$ assigns hamiltonian vector fields $Jdf$ to functions $f$ (functionals on the space of curves in this case). The velocity field $k\vec{b}$ of the Filament Equation can be rewritten in the form

$$\frac{d\vec{\gamma}}{ds} \times \frac{d^2\vec{\gamma}}{ds^2}.$$

i.e. it is the hamiltonian vector field for the hamiltonian functional

$$H[\mathcal{C}] = \int_{\mathcal{C}} \|\frac{d\vec{\gamma}}{ds}\| ds = l(\mathcal{C}), \tag{1.4}$$

the length of the curve $\mathcal{C}$.

In the early '70's, before the hamiltonian formulation of the Filament Equation had been explored, Hasimoto [Has72] found a change of variable which recognized equation (1.1) as a completely integrable soliton equation. The Hasimoto transformation

$$\mathcal{H} : \vec{\gamma} \to \psi = k e^{i \int^{s} \tau du} \tag{1.5}$$

maps $\vec{\gamma}$ to the complex function $\psi$ of the curvature $k$ and the torsion $\tau$. The function $\psi$, which completely determines the shape of the filament, is a solution of the cubic focussing non-linear Schrödinger (NLS) equation.

$$i\psi_t + \psi_{ss} + \frac{1}{2}|\psi|^2\psi = 0. \tag{1.6}$$

The NLS equation can be rewritten as a hamiltonian system on the space of complex-valued functions $\psi$

$$\psi_t = \hat{J}\frac{\delta H}{\delta \psi}, \tag{1.7}$$

with Hamiltonian $H[\psi] = \int_{\mathcal{D}} [\frac{1}{2}\bar{\psi}\psi_{ss} + \frac{1}{8}\bar{\psi}^2\psi^2] du$, ($\mathcal{D}$ the spatial domain). The symplectic operator $\hat{J}$ is multiplication by $i$ and serves to define a Poisson bracket on pairs of functionals of $\psi$ [FA80]. The geometric aspect of the complete integrability of the NLS equation translates into the existence of an infinite sequence of symplectic operators, whose associated hamiltonian flows pairwise commute. Magri [Mag78] related the ordering in the hierarchy of Poisson structures to the existence of a Recursion operator $\hat{R}$ which takes one Poisson operator (and the corresponding bracket) into another

$$\hat{R}^n \hat{J}_0 = \hat{J}_n, \quad n = 0, 1, 2, ... \tag{1.8}$$

It is tempting to suggest ([MW83] did) that the Hasimoto transformation should carry the operation of "multiplication by $i$" on the normal planes to the curve into the "multiplication by $i$" on complex functions. This would imply that the map $\mathcal{H}$ sends the Marsden-Weinstein bracket to the natural hamiltonian structure for NLS.

A few years ago Langer and Perline [LP91] showed that the relation between the two hamiltonian structures is more subtle. They computed the differential of the Hasimoto map and discovered the formula

$$d\mathcal{H}(h\vec{n} + g\vec{b}) = \tilde{R}^2(h + ig)e^{i\int^s \tau du}, \tag{1.9}$$

Where $h\vec{n}+g\vec{b}$ is a tangent vector to $M$ at $\vec{\gamma}$ written with respect to components of the Frenet frame $(\vec{t}, \vec{n}, \vec{b})$ of the curve. Two observations can be made about (1.9):

- The choice of framings of the curve is important. This sounds like a side comment at this stage, but will become the main theme of chapter 2 of this work. If, instead of choosing the Frenet frame as a basis for $TM$, we choose the strange looking frame

$$(\vec{t}, \vec{u}, \vec{v}) = (\vec{t}, e^{-i\int^s \tau du}\vec{n}, e^{-i\int^s \tau du}\vec{b}), \tag{1.10}$$

  then we have the correspondence $(h\vec{u} + g\vec{v}) \rightarrow h + ig$, and $d\mathcal{H}$ becomes the square of the recursion operator with no need to introduce the gauge term $e^{i\int^s \tau du}$. Later we will show that the orthonormal frame $(\vec{t}, \vec{u}, \vec{v})$ arises in a very natural context (see the description of ch.2 in this introduction and ch.2 itself).

- $\mathcal{H}$ is a Poisson map; that is it maps Poisson brackets to Poisson brackets (and therefore preserves the fundamental property of integrability). Moreover there is a shift in Poisson structures: the Marsden-Weinstein Poisson bracket is mapped to the fourth Poisson bracket for the NLS equation.

In this thesis we study a dynamical system which is intermediate between the Filament Equation and the Non-Linear Schrödinger equation. Since the arclength is locally preserved by the approximated vortex filament dynamics, we can differentiate both sides of equation (1.1) with respect to the arclength parameter and derive the evolution equation of the unit tangent vector $\vec{t}$. We are interested in closed curves, so we will study the equation

$$\begin{aligned} \frac{d\vec{t}}{dt} &= \vec{t} \times \frac{d^2\vec{t}}{ds^2}, \qquad \|\vec{t}\| = 1 \\ \vec{t}(0,t) &= \vec{t}(2\pi, t) \end{aligned} \tag{1.11}$$

with periodic boundary conditions. This equation was derived by Lakshamanan [Lak77] and it is known as the Continuous Heisenberg Model (HM). It describes the evolution of the continuum approximation of a discrete chain of spins interacting just with their nearest neighbors (the analogue of the localized interaction for the vortex filament dynamics). It is related to the NLS equation via a gauge transformation which is defined in chapter 4 and which is given a new interpretation as a consequence of the results in chapter 2 (section 2.2.5).

It is of interest to study (1.11) in the context of the integrable evolution of curves for several reasons. We present these reasons while outlining the content of the various chapters.

**Chapter 2.** The unit tangent vector of an arclength parametrized curve lives on the 2-dimensional unit sphere. As anticipated earlier, this chapter justifies the assertion that the choice of frames is important. For the Heisenberg Model it is most natural to introduce orthonormal frames adapted to the sphere, since $\vec{t}$ evolves on $S^2$. The choice of a frame becomes the choice of a unit tangent vector field along the curve traced out by $\vec{t}$. In other words, let $T_1S^2$ be the circle bundle of $S^2$, i.e. the space of all unit tangent vectors at some point. Then the frame of a curve is a lifting of the associated spherical $\vec{t}$-curve into the circle bundle.

As a first consequence of this set-up we show that the Frenet frame is always a lifting to a Legendrian curve. In order to do this, we construct a natural contact structure on $\mathcal{T}_1S^2$ and show that the velocity field of the Frenet lifting is contained at every point in the distribution of contact planes. The applications of this result presented here are a classification of the generic singularities of the spherical $\vec{t}$-curve and a description of the invariants of the curve in $R^3$ which are related to topological invariants of the corresponding Legendrian curve in $\mathcal{T}_1S^2$. This approach promises a better understanding of the topology of the surface swept out by the Legendrian curve in time [Eli94] and the possible discovery of new invariants of $\vec{\gamma}$ associated to invariants of Legendrian knots [Ben89]. But this is for future work (see conclusions in ch. 5).

The frame introduced in equation (1.10) becomes completely natural in this context: it is the horizontal lifting with respect to the canonical invariant connection on $\mathcal{T}_1S^2$. Its expression allows us to give a geometrical interpretation of the relation between HM and NLS and to construct a Poisson map between the respective phase spaces. We show that there is a shift in Poisson structures, as there is one between the Filament Equation and the NLS equation, and that the differential of this map takes the Marsden-Weinstein bracket for HM to the second bracket for NLS. We also show that the origin of the second Poisson operator for the NLS equation is related to a natural symplectic operator on the space of loops in $SO(3, R)$.

**Chapter 3.** Another characteristic feature of completely integrable equations is the presence of an infinite number of invariants: enough for the equation to be "solvable". The notion of integrability for a PDE is an infinite-dimensional analogue of the integrability of finite-dimensional hamiltonian systems. This requires that the constants of motion are in involution, or in other terms that their associated flows pairwise commute. Moreover their gradients need to be linearly independent. A list of such invariants for curves which solve the Filament Equation was given in [LP91]. The first few are

$$\int ds, \quad \int \tau\, ds, \quad \int k^2\, ds, \quad \int k^2 \tau\, ds. \tag{1.12}$$

An infinite number of constraints forces the solutions to live on a much smaller (but still infinite-dimensional in general) subset of the phase space. Finite-dimensionality appears in a special, but large, class of solutions. They are the "$N$-solitons" which move without change of shape in unbounded domains, or the $N$-phase solutions in the periodic (and quasi-periodic) problem. When performing numerical experiments on an equation which "looks" integrable, these solutions are the ones to look for: they interact non-linearly in a particle-like manner. Another characterization of the $N$-phase (or soliton) solutions is that they are critical points of a linear combination of the first $N+1$ constants of motion (the phases are related to Lagrange multipliers). Elastic curves, extremizing the total squared curvature, are 2-phase solutions. The closed elastica in $R^3$ are special curves: planar circles and torus knots [LS84]. In this chapter we construct $N$-phase curves. The form of the pair of linear systems associated to the Heisenberg Model suggests a simple way to construct the position vector of the curve in terms of the eigenfunction matrix. The eigenfunction for the Lax pair of the Heisenberg Model associated to an $N$-phase solution can be constructed from a set of data on a hyperelliptic Riemann surface with a technique analogous to the one developed by Krichever [Kri77] and extended in Previato [Pre85] to the cubic NLS equation. The effect of normalizing the eigenfunction at a point different from its essential singularity leads to our construction of a gauge transformation which relates HM and NLS.

**Chapter 4.** We get to the "last" (concerning just this introduction) main feature of integrability: Bäcklund transformations. Typically, whenever a system is completely integrable, Bäcklund transformations, conservation laws and an inverse spectral transform are present as three aspects of the same phenomenon [WSK75].

A Bäcklund formula for the Heisenberg Model is derived and applied in this chapter, which is the most experimental. The Bäcklund transformation is used in two contexts: symmetries and instabilities of $N$-phase curves. The theoretical papers of [EFM87a, EFM87b] taught us that two variations of the basic transformation can be applied to a given potential to create solutions with very different features. On one hand an iterated Bäcklund transformation constructs a large family of symmetries of the original solution. On the other hand we can construct homoclinic orbits of higher and higher dimensions. Analytically the only example we were able to work out in complete detail was the Bäcklund transformation of a planar circle. The homoclinic orbits to the planar circle are all immersed knots, with transversal intersections which persist during the time evolution, as many as the number of fundamental instabilities of the original solution. Homoclinic orbits are dynamical separatrices in phase space: the fact that we observe singular knots suggests that they might play a role in distinguishing knots which are not ambiently isotopic.

# Chapter 2 Orthonormal Frames

## 2.1 Introduction

If the filament equation describes the evolution of an arclength parametrized curve in $R^3$, then the continuous Heisenberg Model describes the corresponding evolution of its unit tangent vector. However, in the course of this chapter we will explain why, in order to understand the geometry of the solution space of the HM, we need to consider not just the unit tangent vector, but orthonormal framings of the original curve.

The space of unit tangent vectors of space curves is the 2-dimensional sphere of radius 1. The space of their orthonormal frames is its unit tangent bundle $\mathcal{T}_1S^2$. Then, for a given curve in $R^3$, choosing a frame means choosing a way to lift the spherical curve described by the unit tangent vector into the bundle. This chapter studies two such liftings.

(a) The horizontal lifting associated to the canonical connection on $\mathcal{T}_1S^2$ is the geometrical realization of the gauge transformation from the HM to the cubic NLS equation and of the corresponding shift in Poisson structures. It also suggests where the second Poisson structure for NLS comes from.

(b) The lifting to the Frenet frame of the original curve is a Legendrian submanifold of $\mathcal{T}_1S^2$ with respect to a natural contact structure. This realizes the curve on the sphere as the image of a Legendre map (i.e as a wave front) and enables us to describe its generic singularities. Legendrian curves possess Legendrian invariants; we compute a few of them and relate them to total geodesic curvatures of associated spherical curves.

We organized the chapter in the following way. Sections (2.2.1) and (2.2.2) introduce the circle bundle and its canonical connection. Section (2.2.3) constructs the contact structure in $\mathcal{T}_1S^2$ and proves a result about the correspondence between

Legendrian curves and curves in $R^3$. Section (2.2.4) presents the spherical curve as a wave front and describes its generic singularities. Section (2.2.5) defines the horizontal lifting and interprets the gauge transformation between HM and NLS. Section (2.2.6) compares the Poisson structures of HM and NLS and discusses the origin of the second Poisson structure for NLS. Section (2.2.7) concludes the chapter by computing a few invariants of Legendrian curves.

Before turning our attention to frames we briefly characterize the spherical curves which are of interest to us.

They are described by the tangents of closed differentiable curves in $R^3$, which are parametrized according to arclength. We define the phase space for HM to be the space of loops in the 2-sphere $S^2$, i.e.

$$\mathcal{P} = \left\{ \vec{t} : S^1 \to S^2 \,\middle|\, S \in \mathcal{C}^k(S^1) \right\}. \tag{2.1}$$

The requirement that the original curve in $R^3$ be closed restricts the consideration to those loops on $S^2$ which satisfy the "zero mean condition":

$$\int_0^{2\pi} \vec{t}(u)du = 0. \tag{2.2}$$

This is obtained by writing the position vector $\vec{\gamma}$ of the curve as the antiderivative of $\vec{t}$ with respect to the arclength parameter $s$, $\vec{\gamma}(s) = \int_{s_0}^{s} \vec{t}(u)du$, with $s_0$ a base point on the circle. The condition $\vec{\gamma}(s + 2\pi) = \vec{\gamma}(s)$ for every fixed $s$ becomes condition (2.2) for the unit tangent vector $\vec{t}$.

**Remark:** The space $\mathcal{P}_0$ of zero-mean loops is preserved by the HM dynamics, in fact

$$\frac{d}{dt}\int_0^{2\pi} \vec{t}(u)du = \int_0^{2\pi} \frac{d}{du}\left(\vec{t} \times \frac{d\vec{t}}{du}\right) du = 0.$$

We also observe that simple (i.e. non self-intersecting) closed curves in $\mathcal{P}_0$ possess the following geometric characterization: if $A$ is the area of the portion of the sphere enclosed by $\vec{t}(S^1)$ and $\tau$ is the torsion of the curve in $R^3$ with tangent vector $\vec{t}$, then

$$2\pi = A + \oint_{S^1} \tau(s)ds. \tag{2.3}$$

In order to show this, we need one definition and one theorem.

**Definition 1** *Let $c(x)$ be a regular oriented curve which lies on an oriented surface and $x$ be its arclength parameter. The geodesic curvature of $c$ at $x$ is the length of the tangential component of $c''(x)$ (in other words, it is the covariant derivative of $c'(x)$).*

For an arclength parametrized curve $\vec{t}(x)$ on the sphere, its geodesics curvature can be expressed as $k_g = \vec{t}\,'' \cdot (\vec{t} \times \vec{t}\,')$.

**Theorem 1 (Gauss-Bonnet)** *If $C$ is a closed simple curve on the unit 2-sphere which bounds a region $\mathcal{D}$, and $k_g$ is its geodesic curvature, then*

$$2\pi = \int_{\mathcal{D}} d\omega + \int_{\partial\mathcal{D}} k_g \, dx, \tag{2.4}$$

*where $d\omega$ is the area form on the sphere.*

Let $s$ be the arclength parameter for the corresponding curve in $R^3$.
Then, using the Frenet equations for the frame $(\vec{t}, \vec{n}, \vec{b})$ we obtain $\vec{t}\,' = k\vec{n}\frac{ds}{dx}$, $\vec{t}\,'' = (-k^2\vec{t} + k\tau\vec{b})\left(\frac{ds}{dx}\right)^2$, with $\frac{ds}{dx} = \frac{1}{k}$. Therefore $k_g = \frac{\tau}{k} = \tau\frac{ds}{dx}$ and (2.3) follows. ∎

**Consequence :** The area enclosed by the curve $\vec{t}(S^1)$ is locally preserved by the time evolution (since the total torsion is a conserved quantity).

## 2.2 The Frame Space

The Heisenberg Model describes the evolution of the unit tangent vector to the curve and it also describes the dynamics of its orthonormal frames. For this reason, we take the natural configuration space for orthonormal frames to be the unit tangent bundle to the 2-sphere $T_1S^2$.

In this section we describe two geometrical structures on $TT_1S^2$. The first is the canonical connection, which is a distribution of horizontal planes invariant under the left action of the rotation group $SO(3,R)$. The second is a left-invariant contact structure, i.e. a maximally non-integrable distribution of planes in $TT_1S^2$ which is "adapted" to the Frenet frames of curves (but it is independent of any particular curve!) in the following sense: we show that, for a given a curve in $R^3$, its Frenet frame describes a curve in $T_1S^2$ which has the property of being *Legendrian*, i.e. tangent to the contact distribution at each point.

### 2.2.1 The Circle Bundle of $\mathbf{S}^2$

We start with recalling a few facts about the 2-dimensional sphere and its associated circle bundle. The unit sphere

$$S^2 = \left\{ \vec{x} \in R^3 \mid x_1^2 + x_2^2 + x_3^2 = 1 \right\} \tag{2.5}$$

is an example of a *symmetric homogeneous space*. Given a connected Lie group $G$ and a nontrivial group automorphism $\sigma$ such that $\sigma^2 = \mathrm{Id}$ (involutivity), the associated symmetric homogeneous space is the orbit space $G/H$, where H is the identity component of the set of elements $h \in G$ which are invariant under the action of $\sigma$, i.e. $\sigma(h) = h$.

In the case of the sphere, $G$ is the Lie group $SO(3,R)$ of rotations in $R^3$. Let $T = \begin{pmatrix} \mathrm{I} & \\ & -1 \end{pmatrix}$ and define $\sigma$ to be "conjugation by $T$": $\sigma(g) = T^{-1}gT$, for $g \in SO(3,R)$. Then $H$ is the subgroup of all elements of the form $h = \begin{pmatrix} * & \\ & 1 \end{pmatrix}$ and we have the identification $H \cong SO(2,R)$.

There exists a natural diffeomorphism between $S^2$ and the symmetric homogeneous space $SO(3,R)/SO(2,R)$. In fact, we can construct a transitive action of $SO(3,R)$

on $S^2$ as follows. Let $\{e_1, e_2, e_3\}$ be the standard orthonormal basis in $R^3$, and let us define the map

$$\begin{aligned} \delta : SO(3,R) &\longrightarrow S^2 \\ g &\longrightarrow g \cdot e_3. \end{aligned} \tag{2.6}$$

Since the elements of $H$ fix the vector $e_3$, $\delta$ induces a diffeomorphism between $SO(3,R)/SO(2,R)$ and $S^2$.

Moreover, if $p : SO(3,R) \to SO(3,R)/SO(2,R) \cong S^2$ is the canonical projection, $\sigma$ induces the involution $\sigma_o(px) = p\sigma(x)$, $\forall x \in SO(3,R)$, for which the origin $o = pe$ is an isolated fixed point.

We now define the circle bundle of $S^2$ as the space of all unit tangent vectors to the sphere:

$$T_1S^2 = \left\{(x,v) \ : \ x \in S^2,\ v \in T_xS^2,\ \|v\| = 1\right\}, \tag{2.7}$$

where $\|\cdot\|$ is the euclidean norm in $R^3$.

The following properties will be used later:

1) It is called "circle bundle" because there is a natural action of the unit circle which "fixes" the base point $x$ and rotates $v$ in the tangent plane to $S^2$ at $x$. We have the following smooth map

$$\begin{aligned} S^1 \times T_1S^2 &\longrightarrow T_1S^2 \\ h \cdot (x,v) &\longrightarrow (x, hv). \end{aligned} \tag{2.8}$$

This action is free, i.e. with no fixed points.

2) We introduce the bundle projection $\pi : T_1S^2 \to S^2$, $\pi(x,v) = x$ such that $\pi^{-1}(x) = S^1$.

3) We can always construct a local cross section, i.e. we can find a neighborhood $U$ of each point $x \in S^2$ and a smooth unit vector field $e(U)$ (for example, we can

take the vector field $\frac{\partial}{\partial u}/\|\frac{\partial}{\partial u}\|$ where $(u, v)$ is a local coordinate system). Given such a vector field we define the map $\phi : U \times S^1 \to \pi^{-1}(U)$ by $\phi(x, h) = (x, he(x))$. Since the $S^1$-action is free and $e(x)$ is smooth and never zero in $U$, $\phi$ is smooth and invertible, and so is $\phi^{-1}$.

4) The previous observations define a principal fiber bundle with fiber $S^1$ and base $S^2$. The fact that there does not exist a non-vanishing vector field on $S^2$ (and therefore a global cross-section) says that $T_1S^2$ is a non-trivial bundle.

5) There exists a transitive and free action of $SO(3, R)$ on $T_1S^2$ given by the map $g \cdot (x, v) = (gx, gv)$, $g \in SO(3, R)$, which identifies the space of orthonormal frames with the circle bundle of the 2-sphere: $SO(3, R) \cong T_1S^2$.

### 2.2.2 The Canonical Connection

We summarize the notion of a connection in a principal fiber bundle and the construction of the invariant connection for $T_1S^2$. The content of this section can be found in various references, ([KN63] is comprehensive, see also ch.7 in [ST67] for a discussion of the circle bundle of a 2-dimensional Riemannian manifold).

