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DEVELOPMENT OF A GENERALIZED CONSTITUTIVE MODEL
AND ITS IMPLEMENTATION IN SOIL-STRUCTURE INTERACTION

by

Md. Omar Faruque

A Dissertation Submitted to the Faculty of the
DEPARTMENT OF CIVIL ENGINEERING AND ENGINEERING MECHANICS

In Partial Fulfillment of the Requirements
For the Degree of

DOCTOR OF PHILOSOPHY

In the Graduate College

THE UNIVERSITY OF ARIZONA

1983
As members of the Final Examination Committee, we certify that we have read the dissertation prepared by Md. Omar Faruque entitled DEVELOPMENT OF A GENERALIZED CONSTITUTIVE MODEL AND ITS IMPLEMENTATION IN SOIL-STRUCTURE INTERACTION and recommend that it be accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy.

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ABSTRACT

The general principles of continuum mechanics such as conservation of mass, conservation of momenta, first and second law of thermodynamics are applicable to all materials irrespective of their internal constitutions. These principles alone do not provide sufficient equations to obtain solutions for any boundary value problems. The additional equations are provided by the constitutive laws.

There are many groups of constitutive theories. Of them, the theory of plasticity describes rate independent nonlinear and inelastic behavior of materials.

A plasticity-based constitutive law is proposed herein for geological materials. The model, however, may also be used for other frictional materials. A generalized approach is followed in formulating the proposed constitutive model. The technique can be used to construct plasticity-based constitutive models for any other materials.

A series of laboratory tests are performed on cubical soil specimens using a truly triaxial testing device. The testing device is such that the samples can be subjected to a general three-dimensional state of stress.

The test data is used to determine the material constants associated with the proposed constitutive model. The model is then verified by back-predicting the stress-strain curves obtained from the laboratory.
As a final step, the proposed constitutive model is implemented into a three-dimensional finite element procedure. A number of boundary value problems are analyzed using the proposed model. The results are compared with the observation. It is found that the proposed model can effectively characterize the nonlinear and inelastic response of frictional materials. Although the proposed model is investigated with respect to soils, it can also be applied for concrete, rocks, etc.
CHAPTER 1

INTRODUCTION

General

The principles of continuum mechanics such as conservation of mass, conservation of momenta, first and second law of thermodynamics are applicable to all materials, irrespective of their internal constitutions. These principles alone do not provide sufficient number of equations to solve a boundary value problem. Additional equations are provided by the constitutive laws which relate the response of a material to the imposed action. For stress-deformation problems under isothermal condition, the constitutive laws define the relations between the stress and the strain.

There are several groups of constitutive theories. The purpose of each theory is to describe the behavior of certain class of materials under some ideal conditions. Theory of linear elasticity, for example, describes responses of those materials for which a stress-free initial state exists and stress is a unique linear function of strain. Theory of plasticity, on the other hand, describes rate independent inelastic behavior of materials. In this dissertation, only constitutive laws based on the theory of plasticity are considered.
Numerical methods such as the finite element method are often used to solve a wide variety of soil-structure interaction problems. Since geological materials are highly nonlinear and inelastic, proper constitutive models are necessary to obtain accurate solutions from such numerical schemes. Although numerous constitutive models are available at this time, none of these models is general enough to characterize behavior of a class of materials under complex loading conditions. As a result, many researchers are currently engaged in developing new and improved constitutive models for broad classes of engineering materials.

The primary objective of the present study is to develop a generalized constitutive model based on the theory of plasticity. Although such models can be used for a wide range of materials, in this dissertation their applications to geological materials are emphasized.

**Assumptions**

It is mentioned earlier that every constitutive model is based upon certain assumptions and is applicable for only a limited class of materials under certain specified conditions. In the present study, the important assumptions made are the following:

i) The material is initially isotropic and remains isotropic during plastic deformation.

ii) The rate of loading is slow enough to disregard the inertia effect. Thus, the deformation can be considered as quasi-static.
iii) The system under consideration is in an isothermal condition. Thus, the temperature does not enter into the constitutive equations as an explicit state variable.

iv) Deformations are small enough to disregard the nonlinear terms in the strain displacement relations; that is, no geometric nonlinearity is considered in the formulation of the constitutive theory.

v) Elastic and plastic deformations are uncoupled. This may be true when small deformations are assumed. Thus, the total deformation at any material point can be linearly decomposed into elastic and plastic parts.

Scope and Objective of the Present Study

Most soil-structure interaction problems are complex and in many situations the closed-form solutions are intractable. This is mainly due to the nonlinear and inelastic behavior of soils and the complex interaction effects between the structures and the foundations. As a result, numerical methods are often used for the analyses of this class of problems. Among various numerical solution schemes, the finite element technique is widely used because the material nonlinearities and the interaction effects can be easily included in such procedure. To obtain reasonable solutions from any numerical scheme, appropriate constitutive models are necessary along with the proper nonlinear solution techniques.
Constitutive laws based on the theory of linear and nonlinear elasticity cannot describe the behavior of geological materials with reasonable accuracy. This is due to the fact that the behavior of geological materials is highly nonlinear and depends upon the history of deformations. The theory of plasticity describes rate independent nonlinear and inelastic responses of materials. Thus, constitutive laws based on the theory of plasticity can be effectively used to characterize the behavior of geological materials.

The accuracy of the solutions of boundary value problems using any constitutive model is highly dependent upon the values of the material constants associated with the model. As a result, appropriate laboratory tests are necessary to determine these material constants.

The objectives of the present work are summarized below:

i) To develop a generalized constitutive model based on the theory of plasticity which eliminates some of the drawbacks of the available plasticity models. The model describes the rate independent inelastic behavior of materials which are isotropic initially as well as during plastic deformation. The model is developed using a general procedure suggested by Desai (1980).

ii) To determine the material constants associated with the model using laboratory test data for a number of different soils. The testing device used is a stress-controlled cubical triaxial device (Desai et al. 1980; Sture and Desai,
1979). A number of conventional tests are performed on an artificial soil (Desai et al. 1982).

iii) To implement the proposed constitutive model into a three-dimensional (3-D) finite element procedure for solving soil-structure interaction problems.

iv) To verify the proposed constitutive model with respect to the laboratory test data. It is mentioned earlier that the material constants associated with the model are determined from the laboratory tests. The proposed model is then used to back-predict the stress-strain response curves from which the material constants are determined.

v) To solve some boundary value problems using the 3-D finite element procedure with the proposed constitutive model and compare the results with the observations.

**Summaries of Various Chapters**

Chapter 2 of this dissertation is devoted to describing some fundamentals of the theory of plasticity and to reviewing some of the existing plasticity-based constitutive models for soils and other engineering materials.

Chapter 3 is devoted to explaining the proposed constitutive model based on the theory of plasticity. First of all, the yield condition is assumed using a generalized technique proposed by Desai (1980). This is followed by the description of the hardening behavior during inelastic
deformation. Although the proposed constitutive model may be useful for a wide range of materials, in this dissertation its application to geological materials is emphasized.

Chapter 4 describes the general procedure to determine the material constants associated with the proposed model. A set of experimental stress-strain response curves for an artificial soil (Desai et al. 1982) are presented along with the details of the testing device and the techniques of sample preparation. Material constants are determined for three different types of soils.

Chapter 5 is devoted to verifying the model with respect to the laboratory test data. To do this, an integration routine is developed using the proposed constitutive model. Details of the numerical techniques are also described in this chapter. Chapter 6 describes the general formulation of the finite element equations and the nonlinear solution techniques.

Chapter 7 is devoted to presenting some applications of the proposed constitutive model. A three-dimensional finite element procedure is used to solve some boundary value problems; namely, the cubical test sample under a variety of loading conditions, a strip footing and a soil-tool system. For each problem, the finite element result is compared with the laboratory observations.

Finally, a summary of the current work is presented in Chapter 8. Recommendations for extensions and modification of the present approach are also given in this chapter.
CHAPTER 2

REVIEW OF SOME EXISTING LITERATURE

Some Fundamentals of Plasticity Theory

Plasticity is that branch of the constitutive theory which describes the time independent nonlinear and inelastic behavior of materials. Formulation of any constitutive law based on the theory of plasticity should, in general, include the following four phases:

i) yield condition

ii) stress-elastic strain relationship

iii) flow rule

iv) hardening rule.

In this section, these phases will be explained in details.

Yield Condition

The yield condition is generally a scalar condition expressed in terms of stress, which when satisfied at a material point, signifies initiation of inelastic response. Consider the uniaxial stress-strain curve as shown in Fig. 2.1. Up to the point A on the stress-strain curve, the behavior is linearly elastic; that is, when unloaded, it retraces the loading curve. When the stress level goes beyond point A, the material experiences irreversible straining. It can be seen from Fig. 2.1 that when the material point is stressed up to point B and unloaded, it follows
Figure 2.1. Typical Uniaxial Stress-Strain Response in Elasto-Plastic Deformation

\[ \sigma_y = \text{Yield Stress} \]
\[ \varepsilon^p = \text{Plastic Strain} \]
\[ \varepsilon^e = \text{Elastic Strain} \]
a path B-C which is obviously different from the loading curve. In this case, the material point does not go back to its initial undeformed state even when the stress is totally removed, and experience a permanent deformation. The point A on the stress-strain curve is called the elastic limit and the corresponding stress level is called the yield stress and is denoted by \( \sigma_y \) (see Fig. 2.1). Thus, for a uniaxial stress-deformation problem, the yield condition can be expressed in the form,

\[
F = \sigma - \sigma_y = 0
\]  

(2.1)

where \( \sigma \) is the state of stress at any stage of loading history. It is evident from Eq. (2.1) that as long as \( F < 0 \), \( \sigma < \sigma_y \), the behavior is elastic. When \( F = 0 \), \( \sigma = \sigma_y \) and the material point starts yielding. Although yield condition for uniaxial case is quite simple, its counterpart for multiaxial stress state is not so simple. One of the main reasons for this is that for the multiaxial case there are, in general, nine components of stress at a material point and while some of these components are increasing, the remainder of them could decrease. The idea of yield condition for uniaxial case is generalized for multiaxial problems by constructing a scalar valued function whose arguments are all the nine components of the stress tensor, \( \sigma_{ij} \). Although the yield condition is explicitly expressed in terms of the state of stress, for materials which show strain hardening (this will be explained later), it may also be a function of plastic work, \( W^p \), history of plastic strain,
\( \varepsilon_{ij}^p \), and other tensor valued as well as scalar valued internal variables (Schapery, 1968). Symbolically, it can be written in the form,

\[
F = F (\sigma_{ij}, \varepsilon_{ij}^p, W^p, a_{ijk}, \ldots, \eta_1, \eta_2, \ldots, \eta_N) = 0
\]  

(2.2)

where \( \sigma_{ij} \) is the stress tensor, \( \varepsilon_{ij}^p \) is the plastic strain tensor, \( a_{ijk} \) is a tensor valued internal state variable and \( \eta_i (i = 1, 2, \ldots, N) \) are scalar valued internal state variables. In Eq. (2.2), the constant \( N \) defines the total number of scalar valued internal state variables. It is important to note here that the physical interpretations of the internal variables, both tensor valued and scalar valued, may be difficult. In the case of solids, these may be related to the growth of microcracks due to the deformation. For example, the density of these microcracks may be treated as a scalar valued internal variable. This is sometimes termed as dislocation density. It is observed that the growth of these microcracks is directional. This progressive growth of the microcracks may be treated as a tensor valued internal variable (Tokuoka, 1983).

The equality given by Eq. (2.2) is the general yield condition for any mechanical system under isothermal condition. For thermomechanical systems, the yield condition is also a function of the temperature.

For a constitutive theory to represent a material adequately, certain physical and mathematical requirements should be satisfied. These are called the axioms of continuum mechanics (Eringen, 1967). Development of a constitutive theory which will satisfy all the axioms of
continuum mechanics is a formidable task. However, certain axioms are basic and should be satisfied by all constitutive models. Details of these axioms are beyond the scope of this dissertation and can be found in the works of Truesdell (1965); Truesdell and Noll (1965); Eringen (1962, 1967). In this dissertation, only the axiom of objectivity (also known as the axiom of spatial isotropy) is used which states that the constitutive equations should be form invariant under all rigid motions of the spatial frame of reference (Eringen, 1967; 1971). Physically, this means that a mechanical process is independent of the choice of the spatial reference frame.

