DIRECT BOUNDARY ELEMENT ANALYSIS OF ANISOTROPIC SOLIDS AND POROUS MATERIALS

by

SHERIF MOHAMED ABDELFATAH ELKHOLY

A Dissertation Submitted to the Faculty of the

DEPARTMENT of CIVIL ENGINEERING AND ENGINEERING MECHANICS

In Partial Fulfillment of the Requirements
For the Degree of

DOCTOR OF PHILOSOPHY
WITH A MAJOR IN CIVIL ENGINEERING

In the Graduate College

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UMI Number: 9604512

THE UNIVERSITY OF ARIZONA
GRADUATE COLLEGE

As members of the Final Examination Committee, we certify that we have read the dissertation prepared by Sherif Mohamed Abdelfatah ElKholy

entitled Direct Boundary Element Analysis of Anisotropic Solids and Poroelastic Materials

and recommend that it be accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy

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Date

Date

Final approval and acceptance of this dissertation is contingent upon the candidate's submission of the final copy of the dissertation to the Graduate College.

I hereby certify that I have read this dissertation prepared under my direction and recommend that it be accepted as fulfilling the dissertation requirement.

Dissertation Director — Date 7/7/95

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ACKNOWLEDGMENTS

I thank Allah (GOD) for allowing me to finish my graduate studies and for surrounding me with people who made this work possible.

I give enormous thank-you from the very bottom of my heart to my advisor, Dr. Dinshaw Contractor, for the time, encouragement, support, and guidance which he freely gave throughout the course of this research. I thank him very much for everything he did with me throughout my graduate studies.

I very much thank Dr. Achintya Halder, Dr. Rajesh Srivastava, Dr. Tribikram Kundu, and Dr. Abhijit Chandra for their guidance, support and for serving as members of committee.

I am deeply grateful to my wife, Dr. Hanan ElSayed Samaha, a person who cared about me, and always put me in front of herself during this difficult period of our life, and for her continued support, patience, and encouragement throughout my graduate studies. A very special thank-you goes to the most wonderful children, Mohamed and Amro, who gave me love and happiness.

Many thanks are given to my parents, my brothers and sisters, and my friends among whom I grew up and who loved me and support me throughout my whole life.

Finally, I am grateful to the Water Research Center and the Egyptian government for giving me the chance and the financial support to pursue my graduate studies in the USA. Their support and encouragement are highly appreciated.

DEDICATION

This dissertation is dedicated with my deepest gratitude to the most caring parents and the most wonderful wife in the whole world, my father: Mohamed Abdel-Fatah ElKholy, my mother: Aidah ElOkdah, and my wife: Dr. Hanan ElSayed Samaha. I pray to Allah (GOD) to unite us in paradise.

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ABSTRACT

The main objective of this dissertation is to investigate the application of the boundary element methods (BEM) to problems with anisotropic material properties in the areas of transient flow, elastostatics, and soil consolidation. The resulting BEM formulations represent a major contribution for this class of problems which has not received much of the researchers' attention.

The BEM formulations developed are entirely in terms of boundary quantities which reduces the dimensionality of the problem by one. This advantage makes the BEM an attractive alternative to the more popular domain-based techniques, such as finite element and finite difference methods for this category of problems. This gives researchers and engineers a wider range of numerical analysis tools to consider when investigating a problem in the field.

The above mentioned formulations have been implemented in two computer programs, TRANSBE and EL10, which can be used for transient flow and elastic analysis of bodies with anisotropic material properties. Also, a new approach, based on these two programs, is presented to perform consolidation analysis in soils with anisotropy in both elastic and flow properties.

Finally, a number of problems, which cover the three

areas of interest, are solved in order to verify the effectiveness and accuracy of the developed computer programs.

1. INTRODUCTION

1.1 General

The boundary element method (BEM) has been gaining popularity among researchers and engineers as a powerful and accurate alternative to the more popular finite element (FEM) method in performing numerical analyses. This is particularly true when the problems can be analyzed with only a boundary discretization with no need for volume discretization as is the case in FEM analysis. In such a case, the dimensionality of the problem is reduced by one, which in turn reduces the complexity of the analysis. Moreover, in the BEM, numerical differentiation is used to evaluate secondary variables such as stresses which is known to be more accurate than the numerical integration which is used in the FEM.

However, not all the problems analyzed by the BEM can be solved with only boundary discretization. In many BEM analyses, a volume discretization is necessary, but this does not have the same effect on the complexity of the analysis as in the case of FEM analysis (Dargush, 1987). Initially, researchers used the BEM for simple problems such as potential flow and linear elastostatics. During the past two decades, the use of BEM has been extended to include more complex problems such as transient elastodynamics, nonlinear elasticity, soil consolidation, and elastoplasticity. In most of these studies the continuum was assumed to be isotropic because of its simplicity. However, most studies concluded with the recommendation to extend the formulation to the more realistic situation which involves anisotropic conditions, especially in case of soils and rocks which are rarely found to be isotropic in the field (Atkinson, 1975; Tomlin and Butterfield, 1974). Therefore, the present work will focus on extending the BEM analysis to the class of problems in which the properties of flow permeability and elastic deformation are assumed to be anisotropic. The relevant engineering problems occur in the fields of thermomechanics and soil mechanics. Since Dargush (1987) proved the complete analogy of the two fields of engineering practice, any boundary element technique developed for one of the two fields can be utilized for the other. These two fields cover a wide range of problems in engineering practice. In the field of thermomechanics, problems include stress analyses of automotive engine blocks, and gas turbine heat transfer analysis are examples of these problems. In the field of soil mechanics, problems such as flow through porous media, soil consolidation, settlement analysis under structures, embankments and dams can be efficiently analyzed using the BEM. In all of these applications, a process of transient flow of water or heat is known to be the dominant phenomenon which affects the behavior of the problem medium.

1.2 Dissertation Outline

The next chapter presents an overview of the different methods available to analyze the problems under investigation. Next, a review of the literature related to this research is presented. In chapter 3 the boundary element formulation for the problem of two-dimensional anisotropic transient flow is presented. The chapter starts with a short discussion of the characteristics of the anisotropic transient flow and the associated governing equations. Next, the fundamental solution of the problem is presented followed by the reciprocal theorem, both of which represent the main elements of a boundary element formulation. Using these elements, the boundary element formulation of the problem is then presented followed by the details of the numerical implementation. In chapter 4 a similar structure is repeated for the two-dimensional elastic analysis of anisotropic bodies. Chapter 5 is devoted to studying the coupling between these two problems which represent the two main phenomena occurring during the consolidation of a block of soil under load. The method of achieving this coupling will be presented. In chapter 6 a description of the different elements of the computer programs developed in this study are presented. Then the validation of these programs will be shown through the use of a selected number of applications. Finally, chapter 7 provides a brief discussion of the results along with the main conclusions of the study. Recommendations for future work are also presented. All the references are cited in alphabetical order in Appendix A. The figures and tables appear at the end of each chapter. Throughout the dissertation, the standard indicial notation is used. Thus, repeated indices imply summation and commas represent differentiation with respect to spatial coordinates.

2. REVIEW OF LITERATURE

2.1 General

This chapter provides a comparative overview of the available solution techniques to handle the problems considered in this research. Next, a review of the literature on the boundary element method and its usage in similar studies is presented.

2.2 Review of solution techniques

The problems of anisotropic elasticity and transient flow in porous medium are far more complex than the usual isotropic ones. Therefore, there are very few closed-form analytical solutions available even for problems with very simple geometry and boundary conditions. In fact, a solution for the coupled problem (which represents soil consolidation) does not even exist in the literature. For problems with complex geometry and boundary conditions, numerical techniques must be used. In that regard, there are three general purpose numerical methods associated with engineering analyses. They are Finite Difference, Finite Element, and Boundary Element methods.

The finite difference methods (FDM) are applied directly on the governing differential equation. In this method, the entire continuum is discretized in a grid composed of nodal points. Next, the difference operators, which approximate the derivatives, are applied at each node. This procedure converts the governing differential equation into a set of algebraic equations. If the problem is time dependent, the time axis is also discretized and the solution is obtained by marching in time. The strength of this method is that the transformation from the governing equation is mostly straightforward and results in a generally simple and numerically efficient set of algebraic equations. However, the main weakness of the method is the relative inaccuracy of the involved numerical differentiation compared to numerical integration. This usually requires a very fine grid in order to get reasonable accuracy. Other drawbacks of the FDM include the limited applicability of the method to problems with generally simple geometry and boundary conditions. The evaluation of secondary quantities such as stresses is not simple and requires additional numerical differentiation which reduces the accuracy even further. Regardless of these disadvantages, the FDM is still very popular among researchers in heat transfer, fluid flow, and other time dependent problems.

In the finite element method (FEM), the entire domain is discretized into elements of relatively simple geometry defined by nodal points. Shape functions are then used to completely describe both the geometry of the elements and the variation of the primary variables within the elements in terms of the nodal values. A variational principle, such as the principle of minimum potential energy, is then applied and the resulting integration is performed numerically to produce a set of algebraic equations. This set of equations is usually banded and symmetric which is easier and more efficient to solve. FEM is capable of handling problems with complex geometry and boundary conditions as well as nonlinear problems. The computer programming is usually straightforward and relatively simple to implement, enhance, and extend to new applications. Therefore, a wide range of general purpose programs are available and they cover a large variety of real engineering problems. These features give the FEM a great appeal among researchers. However, FEM still has its disadvantages, the most important of which is the need to discretize the entire domain. Geometric modeling can become a time and effort consuming process. The method is not good in handling problems with infinite boundaries. Moreover, the secondary quantities such as stresses and strains are usually calculated from numerical differentiation which, as was mentioned before, is significantly less accurate than numerical integration.

With these drawbacks in mind, the boundary element method (BEM) was developed to overcome some of them. The BEM is based on the use of an appropriate reciprocal theorem derived from the governing differential equations along with initial conditions. Then, the associated fundamental solution (also known as Green's Function) is applied; resulting in a boundary integral equation which in most cases involves only surface integral terms (3-D) or line integral terms (2-D). The boundary of the domain is discretized into elements defined by nodal points similar to the procedure followed in FEM but only on the boundary. The integral equation is then applied at each node and the integral terms are evaluated numerically to produce a set of algebraic equations which, despite their lack of null coefficients and symmetry, are simple to solve. The BEM has many advantages, the most important of which is that only surface integration is required in many of the problems. This way the dimensionality of the problem is reduced by one which in turn reduces the computer time and effort in setting up and solving the problem. The governing differential equations are satisfied exactly with no approximation inside the domain through the use of the fundamental solution. The approximation is only used in the representation of geometry and variables on the boundary. The solution is based on a numerical integration rather than numerical differentiation which yields a better accuracy. The BEM is also capable of handling problems with infinite boundaries. The disadvantages of the BEM include the occasional need for some domain integral which usually eliminates most of its advantages. The singularity of the fundamental solutions increases the difficulty of the mathematical preparation and computer programming of the problem. Special numerical techniques are required to evaluate these singular integrations and reduce the errors associated with them. Due to these difficulties, the computer programming of BEM analyses are more complex than FEM. Therefore, general purpose programs have only appeared recently. With the presentation of these programs, researchers has been comparing BEM and FEM with regard to computational efficiency and accuracy. Mukherjee and Morjaria (1982) compared the BEM and FEM for solving the Laplace equation. They found out that for the same level of discretization, the results from BEM were more accurate than those from FEM. This implies that for the same degree of accuracy, BEM requires a coarser mesh i.e. less computer time and effort. Studies by Wilson et al. (1986) and Wanderlingh (1986) show that BEM provides significant savings over FEM for problems with bulk solids. Dargush (1987) and Dargush and Banerjee (1989a) concluded that generally, the comparison between FEM and BEM is problem dependent. Therefore, it is important that any problem be investigated carefully before choosing the method of numerical analysis. In the following section, a review of the literature available on BEM will be presented.

2.3 Review of literature

A review of the history of the BEM shows that Fredholm (1903, 1906) was the first to lay the foundation of the Boundary Integral Equation Method. The details of the main elements of the method were introduced later on by Muskhelishvilli (1953) and Mikhlin (1957). However, the applications of the method were very limited due to the complexity associated with the mathematical operations especially in the absence of high speed digital computers. With the development of computer technology, the BEM has become accepted by researchers during the past two decades. Its applications expanded to include various fields of structural and fluid mechanics (Banerjee and Butterfield, 1981; Brebbia and Worbel, 1982; Brebbia et al., 1985, 1987..etc; Mukherjee, 1982; Dargush, 1987; Dargush and Banerjee, 1989a, b; Deb et al. 1991; Chan and Chandra; 1991, Chandra and Chan, 1992). This review of literature will cover the three areas considered in this research, elasticity problems, transient flow problems, and coupled problems.

Dargush (1987) presented a detailed review of literature for isotropic elasticity and transient flow, therefore only the literature on anisotropic studies will be presented here. Rizzo and Shippy (1970) appear to have been the first to investigate the BEM formulation for anisotropic elasticity. The boundary integral equation was derived by using Betti-Maxwell reciprocal theorem and the fundamental solution for two-dimensional transversely isotropic media. Benjumea and Sikarskie (1972) attempted indirect boundary element approaches for solving the planar orthotropic problems. Tomlin and Butterfield (1974) applied the indirect boundary element formulation for the two-dimensional stress analysis of zoned orthotropic media. Synder and Cruse (1975) used a direct boundary element approach similar to Rizzo and Shippy's for solving problems of cracked anisotropic plates. Krenk (1978) applied the direct boundary element approach to solve problems of stress concentrations in anisotropic sheets with holes. Vogel and Rizzo (1973) presented the first trial of direct boundary element approach to three-dimensional elastostatic stress analysis of generally anisotropic media. They noted that the fundamental solution for that case can not in general be obtained in closed form. They therefore introduced a numerically computed fundamental solution. Wilson and Cruse (1978) continued the work of Vogel and Rizzo (1973) and presented an efficient numerical scheme for the evaluation of that fundamental solution. Deb et. al. (1991) presented an improved and effective computational procedure for the fundamental solution in three-dimensional problems by converting the integral representation into a finite series representation. Deb and Banerjee (1991) presented a boundary-only formulation for the two-dimensional transversely isotropic elastostatic problem by using the particular integral to account for the body forces type loadings. The method of particular integral is too complex for practical use, however, it was very efficient numerically.

In the area of flow of fluids or temperature, Rizzo and Shippy (1977) addressed the three-dimensional steady-state thermoelasticity case; assuming that the complete description of the surface temperature and flux is known a prior. They were able to convert the domain integral term appearing in the integral equation into a surface integral term. Cruse et al. (1977) derived a boundary-only formulation for the axi-symmetric steady-state thermoelasticity case making use of a Galerkin vector approach. Tanaka et al. (1984) obtained the BE solution for the steady-state diffusion problem in three dimensions using the moving heat source solution of Jaeger as a fundamental solution. Their results were in good agreement with the available analytical solution. Dargush (1987), Dargush and Banerjee (1987, 1989a, b, 1990) presented a series of papers for the application of BEM in isotropic steady-state and transient problems of heat and water flow based on a fundamental solution developed by Clearly (1977). Their work was a state-of-the art boundary-only formulation and the verification included many study cases. Curran et al. (1980), Brebbia and Worbel (1982) and Banerjee and Butterfield (1981) used a time-dependent fundamental solution to solve the isotropic diffusion problem. However, the formulation has a domain integral term.

Recently, Chan and Chandra (1991) and Chandra and Chan (1992) investigated the thermal aspects and their design sensitivities in a steady-state machining process which is analogous to the thermoelasticity phenomenon. Their analysis has proved very effective in capturing the sharp temperature gradients. Chan and Chandra (1992) developed a BE formulation for the transient heat conduction-convection problem based on a time-dependent fundamental solution. The formulation contains boundary only integral terms, the variation of variables was linear in both time and space. In all the above mentioned studies, the recommendations were to study the anisotropic behavior since it represents the general behavior in the field.

Moving next to the area of coupling the elasticity and transient flow problems to represent the phenomenon of poroelasticity or soil consolidation. Banerjee and

Butterfield (1981) discussed a staggered procedure for solving the problem. The algorithm requires solving the transient fluid flow equation first followed by a deformation analysis at each time step. The formulation had a domain integral term, however the results obtained were in good agreement with the analytical solution. Figure 2.1 shows the results for the strip footing example solved by Banerjee and Butterfield (1981). The same technique was used by many other researchers such as Kuroki et al. (1982), Aramaki and Yasuhara (1985), and Aramaki (1986). Giuseppe et al. (1987) used the same idea to perform a BE analysis of land subsidence above three-dimensional gas/oil reservoirs. In their study, they assumed that the pore water pressure distribution is well defined (from a transient analysis or field measurements) before doing the deformation analysis. Another approach to the problem is to transfer the entire problem into the Laplace domain in order to get rid of the time dependency of the problem. After the solution is obtained, it is then transformed back to the real domain. Cheng and Liggett (1984) used that technique for two-dimensional quasistatic poroelasticity problems. While this is a boundary-only formulation, the procedure itself is very complicated and the transformation into and back from Laplace domain introduces serious side effects. The solution is very sensitive to the selection of transformation parameter and the numerical inversion of the transform (Dargush, 1987). Nishimura(1985) developed a time domain BE formulation for two-dimensional consolidation problems. He presented a fundamental solution and numerical implementation for a simple one element problem. Dargush (1987) studied that solution and concluded that errors do exist in it and therefore the results presented are questionable. Dargush and Banerjee (1989a, b, 1990) presented a series of papers on a new BE formulation for the consolidation problem based on the fundamental solution developed by Cleary (1977). The formulation was completely coupled in which the two governing differential equations of elasticity and transient flow were solved simultaneously. They presented solutions for the two-dimensional, and the axi-symmetrical problems. Their work can be considered as the best available in the area of isotropic consolidation and thermoelasticity. Figures 2.2 and 2.3 show the result for two examples solved by Dargush and Banerjee (1989a). They concluded that the investigation should be extended to the anisotropic area. However, since the fundamental solution for the anisotropic soil consolidation case does not exist, a different approach is offered in Chapter 5.

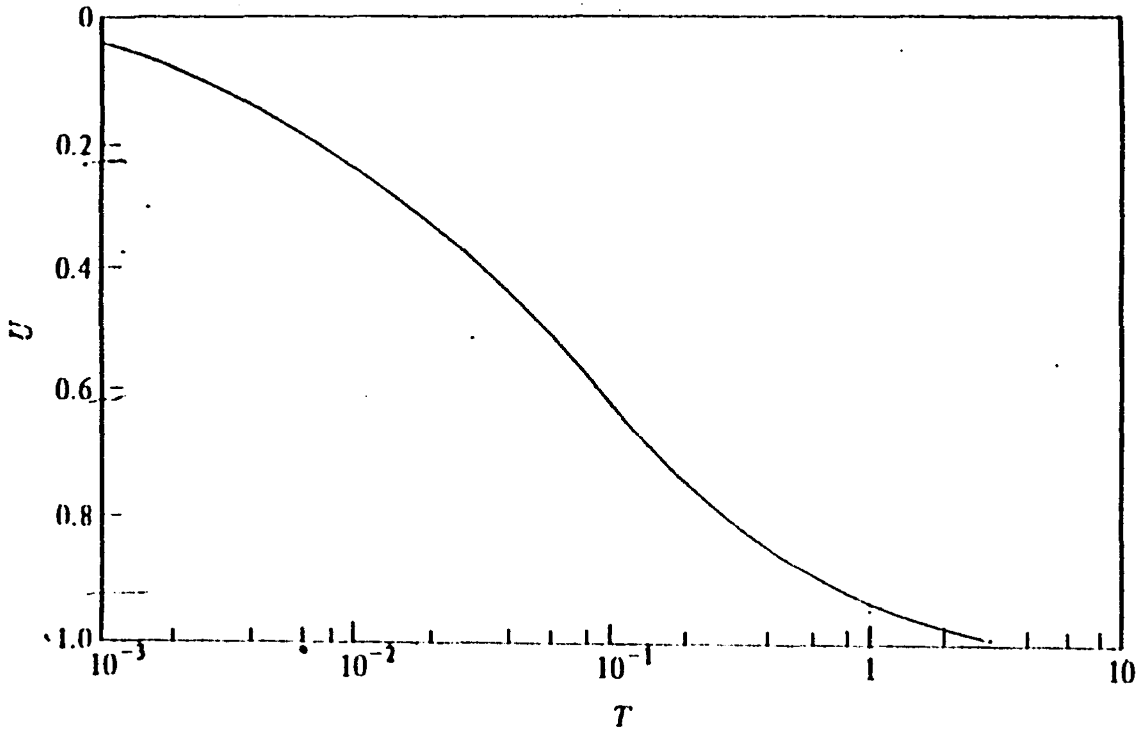


Figure 2.1 Consolidation behavior of soil layer under a strip footing (Banerjee and Buttrefield, 1981)

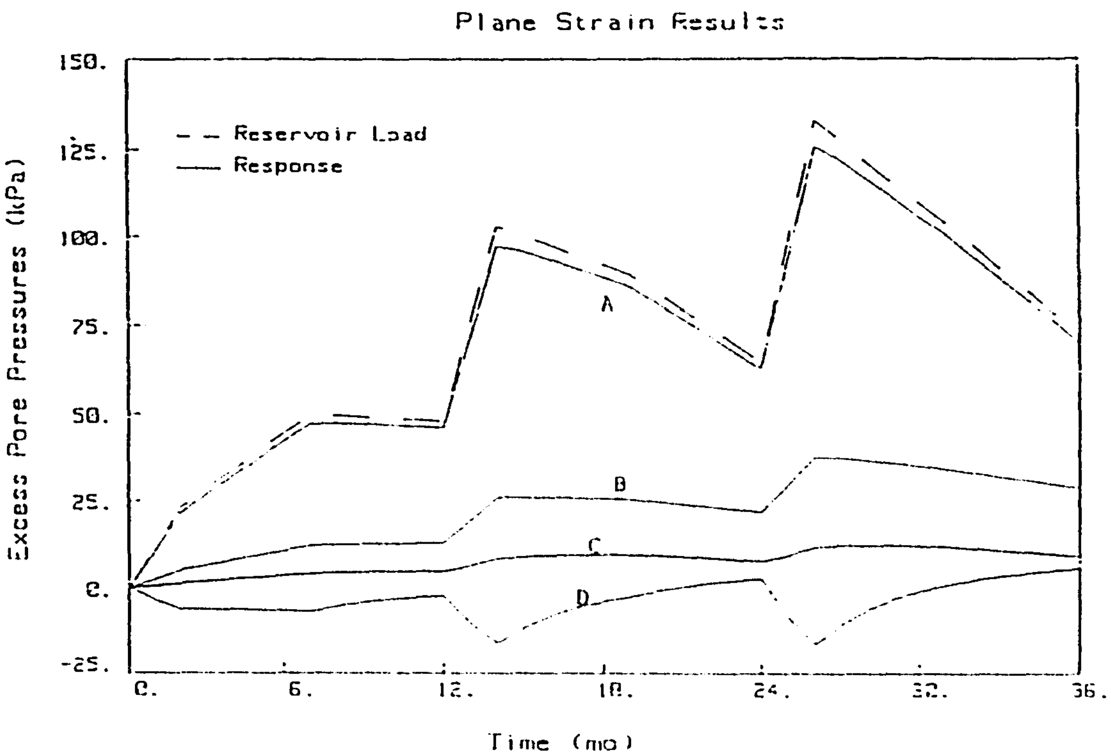


Figure 2.2 Consolidation behavior of soil at the Koyna dam (Dargush and Banerjee, 1989)

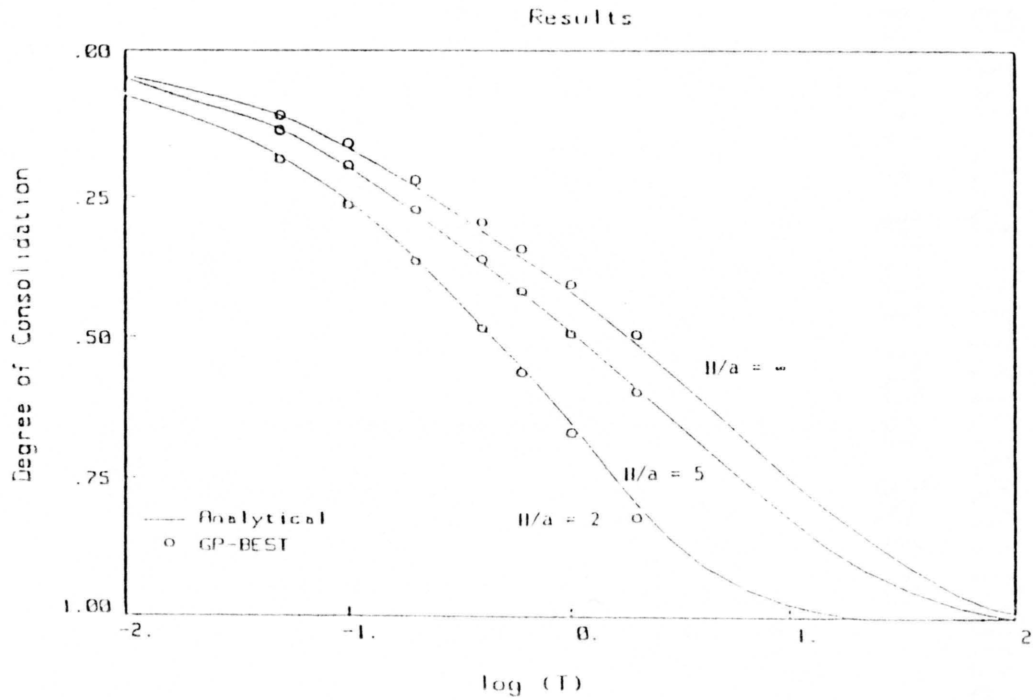


Figure 2.3 Consolidation behavior of soil layer under a strip footing (Dargush and Banerjee, 1989)

3. BOUNDARY ELEMENT FORMULATION FOR 2-D DIFFUSION PROBLEMS

3.1 General

The diffusion problem has a significant importance among researchers since it has applications in two major areas of engineering practice, flow of ground water in porous medium and heat transfer (heat conduction). These two areas share the same governing differential equation with different interpretation of the variables as will be explained in the coming sections.