As in $R^n$ there is a natural way to parallel-translate vectors and to compare tangent vectors at different points, likewise in a general manifold, a choice of a connection prescribes a way to translate tangent vectors "parallel to themselves" and provides us with an intrinsic meaning of directional derivative.
In the case of a principal bundle

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \downarrow{\scriptstyle \pi} \\ & & M \end{array}$$

with structure group $G$ over a manifold $M$, we best explain the role of a connection when thinking of lifting a vector field $v \in TM$ to a vector field $\tilde{v} \in TP$ in a unique way. For each $p \in P$ let $G_p$ be the the subspace of $T_pP$ consisting of all the vectors tangent to the vertical fiber. The lifting of $v$ will be unique if we require

$\tilde{v}$ to lie at each point in a subspace of $T_pP$ complementary to $G_p$. A smooth and $G$-invariant choice of such a complementary subspace is called a "connection" on $P$, more precisely,

**Definition 2** *A connection on $P$ is a smooth assignment of a subspace $H_p \subset T_pP$, $\forall p \in P$ such that:*

*1)* $TP_p = G_p \oplus H_p$

*2)* $H_{gp} = (\mathcal{L}_g)_* H_p, \quad \forall p \in P, \ g \in G$

($\mathcal{L}_g$ is the left-translation in $G$).

Given a connection, the horizontal subspace $H_p$ is mapped isomorphically by $d\pi$ onto $T_{\pi p}M$. Therefore the lifting of $v$ is the unique horizontal $\tilde{v}$ which projects onto $v$.

There is an equivalent way to assign a connection by introducing a Lie algebra valued 1-form $\phi$ (the connection form). If $A \in g$ (the Lie algebra of $G$), let $A^*$ be the vector field on $P$ induced by the action of the 1-parameter subgroup $e^{tA}$. Because the action of $G$ maps each fiber into itself, $A^*$ is tangent to the vertical fiber at each point.

If $X \in T_pP$, $\phi(X)$ is the unique $A \in g$, such that $A^*$ is equal to the vertical component of $X$. Then $\phi(X) = 0$ if and only if $X$ is horizontal.

**Proposition 1** *A connection 1-form $\phi$ has the following properties:*

*(1)* $\phi(A^*) = A$

*(2)* $(\mathcal{L}_g)^*\phi = ad(g)\phi, \quad \forall g \in G$ *(ad is the adjoint representation of $G$).*

The first follows immediately from the definition; for a proof of the second property see [KN63].

We are now ready to construct an invariant connection on $T_1S^2$. As discussed in the previous section, $S^2 \cong SO(3,R)/SO(2,R)$ is a symmetric homogeneous space. We consider the decomposition of the Lie algebra $g = so(3, R)$ given by the automorphism of $g$ induced by $\sigma$ (denoted with the same letter $\sigma$),

$$g = h \oplus k, \tag{2.9}$$

where $h = \{X \in g| \ \sigma X = X\}$ is the subalgebra of the invariant subgroup $H = SO(2, R)$, and $k = \{X \in g| \ \sigma X = -X\}$.

Let $\theta$ be the canonical 1-form of $SO(3, R)$, i.e. the left-invariant $g$-valued 1-form defined by

$$\theta(A) = A, \quad \text{for } A \in g, \tag{2.10}$$

then (see [KN63])

**Theorem 2** *The $h$-component $\phi$ of the canonical 1-form $\theta$ of $SO(3, R)$ defines a left-invariant connection in $T_1 S^2$.*

**Proof:** Because of the left invariance of $\theta$ we can just consider a vector field $X$ on $T_1 S^2$ which is a left-invariant vector field for $SO(3, R)$. Then $\phi(X)$=0 if and only if $X \in k$, and if $Y \in h$ then $\phi(Y) = Y$. Moreover, since $\theta$ is left-invariant under the action of $SO(3, R)$, then it is invariant with respect to the action of its subgroup $H = SO(2, R) \cong S^1$. Therefore $\phi$ has the properties (1) and (2) of a connection form and it is invariant under the full action of $SO(3, R)$. ∎

**Remarks:**

(1) If we identify $g \cong T_e(G)$, the subspace $k$ corresponds to the horizontal subspace at the identity. From the left-invariance of the connection and the definition of $k$ it follows that the connection is invariant under the automorphism $\sigma$, which is an involution of the whole bundle. This will play some role when defining a contact structure on $\mathcal{T} S^2$.

(2) (*Structure equations*) Let $(V, E_1, E_2)$ be the canonical basis for $so(3, R)$ such that $V$ spans the vertical space $h$, and $(E_1, E_2)$ span the horizontal subspace

$k$. We have

$$[V, E_1] = E_2, \quad [V, E_2] = -E_1, \quad [E_1, E_2] = V. \tag{2.11}$$

Setting

$$\theta = \phi V + \omega_1 E_1 + \omega_2 E_2, \tag{2.12}$$

and using the Maurer-Cartan equation, which expresses the exterior derivative of this invariant form as

$$d\theta(X, Y) = -\frac{1}{2}\theta([X, Y]), \quad X, Y \in g, \tag{2.13}$$

we obtain the structure equations for the dual basis $(\phi, \omega_1, \omega_2)$

$$d\phi = -\omega_1 \wedge \omega_2 \tag{2.14}$$

$$d\omega_1 = \phi \wedge \omega_2 \qquad d\omega_2 = -\phi \wedge \omega_1. \tag{2.15}$$

(3) If the Riemannian metric on $S^2$ is the standard one inherited from restricting the euclidean metric in $R^3$ to $S^2$, then the invariant connection constructed above coincides with the Riemannian connection on the frame bundle of $S^2$ (the unique connection which leaves the metric invariant and has zero torsion).
The invariance comes from the left (and right) invariance of the metric in $R^3$. Moreover, since the connection is invariant with respect to the induced involution $\sigma_o$ on $S^2$, also the torsion tensor is invariant under $\sigma_o$. At the fixed point $o$ of the involution $\sigma$ we have

$$T(X, Y) = \sigma_o(T(\sigma_o X, \sigma_o Y)) = -T(-X, -Y) = -T(X, Y), \quad \forall\, X, Y \in T_o S^2.$$

So $T$ is 0 at one point and therefore everywhere. Since the Riemannian connection is unique, it must coincide with the one constructed above.

(4) We can give an invariant definition of the 1-forms $\omega_1$, $\omega_2$ by using the Riemannian structure, the left-invariance of the connection and the isomorphism $d\pi : H_{(x,v)} \to T_x S^2$ between the horizontal subspace $H$ at $(x, v)$ and the tangent

space to the 2-sphere at $x$.

At the point $o$ we can identify $T_oS^2$ with the horizontal subspace $k$. Then $E_1$ and $E_2$ are identified with the orthonormal basis $e_1$ and $ie_1$ of $T_oS^2$, where $ie_1$ is obtained by rotating $e_1$ by $90°$ within the tangent plane.

Then for $t \in T_{(o,e_1)}T_1S^2$, $\omega_1|_{(o,e_1)}(t)$ and $\omega_2|_{(o,e_1)}(t)$ are the components of the projection $d\pi(t)$ relative to the orthonormal basis $(e_1, ie_1)$, we have for example $\omega_1|_{(o,e_1)}(t) =< d\pi(t), e_1 >$. The left invariance of $\omega_{1,2}$ by $S^1$-action allows one to define them for all tangent vectors at $o$, which are of the form $v = he_1$ for some $h \in S^1$. The left invariance of the metric under the full group action defines them everywhere. Finally we have

$$\omega_1|_{(x,v)}(t) =< d\pi(t), v >, \quad \omega_2|_{(x,v)}(t) =< d\pi(t), iv >, \quad t \in T_{(x,v)}T_1S^2. \tag{2.16}$$

### 2.2.3 The Contact Structure and the Legendrian Lifting

We can now define a contact structure on $T_1S^2$ and show that there exists a 1-1 correspondence (up to some trivial symmetries) between curves in $R^3$ with nowhere vanishing curvature and Legendrian curves in the contact manifold $T_1S^2$.

A contact manifold $M$ is an odd-dimensional manifold together with a smooth 1-form $\alpha$ whose kernel defines a maximally non-integrable distribution of hyperplanes in $TM$. This means that there does not exist an integral surface for the distribution, i.e. a surface which is everywhere tangent to a given plane of the distribution. The measure of the degree of non-integrability is given in terms of the exterior derivative of the 1-form $d\alpha$, the non-degeneracy condition is

$$d\alpha \wedge \alpha \neq 0 \tag{2.17}$$

The canonical example to keep in mind is $R^3$ together with the 1-form $\alpha = dz - ydx$. The form $d\alpha \wedge \alpha = dx \wedge dy \wedge dz$ is the volume form in $R^3$ and does not vanish at any point, therefore the kernel of $\alpha$ is a non-degenerate distribution of 2-planes.

In fact, all the contact forms looks like the canonical one in an appropriate local coordinate system; i.e. all contact manifolds locally look alike:

**Theorem 3** *(Darboux) Given a non-degenerate contact form $\alpha$ on a $(2n+1)$-dimensional manifold, there exists a set of local coordinates $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$ and $z$, such that $\alpha$ can be written locally as*

$$\alpha = dz - ydx. \tag{2.18}$$

A distribution of contact planes in $TT_1S^2$ is introduced by means of the 1-form $\omega_2$ which, together with $\omega_1$ and the connection form $\phi$, forms a basis for $T^*T_1S^2$ . We recall its definition: for $y \in T_{(x,v)}T_1S^2$, $d\pi(y)$ is its projection onto $T_xS^2$. Then

$$\omega_2|_{(x,v)}(y) = < d\pi(y), iv > . \tag{2.19}$$

The contact planes are the kernel of $\omega_2$,

$$\mathcal{L}_{(x,v)} = \left\{ y \in T_{(x,v)}T_1S^2 \mid \omega_2(y) = 0 \right\}, \tag{2.20}$$

$\mathcal{L}_{(x,v)}$ contains the tangent line to the vertical space, therefore the contact distribution is everywhere transversal to the horizontal distribution defined by the Riemannian connection. The non-degeneracy property follows from the first structural equation

$$d\omega_2 = -\phi \wedge \omega_1 \tag{2.21}$$

which gives

$$d\omega_2 \wedge \omega_2 = -\varphi \wedge \omega_1 \wedge \omega_2, \tag{2.22}$$

which is non-degenerate since $\omega_1 \wedge \omega_2$ is the pull-back of the area form on $S^2$.

Because $\omega_2$ is a left-invariant form on $TT_1S^2$, the distribution of planes is left-invariant with respect to the action of $SO(3,R)$. We can also look directly at the definition of $\omega_2$: $< \ , \ >$ is an invariant metric, $d\pi$ commutes with the left (and right) action of $SO(3,R)$, therefore $(L_g)^*\omega_2 = \omega_2$, $\forall g \in SO(3,R)$.

We now consider a differentiable curve $\vec{\gamma}(s)$ on $R^3$, such that $s$ is its arclength parameter and such that its curvature is nowhere vanishing, i.e. $\frac{d^2\vec{\gamma}}{ds^2} \neq 0$ at every $s \in [0, 2\pi]$. In this case we can define its Frenet frame $(\vec{t}, \vec{n}, \vec{b})$ which satisfies the Frenet equations

$$\begin{aligned} \frac{d\vec{t}}{ds} &= k\vec{n} \\ \frac{d\vec{n}}{ds} &= -k\vec{t} + \tau\vec{b} \\ \frac{d\vec{b}}{ds} &= -\tau\vec{n} \end{aligned} \tag{2.23}$$

where $k$ and $\tau$ are respectively the curvature and the torsion of $\vec{\gamma}(s)$. The tangent vector $\vec{t}(s)$ to $\vec{\gamma}(s)$ describes a smooth curve $c$ on the surface of $S^2$. We show that $c$ has a unique lifting (up to a discrete involution of $\mathcal{T}_1 S^2$) to a curve $\tilde{c}_L$ in $\mathcal{T}_1 S^2$ which is everywhere tangent to the contact distribution.

**Definition 3** *A Legendrian curve in a contact manifold is a curve which is tangent to a given plane of the contact distribution at each point.*

**Remark:** For a 3-dimensional contact manifold, Legendrian curves are the maximal integral manifolds of the distribution.

**The intuitive argument :** the idea behind our assertion is very simple. The invariant 1-forms $(\phi, \omega_1, \omega_2)$ were defined first at the identity element of $SO(3, R)$, then left-translated by the group action and defined everywhere in $\mathcal{T}_1 S^2$. It is clear that the kernel of $\omega_2$ is given by the left translations of vector fields of the form $U = aV + bE_1$, with some coefficients $a$ and $b$. Representing the elements of the Lie algebra $so(3, R)$ by skew-symmetric matrices, such vector fields have the following form in an appropriate basis

$$U = \begin{pmatrix} 0 & a & 0 \\ -a & 0 & b \\ 0 & -b & 0 \end{pmatrix}. \tag{2.24}$$

We compare the form of $U$ with the form of the matrix entering in the expression of the Frenet equations (2.23) and conclude that the meaning of these equations is that their solution curve in $SO(3, R)$ is Legendrian. On the other hand any left-invariant vector field on $SO(3, R)$ of the same form as $U$ defines the Frenet equations of a curve with curvature $a$ and torsion $-b$. Therefore our Legendrian lifting of the curve described by $\vec{t}$ is nothing but the lifting to the Frenet frame of the corresponding curve $\vec{\gamma}$.

**The formal proof :**

**Existence:** we define the lifted curve to be $\tilde{c}_L(s) = (\vec{t}(s), \vec{n}(s)) \in T_1S^2$, where $\vec{n} = \frac{d\vec{t}}{ds}/\|\frac{d\vec{t}}{ds}\|$ is the unit normal of the curve $\vec{\gamma}$. We need to show that $\tilde{c}_L$ is Legendrian with respect to the contact structure, i.e. that its tangent vector field lies in the distribution of planes. In order to compute the tangent vector to $\tilde{c}_L$ at $\tilde{c}_L(s)$, we work locally in a coordinate patch $U \subset S^2$ and use the map $\phi : U \times S^1 \to \pi^{-1}(U)$ to identify $\pi^{-1}(U)$ with the product space $U \times S^1$. Then there exists a smooth function $\delta : [0, 2\pi] \to R$, such that

$$\tilde{c}_L(s) = (\vec{t}(s), h(s)), \quad \text{with } h(s) = e^{i\delta(s)} \in S^1. \tag{2.25}$$

If $\frac{\partial}{\partial\alpha}$ is the unit tangent vector field on $S^1$ and $r : R \to S^1$ is the exponential map $r(\alpha) = e^{i\alpha}$, then $\frac{dh}{ds} = \frac{d\delta}{ds}dr\left(\frac{d}{ds}\right) = \frac{d\delta}{ds}\frac{\partial}{\partial\alpha}$. Therefore the tangent vector has the following expression

$$\frac{d\tilde{c}_L}{ds}(s) = \left(\frac{d\vec{t}}{ds}(s), \frac{d\delta}{ds}(s)\frac{\partial}{\partial\alpha}\right). \tag{2.26}$$

The second component of $\frac{d\tilde{c}_L}{ds}$ is tangent to the vertical fiber and so is annihilated by the projection $d\pi$. The tangent vector $\vec{t}$ to the curve satisfies the Frenet equation $\frac{d\vec{t}}{ds}(s) = k(s)\vec{n}(s) \in T_{\vec{t}(s)}S^2$, where $k(s)$ is the curvature of $\vec{\gamma}$ at $s$. Therefore we have

$$d\pi\left(\frac{d\tilde{c}_L}{ds}(s)\right) = k(s)\vec{n}(s). \tag{2.27}$$

Since $i\vec{n} = \vec{b}$, the binormal vector to $\vec{\gamma}$, we obtain

$$\omega_2|_{(\vec{t},\vec{n})}\left(\frac{d}{ds}\tilde{c}_L(s)\right) =< k(s)\vec{n}(s), \vec{b}(s) >= k(s) < \vec{n}(s), \vec{b}(s) >= 0. \tag{2.28}$$

This shows that the lifted curve $\tilde{c}_L$ is Legendrian.

**uniqueness**: suppose that there exists another Legendrian lifting $\tilde{c}_1(s) = (\vec{t}(s), \vec{v}(s)) \in T_1S^2$ such that $\pi \circ \tilde{c}_1 = c$. Then, $\vec{v}(s) = e^{i\beta(s)}\vec{n}(s)$ for a smooth function $\beta : [0, 2\pi] \to R$, and

$$\tilde{c}_1(s) = (c(s), e^{i\beta(s)}h(s)) = (c(s), e^{i(\beta(s)+\delta(s))}). \tag{2.29}$$

The tangent vector to $\tilde{c}_1$ at $\tilde{c}_1(s)$ is

$$\frac{d\tilde{c}_1}{ds}(s) = \left(\frac{d\vec{t}}{ds}(s), \left(\frac{d\beta}{ds} + \frac{d\delta}{ds}\right)(s)\frac{\partial}{\partial\alpha}\right), \tag{2.30}$$

so the choice of a different lifting just modifies the vertical component of the tangent vector field and $d\pi$ "sees" the same vector, i.e. $d\pi(d\tilde{c}_1/ds) = d\pi(d\tilde{c}_L/ds) = k\vec{n}$. Moreover, since $\tilde{c}_1$ is Legendrian, we have the equation

$$0 = \omega_2|_{(\vec{t},\vec{v})}(\frac{d\tilde{c}_1}{ds}) = k < \vec{n}, e^{i\beta}i\vec{n} >= k < \vec{n}(s), -\sin(\beta)\vec{n} + \cos(\beta)i\vec{n} >= -k\sin(\beta), \tag{2.31}$$

which is satisfied if $\beta = 0$ or $\pi$. Correspondingly we have the two liftings $\tilde{c}_L$ and $\tilde{c}_1 = (\vec{t}, -\vec{n})$.

**Remark:** $\tilde{c}_L$ and $\tilde{c}_1$ are related by the following involution $j : T_1S^2 \to T_1S^2$, $j(x, v) = (x, -v)$ which preserves the contact structure. In fact, for $t \in T_{(x,v)}T_1S^2$, $j_*\,\omega_2|_{(x,v)}(t) = \omega_2|_{(x,-v)}(dj(t)) =< d\pi o dj(t), -v >= - < d\pi(t), v >= -\,\omega_2|_{(x,v)}(t)$. Therefore, if $t$ belongs the contact distribution, so does $dj(t)$.

This involution is the bundle involution induced by the group automorphism $\sigma$ discussed in the previous section. In fact $\sigma$ acts like the map $v \to -v$ on the vertical fiber which is a subspace of the contact planes.

We choose the lifting corresponding to $\beta = 0$ and call it the "Frenet lifting" of the curve $\vec{\gamma}$ into $T_1S^2$.

On the other hand, if we have a Legendrian curve in $\mathcal{T}_1 S^2$, its velocity field satisfies the Frenet equations (2.23) for some $k$ and $\tau$ and its projection onto $S^2$ can be regarded as the unit tangent vector to some arclength parametrized curve $\vec{\gamma}$ in $R^3$, unique up to a translation. We summarize all these observations in the following

**Proposition 2** *There exists a left-invariant contact structure on the frame space $SO(3, R)$ such that the Frenet lifting of every curve in $R^3$ with nowhere vanishing curvature is a Legendrian curve. Conversely, every Legendrian curve in $SO(3, R)$ descends to a curve in $R^3$, unique up to rigid motions.*

### 2.2.4 A Remark on Legendrian Singularities

In the Legendrian framework we can state what are the generic singularities of the curve described by the unit tangent vector of a space curve. In fact, we can conclude that the curve swept out by $\vec{t}$ is a *wave front*.

A wave front is abstractly defined in the following way. Let $P$ be a contact manifold which is the total space of a Legendrian fibration (all whose fibers are Legendrian submanifolds), let $L$ be a Legendrian submanifold. The projection of $L$ onto the base manifold is called a *Legendrian map* and its image the *wave front* of $L$.

In our case, the total space is the bundle $\mathcal{T}_1 S^2$ with the contact structure associated to the 1-form $\omega_2$. Clearly the $S^1$-fibers are Legendrian submanifolds, in fact they have maximal dimension (1, in the 3-dimensional case) and every vertical vector is annihilated by $d\pi$ and thus belongs to the kernel of $\omega_2$. Therefore $(\mathcal{T}_1 S^2, \omega_2, \pi)$ is a Legendrian fibration and $\pi$ a Legendrian map. From this observation and Proposition 2 we can conclude that

**Corollary 1** *The curve described by the unit tangent vector of an arc-length parametrized curve in $R^3$ is the wave front of a Legendrian map.*

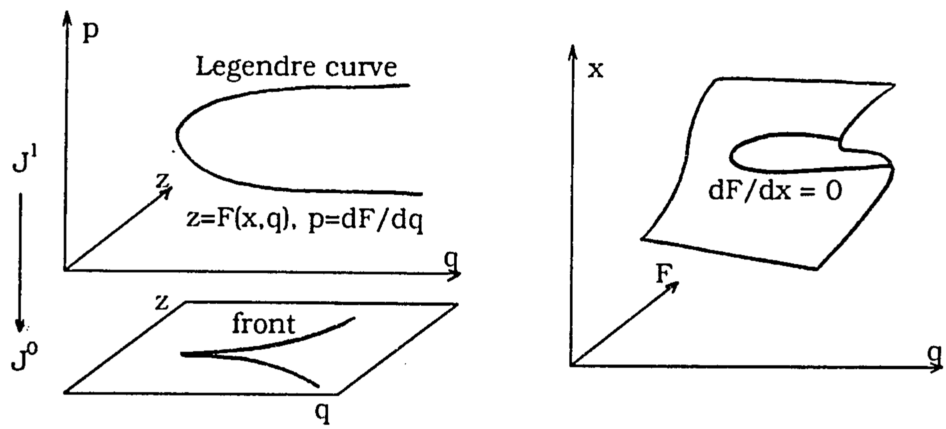


Figure 2.1, The semicubical cusp and its generating function

A singularity of the front is a critical point for the projection $\pi$. In Arnol'd's discussion of Legendre singularities, our situation is the simplest one. We have the following result (see [AN80])

**Proposition 3** *The generic singularities of planar fronts are*

- *semicubic cusps [the Legendrian curve is tangent to the vertical fiber] and*
- *points of transversal self-intersections [the Legendrian curve is branched over the singular point]*

*These singularities are stable to small deformations of the Legendrian curve and of the base manifold.*

Unlike projections of space curves onto a 2-dimensional plane, whose only generic singularities are self-intersections (cusps can be removed by small deformations), cusps appear generically and are stable singularities of Legendrian maps. To explain the cusp formation (and its stability) we can use the fact that locally all contact manifolds look the same and take as a representative an example in which the normal form of the singularity is readily computed. Such an example is the contact manifold $J^1(M,R)$ of all the 1-jets of functions on a manifold $M$. The 1-jet of a function $f$ is the "beginning" of its Taylor series: $f(x) = z + p(x-q) + \dots$ . $J^1(M,R)$ has coordinates $(z,p,q)$ ($z$ is the value of the function, $p$ its derivative at the $q \in M$) and the natural contact structure given by the form $dz - pdq$.Then, the 1-graph of a function $f$, i.e. the set $L = \{z = f(q), p = \frac{df}{dq}\}$ is a Legendrian submanifold. It is also clear that the fibration $\pi : J^1 \to J^0$, given by the projection on the coordinate plane $(z,q)$, is a Legendrian fibration (since the vectors tangent to the vertical fiber belong to the contact planes).