For an initially isotropic material, there is no preferred direction. Thus, when the axiom of form invariance is invoked, any constitutive functional should remain invariant under all orthogonal transformations (Eringen, 1971). Hence, the yield function, $F$, should obey the following condition:

\[ F (\sigma_{ij}, \varepsilon_{ij}, W^p, a_{ijk}, \eta_1, \eta_2, \ldots, \eta_N) = F (\sigma_{ij}', \varepsilon_{ij}', W^p, a_{ijk}', \eta_1, \eta_2, \ldots, \eta_N) \]

(2.3)

where the prime denotes the tensor quantities in the rotated reference system. Equation (2.3) may be too general to formulate any practically useful constitutive law. For simplicity, it is assumed that the yield function, $F$, does not depend on any tensor valued internal variables. Thus, Eq. (2.3) can be rewritten as,
In Eq. (2.4), $F$ is a scalar valued function of two tensor valued arguments. According to the theory of scalar valued functions of tensor arguments, Eq. (2.4) will be satisfied when $F$ is expressed in terms of the direct as well as the joint invariants of stress tensor, $\sigma_{ij}$, and the plastic strain tensor, $\varepsilon_{ij}^p$ (Rivlin and Ericksen, 1955; Eringen, 1962; Leigh, 1968; Spencer, 1971; Spencer and Rivlin, 1959; Baker and Desai, 1981; Desai and Siriwardane, 1983); that is,

$$F \equiv F \left( J_i, I_i^p, K_j, W^p, \eta_1, \eta_2, \ldots \eta_N \right)$$  \hspace{1cm} (2.5)$$

where

$J_i \ (i = 1, 2, 3) = $ Direct invariants of the stress tensor, $\sigma_{ij}$,

$I_i^p \ (i = 1, 2, 3) = $ Direct invariants of the plastic strain tensor, $\varepsilon_{ij}^p$,

$K_j \ (j = 1, 2, 3, 4) = $ Joint invariants of the stress and the plastic strain tensor.

For any two second order tensors, total number of independent invariants are ten, six direct invariants and four joint invariants. This can be
proved by invoking the Cayley-Hamilton theorem (Spencer, 1971; Eringen, 1967 and 1969; Leigh, 1968). Using index notation, $J_1$, $I_1^p$ and $K_j$ can be explicitly expressed as,

$$J_1 = \sigma_{ii}$$

$$J_2 = \frac{1}{2} \sigma_{ij} \sigma_{ji}$$

$$J_3 = \frac{1}{3} \sigma_{ij} \sigma_{jk} \sigma_{ki}$$

$$I_1^p = \epsilon_{ii}^p$$

$$I_2^p = \frac{1}{2} \epsilon_{ij}^p \epsilon_{ji}^p$$

$$I_3^p = \frac{1}{3} \epsilon_{ij}^p \epsilon_{jk}^p \epsilon_{ki}^p$$

$$K_1 = \sigma_{ij} \epsilon_{ji}^p$$

$$K_2 = \sigma_{ij} \sigma_{jk} \epsilon_{ki}^p$$

$$K_3 = \sigma_{ij} \epsilon_{jk}^p \epsilon_{ki}^p$$

$$K_4 = \sigma_{ij} \sigma_{jk} \epsilon_{ki}^p \epsilon_{li}^p$$

An initially isotropic material, when deformed plastically, develops anisotropy which is called the induced anisotropy. This is because plastic flow is directional and generates a new material at every stage of loading history. To include such induced anisotropy in the
formulation, the yield function should be made a function of the joint invariants, \( K_j \). This is necessary because the joint invariants reflect the mutual orientation of the stress and the plastic strain. For example, in a nine-dimensional space, \( K_1 \) is the scalar product of two vectors, \( \sigma_{ij} \) and \( \varepsilon_{ij}^p \), and thus represents the angle between these two vectors. The other joint invariants may also be explained. It should be mentioned here that when the joint invariants are not included, the principal directions of the stress and the plastic strain tensor will be coincident and represents a material which is isotropic initially as well as during plastic deformation.

**Stress-Elastic Strain Relationship**

According to the definition of plasticity, the plastic deformation occurs while the stress level remains unchanged. Thus, at any point, the stress is related to the elastic part of the total strain only. Since the stress-strain relations are nonlinear and inelastic, piecewise linearity is assumed and the incremental stress is related to the incremental elastic strain through the generalized Hooke's Law:

\[
d\sigma_{ij} = C_{ijkl} d\varepsilon_{kl}^e
\]

(2.7)

where \( d\sigma_{ij} \) = incremental stress,

\( d\varepsilon_{ij}^e \) = elastic part of the total incremental strain, \( d\varepsilon_{ij} \), and

\( C_{ijkl} \) = elastic constitutive relation tensor.
When small deformation concept is used, the elastic recovery may be assumed to be independent of any inelastic deformation. For such case, the total incremental strain can be linearly decomposed into elastic part and plastic part (Hill, 1950). That is,

\[ d\varepsilon_{ij} = d\varepsilon_{ij}^e + d\varepsilon_{ij}^p \]  \hspace{1cm} (2.8)

where superscripts 'e' and 'p' refer to elastic and plastic strain, respectively. Using Eq. (2.8) in Eq. (2.7), the incremental stress, \( d\sigma_{ij} \), can be written in the following form:

\[ d\sigma_{ij} = C_{ijkl} (d\varepsilon_{kl} - d\varepsilon_{kl}^p) \]  \hspace{1cm} (2.9)

Equation (2.9) will be used later on to derive explicit forms of constitutive equations.

Flow Rule

Flow rule defines the incremental plastic strain, \( d\varepsilon_{ij}^p \). The word 'flow' is justified because for an elastic-perfectly plastic material, when the stress level exceeds the elastic limit by an infinitesimal amount, the material is assumed to flow like a fluid maintaining a constant state of stress.

In the theory of plasticity, incremental plastic strain is defined by assuming the existence of a function, \( Q \), called the plastic potential. In general, the plastic potential, \( Q \), is a function of the state of stress, \( \sigma_{ij} \), and the incremental plastic strain is proportional
to the gradient of Q. Introducing a constant of proportionality, \( \lambda \),
\[ \text{de}_{ij}^p = \lambda \frac{\partial Q}{\partial \sigma_{ij}} \] (2.10)

where \( \lambda > 0 \).

It may be noted that the plastic potential function, Q, is similar to the fluid potential, h. According to the Darcy's law, the velocity of fluid flow through a porous medium is proportional to the gradient of the fluid head (potential); that is,
\[ v = k \frac{\partial h}{\partial \ell} \] (2.11)

where \( v \) is the velocity of fluid and \( k \) is the coefficient of permeability of the porous medium. It is evident that Eq. (2.10) is analogous to Eq. (2.11).

**Plastic Stability.** For a uniaxial case, a material is said to be stable in the plastic range when the stress is a monotonically increasing function of the plastic strain (Fung, 1965). Figure 2.1 shows a uniaxial stress-strain curve. This can be replotted as stress-plastic strain curve when the elastic strains are subtracted from the total strain. Figures 2.2 and 2.3 show two typical plots of stress and plastic strain. A material point is stable in the plastic range when the plastic work done by an increment of stress, \( d\sigma \), is positive. Referring to Fig. 2.2, the area ABC should be positive. Mathematically, this can be expressed as
The case of unstable plastic state is shown in Fig. 2.3 where \( d\sigma \cdot de^P < 0 \). The case of \( d\sigma \cdot de^P = 0 \) represents a perfectly plastic state and is the limiting case of stable material.

This idea of plastic stability was generalized by Drucker (1951, 1956, 1959) for multiaxial state of stress through his famous stability postulates. Details of Drucker's definition of stability and its consequences will be explained in this section.

**Drucker's Stability Postulates.** Consider a body in equilibrium with a homogeneous state of stress, \( \sigma_{ij} \). Now let an external agency slowly apply an incremental stress, \( d\sigma_{ij} \), and then slowly remove them. The original configuration of the body may or may not be restored after the cycle of application and removal of stresses (Malvern, 1969). During loading, the incremental stress, \( d\sigma_{ij} \), causes an incremental strain, \( de_{ij} \). If the material is assumed to be deformed inelastically under that stress, upon unloading, only incremental elastic strain, \( de^E_{ij} \), will be recovered. Now Drucker's stability postulates can be phrased as follows:

1. Work done by the external agency during application of the stress increment, \( d\sigma_{ij} \), is positive; i.e.,

\[
d\sigma_{ij} \cdot de_{ij} > 0
\]  

(2.13)

2. Net work performed by the external agency over the cycle of application and removal of stress is nonnegative. It may be noted that
Figure 2.2. Uniaxial Stress-Plastic Strain Curve for Stable Plastic Material

Figure 2.3. Uniaxial Stress-Plastic Strain curve for Unstable Plastic Material
when the stress is brought back to the original state, the elastic
strain is recovered fully if small deformation is assumed. Thus, the
unrecovered strain \( (\varepsilon_{ij} - \varepsilon_{ij}^e) \) may be called the incremental plastic
strain, \( \varepsilon_{ij}^p \). Thus, the second postulate can be represented by the
following inequality:

\[
d\sigma_{ij} \cdot \varepsilon_{ij}^p > 0
\]  

(2.14)

In a nine-dimensional Euclidean space, the product \( d\sigma_{ij} \cdot \varepsilon_{ij}^p \) is a
scalar product of the stress and the plastic strain vectors and is a
measure of the plastic work done in a multiaxial state of stress. Thus,
the product \( d\sigma_{ij} \cdot \varepsilon_{ij}^p \) is analogous to \( d\sigma \cdot d\varepsilon^p \) for the uniaxial case.

Consequences of Drucker's Postulates. There are two major con-
sequences of the stability postulates defined above. They are

1. The yield surface, when plotted in the stress hyper space,
   must be a convex hyper surface. The word hyper is used above to indicate
   that the stress space is composed of more than three independent coordi-
nates. For example, when the stress tensor, \( \sigma_{ij} \), is symmetric, that is,

\[
\sigma_{ij} = \sigma_{ji}
\]

(2.15)

the stress hyper space is a six-dimensional Euclidean space.

2. The incremental plastic strain, \( \varepsilon_{ij}^p \), should be normal to the
   yield surface at the loading point.
The proof of these consequences is straightforward. Consider an incremental stress, $d\sigma_{ij}$, and the corresponding incremental plastic strain, $d\varepsilon_{ij}^P$, at a material point. In a nine-dimensional space, $d\sigma_{ij}$ and $d\varepsilon_{ij}^P$ can be viewed as vectors as shown in Fig. 2.4. Denoting the included angle to be $\psi$, the scalar product of $d\sigma_{ij}$ and $d\varepsilon_{ij}^P$ can be written as

$$d\sigma_{ij} \cdot d\varepsilon_{ij}^P = \|d\sigma_{ij}\| \cdot \|d\varepsilon_{ij}^P\| \cos \psi$$  \hspace{1cm} (2.16)

where $\| \cdot \|$ denotes the norm. It is evident from Eq. (2.16) that the sign of $(d\sigma_{ij} d\varepsilon_{ij}^P)$ is controlled by the sign of $\cos \psi$. Thus, to satisfy inequality in Eq. (2.15), $\cos \psi$ has to be nonnegative; i.e.,

$$\cos \psi \geq 0$$ \hspace{1cm} (2.17)

This implies that the included angle, $\psi$, should be an acute angle; i.e.,

$$-\frac{\pi}{2} \leq \psi \leq \frac{\pi}{2}$$ \hspace{1cm} (2.18)

Let $P$ be a point on the yield surface as shown in Fig. 2.4. The incremental stress, $d\sigma_{ij}$, and the incremental plastic strain, $d\varepsilon_{ij}^P$, are shown as vectors at the point $P$. Now consider a hyper plane $AB$ normal to the vector, $d\varepsilon_{ij}^P$, at the point $P$. It is evident from Fig. 2.4 that as long as the stress vectors, $d\sigma_{ij}$, lie below $AB$, the inequality (2.17) will always be satisfied. It may be mentioned here that the incremental stress vector may be anywhere within the yield surface. Thus, to satisfy
Figure 2.4. Schematic of the Incremental Stress and the Incremental Plastic Strain Vectors in the Stress Hyper Space
the requirement that all the stress vectors (in the stress hyper space),
\(d\sigma_{ij}\), be bounded by the hyper plane \(AB\), the yield surface has to be a
convex surface and be tangential at the point \(P\) to \(AB\), which proves that
\(d\varepsilon_{ij}^P\) is normal to the yield surface at the loading point. This is called
the normality rule of plasticity. Thus, if \(F\) is a yield surface, the incremental plastic strain,
\(d\varepsilon_{ij}^P\) is proportional to the gradient of \(F\); that is,
\[
d\varepsilon_{ij}^P = \lambda \cdot \frac{\partial F}{\partial \sigma_{ij}}
\]  
(2.19)
Comparing Eq. (2.19) with Eq. (2.10), it is found that \(Q = F\).