This chapter reviews the governing differential equations and the effect of anisotropy of the medium on the flow of ground water through it. Then, the basic elements of BEM formulation, the fundamental solution and reciprocal theorem will be presented followed by developing the boundary integral equation. Next, the discretization of the time as well as the body boundary into elements and the methodology for numerical implementation schemes are presented. This includes discussing methods of evaluating terms with singular integration. Finally, the method of evaluating the potential at any internal point in the body is presented; followed by the summary. The computer program developed for this problem is presented in chapter 6.

3.2 Governing differential equations

The general governing differential equation for the diffusion problem can be written in a general form as (Chang, 1973):

$$C_{ij} \frac{\partial^2 p(x,t)}{\partial x_i \partial x_j} - \frac{\partial p(x,t)}{\partial t} = 0 \tag{3.1}$$

with the boundary conditions defined as:

$$\begin{aligned} p(x,t) &= P(x,t) && \text{on } S_1 \\ q(x,t) &= \frac{\partial p}{\partial n^*} = Q(x,t) && \text{on } S_2 \end{aligned} \tag{3.2}$$

Notice that:

$$\frac{\partial}{\partial n^*} = \sum_{i,j=1}^{2} C_{ij} \cos(n, x_i) \frac{\partial}{\partial x_j}$$

where n is the normal to the domain surface. Since it is a time dependent problem, some initial conditions at time t = t_o must also be specified as:

$$p(x,t) = P_o(x,t) \tag{3.3}$$

The variables of these equations can be defined for the two kinds of problems mentioned above as follows:

p(x,t): either the temperature or fluid potential field at any point X_i within the domain V at time t,

q(x,t): the temperature gradient (heat flux) or potential gradient (hydraulic flux) at any point X_i within the domain V at time t,

P(x,t), Q(x,t): are arbitrary functions representing the specified boundary conditions on the segments of the boundary S_1 and S_2 which constitute the total boundary S of the body

C_{ij}: the general form of thermal diffusivity or fluid hydraulic conductivity tensor.

Notice that the term C_{ij} may include, based on the problem under consideration, terms such as the depth of the aquifer, the storage coefficient, specific weight of fluid, and the permeability of the medium. The problem presented by the above system of equations will be recast into a boundary-only integral equation for the unknown function p. Solving this integral equation determines the values of p or q at any point on the boundary surface; from which the value of p at any point inside the body will be computed.

3.3 Anisotropy

Materials occurring in nature are seldom isotropic and even less frequently homogeneous. Therefore, anisotropy is a common phenomenon in porous medium such as soils and rocks. Its effect on flow of ground water through porous media has been of particular interest to both hydrologists and geotechnical engineers. Because the directions of the flow and of the hydraulic gradient in an anisotropic porous medium are generally not parallel, two different methods exist to define the directional permeability (Marcus, 1962). The more widely used is the one in which the directional permeability is defined as the ratio of the flow velocity to the component of the potential gradient in the direction of flow. Scheidegger (1960) presented the second definition in which the direction permeability is expressed as the ratio of the component of the velocity in the direction of the potential gradient to the magnitude of the potential gradient. These two definitions do not produce the same results. The difference depends on the anisotropy of the medium and the direction of flow or flow gradient relative to the directions of principal permeability (Marcus and Evenson, 1961). Marcus (1962) found out that these two definitions represent the two extremes between which the lab-measured permeability values exist.

In the most general form, the permeability matrix takes the form:

$$K_{ij} = \begin{bmatrix} K_{xx} & K_{xy} \\ K_{yx} & K_{yy} \end{bmatrix} \qquad (3.4)$$

where, K_{xx} and K_{yy} are the permeabilities in the x and y directions by the first definition mentioned above. Assuming that K_{xy} is very small compared to both K_{xx} and K_{yy} and for simplicity taking the x-y coordinate system to be the principal axes, then the permeability tensor takes the form:

$$K_{ij} = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} \qquad (3.5)$$

where K_1 and K_2 are the major and minor permeability values respectively.

3.4 Boundary element formulation

3.4.1 Fundamental solution of the anisotropic diffusion problem

The fundamental solution for this problem was developed by Chang et al. (1973). It is simply the solution of the governing differential equations for a body in an infinite domain loaded with a concentrated flow source acting at point ξ in the j-direction. This source can be represented by the following function:

$$\phi_j(\xi, t) = \delta(\xi_1)\ \delta(\xi_2)\ H(t) \tag{3.6}$$

where $\delta(\xi_i)$ and H(t) are the standard Dirac delta and the Heaviside step functions. This source produces the following potential and flux at any point x:

$$\begin{aligned} p(x,t) &= P^*\ (x,t;\xi,t_F) \\ q(x,t) &= Q^*\ (x,t;\xi,t_F) \end{aligned} \tag{3.7}$$

where:

$$P^* = \frac{|K^{ij}|^{1/2}}{4\pi(t_F-t)} \exp\left(\frac{-R^2}{4(t_F-t)}\right)$$

$$Q^*(x,t;\xi,t_F) = \frac{D_s\ |K^{ij}|^{1/2}}{8\pi(t_F-t)^2} \exp\left(\frac{-R^2}{4(t_F-t)}\right) \tag{3.8}$$

where:

$|K^{ij}|$: is the determinant of the inverse of the K_{ij} matrix

R : is the Euclidean distance from the source point to the field point defined as;

$$R^2 = \sum_{i,j=1}^{2} K^{ij}\ (X_i-\xi_i)\ (X_j-\xi_j) \tag{3.9}$$

D_s : defined as;

$$D_s=\cos(n,X_1)K_{11}K^{11}(X_1-\xi_1)+\cos(n,X_2)K_{22}K^{22}(X_2-\xi_2) \quad (3.10)$$

t_f, t : the final and current time respectively

X_i, ξ_i : the field and source point respectively

exp : the exponential function

3.4.2 Reciprocal theorem

The reciprocal theorem for a body of volume V and surface S with zero initial conditions can be written as (Ionescu-Cazimer, 1964):

$$\int_S [q^{(1)}*p^{(2)}-p^{(1)}*q^{(2)}]\,dS+\int_V [\phi^{(1)}p^{(2)}-p^{(1)}\phi^{(2)}]\,dV=0 \quad (3.11)$$

where superscripts (1), (2) denote any two independent states, defined by ($p^{(1)}$, $q^{(1)}$, $\phi^{(1)}$) and ($p^{(2)}$, $q^{(2)}$, $\phi^{(2)}$) respectively, existing in the body. Note that, ϕ is any existing flow source, and the symbol * indicates a Riemann convolution integral, where, for example;

$$q^{(1)}*p^{(2)}=\int_0^t q^{(1)}(x,t_F-t)\,p^{(2)}(x,t)\,dt=\int_0^t q^{(1)}(x,t)\,p^{(2)}(x,t_F-t)\,dt$$

This reciprocal theorem will lead to a boundary element formulation with only the potential and flux as the primary and only variables; and no secondary quantities (Dargush, 1987).

3.5 Boundary element equation

The reciprocal theorem mentioned above involves any two independent states existing in the body under consideration. Let one of these states, say state (2), be the, as yet, unknown solution to a given boundary value problem defined by the governing differential equations. Then the remaining state, state (1), may be chosen arbitrarily. We will select that state to be the infinite region response to a unit step source acting at the point ξ within the domain V which is the fundamental solution. This process will eliminate the domain integral which exists in the reciprocal theorem as will be shown later on. For simplicity, it is assumed that no actual sources exist in the actual boundary value problem. Substituting in the reciprocal theorem results in:

$$\int_S [Q^*(x,t;\xi,t_F) * p^{(2)}(x,t) - P^*(x,t;\xi,t_F) * q^{(2)}(x,t)]\, dS(x) + \int_V [\delta(x-\xi)\,\delta(t)\,\delta_{ij} * p^{(2)}(x,t)]\, dV = 0 \tag{3.12}$$

The volume integral in this equation may be simplified as follows:

$$\int_V [\delta(X-\xi)\,\delta(t)\,\delta_{ij} p^{(2)}(x,t)]\,dV(X) = \delta_{ij} p^{(2)}(\xi,t) \qquad (3.13)$$

Dropping the superscript (2) from the equation, the BEM equation can be written as follows:

$$\delta_{ij}\; p(\xi,t) \;=\; \int_S [P^*(x,t;\xi,t_f)\;\; q(x,t;\xi,t_f) - Q^*(x,t;\xi,t_f)\;\; p(x,t;\xi,t_f)]\;\; dS(x) \qquad (3.14)$$

This equation computes the potential at any instant t, at any interior point ξ within the domain V as a function of the boundary variables p_i and q_i. The equation involves only a boundary integral and is an exact statement with no approximation. However, in order to determine the interior variables, the entire history of the boundary values of p_i and q_i must be known. In a well-posed boundary value problem, only half of that information will be available, at any instant of time, as boundary conditions. In order to solve the problem and determine the missing information, the point ξ has first to be moved to the boundary. The process of moving the point ξ to the boundary has been explained in almost every text book on boundary element methods, therefore it will not be repeated here. The end result of this process is the following boundary element equation for the 2-D diffusion equation:

$$C(\xi)\ p(\xi, t_F) = \int_{t_0}^{t_F} dt \int_S [P^*(x, t; \xi, t_F)\ q(x, t) - Q^*(x, t; \xi, t_F)\ p(x, t)]\ dS(x) \tag{3.15}$$

C(ξ) is a function only of the local geometry of the boundary at the point ξ and is reduced to ½ only along a smooth surface. For complex surfaces the explicit calculation of C(ξ) is not always necessary and in many problems it can be evaluated indirectly (Brebbia et al., 1984 and Dargush, 1987). Solving this equation with the given boundary conditions results in the determination of the unknown terms of the vectors p_i and q_i on the boundary which can then be used to determine the value of the potential at any interior point within the domain V.

3.6 Numerical implementation

3.6.1 Introduction

Although the developed boundary element equation (3.15) is an exact statement with no approximation, applying it to the solution of engineering problems requires some approximation to represent the geometry of the problem as well as the variables p_i and q_i in a similar way to the techniques used in the finite element method. The techniques which will be discussed here were developed previously for isotropic elastostatic and transient flow problems (Dargush, 1987; Brebbia, 1984; and Banerjee and Butterfield, 1981).

In the following subsections we will discuss, in order, the time discretization, the spatial discretization, integral singularities, numerical integration, assembly, solution, and the evaluation of interior quantities.

3.6.2 Time discretization

The time variation of the function p and q is not known a prior, therefore, a time stepping technique has to be introduced. In the literature, three different time-marching techniques are used for the numerical solution of equation 3.15. The first, treats each time step as a new and individual problem. Therefore, the initial conditions at time t_o must be updated at the end of each time step to be used for the subsequent one. In the second technique, the time integration process always starts from the actual initial time t_o. This technique saves the time and computer effort required to update the initial conditions at the end of each time step. However, the number of matrices used increases to take into account the effect of all the time steps from time t_o to the current time t_n. The effect of this disadvantage can be minimized during the final time steps where such effects can be ignored as one progresses away from the initial time (Brebbia and Wrobel, 1982). A third technique was proposed by Taigbenu and Liggett (1985) in which they discretized the time into a number of time steps with large value, ΔT, which is then subdivided into smaller sub-steps $\Delta t = \Delta T/N_s$. Taigbenu and Liggett (1985) studied the three techniques and recommended that the second technique has the best overall advantage with respect to time and computer effort. This technique is the one used in this study.

Starting with the boundary integral equation (3.15), discretizing the time from t_o to t_F into $N\Delta t$, we can rewrite the equation in the form:

$$C(\xi)p(\xi,t_F) = \sum_{n=1}^{F} \left(\int_S dS \int_{t_{n-1}}^{t_n} (P^* q^n - Q^* p^n)\ dt\right) \quad (3.16)$$

Assuming that the functions p and q remain constant during each time step, therefore, they can be brought out of the integral. Thus, the equation has only the following two forms of time integral:

$$I1 = \int_{t_{n-1}}^{t_n} P^* dt \quad , \quad I2 = \int_{t_{n-1}}^{t_n} Q^* dt \qquad (3.17)$$

These time integrals can be carried out analytically. The details of that process are listed in the appendix. It yields the following results:

$$I1 = \frac{|K^{ij}|^{1/2}}{4\pi} [E_1(a_{n-1}) - E_1(a_n)] \qquad (3.18.a)$$

and

$$I2 = \frac{D_s |K^{ij}|^{1/2}}{2\pi R^2} [\exp(-a_{n-1}) - \exp(-a_n)] \qquad (3.18.b)$$

where:

$$a_{n-1} = \frac{R^2}{4(t_f - t_{n-1})}, \qquad a_n = \frac{R^2}{4(t_f - t_n)} \qquad (3.19)$$

The variables in these equations were previously defined. Note that t_F is the final time step at which the problem is being solved and n varies from 1 to F. The terms, exp and E_1, are respectively, the exponential function and the

exponential-integral function. E_1 is defined as:

$$E_1(Z) = \int_Z^{\infty} \frac{e^{-X}}{X}\, dX$$

Applying this time integration to the boundary integral equation (3.16) leads to:

$$C_i P_i^{F} = \sum_{n=1}^{F} \int_S [I1_S^{F+1-n} q(x)^n - I2_S^{F+1-n} p(x)^n]\, dS(x) \quad (3.20)$$

<u>3.6.3 Spatial discretization</u>

This process begins by subdividing the entire surface of the body into a number of individual elements, M, of relatively simple shape defined by nodal points. The geometry of each one of those elements,m, is defined by the coordinates of the nodal points and the associated shape functions. That is:

$$X(\xi) = X_i(\xi) = N_w(\xi)\ X_{iw} \quad (3.21)$$

where:

ξ : intrinsic coordinates

N_w : shape functions

X_{iw} : nodal coordinates

and w is an integer varying from one to W, the number of geometric nodes in the element. Next, the same kind of representation is used, within the element, to describe the primary variables. Thus,

$$p_i(\xi) = N_w(\xi)\ p_{iw} \tag{3.22.a}$$

$$q_i(\xi) = N_w(\xi)\ q_{iw} \tag{3.22.b}$$

in which p_{iw} and q_{iw} are the nodal values of the potential and flux respectively at node w of the element i. Note, that it is not necessary to use the same number of nodes, and consequently shape functions, to represent both the geometry of the problem as well as the primary variables. However, it will be simpler and computationally-efficient if we did. In the present work, both geometry as well as the primary variables were described by linear shape functions, for which two-noded elements are employed. The linear shape functions used here are exactly the same as in the finite element method.

Once the spatial discretization has been accomplished and the body surface has been subdivided into N elements, the boundary element equation can be rewritten for any boundary node i, at any instant of time n, as:

$$C_i p_i = \sum_{n=1}^{F} \sum_{m=1}^{M} \int_{S_m} N_w^T \; [\,(I1_{im})^{F+1-n} \; q_{iw}{}^n - (I2_{im})^{F+1-n} \; p_{iw}{}^n] \; dS(X(\xi)) \qquad (3.23)$$

where:

$$S = \sum_{m=1}^{N} \int S_m$$

In the above equation, q_{iw} and p_{iw} are the nodal quantities which can be brought outside the integral. Thus:

$$C_i p_i{}^F = \sum_{n=1}^{F} \sum_{m=1}^{M} [\; q_{iw}{}^n (\int_{S_m} (I1(\xi))_{im}^{F+1-n} N_w(\xi))] - \sum_{n=1}^{F} \sum_{m=1}^{M} [\; p_{iw}{}^n (\int_{S_m} (I2(\xi))_{im}^{F+1-n} N_w(\xi))] \; dS(x(\xi))] \qquad (3.24)$$

This equation may be written in the matrices form as:

$$C_i p_i{}^F = \sum_{n=1}^{F} G_{nF} \; q^n - \sum_{n=1}^{F} H_{nF} \; p^n \qquad (3.25)$$

If we include the term on the left hand side of the equation in the H matrix term, the equation becomes:

$$\sum_{n=1}^{F} G_{nF} q^{n} - \sum_{n=1}^{F} H_{nF} p^{n} = 0 \tag{3.26}$$

where:

$$\begin{aligned} G_{nF} &= \sum_{n=1}^{F} \sum_{m=1}^{M} (I1_{im})^{F+1-n} N_{w}(\xi) \; dS(X(\xi)), \\ H_{nF} &= \sum_{n=1}^{F} \sum_{m=1}^{M} (I2_{im})^{F+1-n} N_{w}(\xi) \; dS(X(\xi)) \end{aligned} \tag{3.27}$$

Taking the nodal quantities outside the integral resulted in the integral equation containing only known functions and thus the integral can be numerically performed. Before discussing that, the integral singularities will be examined next.

3.6.4 Integral singularities

In order to study the integration singularities, we need to examine the fundamental solution of the diffusion problem (equation 3.8). It contains only a singularity in time which occurs when $t = t_F$. However, after performing the time integration analytically, a space singularity appears in one of the two resulting integrals, I2 (equation 3.18.b). That is if R=0 (when the field and source points coincide), I2 goes to infinity. First, we study the time integration which is found in the two integral I1, and I2. When $t = t_F$, the denominator of a_n will be zero, and a_n goes to infinity. Thus, the exponential integral function $E_1(a_n)$ goes to zero. Similarly, in I2 integral, $\exp(-a_F)$ becomes $\exp(-\infty)$ which is equal to zero. Since the terms $E_1(a_{n-1})$ and $\exp(a_n)$ have no problems when $t = t_F$, the time singularity will not affect the integration process.

The second type of singularity (spatial singularity) which appears as mentioned above when R=0, is a stronger singularity, of the type $1/R^2$. This problem can be overcome by expanding the two exponential functions in a Taylor series (Lim et al., 1994).

$$\exp(a_n) = 1+\sum_{k=1}^{\infty} \frac{[a_n]^k}{k!} \quad , \quad \exp(a_{n-1}) = 1+\sum_{k=1}^{\infty} \frac{[a_{n-1}]^k}{k!}$$

Substituting these series in I2, the first term will cancel and the remaining terms will cancel the R^2 in the denominator and eliminate the space singularity.

From the discussion presented above, we note that all the integrations in the boundary element equation are regular in both time and space. They can be evaluated in a closed form and numerically (using Gaussian quadrature) respectively. The only exception is when the two singularities occur at the same time. That is when R=0 at the final time, step when $t=t_F$, because $E_2(a_F)$ will be zero and the expansion of the terms will not cancel the R^2 in the denominator. In this case, special care has to be taken in the numerical evaluation of the I2 integral making use of the definition of the term D_s. From that definition, it is clear that D_s will be zero at one of the two nodes of the linear element which makes I2 approach zero, while for the second node of the element, R does not equal zero which takes care of the singularity. Therefore, assembling the terms of the G and H matrices has to done carefully in order to accommodate the spacial singularity.

3.6.5 Assembly and solution

The complete discretization of the boundary integral equation, in both time and space has been described, as well as the techniques required for numerical implementation. Applying that equation at each node on the surface yields a system of algebraic equations which are then solved to directly determine the unknowns, p and q, at the final time step t_F, without the need to solve at the intermediate time steps. Note, that in order to solve the consolidation problem, we need the solution at the intermediate time step in order to use it to determine the deformations and stresses at any instant of time t_n, n = 1,.....,F as will be shown in chapter 5. For that case, each time step t_n will be considered as a final time t_F, but the matrices will be computed at each time step and stored to be used for the following time steps as will be explained next.

In order to understand how to assemble the matrices, we need to rewrite the boundary element equation for each node i, i = 1,....,N, in matrix form as:

$$\sum_{n=1}^{F} \left([G^{F+1-n}]\{q^n\} - [H^{F+1-n}]\{p^n\} \right) = 0 \qquad (3.28)$$

Consider, now, the first time step, equation 3.28 becomes

$$[\, G^1 \,]\ \{\, q^1 \,\} - [\, H^1 \,]\ \{\, p^1 \,\} = 0 \qquad (3.29)$$

In a well-posed problem, half of the values of the two vectors q^n, p^n, will be known as boundary conditions and the other half will be unknown. Applying these boundary conditions and rearranging the equation results in:

$$[\, A^1 \,]\ \{\, X^1 \,\} = \{\, B^1 \,\} \qquad (3.30)$$

in which:

X^1: the unknown components of both q^1, p^1 (Nx1)

B^1: the known vector resulting from applying boundary conditions and rearranging the terms (Nx1)

A^1: the associated coefficient matrix (NxN)

This system of equations is solved by the Gaussian elimination method with single pivoting which reduces the matrix A to an upper triangular form which is stored for reuse in subsequent time steps. Using backsubistitution, the unknown values of q^1, and p^1, are computed which completes the solution at the first time step.

At the end of this first time step, the entire nodal response vectors, q^1, and p^1 at time Δt are now known. For the subsequent time steps, a simple marching procedure is employed. Thus for the second time step, the boundary element equation looks like:

$$[G^2]\{q^1\} - [H^2]\{p^1\} + [G^1]\{q^2\} - [H^1]\{p^2\} = 0 \quad (3.31)$$

Assuming that the same boundary conditions still apply, this equation can be reformulated as:

$$[A^1]\{X^2\} = [B^1] - [G^2]\{q^1\} + [H^2]\{p^1\} \quad (3.32)$$

Since, the right-hand side contains only known quantities which can be easily calculated to represent the B^2 vector, this equation can be solved to determine the values of X^2. Notice that the decomposed matrix A^1 , and the vector B^1, were calculated and stored from the first time step, so only the relatively inexpensive backsubstitution phase of the solution is required here.

The generalization of equation 3.32 to any time step F is simply:

$$[A^1]\{X^F\} = [B^1] - \sum_{n=1}^{F-1} \left([G^{F+1-n}]\{q^n\} - [H^{F+1-n}]\{p^n\} \right) \qquad (3.33)$$

in which the summation represents the effect of the past events. By systematically storing all of the matrices and nodal response vectors computed during the marching process, little computing time is required at each time step. In fact, for any time step beyond the first, the only major computational task is the integration needed to form the matrices $[G^F]$ and $[H^F]$. Also, as the time marches on, the effect of the events that occurred during the first time steps diminishes. Consequently, the terms containing $[G^F]$ and $[H^F]$ will eventually become insignificant compared to those associated with recent time steps (Dargush, 1987).

Once this point is reached, further integration is unnecessary, and a significant reduction in the computing time and can be achieved.

It should be emphasized that the entire boundary element method developed in this chapter, has involved only surface quantities without the need for any volume discretization. We need to use this solution to determine the values of q and p at any interior point which is discussed in the next section.

3.6.6 Evaluation of interior quantities

When the final system of equations (3.15) is solved, the entire set of primary nodal values (p and q) on the boundary will be known. Consequently, the potential at any point inside the body can be directly determined at any instant of time. At any interior point (ξ), the potential can be determined from the integral equation 3.14 which can be rewritten as:

$$\delta_{ij}\ p(\xi,t) = \int_S [P^*_{ij}(x,t;\xi,t_f)\ q(x,t;\xi,t_f) - Q^*_{ij}(x,t;\xi,t_f)\ p(x,t;\xi,t_f)]\ dS(x)$$

Since all the nodal variables on the right hand side of the equation are now known and the point (ξ) is not on the boundary, all the integrals in this equation are non-singular. However, if the point (ξ) is on the boundary, the potential and flux can be easily determined using the elemental shape functions. Thus, for the boundary points the desired values are simply:

$$p_i(\xi) = N_w(\xi)\ p_{iw}$$

$$q_i(\xi) = N_w(\xi)\ q_{iw}$$

where the variables in these equations are as defined before.

3.7 Summary

In this chapter, a boundary element formulation was developed for the 2-D flow diffusion equation in an anisotropic medium. This formulation can be applied to the problem of groundwater flow in porous media as well as the problem of heat conduction. The boundary integral equation was derived from the appropriate reciprocal theorem and the fundamental solution. A numerical implementation was then discussed to permit the solution of practical engineering problems involving arbitrary geometry and boundary conditions. The validity of the formulation will be demonstrated in chapter 6 through the use of some examples.

4. BOUNDARY ELEMENT FORMULATION OF 2-D ANISOTROPIC ELASTICITY

4.1 General

This chapter is devoted to presenting the boundary element formulation of 2-D anisotropic elasticity. The chapter starts by reviewing the small strain theory of anisotropic elasticity and the associated governing differential equations. The boundary element equation will then be presented based on the same procedure described in the previous chapter for the flow diffusion problem. Next, the discretization of the domain boundary into elements and the methodology for numerical implementation schemes are presented. This includes discussion of the method used to evaluate the terms with singular integrals. Evaluation of the displacement at internal points is then presented followed by the evaluation of secondary quantities such as, strains, and stresses and a summary. The computer program developed for this problem will be presented in chapter 6.

4.2 Theory of anisotropic elasticity

Until recently the main assumption of almost all studies in structural engineering and engineering mechanics (Lekhnitskii, 1963) has been that the materials investigated are homogeneous and isotropic. This assumption is realistic for common structural materials such as metals and concrete.

However, for materials such as rocks, soils or the newly developed materials such as fiber-reinforced composites, the more complicated property of anisotropy must be introduced (Deb, 1990).