The classification of the singularities of $\pi$ (points where $d\pi$ vanishes) is then reduced to the study of families of curves in the base manifold $(z,q)$. A family of functions on the base manifold $z = F(x,q)$, depending on the parameter $x$, is called a *generating function* for the Legendre submanifold $L$ if

$$L = \{(z,p,q) \mid \exists\, x, s.t.\ z = F(x,q), p = \frac{\partial F}{\partial q}, \frac{\partial F}{\partial x} = 0\}. \tag{2.32}$$

The first two conditions just say that $z = F(x,q)$ is the image of $L$ via the Legendrian projection. The third condition is less intuitive and is related to the stability of the family (with respect to small variations of the parameters). This becomes clear in the following physical picture: suppose that $x$ parametrizes a curve of point light sources in the base manifold $(z,q)$. Let $F(x,q)$ be the time of propagation of the signal from $x$ to $q$. Then the condition for the minimum time of propagation of light to be equal to $z$ is $z = F(x,q), \frac{\partial F}{\partial x} = 0$: this defines the front. For a planar front with a caustic singularity, the generating family is given by $F(x,q) = x^3 + qx$, which is the front of the Legendrian submanifold $L$

$$z = -2x^3, \quad p = x, \quad q = -3x^2. \tag{2.33}$$

$L$ is a smooth curve in the $(z,p,q)$ space, but its projection has a semicubical cusp at the origin (see fig. 2.1). Observe that small perturbations of the generating family will not remove the cusp, which is therefore a generic singularity.

**Remark:** The time evolution causes the front to move, so its singularities may change. Singularities of moving fronts are classified as singularities of fronts one dimension higher (time has been added). In the simplest case of planar fronts, swallowtails appear in space-time as generic singularities.

### 2.2.5 Horizontal Lifting, Natural Curvatures, Gauge Transformation

In this subsection we look at the horizontal lifting of the curve $c(s) = \vec{t}(s)$ to the bundle $T_1S^2$. We will rediscover the natural curvatures introduced in [LP91] and we will give a geometric interpretation of the gauge transformation which relates HM and the cubic NLS and which is described in chapter 4 (section 4.2).

Given the curve $c(s) = \vec{t}(s)$ in $S^2$, we define the unique horizontal lifting $\tilde{c}_o = (\vec{t}, \vec{v})$ with $\vec{v}(0) = \vec{n}(0)$ (so that the Frenet lifting and $\tilde{c}_o$ agree at $s = 0$). Its construction is very similar to the one of the Legendrian lifting. In a local patch, $\vec{v}(s)$ can be written as $\vec{v}(s) = e^{i(\beta(s)+\delta(s))}$ for a smooth real function $\beta(s)$; here $\tilde{c}_L = (\vec{t}, e^{i\delta})$ is the Frenet lifting. The tangent vector field of $\tilde{c}_o$ is

$$\frac{d\tilde{c}_o}{ds} = \left(k\vec{n}, \left(\frac{d\beta}{ds} + \frac{d\delta}{ds}\right)\frac{\partial}{\partial\alpha}\right), \tag{2.34}$$

where $\dfrac{\partial}{\partial\alpha}$ is, as before, the unit tangent vector to the vertical fiber $S^1$.
Then $\tilde{c}_o$ is horizontal (i.e. it lies in the horizontal distribution) if and only if $\dfrac{d\beta}{ds} = -\dfrac{d\delta}{ds}$.
In the remark following Lemma 1 of section 2.2.6 we will show that $\dfrac{d\delta}{ds} = \tau$. We then obtain (taking count of the initial condition)

$$\beta(s) = -\int_0^s \tau(u)du. \tag{2.35}$$

The corresponding horizontal lifting is given by

$$\tilde{c}_o(s) = \left(\vec{t}(s), e^{-i\int_0^s \tau(u)du}\vec{n}\right) = \left(\vec{t}(s), \cos(\int_0^s \tau du)\vec{n} - \sin(\int_0^s \tau du)i\vec{n}\right). \tag{2.36}$$

It defines a new orthonormal framing of the curve

$$\vec{u} = \cos\left(\int_0^s \tau du\right)\vec{n} - \sin\left(\int_0^s \tau du\right)\vec{b} \tag{2.37}$$

$$\vec{v} = \sin\left(\int_0^s \tau du\right)\vec{n} + \cos\left(\int_0^s \tau du\right)\vec{b}. \tag{2.38}$$

This is the "natural frame" defined in [LP91] (see [Bis75] for some history and discussion); it is indeed "natural" in the sense that it corresponds to the lifting of a parallel vector field along $\vec{t}(s)$ to the orthonormal frame bundle with its canonical connection. The natural frame varies along the curve according to the following system of linear equations

$$\begin{aligned} \frac{d\vec{t}}{ds} &= k\cos\left(\int_0^s \tau du\right)\vec{u} + k\sin\left(\int_0^s \tau du\right)\vec{v} \\ \frac{d\vec{u}}{ds} &= -k\cos\left(\int_0^s \tau du\right)\vec{t} \\ \frac{d\vec{v}}{ds} &= -k\sin\left(\int_0^s \tau du\right)\vec{t}. \end{aligned} \tag{2.39}$$

Correspondingly, the components of the projection of the vector field tangent to the horizontal lifting with respect to $\vec{u}$ and $\vec{v}$ are called the "natural curvatures",

$$\begin{aligned} k_u &= \omega_1|_{\tilde{c}_o}\left(\frac{d\tilde{c}_o}{ds}\right) = < k\vec{n}, \cos\left(\int_0^s \tau du\right)\vec{n} - \sin\left(\int_0^s \tau du\right)i\vec{n} >= k\cos\left(\int_0^s \tau du\right) \\ k_v &= \omega_2|_{\tilde{c}_o}\left(\frac{d\tilde{c}_o}{ds}\right) = < k\vec{n}, \cos\left(\int_0^s \tau du\right)i\vec{n} + \sin\left(\int_0^s \tau du\right)\vec{n} >= k\sin\left(\int_0^s \tau du\right). \end{aligned}$$

Now we can interpret the gauge transformation which relates HM and NLS in a geometric way: the procedure described above defines a map from curves in $S^2$ to curves in the complex plane

$$f: \ \vec{t}(s) \longrightarrow \psi(s) = k(s)e^{i\int_0^s \tau du}. \tag{2.40}$$

The real and imaginary component of the complex function $\psi$ are the components of the projection of the horizontal vector field onto $T_{\vec{t}(s)}S^2$. If $\vec{t}$ satisfies the Heisenberg Model equation, then the complex function $\psi(s) = k(s)e^{i\int_0^s \tau du}$ is a solution of the cubic non-linear Schrödinger equation.

Thus, for frozen $t$ the gauge transformation is the composition of the parallel-transport of a vector $\vec{v}(0)$ along the curve $\vec{t}$ with the projection of the tangent vector field onto the tangent space to $S^2$.

**Remarks:**

(a) If we decide to parallel transport a different vector (there is a whole circle of choices) the change is reflected in a constant phase factor. The corresponding complex function is of the form $\psi = ke^{i\int_0^s \tau du + i\beta_0}$ for some real constant $\beta_0$, which is still a solution of NLS. So, making a particular choice means quotienting out the phase symmetry of the NLS solution.

(b) A "natural frame" (a choice of a parallel vector field) is always defined, even when the curvature $k(s)$ vanishes, unlike the Frenet frame. In fact, once the initial vector $\vec{t}(0)$ is chosen, its parallel lifting is unique.

We notice that while the Frenet lifting of a closed curve is always closed, the horizontal lifting need not be. Its holonomy is the element of the fibre $S^1$ which takes the initial value of the lifted curve to its value at $s = 2\pi$. We have

$$\mathrm{hol}(\dot{c}_o) = e^{i\oint \tau du}. \tag{2.41}$$

Therefore the condition for a closed lifting (or for trivial holonomy) is that the torsion must be quantized,

$$\oint \tau du = 2\pi j \text{ for } j \in Z. \tag{2.42}$$

Observe that all the closed liftings of parallel vectors along a solution of HM correspond to periodic solutions of the NLS equation, since $\psi(2\pi) = \psi(0)e^{i\oint \tau du}$; a general lifting corresponds instead to a quasi-periodic solution. In particular, if the curve $\vec{\gamma}$ itself lives on a sphere of radius $r$, its total torsion vanishes [Car76] and therefore it has a periodic "natural frame".

### 2.2.6 Comparison between the HM and NLS Phase Spaces

We have seen that the horizontal lifting of a closed curve in $S^2$ to $T_1S^2$ is a map from the circle into $T_1S^2$ which is "not quite" closed, and that the failure to be closed is measured in terms of the total torsion of the corresponding curve in $R^3$. Since the HM flow leaves the total torsion invariant, we restrict our attention to one of the spaces $\mathcal{L}(S^2)_k = \left\{\vec{t} : S^1 \to S^2 \mid \oint \tau du = 2\pi k\right\}$. In this case every horizontal lifting is a loop in $SO(3, R)$ and the corresponding complex-valued function $\psi$ is periodic.

We need some notation. Let $\mathcal{L}(SO(3, R))$ be the loop space of the group $SO(3, R)$; let $\mathcal{L}_o(SO(3, R))$ be the set of all loops horizontal with respect to the direct vector space decomposition $so(3, R) = h \oplus k$ defined in the previous sections, and let $\Psi = \{\psi : S^1 \to C\}$ be the space of periodic complex-valued functions. Moreover, since the lifting is defined up to the choice of an initial unit tangent vector, we introduce the quotient space $\Psi/S^1$. In what follows we explain the diagram

$$\begin{array}{ccc} \mathcal{L}_o(SO(3,R)) & \xrightarrow{\mathcal{H}_2} & \Psi \\ \uparrow{\scriptstyle \mathcal{H}_1} & & \downarrow{\scriptstyle \pi} \\ \mathcal{L}(S^2)_k & & \Psi/S^1 \end{array}$$

where the maps $\mathcal{H}_1$ and $\mathcal{H}_2$ are defined as follows:

(1) In a local product representation

$$\mathcal{H}_1(\vec{t}) = (\vec{t}, e^{-i\int^s \tau}\vec{n}) \tag{2.43}$$

is the horizontal lifting of the curve described by $\vec{t}$.

(2) For every element $O \in \mathcal{L}_o(SO(3, R))$

$$\mathcal{H}_2(O) = \omega_1\left(O^{-1}\frac{dO}{ds}\right) + i\omega_2\left(O^{-1}\frac{dO}{ds}\right) \tag{2.44}$$

is the complex function built out of the components of the left invariant tangent field of $O$ with respect to the basis $(E_1, E_2)$ of the horizontal subspace $k$.

Firstly, we compute the differential of $d\mathcal{H}_1$ and show that the natural Poisson structure on $\mathcal{L}(SO(3,R))$ induces a Poisson structure on $\mathcal{L}(S^2)$ which is compatible with the Marsden-Weinstein bracket.

Secondly, we compute the differential of the composed map $\pi \circ \mathcal{H} = \pi \circ \mathcal{H}_2 \circ \mathcal{H}_1$ following a procedure contained in [LP91], where a compact expression is given for the differential of the Hasimoto map. We will show that $d\mathcal{H}$ can be expressed in terms the second symplectic operator for the NLS equation, which is now interpreted as coming from the natural Poisson structure on the loop space of $\mathcal{L}(SO(3,R))$. As a consequence we recover that the gauge transformation (the map $\mathcal{H}$) sends the Marsden-Weinstein Poisson structure for HM into the second Poisson structure for NLS, and that there is a corresponding shift in the hierarchy of hamiltonian vector fields.

We recall a few facts about Poisson brackets.

**Definition 4** *A Poisson bracket on a manifold $M$ is a bilinear skew-symmetric operation which endows the space of smooth functions on $M$ with a Lie algebra structure; i.e. it is a bilinear map*

$$\{\ ,\ \}:\ \mathcal{C}^{\infty}(M) \times \mathcal{C}^{\infty}(M) \longrightarrow M \tag{2.45}$$

*which possesses the following properties:*

(*i*) $\{f,g\} = -\{g,f\}$ (*Skew symmetry*)

(*ii*) $\{f,gh\} = g\{f,h\} + \{f,g\}$ (*Leibnitz rule*)

(*iii*) $\{f,\{g,h\}\} + \{h,\{f,g\}\} + \{g,\{h,f\}\}$ (*Jacobi Identity*).

If there is an inner product $< \ ,\ >_m$ on each tangent space $T_mM$, (i.e. $M$ is a Riemannian manifold) then a skew-symmetric linear operator $J(m)$ on tangent vector fields induces a skew-symmetric bilinear map in the following way

$$\{f,g\}(m) = < J\nabla f, \nabla g > (m), \quad m \in M. \tag{2.46}$$

(condition $(ii)$ is then automatically satisfied).

**Remark:** If the operator $J$ is associated with a non-degenerate Poisson bracket defined as in (2.46), then it induces a symplectic structure on $M$ (condition $(iii)$ is equivalent to the closure of the 2-form $< J\cdot\, ,\, >$).

**Examples:**

(a) On $\mathcal{L}(S^2)$ we introduced the Marsden-Weinstein Poisson structure associated with the operator

$$J|_{\vec{t}}X = \vec{t}\times X \qquad \vec{t}\in S^2,\ X\in T_{\vec{t}}\mathcal{L}(S^2), \tag{2.47}$$

and with the inner product

$$< X, Y >_{\mathcal{L}(S^2)} (\vec{t}) = \frac{1}{2\pi}\int_0^{2\pi} < X, Y > (s)ds \tag{2.48}$$

($< X, Y >$ is the scalar product in $R^3$).

(b) On $\Omega = \mathcal{L}(SO(3,R))$ we can define a natural Poisson bracket (see [Pre82]). Let $L(\Omega)\cong \mathrm{Map}(S^1,R)\otimes so(3,R)$ be the Lie algebra of $\Omega$. Then, for $X$, $Y$ in $L(\Omega)$, we introduce the bilinear form

$$S(X,Y) = \frac{1}{2\pi}\int_0^{2\pi} < \frac{dX}{ds}, Y > ds, \tag{2.49}$$

where $< X, Y >= \frac{1}{2}Tr(XY)$ is the standard inner product on $so(3,R)$. Integration by parts shows that $S(\ ,\ )$ is a skew-symmetric form. The Leibnitz rule follows directly from the fact that $J = \dfrac{d}{ds}$ is a derivation. We need to check the Jacobi identity. To this end, we verify that the 2-form $S$ is closed. We compute

$$\begin{aligned} dS(X,Y,Z) &= X\cdot S(Y,Z) + Y\cdot S(Z,X) + Z\cdot S(X,Y) \\ &- S([X,Y],Z) - S([Z,X],Y) - S([Y,Z],X). \end{aligned} \tag{2.50}$$

Since the form is left-invariant, its derivative in the direction of any left-invariant vector-field vanishes, and we are left with

$$dS(X,Y,Z) = \frac{1}{2\pi}\int_0^{2\pi} (< Y', [Z,X] > + < X', [Y,Z] > + < Z', [X,Y] >)\, ds. \tag{2.51}$$

We now use the left-invariance of the inner product on $so(3,R)$ to write

$$
\begin{aligned}
< Y', [Z,X] > &= < Y, [Z,X] >' - < Y, [Z',X] > - < Y, [Z,X'] > \quad (2.52)\\
&= < Y, [Z,X] >' - < Z', [X,Y] > - < X', [Y,Z] > .
\end{aligned}
$$

Inserting this relation into equation (2.51) we find that $dS = 0$.

**The second Poisson structure for HM:** given the map $\mathcal{H}_1$, we compute the pull-back of the natural Poisson structure on $\mathcal{L}(SO(3,R))$ to $\mathcal{L}(S^2)$. In order to do that we need to understand how the action of the differential $d\mathcal{H}_1$ carries vector fields on $\mathcal{L}(S^2)$ to vector fields on $\mathcal{L}(SO(3,R))$.

**Remark:** In general, if $f : M \to N$ is a smooth map, then an explicit formula for its differential can be obtained from the definition of a tangent vector field as a directional derivative along a curve.

Let $g : N \to R$ be a function on $N$, then $g \circ f : M \to R$ is a function on $M$. A vector field $V \in TM$ acts on it and gives the number $V[g \circ f]$ (the directional derivative of $g \circ f$ along $V$). Then the differential of $f$ is defined by the following expression:

$$df(V)[g] = V[g \circ f]. \quad (2.53)$$

In order to compute the differential of $\mathcal{H}_1$ we need to derive some variational formulas.

**Lemma 1** *Consider a family of spherical curves $\vec{t}(w,s) : (-\epsilon,\epsilon) \times S^1 \to S^2$ representing the unit tangent vectors of a family of space curves $\gamma(w,s)$. Let $k$ and $\tau$ be the curvature and torsion of $\gamma(0,s)$. If $W = d\vec{t}/dw\Big|_{(0,s)}$ is the variation vector field along the curve $\vec{t}$ (belonging to $T_{\vec{t}(0,s)}S^2$ at each $s \in S^1$), then the variation of $k$ and $\tau$ along $W$ are*

$$
\begin{aligned}
W(k) &= < \frac{d}{ds} W, \vec{n} > \quad (2.54)\\
W(\tau) &= \frac{d}{ds}\Big(< \nabla_{\vec{n}} W, \vec{b} >\Big) + k < W, \vec{b} > \quad (2.55)
\end{aligned}
$$

**Proof:** Let $V = \frac{d\vec{t}}{ds}\Big|_{(0,s)} = k\vec{n}$ be the velocity field of the curve $\vec{t}$. $V$ commutes with the variation $W$, i.e.

$$0 = [W, V] \equiv \nabla_W(k\vec{n}) - \nabla_{k\vec{n}}W = W(k)\vec{n} + k\nabla_W\vec{n} - k\nabla_{\vec{n}}W. \tag{2.56}$$

Therefore,

$$\nabla_W\vec{n} = \nabla_{\vec{n}}W - \frac{W(k)}{k}\vec{n}, \tag{2.57}$$

and

$$[W, \vec{n}] = -\frac{W(k)}{k}\vec{n}. \tag{2.58}$$

Now we observe that $k^2 =< k\vec{n}, k\vec{n} >$; we differentiate this expression in the direction of $W$, use the commutation relation (2.56) and obtain

$$2kW(k) = 2 < \nabla_W(k\vec{n}), k\vec{n} >= 2k < \nabla_{k\vec{n}}W, \vec{n} >= 2k^2 < \nabla_{\vec{n}}W, \vec{n} > .$$

In summary,

$$\begin{aligned} W(k) &= k < \nabla_{\vec{n}}W, \vec{n} > && (2.59)\\ [W, \vec{n}] &= - < \nabla_{\vec{n}}W, \vec{n} > \vec{n} && (2.60)\\ \nabla_W\vec{n} &= \nabla_{\vec{n}}W - < \nabla_{\vec{n}}W, \vec{n} > \vec{n} =< \nabla_{\vec{n}}W, \vec{b} > \vec{b}. && (2.61) \end{aligned}$$

As for $W(\tau)$ we write $\tau^2 =< \tau\vec{b}, \tau\vec{b} >$, where $\tau\vec{b} = \nabla_{k\vec{n}}\vec{n}$. Using the chain rule and formula (2.56) we compute

$$\begin{aligned} 2\tau W(\tau) &= 2\tau < \nabla_W(\nabla_{k\vec{n}}\vec{n}), \vec{b} > \\ &= 2\tau < \left(\nabla_{k\vec{n}}\nabla_W + \nabla_{[W,k\vec{n}]} + R(W, k\vec{n})\right)\vec{n}, \vec{b} > . && (2.62) \end{aligned}$$

We have used equation

$$\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z = R(X, Y)Z, \tag{2.63}$$

where $R(X, Y)$ is the Riemann curvature tensor. For the unit sphere (which has constant sectional curvature equal to 1) $R(X, Y)$ has the form

$R(X,Y)Z =< Z,Y > X- < Z,X > Y$. Therefore,

$$\begin{aligned} W(\tau) &\overset{(2.63)}{=} < \nabla_{k\vec{n}}\nabla_W\vec{n} + kW- < \vec{n},W > k\vec{n},\vec{b} > \\ &\overset{(2.61)}{=} < \nabla_{k\vec{n}}\left(< \nabla_{\vec{n}}W,\vec{b} > \vec{b}\right),\vec{b} > +k < W,\vec{b} > \\ &= k\vec{n}(< \nabla_{\vec{n}}W,\vec{b} >) + k < W,\vec{b} > \\ &= \frac{d}{ds} < \nabla_{\vec{n}}W,\vec{b} > +k < W,\vec{b} > . \end{aligned} \tag{2.64}$$

∎

**Remark:** In our situation $\vec{n}$ represents a point in the vertical fiber over $\vec{t}$ in $\mathcal{T}_1S^2$. As has been discussed in section 2.2.3, in a local representation we can identify $\vec{n}$ with an element $e^{i\beta}$ of $S^1$ for a smooth real function $\beta$. Then, the action of the vector field $W$ on $\vec{n}$ can be written as $\nabla_W\vec{n} \simeq W(\beta)\frac{\partial}{\partial\alpha}$, where $\frac{\partial}{\partial\alpha}$ is the unit tangent vector field along the vertical fiber. We can compare this expression with formula (2.61) and observe that $\vec{b}$ is a unit vector tangent to the fiber at $\vec{n}$. Then $\vec{b}$ can be identified with $\frac{\partial}{\partial\alpha}$ and we deduce that $W(\beta) =< \nabla_{\vec{n}}W,\vec{b} >$. In particular, if $W = V = k\vec{n}$, the velocity field of the curve, we have $V(\beta) = \frac{d}{ds}\beta = \tau$.