It is important to note here that, in general, a real material
may not be stable in the sense of Drucker's definition. Thus, the plastic potential, \(Q\), may not be identical to the yield function, \(F\). This
results in two kinds of plasticity formulations: (a) associative plasticity where it is assumed that \(Q = F\), and (b) nonassociative plasticity
where \(Q \neq F\) (Mroz, 1963; Lade, 1975; Prevost, 1977; Ohashi et al. 1975;
Shiratori et al. 1979, Desai and Siriwandane, 1980; Baker and Desai,
1982). In this dissertation, only associative plasticity concept is
utilized.

Hardening Rule

As stated earlier, a material point is plastically stable if the
stress at that point is a monotonic function of the strain (Fung, 1965).
This is shown schematically in Fig. 1.1. Unlike a perfectly plastic
material, a real material shows some increase in stress even after the elastic limit. This is called the work or strain hardening. Such behavior may be attributed to the microstructural rearrangements as well as locking of crystals during plastic deformations (Hill, 1950). Hardening rules are intended to define the process of strength gain in a material due to permanent straining in comparison to perfectly plastic response. In the theory of plasticity, there are three kinds of hardening rules:

1. Isotropic hardening
2. Kinematic hardening
3. Anisotropic hardening.

In this section, these hardening rules will be explained in detail.

**Isotropic Hardening.** Isotropic hardening concept dates back to the development of work-hardening plasticity theory (Hill, 1950). In this rule, it is assumed that the yield surface expands uniformly in the stress space. There are no rigid rotations or translations of the yield surface. Referring to the uniaxial stress-strain response, as shown in Fig. 2.5, this rule can be explained as follows:

The point A on the stress-strain curve represents the elastic limit. Thus, the initial yielding occurs when the stress state reaches A and the corresponding yield surface is denoted by $F_0$. From A to B, the material point hardens as plastic deformation continues. The yield surface $F_0$ expands to $F_1$ at point B. When unloaded from point B, and continued in the reverse direction, the material point responds elastically until it reaches the point C, such that $|\sigma_B| = |\sigma_C|$. When the
Figure 2.5. Schematic of the Isotropic Hardening Rule for Uniaxial Stress-Strain Response

\[ \sigma_A = \sigma_y \text{ (Yield Stress)} \]

\[ \sigma_B = \sigma_C \]
reverse loading is continued from C up to a point D, again plastic deformation occurs and the yield surface $F_1$ changes to $F_2$ at point D. Thus, as plastic deformation continues, the elastic limit increases, and the yield surface expands isotropically without changing its origin. A real material, however, starts yielding in the reverse loading far before the stress level reaches the point C. This phenomenon is known as the Bauchinger effect (Hill, 1950; Fung, 1965; Malvern, 1969) and is caused by the anisotropy induced by the plastic deformation. The major drawback of isotropic hardening rule is that it cannot characterize the Bauchinger effect when load reversals occur. However, for monotonic loading process, this hardening rule may be adequate.

**Kinematic Hardening.** This hardening rule was introduced by Ishilinsky (1959) and Prager (1955, 1956) and is known as Prager's kinematic hardening rule. It is assumed that the yield surface does not change its size but translates in the stress space as a rigid body during the inelastic straining process. Figure 2.6 shows uniaxial representation of the kinematic hardening rule. Initial yielding starts when the stress reaches point A and the corresponding yield surface position is denoted by $F_0$. During loading from A to B, the yield surface translates as a rigid body and becomes $F_1$ at point B. When the system is unloaded from point B and loaded in the reverse direction, yielding starts at point C as shown in Fig. 2.6. In this case $|\sigma_C| < |\sigma_B|$. During further reverse loading from point C to D, the material deforms plastically and the yield surface $F_1$ moves downward and becomes $F_2$ at point D. Although
Figure 2.6. Schematic of the Kinematic Hardening Rule for Uniaxial Stress-Strain Response
Kinematic hardening rules predict Bauchinger effect, the yielding in reverse loading starts too soon which may not be the case with real materials.

**Anisotropic Hardening.** This rule is the combination of isotropic hardening rule and the kinematic hardening rule. It is assumed that the yield surface expands as well as translates in the stress space. However, the shape of the yield surface remains the same. Figure 2.7 shows the uniaxial representation of the anisotropic hardening rule. As usual, the material yields at point A and the corresponding yield surface is $F_0$. During loading from A to B, the yield surface expands as well as translates upward and becomes $F_1$ at point B. During reverse loading, the material yields at C. However, from B to C, material behavior is purely elastic and thus the yield surface remains unchanged up to point C. When the reverse loading is continued, the yield surface expands further and, at the same time, translates downward. At point D, the yield surface becomes $F_2$. Since plastic deformation is irreversible, the expansion of the yield surface is monotonic; i.e., the sizes of the yield surfaces monotonically increase. Referring to Fig. 2.7, it may be represented by the following condition:

$$R_0 < R_1 < R_2 \ldots < R_N$$  \hspace{1cm} (2.20)

where $R_0$, $R_1$, ..., $R_N$ denote the sizes of the yield surfaces $F_0$, $F_1$ ..., $F_N$, respectively. The shape of the yield surfaces, however, remains the same.
Figure 2.7. Schematic of the Anisotropic Hardening Rule for Uniaxial Stress-Strain Response
In general, anisotropic hardening rule describes Bauchinger effect better than kinematic hardening rule. However, the anisotropic hardening rule described above is the simplest possible anisotropic rule. Improved anisotropic models are proposed by many researchers such as Mroz (1963, 1967, 1972, 1978, 1979); Baltov and Sawczuk (1965); Iwan (1967); Prevost (1977, 1979); Dafalias and Popov (1978); Baker and Desai (1982); and many others.

**General Plasticity Models**

Although the theory of plasticity was originally evolved to describe behavior of metals in the inelastic range, its application to geological materials is also extensive. In the first part of this section, a review of general plasticity models will be described. This will be followed by the plasticity models developed to characterize nonlinear and inelastic behavior of geological materials.

**General Developments**

Tresca (1868) proposed that the metals yield when the maximum shear stress reaches the value of shear stress for yielding in uniaxial loading. One of the major drawbacks of Tresca yield criterion is that it does not include the effect of the intermediate principal stress. Moreover, when geometrically represented in two or three-dimensions, Tresca (1868) yield criterion plots as a discontinuous surface which poses difficulty when associated plasticity theory is adopted. Because at the point of discontinuity the gradient vector is nonunique, the direction of the incremental plastic strain vector cannot be determined uniquely.
Von Mises criteria (1913) may be considered as a generalization of the Tresca criteria which states that a material point under multiaxial state of stress yields when the second invariant of the deviatoric stress tensor (defined below), $J_{2D}$, becomes equal to a constant, $k^2$; that is,

$$J_{2D} = k^2$$ (2.21)

where $k$ can be obtained from uniaxial tension or compression test or from pure shear tests. If a pure shear test is performed, the constant $k$ is equal to the shear stress at yield. For uniaxial tension or compression test, $k$ is equal to $\sigma_y/\sqrt{3}$, where $\sigma_y$ is the yield stress in uniaxial tension or compression. Figure 2.8 and 2.9 show geometric representation of Tresca and von Mises yield conditions in two and three dimensions, respectively. The term $J_{2D}$ in Eq. (2.21) is defined as

$$J_{2D} = \frac{1}{2} S_{ij} S_{ij}$$ (2.22)

where $S_{ij}$ is the deviatoric stress tensor. Denoting the total stress tensor by $\sigma_{ij}$, $S_{ij}$ can be expressed as

$$S_{ij} = \sigma_{ij} - \frac{1}{3} \sigma_{kk} \delta_{ij}$$ (2.23)

where $\delta_{ij}$ = Kronecker's delta.
Figure 2.8. Schematic of Tresca and von Mises Yield Conditions in Two-Dimensional Stress Space

Figure 2.9. Schematic of Tresca and von Mises Yield Conditions in the Principal Stress Space
\[ \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad (2.24) \]

As mentioned earlier, the yield condition for an initially isotropic material, in general, can be expressed in terms of three invariants of the stress tensor. von Mises's yield condition, however, utilizes only the second invariant of the deviatoric stress tensor. A more general form may be obtained when the yield condition is expressed in terms of the second and the third invariants of the stress tensor (Osgood, 1947; Fung, 1965). It may be mentioned that metals are almost frictionless materials; that is, shear strengths of metals are not dependent on the applied mean pressure. Thus, the yield conditions for metals should be independent of the first invariant of the stress tensor, \( J_1 \), where \( J_1 \) is defined as

\[ J_1 = \sigma_{ii} \quad (2.25) \]

The yielding of frictional materials, on the other hand, is dependent largely on the mean (confining) pressure, \( J_1/3 \).

Although a continuous yield surface is necessary when normality rule is adopted, it is observed experimentally that at an active loading point the yield surface can be discontinuous. In the past, attempts were made to include such discontinuity in the formulation (Koiter, 1953; Budianski, 1959; Kliushnikov, 1959).

It is implicitly assumed in the yield conditions of Tresca and von Mises that the yield strength of materials is the same in both
tension and compression. In fact, any yield condition which is a quadratic in terms of stresses always implies this condition unless some additional constraints are imposed. This behavior may be valid for metals. Geological materials, on the other hand, may show entirely different responses in tension and compression. For example, the strength of dry cohesionless soil under uniaxial tension may be almost zero while in compression it is significant. The yield conditions of Tresca and von Mises, when applied to strain-hardening materials, always imply isotropic hardening rule. Thus, Bauchinger effect, which is commonly observed in metals, cannot be represented by these models. As mentioned earlier, Ishilinsky (1954) and Prager (1956) introduced the idea of kinematic hardening where the yield surface translates in the stress space as a rigid body. Mathematically, such yield condition is expressed as

\[ F(\sigma_{ij} - \alpha_{ij}) = k^2 \]  

where \( \alpha_{ij} \) is the position vector of the center of the yield surface in a nine-dimensional space formed by the components of the stress tensor, \( \sigma_{ij} \), and \( k \) is a material constant which defines the size of the yield surface. To complete the formulation, a translational rule for the yield surface is to be specified. Prager (1956) proposed that the yield surface translates in the direction of incremental plastic strain, \( \varepsilon_{ij}^P \). Introducing a constant of proportionality, \( c \), Prager expressed \( d\alpha_{ij} \) as
Figure 2.10 shows a schematic of the Prager's kinematic hardening rule. The translational rule given by Eq. 2.27 is not invariant with respect to reductions in dimensions of the stress space (Perrone and Hodge, 1957; Shield and Ziegler, 1958). In other words, if the yield surface in nine-dimensional stress space translates normal to the yield surface (Eq. 2.27 implies this when associated plasticity concept is used), the two-dimensional yield surface in biaxial state of stress does not do so (Ziegler, 1959). Ziegler (1959) modified Prager's translational rule by assuming that the instantaneous translation of the yield surface occurs in the direction of the radius vector connecting the center of the yield surface with the stress point (Fig. 2.11). Mathematically, this can be written as

\[ \delta \alpha_{ij} = d\mu (\sigma_{ij} - \alpha_{ij}) \]  

where \( d\mu \) is a scalar constant of proportionality and governs the magnitude of instantaneous translation. To ensure that the movement of the yield surface occurs in the direction of the radius vector \( (\sigma_{ij} - \alpha_{ij}) \), \( d\mu \) should be always positive. The scalar constant, \( d\mu \), in Eq. 2.28 is determined from the condition that a stress point in loading always remains on the yield surface (Ziegler, 1959). In other words, loading from one plastic state must lead to another plastic state (Fung, 1965). The
Figure 2.10. Schematic of Prager's Kinematic Hardening Rule
Figure 2.11. Ziegler's Modified Translational Rule
kinematic hardening concept of Prager is extended by Hodge (1957) to include simultaneous expansion and translation of the yield surface.