The complexity of anisotropy comes from the number of independent constants required to describe the stress-strain constitutive relationship which is 21 compared to only 2 for the isotropic case. This constitutive relationship can be written in a Cartesian frame of reference as:

$$\epsilon_i = b_{ij}\ \sigma_j \tag{4.1}$$

Where, ϵ_i and σ_i are the strain and stress vectors which can be written as:

$$\left\{\begin{bmatrix}\epsilon_1\\ \epsilon_2\\ \epsilon_3\\ \epsilon_4\\ \epsilon_5\\ \epsilon_6\end{bmatrix}\right\} = \left\{\begin{bmatrix}\epsilon_{11}\\ \epsilon_{22}\\ \epsilon_{33}\\ 2\epsilon_{23}\\ 2\epsilon_{13}\\ 2\epsilon_{12}\end{bmatrix}\right\} \quad \text{and} \quad \left\{\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ \sigma_5\\ \sigma_6\end{bmatrix}\right\} = \left\{\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix}\right\}$$

and the b_{ij} is the compliance matrix constructed of the 21 independent elastic coefficients written as:

$$
\begin{bmatrix}
b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} \\
 & b_{22} & b_{23} & b_{24} & b_{25} & b_{26} \\
 & & b_{33} & b_{34} & b_{35} & b_{36} \\
\text{Symmetric} & & & b_{44} & b_{45} & b_{46} \\
 & & & & b_{55} & b_{56} \\
 & & & & & b_{66}
\end{bmatrix}
$$

4.2.1 Types of anisotropy

Due to this large number of independent elastic constants, it is very difficult, mathematically, to handle the general case of anisotropy (Brebbia and Dominguez, 1989). However, by assuming the existence of some planes of symmetry inside the domain within which the material has the same elastic constants in any direction we can dramatically reduce the number of independent elastic constants. In the current study, the material is assumed to have three mutually perpendicular planes of elastic symmetry passing through every point of it. This type of anisotropic material is known as orthogonal-anisotropic or simply, orthotropic. The number of independent elastic constants is now only nine and five for 3-D and 2-D problems, respectively. This type of anisotropy is considered to be a suitable description of the behavior of soil in the field, especially of overconsolidated clays (Atkinson, 1975).

Assuming the coordinate system x,y,z are normal to the planes of symmetry, the stress-strain constitutive relationship, also known as generalized Hooke's law (Lekhnitskii, 1968), for this material can be written as:

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & 0 & 0 & b_{16} \\ & b_{22} & b_{23} & 0 & 0 & b_{26} \\ & & b_{33} & 0 & 0 & b_{36} \\ & & & b_{44} & b_{45} & 0 \\ & \text{Symmetric} & & & b_{55} & 0 \\ & & & & & b_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} \tag{4.2}$$

For two-dimensional orthotropic bodies, $b_{16} = b_{26} = 0$, and the constitutive relation thus becomes:

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & 0 \\ b_{12} & b_{22} & 0 \\ 0 & 0 & b_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} \tag{4.3}$$

where in terms of engineering constants the independent elastic constants are defined as:

$$b_{11} = 1/E_x \quad , \quad b_{12} = -\upsilon_x/E_x = -\upsilon_y/E_y$$

$$b_{22} = 1/E_y \quad , \quad b_{66} = 1/G$$

A second type of anisotropy is known as transverse isotropy or cross isotropy. In this type the material has only one plane of symmetry say the x-y plane and the z axis being perpendicular to that plane. In such a case, the number of independent elastic constants is reduced to only five in 3-D problems. Figure 4.1 shows the orientation of the coordinate axes for the two types of anisotropy. In the current study the behavior of the soil is assumed to be orthotropic.

4.3 Boundary element formulation for orthotropic elasticity

4.3.1 Problem description

In the absence of body forces the governing differential equation of equilibrium can be written as:

$$\sigma_{ij,j} = 0 \tag{4.4}$$

which represents a system of two equations of the form:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0 \qquad (4.5.a)$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = 0 \qquad (4.5.b)$$

Hooke's law can be written as:

$$\epsilon_i = b_{ij}\sigma_j \tag{4.6}$$

and the strain-displacement equations are:

$$\epsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad (4.7)$$

The system of equations 4.4,4.6 and 4.7 represent the boundary value problem we need to solve for an orthotropic body subjected to certain boundary conditions which can be defined as:

$$\begin{aligned} u_i(x) &= g_i(x) && \text{on } S_1 \\ t_i(x) &= \sigma_{ij}(x)\, n_j = h_i(x) && \text{on } S_2 \end{aligned} \qquad (4.8)$$

where u_i and t_i are the displacement and traction vectors respectively and $g_i(x)$ and $h_i(x)$ are arbitrary functions representing the specified boundary conditions on the segments of the boundary S_1 and S_2 which constitute the total boundary S of the body. Solving this system of equations results in a complete determination of all stresses and strain components at any point of the body.

4.3.2 Fundamental solution of 2-D orthotropic elasticity

As mentioned before for diffusion, the fundamental solution is simply the solution of the governing differential equations for a body in an infinite domain loaded with a concentrated point load (instead of a point source as used in the previous chapter) acting at point ξ in the j-direction. This force produces the following displacement and traction at any point x:

$$\begin{aligned} u_k &= G_{jk}\ e_j \\ t_k &= F_{jk}\ e_j \end{aligned} \tag{4.10}$$

e_j is a unit vector, and

$$G_{jk} = 2Re[A_{k1}B_{j1}\ Log_e\ Z_1 + A_{k2}B_{j2}\ Log_e\ Z_2] \tag{4.11.a}$$

$$F_{jk} = 2Re[Q_{k1}(\mu_1 n_1 - n_2)B_{j1}/Z_1 + Q_{k1}(\mu_2 n_1 - n_2)B_{j2}/Z_2) \tag{4.11.b}$$

The "Re" term in these equations stands for the "real part of". The rest of the coefficients are defined as shown in the following set of equations. The complex constants A_{k1} are given as:

$$\begin{matrix} A_{1k} \\ A_{2k} \end{matrix} = \begin{matrix} b_{11}\mu_k^2 & + & b_{12} \\ b_{12}\mu_k & + & \dfrac{b_{22}}{\mu_k} \end{matrix} \qquad (k=1,2) \tag{4.12}$$

The complex constants B_{kl} are the solutions of the following system of equations:

$$
\begin{aligned}
\mu_1 B_{j1} - \overline{\mu}_1 \overline{B}_{j1} + \mu_2 B_{j2} - \overline{\mu}_2 \overline{B}_{j2} &= \frac{\delta_{j1}}{2\pi i} \\
B_{j1} - \overline{B}_{j1} + B_{j2} - \overline{B}_{j2} &= \frac{\delta_{j2}}{2\pi i} \\
A_{11} B_{j1} - \overline{A}_{11} \overline{B}_{j1} + A_{12} B_{j2} - \overline{A}_{12} \overline{B}_{j2} &= 0 \\
A_{21} B_{j1} - \overline{A}_{21} \overline{B}_{j1} + A_{22} B_{j2} - \overline{A}_{22} \overline{B}_{j2} &= 0 \\
(j = 1,2, \quad i = \sqrt{-1})
\end{aligned}
\tag{4.13}
$$

The complex variables Z_k are defined as:

$$
Z_k = (X_1 - \xi_1) + \mu_k (X_2 - \xi_2) \tag{4.14}
$$

Note that $\bar{A}_{ij}$ and $\bar{B}_{ij}$ are the conjugate values of A_{ij} and B_{ij} respectively. Finally, μ_k and $\bar{\mu}_k$ are the complex conjugate roots of the following characteristic equation (Lekhnitskii; 1968):

$$
b_{11}\,\mu^4 + 2\,b_{12}\,\mu^2 + b_{22} = 0 \tag{4.15}
$$

We denote the roots as $\mu_k = a_k + ic_k$ and $\bar{\mu}_k = a_k - ib_k$ ($k = 1,2$), and Q_{ik} as:

$$[\,Q_{ik}] = \begin{matrix} \mu_1 & \mu_2 \\ -1 & -1 \end{matrix} \qquad (4.16)$$

4.3.3 Reciprocal theorem

The reciprocal theorem for a body of volume V and surface S with zero initial conditions can be written as (Ionescu-Cazimer, 1964):

$$\int_S [\, t_i^{(1)} u_i^{(2)} - u_i^{(1)} t_i^{(2)} \,]\, dS + \int_V [\, f_i^{(1)} u_i^{(2)} - u_i^{(1)} f_i^{(2)} \,]\, dV = 0 \qquad (4.17)$$

where superscripts (1), (2) denote any two independent states, defined by ($u_i^{(1)}$, $t_i^{(1)}$, $f_i^{(1)}$) and ($u_i^{(2)}$, $t_i^{(2)}$, $f_i^{(2)}$) respectively, existing in the body. Note that, f is the existing body force. This reciprocal theorem will lead to a boundary element formulation with only the displacement and traction as primary and only variables and no secondary quantities such as displacement and traction rates (Dargush, 1987).

4.4 Boundary element equation

The same procedure used for the diffusion problem in the previous chapter will be followed here to develop the boundary element equation. Note that the anisotropic elasticity is a time independent problem, thus there is no need for the convolution integral here which simplifies the problem even more. Since body forces do not exist in the actual boundary value problem, substitution in the reciprocal theorem results in:

$$\int_S [F_{ij}(x,\xi)\, e_j u_i^{(2)}(x) - G_{ij}(x,\xi)\, e_j t_i^{(2)}(x)]\, dS(x) + \int_V [\delta(X-\xi)\, \delta_{ij} e_j u_i^{(2)}] = 0 \qquad (4.18)$$

The volume integral in this equation may be simplified as follows:

$$\int_V [\delta(x-\xi)\ \delta_{ij}\ u_i^{(2)}(x)]\ dV = \delta_{ij}\ u_i^{(2)}(\xi)$$

e_j is a common constant which may be canceled from the equation and dropping the superscript (2), the BEM equation can be written as follows:

$$\delta_{ij} u_i(\xi) = \int_S [G_{ij}(x,\xi)\, t_i(x) - F_{ij}(x,\xi)\, u_i(x)]\, dS(x) \tag{4.19}$$

This equation computes the displacement at any interior point ξ within the domain V as a function of the boundary variables u_i and t_i. Similar to the diffusion problem in the previous chapter, this equation involves only boundary integrals and is an exact statement with no approximation at all. However, in order to determine the interior variables, the values of u_i and t_i on the entire boundary must be known. In order to do that, the point ξ has first to be moved to the boundary. The end result of this process is the following boundary element equation for 2-D elastostatics in orthotropic bodies:

$$C_{ij}\; u_j(\xi) = \int_S \; [G_{ij}\; t_i \; - \; F_{ij}\; u_i] \;\; dS(x) \tag{4.20}$$

C_{ij} is a matrix which is a function only of the local geometry of the boundary at the point ξ and is reduced to $\delta_{ij}/2$ along a smooth surface. The explicit calculation of C_{ij} is not always necessary and in most of the problems it can be evaluated indirectly using the rigid body concept (Banerjee and Butterfield, 1982; Brebbia et al., 1984; Dargush, 1987). Solving this equation with the given boundary conditions results in the determination of the unknown terms of the vectors u_i and t_i on the boundary which then can be used to determine the values of the displacement, strains and stresses at any interior point within the domain V.

4.5 Numerical implementation

4.5.1 Introduction

Although the developed boundary element equation 4.20 is an exact statement with no approximation, applying it for the solution of engineering problems requires some approximation to represent the geometry of the problem as well as the variables u_i and t_i in a similar way to the techniques used in the finite element method. The following subsections discuss, in order, the spatial discretization, integral singularities, numerical integration, assembly, solution, evaluation of the displacements at any interior point, and evaluation of secondary quantities (strains and stresses).

4.5.2 Spatial discretization

The process has been described in the previous chapter, therefore it is not repeated here. Replacing the potential and flux with displacement and traction respectively, the boundary element equation can be rewritten as:

$$C_i(\xi)\ u_i(\xi) = \sum_{m=1}^{M} \int_{S_m} [\ G_{ij}(x(\xi)-\xi)\ N_w(\xi)\ t_{iw} - F_{ij}(x(\xi)-\xi)\ N_w(\xi)\ u_{iw}\ ]\ dS(x(\xi)) \qquad (4.21)$$

where:

$$S = \sum_{m=1}^{M} \int d\, S_m$$

In the above equation, t_{iw} and u_{iw} are the nodal quantities which can be brought outside the integrals, which leads to:

$$C_i\ (\xi) u_i(\xi) = \sum_{m=1}^{M} [t_{iw} \int_{S_m} G_{ij}(x(\xi)-\xi)\ N_w(\xi) - u_{iw} \int_{S_m} F_{ij}(x(\xi)-\xi)\ N_w(\xi)]\ dS(x(\xi)) \qquad (4.22)$$

Taking the nodal quantities outside the integrals will result in the integral equation containing only known functions and thus the integrals can be numerically evaluated. Before discussing that, the integral singularities will be examined.

4.5.3 Integral singularities

Taking a look at the fundamental solution mentioned above, equations 4.11, show that, as long as Z does not equal zero, the integrals are regular and can be evaluated numerically using Gaussian quadrature formulas. These integrals constitute the off-diagonal terms of the assembled G and F matrices. Unfortunately, the integrals become singular when Z=0, that is, when the load point (ξ) and field point (X) coincide. This is true for both G_{ij} and F_{ij}. These singularities constitute the diagonal terms of the two matrices G and H respectively. The singularity of G_{ij} is a weak one since it is logarithmic singularity (Brebbia and Dominguez, 1989). It can be integrated using the special semi-analytical method described by Brebbia and Dominguez (1989). The singularity of F_{ij}, on the other hand, is a stronger one since it is a 1/Z singularity (Dargush, 1987). This singularity is more difficult to integrate. However, the diagonal terms of the H matrix can still be indirectly-determined using the rigid body principle (Brebbia et al., 1984 and Banerjee and Butterfield, 1981).

4.5.4 Assembly and solution

After developing the boundary element equation, we come to this last step. Applying equation 4.22 at each node on the surface, and using the discretization method mentioned above, will result in a system of algebraic equations of the form:

$$C_i u_i = \sum_{m=1}^{M} [\int_{S_m} G_{ij} N_w dS] \ t_{iw} - \sum_{m=1}^{M} [\int_{S_m} F_{ij} N_w dS] \ u_{iw} \quad (4.23)$$

which represent the displacement at each node i of the discretized surface. The matrices G_{ij} and F_{ij} are 2x2 for two dimensional problems. Writing this equation at each node i and assembling the contribution of the two elements, i+1 and i-1, which meet at node i leads to this global system of equations:

$$G\ T - F\ U = 0 \tag{4.24}$$

Notice that the constant C_i will be absorbed in the diagonal elements of the matrix F. Notice in this equation, F is a square matrix of the order NxN, while G is a rectangular one of the order Nx2N, where N is the total number of boundary nodes. That is so because each of the nodes has only one value of displacement (continuous displacement mode) while the value of traction can be different before and after the node. The vectors U and T represent the displacement and traction variables at the nodes before applying the boundary conditions. In a well-posed boundary value problem, half of these values must be known as boundary conditions and the other half are the unknown values to be determined from the solution. Introducing these boundary conditions into the global system of equations and rearranging it to separate the known and the unknown terms yield this final system of equations:

$$[A] \quad \{X\} \quad = \quad \{F\} \tag{4.25}$$

where, X is the vector of unknown terms while A and F are the coefficient matrix and vector resulting from applying the boundary conditions and rearranging the system of equations. They are of the order NxN and Nx1 respectively.

4.5.5 Interior displacement

When the final system of equations, 4.20, is solved, the entire set of primary nodal values (u and t) on the boundary will be known. Consequently, the displacement at any point inside the body can be directly determined. At any interior point (ξ), the displacement can be determined from the integral equation, 4.19 That is:

$$u_i(\xi) \quad = \int_S [G_{ij}(x,\xi)\, t_i(x) - F_{ij}(x,\xi)\, u_i(x)]\, dS(x) \tag{4.26}$$

Since all the nodal variables on the right hand side of the equation are now known and the point (ξ) is not on the boundary, all the integrals in this equation are non-singular. However, if the point (ξ) is on the boundary, the displacement or traction can be determined using the elemental shape functions as was explained in chapter 3.

4.5.6 Evaluation of the secondary quantities

In many practical engineering problems, additional quantities such as strains and stresses, are also important. The integral equation for strains at any interior point can be developed by the differentiation of equation 4.20 at the field point as follows:

$$\frac{\partial u_i}{\partial x_k} = \int_S \frac{\partial F_{ij}}{\partial x_k} u_j \, dS - \int_S \frac{\partial G_{ij}}{\partial x_k} t_j \, dS \quad (4.27)$$

The strain vector then is given from the strain displacement equation (4.7), which then leads to the following equation for interior strain:

$$2\epsilon_{jk} = \int_S \left[\frac{\partial F_{ji}}{\partial x_k} + \frac{\partial F_{ki}}{\partial x_j}\right] u_i dS - \int_S \left[\frac{\partial G_{ji}}{\partial x_k} + \frac{\partial G_{ki}}{\partial x_j}\right] t_i \, dS \quad (4.28)$$

or in other words:

$$2\epsilon_{jk} = \int_S S_{jki} \, u_i \, dS - \int_S D_{jki} \, t_i \, dS \quad (4.28)$$

where S_{jki} and D_{jki} are given by:

$$D_{jki} = 2Re[R_{k1}A_{i1}B_{j1} / Z_1 + R_{k2}A_{i2}B_{j2} / Z_2] + 2Re[R_{j1}A_{i1}B_{k1} / Z_1 + R_{j2}A_{i2}B_{k2} / Z_2] \quad (4.29)$$

$$S_{jki} = - 2Re[R_{k1}Q_{i1} (\mu_1 n_1 - n_2) B_{j1}/Z_1^2 + R_{k2}Q_{i2} (\mu_2 n_1 - n_2) B_{j2}/Z_2^2] - 2Re[R_{j1}Q_{i1} (\mu_1 n_1 - n_2) B_{k1}/Z_1^2 + R_{j2}Q_{i2} (\mu_2 n_1 - n_2) B_{k2}/Z_2^2] \quad (4.30)$$

where:

$R_{1k} = 1$

$R_{2k} = \mu_k$

and the rest of notations are as defined before. The stresses can then be determined by substituting in the constitutive stress-strain relationship (equation 4.6).

4.6 Summary

In this chapter, a boundary element formulation was developed for 2-D linear elasticity in orthotropic bodies. The boundary integral equation was derived from the appropriate reciprocal theorem and the fundamental singular solution. A numerical implementation was then discussed to permit the solution of practical engineering problems involving arbitrary geometry and boundary conditions. The validity of the formulation will be demonstrated in chapter 6 through the use of some simple examples with known solutions.

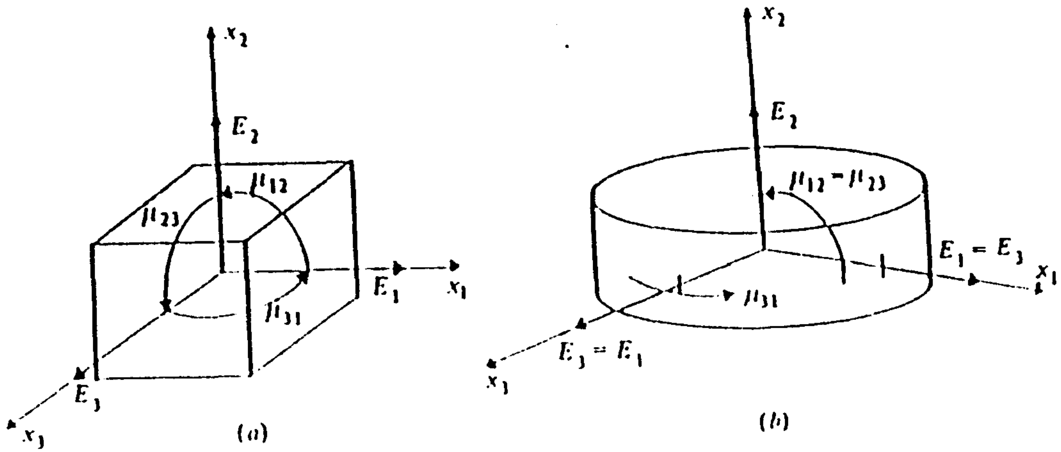


Figure 4.1 The special cases of anisotrpy in elastic bodies a) orthotropic b) transversly isotropic (Banerjee and Butterfield, 1981)

5. COUPLING THE PROBLEMS OF ANISOTROPIC ELASTICITY AND DIFFUSION TO STUDY THE CONSOLIDATION OF ANISOTROPIC SOILS

5.1 Introduction

The consolidation process of soils is one of the most important phenomena in engineering practice. This is because it has a major impact on understanding and studying many problems in different areas of engineering field work. This includes, the settlement of foundations under any structure, the soil subsidence resulting from pumping of ground water, failure of highway shoulders or earth retaining structures due to seepage problems, to mention a few. Therefore, consolidation of soils is one of the engineering problems that seems to continuously interest researchers all the time.

In this chapter, a new approach to perform a boundary element analysis of the consolidation process in anisotropic soils will be presented. As was mentioned in chapter 2, one of the main elements of a boundary element analysis is missing for this problem, which is the fundamental solution of the governing differential equation, due to its mathematical complexity. Therefore, a new approach has to be tried to overcome this difficulty. The chapter starts by introducing a preview of the consolidation process and its governing differential equations. A discussion of the use of the solution of the anisotropic elasticity and diffusion problems to study the consolidation of anisotropic soils is presented.

Next, the steps of solving a consolidation problem in anisotropic soils using the boundary element method will be presented. The effectiveness of that approach as well as its advantages and disadvantages will be demonstrated in chapter 6 by comparing its numerical results with some simple problems with known solutions.

5.2 Consolidation

The theoretical basis of the process of consolidation was established in 1923 by Terzaghi. Although this initial effort by Terzaghi was restricted to only a one-dimensional fully-saturated soil layer, satisfactory results were obtained for a large number of practical consolidation problems. The core of Terzaghi's theory was the principle of effective stress which states that the total normal stress, σ_{ii}, imposed on a soil column is shared by the soil particles, in terms of effective stresses, $\bar{\sigma}_{ii}$, and the pore water pressure (P) developed by the free-flowing fluid. This is expressed mathematically as:

$$\sigma_{ii} = \sigma^{-}_{ii} + \delta_{ij}P \qquad (5.1)$$

Note that stresses are usually positive in tension while pore pressures are positive in compression. Biot (1941a) generalized the theory to the three-dimensional case and extended it to include the partially saturated soil too. In order to write the governing differential equations for consolidation (Biot 1941a), consider a small cubic element of consolidating soil with its sides being parallel with the coordinate axes. The total stress tensor, σ_{ii}, is decomposed into an effective stress, $\bar{\sigma}_{ii}$, and a stress acting on the fluid portion of the soil column, σ, such that:

$$\sigma_{ij} = \sigma^{-}_{ij} + \delta_{ij}\sigma \tag{5.2}$$

This stress, σ, is related to the pore pressure developed, P, according to:

$$\sigma = -fP \tag{5.3}$$

where f is the porosity denoted by the ratio between the volume of pores, V_p, to the bulk volume, V_b, in a sample of porous medium,

$$f = \frac{V_p}{V_b} \tag{5.4}$$

Neglecting any body forces, and considering only static conditions the stresses must satisfy the equilibrium equation,

$$\sigma^{-}_{ij,j} + \sigma_{,i} = 0 \tag{5.5}$$

Although, assuming the soil to be an isotropic material usually has produced satisfactory results, in many field problems, soil must be considered to be an anisotropic material with different properties in each direction. This anisotropy may result for instance because of its history of deposition or the way the soil is loaded (stress-induced anisotropy) and it surely affects the behavior of soil in the field.

Sandhu and Wilson (1969) have described the clayey soil in the field to have either elastic or hydraulic anisotropy or both at the same time. If only the first type exists, the deformation analysis will be done using the anisotropic fundamental solution and the pore pressure analysis will use the regular isotropic fundamental solution and vice versa if only the second type exists. For these two cases the two BEM programs in the previous two chapters can be used. However, the more difficult situation arises if both types exist in the soil body, which is generally the case in many engineering problems in the field. For that case, special care has to be given to the analysis procedure in order to consider the effect of change in pore pressure on the deformations in the soil body as explained next.

5.3 Boundary element analysis of consolidation

5.3.1 The analysis procedure

The approach used here was used before by Banerjee and Butterfield (1981) to study the consolidation process of isotropic soil for the first time. Giuseppe et al. (1987) used the same approach to study land subsidence above storage tanks of oil/gas. It was also used in a finite element study of the consolidation problem by Eisenstein et al (1978). All these studies showed that it is an effective alternative tool to solve these kinds of problems in cases where solving the problem directly in one step, including deformation and pore pressure analysis, seems to be difficult as in our case where the fundamental solution does not exist.

In order to solve the problem, we will assume that the consolidation process can be uncoupled into two mechanisms taking place alternately, deformation of soil particles and pore pressure dissipation. The deformations can be evaluated from an elasticity analysis and the pore pressure values may be determined from a diffusion analysis. Since these two problems have well known fundamental solutions and the boundary element formulations for both of them were introduced in the previous two chapters, we can make use of them to solve the consolidation problem. When the load is first applied, an immediate deformation of soil particles takes place while the pore pressure builds up inside the voids. As soon as the pore water starts to dissipate, gradually, the pore pressure is reduced moving the load to the soil skeleton in a form of increase in effective stress with time as this process continues. This process is illustrated in figure 5.1. The algorithm followed to solve the problem and determine both the deformation and pore pressure distributions is introduced next.