Next we construct the pull-back of the natural Poisson structure on $\mathcal{L}(SO(3,R))$ to $\mathcal{L}(S^2)$. It is defined through the following pulled-back 2-form

$$\Omega|_{\vec{t}}(V,W) = S|_{\mathcal{H}_1(\vec{t})}(\mathrm{d}\mathcal{H}_1(V),\mathrm{d}\mathcal{H}_1(W)), \qquad V,W \in T|_{\vec{t}}\mathcal{L}(S^2). \tag{2.65}$$

We now write the explicit expression of the differential $d\mathcal{H}_1$; we abuse the notation slightly and work again in a local product representation, then

$$\mathcal{H}_1(\vec{t}) = \left(\vec{t}, e^{-i\int^s \tau + i\beta}\right) \in U \times S^1.$$

We use the definition (2.53) (the function $g$ is taken to be the components $\mathcal{H}_1(\vec{t})$ in the product representation) and the remark at the end of Lemma 1 to compute

$$\mathrm{d}\mathcal{H}_1(W) \quad = \quad \left(W, W(e^{-i\int^s \tau + i\beta})\right)$$

$$\begin{aligned}
&= \left(W, (-W(\int^s \tau) + W(\beta))\frac{\partial}{\partial\alpha}\right) \\
&= \left(W, (-\int^s W(\tau) + < \nabla_W \vec{n}, \vec{b} >)\frac{\partial}{\partial\alpha}\right) \qquad (2.66) \\
&\overset{(2.65)}{=} \left(W, (- < \nabla_{\vec{n}} W, \vec{b} > - \int^s k < W, \vec{b} > du + < \nabla_W \vec{n}, \vec{b} >)\frac{\partial}{\partial\alpha}\right) \\
&= \left(W, (-\int^s k < W, \vec{b} > du + c)\frac{\partial}{\partial\alpha}\right)
\end{aligned}$$

with $c$ a constant of integration. Then the pull-back of $S$ to $T\mathcal{L}(S^2)$ has the following expression

$$\begin{aligned}
\Omega(V, W) &= \frac{1}{2\pi}\int_0^{2\pi}\left(< \frac{dV}{ds}, W > + k < V, \vec{b} > \int^s k < W, \vec{b} > du\right) ds \\
&= \frac{1}{2\pi}\int_0^{2\pi} < \frac{dV}{ds} - k\vec{b}\int^s k < V, \vec{b} > du, W > ds. \qquad (2.67)
\end{aligned}$$

We have shown that $\Omega$ is a second Poisson structure on $T\mathcal{L}(S^2)$ associated with the following operator:

$$L_{\vec{t}}(V) = \nabla_{k\vec{n}} V - k\vec{b}\left[\int_{s_0}^s k < V, \vec{b} > du + c\right].$$

**Remark:** $\frac{dW}{ds}$ has in general a component along $\vec{t}$, which we subtracted in the expression of the linear operator ($\frac{dW}{ds} - < \frac{dW}{ds}, \vec{t} > \vec{t} = (\vec{t} \times \frac{d}{ds})(\vec{t} \times W)$ is still a skew-symmetric operator). Equivalently, in the loop space of $SO(3, R)$, the operator $\frac{d}{ds}$ takes a horizontal vector field to a vector field which has in general a vertical component. The Poisson bracket is on the other hand well-defined on the restriction of $\mathcal{L}(SO(3, R))$ to horizontal loops, in fact expression (2.67) shows that the component of $\frac{dW}{ds}$ in the direction of $\vec{t}$ is annihilated when taking the inner product with a vector field tangent to the sphere.

The constant of integration $c$ can be chosen so that $L_{\vec{t}}$ is a skew-symmetric operator. Letting $\mathcal{P}_W(s)$ be any primitive of $k < W, \vec{b} >$, the correct choice is $c = \frac{1}{2}(\mathcal{P}(0) + \mathcal{P}(2\pi))$, corresponding to the operator

$$L_{\vec{t}}(V) = \nabla_{k\vec{n}} V - \frac{1}{2}k\vec{b}\left(\int_0^s + \int_{2\pi}^s\right) k < V, \vec{b} > du. \qquad (2.68)$$

In fact, let $I(W) = \frac{1}{2}k\vec{b}\left(\int_0^s + \int_{2\pi}^s\right) k < W, \vec{b} > du$. We just need to check that $I$ is a skew-symmetric operator with respect to the inner product on $T\mathcal{L}(S^2)$. Integration by parts gives

$$
\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} < V, I(W) > ds = \frac{1}{2\pi}\int_0^{2\pi} k < V, \vec{b} > \left(\int_0^s + \int_{2\pi}^s\right) k < W, \vec{b} > du\, ds \\
= \;& \frac{1}{2\pi}\mathcal{P}_V(s)\left(\int_0^s + \int_{2\pi}^s\right) k < W, \vec{b} > du\Big|_0^{2\pi} - \frac{1}{2\pi}\int_0^{2\pi} k < W, \vec{b} > \mathcal{P}_V(s)\, ds \\
= \;& \frac{1}{2\pi}\int_0^{2\pi} k < W, \vec{b} > (\mathcal{P}_V(2\pi) + \mathcal{P}_V(0))ds - \frac{1}{2\pi}\int_0^{2\pi} k < W, \vec{b} > \mathcal{P}_V(s)\, ds \\
= \;& -\frac{1}{2\pi}\int_0^{2\pi} k < W, \vec{b} > \left(\int_0^s + \int_{2\pi}^s\right) k < V, \vec{b} > du = -\frac{1}{2\pi}\int_0^{2\pi} < I(V), W > ds
\end{aligned}
$$

In order to understand the relation between the two Poisson operators $J_{\vec{t}}$ and $L_{\vec{t}}$, we observe that we can rewrite the right-hand side of the Heisenberg Model equation with respect to two different hamiltonian decompositions. In fact, let

$$
V_0 = \frac{\frac{\partial^2 \vec{t}}{\partial s^2} \times \frac{\partial \vec{t}}{\partial s} \cdot \vec{t}}{\frac{\partial \vec{t}}{\partial s} \cdot \frac{\partial \vec{t}}{\partial s}}\vec{t} - \vec{t} \times \frac{\partial \vec{t}}{\partial s} = -\tau\vec{t} - k\vec{b} \tag{2.69}
$$

$$
V_1 = \frac{\partial^2 \vec{t}}{\partial s^2} = -k^2\vec{t} + k_s\vec{n} + k\tau\vec{b} \tag{2.70}
$$

be the gradients of the first two conserved quantities (see [FA80] for a derivation of the generating function of the conservation law by means of the inverse scattering transform)

$$
I_0 = \frac{1}{2\pi}\int_0^{2\pi} \frac{\frac{\partial^2 \vec{t}}{\partial s^2} \times \frac{\partial \vec{t}}{\partial s} \cdot \vec{t}}{\frac{\partial \vec{t}}{\partial s} \cdot \frac{\partial \vec{t}}{\partial s}} ds = \frac{1}{2\pi}\int_0^{2\pi} \tau ds \tag{2.71}
$$

$$
I_1 = \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial \vec{t}}{\partial s} \cdot \frac{\partial \vec{t}}{\partial s} ds = \frac{1}{2\pi}\int_0^{2\pi} k^2 ds. \tag{2.72}
$$

Then we can write the evolution equation as

$$
\frac{d\vec{t}}{dt} = \vec{t} \times L_{\vec{t}}(\vec{t} \times V_0) \tag{2.73}
$$

$$
= J_{\vec{t}} V_1. \tag{2.74}
$$

In other words, the operators $J_{\vec{t}}$ and $K_{\vec{t}} = (J \cdot L \cdot J)_{\vec{t}}$ have a hamiltonian vector field in common. In Appendix A, we check that $J_{\vec{t}}$ and $K_{\vec{t}}$ are compatible Poisson operators in the sense described by Magri [Mag78] (i.e. $J + K_{\vec{t}}$ defines a Poisson bracket). As a consequence, $J_{\vec{t}}$ and $K_{\vec{t}}$ have in common an infinite sequence of hamiltonian vector fields $\{J_{\vec{t}} V_k\}$ which are associated with an infinite family of conservation laws. We summarize this discussion in the following

**Proposition 4** *The Poisson operator*

$$K_{\vec{t}} V = \nabla_{k\vec{n}} V + \frac{1}{2} k\vec{n} \left( \int_0^s + \int_{2\pi}^s \right) < V, k\vec{n} > du, \tag{2.75}$$

*and the Marsden-Weinstein Poisson operator define a bihamiltonian structure for the HM equation. The corresponding recursion operator*

$$R_{\vec{t}} = K_{\vec{t}} J_{\vec{t}}^{-1} \tag{2.76}$$

*generates the hierarchy of commuting hamiltonian vector fields*

$$J_{\vec{t}} V_k = (R_{\vec{t}})^k J_{\vec{t}} V_0. \tag{2.77}$$

**The relation with NLS:** next we compute the differential of the map $\pi \circ \mathcal{H}$ and show its relation to the second Poisson structure for NLS. We first recall the following results which can be found in [Mag78] and [FA80]:

**Theorem 4** *There exist two compatible symplectic operators for the periodic NLS equation:*

$$\begin{aligned} \tilde{J}\phi &= i\phi & (2.78) \\ \tilde{K}_\psi \phi &= \frac{d\phi}{ds} + \frac{1}{4}\psi \left( \int_0^s + \int_{2\pi}^s \right) [\phi\bar{\psi} - \bar{\phi}\psi] du, & (2.79) \end{aligned}$$

*with respect to the inner product*

$$< \phi_1, \phi_2 >|_\psi = \frac{1}{2\pi} \int_0^{2\pi} (\phi_1 \bar{\phi}_2 + \bar{\phi}_1 \phi_2) du \quad \phi_1, \phi_2 \in T_\psi \Psi. \tag{2.80}$$

*The associated recursion operator $\tilde{R} = \tilde{K}_\psi \tilde{J}^{-1}$ generates the following infinite hierarchy of Poisson structures*

$$\{f, g\}_n = < \tilde{R}^n \tilde{J} \nabla f, \nabla g >, \quad f, g \in \Psi. \tag{2.81}$$

Using once more the chain rule and the variational formulas for $k$ and $\tau$, we compute the differential of $\mathcal{H}$ and obtain

$$\begin{aligned} d\mathcal{H}(W) &= W(k)e^{i\int^s \tau du} + ik\left(\int^s W(\tau)du\right)e^{i\int^s \tau du} \\ &= \left(< \frac{dW}{ds}, \vec{n} > +i < \frac{dW}{ds}, \vec{b} >\right)e^{i\int^s \tau du} + i\psi \int^s < W, k\vec{b} > du + ic\psi \end{aligned} \tag{2.82}$$

$c$ is a constant of integration, and since the kernel of $d\pi$ contains vector fields of the form $c\psi$, we can write equivalently

$$d\mathcal{H}(W) = \left(< \frac{dW}{ds}, \vec{n} > +i < \frac{dW}{ds}, \vec{b} >\right)e^{i\int^s \tau du} + \frac{i}{2}\psi\left(\int_0^s + \int_{2\pi}^s\right) < W, k\vec{b} > du. \tag{2.83}$$

We write $W = g\vec{u} + h\vec{v}$ on the basis of the natural frames and introduce the complex vector field $\eta(W) = g + ih$. Using the evolution equation (2.39) for the components of the natural frame, we can rewrite the above expression as

$$d\mathcal{H}(W) = \frac{d}{ds}\eta(W) + \frac{1}{4}\psi\left(\int_0^s + \int_{2\pi}^s\right)[\eta(W)\bar{\psi} - \bar{\eta}(W)\psi]du. \tag{2.84}$$

The right-hand side is nothing but the second Poisson operator for the NLS equation $\tilde{K}_\psi$ acting on the tangent vector $\eta(W)$; we can then write the following formula for the differential of $h = \pi \circ \mathcal{H}$:

$$dh = d\pi \circ \tilde{K}_{h(\vec{t})} \circ \eta. \tag{2.85}$$

This expression contains two pieces of information. On one hand, given our previous discussion about the origin of the second Poisson structure for HM, we find a connection between the second Poisson structure for NLS and the natural Poisson bracket on $\mathcal{L}(SO(3,R))$. Moreover, with a few more steps we can show that (2.85) realizes the known fact that there is a shift in Poisson structures between HM and NLS (see for example [FA80]).

We check that $h$ is a Poisson map and carries the Marsden-Weinstein Poisson bracket into the second one for NLS. Two formulas are needed (their derivation is a one line computation, in a similar context see [LP91]:

$$\nabla(f\circ h)|_{\vec{t}} = dh^*(\nabla f)|_{h(\vec{t})}, \quad f\in\Psi/S^1 \tag{2.86}$$

$$\eta\circ J\circ\eta^* = \tilde{J}, \tag{2.87}$$

where $^*$ indicates the adjoint with respect to the inner product. The shift in Poisson structures (see (2.81)) is shown by the following computation:
let $f,g\in\Psi/S^1$, then

$$\begin{aligned}
&\{f\circ h, g\circ h\}(\vec{t}) \equiv < J\nabla(f\circ h), \nabla(g\circ h) > (\vec{t}) && (2.88)\\
=\ & < Jdh^*\nabla f, dh^*\nabla g > (h(\vec{t}))\\
=\ & < dhJdh^*\nabla f, \nabla g > (h(\vec{t}))\\
=\ & < d\pi\circ \hat{K}\phi J\phi^*\circ\hat{K}^*\circ d\pi^*\nabla f, \nabla g > (h(\vec{t}))\\
=\ & < \hat{K}\tilde{J}\hat{K}^* d\pi^*\nabla f, d\pi^*\nabla g > (h(\vec{t}))\\
=\ & < \hat{R}^2\nabla(f\circ\pi), \nabla(g\circ\pi) > (\mathcal{H}(\vec{t}))\\
=\ & \{f\circ\pi, g\circ\pi\}_2(\mathcal{H}(\vec{t})) = \{f,g\}_2(h(\vec{t})). && (2.89)
\end{aligned}$$

Thus, we can view NLS and HM as the same hamiltonian system, but with respect to two different Poisson structures which belong to the same hierarchy.

### 2.2.7 Invariants of Legendrian Curves

An invariant of a Legendrian curve is an integer-valued function whose value does not change if the curve moves staying Legendrian.

The Legendrian liftings of the curve $\vec{t}\in S^2$ remain Legendrian during the time evolution. Therefore any of their invariants is preserved in time until the projection of the Legendrian curve onto the base manifold acquires a singularity. As a consequence, a Legendrian invariant of a lifting of $\vec{t}$ to $T_1S^2$ is an invariant of the corresponding curve in $R^3$.

Next we compute the Maslov index of the Frenet lifting of $\vec{t}$. We will define it and place it in a more general context while we compute it for the special case that interests us.

Let us consider the manifold $T_1S^2 \cong SO(3,R)$, together with its tangent vector fields $(E_1, E_2, V)$ and the associated dual basis of 1-forms $(\omega_1, \omega_2, \phi)$ introduced in Section (2.2). The structure equations (2.14),(2.15) guarantee that each element of the dual basis defines a contact structure on $T_1S^2$.
The contact distribution of 2-planes associated with $\omega_2$ is given at each point by $B_1 = span\{(E_1, V)\}$. Then $(B_1, d\omega_2)$ is a symplectic vector bundle of rank 2, i.e. a vector bundle $B_1 \longrightarrow T_1S^2$ together with a symplectic form on each fiber, which varies differentiably with the base point.
Let now $c$ be a Legendrian curve, and let $c'$ be its unit tangent vector field. Then $L_1 = span\{c'\}$ is a Lagrangian subbundle of $B_1|_c$ (it is of maximal dimension 1 and therefore the symplectic form $d\omega_2$ vanishes on it). The subspace generated by $E_1$ defines another Lagrangian subbundle $L_0 = span\{E_1\}|_c$.

Given two Lagrangian subbundles $L_0$, $L_1$ of a symplectic vector bundle $(M, E, \omega)$, we associate an invariant class and an integer in the following way (see [Vai87]):
Let $J$ be a complex structure on $E$ which induces a positive metric
$g(X,Y) = \omega(JX,Y)$. Then, in a local trivialization $U \subset M$ of the bundles $L_0, L_1$, we can define two fields of unitary frames

$$u_k^a = \frac{1}{\sqrt{2}}(e_k^a - iJe_k^a), \quad a = 0, 1 \tag{2.90}$$

where $\{e_k^a\}$ is an orthonormal basis of $L_a, a = 0, 1$ with respect to $g$. Then the two subbundles are related through their unitary frames

$$u^1 = u^0 A_U, \tag{2.91}$$

with $A_U$ a unitary matrix. Taking a covering $\{U\}$ of $M$, since a change of orthonormal bases is realized by multiplication by an orthonormal matrix, we obtain a total mapping $A(L_0, L_1) : M \to U(n)/O(n)$. The mapping $A(L_0, L_1)$ has the following invariant

$$\phi(L_0, L_1) = Det^2(A(L_0, L_1)) : M \to S^1 \tag{2.92}$$

($Det^2$ is the square of the determinant, well defined since orthogonal matrices have determinant $\pm 1$). The degree of the map $\phi(L_0, L_1)$ associates an invariant to any closed curve in $M$, we state the following result (for the proof see [Vai87]):

**Proposition 5** *The Maslov class of* $(L_0, L_1)$

$$m(L_0, L_1) = \phi^*(L_0, L_1)\left(\frac{dz}{2\pi i z}\right) \in H^1(M, R), \tag{2.93}$$

*and the Maslov index of a curve* $c : S^1 \to M$

$$m^c_{L_0 L_1} = \int_c m(L_0, L_1) = deg\,(\phi(L_0, L_1), c) \tag{2.94}$$

*are respectively an integral cohomology class and an integer number independent of the complex structure* $J$.

Back to our example: here the Lagrangian submanifolds are defined by the Legendrian curve $c$, therefore its Maslov index will be an integral invariant of the curve itself. On $B_1$ we define the complex structure

$$JE_1 = V, \qquad JV = -E_1. \tag{2.95}$$

Then $d\omega_2(J\cdot, \cdot)$ defines a positive metric on $B_1$, (it can be checked directly using the expression $d\omega_2 = -\phi \wedge \omega_1$). The orthonormal bases for $L_0$ and $L_1$ are the vectors $c' = \cos(\theta)E_1 + \sin(\theta)V$ and $E_1$ respectively. Then $Jc' = -\sin(\theta)E_1 + \cos(\theta)V$, $u^1 = \frac{1}{\sqrt{2}}(E_1 - iV)$ and

$$u^0 = e^{i\theta}u^1. \tag{2.96}$$

Therefore, the Maslov index of the curve $c$ is twice the rotation number of the angle $\theta$. For the Frenet lifting $c_f = (\vec{t}, \vec{n})$, with $\frac{d}{ds}c_f = (kn, \tau\frac{\partial}{\partial\alpha})$, we obtain

$e^{i\theta} = \dfrac{k+i\tau}{\sqrt{k^2+\tau^2}}$. Twice its degree is the Maslov index of $c_f$:

$$m(c_f) = 2\int_0^{2\pi} \frac{d}{dx}\arctan(\tau/k)dx. \tag{2.97}$$

**Remark:** We observe that the Maslov index is twice the total geodesic curvature of the spherical curve described by the normal vector $\vec{n}$, which is now seen to be a Legendrian invariant.

Similarly, we can compute the Maslov index of a closed horizontal lifting $c_o = (\vec{t}, e^{-i\int \tau du}\vec{n})$ of a curve with $\oint \tau du = 2\pi k$. An analogous computation (the symplectic vector bundle is now $span\{(E_1, E_2)\}$), shows that that the Maslov index of $c_o$ is twice the total torsion

$$m(c_o) = 2\int_0^{2\pi} \tau du, \tag{2.98}$$

and that the quantization of the Maslov index coincides with the condition for the lifting to be closed. In summary,

**Proposition 6** *The Maslov index of a closed Legendrian curve in the contact manifold $(T_1S^2, \omega_2)$ is twice the total geodesic curvature of the unit normal of the corresponding curve in $R^3$.*

*The Maslov index of a closed Legendrian curve in the contact manifold $(T_1S^2, \omega_1)$ is twice the total torsion of the associated curve in $R^3$.*

*Both are invariants of the curve in $R^3$ and are preserved by the HM flow. In particular, the total torsion is preserved for all times, while the total geodesic curvature is locally preserved since $k$ can vanish during the evolution.*

# Chapter 3 Construction of N-phase Curves

## 3.1 Introduction

In this chapter we consider the class of $N$-phase solutions of the HM.

From the point of view of the integrable theory they are the analogues of the $N$-solitons in the case of quasi-periodic boundary conditions. They are dense in the space of all periodic potentials; they lie on low-dimensional sets which topologically are $N$-tori in function space and which foliate the phase space in a way similar to a finite-dimensional integrable system. Moreover formulae for $N$-phase solutions can be constructed explicitly.