In all the models described so far, it is assumed that the initial shape of the yield surface remains invariant during the entire hardening process. However, it is observed experimentally that the yield surfaces not only expand and translate in the stress space but also change their shapes continuously (Philips and Sierakowski, 1965; Philips and Eisenberg, 1966; Justusson and Philips, 1966; Philips, 1973; Philips and Weng, 1975; Baltov and Sawczuk, 1965; Ohashi, Kawashima and Yokochi 1975; Shiratori, Ikeyami and Yoshida, 1979). Many researchers proposed different models to include such shape changes. Baltov and Sawczuk (1965) introduced the idea of an anisotropic tensor whose coefficients are functions of the history of plastic deformation. Using a von Mises-type yield condition with kinematic hardening, Baltov and Sawczuk (1965), proposed that,

\[ F = N_{ijkl} (s_{ij} - \alpha_{ij}) (s_{kl} - \alpha_{kl}) - 1 = 0 \]  

(2.29)

where \( N_{ijkl} \) is the anisotropic tensor. Although Eq. 2.29 allows for the change of shape of the yield surface, it is not quite general in form. For example, if a material is initially isotropic, Eq. (2.29) represents an initial yield surface which is a hyper sphere. Subsequently, when plastic strains are induced, the hyper sphere changes to a hyper ellipsoid. The idea of Baltov and Sawczuk (1965) is extended later by Backhaus (1968) and Williams and Svensson (1970). Experimental results
obtained by Williams and Svensson (1970) show that the shapes of the subsequent yield surfaces are dependent upon the values of the plastic strains. Thus, both the size and the shape of the yield surface should be made functions of the deformation history. Ohashi et al. (1975) and Shiratori et al. (1979) presented a formulation utilizing the idea of Ilyushin (1954) who proposed a method of formulating the plastic behavior of materials by means of a functional relation in tensor space. Details of this approach are beyond the scope of the present research.

The kinematic hardening rule introduced by Ishilinsky (1954) and Prager (1956) is called a linear strain hardening law because the size of the yield surface remains invariant. Mroz (1967) and Iwan (1967) generalized this for nonlinear strain-hardening by introducing the concept of a field of work-hardening moduli associated with a family of nested yield surfaces. Figure 2.13 shows a schematic of the nested yield surfaces to describe the uniaxial cyclic stress-strain response.

Conceptually, since a material hardens continuously, an infinite number of nested yield surfaces is necessary to describe the deformation process. For practical purposes, a finite number of yield surfaces is used, assuming the stress-strain response to be piecewise linear. The idea of Mroz (1967) and Iwan (1967) has significantly influenced the developments in modern plasticity. Many researchers adopted this concept and presented formal plasticity models for metals and geologic materials (Prevost, 1977, 1978, 1979; Dafalias and Popov, 1975, 1977; Krieg, 1975). Prevost (1977) formalized the nested yield surface
Figure 2.12. Nested Yield Surfaces

Figure 2.13. Piecewise Linear Cyclic Uniaxial Stress-Strain Response Curve
concept for clay under undrained conditions. Subsequently, Prevost (1978, 1979) also extended and modified it to describe the behavior of soils under drained conditions. Dafalias and Popov (1975, 1977, 1979) introduced the idea of bounding surface plasticity model which can be considered a modification of Mroz's anisotropic hardening concept. However, only two surfaces are considered in this formulation and the field of work-hardening moduli is represented by a single function which depends on the history of deformation. Thus, the work-hardening modulus changes continuously in the bounding surface models. A similar idea is also proposed by Krieg (1975).

It is mentioned earlier that the yield conditions for initially isotropic materials should be expressed in terms of six direct invariants and four joint invariants of the stress and the plastic strain tensors. These invariants are defined in Eq. (2.6). Shrivastava et al. (1972) and Baker and Desai (1981) presented formulations of anisotropic hardening theories using the joint invariants, $K_j$ ($j = 1, \ldots, 4$) as defined in Eq. (2.6). Baker and Desai (1981) proved that the maximum induced anisotropy possible in an initially isotropic material is orthotropic anisotropy and such anisotropy can be included in the formulation if the yield function is expressed in terms of the joint invariants as well as the direct invariants.

Selection of an appropriate form for the yield function may be difficult for many engineering materials. Desai (1980) suggested a general procedure to choose the yield functions and the plastic potential
functions from a general polynomial in terms of the stress invariants, $J_i$ ($i = 1, 2, 3$), as defined by Eq. (2.6).

Associated plasticity laws can adequately describe the behavior of materials which can be assumed to be almost frictionless. Metals are good examples of such materials. It is observed experimentally that for frictional materials such as dry sand, the incremental plastic strain vector is not normal to the yield surface (Lade, 1975). Thus, non-associated plasticity laws should be used for this class of materials. In the past, many researchers have developed constitutive models using non-associated plasticity theory (Lade, 1979; Prevost, 1978; Desai and Siriwardane, 1980; Baker and Desai, 1982; Desai and Faruque, 1983). Details of these developments are given by Desai and Siriwardane (1983).

It is important to note that the yield surface may also be viewed as a closed surface in the strain space. Such idea was originally proposed by Green and Naghdi (1965). Subsequently, Naghdi and Trapp (1975) and others have developed formal plasticity models using the strain space instead of stress space. Recently Yoder and Iwan (1981) developed a model using the concept of nested yield surfaces in a strain space. In this dissertation, strain-space plasticity formulations are not considered and thus no details of these models are given.

All the models discussed so far are based on the assumption that at a material point a closed yield surface exists, separating the elastic and plastic states. However, the concept of a yield condition and its representation as a surface in the stress space (Prager, 1951) may not exist physically. Thus, it should also be possible to construct
constitutive laws to describe rate independent inelastic behavior of materials without assuming the existence of a yield surface. Valanis (1971) presented such a model where history dependency in the formulation enters through a pseudo time scale known as intrinsic time. By definition, intrinsic time is a combination of physical time and a derived time related to the permanent deformation. For rate independent behavior, time drops out. The intrinsic time is defined in such a way that it is a monotonic function of deformation. Thus, this can uniquely identify each state of the material for loading as well as unloading and reverse loading conditions. The theory proposed by Valanis (1971, 1975) is known as endochronic theory. Although Valanis first used the intrinsic time in a formal constitutive model, it was Schapery (1966, 1969) who originally introduced the idea of intrinsic time in a slightly different form. Endochronic theory is also extended and used for concrete and for geological materials (Bazant and Bhat, 1976; Valanis and Read, 1980; Bazant and Krizek, 1976).

**Plasticity Models for Geological Materials**

Coulomb (1773) proposed that the granular materials fail by developing a plane of failure when the shear stress on that plane becomes equal to a limiting value, $\tau_L$, defined as

$$\tau_L = c + \left(\frac{\sigma_1 - \sigma_3}{2}\right) \tan \phi \quad (2.31)$$

Equation (2.31) is the well-known Mohr-Coulomb criteria. It may be noted that Eq. (2.31), when plotted in the principal stress space, represents
as irregular hexagonal pyramid. Figure 2.14 shows the cross-section of this surface on the π-plane which passes through the origin of the principal stress space. It is evident from Eq. 2.31 that the Mohr-Coulomb criteria does not include the effect of the intermediate principal stress. A generalization of Mohr-Coulomb criteria is proposed by Drucker and Prager (1952) where all the principal stresses are considered. Drucker-Prager criteria can be expressed as

$$ F = \sqrt{J_{2D}} - \alpha J_1 - k = 0 \quad (2.32) $$

where α and k are the material constants. k has a meaning similar to cohesion and α is similar to the term tan φ in the Mohr-Coulomb criterion. Equation (2.32) represents a right circular cone whose axis is the space diagonal as shown in Fig. 2.15. On the π-plane, it plots as a circle and is shown in Fig. 2.14. Equation (2.32) may also be plotted on $\sqrt{J_{2D}} - J_1$ plane where it plots as a straight line with a slope of α and intersection of k with the $\sqrt{J_{2D}}$ axis. This is shown in Fig. 2.16. Drucker-Prager criterion may also be thought of as a generalization of von Mises criteria where the effect of mean pressure is included.

Although Drucker-Prager criterion includes the effect of mean pressure, it has severe limitations. For example, the model does not permit any plastic volume change under pure isotropic compression state of stress. Also, under a combined stress path, it always predicts plastic volumetric expansive strains if normality rule is adopted. Consequently, this model may overpredict the observed dilatant behavior of some granular materials.
Figure 2.14. Mohr-Coulomb and Drucker-Prager Yield Criteria on π-Plane
Figure 2.15. von Mises and Drucker-Prager Yield Criteria in Principal Stress Space
To predict the plastic volume change under pure hydrostatic compression, Drucker, Gibson and Henkel (1957) introduced the idea of a circular yield cap in addition to the failure surface. This is shown in Fig. 2.17. In addition to $\sqrt{J_2 D} - J_1$ space, the yield surface can also be plotted on triaxial plane. Triaxial plane is a plane in the principal stress space which makes equal angles with the $\sigma_2$- and $\sigma_3$- axes and contains $\sigma_1$- axis. $\sigma_1$, $\sigma_2$, $\sigma_3$, described above, are the principal stresses such that

$$\sigma_1 \geq \sigma_2 \geq \sigma_3$$

(2.33)

Sometimes, plotting of the yield surface on the triaxial plane may be more meaningful. Figure 2.18 shows a schematic of the Drucker, Gibson and Henkel's (1957) yield condition on triaxial plane. It is seen that the circular yield cap plots as a dome on this plane. It may be mentioned here that the hardening behavior of materials can be represented by this model by allowing the caps to expand in the stress space. Thus, conceptually, the addition of the cap allows for plastic volume change during pure hydrostatic compression. However, experimentally, it is observed that the linear failure surface and the circular shape of the yielding cap may not compare with the experimental data with reasonable accuracy. Subsequently, several extensions and modifications of the cap model are proposed for geological materials (DiMaggio and Sandler, 1971; Sandler and Rubin, 1979; Roscoe and Burland, 1958; Roscoe, Schofield and Wroth, 1958; Lade, 1977; Desai, Phan and Sture, 1981).
Figure 2.16. Drucker-Prager Yield Criterion in $\sqrt{J_{2D}} - J_1$ Space

$F \equiv \sqrt{J_{2D}} - \alpha J_1 - k = 0$

Figure 2.17. Drucker, Gibson and Henkel's Yield Criteria in $\sqrt{J_{2D}} - J_1$ Space

$F \equiv \sqrt{J_{2D}} - \alpha J_1 - k = 0$

Circular Yield Cap
Figure 2.18. Schematic of Drucker, Gibson and Henkel's Yield Criteria on Triaxial Plane
Roscoe and his co-workers (1958) developed a plasticity model based on the findings of Rendulic (1936) and Hvorslev (1936, 1938), and it is widely known as the critical state model. The underlying foundation of this model is the concept of critical void ratio or critical density. When a soil sample is subjected to a general loading, it goes through phases of progressive yielding. This process of yielding continues until the material reaches a critical void ratio, after which the void ratio remains constant and the sample experiences large shearing strain and finally collapses. The critical state points from a variety of different stress paths when plotted on q-p plane fall on a straight line known as the critical state line. The slope of this line is a material constant. Figure 2.19 shows a typical critical state line on q-p plane where q is the difference between the major and the minor principal stresses, and p is the effective mean (confining) pressure. Although critical state line defines the ultimate state of a material point in combined loading, for representing plastic volume change behavior under pure hydrostatic compression, the critical state model utilizes the idea of the cap, as mentioned earlier.

A number of constitutive models are developed using the concept of critical state of granular materials; for example, granular gravel, cam clay and modified cam clay models (Schofield and Wroth, 1968; Desai and Siriwardane, 1983). In each of these models, the shape of yield cap is different. In the modified Cam Clay model, the assumed cap plots as an ellipse on q-p plane (Fig. 2.19) (Schofield and Wroth, 1968).
Figure 2.19. Critical State Line and the Yielding Caps on $q - p$ Plane
DiMaggio and Sandler (1971) proposed a modified form of cap model for geological materials. As mentioned earlier, the linear failure surface in the $\sqrt{J_{2D}} - J_1$ plane overpredicts the dilatancy for granular materials. DiMaggio and Sandler (1971) proposed an exponentially decaying failure surface as given below:

$$F_f = \sqrt{J_{2D}} - A + C e^{-BJ_1} = 0$$

(2.34)

where $F_f$ is the failure condition and $A$, $B$ and $C$ are the material constants for failure. Figure 2.20 shows the geometric representation of Eq. (2.34). It is evident from Eq. (2.34) that for large positive (compressive) value of $J_1$, the exponential term becomes negligible compared to the constant $A$ and thus the failure surface becomes asymptotic to a limiting surface given by

$$\sqrt{J_{2D}} = A$$

(2.35)

Equation (2.35) is similar to the von Mises yield condition for elastic-perfectly plastic materials. That is why the asymptotic limit of the failure surface is termed as Mises limit. To represent the plastic volumetric changes under hydrostatic compression, DiMaggio and Sandler (1971) proposed an elliptical yielding cap. This is shown in Fig. 2.20. The yielding cap starts from the $J_1$-axis and ends at the failure surface where its slope is zero. The cap intersects the $J_1$-axis perpendicularly. This is necessary to ensure that a state of pure hydrostatic stress will
Figure 2.20. Schematic of the Cap Model Proposed by DiMaggio Sandler on $\sqrt{J_{2D}} - J_1$ Space
cause only volumetric plastic deformation when associated plasticity theory is used. For hardening materials, the cap expands in such a way that the ratio of the major and the minor axes of the ellipse remains invariant. This ratio is known as the shape factor and is a material constant. It is important to note here that in this model the hardening behavior of geological materials is described by the history of volumetric plastic strain only. DiMaggio and Sandler (1971) assumed an exponential relation between the mean pressure, \( p \), and the volumetric plastic strain, \( \varepsilon^P_\text{V} \); i.e.,

\[
\varepsilon^P_\text{V} = W (1 - e^{-3pD})
\]  

(2.36)

where \( W \) and \( D \) are the constants that describe hardening. In Eq. (2.36), it is assumed that the mean pressure, \( p \), is positive when compressive. The constants \( W \) and \( D \) are determined entirely from a hydrostatic compression test.