5.3.2 Steps of the analysis of consolidation in anisotropic soil

First, we solve the orthotropic elasticity problem which is represented by the equation:

$$C_{ij}\ u_j(\xi)\ =\ \int_S\ [G_{ij}\ t_i\ -\ F_{ij}\ u_i]\ dS(x) \qquad (5.6)$$

From this solution, the total stresses σ_{ij} are determined at any point at the initial time t_o. Notice that equation 5.6 is solved in terms of total stresses, therefore G_{ij} and F_{ij} have to be evaluated with the terms of b_{ij} corresponding to total stresses (Banerjee and Butterfield, 1981). The effective stresses σ_{ij}^- at time t_o can be determined from the constitutive stress-strain equation 4.1. Thus, the initial excess pore pressure (P at time t_o) is equal to:

$$\delta_{ij}P = \sigma_{ii} - \sigma^-_{ii} \tag{5.7}$$

which is simply shown to be equal to $\sigma_{ij}/3$. Next, this initial pore pressure distribution is applied in a diffusion-setup problem for the same body which is then solved to determine the history of pore pressure dissipation from time t_o to time t_F as was shown in chapter 3. Notice that this second step results in the distribution of the excess pore water pressure at nodes at the end of successive time steps. Then, In order to include the effect of this developed pore pressure on the deformations, the resulting change in pore pressure is set to equal in magnitude the increase in effective stresses after each time increment. Finally this increase in effective stress is reapplied on the body as the new boundary conditions for the new elasticity analysis. The elasticity problem is then resolved with the new boundary conditions and the G_{ij} and F_{ij} evaluated using effective stress engineering properties to determine the new deformations resulting from the increase in effective stresses with time due to the continuous water dissipation. The total deformation is thus the sum of the immediate deformation calculated from the first stage (total stress analysis) and those calculated after each time step from the increase in effective stresses (effective stress analysis).

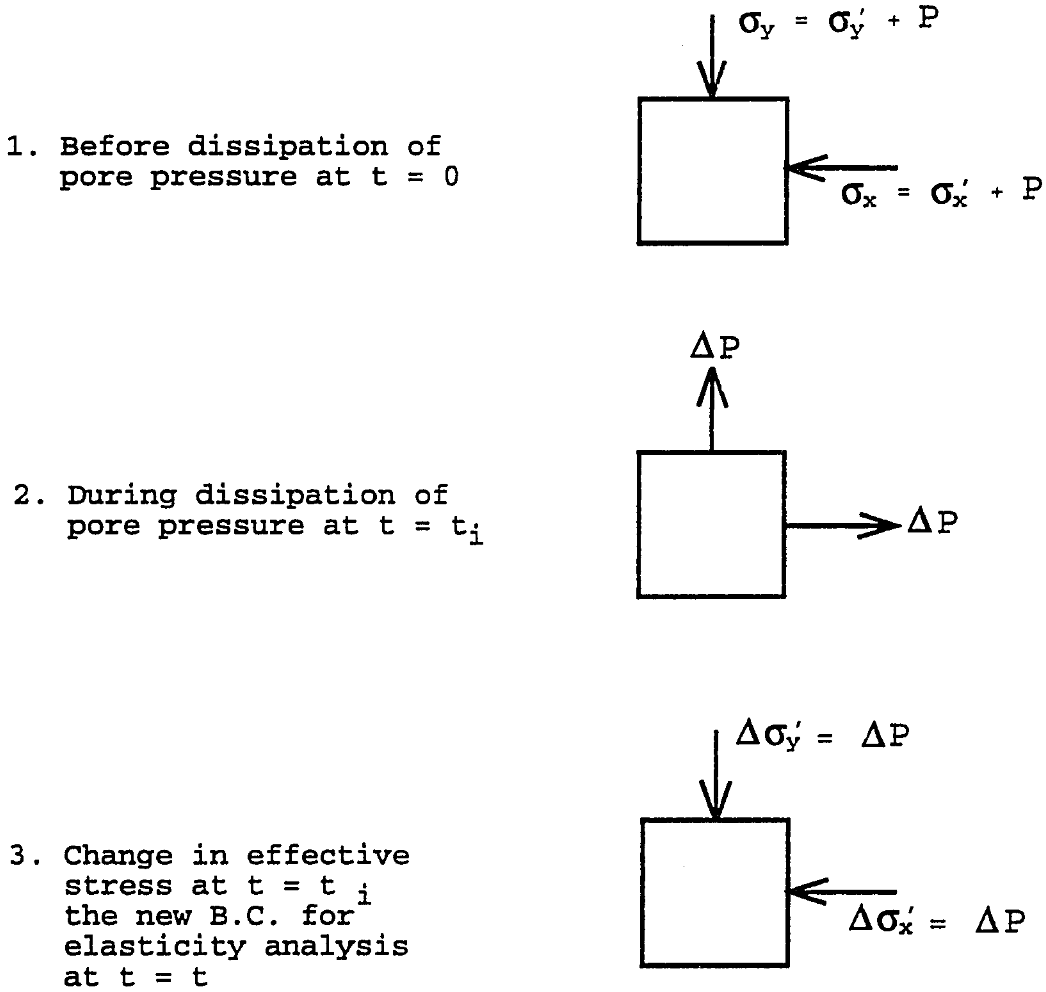


Figure 5.1 Pore pressure dissipation and the analysis procedure of the consolidation process

6. COMPUTER PROGRAMS AND VALIDATION

6.1 General

In the previous three chapters, the boundary element formulations and the numerical implementations for orthotropic elasticity, transient flow and soil consolidation were presented. In this chapter, the structure of the computer programs developed and the analysis procedure are presented. Next, the validity of these programs will be demonstrated by using them to solve some example problems for which there exist closed form or numerical solutions so as to verify the results.

6.2 Computer program for the flow diffusion problem

The computer program for the diffusion problem (TRANSBEM) is written in FORTRAN 77 to numerically solve the boundary element equation (3.15). The program uses a linear variation for both geometry and the primary variables (p and q) in any element. It can be run on IBM computers or compatibles. The program in the current version allows for the solution of problems with a maximum number of 50 elements and 6 time increments. This, however, can be increased by a simple modification of the size of the arrays in the program and running the program on computer systems of larger capacities such as the VAX or CONVEX systems.

6.2.1 Structure of the program

The flow chart of the program is shown in figure 6.1. The main program opens the required input and output files and calls different subroutines to execute the different tasks of the program. These subroutines are named, INPUTPL, GHMATPL, ASSEMBLE, SLNPD, INTERPL, and OUTPTPL. The routine INPUTPL reads the input required for the boundary element formulation. The data file should contain, a title line, the number of elements, the number of internal points, number and size of the time increments, nodal coordinates, boundary conditions and the coordinates of the internal points at which the potential is to be determined. The data file can be written in free FORMAT. The boundary conditions may be specified based on the smoothness of the boundary at the considered node. If the boundary at the node is smooth, both fluxes before and after the node are assumed to be the same unless they are specified different, but in any case, only one variable, per node, will be unknown; either the potential or the unique flux. If the node is at a corner, one of four different combinations of boundary conditions may be used. They are:

a. Known values: the fluxes before and after the corner.

   Unknown values: the unique potential

b. Known values: the unique potential, and the flux before the corner.

Unknown values: the flux after the corner.

c. Known values: the unique potential, and the flux after the corner.

Unknown values: the flux before the corner.

d. Known values: the unique potential.

Unknown values: the flux before, and after the corner.

Then, the program loops over time increments. The routine GHMATPL constructs the G^1 and H^1 matrices where the superscript 1 is the number of the time increment. It loops over the boundary nodes to construct the local g and h terms required for the global G and H matrices. Based on the nature of the integrations in g or h terms, the routine GHMATPL calls the subroutine, EXTENPL (regular integrations) or LOCINPL (singular integrations). GHMATPL, then does the assembling of the system of algebraic equations in the form, $G^1 q^1 - H^1 p^1 = 0$. The boundary conditions are then applied and the terms of the equations are rearranged to transform the system into $A^1 x^1 = B^1$ ready to be solved. The routine SLNPD is then used to solve the system of equations using Gauss elimination method. The routine, INTERPL, is then used to determine p^1 at the desired internal points. This routine first sorts all the, now known, boundary terms, potential and flux, to be used next in the solution at the internal points. The routine INTERPL calls the routine EXTENPL to do the required integration. Note that all the integrations here are regular and done using Gaussian Quadrature. At this step the solution of the first time increment is thus completed and the program then loops to the next time increment. The routine GHMATPL now calculates the matrices associated with the new time increment G^2 and H^2 The routine ASSEMBLE calculates the effect of the previous time increments to be used in the boundary element equation (3.32). Note, that as we mentioned in chapter 3 the matrix A^1 remains constant for all the time increments, while the vector B is updated after each one to include the effect of the accumulated time increments. The routine SLNPD now solves for the unknown X^2 and then the internal quantities are evaluated at the end of the second time increment. This process is repeated for every time step. The routine OUTPTPL is used to print out the results of the problem, p and q, at the boundary nodes as well as the internal points after the desired time increments.

6.2.2 Illustrative examples

Due to the complexity of the governing differential equations for the anisotropic diffusion, problems with closed form solutions are not available, therefore, the implementation of the computer program will be validated by comparing the numerical solutions obtained by TRANSBE to those obtained by the finite element method. The examples are intended to validate the theoretical work as well as to demonstrate the utility of the numerical implementation.

6.2.2.1 Two dimensional transient flow diffusion through a rectangular region

The first example deals with a typical problem in heat transfer and groundwater flow, the flow of heat or water into an anisotropic rectangular domain (or confined aquifer). The ratio of horizontal to vertical thermal conductivity (permeability) is K_1:K_2 is 5:1. The region is subject to Dirichlet type of boundary conditions in which the potential is the prescribed variable. The initial conditions are constant temperature (potential) for the entire region, then suddenly the temperature is raised along two of the sides while the other two sides are kept insulated. The program TRANSBE was used to solve the problem. The boundary is divided into 36 linear elements and the time increment, T, is chosen as 0.005 (non-dimensional, equal to K_1t/L). From the boundary solution obtained, the values of p at scattered internal points were also obtained. The same problem was solved using a FEM program developed by Hassan, 1995. The finite element model used 100 square elements and a time increment of 0.0005. Figure 6.2 shows the schematic diagram of the problem and the two grids used for the BEM and FEM as well as the applied boundary conditions. The results from the TRANSBE and FEM are shown in figure 6.3 in the form of the contour lines of equal pressure inside the region at T=0.005. Figure 6.4 shows the distribution of pressure along a y=50 line, at x=10 increments, after three time increments. The figures show good agreement between the two methods. Notice that the TRANSBE solution was obtained with less number of elements and computer effort in preparing the data file.

6.2.2.2 Dissipation of excess pore pressure under a dam

This is another typical problem in geotechnical engineering, the dissipation of excess pore-water pressure under the foundation of a building or a concrete dam. Figure 6.5 shows the domain of the problem surrounded by impervious boundaries, on which Neuman-boundary conditions are prescribed $(q(0,y,t) = q(160,y,t) = q(x,0,t) = 0)$. The ratio of permeability in the horizontal and vertical directions is 2:1. The two grids used for BEM and FEM are shown in figure 6.5 (32 square elements and 28 linear elements respectively). The time increments used are 0.005 and 0.0005 for TRANSBE and FEM respectively. The contour lines of the pore pressure are shown in figure 6.6, while figure 6.7

shows a comparison between the pore pressure distribution for a horizontal section midway below the surface obtained from the two solutions. A good agreement between the two solutions is evident. The contours shown in figure 6.6 are the result of the output of the graphics software and reflex the effect of the interpolation process in that software. This problem will be reexamined in the consolidation section of this chapter to study the deformation history due to the dissipation of this excess pore water pressure.

The same problem is solved next under the opposite type of boundary conditions. The boundaries of the domain, except for the bottom base, are now surrounded with pervious layers (P=0), such as the filters used in the earth dams in the field. The set up for the solutions by the FEM and BEM is exactly the same as in the example previously shown in figure 6.5. The contour lines, from TRANSBE and FEM are shown in figure 6.8 and the comparison between the two solutions at a horizontal section is shown in figure 6.9. A good agreement between the two solutions is shown.

6.3 Verification of the computer program for orthotropic elasticity

The computer program for orthotropic elasticity (EL10) 0is also written in FORTRAN 77 for solving the boundary element equation (4.20). The program uses quadratic representation for both geometry and variables (i.e. quadratic elements). It can be run on IBM personal computers or compatibles. The program in the current version allows for the solution of problems with a maximum number of 100 unknowns. This can be increased by increasing the size of the different arrays in the program and using computer systems with larger capacities.

6.3.1 Structure of the program

The flow chart diagram of the EL10 boundary element program is shown in figure 6.10. The main program opens the required input and output files and defines the size of all the arrays. It then calls the following subroutines, COEFF, INPUTEQ, GHMATEQ, EXTENEQ or LOCINEQ, SLNPD, INTEREQ, SIGMAEQ, AND OUTPTEQ. In the COEFF subroutine, the user inputs the compliance coefficients, b_{ij}, of the material, and the routine solves the characteristic equation (4.15) and the system of equations (4.13) to yield the complex coefficients required for the fundamental solution (4.11). The routines INPUTEQ, GHMATEQ, EXTENEQ, LOCINEQ, SLNPD, and OUTPTEQ perform similar tasks to their counterparts in the diffusion program (TRANSBE). The routine, INTEREQ, is used to determine the required quantities at the internal points (displacements, strains and stresses). The routine first sorts all the, now known, boundary terms, displacements and tractions, to be used next in the solution at the internal points. The routine INTEREQ calls the routine SIGMAEQ to do the required integration for the evaluation of internal strains and stresses. Note that all the integrations here are regular and evaluated using Gaussian Quadrature. The boundary conditions (displacements and tractions) at the corners can be handled the same way as in the TRANSBE program. Note that the displacement at a node along a smooth boundary is the variable that has to be continuous, while the traction can be different before and after the node.

6.3.2 Illustrative examples

In chapter 4 the boundary element formulation and a numerical implementation were developed for 2-D linear elasticity for orthotropic bodies. This implementation will now be validated by applying it to some simple problems with known closed form or numerical solutions. The examples are intended to validate the theoretical work as well as demonstrate the utility of the numerical implementation.

6.3.2.1 Bending of an orthotropic beam under a uniformly distributed load

This example is taken from Lekhnitskii (1968). The reference gives the stress distributions and maximum deflection of an orthotropic beam bent by a uniformly distributed load. The stress distribution is obtained using a stress function taken in the form of a polynomial of the fifth order. The details of the solution will not be repeated here but the resulting equations are presented next for two types of end supports.

a) Simply supported beam

An orthotropic beam with a rectangular cross section is loaded with a normal load q (per unite length), as shown in figure 6.11(a). The following formulas represent the stress distribution at any arbitrary cross section x of the beam, as taken from Lekhnitskii (1963):

$$\sigma_x = \frac{q}{2J}\,(L^2-X^2)\;Y + \frac{q}{h}\,[2\,(\frac{2b_{12}+b_{66}}{4b_{11}})\,(\frac{4Y^3}{B^3}-\frac{3Y}{5B})]$$

$$\sigma_y = \frac{q}{2Bh}\;Y\;(1-4\frac{Y}{B^2})$$

$$\tau_{xy} = -\frac{qX}{2J}\;(\frac{B^2}{4}-Y^2)$$

The equation for the maximum deflection is:

$$\Delta = \frac{5qb_{11}L^4}{24J} + \frac{qB^2L^2}{80J}(3b_{12}+4b_{66})$$

where h and B are the beam height and width respectively. The rest of the variables are defined as in chapter 3.

<u>b) Cantilever beam</u>

A beam similar to the one mentioned above is loaded as a cantilever and is shown in figure 6.11.b. The following solution is presented:

$$\sigma_x = -\frac{qX^2Y}{2J} + \frac{q}{h}\left(\frac{2b_{12}+b_{66}}{2b_{11}}\right)\left(\frac{4Y^3}{B^3}-\frac{3Y}{5B}\right)$$

$$\sigma_y = \frac{q}{2h}\left(-1 + 3\frac{Y}{B} - 4\frac{Y^3}{B^3}\right)$$

$$\tau_{xy} = -\frac{qX}{2J}\left(\frac{B^2}{4}-Y^2\right)$$

$$\Delta = \frac{qb_{11}L^4}{8J} - \frac{qB^2L^2}{80J}(3b_{12}+4b_{66})$$

The two problems are solved using the EL10 program with the boundary divided into 22 quadratic elements. The elastic properties were taken as: E_1 = 1.31x10^5 MPa, E_2 = 0.13x10^5 MPa, G_{12} = 0.064x10^5 MPa, and ν_{12} = 0.038. The results were obtained in the form of maximum deflection, the shear and bending stress distributions. Tables 6.1 and 6.2 present a comparison between the analytically-calculated maximum deflection and the vertical displacements at the corresponding nodes for the two loading cases. For the cantilever beam, the values of bending stress (σ_x) and shear stress (τ_{xy}) on a section near the fixed end were evaluated from the boundary solution. These values are plotted versus the analytical values and shown in figures 6.12 (a) and (b). A good agreement between the two solutions is shown.

6.3.2.2 Uniaxial tension of an orthotropic plate with a circular cutout

This problem is also solved analytically by Lekhnitskii (1968). Figure 6.13 shows the schematic of the problem, the applied boundary conditions and the BEM mesh used to solve it. The analytical solution for this problem is given by Lekhnitskii (1968) in the form of the stress distribution on the perimeter of the cutout, σ_θ as,

$$\sigma_\theta = p\,\frac{E_\theta}{E_1}\,[-\,k\cos^2\theta + (1+n)\sin^2\theta]$$

where:

$$k = -\,\mu_1\,\mu_2 = \sqrt{\frac{E_1}{E_2}}\,, \quad n = -\,i\,(\mu_1 + \mu_2) = \sqrt{2\left(\frac{E_1}{E_2} - \nu\right) + \frac{E_1}{G}}$$

$$\frac{1}{E_\theta} = \frac{\sin^4\theta}{E_1} + \left(\frac{1}{G} - \frac{2\nu}{E_1}\right)\sin^2\theta\cos^2\theta + \frac{\cos^4\theta}{E_2}$$

and θ is measured as shown in figure 6.13.

The problem was solved by the EL10 program using 24 boundary elements, of which, 4 elements were used to represent the circular cutout. The elements are not equally spaced, they are concentrated closer to the cutout than away from it. The material elastic properties were taken as: E_1 = 1.2×10^5 kg/cm^2, E_2 = 0.6×10^5 kg/cm^2, G_{12} = 0.07×10^5 kg/cm^2, ν_{12} = 0.071. Figure 6.14 shows the boundary stress distribution along the perimeter of the cutout (tension is positive) obtained from both methods. Table 6.3 shows the values of the stresses obtained at different angles θ compared to those obtained from the analytical solution. A very good agreement is observed in table 6.3. and figure 6.14.

6.3.2.3 Hydrostatic tension of an orthotropic plate with a circular cutout

A plate similar to the one used in the previous example is now subjected to tension in the two principal directions x and y by equal forces p (equivalent to hydrostatic tension). Figure 6.13 (case b) shows the problem and the BEM mesh used to solve it. The analytical solution is given by,

$$\sigma_\theta = p \frac{E_\theta}{E_1} [- k + k (k + n) \cos^2\theta + (1 + n) \sin^2\theta]$$

and the variables are as defined in the previous example. The distributions of σ_θ along the perimeter of the cutout are shown in figure 6.15. The values of the stresses obtained from the EL10 solution are compared to those obtained from the analytical solution in Table 6.4. A good agreement is also observed.

The stresses computed in the above examples (internal and boundary) are computed directly from displacements and tractions determined from the BEM solutions, therefore they provide a good means of evaluating the accuracy of the solution obtained by the boundary element method using the EL10 program.

6.4 Verification of the new approach to study consolidation problems

The approach presented in chapter 5 to study the consolidation process in anisotropic soils will now be tested for some simple problems with known solutions in order to investigate its effectiveness. However, to the author's knowledge, there have been no closed form or numerical solutions in the literature for the problem of consolidation in an anisotropic soil. Therefore, some isotropic problems are chosen and treated as special cases of anisotropic problems so that the solutions obtained are compared to those obtained by the new approach.

6.4.1 Unidirectional consolidation

The first problem is a typical one in geotechnical engineering, in which a single unit of soil layer is under traction load at the top. Boundary conditions consist of a uniform load of 200 Pa applied at the top surface where the pore water is allowed to drain. The rest of the boundary is impermeable, along which only vertical displacement is allowed. This problem is taken from Kuroki et al. (1982) in which it was solved using the finite element method. Figure 6.16 shows the geometry of the problem together with the boundary element and finite element meshes. The BEM mesh consists of 20 elements and the time increment used is 1.5×10^4 (s).

In order to solve the problem using the new approach, as described in chapter 5, the elasticity problem will be solved first, using EL10 program. A total stress analysis is assumed, and the modulus of elasticity in both x and y directions, E_1 and E_2 are taken equal (0.106×10^9 Pa). From this solution, the immediate deformations and the boundary stresses are obtained. Then, the developed pore water pressure on the boundaries are calculated, as was shown in chapter 5. Using this developed pore pressure, the diffusion problem is then solved, using TRANSBE, with the permeability coefficients assumed to be equal in both x and y directions (0.4×10^{-2} m/s). The history of pore pressure dissipation with time is thus obtained. The values of pore water pressure at each time increment are stored in an intermediate data file. The increments of effective stresses (due to pore pressure dissipation) are then used to resolve the problem, using the EL10 and effective stress properties, to determine the corresponding deformations. The calculated deformations are then added to the immediate deformation (obtained from the first elasticity analysis) to calculate the total deformation.

Figure 6.17 shows the comparison between the vertical displacement at the two selected points, A and B, shown in figure 6.16 as determined from both FEM and the current approach. The normalized pore pressure, along the vertical boundaries, at three different times are compared in figure 6.18 to those obtained using the finite element method. Excellent agreement is shown in both figures.

6.4.2 Two dimensional consolidation

This problem is taken from the same reference mentioned in the above example. The schematic of the problem with the BEM and FEM meshes used to solve it are similar to those of the previous example. A traction of 200 Pa is applied on the left one third of the top surface and the water is allowed to drain from the right two third parts of the surface. The boundary conditions are similar to those of the previous example, except that the vertical sides are free to move in x and y directions. Similar assumptions to those made in the previous example are made here to solve the problem using EL10 and TRANSBE programs. Figure 6.19 shows the vertical displacement at the two selected points, A and B, shown in figure 6.16 as determined from both FEM and the current approach. No significant difference is observed.

6.4.3 Rectangular clay core with side drains

This is a practical example taken from Eisenstein et al. (1978). A rectangular saturated clay core with incompressible side drains is studied. The configuration of the problem is shown in figure 6.20 as well as the BEM mesh used to solve it. A surface traction p=2 Ksf is applied on the core and the water is drained from the top surface as well as from the side drains. The material properties are taken as, $E_1 = E_2 = 500$ Kips/ft^2, $\nu = 0.3$, and $K_1 = K_2 = 2.3\times10^{-7}$ ft/min. It is assumed that there is no friction between the clay core and the sides of the drains. Therefore, only vertical displacements are allowed in the elasticity analysis. For the diffusion part of the analysis, the water is allowed to drain from the top surface as well as from the drains on the side, while the bottom base of the core is impervious i.e. flux is zero (2-way drainage). The results from solving this problem using the current approach is shown in figure 6.21 in the form of the degree of consolidation, at the point A (shown in figure 6.20), compared to the results obtained from the FEM study by Eisenstein et al. (1978). The time factor $T = C_v t/L^2$, where C_v is the coefficient of consolidation and L is the thickness of the clay layer. The degree of consolidation is calculated as the ratio of the displacement at any time to the final displacement. A good agreement between the results from the two solutions is observed in the figure.

6.4.4 Consolidation under a strip foundation

A typical example in geotechnical engineering is shown in Figure 6.22 of an impermeable footing laying on a porous, water saturated clay soil subject to a unit traction load. The BEM mesh as well as the flow and elastic boundary conditions used to solve the problem are shown in figure 6.22. Again the problem is solved as a special case of an anisotropic one. The material parameters are taken in non-dimensional forms as, $E_1 = E_2 = 1.0$, $\nu = 0.5$, and $K_1 = K_2 = 1.0$. The degree of consolidation at the point A shown in figure 6.22 as calculated from the solution using the current approach is compared to the solution given by Dargush and Banerjee (1989a) in figure 6.23. No significant difference is observed in the figure.

6.5 Study of anisotropic effect on the results

It has always been more convenient to assume isotropic conditions for many of the engineering problems. This has been supported by the fact that the results have been shown to be reasonably and satisfactory without having to consider the more complex anisotropic conditions. Obviously, this assumption cannot be correct with the rising structural use of materials such as fiber-reinforced composite materials which are generally anisotropic. In the area of flow through porous media, soils are always more permeable in the horizontal direction than in the vertical one because of its history of deposition. Researchers have shown that the potential flow under irrigation structures is highly affected by this fact. If this anisotropy is considered in transient flow as well, this could be more economical in the design of earth dams, embankments, etc since the consolidation process will be affected as well. In addition, triaxial testing on clay samples loaded in different directions has always shown that the consolidation curve for anisotropic clay samples is affected by changing the direction of loading (anisotropic behavior).

In order to study that effect in time-dependent problems, some of the example problems discussed above are re-run with various ratios of anisotropy (k_1:k_2 and E_1:E_2). The first problem is the diffusion through a rectangular region. The same problem was solved with the ratios, 1:1, 2:1, and 5:1, keeping k_1 constant, under similar conditions to those mentioned in section 6.2.2. Figures 6.24 and 6.25 show the pore pressure distributions obtained using TRANSBE with the three ratios after the first and sixth time steps respectively. It is clear that the results are totally different for the three ratios. However as the solutions approach the steady state conditions, the difference in pressure values is reduced.

Next, two consolidation problems are studied, a rectangular clay core of earth dam and consolidation under a strip footing. The ratios of $k_1:k_2$ and $E_1:E_2$ are changed (1:1, 2:1, and 5:1), with k_1 and E_1 kept constant, and the same curves in figures 6.21, and 6.23 are regenerated using the three ratios. Figures 6.26 and 6.27 show the results of each problem using the three ratios of anisotropy. While these results can not be verified, it is clear that the results of the 1:1 ratio cannot be considered to be representative of the other two ratios. This part of the study needs to be investigated more in order to verify the results, specially to study which part of the anisotropy (elastic or flow) has the most effect on consolidation settlement. These results may affect the methods used by engineers to design structures in which the consolidation process is the predominant factor.