The corresponding curves are also interesting from a geometric point of view. They are critical points of the geometric invariants $\int_0^{2\pi} \tau$, $\int_0^{2\pi} k^2$,$\int_0^{2\pi} k^2\tau$, ... , and they promise to be rather special curves. In the trivial case of 1-phase curves, we have planar circles. The next case is much more interesting: the critical points of the total squared curvature are the elastic curves. The closed elastica in $R^3$ have been classified by [LS84], who showed that they lie on tori of revolution and belong to the class of torus knots. The evolution of an elastic curve under the filament flow is trivial: it is the composition of a rigid translation and a slide motion along itself (which therefore does not change the knot type).

In the next section we construct a large family of quasi-periodic solutions of HM and the corresponding curves using methods of algebraic geometry. We find the construction itself interesting, since it gives another way to interpret the gauge transformation between the HM and the NLS equation.

The chapter is structured in the following way: section (3.2) deals with the construction of $N$-phase curves. We start with deriving the expression of the position vector in terms of the eigenfunction of the associated linear problem (section (3.2.1)). Section (3.2.2) reviews a method of construction of the eigenfunction which uses Riemann surface techniques. Section (3.2.3) discusses the normalization of the eigenfunction. Section (3.2.4) constructs the $N$-phase curves.

In the second part of this chapter we discuss the role of the squared eigenfunctions. Section (3.3) relates the squared eigenfunctions to the Frenet frame of the curve. We also construct an infinite hierarchy of commuting flows, and write down the generating function for the conserved quantities.

## 3.2 N-Phase Curves

We will be working with matrices rather than vectors. The unit tangent vector $(t_1, t_2, t_3)^T$ is represented by the hermitian matrix

$$S = \begin{pmatrix} t_3 & t_1 - it_2 \\ t_1 + it_2 & -t_3 \end{pmatrix}, \quad S^2 = Id. \tag{3.1}$$

Then the HM is rewritten in the following form

$$\frac{\partial S}{\partial t} = \frac{1}{2i}[S, \frac{\partial^2 S}{\partial x^2}]. \tag{3.2}$$

Equation (3.2) is the compatibility condition of the following pair of linear systems

$$\frac{\partial \vec{F}}{\partial x} = \mathrm{i}\lambda S\vec{F} \tag{3.3}$$

$$\frac{\partial \vec{F}}{\partial t} = (2\mathrm{i}\lambda^2 S - \lambda\frac{\partial S}{\partial x}S)\vec{F}, \tag{3.4}$$

where $\vec{F}$ is an auxiliary complex-valued vector function and $\lambda$ is the spectral parameter.

### 3.2.1 The Reconstruction of the Curve.

Instead of taking the anti-derivative of the tangent vector, we derive a formula for the position vector of the curve in terms of the eigenfunction of the associated linear problem. In the context of $N$-phase solutions, this provides one with a very direct way to obtain the curve. A similar formula (involving the NLS eigenfunction) can be found in [A.S88].

We start with the following result:

**Proposition 7** *The solution of the linear problem*

$$\frac{\partial \vec{F}}{\partial x} = i\lambda S\vec{F} \tag{3.5}$$

*is analytic in $\lambda \in \boldsymbol{C}$ for an analytic initial condition $\vec{F}(x=0)$.*

The conclusion follows from standard ODE theory. Below we write a proof which makes use of the boundedness of the matrix $S$.

**Proof:** Given the analytic initial data $\vec{F}(0) = \sum_{i=0}^{\infty} \vec{F}_i^{(0)} \lambda^i$ we construct the solution of the following sequence of coupled linear systems using Picard's iteration,

$$\begin{aligned} \frac{d\vec{G}_k}{dx} &= iS\vec{G}_{k-1} \\ \vec{G}_0 = \vec{F}_0^{(0)} \quad &, \quad \vec{G}_k(0) = \vec{F}_k^{(0)}, \quad k = 1, 2, \dots . \end{aligned} \tag{3.6}$$

The outcome is the following formal series for the solution $\vec{F}$ of (3.5),

$$\vec{F}(x,\lambda) = \left(\sum_{i=0}^{\infty} \lambda^i R_i\right) \vec{F}(0), \tag{3.7}$$

with

$$R_0 = Id, \ R_k = (i)^k \int_0^x S(s_1) \int_0^{s_1} S(s_2) \dots \int_0^{s_{k-1}} S(s_k)\, ds_k \dots ds_1. \tag{3.8}$$

$S$ is bounded in norm, for example in the $l^1$-norm

$$\|S\|_1 = \max_{k=1,2} \sum_{i=1}^{2} |S_{ik}| = \sqrt{s_1^2 + s_2^2} + |s_3| \le 1 + 2\sqrt{2}. \tag{3.9}$$

For $\|S\|_1 \le C$, $C$ some positive constant, we have

$$\|R_k\|_1 \le C^k \int_0^x \int_0^{s_1} \dots \int_0^{s_{k-1}} ds_k \dots ds_1 = \frac{C^k x^k}{k!}, \tag{3.10}$$

and

$$\left\| \sum_{i=0}^{\infty} R_i \lambda^i \right\|_1 \le \sum_{i=0}^{\infty} \|R_i\|_1 |\lambda|^i \le \sum_{i=0}^{\infty} \frac{C^i |\lambda|^i x^i}{i!} = \exp(C|\lambda| x), \tag{3.11}$$

which shows the convergence of the formal series and thus the analyticity of the eigenfunction.

∎

Let $\Phi$ be the fundamental solution matrix of (3.5); because it is analytic with respect to the eigenvalue parameter, we can differentiate both sides of the linear system with respect to $\lambda$ and evaluate them at $\lambda = 0$, obtaining the following formula for $S$:

$$S = -i \frac{\partial}{\partial x} \frac{\partial \Phi}{\partial \lambda}\bigg|_{\lambda=0} \Phi|_{\lambda=0}^{-1}. \tag{3.12}$$

This expression is a perfect derivative since the eigenfunction at $\lambda = 0$ is constant in $x$ and $t$, so we can integrate it with respect to $x$ and obtain the hermitian matrix $\Gamma$ which represents the position vector of the curve,

$$\Gamma \stackrel{\text{def}}{=} \int^x S(s)ds = -i \frac{\partial \Phi}{\partial \lambda}\bigg|_{\lambda=0} \Phi|_{\lambda=0}^{-1}. \tag{3.13}$$

We have just discovered that both the position vector of the curve and its tangent vector can be expressed in terms of the eigenfunction matrix of the associated linear problem and its $x$-derivative. In the next section we construct the fundamental solution of the linear system for the Heisenberg Model in the space of quasi-periodic functions. We follow a construction by Krichever which uses methods of algebraic geometry.

### 3.2.2 The Baker-Akhiezer Function

Quasi-periodic solutions are associated to a set of data on a Riemann surface. We start with a hyperelliptic Riemann surface $\Sigma$ of genus $g$ described by the equation

$$y^2 = \prod_{i=1}^{2g+2} (\lambda - \lambda_i). \tag{3.14}$$

We mark the two points $\infty_+$, $\infty_-$, which are permuted by the involution $\tau(\lambda, y) = (\lambda, -y)$ exchanging the two sheets.

We also choose a set of $g+1$ distinct points $\mathcal{D} = \{P_1, \ \dots P_{g+1}\}$ placed in a generic position (a non-special divisor) and not containing $\infty_\pm$.

Let $\lambda(P) : \Sigma \to \mathbb{C} \cup \{\infty\}$ be the hyperelliptic projection to the Riemann sphere; it can also be seen as one of the meromorphic functions on $\Sigma$ whose pole divisor is $\infty_+ + \infty_-$. In neighborhoods of $\infty_\pm$ we choose the local parameters $k_\pm$ such that

$$(k_\pm)^{-1} = \pm(\lambda(P))^{-1} \tag{3.15}$$

at $\infty_\pm$ respectively.

The main idea of Krichever is to construct a function $\psi(\lambda)$ on $\Sigma$ which is uniquely defined by a prescribed behavior at its singularities and which turns out a posteriori to solve to a pair of commuting linear systems. The compatibility condition, i.e. the zero curvature representation of these two linear operators will be a completely integrable non-linear equation for the coefficients. So, the construction of such a function provides one both with the non-linear equation, an initial condition and its solution.

**Definition 5** *A Baker-Akhiezer function associated to* $(\Sigma, \mathcal{D}, \infty_\pm)$ *is a function* $\psi(\lambda)$ *which:*

- *is meromorphic everywhere except at* $\infty_\pm$ *and whose poles on* $\Sigma \backslash \{\infty_\pm\}$ *are contained in* $\mathcal{D}$*,*
- *has an essential singularity at* $\infty_\pm$ *which locally is of the form* $\psi(k) \sim const \exp(p(k))$*, with* $p(k)$ *an arbitrary polynomial with complex coefficients.*

For our purposes, we recall the following result ([Kri77]),

**Proposition 8** *Suppose that the following technical condition holds:*

*Condition A: The divisor* $P_1 + \cdots + P_{g+1} - \infty_+ - \infty_-$ *is not linearly equivalent to a positive divisor.*

*Then, if $p(k) = ikx + Q(k)t$, and $x$ and $t$ are complex parameters with $|x|, |t|$ sufficiently small and $Q(k)$ is a given polynomial, the linear vector space of Baker-Akhiezer functions associated to $(\Sigma, \mathcal{D}, k_\pm)$ is 2-dimensional and it has a unique basis $\psi^1, \psi^2$ with the following normalized expansion at $\infty_\pm$:*

$$\psi^j_\pm(x,t,\lambda) = \exp(ik_\pm x + Q(k_\pm)t)\left(\sum_{n=0}^{\infty} \zeta_n^{j\pm}(x,t)k_\pm^{-n}\right), \quad j = 1,2 \tag{3.16}$$

*with $\zeta_0^{1+} = 1$, $\zeta_0^{1-} = 0$, $\zeta_0^{2+} = 0$, $\zeta_0^{2-} = 1$.*

**Remarks:**

We make two remarks before proving the proposition.

1) For a non-special divisor of degree $d$ (a formal integer linear combination of points on $\Sigma$ counted with multiplicity) the Riemann-Roch formula (see for example [GH78] for the proof), states that the dimension $h^0(\mathcal{D})$ of the linear space of meromorphic functions on $\Sigma$ whose pole divisor lies in $\mathcal{D}$ is

$$h^0(\mathcal{D}) = \begin{cases} 1, & d \leq g, \\ d-g+1, & d > g. \end{cases} \tag{3.17}$$

2) Condition A means that there exists no non-constant meromorphic function with pole divisor $P_1 + \ldots + P_{g+1}$ which vanishes simultaneously at $\infty_+$ and at $\infty_-$.

**Proof:**

(1) Uniqueness: Suppose there are two functions $\Psi_1$ and $\Psi_2$ which satisfy the prescription; then their ratio $\Psi_1/\Psi_2$ is a meromorphic function (the essential singularities mutually cancel) whose poles are contained in the zero divisor of $\Psi_2$. The condition of "non-speciality" assures that the dimension of the space of such functions is $2(= g+1-g+1$, according to the Riemann-Roch formula). For the normalization, since $h^0(\mathcal{D}-\infty_+) = h^0(\mathcal{D}-\infty_-) = 1$, we can choose two such functions vanishing at $\infty_+$ and at $\infty_-$ respectively; because $h^0(\mathcal{D}-\infty_+-\infty_-) = 0$ (Condition A), they must be independent.

(2) Existence: We follow an argument given in ([Pre85]) and exhibit a pair of independent functions on $\Sigma$ with the correct singularities. Functions on a Riemann surface are built as ratios of Riemann Theta functions (a nice survey of the features which are relevant in this context is in ([Dub81]). Let

$$a_1, \ \dots, a_g, \quad b_1, \ \dots, b_g, \tag{3.18}$$

be an arbitrary homology basis for the Riemann surface $\Sigma$ and

$$\omega_1, \ \dots \omega_g, \qquad \oint_{a_i} \omega_j = \delta_{ij}, \quad i, j = 1, \ \dots, \ g \tag{3.19}$$

be $g$ normalized holomorphic differentials.

We construct the period matrix $B$,

$$B_{ij} = \oint_{b_i} \omega_j, \quad i, j = 1, \ \dots, g \tag{3.20}$$

which defines the Riemann Theta function

$$\theta(z) = \sum_{n \in \mathcal{Z}^g} \exp i\pi \left(< n, Bn > +2 < n, z >\right), \quad z \in \mathbb{C}^g, \tag{3.21}$$

and we choose a base point $P_0 \in \Sigma$.

The essential behavior at $\infty$ is introduced by means of the unique normalized differentials of the second kind $\eta$ and $\zeta$, which satisfy the following conditions:

1) $\eta$ and $\zeta$ have a single pole at $\infty$ with local expansions

$$\eta \sim dk_\pm, \qquad \zeta \sim dQ(k_\pm) \tag{3.22}$$

(these are dictated by the polynomial dependence of the exponent on the local parameter).

2)(normalization)

$$\oint_{a_i} \eta = 0, \qquad \oint_{a_i} \zeta = 0 \tag{3.23}$$

We can now build the following function of $P \in \Sigma$

$$\tilde{\psi} = \exp\left(x\int_{P_0}^{P}\eta + t\int_{P_0}^{P}\zeta\right)\frac{\theta(\mathcal{A}(P) + Ux + Wt - K - \mathcal{A}(\tilde{\mathcal{D}}))}{\theta(\mathcal{A}(P) - K - \mathcal{A}(\tilde{\mathcal{D}}))}. \tag{3.24}$$

In this expression $K$ is the Riemann constant and

$$\mathcal{A}(\sum_{k=1}^{g} P_k) = \sum_{k=1}^{g}\int_{P_0}^{P_k}\vec{\omega} \tag{3.25}$$

is the Abel map. $\mathcal{A}$ associates a divisor $\sum_{k=1}^{g} P_k$ on $\Sigma$ to a point of $Jac(\Sigma) = \mathbb{C}^g \backslash \Lambda$, where $\Lambda$ is the $2g$-dimensional lattice spanned by the columns of $(Id; B)$.

The vectors $U$ and $W$ are introduced to make $\tilde{\psi}$ a well-defined function on $\Sigma$. The only indeterminacy is the path of integration which can be modified by adding any integer combination of homology cycles. This produces the overall factor (taking count of the vanishing condition for $\eta$ and $\zeta$)

$$\exp\left(\sum_{k=1}^{g}\left[m_k\left(x\oint_{b_k}\eta + t\oint_{b_k}\zeta\right) - m_k\left(xU_k + tW_k\right)\right]\right). \tag{3.26}$$

This is 1 if we define the components of the "frequency vectors" $U$ and $W$ to be given by

$$U_k = \oint_{b_k}\eta, \qquad W_k = \oint_{b_k}\zeta. \tag{3.27}$$

At last we are left to choose the pole divisor $\tilde{\mathcal{D}}$ of degree $g+1$ and to fix the normalization. As discussed above, two independent functions are uniquely picked by requiring that one vanishes at $\infty_+$ and the other at $\infty_-$; for this purpose we introduce the following

**Definition 6** *$\mathcal{D}_\pm$ is the unique positive divisor which is linearly equivalent to $\mathcal{D} - \infty_\pm$.*

The choice $\mathcal{D} = \mathcal{D}_+$ and $\mathcal{D} = \mathcal{D}_-$ in formula (3.24) gives two independent functions, whose poles are in $\mathcal{D}_+$ and $\mathcal{D}_-$ respectively. In order to make the pole divisor be $\mathcal{D}$ we multiply $\tilde{\psi}^\pm$ by a meromorphic function $g_\pm(P)$ whose zeros lie in $\mathcal{D}_\pm + \infty_\mp$ and whose poles lie in the original divisor $\mathcal{D}$.

We finally obtain the correct Baker-Akhiezer functions

$$\psi^{\pm} = \exp\left(x(\int_{P_0}^{P}\eta - \eta_{\pm}^{\infty}) + t(\int_{P_0}^{P}\zeta - \zeta_{\pm}^{\infty})\right)\frac{\theta(\mathcal{A}(P)+Ux+Wt-K-\mathcal{A}(\mathcal{D}_{\pm}))}{\theta(\mathcal{A}(P)-K-\mathcal{A}(\mathcal{D}_{\pm}))}\cdot$$
$$\frac{\theta(\mathcal{A}(\infty_{\pm})-K-\mathcal{A}(\mathcal{D}_{\pm}))}{\theta(\mathcal{A}(\infty_{\pm})+Ux+Wt-K-\mathcal{A}(\mathcal{D}_{\pm}))}\cdot\frac{g_{\pm}(P)}{g_{\pm}(\infty_{\pm})}. \tag{3.28}$$

The constant terms $\eta_{\pm}^{\infty}$ and $\zeta_{\pm}^{\infty}$ in the expansion of the argument of the exponential at $\infty_{\pm}$ have been subtracted to make the leading coefficient of the meromorphic part of the eigenfunction matrix be the identity.

∎

In the following section we will understand better the importance of the normalization at $\infty$: different normalizations produce different (but gauge related) Lax pairs and so different associated non-linear equations.

### 3.2.3 The Normalization

Given the unique basis of the vector space of Baker-Azkhiezer functions guaranteed in the previous theorem, we can build unambiguously a function and a corresponding pair of linear operators which will be identified with the Lax pair for the Continuous Heisenberg Model. We have the following result:

**Proposition 9** *If $Q(k) = 2ik^2$ and $\Psi(x,t,\lambda)$ is the matrix of Baker-Akhiezer functions*

$$\Psi(x,t,\lambda) = \begin{pmatrix} \psi_+^1(x,t,\lambda) & \psi_-^1(x,t,\lambda) \\ \psi_+^2(x,t,\lambda) & \psi_-^2(x,t,\lambda) \end{pmatrix}, \tag{3.29}$$

*then the columns of the matrix*

$$\Phi(x,t,\lambda) = \Psi(x,t,0)^{-1}\Psi(x,t,\lambda) \tag{3.30}$$

*are linearly independent simultaneous solutions of the following pair of linear systems*

$$\frac{\partial \vec{F}}{\partial x} = i\lambda S\vec{F} \tag{3.31}$$

$$\frac{\partial \vec{F}}{\partial t} = (2i\lambda^2 S - \lambda \frac{\partial S}{\partial x} S)\vec{F}, \tag{3.32}$$

*where the matrix $S$ is independent of $\lambda$ and is given by*

$$S(x,t) = \Psi(x,t,0)^{-1}\sigma_3\Psi(x,t,0), \tag{3.33}$$

*with* $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

**Observation:** the value of $\Psi$ at $\lambda = 0$ defines the normalization of the eigenfunction at $\Phi$ the essential singularities. We have in fact

$$\lim_{\lambda\to\infty} \Phi(x,t,\lambda) = \Psi(x,t,0)^{-1}. \tag{3.34}$$

We can then interpret the gauge transformation as a change in the point at which the Baker-Akheizer eigenfunction is normalized.

This also shows that the potential $S$ is the leading term in the asymptotic expansion in $\lambda$ of the matrix of squared eigenfunctions,

$$S(x,t) = \lim_{\lambda\to\infty} \Phi(x,t,\lambda)\sigma_3\Phi(x,t,\lambda)^{-1}. \tag{3.35}$$

This fact will be useful later on, when explaining the relation between squared eigenfunctions and orthonormal frames.

The proof of Proposition (3) is contained in the following lemma,

**Lemma 2** *Given the Baker-Akhiezer matrix function $\Phi(x,t,\lambda)$ normalized as in Proposition 2, there exists a unique pair of matrix differential operators $L_1$ and $L_2$ of the following form*

$$L_1 = \sum_0^1 U_\alpha(x,t)\frac{\partial^\alpha}{\partial x^\alpha}, \quad L_2 = \sum_0^2 V_\beta(x,t)\frac{\partial^\beta}{\partial x^\beta}, \tag{3.36}$$

*such that $\Phi(x,t,\lambda)$ solves simultaneously the linear systems*

$$L_1\Phi = \lambda\Phi, \quad L_2\Phi = \frac{\partial\Phi}{\partial t}. \tag{3.37}$$

**Proof:**

In order to determine the coefficients of the linear operators $L_1$ and $L_2$ we consider the behavior of both columns of $\Phi$ at $\infty_\pm$.

We define the quantity $A(x,t) = \Psi^{-1}(x,t,0)$ and look at an expansion for $\Phi$ (in the global coordinate $\lambda$) of the form

$$\Phi(x,t,\lambda) \sim \left(A(x,t) + \sum_{n=1}^{\infty} \frac{1}{\lambda^n} X_n(x,t)\right) \begin{pmatrix} \exp(i\lambda x + 2i\lambda^2 t) & 0 \\ 0 & \exp(-i\lambda x - 2i\lambda^2 t) \end{pmatrix} \tag{3.38}$$

The operator $L_1$ is uniquely determined by the requirement

$$(L_1 - \lambda Id)\Phi(x,t,\lambda) = O(\frac{1}{\lambda}) \begin{pmatrix} \exp(i\lambda x + 2i\lambda^2 t) & 0 \\ 0 & \exp(-i\lambda x - 2i\lambda^2 t) \end{pmatrix}. \tag{3.39}$$

In fact, by substituting expression (3.38) in equation (3.39) and requiring that the $O(1)$ and $O(\lambda)$ terms are equal to zero, one finds the following recursive system of equations:

$$U_1 = -iA\sigma_3 A^{-1} \tag{3.40}$$

$$U_0 = X_1 A^{-1} - U_1 A_x A^{-1} - iU_1 X_1 \sigma_3 A^{-1} \tag{3.41}$$

which determines $U_0$ and $U_1$ as functions of $A$ and $X_1$.

Since $\Phi(x,t,0) = Id$, the multiplicative term $U_0$ in the expression of $L_1$ must vanish. We check this in the following way.