Lade (1975, 1977) proposed a form of cap model to characterize the inelastic behavior of cohesionless materials based on laboratory test data. Again, this model also consists of a curved failure surface and a circular cap. Figure 2.21 shows the curved failure surfaces as well as the yielding caps on the triaxial plane. The curved failure surface is expressed as a function of \( J_1 \) and \( J_3 \) only. It may be noted that the third stress invariant, \( J_3 \), reflects the combined effect of \( J_1 \) and \( J_2 \). Thus, it may be justified to form a failure surface without using the
Figure 2.21. Location of Yield Cap Relative to Conical Yield Surface Shown in Triaxial Plane
second invariant of the stress tensor, $J_2$. It is assumed that during hardening, both the failure surface and the yielding cap expands. Lade (1975) used the plastic work, $W^P$, as the hardening parameter, where $W^P$ is defined as

$$W^P = \int \sigma_{ij} \, d\epsilon_{ij}^p$$

(2.37)

Figure 2.22 shows a schematic of the failure surface on the octahedral plane which is more like a triangle with rounded corners and curved edges. This is consistent with the observed yield surfaces for brittle materials (Lade, 1983; Schreyer and Babcock, 1983). However, the cross-section of a yield surface on the octahedral plane is always circular if the third invariant, $J_3$, is not used to form the equation of the yield surface. Use of $J_3$ is also found in the works of Matsuoka and Nakai (1974), Davis and Mullenger (1979), Desai (1980), Desai and Faruque (1983).

The models which are described above constitute only a small part of the entire class of plasticity models. Developmental work in this area is still continuing and new models are being proposed for metals, geological materials, concrete and other materials (Ghaboussi and Momen, 1982; Mroz, et al., 1981; Pietruszczak and Mroz, 1979; Smith and Cheatham, 1980; Desai and Siriwardane, 1980; Desai and Faruque, 1983). A detailed account of all these works is beyond the scope of this dissertation. A comprehensive review of all these models is given by Desai and Siriwardane (1983).
Figure 2.22. Sections of the Conical Yield Surface and the Yield Cap on the Octahedral Plane
CHAPTER 3

PROPOSED CONSTITUTIVE LAWS

General

In classical plasticity, the yield condition is represented as a surface in the stress space. Although alternative formulations such as endochronic theory and strain-space plasticity are available, the classical concept is still extensively used. In the classical plasticity, the yield function is explicitly represented in terms of the state of stress while the hardening behavior is characterized by irreversible strain and other internal variables. When initial material isotropy is assumed, the yield function can be expressed in terms of the three invariants of the stress tensor.

The idea of expressing the yield function as a polynomial in terms of certain powers of the stress invariants was proposed by Desai (1980). It was shown that a majority of the available yield criteria, both classical and recent, are derivable from this polynomial concept. Thus, a general polynomial representation of the yield condition is a valid and rational approach. The greatest advantage of this technique is that it gives one the flexibility of forming appropriate yield conditions for different materials. This can be achieved by using various truncated forms of the polynomial or by selecting appropriate terms of the polynomial using an optimization procedure.
The necessity of new developments in the area of constitutive modelling comes from the fact that none of the available models is general enough to provide a basis to characterize the behavior of different classes of materials under a variety of loading conditions. Thus, different constitutive models are required for different materials as well as for different histories of loadings. The technique proposed by Desai (1980) can provide significant flexibility in terms of choosing appropriate yield conditions for plasticity-based constitutive laws. It should be noted here that some of the terms in the polynomial can be discarded at the outset, depending upon particular characteristics of given material(s). For example, the yielding of metals is not affected by the presence of moderately high isotropic compression or tension. Thus, the yield conditions for metals can exclude terms containing the first invariant of the stress tensor. On the other hand, the yielding of geological materials is highly dependent on the mean pressure. Dry cohesionless soil (sand), for example, derives most of its shear strength from the applied mean pressure. The yield criteria for such materials should include terms containing first invariant of the stress tensor. This is necessary to properly simulate the material behavior under a combined isotropic and deviatoric stress condition.

Polynomial Representation of the Yield Condition

As stated earlier, the yield conditions, \( F \), for an initially isotropic material can be expressed in terms of the invariants of the stress tensor. Mathematically,
where $J_1$, $J_2$, $J_3$ are the three stress invariants given by

$$J_1 = \sigma_{ij}\sigma_{ij}$$

$$J_2 = \frac{1}{2} \sigma_{ij}\sigma_{ij}$$

$$J_3 = \frac{1}{3} \sigma_{ij}\sigma_{jk}\sigma_{ki}$$

$\sigma_{ij}$ is the stress tensor, and $F (J_1, J_2, J_3)$ is the yield function. Equation (2.1a), when plotted in the stress space, generates a closed surface. The interior of that surface represents elastic states. Any state of stress which falls on the surface causes incipient plastic flow.

The yield function, $F (J_1, J_2, J_3)$, can be represented as a polynomial in terms of the stress invariants $J_1$, $J_2$ and $J_3$. However, a more appropriate form can be obtained when $F (J_1, J_2, J_3)$ is expressed as a polynomial in terms of $J_1$, $J_2^{1/2}$ and $J_3^{1/3}$. This is originally suggested by Desai (1980). The advantage of using the set $J_1$, $J_2^{1/2}$ and $J_3^{1/3}$ is that one can obtain consistent dimensions of the terms in the yield function. Following Desai (1980), the yield function $F (J_1, J_2^{1/2}, J_3^{1/3})$ can be written as
\[
F (J_1, J_2^{1/2}, J_3^{1/3}) \equiv \alpha_0 + \alpha_1 J_1 + \alpha_2 J_2^{1/2} + \alpha_3 J_3^{1/3} + \\
\alpha_4 J_1^2 + \alpha_5 J_1 J_2^{1/2} + \alpha_6 J_2 + \\
\alpha_7 J_1 J_3^{1/3} + \alpha_8 J_2^{1/2} J_3^{1/3} + \\
\alpha_9 J_3^{2/3} + \alpha_{10} J_1^3 + \alpha_{11} J_1 J_2^{1/2} + \\
\alpha_{12} J_1^2 J_3^{1/3} + \alpha_{13} J_1 J_2 + \\
\alpha_{14} J_1 J_3^{2/3} + \alpha_{15} J_2 J_3^{1/3} + \\
\alpha_{16} J_2^{3/2} + \alpha_{17} J_2^{1/2} J_3^{2/3} + \alpha_{18} J_3 \\
+ \alpha_{19} J_1 J_2^{1/2} J_3^{1/3} + \ldots \ldots .
\]

(3.2)

It may be noted that the yield function can be expanded up to any desired order. The constants, \( \alpha_i \) \((i = 1, 2, \ldots, n)\), in Eq. (3.2) are material response functions. In general, all the material response functions can be expressed in terms of the history of deformation as well as a number of other tensor-valued and scalar-valued internal variables. Thus, \( \alpha_i \) \((i = 1, 2, \ldots, n)\) can be expressed as,

\[
\alpha_i = \alpha_i (\sigma_{mn}^p, W^p, \dot{a}_{kl}, \ldots, n_1, n_2, \ldots, n_N)
\]

(3.3)
where $\int \! d\varepsilon_{ij}^P$ = history of plastic strain, $W^P = \int \sigma_{ij} \, d\varepsilon_{ij}^P$ = plastic work, $a_{ij}$ = tensor-valued internal state variables, and $n_1, n_2, \ldots, n_N$ = scalar-valued internal state variables. It should be emphasized that, for implementation purpose, explicit forms of all chosen response functionals are required. However, selection of appropriate forms can be tedious and, at the same time, difficult. Thus, usual practice is to use only a few material response functions while a majority of $\alpha_i$'s will be assumed to be material constants.

The yield function given by Eq. (3.2) is expanded in terms of the invariants of the total stress tensor. A more convenient form may be obtained if $J_2$ and $J_3$ in Eq. (3.2) are substituted by $J_{2D}$ and $J_{3D}$ where $J_{2D}$ and $J_{3D}$ are the second and third invariants of the deviatoric stress tensor, respectively. Using index notations, $J_{2D}$ and $J_{3D}$ can be expressed in the form,

$$J_{2D} = \frac{1}{2} S_{ij} S_{ji}$$

$$J_{3D} = \frac{1}{3} S_{ij} S_{jk} S_{ki}$$

and $S_{ij} = \sigma_{ij} - \frac{1}{3} \sigma_{kk} \delta_{ij}$ (deviatoric stress tensor).

Desai (1980) chose a number of truncated forms of $F(J_1, J_2, J_3)$ and evaluated them at ultimate (stress) condition for four different materials and for different stress paths. It may be noted that the ultimate condition is defined by the asymptotic value of the stress in the stress-strain response curve, as shown in Fig. 3.1. Of all the forms Desai (1980) used, the form
Figure 3.1. Ultimate Condition Shown for a Typical Stress-Strain Response
\[ \frac{J_1 J_3^{1/3}}{J_2} = C \]  

(3.5)

essentially showed an invariant value of \( C \) at the ultimate conditions for all the materials. It is important to note here that the form given by Eq. (3.5) is not the only form which assumes an invariant value at ultimate condition. It is possible to construct a number of other such forms by choosing appropriate terms from the polynomial given by Eq. (3.2).

Lade (1975, 1977) and Matsuoka and Nakai (1974, 1982) have proposed similar forms guided by the experimental behavior of cohesionless soil. Davis and Mullenger (1979) derived a failure condition for granular materials using hypoelasticity (Truesdell, 1965; Eringen, 1962; Desai and Siriwardane, 1983) and critical state (Roscoe et al., 1958) concept.

However, all these conditions can be derived from the general polynomial (Eq. 3.2) by selecting appropriate terms.

**Proposed Constitutive Model**

A large number of plasticity models for geological materials are available at the present time. Of them, critical state (Roscoe et al., 1958, 1963, 1968; Schofield and Wroth, 1968) and Cap Models (DiMaggio and Sandler, 1971; Sandler et al., 1976; Sandler, 1976; Sandler, 1983) are widely used. The model proposed by Lade (1975, 1977) also allows for work-hardening and nonassociative behavior. One of the major drawbacks of all these models is that the yielding is controlled by two separate yield functions which intersect each other with a slope discontinuity.
This results in nonuniqueness of the normals at the point of intersection. In the associated theory of plasticity, the incremental plastic strain is assumed to be normal to the yield surface at the loading point. Thus, the direction of incremental plastic strain is not defined at the point of intersection of the two yield surfaces (see Fig. 3.2). This problem can be eliminated if a single yield surface (function) is used instead of two or multiple yield surfaces.

On the basis of the form given by Eq. (3.5) (Desai, 1980; Desai and Faruque, 1983), the following form of the yield function is proposed:

\[ F = J_{2D} + \alpha J_1^2 - \beta J_1 J_3^{1/3} - \gamma J_1 - k^2 = 0 \]  

(3.6)

where \( J_{2D} \) = second invariant of the deviatoric stress tensor, \( J_1, J_3 \) = first and third invariants of the total stress, respectively, \( k \) = measure of the cohesive strength of the material, and \( \alpha, \beta, \gamma \) = material response functions.