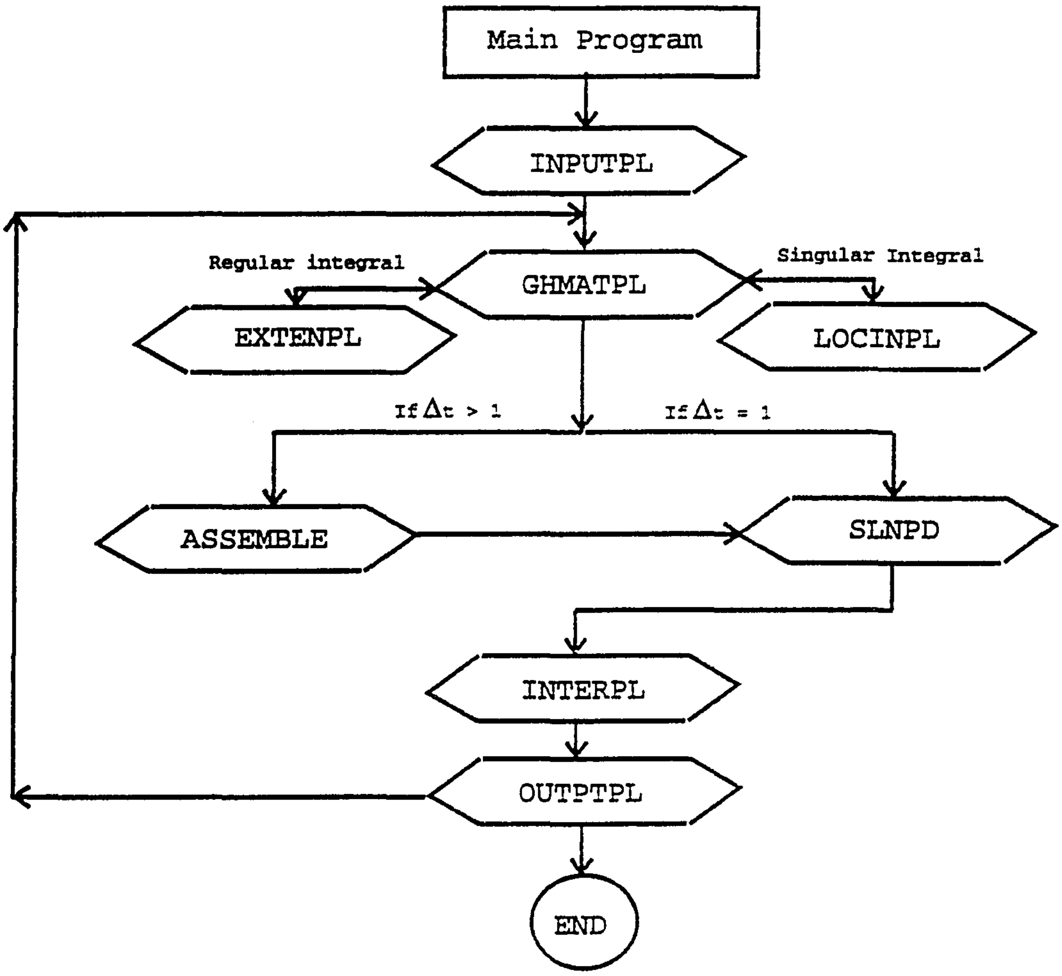


Figure 6.1 Flow chart of the TRANSBE program

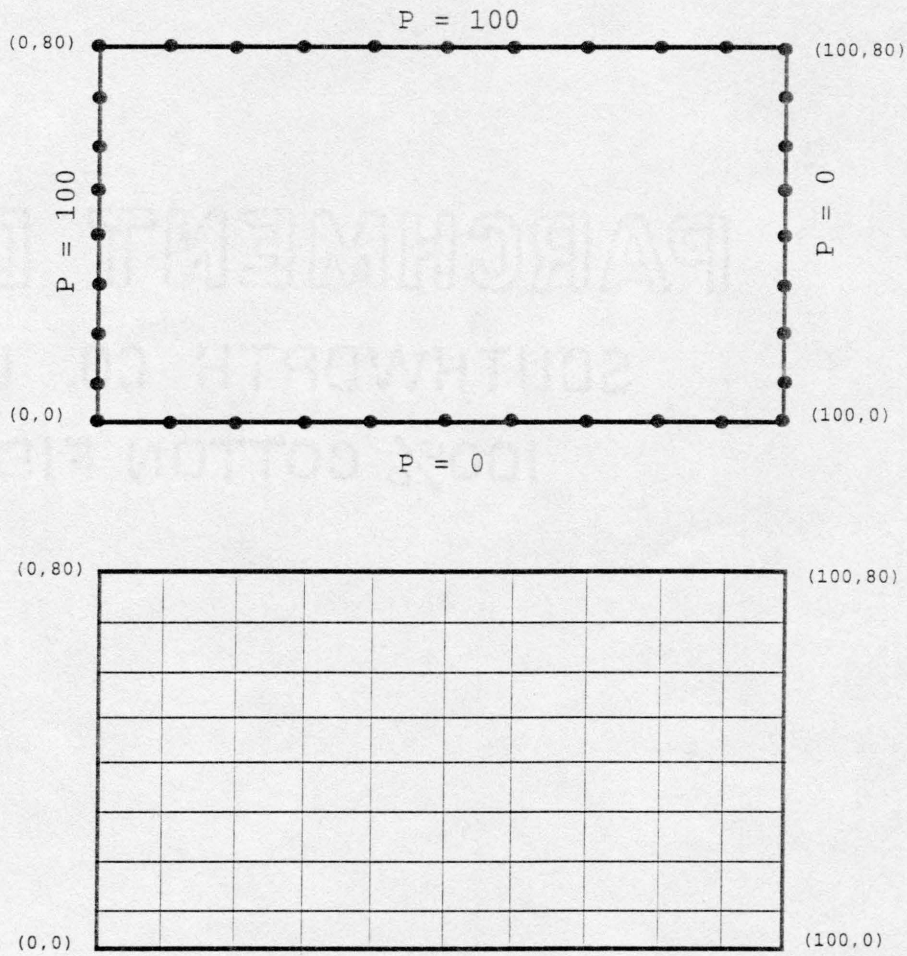


Figure 6.2 BEM and FEM meshes for 2-D diffusion through a rectangular region

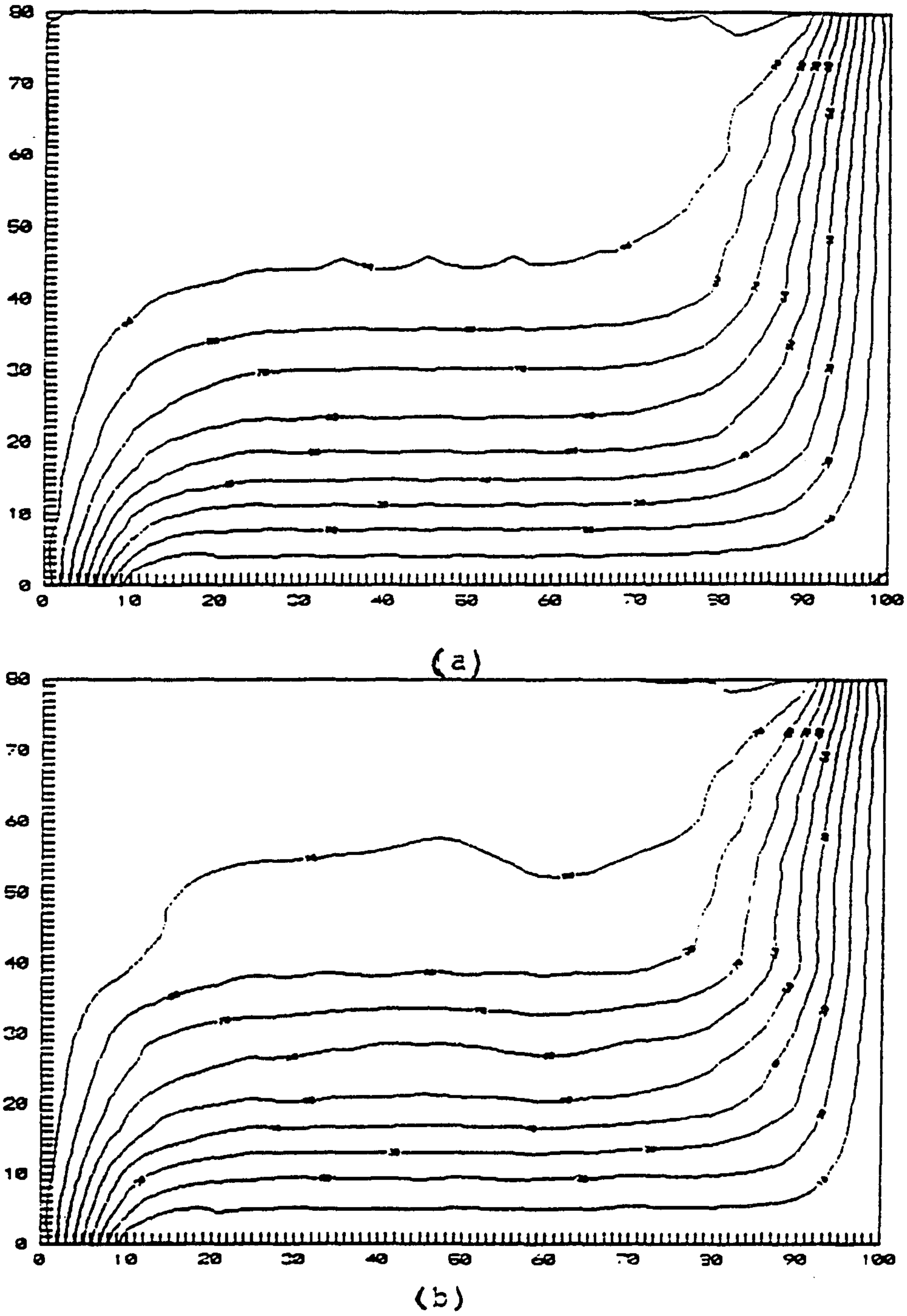


Figure 6.3 Contour lines of equal pore pressure in a 2-D rectangular region a) BEM b) FEM

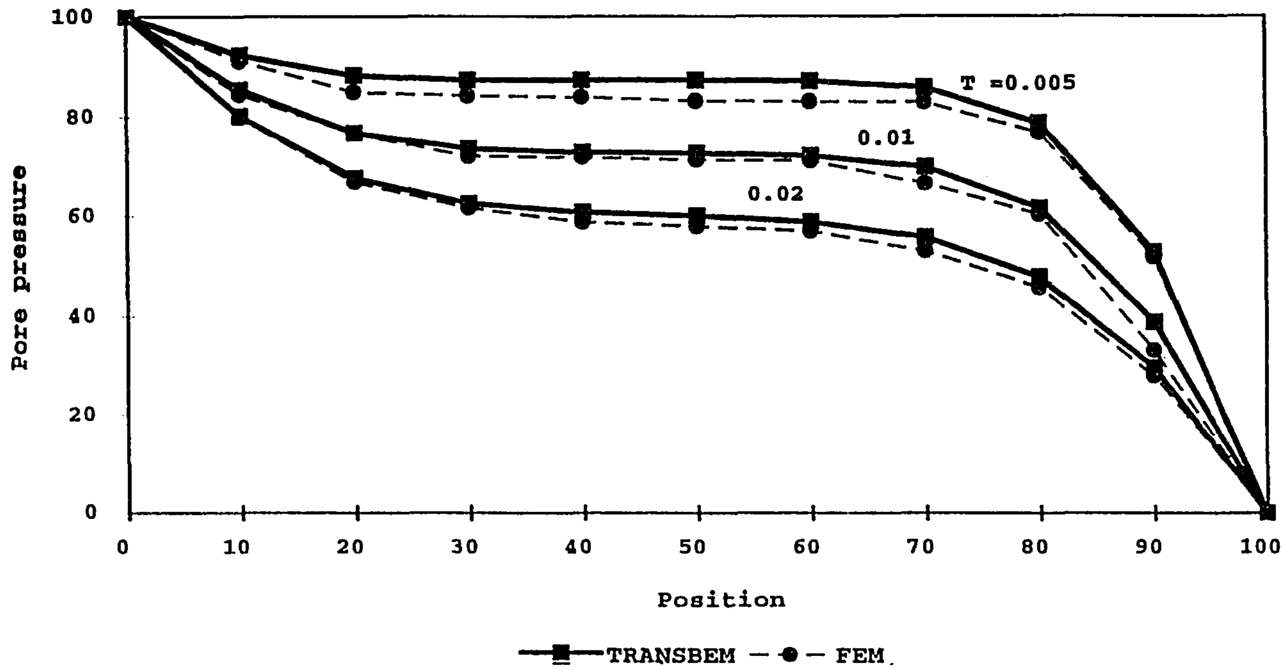


Figure 6.4 pore pressure distribution along the horizontal line Y=50 at three different time increments

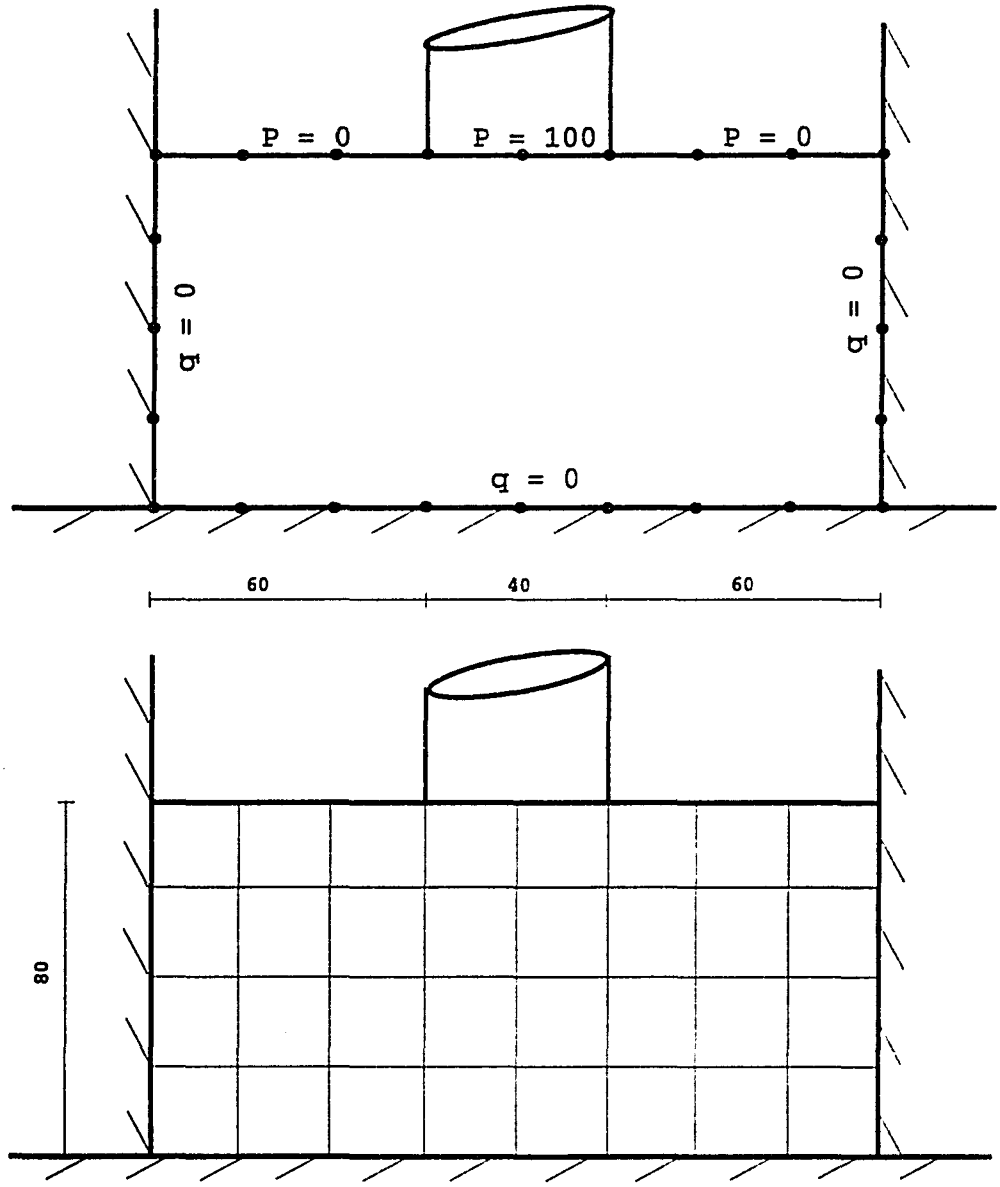
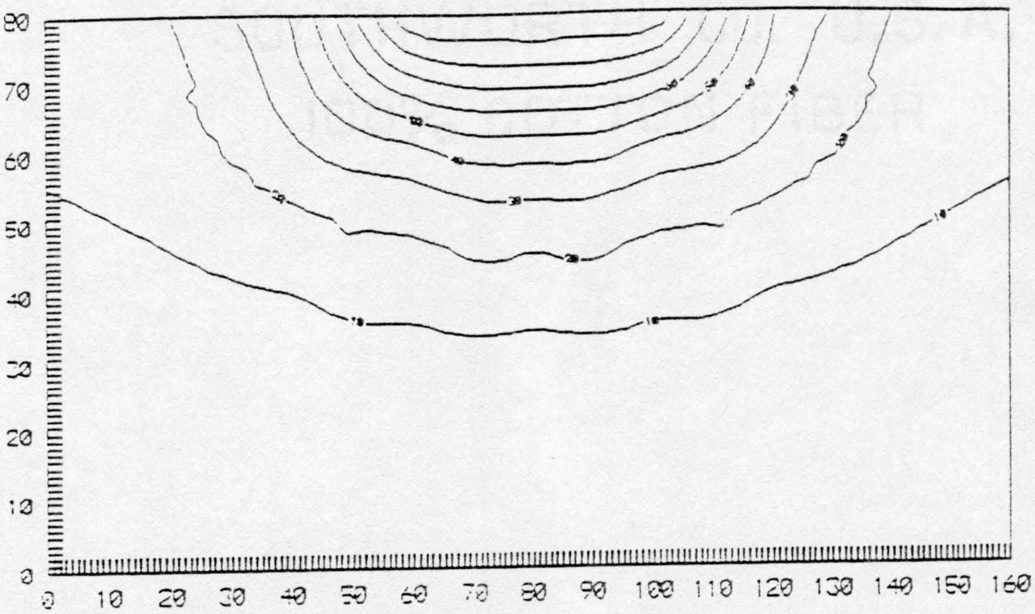
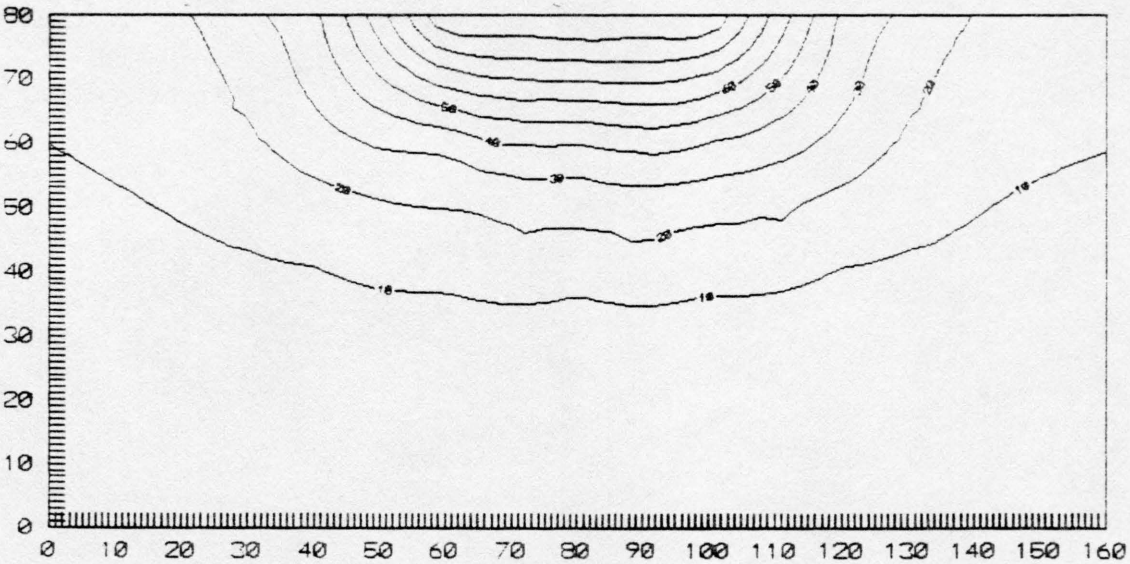


Figure 6.5 BEM and FEM models for pore pressure dissipation under a concrete dam



(a)



(b)

Figure 6.6 Contour lines of equal pore pressure under a dam a) BEM b) FEM

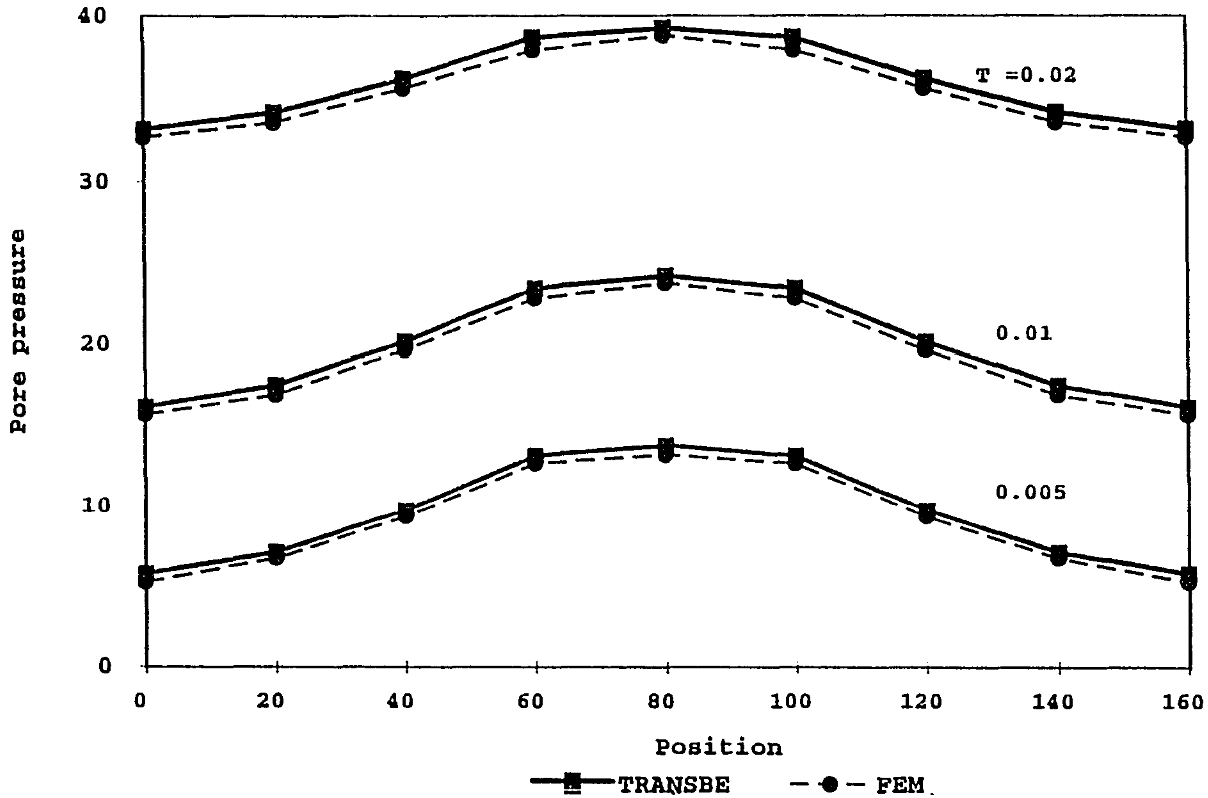


Figure 6.7 Pore pressure distribution along the horizontal line Y=50 at three different time increments