We consider the matrix of normalized Baker-Akhiezer functions $\Psi(x,t,\lambda)$ constructed in Proposition 2 and its asymptotic expansion at $\infty$

$$\Psi(x,t,\lambda) \sim \left(Id + \sum_{n=1}^{\infty} \frac{1}{\lambda^n} Z_n(x,t)\right) \begin{pmatrix} \exp(i\lambda x + 2i\lambda^2 t) & 0 \\ 0 & \exp(-i\lambda x - 2i\lambda^2 t) \end{pmatrix}, \tag{3.42}$$

Because $\Phi$ is obtained by normalizing $\Psi$ as in (3.33), then $Z_n = A^{-1}X_n$, $n = 1, \ldots$. If we substitute the asymptotic expansion of $\Psi$ into an equation of the same form as (3.39), we can show that $\Psi$ satisfies the following linear eigenvalue problem,

$$-i\sigma_3 \frac{\partial \Psi}{\partial x} - (\sigma_3 Z_1 \sigma_3 - Z_1)\Psi = \lambda \Psi, \tag{3.43}$$

In particular, the coefficient $A^{-1}(x,t) = \Psi(x,t,0)$ satisfies (3.43) at $\lambda = 0$ which we rewrite as

$$\frac{dA^{-1}}{dx} - i[Z_1,\sigma_3]A^{-1} = 0. \tag{3.44}$$

We now solve for $U_0$ in the system (3.41), use equation (3.44) and obtain

$$U_0 = -A\left(\frac{dA^{-1}}{dx} - i[Z_1,\sigma_3]A^{-1}\right) = 0. \tag{3.45}$$

The coefficients of $L_2$ are determined in the same way. The requirement

$$(L_2 - \frac{\partial}{\partial t})\Phi(x,t,\lambda) = O(\frac{1}{\lambda})\begin{pmatrix} e^{(i\lambda x+2i\lambda^2 t)} & 0 \\ 0 & e^{-(i\lambda x+2i\lambda^2 t)} \end{pmatrix}, \tag{3.46}$$

together with the asymptotic expansion (3.38), determines the following recursive system of equations for the coefficients $V_i$:

$$\begin{aligned} V_2 &= -2iA\sigma_3 A^{-1} && (3.47)\\ V_1 &= 2X_1A^{-1} - 2V_2A_xA^{-1} - iV_2X_1\sigma_3A^{-1} && (3.48)\\ V_0 &= A_t - V_1(A_x + iX_1\sigma_3) - V_2(A_{xx} + 2iX_{1x}\sigma_3 - X_2). && (3.49)\end{aligned}$$

In order to simplify the expressions for $V_1$ and $V_2$ we use the the equation for the time evolution of $\Psi(x,t,\lambda)$, which we can derive by substituting the asymptotic expansion of $\Psi$ in the time linear system (3.36),

$$-2i\sigma_3\frac{\partial^2\Psi}{\partial x^2}+2[Z_1,\sigma_3]\sigma_3\frac{\partial\Psi}{\partial x}+2i[Z_2,\sigma_3]-2i[Z_1,\sigma_3]\sigma_3Z_1\sigma_3-2\sigma_3Z_{1x}\sigma_3 = \frac{\partial\Psi}{\partial t}. \tag{3.50}$$

Observing that also $A^{-1}(x,t) = \Psi(x,t,0)$ satisfies equation (3.50) (which is $\lambda$ independent), and inserting the expression for its time derivative in (3.49), we obtain

$$\begin{aligned} V_2 &= -2iA\sigma_3A^{-1} && (3.51)\\ V_1 &= -i(A\sigma_3A^{-1})_x && (3.52)\\ V_0 &= 0. && (3.53)\end{aligned}$$

For this choice of $U_i$'s and $V_i$'s the right hand side of the relation

$$(L_j - \lambda Id)\Phi(x,t,\lambda) = O(\frac{1}{\lambda})\begin{pmatrix} \exp(i\lambda x + 2i\lambda^2 t) & 0 \\ 0 & \exp(-i\lambda x - 2i\lambda^2 t) \end{pmatrix}, \quad (3.54)$$

gives a pair of Baker-Akhiezer functions such that the leading coefficients of their asymptotic expansions vanish at $\infty_{\pm}$. By uniqueness, they must be identically zero. Therefore $\Phi(x,t,\lambda)$ solves simultaneously the equations

$$(L_j - \lambda Id)\Phi(x,t,\lambda) = 0, \quad j = 1,2 \quad (3.55)$$

which, after introducing the quantity $S = A\sigma_3 A^{-1}$, are equivalent to the linear system for the Continuous Heisenberg Chain.

∎

### 3.2.4 The Theta Function Representation of the Knot.

In the previous section, we built a basis of Baker-Akhiezer eigenfunctions for the linear problem of the Heisenberg Model and we expressed the potential $S$ purely in terms of the leading term of the expansion of its meromorphic part at $\lambda = \infty$.

We now derive an expression for the corresponding $N$-phase curves. It is clear from what we have discussed so far that the Baker-Akhiezer function for the Heisenberg Model is specified by $g+1$ poles, two essential singularities over $\infty$ and its normalization at 0; it is also clear that normalizing at a different point (so far as it does not coincide with one of the poles or the zeros) will affect neither the essential singularity nor the divisor of the meromorphic part.

Moreover if the pole divisor $\mathcal{D}$ is in general enough position, the condition A can be replaced by the following equivalent

*Condition A': The divisor* $P_1 + \ldots + P_{g+1} - 0_+ - 0_-$ *is not linearly equivalent to a positive divisor.*

Here $0_+$ and $0_-$ are the two points on $\Sigma$ over $\lambda = 0$ which are exchanged by the hyperelliptic involution: $0_- = \tau(0_+)$

Condition A' and $\mathcal{D}$ being non-special assure that, among the meromorphic functions with poles in $\mathcal{D}$, there is only one which vanishes at $0_+$ and just one which vanishes at $0_-$ (this gives the existence of two distinct functions with poles in $\mathcal{D}$), but there exists no non-constant function which vanishes at both points (this guarantees the independence of the two and so the ability to realize any normalization at $\lambda = 0$).

We introduce the following quantities:

1) $\mathcal{D}^0_+$ (resp. $\mathcal{D}^0_-$), the unique effective divisor which is linearly equivalent to $\mathcal{D} - 0_-$ (resp. $\mathcal{D} - 0_+$).

2) $h_\pm(P)$, a meromorphic function whose divisor is $(h_\pm) = \mathcal{D}^0_\pm + 0_\mp - \mathcal{D}$.

By an argument identical to the one described in Proposition 2, we obtain the following result,

**Proposition 10** *The linear problem associated to the Heisenberg Model is solved by a vector function with the following components*

$$\begin{aligned}\phi^\pm &= \exp\left(x(\int_{P_0}^P \eta - \eta^0_\pm) + t(\int_{P_0}^P \zeta - \zeta^0_\pm)\right) \frac{\theta(\mathcal{A}(P) + Ux + Wt - K - \mathcal{A}(\mathcal{D}^0_\pm))}{\theta(\mathcal{A}(P) - K - \mathcal{A}(\mathcal{D}^0_\pm))} \cdot \\ &\quad \frac{\theta(\mathcal{A}(0_\pm) - K - \mathcal{A}(\mathcal{D}^0_\pm))}{\theta(\mathcal{A}(0_\pm) + Ux + Wt - K - \mathcal{A}(\mathcal{D}^0_\pm))} \cdot \frac{h_\pm(P)}{h_\pm(0_\pm)}. \end{aligned} \tag{3.56}$$

*Moreover, the corresponding eigenfunction matrix*

$$\Phi(P) = \begin{pmatrix} \phi^+(P) & \phi^+(\tau(P)) \\ \phi^-(P) & \phi^-(\tau(P)) \end{pmatrix} \tag{3.57}$$

*is normalized to be the identity matrix at $\lambda = 0$.*

In (3.56) $(\eta, \zeta, P_0, \vec{\omega})$ are the same as in Proposition 2; the terms $\eta^0_\pm = \int_{P_0}^{0_\pm} \eta$ and $\zeta^0_\pm = \int_{P_0}^{0_\pm} \zeta$ are introduced to give the wanted normalization at $\lambda = 0$.

Finally we construct the $N$-phase curves. We use the reconstruction formula derived in section 3.1

$$\Gamma(x,t) = -i\frac{\partial\Phi(x,t,\lambda)}{\partial\lambda}\bigg|_{\lambda=0}. \tag{3.58}$$

The derivatives of the entries of $\Phi$ are given in terms of the following functions (and by the sheet information)

$$\frac{\partial\phi^{\pm}}{\partial\lambda}\bigg|_{\lambda=0} = a_{\pm}x + b_{\pm}t + \frac{\partial}{\partial P}\log\theta(\mathcal{A}(P) + Ux + Wt - K - \mathcal{A}(\mathcal{D}^0_{\pm}))\bigg|_{P=0_{\pm}} + c^{\pm}, \tag{3.59}$$

where the $c^{\pm}$'s are constant in $x$ and $t$ depending on the Riemann surface data, $a_{\pm} = \pm\eta(0)$, and $b_{\pm} = \pm\zeta(0)$.

We show that both coefficients of the linear terms in $x$ and $t$ are zero. Let $\lambda$ be real for simplicity (we are just interested in what happens at $\lambda = 0$). In this case it can be shown that the fundamental solution $\Phi$ of the spatial linear problem is a unitary matrix. We introduce the transfer matrix

$$T(x,\lambda) = \Phi(x+2\pi,\lambda)\Phi^{-1}(x,\lambda) \tag{3.60}$$

which takes the solution at a given $x$ to its value after the translation by one spatial period. It can be shown [FA80] that its trace

$$\Delta(\lambda) = Tr\,[T(x,\lambda)] \tag{3.61}$$

is an invariant both with respect to $x$ and $t$, and that it determines the spectrum of the spatial linear operator. If we define the *Floquet Eigenfunctions* by means of the following formula,

$$\vec{\phi}^{\pm}(x,\lambda) = \exp\left(i\rho(\lambda)x\right)\vec{f}^{\pm}(x,\lambda), \tag{3.62}$$

where $\vec{f}^{\pm}(x,\lambda)$ are bounded, periodic functions, it is easy to check the following relation between the *Floquet exponent* $\rho(\lambda)$ and $\Delta(\lambda)$:

$$\frac{d\rho}{d\lambda} = \frac{1}{2\pi}\frac{\frac{d\Delta}{d\lambda}}{\sqrt{\Delta^2 - \lambda}}. \tag{3.63}$$

By comparing the expression of the Baker-Akheizer eigenfunction with the one for the Floquet eigenfunction, we can identify the differential $\eta$ with the derivative of the Floquet exponent,

$$\eta(\lambda) = \frac{d\rho}{d\lambda}(\lambda). \tag{3.64}$$

Now we use the symmetry of the solution with respect to the transformation $\lambda \to -\lambda$. From the linear system we see that

$$\Phi(x, -\lambda) = \Phi(-x, \lambda) \tag{3.65}$$

and thus $\Phi(0, -\lambda) = \Phi(0, \lambda)$ and $\Phi(2\pi, -\lambda) = \Phi(-2\pi, \lambda)$; for real $\lambda$ we deduce

$$T(2\pi, -\lambda) = T^{-1}(2\pi, \lambda) = T^T(2\pi, \lambda). \tag{3.66}$$

Since the trace of a unitary matrix is real, we have

$$\Delta(-\lambda) = \Delta(\lambda), \quad \lambda \in R. \tag{3.67}$$

So, $\Delta(\lambda)$ is an even function of $\lambda$ and therefore the Floquet exponent $\rho = \eta$ vanishes at 0. As regards the coefficient $b$, we use the same symmetry property and deduce that $\frac{d\phi}{d\lambda}(x, -\lambda) = -\frac{d\phi}{d\lambda}(-x, \lambda)$, where $\phi$ is any of the components of the eigenmatrix. This formula must be valid for all times, therefore for $t \sim \infty$, the leading order terms give $bt = -bt$. It follows that $b = 0$.

Finally we obtain the following compact formula for the components of the matrix of the position vector,

$$\Gamma_\pm = -i \frac{\partial}{\partial P} \log \theta(\mathcal{A}(P) + Ux + Wt)\Big|_{P=0_\pm}, \tag{3.68}$$

where we absorbed the information about the divisor and the Riemann constant in the initial condition.

## 3.3 Squared Eigenfunctions and Frames

The role of the squared eigenfunctions is explored here. Their linear equations will lead to the generalization of the evolution equation for $S$ to an infinite sequence of commuting flows. As a consequence of the asymptotic behavior of the squared eigenfunctions, we can relate the coefficients of the asymptotic expansion to the components of the Frenet frame of the correspondent curve. The coefficients of the asymptotic series in $\lambda$ turn out to be the vector fields of the integrable hierarchy. We give a formula for the generating function of the conservation laws.

We define the matrix of squared eigenfunctions

$$Q(x,t,\lambda) = \Phi(x,t,\lambda)\sigma_3\Phi^{-1}(x,t,\lambda), \tag{3.69}$$

and recall its asymptotic expansion at $\lambda = \infty$

$$Q = S + \frac{1}{\lambda}Q_1 + \frac{1}{\lambda^2}Q_2 \ \dots \ . \tag{3.70}$$

**Remark:** the fact that the leading coefficient of the asymptotic expansion $Q_0 = S$ is the solution of the non-linear equation plays an important role in what follows. It is responsible for two facts: 1) The evolution equation for $S$ arises in a very natural way from the hierarchy of hamiltonians flows. 2) It allows us to find the relation between the coefficients of the expansion of $Q$ and the Frenet frame.

We adjoin an infinite sequence of flows to the pair of linear systems associated with HM.

For a matrix $X \in sl(2,C)$ which can be expressed as a formal power series of the form

$$X = \sum_{i=-\infty}^{N} X_i\lambda^i, \quad N < \infty, \tag{3.71}$$

we define $X_+$ to be

$$X_+ = \sum_{i=1}^{N} X_i\lambda^i = X_1\lambda + X_2\lambda^2 + \ \dots \ X_N\lambda^N. \tag{3.72}$$

We observe that the pair of of linear systems can be rewritten as

$$\begin{aligned} \Phi_x &= i\lambda Q_0\Phi \\ \Phi_t &= 2i(\lambda^2 Q)_+\Phi. \end{aligned} \tag{3.73}$$

($\Phi$ is the eigenmatrix constructed in the previous section). In order to show that, we substitute the asymptotic expansion (3.70) in the first equation of the linear system,

$$\begin{aligned} Q_x &= [i\lambda S, Q] \\ Q_t &= [2i\lambda^2 S - \lambda S_x S, Q], \end{aligned} \tag{3.74}$$

and obtain in particular

$$\begin{aligned} Q_{0x} &= i[Q_0, Q_1] \\ Q_{kx} &= i[Q_0, Q_{k+1}] \quad k = 1, 2, \ \ldots . \end{aligned} \tag{3.75}$$

Also observe that, since $\sigma_3^2 = Id$, the following must hold

$$\begin{aligned} Id = Q^2 &= Id + \frac{1}{\lambda}(SQ_1 + Q_1 S) + \frac{1}{\lambda^2}(SQ_2 + Q_2 S + Q_1^2) \\ &+ (SQ_3 + Q_3 S + Q_1 Q_2 + Q_2 Q_1) + \ldots \end{aligned} \tag{3.76}$$

This gives the infinite set of relations

$$\sum_{i+j=k} Q_i Q_j = 0 \quad k = 1, 2, \ \ldots . \tag{3.77}$$

The first few are given below

$$\begin{aligned} SQ_1 + Q_1 S &= 0 \\ SQ_2 + Q_2 S + Q_1^2 &= 0 \\ SQ_3 + Q_3 S + Q_1 Q_2 + Q_2 Q_1 &= 0 \\ &\ldots . \end{aligned} \tag{3.78}$$

To show that $2i\lambda^2 S - \lambda S_x S = 2i(\lambda^2 Q)_+$ is a direct computation. From (3.75), using the relations (3.77), we get

$$S_x = 2iSQ_1 \tag{3.79}$$

Since $S^2 = Id$, we have

$$2i(\lambda^2 Q)_+ = 2i(\lambda^2 Q_0 + \lambda Q_1) = 2i(\lambda^2 S + \frac{\lambda}{2} i S_x S) = 2i\lambda^2 S - \lambda S_x S. \tag{3.80}$$

If we set $t_1 = x, t_2 = t$, we can write the following infinite hierarchy of flows in which no distinguished role is played by the original space variable $x$ and by the time $t$,

$$-iQ_{t_n} = n[(\lambda^n Q)_+, Q], \quad n = 1, 2, 3 \dots . \tag{3.81}$$

We show that (3.81) gives an infinite sequence of commuting hamiltonian equations for the potential $S$.

We consider the corresponding eigenvalue problems

$$\begin{aligned} \Phi_x &= i\lambda Q_0 \Phi \\ -i\Phi_{t_n} &= n(\lambda^n Q)_+ \Phi \end{aligned} \tag{3.82}$$

and impose commutativity of the flows at each level,

$$\frac{d^2}{dx dt_n}\Phi = \frac{d^2}{dt_n dx}\Phi, \quad n = 2, 3, \dots , \tag{3.83}$$

obtaining the equations

$$Q_{0t_n} = nQ_{n-1x} \tag{3.84}$$

$$Q_{kx} = i[Q_0, Q_{k+1}], \; k = 0, \dots, n-2. \tag{3.85}$$

(3.85) is a subset of the equations (3.75) which we can use to re-express the right-hand side of the evolution equations for $Q_0 = S$ and thus obtain

$$S_{t_n} = in[S, Q_n], \quad n = 1, 2 \dots \tag{3.86}$$

It is an easy check to derive the original non-linear equation (HM) for $n = 2$. Notice that these equations are an infinite hierarchy of hamiltonian flows with the following hamiltonian densities

$$F_k(S) = \frac{k}{2} Tr(SQ_k) \tag{3.87}$$

We show the commutativity of these flows directly, it is interesting that this leads to a sequence of zero-curvature conditions. We compute the quantity

$$\frac{d^2}{dt_n dt_m} S - \frac{d^2}{dt_m dt_n} S = i \left[ S, \frac{d(nQ_n)}{dt_m} - \frac{d(mQ_m)}{dt_n} - i[nQ_n, mQ_m] \right]. \tag{3.88}$$

By substituting the asymptotic expansion for $Q$ into the evolution equations (3.81) we obtain

$$-i \sum_{k=0}^{\infty} Q_{kt_n} \lambda^{-k} = n \sum_{k=0}^{\infty} \left( \sum_{j=0}^{k} [Q_j, Q_{n+k-j}] \right) \lambda^{-k}, \tag{3.89}$$

which gives, for $m < n$,

$$\frac{d(nQ_n)}{dt_m} - \frac{d(mQ_m)}{dt_n} = inm \sum_{j=m+1}^{n} [Q_j, Q_{n+m-j}] = inm[Q_n, Q_m], \tag{3.90}$$

equivalent to the zero curvature equations

$$\frac{d(nQ_n)}{dt_m} - \frac{d(mQ_m)}{dt_n} - i[nQ_n, mQ_m] = 0, \tag{3.91}$$

and to the commutativity of the corresponding flows.

We now find the relationship between the squared eigenfunctions and the Frenet frame of the curve associated to $S$.

Since $S = Q_0$ is the hermitian matrix which represents the unit tangent vector to the curve, its $x$-evolution is the first one of the Frenet equations.

$$\frac{dS}{dx} = kN, \tag{3.92}$$

where $k$ is the curvature of the curve and $N$ is the unit normal. By comparing with the first of the sequence of equations (3.86), we get

$$kN = i[S, Q_1] \tag{3.93}$$

$$kB = \frac{1}{2i}[S, kN] = \frac{1}{2}[S, [S, Q_1]] = 2Q_1. \tag{3.94}$$

We have obtained the following expression for the frame in terms of squared eigenfunctions,

$$T = Q_0, \quad , \quad N = \frac{1}{2i}[\frac{Q_1}{\sqrt{\frac{1}{2}Tr(Q_1^2)}}, Q_0] \quad , \quad B = \frac{Q_1}{\sqrt{\frac{1}{2}Tr(Q_1^2)}}, \tag{3.95}$$

with $k = \sqrt{2Tr(Q_1^2)}$.

If we re-express the $Q_i's$ in terms of the frame and the curvatures, we already have

$$Q_1 = \frac{k}{2}B. \tag{3.96}$$

Using (3.96) and the Frenet equation for $B$ to obtain

$$\frac{2}{k}\frac{dQ_1}{dx} - \frac{1}{k^2}\frac{dk}{dx}Q_1 = -\frac{\tau}{k}i[S, Q_1], \tag{3.97}$$

Substituting the expression for $\frac{dQ_1}{dx}$ and using the constraints (3.77), we derive the following

$$Q_2 = -\frac{1}{8}k^2 S - \frac{1}{16}k\frac{dk}{dx}N + \frac{1}{4}k\tau B. \tag{3.98}$$

Observe that $Q_0 = S$, $Q_1$ and $Q_2$ are the first three vector fields of the integrable hierarchy, expressed in terms of the frame (compare their expression to the one appearing in [LP91]); iterating the procedure we obtain them all.

Now we can express the Hamiltonian densities (3.87) in a more transparent way in terms of the curvature and the torsion of the curve and their derivatives.

$$\sum_{i+j=k} Q_i Q_j = 0 \quad k = 1, 2, \ \ldots, \tag{3.99}$$

we can rewrite the densities $F_k(Q)$ as

$$F_k(Q) = \frac{k}{2}Tr(Q_0 Q_k), \quad k = 1, 2, \ \ldots. \tag{3.100}$$

Using the set of relations (3.77) and the expressions for the $Q_k's$ in terms of the frame, we write the first few,

$$F_1 \quad = \quad \frac{1}{2}Tr(SQ_1) = 0$$

$$\begin{aligned} F_2 &= Tr(SQ_2) = -\frac{1}{2}Tr(Q_1^2) = -\frac{1}{4}k^2 \\ F_3 &= \frac{3}{2}Tr(SQ_3) = -\frac{3}{2}Tr(Q_1Q_2) = -\frac{3}{16}k^2\tau \\ & \dots \end{aligned} \tag{3.101}$$

which agree with the ones listed in [LP91].