As discussed earlier, generally, all the response functions should be made functions of the history of deformation and other internal variables. Explicit forms of all the response functions are difficult to choose. Thus, only a few of the response functions will be made history dependent and the remaining will be treated simply as material constants. In this study, only \( \beta \) will be made function of the history of deformation while \( \alpha, \gamma, k \) are assumed to be material constants. It may be noted that \( \alpha, \gamma \) and \( k \) will be evaluated considering the ultimate condition of a material. The response function \( \beta \) is called
Figure 3.2. Typical Example of Two Yield Surface Plasticity Model
the growth or evolution function. In general, the functional form of $\beta$ can be written in the following form:

$$\beta = \beta \left( \int \epsilon_i^p \epsilon_j^p, w^p, a_{ijk}, \ldots, \gamma_1, \gamma_2, \ldots, \gamma_N, T \right)$$  \hspace{1cm} (3.7)

The arguments of $\beta$ are defined in Eq. (3.3). The variable $T$ in Eq. (3.7) is the temperature. Assuming isothermal condition, the dependence of $\beta$ on temperature can be eliminated. In general, a large list of history dependent parameters can be chosen to express the growth function $\beta$. In this study, however, $\beta$ will be made function of a single parameter, $\xi$, defined as,

$$\xi = \int (\epsilon_i^p \epsilon_j^p)^{1/2}$$  \hspace{1cm} (3.8)

It is evident from Eq. (3.8) that $\xi$ is the trajectory of the plastic strain in a nine-dimensional Euclidean space formed by the components of the plastic strain tensor, $\epsilon_i^p$. The parameter $\xi$ will be termed as the growth parameter. Since the scalar product of two vectors is always positive (provided the coefficients are all real), the growth parameter $\xi$ is always positive. Thus, $\xi$ will be unique for a particular plastically deformed state of a material point. The total plastic strain trajectory, $\xi$, may be expressed in terms of spherical and deviatoric components, $\xi_w$ and $\xi_D$, respectively. These are related to each other through the following equation:
\[ \xi = \int \left\{ (d\varepsilon_D)^2 + (d\varepsilon_V)^2 \right\}^{1/2} \]  

(3.9)

where \((d\varepsilon_D)^2 = de_{ij}^p de_{ij}^p\)

\[(d\varepsilon_V)^2 = \frac{1}{3} (d\varepsilon_{kk}^p)^2\]

\[de_{ij}^p = de_{ij}^p - \frac{1}{3} de_{kk}^p \delta_{ij}\] is the incremental deviatoric plastic strain tensor and \(de_{kk}^p\) is the incremental volumetric plastic strain.

Thus, \(\xi_V\) and \(\xi_D\) can be explicitly written as

\[\xi_V = \frac{1}{\sqrt{3}} \int d\varepsilon_{kk}^p\]  

(3.10)

\[\xi_D = \int \left( de_{ij}^p de_{ij}^p \right)^{1/2}\]  

(3.11)

It may be noted that the combinations of \(\xi, \xi_V\) and \(\xi_D\) will form a better basis compared to \(\xi\) alone to describe the growth function \(\beta\) for plastically hardening materials. As an alternative to \(\xi\), the plastic work, \(W^p\), may also be used. This parameter, however, is not investigated in this study.

Properties of the Proposed Yield Function

According to the stability postulates of Drucker (1951), material point is stable in the plastic range when the plastic work done by an increment of stress, \(d\sigma_{ij}\), is nonnegative; i.e.,
The above inequality forces the yield surface to be a convex surface in the stress space. In this study, only stable plastic materials will be considered and, thus, the yield function should plot as a convex surface in the stress space. When the stress tensor is symmetric, there are six independent components of stress, and the yield surface is a hyper surface in a six-dimensional hyper space made up of the six components of stress. Figure 3.3 shows these six components of stress in a three-dimensional Euclidean space. It is difficult to visualize a hyper surface in a six-dimensional space. Thus, the yield surface will be mapped onto different two-dimensional spaces to investigate its convexity. Although theoretically infinite number of such planes may be constructed, only three planes will be considered here; namely, triaxial plane, a plane formed by $\sqrt{J_{2D}}$ and $J_1$ and the octahedral plane. It may be noted that the octahedral plane is normal to the plane formed by $\sqrt{J_{2D}}$ and $J_1$.

Figure 3.4 shows a schematic of the proposed yield function on the $\sqrt{J_{2D}}$ - $J_1$ space. It is seen that the yield surface is convex. Moreover, it intersects the $J_1$-axis at right angles. This is consistent with the definition of associated plasticity because for an initially isotropic material, a state of hydrostatic compression should cause only volumetric plastic strain. The plot is given for an ideal cohesionless material, that is assuming $k = 0.0$. Obviously the convexity of the proposed yield surface is largely dependent upon the values of the material
Figure 3.3. Cartesian Components of the Stress Tensor
\[ F = J_{2D} + \alpha J_1^2 - \beta J_1 J_3^{1/3} - \gamma J_1 - k^2 = 0 \]

Figure 3.4. Plot of the Proposed Yield Function in $\sqrt{J_{2D}} - J_1$ Space for $J_{3D} = 0$
constants \(\alpha, \gamma\) and the growth function \(\beta\). Appropriate ranges can be determined by considering the length of intersection with the \(J_1\)-axis. At the point of intersection with the \(J_1\)-axis, the state of stress is purely spherical. Thus, the deviatoric stress invariants \(J_{2D}\) and \(J_{3D}\) vanish. When specialized for this case, the yield condition takes the form,

\[
\alpha J_1^2 - \frac{\beta J_1^2}{3} - \gamma J_1 - k^2 = 0
\]  

(3.13)

and the length of the intersection is given by

\[
J_1 = \frac{3\gamma}{2(3\alpha-\beta)} + \sqrt{\left[\frac{3\gamma}{2(3\alpha-\beta)}\right]^2 + \frac{3k^2}{(3\alpha-\beta)}}
\]  

(3.14)

For cohesionless materials, \(k = 0\) and Eq. (3.14) reduces to the form:

\[
J_1 = \frac{3\gamma}{(3\alpha-\beta)}
\]  

(3.15)

It is evident from the Eqs. (3.14) and (3.15) that the intercept is positive as long as \((3\alpha-\beta) > 0\) for all \(\gamma > 0\). When \((3\alpha-\beta) = 0\), the intercept becomes infinite. When \((3\alpha-\beta) < 0\) for \(\gamma > 0\), the yield function plots as a concave surface in \(\sqrt{J_{2D}} - J_1\) space. Thus, the ranges of \(\alpha, \gamma, \beta\) for a convex yield surface are the following:

\[
\begin{align*}
\gamma &> 0 \\
\alpha &> 0 \\
\beta &\leq 3\alpha
\end{align*}
\]  

(3.16)
The case of $S = 3\alpha$ represents the ultimate yielding of a material point, and defines the upper limit of the growth function $\beta$. The lower limit of $\beta$ is associated with the elastic limit of a material. For materials whose elastic limit does not exist, that is the elasto-plastic deformation starts from the very beginning, the theoretical lower limit of $\beta$ is $-\infty$. When $\beta = -\infty$, the yield surface becomes a singular surface. Thus, the range of $\beta$ is

$$-\infty \leq \beta \leq 3\alpha \tag{3.17}$$

Plots of the proposed yield function on the octahedral plane and the triaxial plane are shown in Figs. 3.5 and 3.6, respectively. It is seen that it plots as a convex surface in all the spaces and thus satisfies the convexity requirements.

Over the past few years, it has been argued that a yield surface which is convex in different sub spaces may not be convex in a general six-dimensional stress space. However, there are theorems to prove the convexity of a function of several variables mathematically (Yang, 1980). The yield function, $F$, will be a convex function in the six-dimensional stress space if the Hessian matrix of $F$ is positive semi-definite. Mathematically, this can be expressed as

$$\text{Hessian of } F = H_{\sigma}(F) = \frac{\partial^2 F}{\partial \sigma_i \partial \sigma_j} \geq 0 \tag{3.18}$$

where the subscripts $i$ and $j$ range from 1 to 6. It should be noted that in Eq. (3.18), $\sigma_1$, $\sigma_2$, $\sigma_3$ represents the three normal stresses while
Figure 3.5. Typical Sections of the Yield Surface on the Octahedral Plane for Various $J_1$
Figure 3.6. Plot of the Proposed Yield Function in Triaxial Plane (k = 0.0)

\[ F = J_{20} + aJ_1^2 - bJ_1J_3^{1/3} - \gamma J_1 - k^2 = 0 \]
\(\sigma_4, \sigma_5, \sigma_6\) represents three shear stresses. The condition given by Eq. (3.18) is analogous to the second derivative of a single variable function. It is well known that such a function is convex when the second derivative of that function is positive semi-definite.

Evolution Function, \(\beta\)

As mentioned earlier, \(\xi, \xi_V\) and \(\xi_D\) will be used as growth parameters. Various forms of \(\beta (\xi, \xi_V, \xi_D)\) will be assumed and investigated in the light of experimental observations for a number of (cohesionless) soils.

The selection of an appropriate form for \(\beta (\xi, \xi_V, \xi_D)\) cannot be made in an arbitrary manner. First of all, the variation of \(\beta\) with respect to its arguments should be observed from actual test data. Any particular form of \(\beta (\xi, \xi_V, \xi_D)\) should then be selected guided by the observed behavior. Before adopting any particular form of \(\beta (\xi, \xi_V, \xi_D)\), some terminologies should be defined.

**Ultimate Yield.** This is the state of a material point after experiencing infinite amount of shearing strain. For all practical purposes, this state will be considered to occur at a large value of shearing strain. This condition can be stated by the following inequality:

\[
\xi_D \geq M \tag{3.19}
\]

where \(M\) is a large positive number. Inequality (3.19) can be restated in the form,
where \( \bar{M} \) is similar to \( M \). Throughout this dissertation, the second statement of the ultimate yielding will be used.

It should be noted that the terms 'failure' and 'critical state' may or may not coincide with the ultimate yielding condition.

**Hardening.** When a material point is subjected to a load level beyond the elastic limit, inelastic deformations set in. Elastic-perfectly plastic materials essentially flow under a constant stress level at which the inelasticity starts. However, most materials exhibit some increase in strength even after inelastic deformation. This process of strength gain is termed as hardening. In crystalline materials, hardening is caused due to the reorientations and locking of the crystals during permanent straining.

As discussed in Chapter 2, basically there are three kinds of hardening models available in the theory of plasticity; namely, isotropic hardening model, kinematic hardening model and anisotropic hardening model. For quasi-static monotonic loading process, the isotropic hardening model can work well. However, when load reversals occur, kinematic or anisotropic hardening models may be more appropriate. In this study, only isotropic strain hardening concept will be utilized.

As mentioned earlier, the growth function can be plotted from actual test data before adopting any particular form. Figure 3.7 shows the typical variation of \( \beta \) as a function of the growth parameter, \( \xi \). To fit this observed behavior, a hyperbolic relationship between \( \beta \) and \( \xi \) is assumed:

\[
\xi \geq \bar{M}
\] (3.20)
Figure 3.7. Typical Variation of Hardening Function, $\beta$, with Respect to the Hardening Parameter, $\xi$. 

$\beta = 3\alpha$ at Ultimate Yield
\[ \beta = \beta (\xi) = \beta_u \left[ 1 - \frac{\bar{\beta}_a}{i + f(\xi)} \right] \]  

(3.21)

where \( \beta_u \) = value of \( \beta \) at ultimate yielding = 3a, \( \bar{\beta}_a \) = material constant for hardening, \( i \) = material constant which identifies the initial size of the yield surface, and \( f(\xi) \) = function defining hardening.

The following restrictions should have to be imposed on \( f(\xi) \):

\[ f(\xi) = 0, \text{ when } \xi = 0 \]
\[ f(\xi) + \infty, \text{ when } \xi + \infty \]  

(3.22)

\[ f(\xi) \geq 0, \text{ when } \xi \geq 0 \]

Thus, when \( f(\xi) + \infty \), \( \beta(\xi) + \beta_u \), which is the ultimate value of the growth function, \( \beta(\xi) \). An additional restriction is necessary if softening is to be excluded from the formulation. This additional restriction is such that \( f(\xi) \) has to be a monotonically increasing function of \( \xi \). A variety of different forms are possible for \( f(\xi) \). For simplicity, it will be assumed,

\[ f(\xi) = \xi^n \]  

(3.23)

where \( n \) is a material constant defining hardening. Thus, the explicit form of the growth function \( \beta \) can be expressed as,

\[ \beta = \beta (\xi) = \beta_u \left[ 1 - \frac{\bar{\beta}_a}{i + \xi^n} \right] \]  

(3.24)

The materials for which elastic limit do not exist, the parameter \( i = 0 \). Most soils exhibit inelastic response from the very beginning of the loading process. Thus, for soils, it may be justified to assume \( i = 0 \).
Equation (3.24) may not be suited for materials which exhibit very small volume change compared to shearing. As discussed earlier, when $\beta$ is expressed in terms of $\xi$, $\xi_V$ and $\xi_D$, a more general form of the growth function can be obtained. Since a large number of combinations of $\xi$, $\xi_V$ and $\xi_D$ are possible, an experimental study should be done using these parameters before selecting the form of the growth function, $\beta(\xi, \xi_V, \xi_D)$.