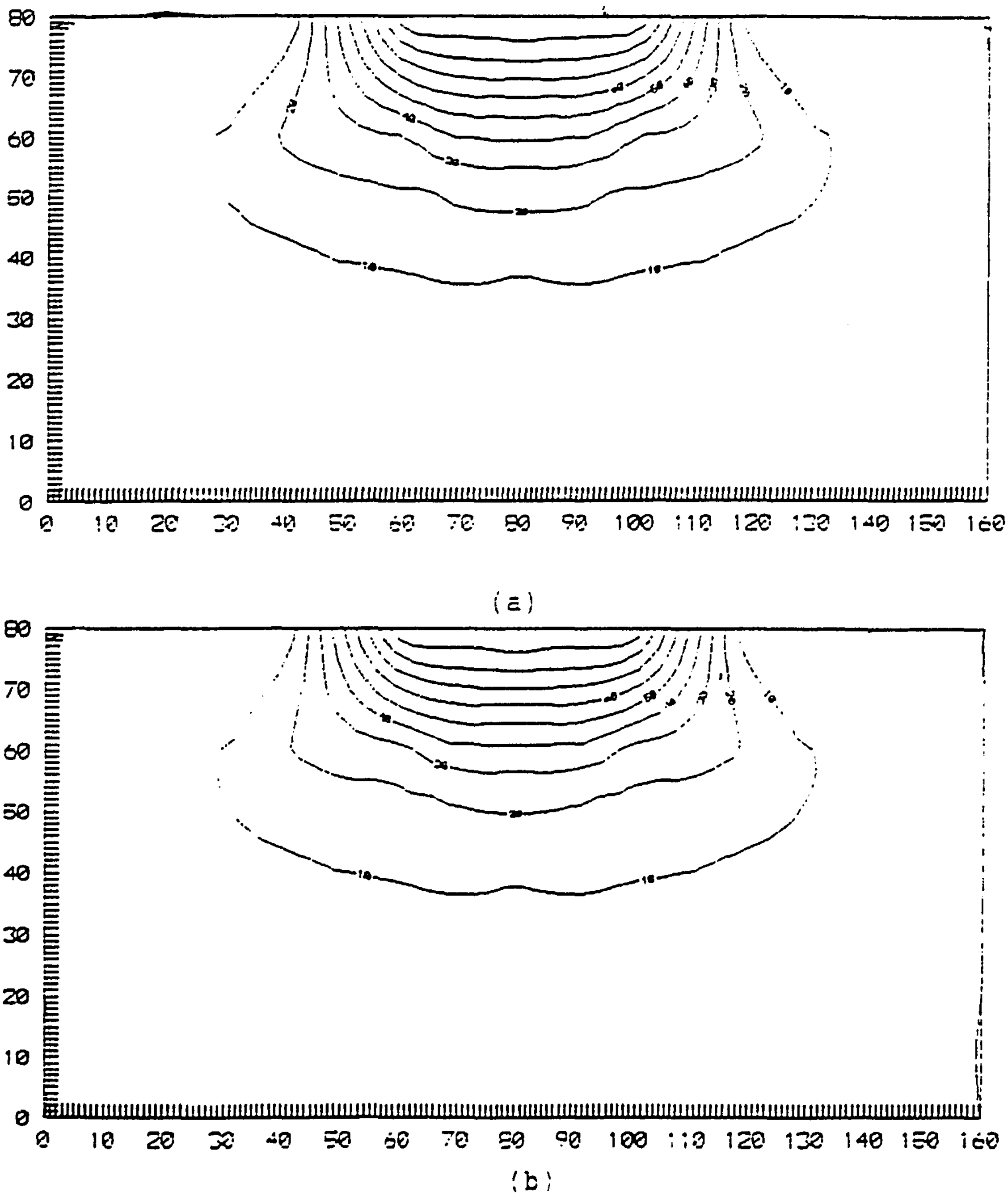


Figure 6.8 Contour lines of equal pore pressure under a dam a) BEM b) FEM

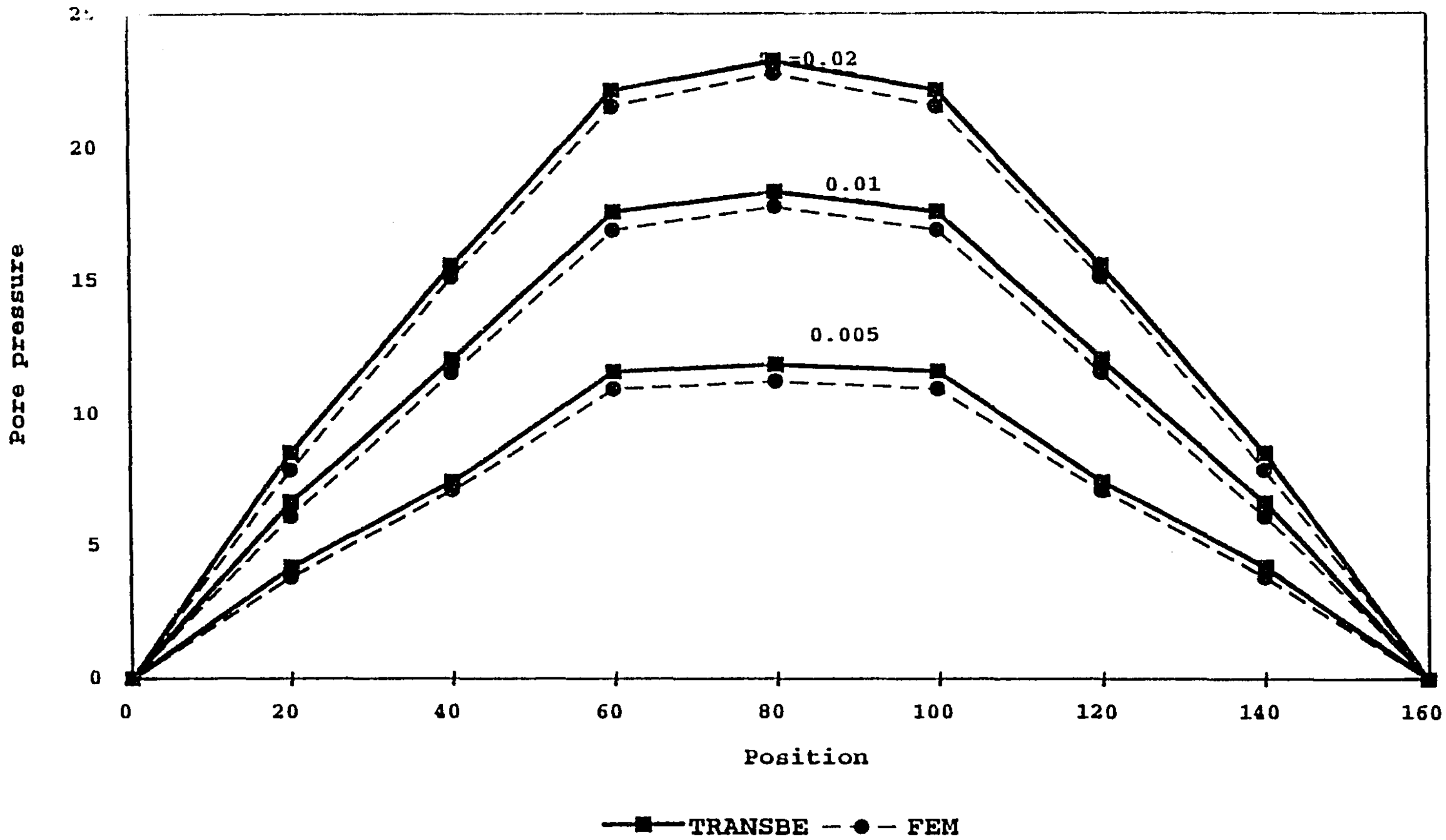


Figure 6.9 Pore pressure distribution along the horizontal line Y=50 at three different time increments

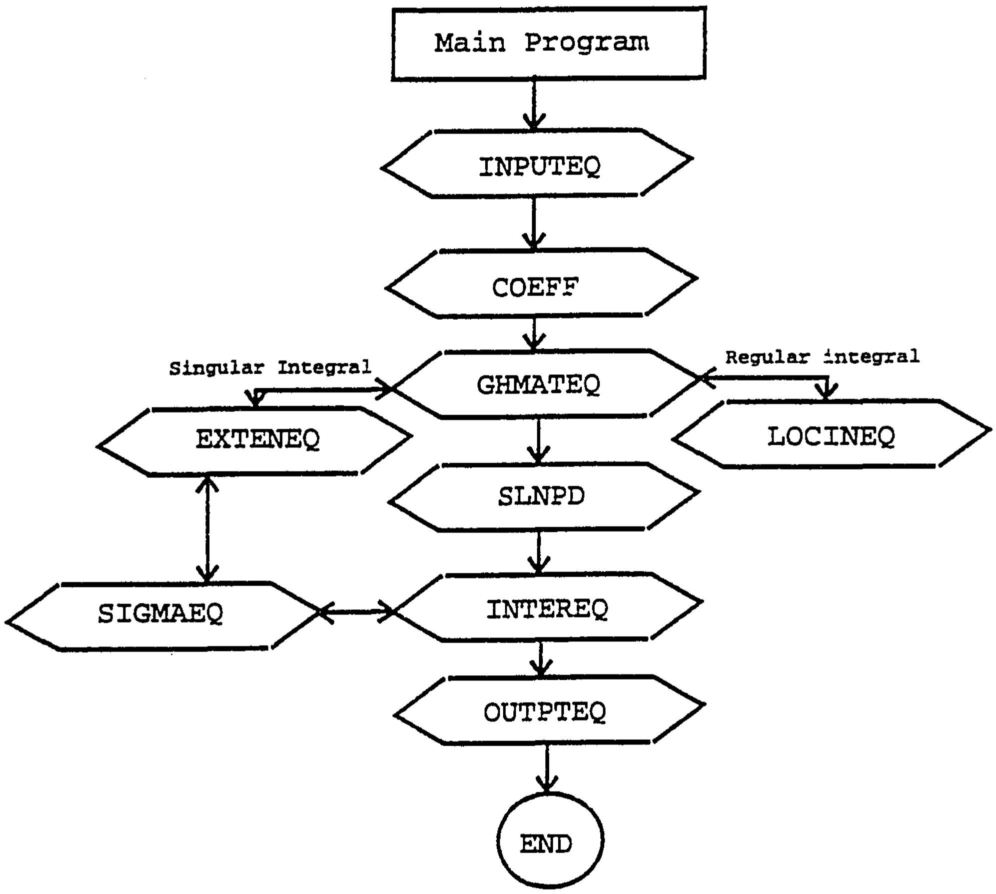


Figure 6.10 Flow chart of the EL10 program

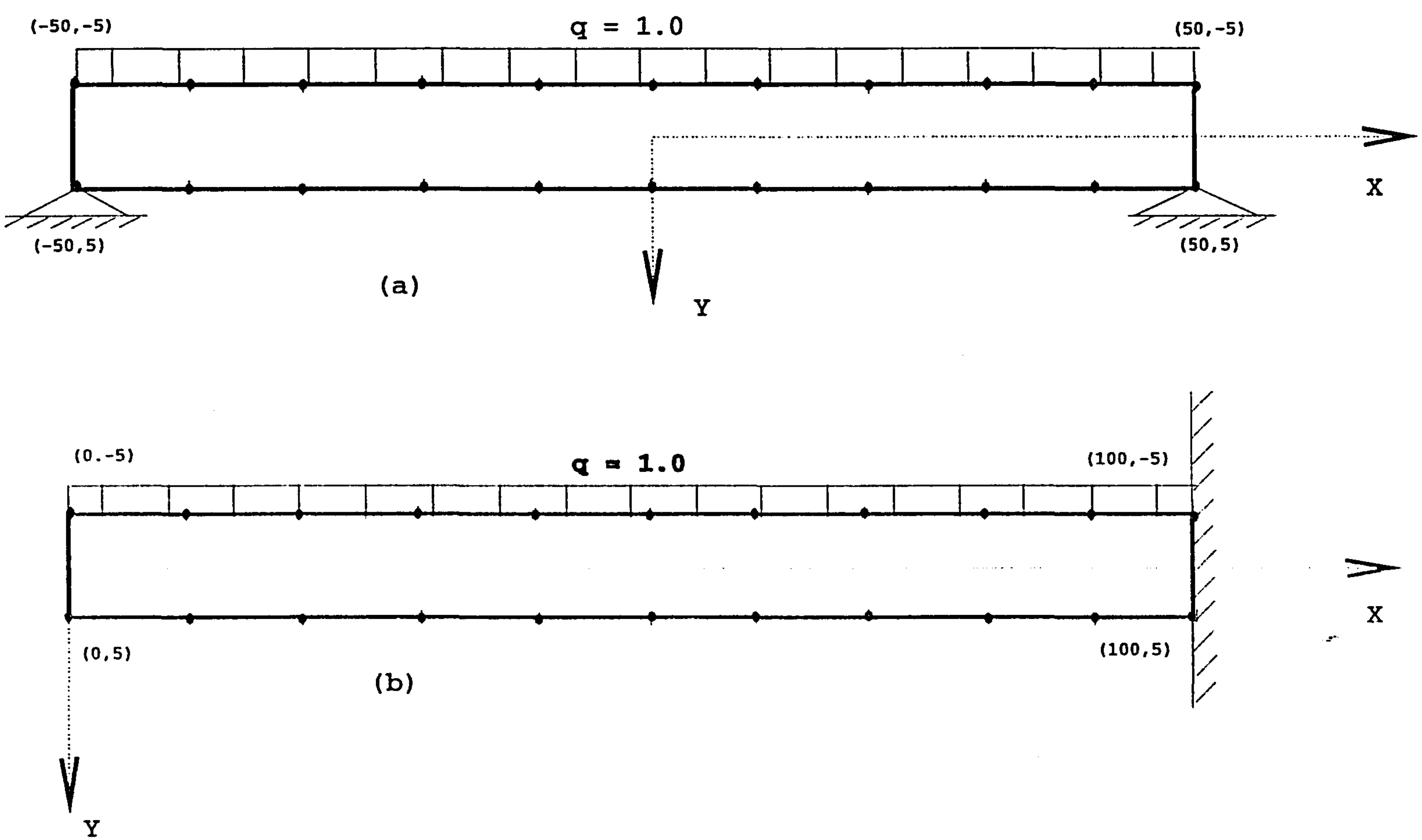


Figure 6.11 Boundary element model for beams with uniformly-distributed load

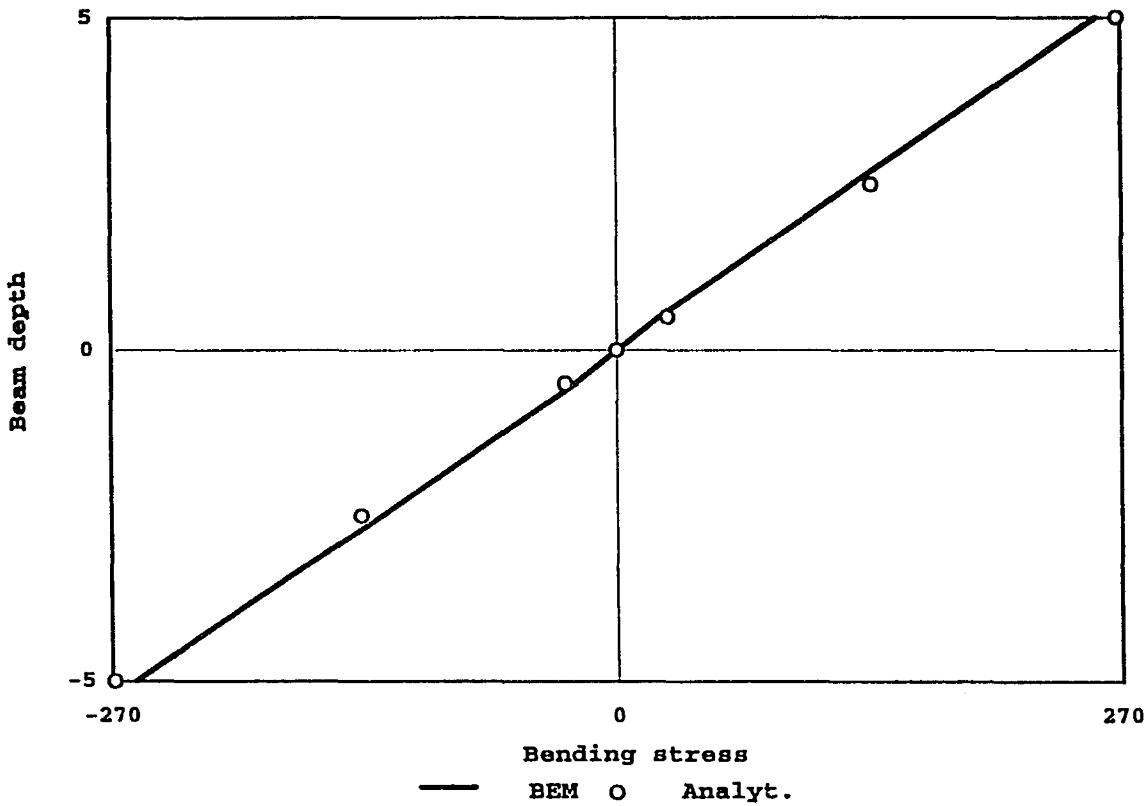


Figure 6.12.a Bending stress distribution near the fixed end of a cantilever beam

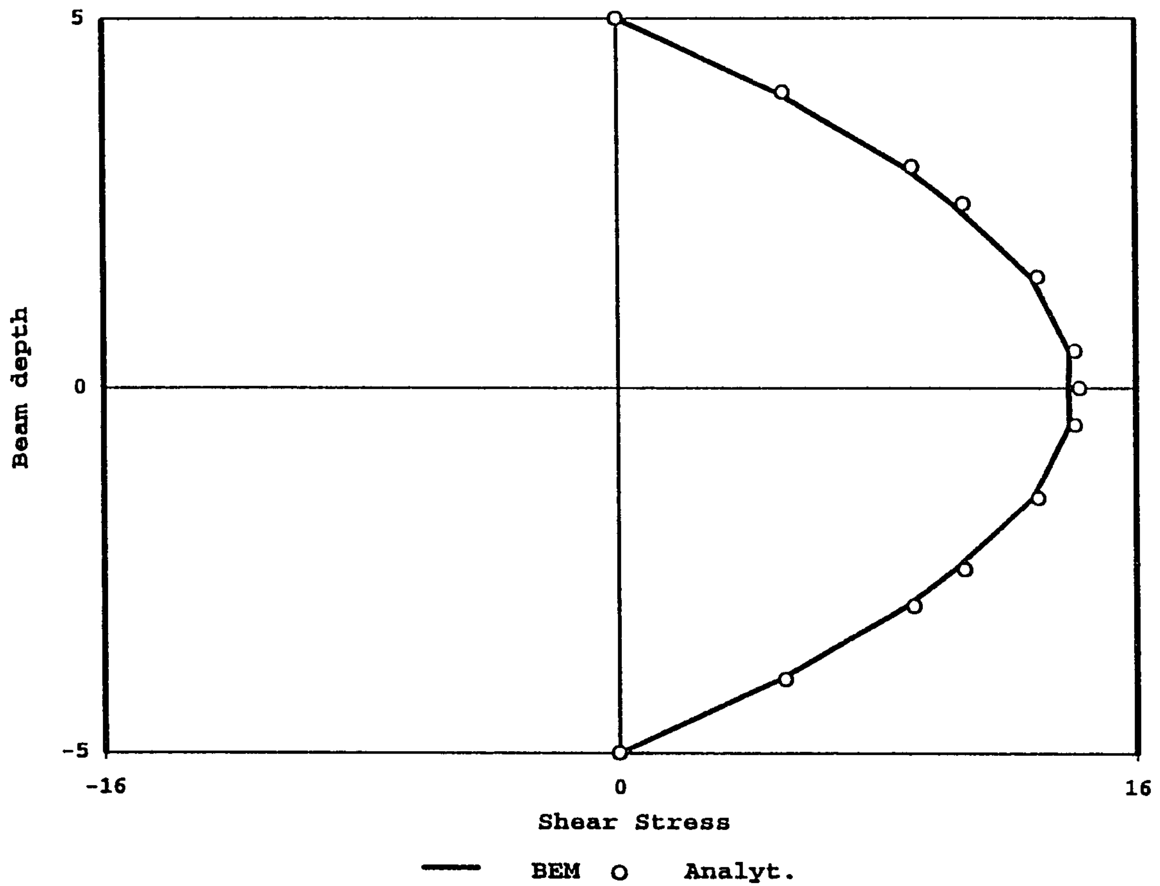


Figure 6.12.b Shear stress distribution near the fixed end of a cantilever beam

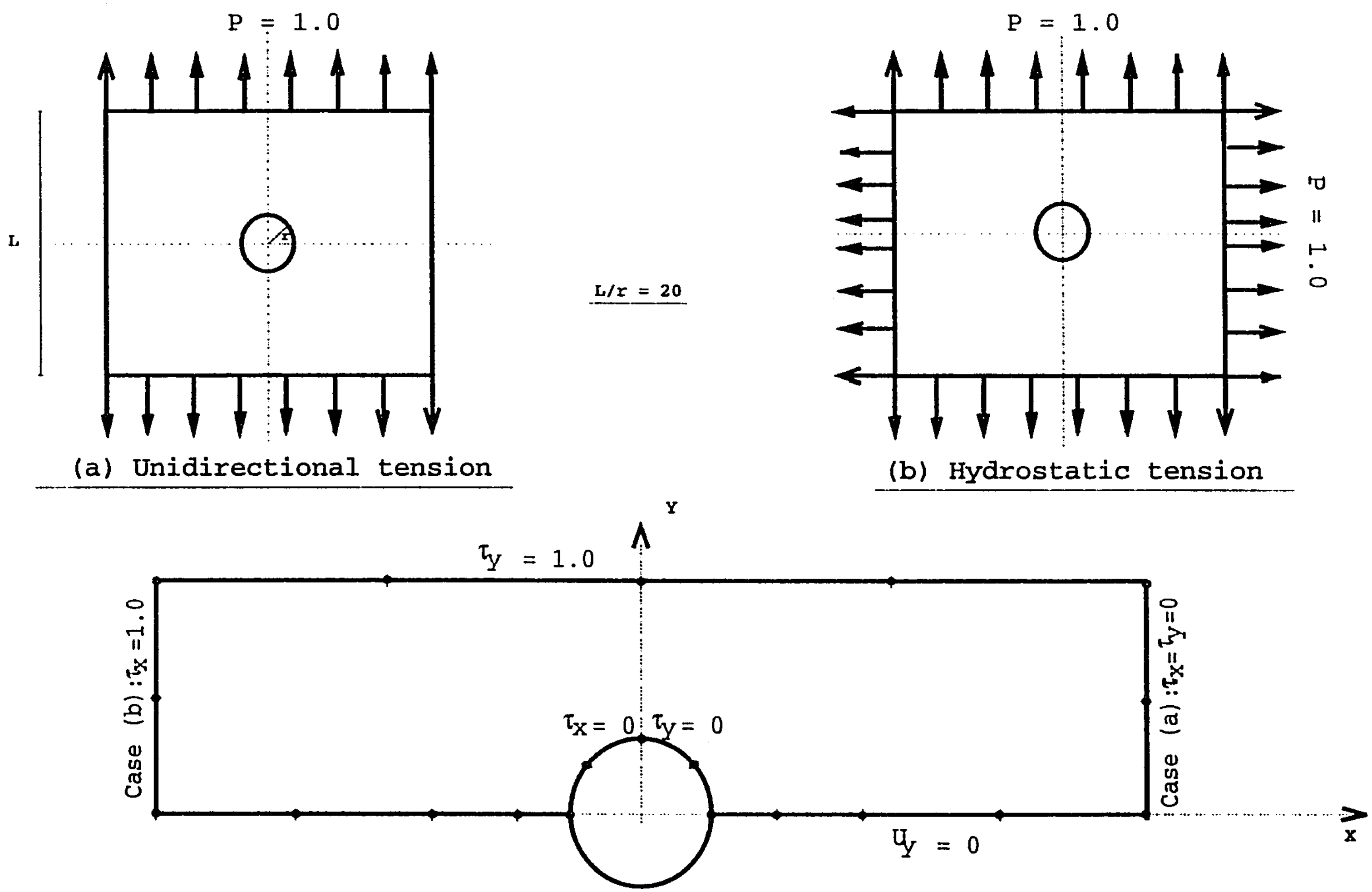


Figure 6.13 Boundary element model for an orthtropic plate with a circular cutout under tension

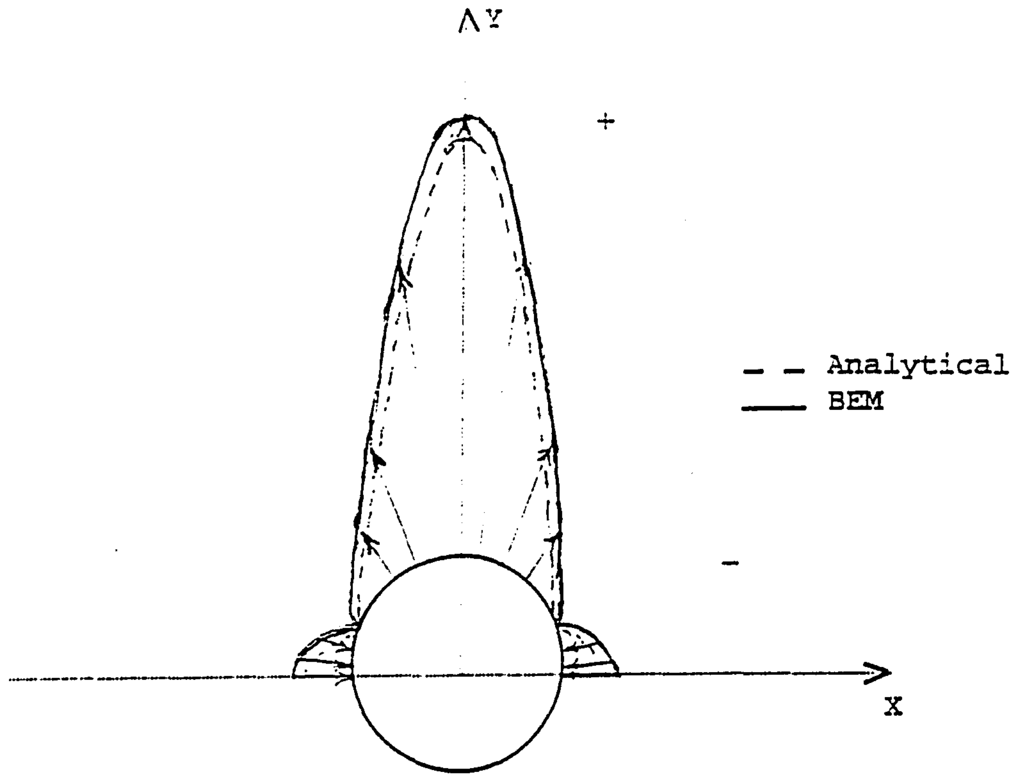


Figure 6.14 Distribution of σ_θ along the perimeter of a circular cutout in orthotropic plate under unidirectional tension