# Chapter 4 Bäcklund Transformations, Immersed Knots

## 4.1 Introduction

In this chapter we present an application of Bäcklund transformations to the curve dynamics described by the HM equation.

A Bäcklund transformation takes a solution $S$ of the non-linear equation and a solution of the linear systems

$$\frac{\partial \vec{F}}{\partial x} = i\nu S\vec{F} \tag{4.1}$$

$$\frac{\partial \vec{F}}{\partial t} = (2i\nu^2 S - \nu\frac{\partial S}{\partial x}S)\vec{F} \tag{4.2}$$

at the pair $(S, \nu)$, and creates a new solution $S^{(1)}$.

Let us assume that $S$ is a $N$-phase solution as defined in ch.3. In [EFM87b] two different types of Bäcklund transformations are discussed:

(I) (type 1) If $\vec{\phi}$ is a single Baker-Akheizer eigenfunction at a general $\nu \in C$, then the new solution $S^{(1)}$ is periodic in $x$ and is an $N$-phase solution.

(II) (type 2) If $\vec{\phi}$ is a general complex linear combination of Baker-Akheizer eigenfunctions then, for a generic $\nu$, $S^{(1)}$ may not be periodic. However, if $\nu$ is a double point of the spectrum of (4.1) (these notions will be defined in a specific example later on), then $S^{(1)}$ is a homoclinic orbit of $S$.

For the general treatment of these facts we refer to [EFM87b]. We will present here the simplest example of Bäcklund transformations of a planar circle and give the necessary definitions in the course of the illustration.

The two distinct types of Bäcklund formula suggest two ideas when thinking of the corresponding curves.

(I) The type 1 transformation is a symmetry of the family of $N$-phase solutions to which $S$ belongs; therefore iterated Bäcklund transformations of type 1 should exhaust the symmetries of the corresponding class of curves. In the case of the planar circle we find that the vector field corresponding to an infinitesimal Bäcklund transformation is a Killing field (i.e. the generator of an infinitesimal rigid motion).

(II) The type 2 transformation generates homoclinic orbits and can be used to produce dynamics bifurcations of knot types. In our example immersed knots with stable self-intersections bifurcate from (degenerate) multiple copies of a planar circle.

In section 4.2 we give a derivation of the Bäcklund formula for HM, which makes use of the gauge transformation between HM and NLS and of the Bäcklund transformation for the NLS equation. In section 4.3 we apply this result to degenerate planar circles, and discuss the outcome of the two types of Bäcklund transformation.

## 4.2 The Bäcklund Formula.

In order to derive a Bäcklund formula we begin with the observation that an appropriate gauge transformation will reduce the linear problem for the HM to the linear problem for the focussing cubic nonlinear Schrödinger equation. Following [FA80], we introduce a unitary matrix $V$ which satisfies the following two conditions:

$$\begin{aligned} VSV^{-1} &= \sigma_3, && (4.3) \\ \frac{\partial V}{\partial x}V^{-1} &= \begin{pmatrix} 0 & iq \\ i\bar{q} & 0 \end{pmatrix}, && (4.4) \end{aligned}$$

where $q$ is a complex-valued function of $(x,t)$.

Then, if $\vec{F}$ solves the linear problem for HM

$$\frac{\partial \vec{F}}{\partial x} = i\lambda S\vec{F} \tag{4.5}$$

at $(S,\lambda)$, the vector $\vec{F}_{NS} \stackrel{\text{def}}{=} V\vec{F}$ solves the spatial linear problem for the NLS equation

$$\frac{\partial \vec{F}_{NS}}{\partial x} = \left[ i\lambda\sigma_3 + i \begin{pmatrix} 0 & q \\ \bar{q} & 0 \end{pmatrix} \right] \vec{F}_{NS} \tag{4.6}$$

at $(q,\lambda)$.

The Bäcklund transformation for the Nonlinear Schrödinger Equation is obtained by means of the following procedure (see [SZ87]):

Let $(\vec{\psi}_+, \vec{\psi}_-)$ be two independent solutions of the linear system (4.6) at $(q,\nu)$. We construct the following quantities

$$\vec{\psi} = c_+\vec{\psi}_+ + c_-\vec{\psi}_- \tag{4.7}$$

$$N_{NS} = \begin{pmatrix} \psi_1 & -\bar{\psi}_2 \\ \psi_2 & \bar{\psi}_1 \end{pmatrix} \tag{4.8}$$

$$G_{NS} = N_{NS} \begin{pmatrix} \lambda-\nu & 0 \\ 0 & \lambda-\bar{\nu} \end{pmatrix} N_{NS}{}^{-1}. \tag{4.9}$$

Then

$$\vec{F}^{(1)}_{NS}(x,t,\lambda,\nu) = G_{NS}(\lambda,\nu,\vec{\psi})\vec{F}_{NS}(x,t,\lambda) \tag{4.10}$$

solves equation (4.6) at $(Q^{(1)},\lambda)$, where

$$Q^{(1)} = q + 2(\nu-\bar{\nu})\frac{\phi_1\bar{\phi}_2}{|\phi_1|^2+|\phi_2|^2} \tag{4.11}$$

is the corresponding new solution of the NLS equation. The relation (4.10) between the old and the new NLS eigenfuctions, together with the change of gauge which carries (HM) to (NLS), allows one to obtain the Bäcklund formula for the Heisenberg Model with no dependence on the gauge transformation. This is stated in the following:

**Proposition 11 (HM Bäcklund Transformation)** *Let $\vec{\phi} = c_+\vec{\phi}_+ + c_-\vec{\phi}_-$ be a complex linear combination of linearly independent solutions of equation (4.5) at $(S,\nu)$. We construct the matrix of gauge transformation*

$$G(\lambda,\nu,\vec{\phi}) = N \begin{pmatrix} \frac{\nu-\lambda}{\nu} & 0 \\ 0 & \frac{\bar{\nu}-\lambda}{\bar{\nu}} \end{pmatrix} N^{-1} \tag{4.12}$$

*with*

$$N = \begin{pmatrix} \phi_1 & -\bar{\phi}_2 \\ \phi_2 & \bar{\phi}_1 \end{pmatrix}. \tag{4.13}$$

*Then, if $\vec{F}$ solves the linear system (4.5) at $(\lambda, S)$,*

$$\vec{F}^{(1)}(x,t,\lambda,\nu) = G(\lambda,\nu,\vec{\phi})\vec{F}(x,t,\lambda) \tag{4.14}$$

*solves (4.5) at $(\lambda, S^{(1)})$. The new solution $S^{(1)})$ of the Heisenberg Model is given by the Bäcklund formula*

$$S^{(1)}(x,t) = U^{-1}(x,t;\nu)S(x,t)U(x,t;\nu), \tag{4.15}$$

*with*

$$U = N \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} N^{-1}, \quad e^{i\theta} = \frac{\nu}{|\nu|} \tag{4.16}$$

**Remarks:**

1) $U$ is a unitary matrix, hence the solution $S^{(1)}$ is hermitian and has zero trace (and its determinant is 1). The corresponding vector $\vec{S}^{(1)} \in R^3$ is obtained by a pointwise rotation of the original solution $\vec{S}$ by an angle $\theta = \frac{\nu}{|\nu|}$, depending only on the eigenvalue parameter $\nu$, around the instantaneous axis of rotation $(\frac{2Re(\phi_1\bar{\phi}_2)}{|\phi_1|^2+|\phi_2|^2}, -\frac{2Im(\phi_1\bar{\phi}_2)}{|\phi_1|^2+|\phi_2|^2}, \frac{|\phi_1|^2-|\phi_2|^2}{|\phi_1|^2+|\phi_2|^2})^T$

2) If $\nu \in R$, then $S^{(1)} = S$ and the Bäcklund transformation is the identity. If $\nu$ is complex, the eigenfuction $\vec{\phi}$ is in general quasi-periodic unless $\nu$ belongs to a discrete set of values for which $\vec{\phi}$ is spatially periodic. We will discuss this case in the next section.

**Proof:**

We use the gauge transformation to pull the NLS-Bäcklund formula back to the Heisenberg Model. Given the eigenvector $\vec{F}_{NS} = V\vec{F}$ for the NLS linear system at $(q,\lambda)$, the Bäcklund transformation produces the eigenvector $\vec{F}^{(1)}_{NS}$ corresponding to the pair $(Q^{(1)},\lambda)$. This can be expressed as $\vec{F}^{(1)}_{NS} = V^{(1)}\vec{F}$, where $V^{(1)}$ is the new gauge matrix. On the other hand $V^{(1)}$ solves the linear system for NLS at the new potential $Q^{(1)}$ and at the point $\lambda = 0$, that is $V^{(1)} = G_{NS}|_{\lambda=0}V$. Therefore the new eigenfunction for the Heisenberg model is given by

$$\vec{F}^{(1)} = V^{-1}{G_{NS}}^{-1}|_{\lambda=0}\vec{F}_{NS} = V^{-1}{G_{NS}}^{-1}|_{\lambda=0}G_{NS}V\vec{F} \tag{4.17}$$

The important point is that $F^{(1)}$ is independent of the gauge transformation and so does not depend on the NLS eigenfunctions.

In order to see this, let $\vec{\psi} = V\vec{\phi}$. Since $V$ is a unitary matrix of determinant 1, the following identity is true

$$V^{-1}N_{NS} = V^{-1}\begin{pmatrix} \psi_1 & -\bar{\psi}_2 \\ \psi_2 & \bar{\psi}_1 \end{pmatrix} = \begin{pmatrix} \phi_1 & -\bar{\phi}_2 \\ \vec{\phi}_2 & \bar{\phi}_1 \end{pmatrix} = N. \tag{4.18}$$

Taking $\vec{\phi}$ to be a complex linear combination $\vec{\phi} = c_+\vec{\phi}_+ + c_-\vec{\phi}_-$ of independent eigenfunctions of the linear system (4.5) we obtain the expression (4.14) for the new eigenfunction.

Given $\vec{F}^{(1)}$, we can write the expression for the new squared eigenfunction

$$Q^{(1)} = G(\lambda)QG(\lambda)^{-1}. \tag{4.19}$$

$S^{(1)}$ is the dominant term of its asymptotic expansion at $\lambda = \infty$. For large $\lambda$ we obtain

$$Q^{(1)}(\lambda) \sim U^{-1}SU + O(\frac{1}{\lambda}), \tag{4.20}$$

which completes the proof. ∎

## 4.3 Application: Symmetries and Singular Knots.

Firstly, we use the formula for the reconstruction of the curve in terms of the fundamental solution of the linear problem and derive an expression for the Bäcklund transformed curve. Secondly, we compute a single Bäcklund transformation of types 1 and 2 for the simplest (1-phase) solution.

The fundamental matrix of the linear problem (4.5) at $(S^{(1)}, \lambda)$ (normalized to be the identity matrix at $x = 0$) is

$$\Phi^{(1)}(x,t,\lambda) = G^{-1}(0,\lambda)G(x,\lambda)\Phi(x,t,\lambda), \tag{4.21}$$

where $\Phi$ is the fundamental solution of (4.5) at $(S, \lambda)$. Using formula (3.13) derived at the beginning of ch.3, we compute the matrix of the position vector of the curve

$$\begin{aligned} \Gamma^{(1)}(x,t) &= -\left.\frac{dG(0)}{d\lambda}\right|_{\lambda=0} + \left.\frac{dG(x)}{d\lambda}\right|_{\lambda=0} + \left.\frac{d\Phi(x)}{d\lambda}\right|_{\lambda=0} = \\ &= \Gamma(x,t) + \frac{\sin\theta}{|\nu|}\left[V(x,t) - V(0,t)\right] \end{aligned} \tag{4.22}$$

where $V = N\sigma_3 N^{-1}$, $\nu = |\nu|e^{i\theta}$.

The expression (4.22) shows that the Bäcklund transform is the identity at a real eigenvalue $\nu$ ($\theta = 0$), and that it generates bounded solutions (not necessarily periodic in $x$) for a general complex $\nu$. The following boundedness argument for the norm of the position vector of the curve follows easily from expression (4.22):

**Proposition 12**

$$||\vec{\Gamma}|| - 2\frac{|Im(\nu)|}{|\nu|^2} \leq ||\vec{\Gamma}^{(1)}|| \leq ||\Gamma|| + 2\frac{|Im(\nu)|}{|\nu|^2} \tag{4.23}$$

*where $||\cdot||$ is the Euclidean norm in $R^3$. Therefore the new curve is confined to the interior of a sphere if $\frac{|Im(\nu)|}{|\nu|^2} \geq \max_x ||\Gamma(x)||$, while it is confined to the interior of a spherical shell if $\frac{|Im(\nu)|}{|\nu|^2} < \max_x ||\Gamma(x)||$.*

This reproduces a result which Sym et al. ([JPA86]) obtained by reconstructing the curve from homoclinic orbits of the NLS equation.

### 4.3.1 Example: The Planar Circle.

The simplest solutions which possess homoclinic instabilities are planar circles. Their tangent vector-fields are fixed points of the Continuous Heisenberg Model, so the corresponding reconstructed curve can be chosen to be time-independent. We start with $k$ copies of a circle lying in the $(x, y)$ plane with non zero curvature $k \in Z$. If $x$ denotes the arclength parameter, the matrix corresponding to the unit tangent vector is

$$S_0(x,t) = \begin{pmatrix} 0 & e^{-ikx} \\ e^{ikx} & 0 \end{pmatrix} \tag{4.24}$$

The properties of the level set on which the solution $S_0$ resides can be studied by means of the discriminant $\Delta(S, \lambda)$ of the linear operator

$$L_1(S, \lambda) = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{d}{dx} + i\lambda S, \tag{4.25}$$

which was introduced at the end of section 3.2. We describe the relevant properties of $\Delta(S, \lambda)$ without proofs. These are contained in [EFM87a, EFM87b] and the references therein.

We construct the fundamental solution matrix $\Phi(x; S, \lambda)$ with the following properties:

$$\begin{aligned} \hat{L}(S, \lambda)\Phi &= 0 \\ \Phi(0; S, \lambda) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{4.26}$$

The spectrum of the operator $L_1$ is defined by means of the Floquet discriminant

$$\Delta(S, \lambda) = tr\left[\Phi(x + 2\pi, x; S, \lambda)\right]. \tag{4.27}$$

It can be shown that $\Delta(S, \lambda)$ is analytic in both $S$ and $\lambda$.

For a fixed $\lambda$, the corresponding discriminant $\Delta(S, \lambda)$ is a constant of motion and therefore it can be used to define the level set corresponding to a given solution. We recall the following results:

**Theorem 5** *If $S$ solves the Continuous Heisenberg Model then, for all $\lambda$,*

$$\frac{d}{dt}\Delta(S(t), \lambda) = 0 \tag{4.28}$$

**Theorem 6** *The spectrum of $L_1$ is given by*

$$\sigma(L_1) = \{\lambda \in C|\ \Delta(S,\lambda) \in R,\ -2 \leq \Delta(S,\lambda) \leq 2\} \tag{4.29}$$

In the complex plane we distinguish the following special points of the spectrum:

1) critical points $\lambda_c$:

$$\frac{d}{d\lambda}\Delta(S,\lambda)\bigg|_{\lambda=\lambda_c} = 0, \tag{4.30}$$

2) periodic (antiperiodic) points $\lambda_\pm$:

$$\Delta(S,\lambda)|_{\lambda=\lambda_\pm} = \pm 2, \tag{4.31}$$

3) multiple points $\lambda_m$

$$\begin{aligned} \Delta(S,\lambda)|_{\lambda=\lambda_m} &= \pm 2 \\ \frac{d}{d\lambda}\Delta(S,\lambda)\bigg|_{\lambda=\lambda_m} &= 0 \end{aligned} \tag{4.32}$$

The periodic (antiperiodic) points are associated with periodic (antiperiodic) eigenfunctions. Therefore the Bäcklund transformation at one such point produces a solution which is periodic in $x$.

The multiple points (we will consider only double points) indicate that the level set is saddle-like and the corresponding homoclinic instabilities can be constructed by means of Bäcklund transformations. Here we consider the simplest type of homoclinic instabilities produced by a single iteration of the Bäcklund formula.

The Baker eigenfunctions for the linear problem at $(S_0,\lambda)$ are

$$\vec{\phi}_- = e^{-\frac{1}{2}\delta(x+2\lambda t)}\begin{pmatrix} -2\frac{\lambda}{k+\delta}e^{-\frac{1}{2}kx} \\ e^{\frac{1}{2}kx} \end{pmatrix}, \quad \vec{\phi}_+ = e^{\frac{1}{2}\delta(x+2\lambda t)}\begin{pmatrix} e^{-\frac{1}{2}kx} \\ \frac{\delta-k}{2}e^{\frac{1}{2}kx} \end{pmatrix}, \tag{4.33}$$

with $\delta = \sqrt{k^2+4\lambda^2}$.

The discriminant for the solution $S_0$ can be computed from the corresponding fundamental solution matrix; we obtain

$$\Delta(S_0,\lambda) = \Delta(k,\lambda) = (-1)^k 2\cos(\delta\pi) \tag{4.34}$$

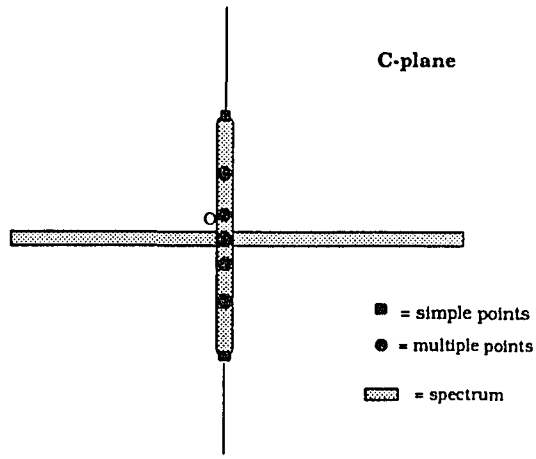


Figure 4.1, The spectral configuration of the planar circle

Thus the spectrum is given by

$$\sigma(S_0,\lambda) = \{\lambda \in C \mid \cos(\delta\pi) \in R\} = R \ \cup \{i\mu \mid \mu \in R\ ,\ -k/2 \le \mu \le k/2\} \quad (4.35)$$

The critical points are the set of zeroes of

$$\frac{d}{d\lambda}\Delta = -\sin(\delta\pi)\frac{4\lambda}{\sqrt{k^2+4\lambda^2}} = 0. \quad (4.36)$$

All of them are multiple points. The origin $\lambda = 0$ is a point of infinite multiplicity, and the complex double points are given by

$$\lambda_n = \pm\frac{i}{2}\sqrt{k^2-n^2} \qquad n = 1,2,...,k-1$$

The spectral configuration of the level set of $S_0$ is shown in figure 4.1:

We now apply the results of the previous sections and construct the Bäcklund transformed solutions. We compute the two types of Bäcklund transformations mentioned in the introduction. They both produce solutions belonging to the level set of $S_0$.

Type 1: we take $\vec{\phi}$ to be a single Baker eigenfunction evaluated at the eigenvalue $\nu$. We obtain a periodic solution for every choice of $\nu \in C$, which belongs to the same class of solutions as $S_0$.

For $\vec{\phi} = \vec{\phi}_+$ we obtain the family of planar circles

$$\Gamma^{(1)}(x) = \begin{pmatrix} 0 & \frac{i}{k}e^{-ikx} \\ -\frac{i}{k}e^{ikx} & 0 \end{pmatrix} + \frac{\mathrm{Im}(\nu)}{|\nu|^2(4|\nu|^2 + |\delta - k|^2)} \begin{pmatrix} 4|\nu|^2 - |\delta - k|^2 & 4\nu(\bar{\delta} - k)e^{-ikx} \\ 4\bar{\nu}(\delta - k)e^{ikx} & 4|\nu|^2 - |\delta - k|^2 \end{pmatrix} \quad (4.37)$$

which is parametrized by the complex parameter $\nu$. As $\mathrm{Im}(\nu) \to 0$ we obtain a family of vector fields of the form

$$\vec{V}(x,\nu) = c_1(\nu)(0,0,1)^T + c_2(\nu)\vec{S}_0 \quad (4.38)$$

(the $c_i$'s are constant depending on $\nu$) which is a family of Killing fields for the circle (a linear combination of a rigid translation and a rotation). We already see in this simple case that the Bäcklund transformation is associated to a group of symmetries of the level set of the solution $S_0$.

Type 2: we take $\vec{\Phi} = c_-\vec{\Phi}_- + c_+\vec{\Phi}_+$ to be a general complex linear combination of Baker eigenfunctions. The resulting solution is homoclinic to the original circle, and it is periodic in $x$ if $\nu$ is one of the complex double points:

$$\nu = i\mu_n = \pm\frac{i}{2}\sqrt{k^2 - n^2} \qquad n = 1, 2, ..., k-1.$$

In this case, introducing the complex parameter $c_+/c_- = \rho e^{i\theta}$, we obtain the following formula:

$$\Gamma^{(2)}(x,t) = \begin{pmatrix} 0 & \frac{i}{k}e^{-ikx} \\ -\frac{i}{k}e^{ikx} & 0 \end{pmatrix} + \frac{1}{\mu_n} \begin{pmatrix} f(x,t) - f(0,t) & \overline{g(x,t)} - \overline{g(0,t)} \\ g(x,t) - g(0,t) & -(f(x,t) - f(0,t)) \end{pmatrix} \quad (4.39)$$

with

$$\begin{aligned} f(x,t) &= -\frac{1}{2\mu_n}\frac{h'(t)}{kh(t) + 8\rho\mu_n{}^2\sin(nx+\theta)} \\ g(x,t) &= -2i\frac{\mu_n}{k}e^{-ikx} - 4\mu_n\rho n\frac{\cos(nx+\theta) + i\frac{n}{k}\sin(nx+\theta)}{kh(t) + 8\rho\mu_n{}^2\sin(nx+\theta)}e^{-ikx} \\ h(t) &= a_n\sinh(n\mu t) + b_n\cosh(n\mu t), \end{aligned} \quad (4.40)$$

and $a_n = (k+n) + (k-n)\rho^2$ and $b_n = (k+n) - (k-n)\rho^2$. Observe that $\lim_{t\to\pm\infty} \Gamma_2(x,t) = \Gamma_0(x)$, thus the new solution is a homoclinic orbit of the original circle. Figures 4.2 and 4.3 illustrate the cases $k=6, n=5$ and $k=5, n=2$.