Motivation for Selecting $\beta(\xi, \xi_V, \xi_D)$. It is mentioned earlier that the parameters $\xi_V$ and $\xi_D$ are the volumetric and the deviatoric part of the total plastic strain trajectory, $\xi$. The relationship between these three parameters, $\xi$, $\xi_V$ and $\xi_D$, is given by Eq. (3.9). Now consider two ratios, $r_V$ and $r_D$, such that,

$$r_V = \frac{\xi_V}{\xi}$$

$$r_D = \frac{\xi_D}{\xi}$$

(3.25)

The ratio, $r_D$, describes the deviatoric behavior of a material point during inelastic straining. An initially isotropic material, when subjected to purely hydrostatic state of stress, experiences only volumetric deformations. For such condition, $\xi_D = 0$ and consequently $r_D = 0$. On the other hand, for purely deviatoric behavior (in the case of non-frictional materials), the parameter $r_D$ will be always unity. It may be noted here that besides these two limiting aspects, $r_D$ has a special
meaning when the critical state concept is used. According to the critical state theory, a frictional material will experience change in volume for a particular mean pressure till the time it reaches a critical volume. Subsequently, there will be no more volume change and the material will experience large shear strains until it fails. Thus, permanent volume change is limited while the deviatoric plastic deformations accumulate in an unbounded manner; that is, at the critical state $\xi_D >> \xi_V$ and the shear intensity factor, $r_D$ approaches unity. Theoretically, when $\xi_D + \infty$, $r_D$ tends to unity. For all practical purposes, however, critical state of a material point (frictional) will be defined by the inequality,

$$
(1 - r_D) \leq \varepsilon
$$

(3.26)

where $\varepsilon$ is positive and $\varepsilon << 1$.

The ratio, $r_V$, describes the plastic volume change behavior under a general loading history. It is well known that metals are elastic in bulk up to a moderately high hydrostatic state of stress. Thus, for metals, $\xi_V = 0$ and $r_V = 0$ and plastic behavior is purely deviatoric. On the other hand, geological materials such as soils show significant plastic volume change under hydrostatic state of stress. Assuming the materials to be initially isotropic, a hydrostatic state of stress will cause only volumetric plastic strain. Under this condition, $\xi_D$ is always zero and the parameter, $r_V$, is always unity.
When the volumetric and deviatoric plastic deformations are coupled, both \( r_V \) and \( r_D \) become less than unity. However, there is a unique relation between these two parameters. Figures 3.8 through 3.12 show \( r_V - r_D \) plots for five different soils and for a variety of tests. It is evident from these figures that during a combined loading process the parameters \( r_V \) and \( r_D \) are uniquely related. Thus, it is possible to use one parameter out of \( r_V \) and \( r_D \) along with \( \xi \) to characterize the actual hardening process. It may be noted that a combination of \( r_V \) and \( r_D \) may also be used instead of \( r_V \) or \( r_D \). For example, it is possible to define a parameter, \( r_t \), such that

\[
 r_t = \sqrt{r_V^2 + r_D^2} \tag{3.27}
\]

The parameter, \( r_t \), in Eq. (3.26) defines the radius vector as shown in Fig. 3.13. It is evident from Fig. 3.13 that the value of \( r_t \) is less than unity as long as both \( r_V > 0 \) and \( r_D > 0 \). When one of these parameters vanishes, \( r_t \) becomes unity.

Although any one parameter among \( r_V \), \( r_D \) and \( r_t \) may be used, in this study the parameter, \( r_D \), is used along with \( \xi \) to express the growth function, \( \beta \). Referring to Fig. 3.7, the following form of \( \beta (\xi, r_D) \) is assumed:

\[
 \beta = \beta(\xi, r_D) = \beta u \left[ 1 - \frac{\beta a}{\eta i + \xi g(r_D)} \right] \tag{3.28}
\]
Figure 3.8. $r_V - r_D$ Plot Obtained from a Number of Triaxial Tests for the Silty Sand
Figure 3.9. $r_V - r_D$ Plot Obtained from a Number of Triaxial Tests for the Artificial Soil
Figure 3.10. $r_V - r_D$ Plot Obtained from a Number of Triaxial Tests for the Agricultural Soil
Figure 3.11. $r_V - r_D$ Plot Obtained from a Number of Triaxial Tests for the Ottawa Sand
Figure 3.12. $r_V - r_D$ Plot Obtained from a Number of Triaxial Tests for the Munich Sand
Figure 3.13. Typical $r_V - r_D$ Plot for Tests where Deviatoric and Volumetric Deformations are Coupled
where $g(r_D)$ is an unknown function defining the shearing behavior. For purely volumetric behavior, $r_D = 0$ and the unknown function, $g(r_D)$ in Eq. (3.28) should be unity. Thus, $g(r_D)$ may be expressed as

$$g(r_D) = 1 - g_1(r_D)$$

(3.29)

where

$$g_1(r_D) = 0 \text{ if } r_D = 0$$

(3.30)

Assuming $i = 0$, Eq. (3.28) can be written in the form,

$$\beta = \beta_u \left[ 1 - \frac{\beta_a}{\xi \eta_1 \{1 - g_1(r_D)\}} \right]$$

(3.31)

where $\beta_a$ and $\eta_1$ are constants for hardening and are determined from isotropic compression tests. Once $\beta_a$ and $\eta_1$ are determined, the unknown function $g_1(r_D)$, called the shear participation function, can be plotted from actual test data for a given $F$. Typical plot of $g_1(r_D)$ on a function of $r_D$ is shown in Fig. 3.14. Based on this observation, $g_1(r_D)$ is assumed in the form,

$$g_1(r_D) = \beta_b \cdot (r_D)^{\eta_2}$$

(3.32)

In Eq. (3.32), $\beta_b$ and $\eta_2$ are constants defining hardening and are determined entirely from tests where volume and shear behavior is coupled, for example, conventional triaxial compression tests.
Figure 3.14. Typical Plot of $g_1$ as a Function of $r_D$
Elasto-Plastic Constitutive Relations

In this section, the relationship between the incremental stress, \( d\sigma_{ij} \) and the incremental plastic strain, \( d\varepsilon_{ij}^p \), will be derived. Index notations will be used throughout the entire section; however, the final relationship will also be given in a matrix form.

The stress-elastic strain relationship can be written in the form

\[
d\sigma_{ij} = C_{ijkl} d\varepsilon_{kl}^e
\]  

(3.33)

where \( C_{ijkl} \) is the elastic constitutive relation tensor. \( d\varepsilon_{kl}^e \) = elastic part of the total incremental strain, \( d\varepsilon_{kl} \). Assuming small strain, the total incremental strain, \( d\varepsilon_{kl} \), can be linearly decomposed into elastic and plastic part as

\[
d\varepsilon_{kl} = d\varepsilon_{kl}^e + d\varepsilon_{kl}^p
\]  

(3.34)

where the superscripts \( e \) and \( p \) refer to elastic and plastic, respectively. Using Eq. (3.34) into Eq. (3.33), the incremental stress, \( d\sigma_{ij} \), can be written as

\[
d\sigma_{ij} = C_{ijkl} (d\varepsilon_{kl}^e - d\varepsilon_{kl}^p)
\]  

(3.35)

Flow Rule. The flow rule defines the incremental plastic strain. When associated plasticity law is considered, the incremental plastic strain vector is normal to the yield surface, \( F \), at the loading point.
(Desai and Siriwardane, 1983). Thus, $\text{d}e_{ij}^p$ is proportional to the gradient of $F$ with respect to stress, $\sigma_{ij}$; that is,

$$
\text{d}e_{ij}^p = \lambda \frac{\partial F}{\partial \sigma_{ij}} \quad (3.36)
$$

where $\lambda > 0$, and is obtained from the condition that the stress point always remains on a yield surface. This condition is known as the consistency condition (Prager, 1951; Desai and Siriwardane, 1983).

Mathematically, it is given by the equality,

$$
dF = 0 \quad (3.37)
$$

In Fig. 3.15, two yield surfaces, $F_1$ and $F_2$, are shown. $F_2$ is the yield surface which is reached from $F_1$ because of an increment of stress, $d\sigma_{ij}$. Thus, $F_2$ can be written as

$$
F_2 = F_1 + dF \quad (3.38)
$$

Initially, the stress state, $\sigma_{ij}^{(1)}$, which is on $F_1$, satisfies $F_1 = 0$. When the stress changes to $\sigma_{ij}^{(2)} = \sigma_{ij}^{(1)} + d\sigma_{ij}$, the yield surface changes to $F_2$. If $\sigma_{ij}^{(2)}$ is to be on $F_2$, the requirement is that $F_2 = 0$. From Eq. (3.38), it is evident that this will be satisfied when $dF = 0$. In the present study, the yield function is a function of both stress as well as the trajectory of the plastic strain, $\xi$. Thus, the consistency condition can be restated as
Figure 3.15. Typical Example of Yield Surfaces for a Hardening Plastic Material
\[ dF = \frac{\partial F}{\partial \sigma_{ij}} \ d\sigma_{ij} + \frac{\partial F}{\partial \xi} \ d\xi = 0 \]  
(3.39)

Substituting \(d\varepsilon_{ij}^P = \lambda \ \frac{\partial F}{\partial \sigma_{ij}}\), \(d\xi\) can be written as
\[ d\xi = \lambda \left( \frac{\partial F}{\partial \sigma_{ij}} \right) \frac{1}{2} \]  
(3.40)

\(\left( \frac{\partial F}{\partial \sigma_{ij}} \right) \frac{1}{2}\) in Eq. (3.40) is the length of the gradient vector in the stress space and is denoted by \(\gamma_F\). Thus,
\[ d\xi = \lambda \ \gamma_F \]  
(3.41)

Utilizing Eqs. (3.35), (3.36), (3.39) and (3.41), the proportionality constant, \(\lambda\), can be expressed as
\[ \lambda = \frac{\frac{\partial F}{\partial \sigma_{ij}} \ C_{ijkl} \ d\varepsilon_{kl}}{\frac{\partial F}{\partial \sigma_{pq}} \ C^{pqrs} \ \frac{\partial F}{\partial \sigma_{rs}} - \gamma_F \frac{\partial F}{\partial \xi}} \]  
(3.42)

Substituting the expression of \(\lambda\) back into Eq. (3.35) along with Eq. (3.36), the incremental stress-strain relationships for a work-hardening elasto-plastic material is obtained as
\[ d\sigma_{ij} = C_{ijkl}^{e-P} \ d\varepsilon_{kl} \]  
(3.43)

where \(C_{ijkl}^{e-P}\) is the elasto-plastic constitutive relation tensor and is given by
In matrix notation, Eq. (3.44) can be written as

\[
[C]^{e-p} = [C] - \left( \frac{\partial F}{\partial \sigma} [\{ \frac{\partial F}{\partial \sigma} \}^T [C] \frac{\partial F}{\partial \sigma} - \gamma_F \frac{\partial F}{\partial \varepsilon} \right)
\]  

(3.45)

When \( F = F(J_1, J_{2D}, J_{3D}) \), the gradient of \( F \) with respect to \( \sigma_{ij} \) can be expressed as

\[
\frac{\partial F}{\partial \sigma_{ij}} = \frac{\partial F}{\partial J_1} \delta_{ij} + \frac{\partial F}{\partial J_{2D}} S_{ij} + \frac{\partial F}{\partial J_{2D}} (S_{ik} S_{kj} - \frac{2}{3} J_{2D} \delta_{ij})
\]

(3.46)

where \( \delta_{ij} \) = Kronecker's delta and \( S_{ij} \) = deviatoric stress tensor.

### Alternative Forms of Yield Conditions

The general polynomial function (Eq. 3.2) permits choice of a number of approximate (truncated) forms. Thus, several alternatives to the proposed yield condition, Eq. (3.6), are possible; a few of these forms are listed below:

\[
F^{(1)} = (J_{2D})^2 + \beta_1 J_1 J_3 - \gamma_1 J_1^3 - k_1^4 = 0
\]

(3.47)

\[
F^{(2)} = (J_{2D})^3 + \beta_2 J_1^3 J_3 - \gamma_2 J_1^5 - k_2^6 = 0
\]

(3.48)
In all these forms, $\beta$ is assumed to be the growth function while $\gamma$, $k$ and $\nu$ are material constants. It is important to note here that the constants $\gamma$, $k$ and $\nu$ are determined considering the stress state at the ultimate condition of a material point.