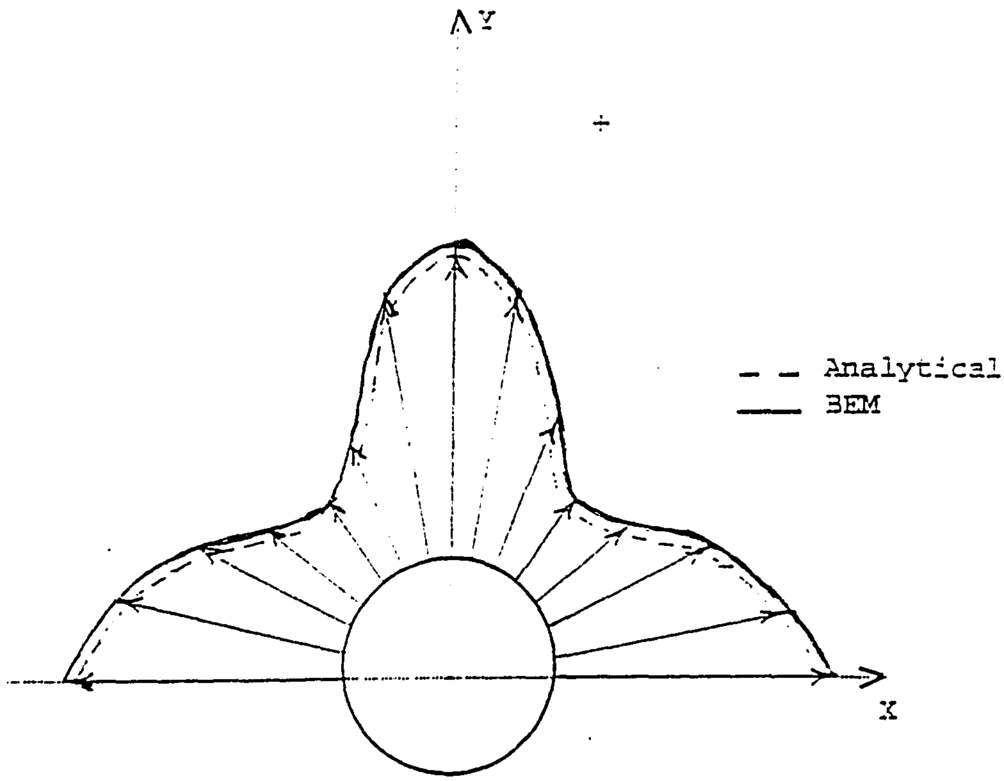


Figure 6.15 Distribution of σ_θ along the perimeter of a circular cutout in orthotropic plate under hydrostatic tension

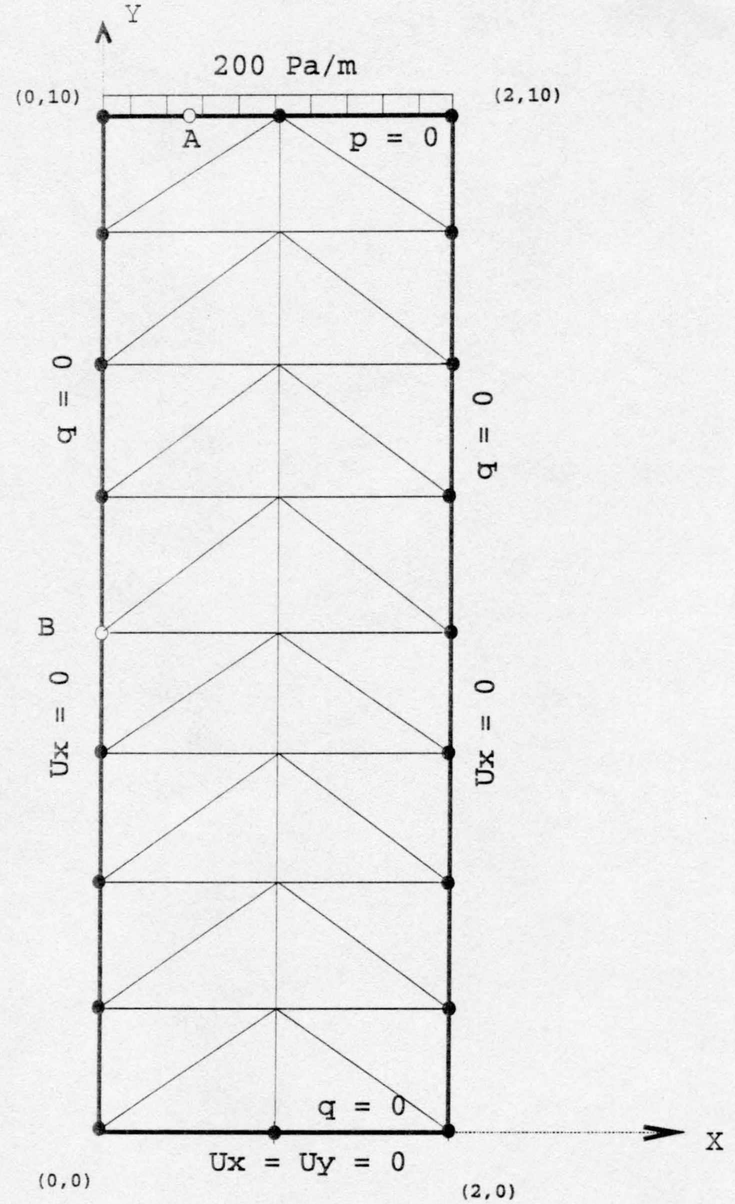


Figure 6.16 Boundary and finite element meshes of a soil layer subject to unidirectional consolidation

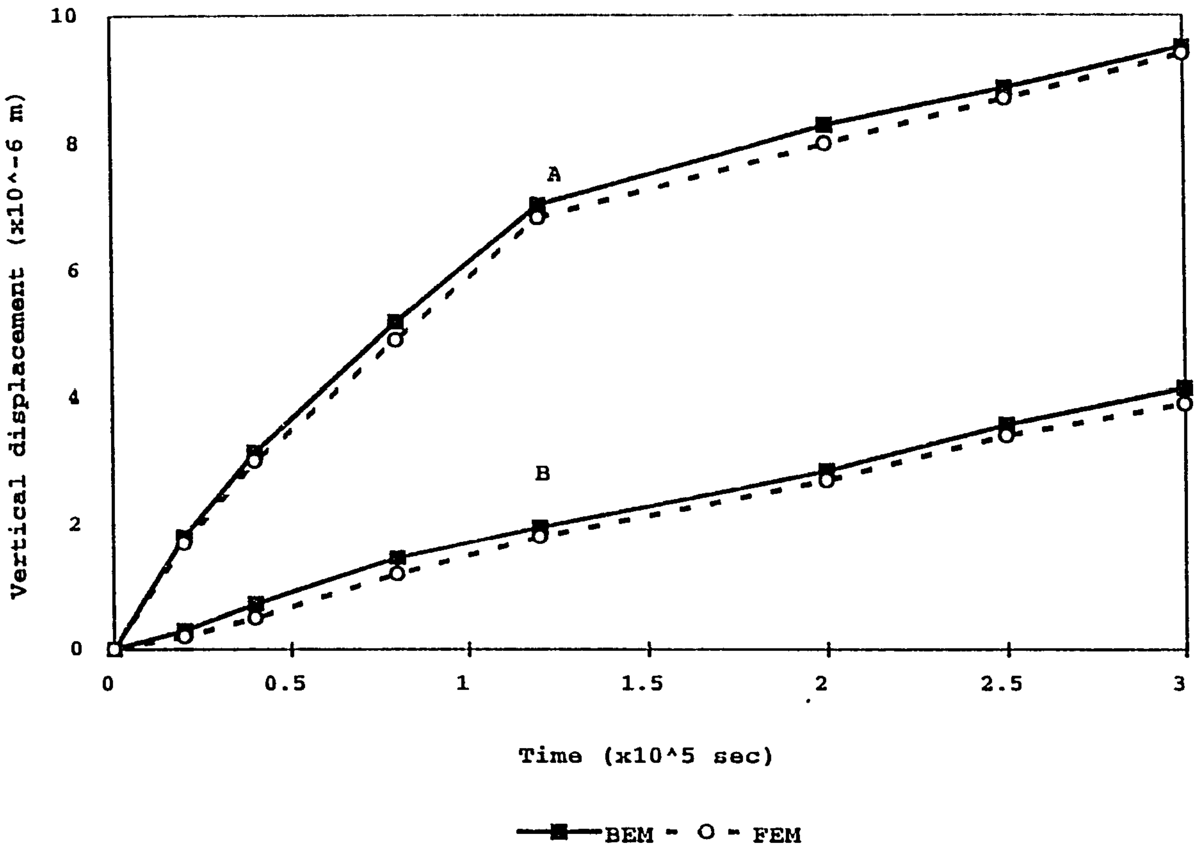


Figure 6.17 Vertical displacement at points A and B in a soil layer under unidirectional consolidation

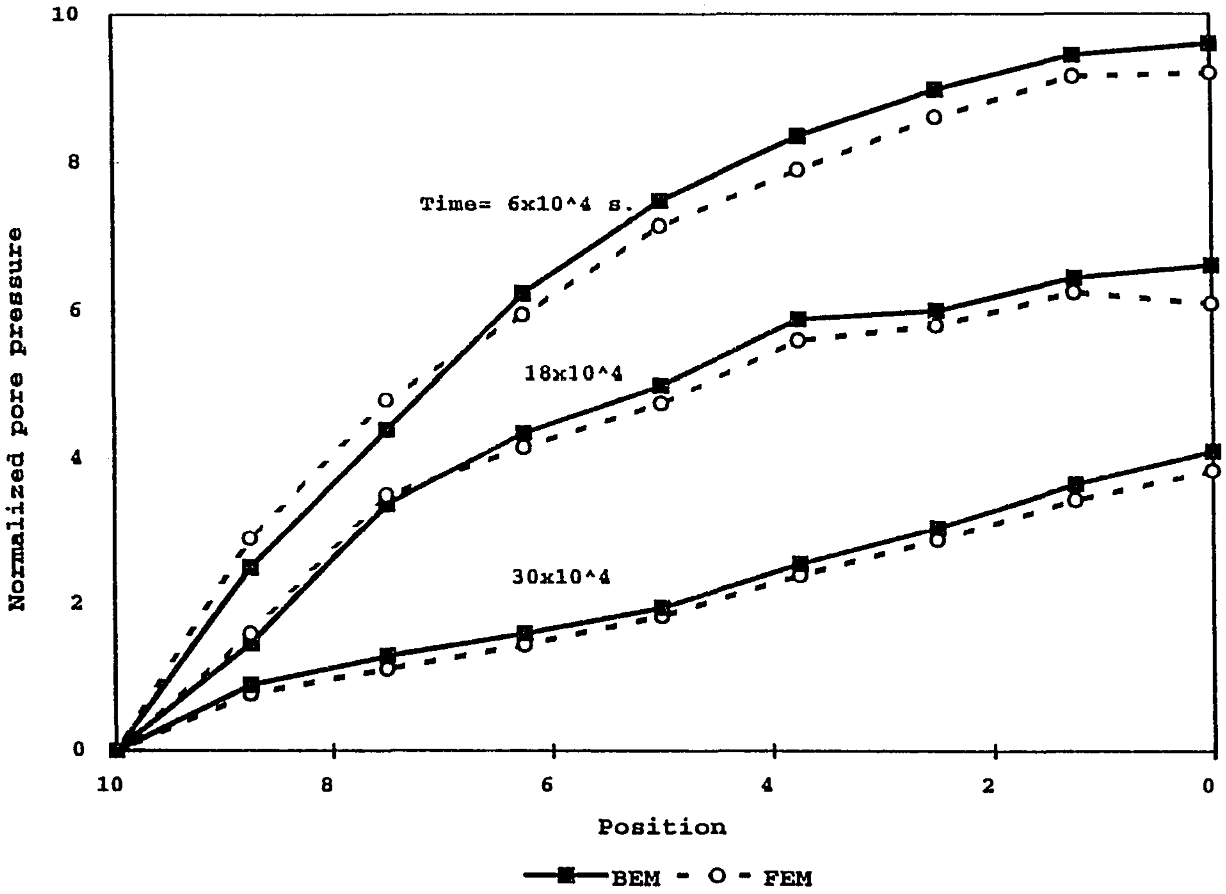


Figure 6.18 Normalized pore pressure distributions in a soil layer under unidirectional consolidation

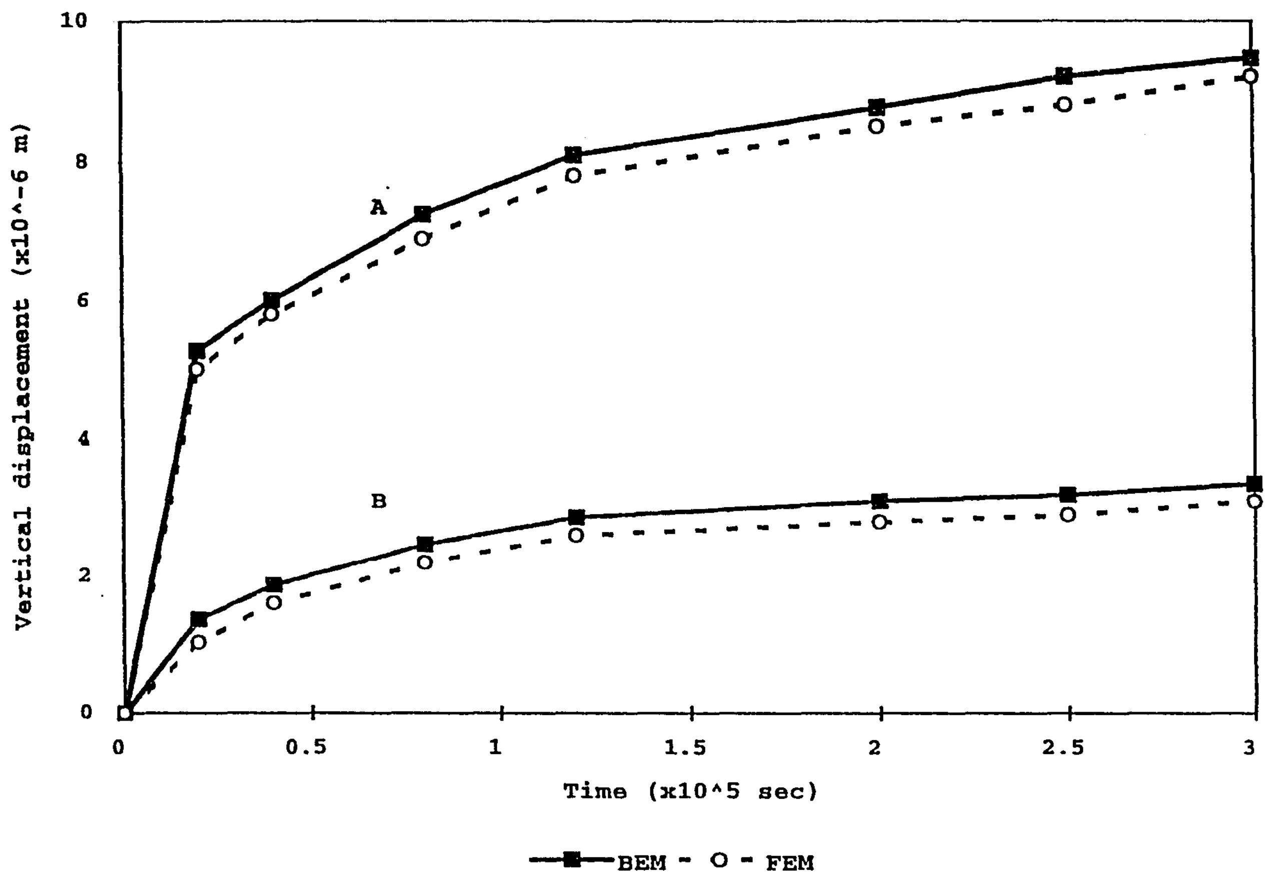


Figure 6.19 Vertical displacement at points A and B in a soil layer under 2-D consolidation