They are time frames of the evolution of orbits homoclinic to degenerate circles (a 6-fold circle and a 5-fold circle respectively). The resulting curves are singular knots which have points of self-intersection which persist throughout the evolution. The number of such "stable" points of self-intersection is the order of the complex double point appearing in the Bäcklund formula. We can see how the resulting curve does not belong to the same class as the original solution. We conjecture that these dynamical separatrices play a role in distinguishing different special classes of knot types.

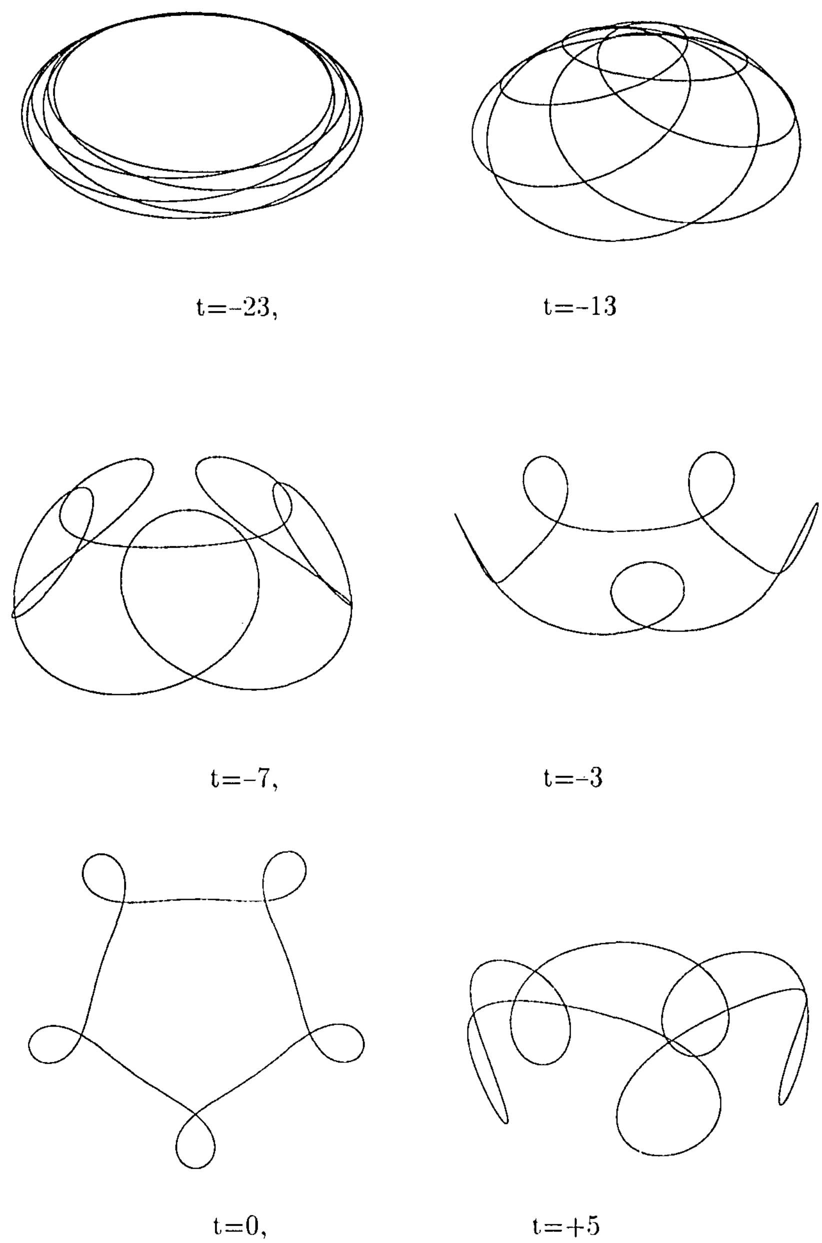


Figure 4.2, Evolution of the Bäcklund transformation of a 6-fold planar circle, $k = 6, n = 5, \rho = 1, \theta = 0$

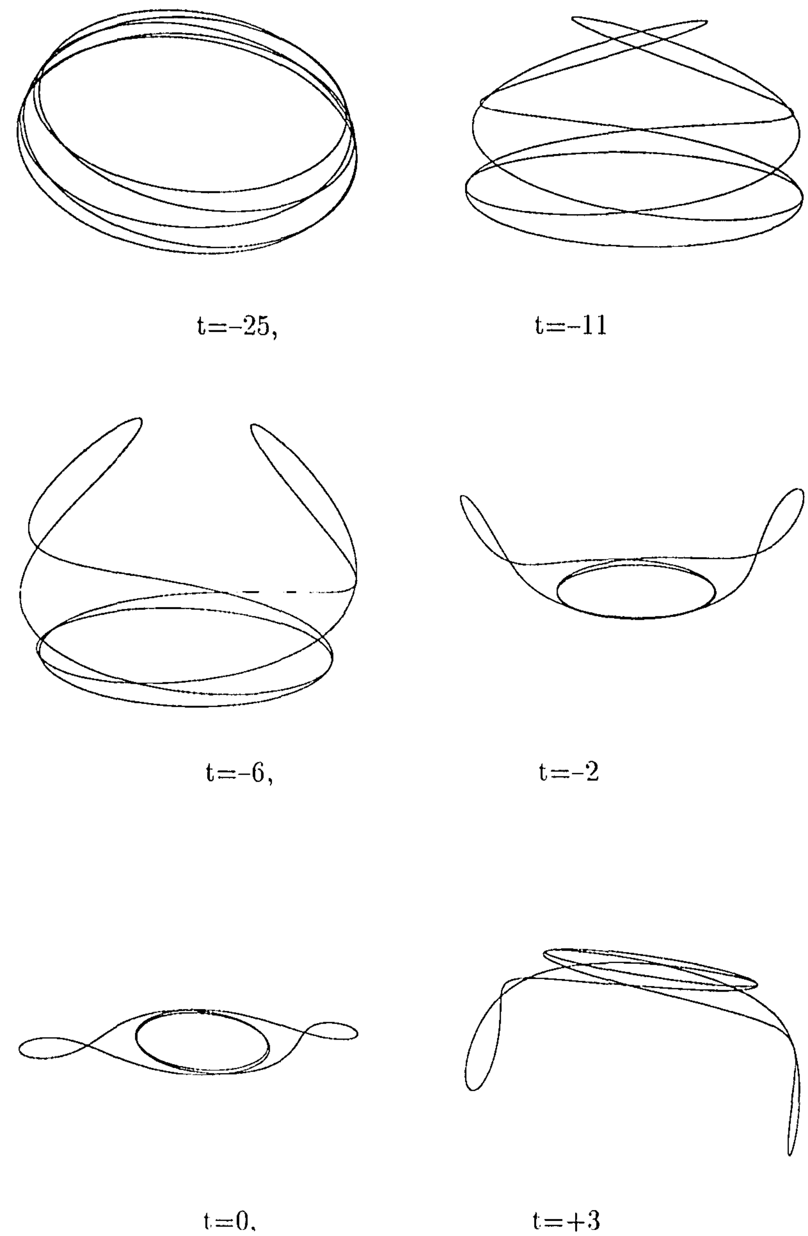


Figure 4.3, Evolution of the Bäcklund transformation of a 6-fold planar circle, $k = 5, n = 2, \rho = 1, \theta = \pi/4$

# Chapter 5 Conclusions

There are several ways to explain the wealth of interest that the dynamics of curves and surfaces modeled by soliton equations has arisen in recent years [Ham65, Lam80, Lak77, A.S85, GP92, LP91, NSW92, DS94].

1) A unifying attitude: many phenomena can be described by the same equations. Soliton equations appear ubiquitously in the description of non-linear phenomena, which include non-linear optics, hydrodynamics, plasmas, biological structures, solid-state physics and matrix models in quantum field theory. Many of these integrable equations enter also in the description of the differential geometry of surfaces and curves: a case is the vortex filament dynamics we have studied. Other examples include the dynamics of vortex patches and the differential geometry of constant negative curvature surfaces.

2) A need of deeper understanding of certain phenomena: often simplicity means that we can access a much richer set of information. The understanding of the fundamental structures of simplified models can provide a roadway to the study of more complex situations. Examples are the modelling of chaotic dynamics, the study of singular limits of PDE's, the theory of large amplitude perturbations of integrable systems.

3) The idea that we can use the relation between the integrability of certain evolution equations and the geometry (or the topology) of curves and surfaces to attack open problems both in soliton theory and in differential geometry or topology. One example is the case of knot theory. Knots have been studied mainly as "static objects": one handles a knot, lists the moves necessary to unknot it and classify it. On the other hand a simple, but rich dynamics on the space of closed curves, which adds extra structure and possesses special classes of solutions, should be a powerful tool to provide some kind of classification of knots and their

invariants.

One of the main themes of this area of research has been the relation between the dynamics of curves described by Filament Flow-type of equations and well-known soliton hierarchies. Successful insights have ranged from finding the correspondence between models of curve dynamics and particular integrable equations [Ham65, Lam80, GP92, DS94], to a deeper exploration of the corresponding Poisson geometry [LP91].

This work starts from these general motivations and considers one soliton equation which has been mentioned in connection to integrable curve evolution [LP92, Lak77], but not investigated in depth. It is the Continuous Heisenberg Model of the evolution of the unit tangent vector to the curve, which provides a bridge between the Filament Flow on curves in $R^3$ and the evolution of a complex wave function under the NLS flow. We studied it for periodic boundary conditions, corresponding to closed curves.

We found it to be a very rich model, because it provides us with a more natural framework to study questions regarding the geometry of the space of curves in connection with integrability. To mention a couple of advantages, it is very natural to reconstruct a curve from its tangent vector and to find conditions on the tangent vector for the corresponding curve to be closed (the "zero mean" condition discussed in ch. 1). For the NLS equation we do not know which subspace of its solutions corresponds to closed curves; moreover the reconstruction of the original curve means solving an inverse problem.

## 5.1 Poisson Geometry

The reconstruction and characterization of closed curves are minor advantages with respect to the ones we earned while studying the Poisson geometry of the Heisenberg Model. In our setting, Poisson geometry deals with Lie algebra structures on the tangent space to the solution manifold. In the case of curves, Poisson brackets are defined on the space of frames. The naturality of the Heisenberg Model is a consequence of our ability to construct a general framework (the circle bundle $\mathcal{T}_1 S^2$ to the 2-sphere) in which frames arise as liftings of the solution curve into a principal bundle over $S^2$. The lifting to curves everywhere tangent to the horizontal subspaces of the canonical connection on $\mathcal{T}_1 S^2$ defines a Poisson map between the phase space for HM (loops in $S^2$) and the space of solutions of NLS (complex functions on $S^1$). This map provides an interpretation of the gauge transformation between HM and NLS and a natural correspondence between their Poisson structures. There is no need of the reparametrization operator introduced by Langer and Perline [LP91]; moreover, the use of the frame corresponding to the horizontal lifting as a basis for the vector fields allows one to identify vector fields for HM with complex vector fields for NLS in a direct way. The Poisson map between HM and NLS given by the horizontal lifting possesses a natural factorization through a Poisson map between $\mathcal{L}(S^2)$ and the loop space of $SO(3, R)$. In terms of this Poisson mappings the meaning of the second Poisson structure for NLS becomes transparent. It is related to the natural Poisson structure on the loop space of $SO(3, R)$. It would be interesting to know if this is a more general feature of soliton equations; that is, whether their Poisson structures become "simpler" and more natural when they can be viewed as reductions from a loop group setting. An open direction within this framework is the role of the recursion operator at the level of the differential geometry of curves. The recursion operator takes one hamiltonian vector field to the next one of the integrable hierarchy, therefore it is a generator of the infinite number of symmetries of the solution. Its relation with the symmetries of the corresponding curve needs to be explored.

## 5.2 Contact Geometry

Another advantage of the Heisenberg Model framework which we constructed is that it is natural to build a (in fact several) contact structure on the frame bundle which is adapted to the Riemannian structure on $S^2$. Bearing in mind the fundamental role of the rotation group $SO(3, R)$ we chose the canonical invariant metric on $S^2$ and built the corresponding canonical connection on $\mathcal{T}_1 S^2$. We showed that one of the elements of the corresponding basis of $T^*\mathcal{T}_1 S^2$ defines a left-invariant contact structure with respect to which the curve described by the Frenet frame of the original curve in $R^3$ (the Frenet Lifting in ch. 2) is a Legendrian knot. As an application of this result, we showed that the curve described by the tangent vector is a wave front. As a consequence, its generic singularities can be easily classified to be cusps or points of self-intersection.

Legendrian knots are more "rigid" than usual knots, being constrained to be everywhere tangent to a distribution of planes; moreover, their invariants provide a refinement of the invariants of knots. Regarding this question, invariants of the Legendrian knot described by the Frenet frame are invariants of the original curve. We computed the Maslov index and found that it is related to geodesics curvatures of associated spherical curves. It is interesting to have obtained geometric invariants from topological invariants of related Legendrian curves. There is more to explore in this direction. On one hand the computation of other invariants, such as the Thurston-Bennequin invariant [Ben89] and their relation with the conserved quantities of the integrable equation. On the other hand, a Legendrian curve evolving in time in $SO(3, R)$ sweeps out a surface which is foliated by Legendrian curves. Eliashberg suggests a way to study its topology through the study of the singularities of the Legendrian foliation [Eli94].

## 5.3 Integrability

In chapter 3 we found that we do not need to make use of the gauge transformation between HM and NLS in order to construct $N$-phase solutions of HM. Or more practically, we find that the gauge transformation amounts to a normalizing factor at the essential singularity of the Baker-Akheizer eigenfunction which we do not even need to compute. From a linear system which is fundamentally equivalent to the one for NLS, we constructed the general family of $N$-phase solutions. We can now hope to plot the corresponding curves. The one-phase solutions are already interesting: cylindrical coordinates have been constructed in $R^3$ which show that they lie on tori of revolution [LS84]. There is no reason why the higher genus ones should reside on such simple surfaces, but their structure should be special since they are critical points of the higher geometric invariants.

Frames were explored more in this chapter, in conjunction with the role of squared eigenfunctions, which were shown to generate the hierarchy of commuting vector field. The squared eigenfunctions should be some sort of universal framing of the solution curve parametrized by the complex spectral parameter. Compatibility of the linear system should mean that the evolution of this family of nearby curves completely determines the integrable dynamics of the framed curve.

## 5.4 Singular Knots

In the last chapter we constructed a Bäcklund transformation for the HM.The simplest example of the Bäcklund transformation of a planar circle indicates that we could use it to explore the symmetries of higher genus solutions (this will be a numerical study, formulae become involved unless one manages work in an abstract setting). The Bäcklund formula applied to the construction of homoclinic orbits suggests a relation between the dynamical separatrices and the the separatrices of different knot types. There is a hope of understanding the existence of self-intersections and their persistence in time in terms of the Plücker coordinates and of the degree theory of corresponding complex algebraic curves [GH78]. The Bäcklund formula for the planar circle defines a rational homogeneous curve (coordinatized by $(\sin x, \cos x, 1)$), whose degree can be expressed in terms of the mode number $n$ and of the integral curvature of the original circle. The degree should provide us with an invariant of the corresponding knot.

# APPENDIX A

We check the compatibility of the following Poisson operators

$$K_{\vec{t}}V = \nabla_{k\vec{n}}V + \frac{1}{2}k\vec{n}\left(\int_0^s + \int_{2\pi}^s\right) V \cdot (k\vec{n})du \tag{A.1}$$

$$J_{\vec{t}}V = \vec{t} \times V, \tag{A.2}$$

The requirement that the operator $J_{\vec{t}} + K_{\vec{t}}$ defines a Poisson bracket with respect to the inner product

$$< V, W > (\vec{t}) = \frac{1}{2\pi}\int_0^{2\pi} V \cdot W ds, \quad V, W \in T_{\vec{t}}\mathcal{L}(S^2), \tag{A.3}$$

is equivalent to the following coupling condition shown by Magri [Mag78]

$$\begin{aligned} & < J'_{\vec{t}}(\phi, K_{\vec{t}}\psi), \chi > + \ \ cyclic\, permutations \qquad\qquad (A.4)\\ = \ & - < K'_{\vec{t}}(\phi, J_{\vec{t}}\psi), \chi > + \ \ cyclic\, permutations, \quad \forall\ (\phi, \psi, \chi) \in T_{\vec{t}}\mathcal{L}(S^2). \end{aligned}$$

The notation $L'_u$ indicates the Gateaux derivative of $L_u$ defined to be

$$L'_u(v, w) = \frac{d}{d\epsilon}\left(L_{u+\epsilon w}v\right)\big|_{\epsilon=0} \cdot \tag{A.5}$$

Using definition (A.5) we compute

$$J'_{\vec{t}}(\phi, K_{\vec{t}}\psi) = \nabla_{k\vec{n}}\psi \times \phi + \frac{1}{2}(k\vec{n} \times \phi)\left(\int_0^s + \int_{2\pi}^s\right) \psi \cdot (k\vec{n})du. \tag{A.6}$$

We have

$$\begin{aligned} & < J'_{\vec{t}}(\phi, K_{\vec{t}}\psi), \chi > \qquad\qquad (A.7)\\ = \ & \frac{1}{2\pi}\int_0^{2\pi}\left[(\nabla_{k\vec{n}}\psi \times \varphi) \cdot \chi + \frac{1}{2}(k\vec{n} \times \varphi) \cdot \chi \left(\int_0^s + \int_{2\pi}^s\right) \psi \cdot (k\vec{n})du\right] ds. \end{aligned}$$

Since $\phi, \psi, \chi$ belong to the span of $\vec{n}$ and $\vec{b}$, then the term $(k\vec{n} \times \phi) \cdot \chi$ (and its cyclic permutations) vanishes. Moreover

$$\begin{aligned} (\nabla_{k\vec{n}}\psi \times \phi) \cdot \chi + \ \ldots \ & = \nabla_{k\vec{n}}(\psi \times \phi \cdot \chi)\\ - \ \ \psi \times \nabla_{k\vec{n}}\phi \cdot \chi \ \psi \times \phi \cdot \nabla_{k\vec{n}}\chi & + \nabla_{k\vec{n}}\chi \times \psi \cdot \phi + \nabla_{k\vec{n}}\phi \times \chi \cdot \psi = 0. \end{aligned}$$

Therefore, the left-hand-side of the coupling condition (A.4) is identically equal to 0. Analogously, we compute

$$< K'_{\vec{t}}(\phi, J_{\vec{t}}\psi), \chi > \;=\; \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{d\phi}{ds}\cdot\vec{t}(\psi\times\vec{t})\cdot\chi + \right.$$

$$\left.\frac{1}{2}\frac{d\vec{t}}{ds}\cdot\chi\left(\int_0^s+\int_{2\pi}^s\right)\phi\cdot\frac{d}{ds}(\vec{t}\times\psi)du \;-\; \frac{1}{2}\frac{d\vec{t}}{ds}\cdot\phi\left(\int_0^s+\int_{2\pi}^s\right)\chi\cdot\frac{d}{ds}(\vec{t}\times\psi)du\right]ds.$$

The show first how to deal with the terms containing the double integral. When we add the cyclic permutations, we obtain three terms of the form

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}\frac{d\vec{t}}{ds}\cdot\chi\left(\int_0^s+\int_{2\pi}^s\right)\left[\phi\cdot\frac{d}{ds}(\vec{t}\times\psi)-\psi\cdot\frac{d}{ds}(\vec{t}\times\phi)\right]duds\\
=\;&\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}\frac{d\vec{t}}{ds}\cdot\chi\left(\int_0^s+\int_{2\pi}^s\right)\frac{d}{ds}(\phi\cdot\vec{t}\times\psi)duds\\
=\;&\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}\frac{d\vec{t}}{ds}\cdot\chi(\phi\cdot\vec{t}\times\psi)ds. \qquad\qquad (A.8)
\end{aligned}$$

On the other hand, integrating by parts we rewrite

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{d\phi}{ds}\cdot\vec{t}(\psi\times\vec{t})\cdot\chi ds = -\frac{1}{2\pi}\int_0^{2\pi}\frac{d\vec{t}}{ds}\cdot\phi(\psi\cdot\vec{t}\times\chi). \qquad (A.9)$$

Therefore

$$< K'_{\vec{t}}(\phi, J_{\vec{t}}\psi), \chi > \;+\ldots\; = -\frac{1}{4\pi}\int_0^{2\pi}\frac{d\vec{t}}{ds}\cdot\chi(\phi\cdot\vec{t}\times\psi)ds \;+\;\ldots \qquad (A.10)$$

Writing $\phi = a\vec{n}+b\vec{b}, \psi = c\vec{n}+d\vec{b}, \chi = e\vec{n}+f\vec{b}$, we compute

$$\begin{aligned}
&\frac{d\vec{t}}{ds}\cdot\chi(\phi\cdot\vec{t}\times\psi)+\frac{d\vec{t}}{ds}\cdot\psi(\chi\cdot\vec{t}\times\phi)+\frac{d\vec{t}}{ds}\cdot\phi(\psi\cdot\vec{t}\times\chi)\\
=\;&k\{e(bc-ad)+c(af-eb)+a(ed-cf)\} = 0. \qquad\qquad (A.11)
\end{aligned}$$

Therefore also the right-hand-side of the coupling condition (A.4) vanishes.

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