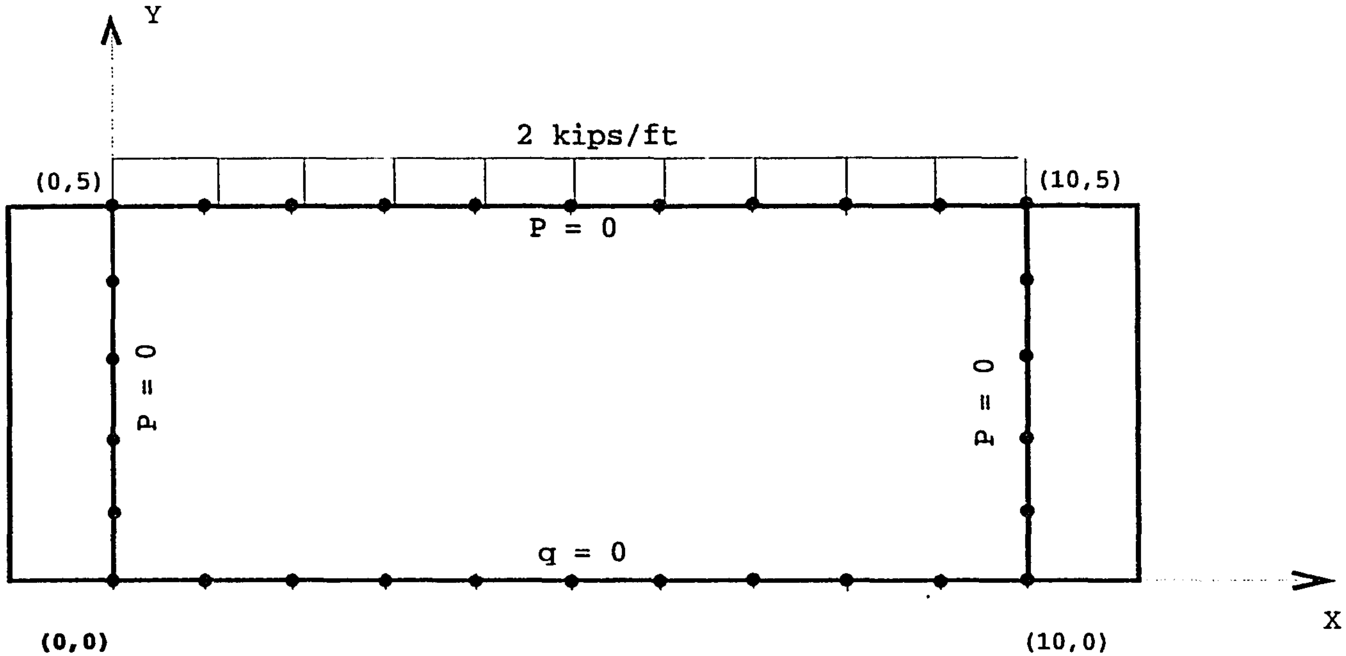


Figure 6.20 Boundary element model of a clay core with side drains

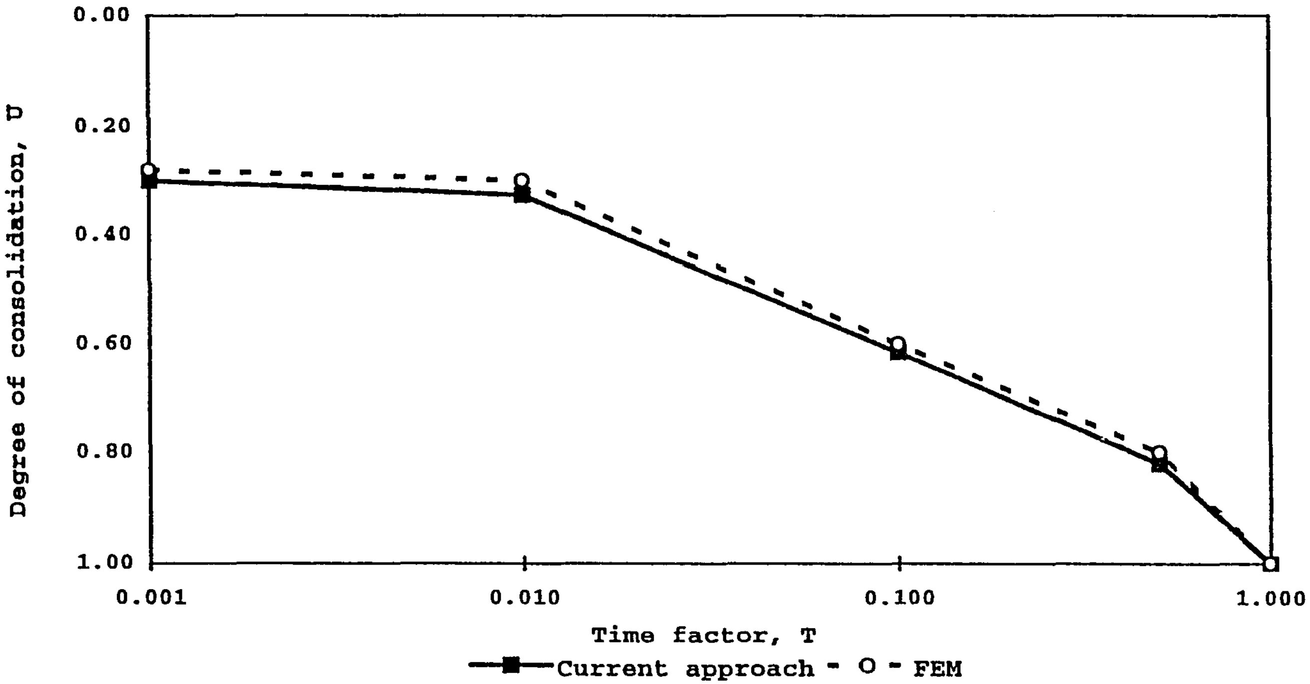


Figure 6.21 Consolidation behaviour of a rectangular clay core

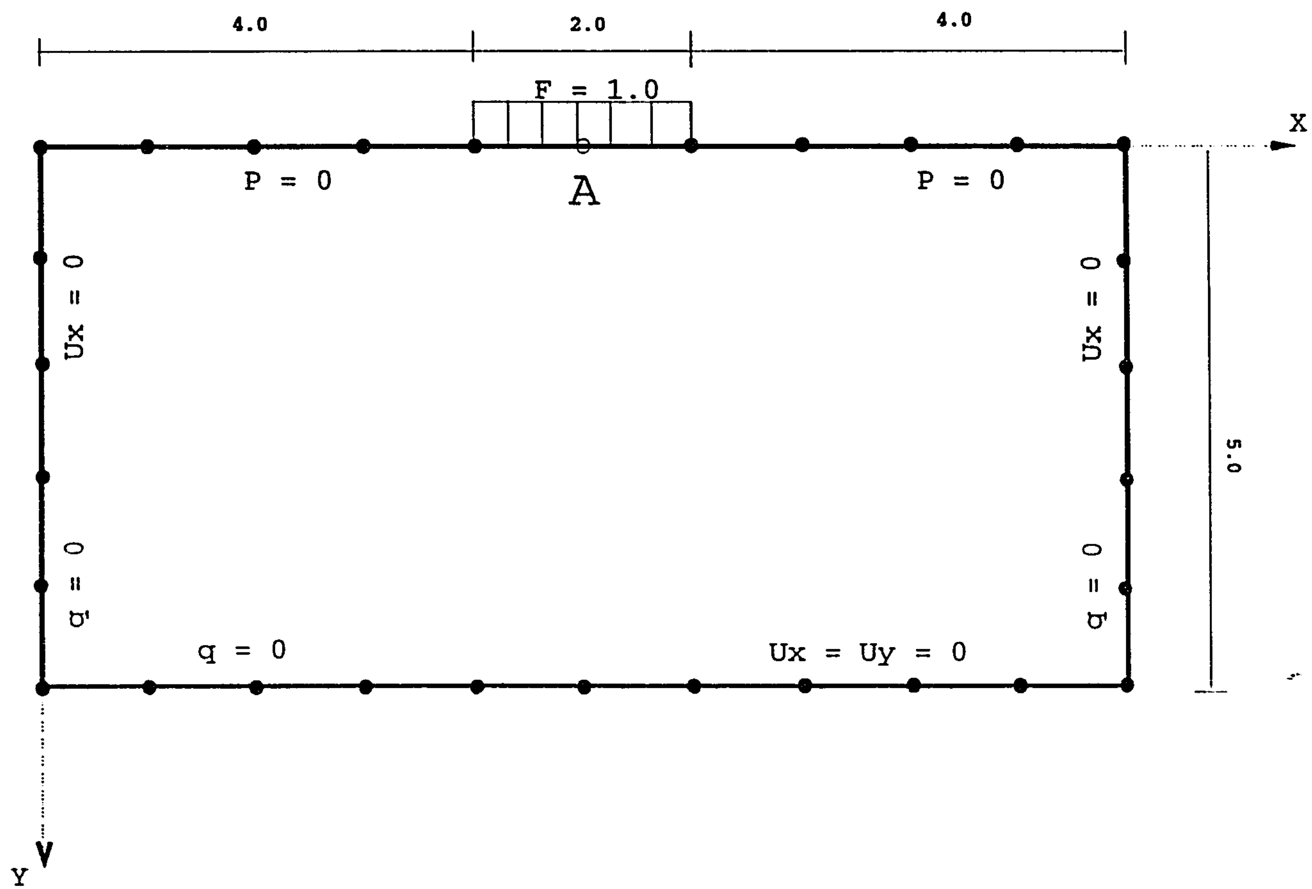


Figure 6.22 Boundary element model of consolidation under a strip footing

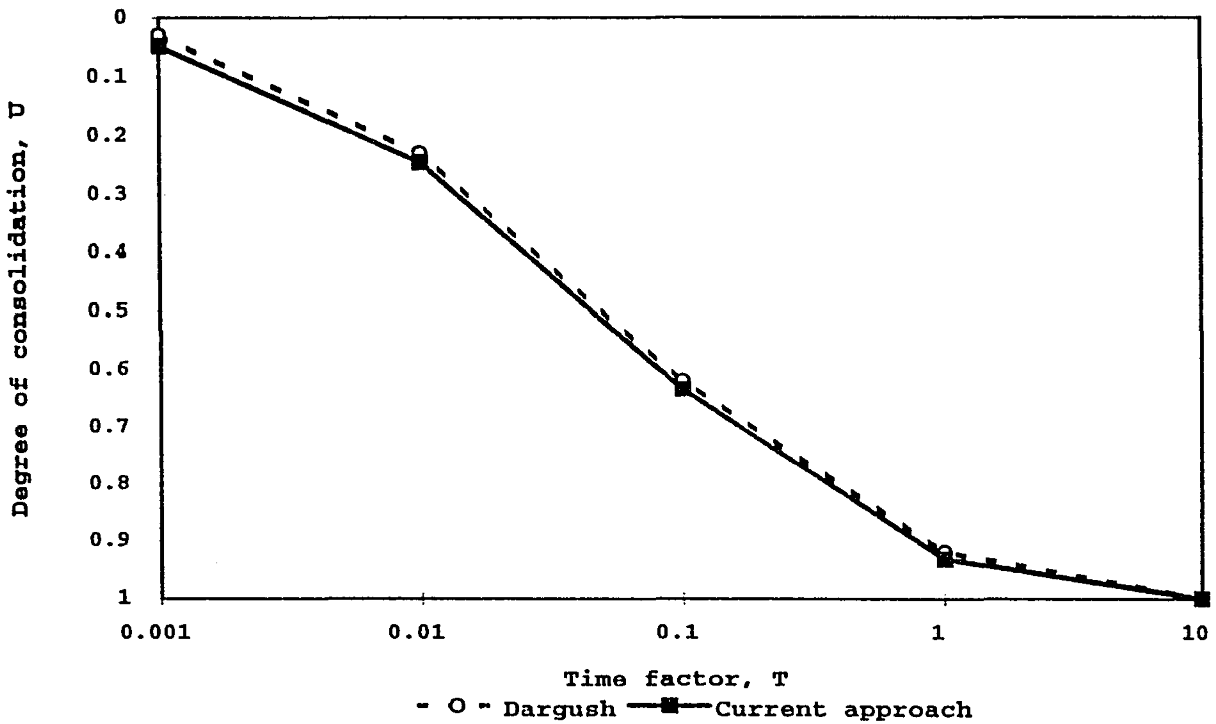


Figure 6.23 Consolidation behaviour of a soil layer under a strip footing

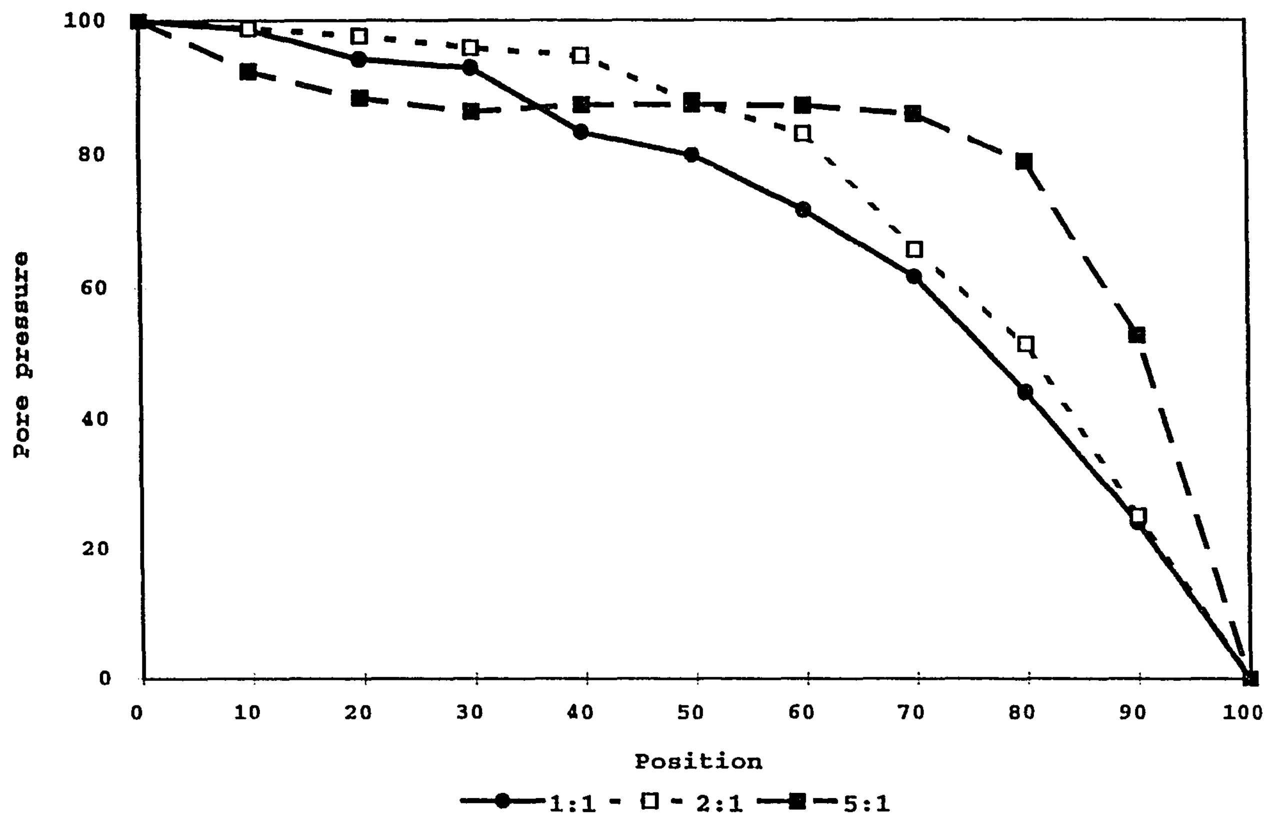


Figure 6.24 Effect of changing the anisotropy ratio (k1:k2) on the pore pressure developed in a rectangular region after one time increment

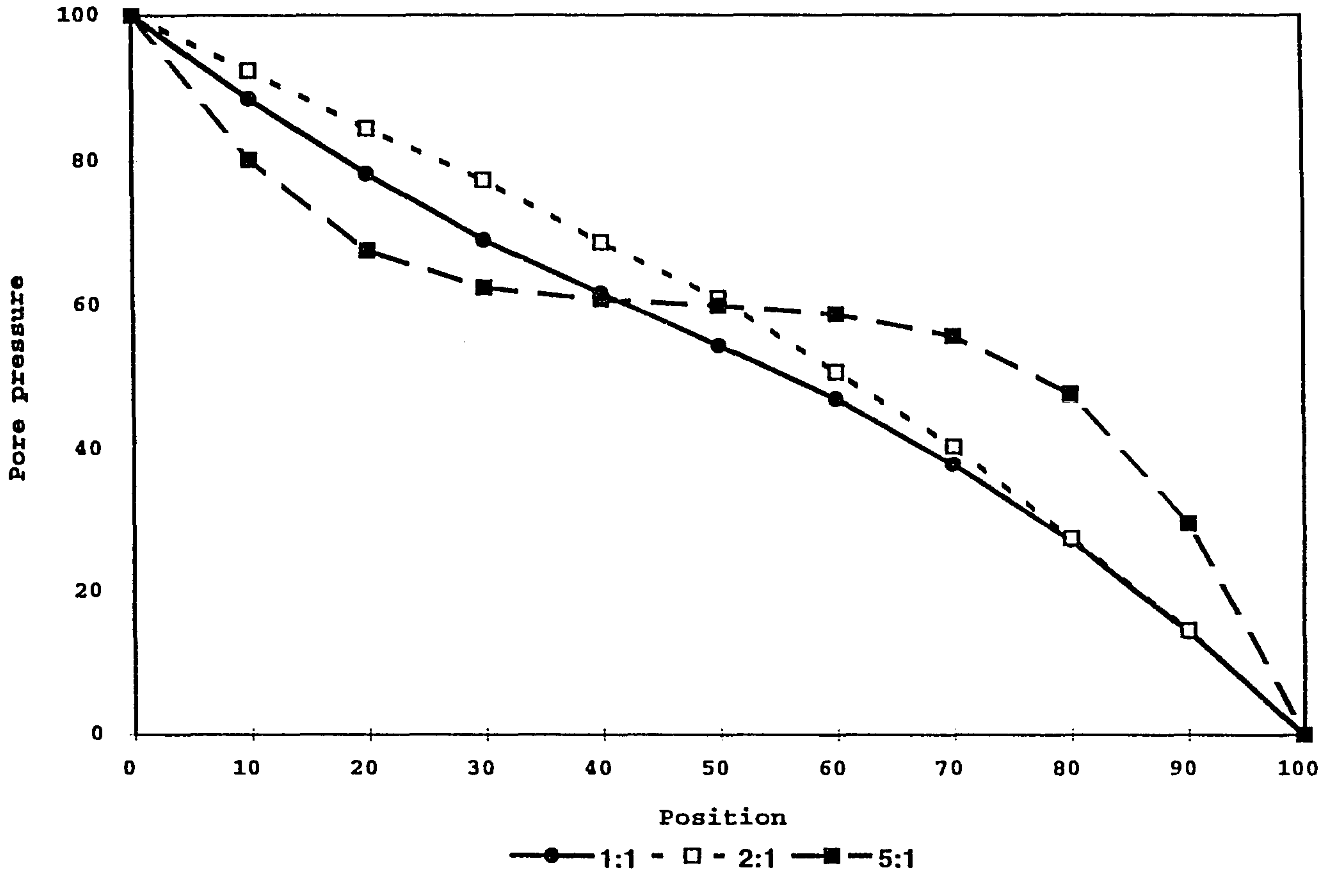


Figure 6.25 Effect of changing the anisotropy ratio (k1:k2) on the pore pressure developed in a rectangular region after six time increments

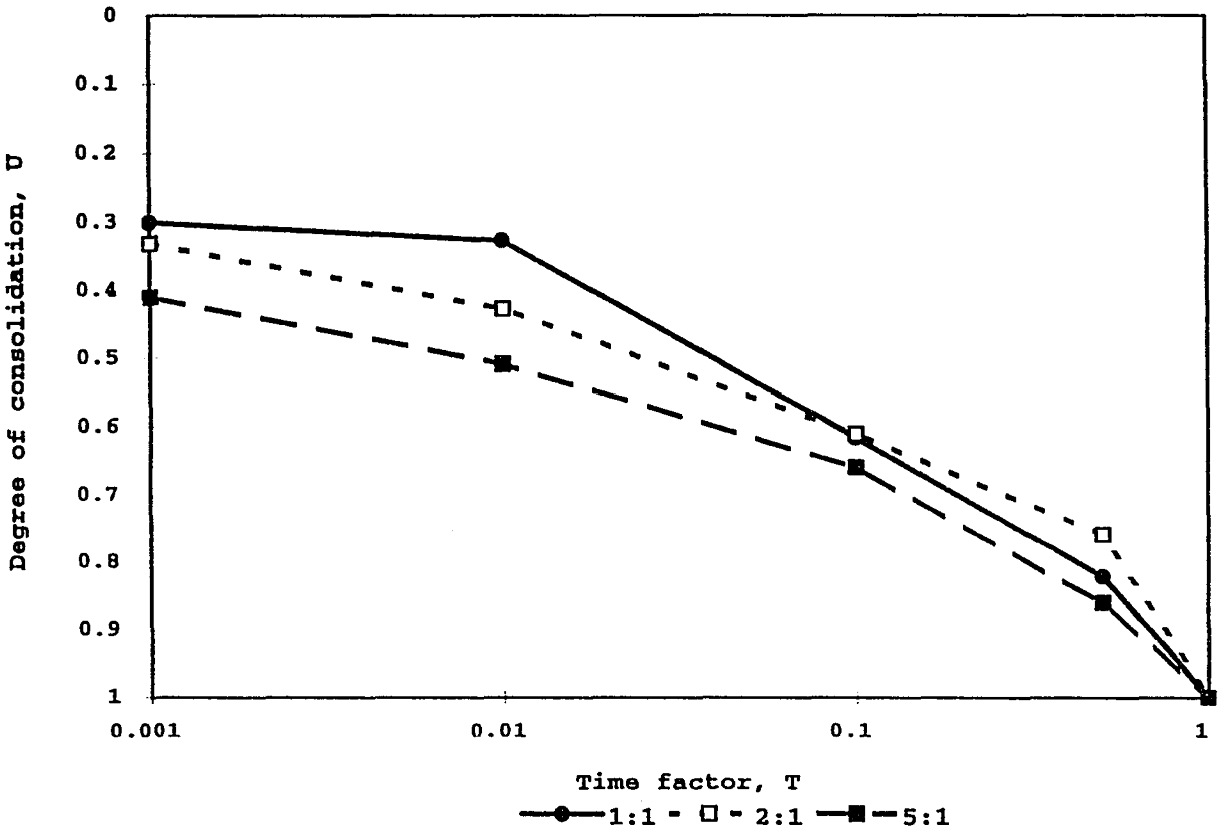


Figure 6.26 Effect of changing anisotropy ratio on the degree of consolidation of a rectangular clay core

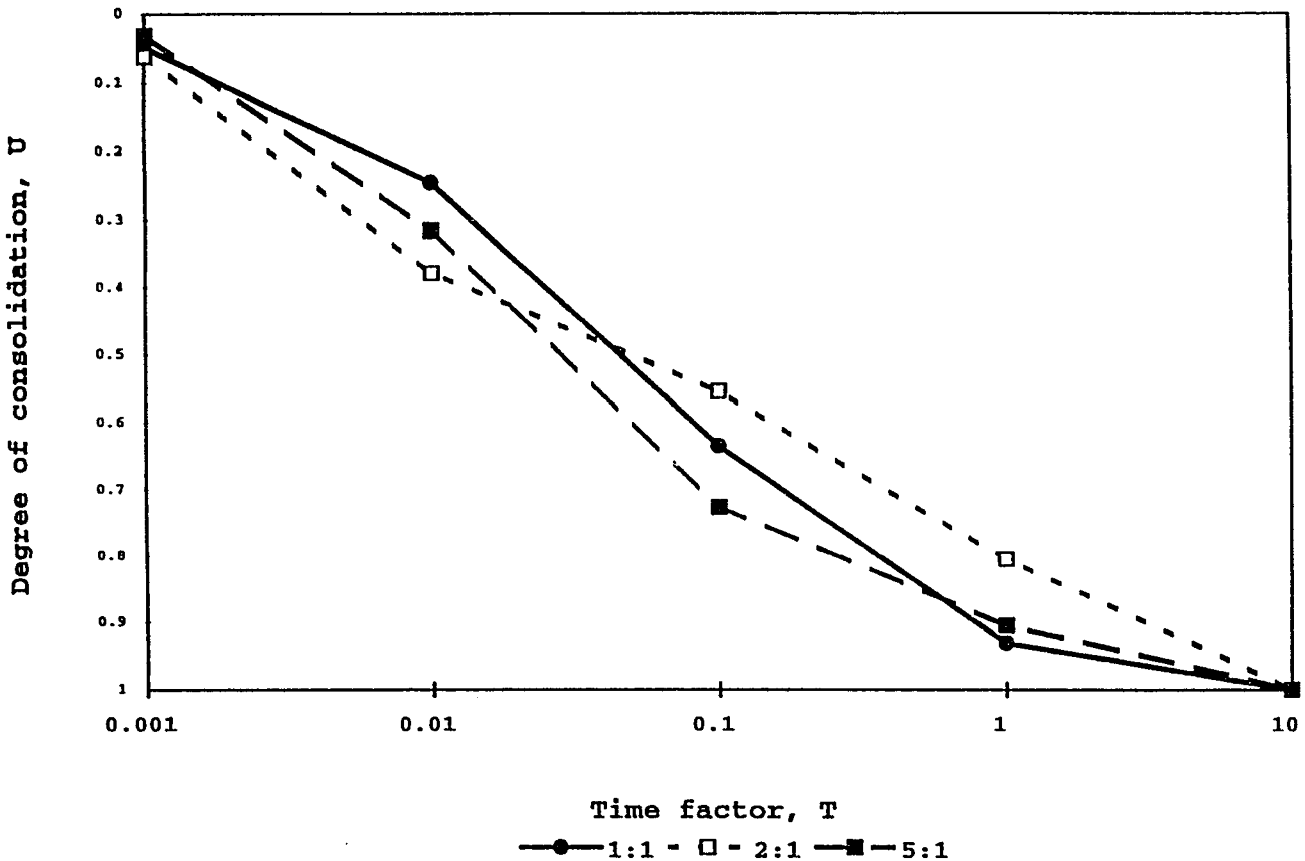


Figure 6.27 Effect of changing anisotropy ratio on the degree of consolidation of soil under a strip footing

Table 6.1 Comparison between the analytical and the BEM-calculated maximum vertical displacement of a simply supported beam under uniformly distributed load.

Node coordinate	EL10 solution	Analytical solution	% Error
(0,-5)	2.0675	2.0	3.23
(0,5)	2.0645	2.0	3.23

Table 6.2 Comparison between the analytical and the BEM-calculated maximum vertical displacement of a cantilever beam under uniformly distributed load.

Node coordinate	EL10 solution	Analytical solution	% Error
(100,-5)	1.0848	1.05	3.39
(100,0)	1.0856	1.05	3.40
(100,5)	1.0847	1.05	3.39

Table 6.3 Comparison of stress distribution from analytical and BEM solutions for orthotropic plate with circular cutout under uniaxial tension.

Angle θ	EL10 solution	Analytical solution	% Error
00	-0.73845	-0.7072	4.42
27	0.00049	0.00047	4.26
54	0.75589	0.7209	4.82
81	4.07136	3.8931	4.58
90	5.71089	5.453	4.73
108	2.20178	2.098	4.84
135	0.46987	0.4487	4.72
153	0.00049	0.00047	4.26
180	-0.73845	-0.7072	4.42

Table 6.4 Comparison of stress distribution from analytical and BEM solutions for orthotropic plate with circular cutout under hydrostatic tension.

Angle θ	EL10 solution	Analytical solution	% Error
00	3.7135	3.568	4.08
27	2.3268	2.2291	4.38
54	1.2289	1.1751	4.58
81	3.1592	3.0265	4.38
90	4.2195	4.039	4.47
108	1.9665	1.8851	4.31
135	1.2658	1.2133	4.33
162	4.6652	4.4738	4.30
180	3.7135	3.568	4.08

7. SUMMARY AND RECOMMENDATIONS FOR FUTURE WORK

7.1 General conclusions

The three problems discussed in this work represent a wide range of field problems in engineering practice. The principal contribution of this work is the development of a time-domain boundary element method for the analysis of heat or water flow in an anisotropic region which is also used in the analysis of the consolidation process in orthotropic soils. This contribution resulted in the form of two boundary element-based Fortran programs (TRANSBE and EL10) to be used to solve these kinds of engineering problems in the field, which are usually difficult to solve specially with arbitrary geometry and/or boundary conditions.

The boundary element formulation of the anisotropic diffusion equation is a new contribution in this area, since all the work in the current literature considered only problems with isotropic conditions. It is also a better alternative to the method usually used to solve the potential problems in which a change of scale of the problem is used to account for the difference in horizontal and vertical permeabilities. The examples studied in chapter 6 showed that the BEM analysis using a time-dependent fundamental solution is accurate and simpler to use than the method mentioned in the literature in which the problem is transferred to the Laplace domain in order to get rid of the time dependency, the problem is solved there and then the solution is transferred back to the real domain. The preparation of data files for the BEM program was shown to take less time and effort than that needed for the FEM-based programs. The range of the problems that the developed computer program can be used to solve, includes applications in both heat transfer and flow of water in porous media.

The elasticity program, EL10, is a useful tool to use in the analysis of elastic orthotropic bodies under any loads. This may include, but is not limited to, structural analysis of beams, plates, earth embankments or excavations and foundations on dry soils. The examples studied in chapter 6 showed that the boundary element method of analysis is accurate and compares very well with analytical solutions. Although these kinds of problems have been analyzed using the more popular numerical analysis tool, the finite element method, it is now clear that BEM can be an excellent alternative because of its simplicity in data preparation. However, this cannot be taken as a general rule and each problem has to be investigated individually to choose the most suitable numerical method to use.

The presented work cannot be considered completely new, since other special cases of anisotropy have been studied before. However, it is an important special case of general anisotropy because it represents the behavior of many practical materials used in everyday engineering practice as was mentioned in chapter 1. Solving this problem is also an important part of solving the problems of consolidation of anisotropic soils using the boundary element method.

The new approach presented for the consolidation process is an important contribution in this area of research. Soil in the field can never be completely isotropic. Due to its history of deposition, permeability in the horizontal direction is usually higher than that in the vertical direction. This phenomenon must have an important effect on the rate of the consolidation which should be taken into consideration in order to get an economical design for foundations or other similar structures. Understanding this phenomenon and making use of the staggered method of solution presented here can also be very useful in planning and building earth dams in the field (Eisenstein et al., 1978). Since, they are usually built in stages to accommodate for the seasonal stops due to maintenance schedules and other field constraints. After each stage, the pore pressure dissipation is studied and the resulting deformations are taken into consideration in the design of the subsequent construction stages.

An obvious shortcoming of this approach is the time it takes to run the two computer programs to solve a consolidation problem. However, it is an important first step towards solving this kind of important engineering problems taking into account more realistic field conditions. Should the time-dependent fundamental solution of the problem be developed, this approach can be modified to solve the problem in one step as was the case with the isotropic problems which were first solved using a similar approach to the one used here before it was later solved in one step using a time-dependent fundamental solution.

It has always been the case that it is easier and more convenient to assume that any body displays isotropic behavior under loads, even though it is rarely the real case in the field, due to either the way it exists or the way it is loaded. The present work shows that bodies with anisotropic properties can be handled easily and accurately.

The present work is an important step towards using the powerful boundary element method to solve more complicated engineering problems with anisotropic conditions which leads to the recommendations for future work which are presented next.

7.2 Recommendations for future work

The recommendations can be divided into two areas. First, the improvement of the present work, and, second, expanding it to include more areas of applications. An important improvement is to investigate the coupling process between the pore pressure and displacement analyses to study the consolidation problem. A parametric study of factors that might affect this coupling to determine when it is important to take the coupling into consideration and when it is convenient, yet accurate, to neglect that coupling. Another improvement is to investigate the gradual change in the void ratio of the soil due to the change in pore water pressure and its effect on the deformation analysis. The current work may be dramatically improved if we can determine a closed form time-dependent fundamental solution for the consolidation problem in anisotropic soils. This will eliminate the need to solve the problem in two stages and greatly reduce the time and computer effort needed to solve the problem as in the isotropic cases. Also, the method of handling the strong singularity which appears in the fundamental solution of the diffusion problem should be further studied.

An expansion of the work may include the introduction of source and sink-type forces in the diffusion analysis. At the same time we have to be careful because this will introduce domain integrals in the boundary element equation. This modification allows for solving problems such as soil subsidence due to dewatering or pumping out of oil or gas. Solving nonlinear problems as well as large deformation problems in anisotropic bodies are some of the areas of research which should be investigated. Dynamic analysis of anisotropic bodies is an important area which did not receive much of the researchers' attention. All of these areas have been investigated before using the finite element method, it is important now to try to take advantage of the boundary element method to solve these problems, and it may be that the results would be as accurate and would be obtained with less time and computer effort to both engineers and researchers.

APPENDIX

ANALYTICAL INTEGRALS IN THE DIFFUSION PROBLEM

Temporal integration of fundamental solution of the diffusion equation:

I1 integral

$$I1 = \int_{t_n}^{t_{n-1}} P^* \, dt$$

$$P^* = \frac{|K^{ij}|^{1/2}}{4\pi\,(t_F - t)} \exp\left(\frac{-R^2}{4\,(t_F - t)}\right)$$

$$I1 = \frac{|K^{ij}|^{1/2}}{\pi R^2} \int_{t_{n-1}}^{t_n} \frac{R^2}{4\,(t_F - t)} \exp\left(\frac{-R^2}{4\,(t_F - t)}\right) dt$$

Define:

$X = R^2/4(t_F - t)$

thus:

$$I1 = \frac{|K^{ij}|^{1/2}}{4\pi} \int_{a_{n-1}}^{a_n} \frac{4\,(t_F - t)}{R^2} \exp\left(\frac{-R^2}{4\,(t_F - t)}\right) dX$$

$$= \frac{|K^{ij}|^{1/2}}{4\pi} \int_{a_{n-1}}^{a_n} \frac{e^{-X}}{X} dX$$

which ends up as:

$$I1 = \frac{|K^{ij}|^{1/2}}{4\pi} [E1(a_{n-1}) - E1(a_n)]$$

where:

$a_{n-1} = R^2/4(t_F - t_{n-1})$

$a_n = R^2/4(t_F - t_n)$

I2 integration:

$$Q^* = \frac{\partial G}{\partial n^*} = \frac{|K^{ij}|^{1/2}}{8\pi(t_F-t)^2}[D_s] \exp\left(\frac{-R^2}{4(t_F-t)}\right)$$

$$I2 = \frac{|K^{ij}|^{1/2}}{2\pi}\left[\frac{D_s}{R^2}\right] \int_{t_{n-1}}^{t_n} \frac{R^2}{4(t_F-t)^2} \exp\left(\frac{-R^2}{4(t_F-t)}\right) dt$$

Define:

$X = R^2/4(t_F-t)$

thus:

$$I2 = \frac{|K^{ij}|^{1/2}}{2\pi}\left[\frac{D_s}{R^2}\right] \int_{a_{n-1}}^{a_n} \frac{R^2}{4(t_F-t)^2} \exp\left(\frac{-R^2}{4(t_F-t)}\right) \frac{4(t_F-t)}{R^2}$$

$$= \frac{|K^{ij}|^{1/2}}{2\pi}\left[\frac{D_s}{R^2}\right] \int_{a_{n-1}}^{a_n} e^{-X}\, dX$$

which ends up as:

$$I2 = \frac{|K^{ij}|^{1/2}}{2\pi}\left[\frac{D_s}{R^2}\right] [-\exp(-a_n) + \exp(-a_{n-1})]$$

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