The requirements for these functions are that i) they should plot as convex surfaces in the stress space, ii) they should intersect the $J_1$-axis at right angles for initially isotropic and associative materials. Fortunately, all the above functions satisfy these two requirements and, thus, can be used as yield functions. Figures 3.16 through 3.23 show the plots of these yield functions in $\sqrt{J_{2D}} - J_1$ space as well as on octahedral plane. It may be noted that the triaxial plane is similar to the $\sqrt{J_{2D}} - J_1$ plane and may be obtained by rotating the $\sqrt{J_{2D}}$ and $J_1$ axes.

It is evident from the Figures 3.16 to 3.23 that all the yield surfaces are convex and intersect the hydrostatic axis ($J_1$-axis) at right angles.

The growth function, $\beta$, in all the alternative forms essentially varies in the same manner as discussed earlier. However, its limits are different. The following section is devoted to explain the growth function, $\beta$, for the yield condition given by Eq. (3.47). The growth functions for the other two forms are similar and have the same limits.
Figure 3.16. Plot of the Alternative Yield Function, $F^{(1)}$, in $\sqrt{J_2} - J_1$ Space
Figure 3.17. Plot of the Alternative Yield Function, $F^{(2)}$, in $\sqrt{J_{2D}} - J_1$ Space
Figure 3.18. Plot of the Alternative Yield Function, $F^{(3)}$, in $\sqrt{J_{2D}} - J_1$ Space
Figure 3.19. Plots of the Alternative Yield Function, $F^{(1)}$, on the Octahedral Plane.
Figure 3.20. Plots of the Alternative Yield Function, $F^{(2)}$, on the Octahedral Plane
Figure 3.21. Plots of the Alternative Yield Function, $F^{(3)}$, on the Octahedral Plane
Figure 3.22. Plot of the Alternative Yield Function, $F^{(1)}$, in Triaxial Plane
Figure 3.23. Plot of the Alternative Yield Function, $F^{(3)}$, in Triaxial Plane
Consider the case of cohesionless soils. Thus the constant \( k_1 = 0 \) and Eq. (3.47) reduces to the form

\[
(J_{2D})^2 + \beta_1 J_1 J_3 - \gamma_1 J_1^3 = 0
\]  

(3.50)

For purely hydrostatic state of stress, \( J_{2D} = 0 \) and \( J_3 = J_1^{2/3} \) and Eq. (3.50) becomes

\[
\frac{\beta_1}{3} J_1^4 - \gamma_1 J_1^3 = 0
\]  

(3.51)

Equation (3.51) defines the intercept with the \( J_1 \)-axis. In terms of \( \gamma_1 \) and \( \beta_1 \), this intercept can be expressed as

\[
J_1 = \frac{3\gamma_1}{\beta_1}
\]  

(3.52)

Equation (3.52) when interpreted with respect to the convexity of the yield function (Eq. 3.47), the following restrictions result:

\[
\beta_1 \geq 0 \text{ if } \gamma_1 > 0
\]  

(3.53)

The condition \( \beta_1 = 0 \) defines the ultimate condition of a material under a complex state of stress. At that point, the intercept with the \( J_1 \)-axis becomes infinite. When \( \beta_1 < 0 \), the yield function plots as a concave surface in \( \sqrt{J_{2D}} - J_1 \) space and, thus, violates the consequences of the stability postulates of Drucker (1951). Considering the above conditions
with respect to the convexity of the yield function, the range of $\beta_1$ can be written as

$$0 \leq \beta_1 < \infty$$  \hspace{1cm} (3.54)

It may be noted here that when $\beta_1 = \infty$, the intercept with the $J_1$-axis vanish and the yield surface becomes a singular surface.

It is mentioned in the previous section that, in general, the growth function can be a function of a large number of history dependent parameters. However, for simplicity, the growth function, $\beta_1$ is expressed as a function of a single parameter, $\xi$, where $\xi$ is the trajectory of the plastic strain vector as defined by Eq. (3.8). To select an appropriate function form of $\beta_1(\xi)$, variation of $\beta_1$ with respect to the growth parameter, $\xi$, should be plotted from actual laboratory test data. A typical plot is shown in Fig. 3.24. As mentioned before, the effect of stress path can be included in the formulation if the growth function is made function of two parameters, $\xi$ and $r_D$ as defined earlier. Guided by the experimental behavior, the following form of $\beta_1(\xi, r_D)$ can be assumed

$$\beta_1 = \frac{\beta_a^{(1)}}{\xi^{n_1^{(1)}} [1 - g_1^{(1)} (r_D)]:}$$  \hspace{1cm} (3.55)

where the superscript $(1)$ is used to emphasize that these material constants are different from the ones used before. The form of the function
Figure 3.24. Typical Variation of $\beta$ as a Function of $\xi$ for the Alternative Yield Function, $f(1)$
\( g_1^{(1)}(r_D) \) can be assumed in a similar manner as defined by Eq. (3.32); that is,

\[
g_1^{(1)}(r_D) = \beta_b^{(1)} r_D^{\eta_2^{(1)}}
\]

(3.56)

where \( \beta_b^{(1)} \) and \( \eta_2^{(1)} \) are the material constants defining the hardening behavior during plastic deformation.
CHAPTER 4

DETERMINATION OF MATERIAL CONSTANTS

General

The proposed constitutive model has a number of material constants including the Young's modulus, E, and the Poisson's ratio, ν. Determination of such constants for any material requires a comprehensive series of laboratory tests with number of loading, unloading and re-loading cycles. The characteristics of the test series depend on the type of material, field conditions and on the available testing facilities.

The proposed constitutive model is, in general, applicable for any frictional material. In this dissertation, however, only geological materials will be considered.

Three different soils are considered here to obtain the material constants associated with the proposed constitutive model. These are i) a silty sand, ii) an artificial soil and iii) an agricultural soil. The stress-strain responses for the artificial soil are presented in the following section. The test data for the silty sand and the agricultural soil are taken from Desai et al. (1982) and Markle (1981).
Laboratory Testing

Testing Device and Stress Paths

The test apparatus is a stress-controlled truly triaxial device where the sample can be subjected to a three-dimensional state of stress (Desai et al. 1981; Sture and Desai, 1979; Desai et al. 1982). The cubical sample is 4x4x4 inches (10.16x10.16x10.16 cm) in size (see Fig. 4.1) and is loaded by air pressure through the six faces of the apparatus. The applied stresses in each direction can be controlled independently. Linear variable differential transformers (LVDT) probes are used to measure the displacements at each face. There are three such probes on each face of the apparatus. An "exploded view" of the device is shown in Fig. 4.2 (Desai et al. 1982; Sture and Desai, 1979).

The testing device is capable of applying loads following any arbitrary stress path. Thus, it is ideally suited for testing geological materials because their stress-strain responses are highly path dependent. As a result, the parameters obtained from such test data would be more representative of the field conditions compared to the conventional cylindrical triaxial tests in which only a limited number of stress paths are possible. Although any arbitrary stress path may be followed, usually some well-defined tests are performed. Figure 4.3 shows the stress paths which are generally used for testing geological materials such as soils (Desai et al. 1981).
Figure 4.1. Schematic of the Cubical Soil Specimen
Figure 4.2. Schematic of the Truly Triaxial Device Showing "An Exploded View" of Arrangements on One Face
Figure 4.3. Schematic of the Commonly Used Stress Paths

a) Stress Paths on the Octahedral Plane

b) Stress Paths in Triaxial Plane
Description of the Soil and Sample Preparation

The device is used to perform a series of tests on an artificial soil. The soil consists of 50% Florida Zircon sand and 50% Fire clay; 10% of No. 5 SAE mineral oil is added to the mixture (Desai et al. 1981; Mould, 1979). Oil is used instead of water to eliminate the influence of moisture changes during prototype tests and sample preparation. Figure 4.4 shows the grain-size distribution for the artificial soil.

The tests are performed under completely drained condition. It is observed that the soil is highly compressible and almost cohesionless.

Sample Preparation. The samples are prepared using a device developed by Desai and Munster; a detailed description of the device is given by Munster (1981). The main aim of the device is to achieve a constant density, $\gamma_0$, for all the materials. To prepare the sample, a desired density, $\gamma_0$, is selected and the corresponding amount of soil is measured. This soil is poured into a 4x4x4 inches (10.16x10.16x10.16 cm) cubical steel mold and vibrated. The vibration helps the soil to settle uniformly in the vertical direction. A hydraulic jack with a 4x4 inches (10.16x10.16 cm) square piston head is then used to compress the soil down to 4 inches (10.16 cm) of depth, which makes an exact 4x4x4 inches (10.16x10.16x10.16 cm) cubical soil sample with a selected density, $\gamma_0$. In the present series of tests with artificial soil, the density used is 2 gm/cc. It may be noted here that upon removing the piston, the soil elastically rebounds. However, such elastic rebound is small for most soils. This can be adjusted by compressing the sample to a height slightly less than 4 inch (10.16 cm). This, however, requires few trials.
<table>
<thead>
<tr>
<th>Gravel</th>
<th>Sand</th>
<th>Fines</th>
</tr>
</thead>
<tbody>
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<td>Coarse to medium</td>
<td>Fine</td>
</tr>
<tr>
<td>U.S. standard sieve sizes</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3/4 in.</td>
<td>No. 4</td>
<td>No. 10</td>
</tr>
</tbody>
</table>

Figure 4.4. Grain Size Distribution Curve for the Artificial Soil
Test Results

The stress-strain responses of the artificial soil, as obtained from the truly triaxial device, are described in this section. Figure 4.5a shows the test results for a hydrostatic compression (HC) test. It is seen from Fig. 4.5a that the strain in the vertical direction is substantially less than the strains in the lateral directions. This may be due to the fact that the sample is made by compressing the soil in the vertical direction. Although an isotropic soil sample is intended, some anisotropy during sample preparation is unavoidable. There are two unloading-reloading cycles in the test. The slopes of the unloading-reloading curves are used to calculate the elastic parameters as well as the plastic part of the total strain. As seen from Fig. 4.5a, the soil is highly compressible. However, the upward bending trend of the stress-strain (see Fig. 4.5a) curves shows that the material hardens significantly under pure hydrostatic compression. Figure 4.5b shows the mean pressure-volumetric strain curve. It may be mentioned here that the volumetric strain is obtained by adding the three normal strains.

Figures 4.6, 4.7 and 4.8 show the stress-strain responses for three conventional triaxial compression (CTC) tests at confining pressures of 10 psi (68.9 kpa), 15 psi (103.35 kpa) and 20 psi (137.8 kpa), respectively. In these tests, the stresses in the lateral directions, $\sigma_2$ and $\sigma_3$ are kept constant (equal to the confining pressure, $\sigma_0$), while the vertical stress, $\sigma_1$, is increased in small increments. It is reported by Desai et al. (1980) and also observed from present series of
\[ \varepsilon_1 = \varepsilon_2 = \varepsilon_x \]
\[ \varepsilon_3 = \varepsilon_y \]

Figure 4.5a. Stress-Strain Response Curves for Hydrostatic Compression Test
Figure 4.5b. Average Stress-Strain Response for Hydrostatic Compression Test

\[ \varepsilon_V = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \]
Figure 4.6. Stress-Strain Response Curves for Conventional Triaxial Compression Test

\( \gamma_0 = 2 \text{ gm/cc} \)

\( \sigma_0 = 10 \text{ psi} \)

1 psi = 6.89 kpa
Figure 4.7. Stress-Strain Response Curves for Conventional Triaxial Compression Test

\( \gamma_0 = 2 \text{ gm/cc} \)

\( \sigma_0 = 15 \text{ psi} \)

1 psi = 6.89 kpa
Figure 4.8. Stress-Strain Response Curves for Conventional Triaxial Compression Test

\[ \gamma_0 = 2 \text{ gm/cc} \]
\[ \sigma_0 = 20 \text{ psi} \]
\[ 1 \text{ psi} = 6.89 \text{ kpa} \]
tests that the artificial soil creeps significantly when left at a particular load level. Since plasticity-based constitutive theories describe only time independent material behavior at each load increment, the waiting time before the reading was only 4-5 minutes. There are two unloading-reloading cycles which are used to calculate the plastic strains during straining.

Figure 4.9 shows the results of a simple shear test (SS) performed at an initial confining pressure, \( \sigma_0 \), of 20 psi (137.8 kpa). The test is carried out by loading the sample hydrostatically up to 20 psi (137.8 kpa) and then shearing it in such a way that the mean pressure remains constant at 20 psi (137.8 kpa). To do this, the stress component, \( \sigma_2 \), is kept constant. The stress component, \( \sigma_1 \), is then increased while \( \sigma_3 \) is decreased simultaneously by the same amount. Again, unloading-reloading cycles are used to calculate the plastic strains and the elastic parameters. The test is terminated when the strain in the vertical direction reached a value of about 8%.

Figure 4.10 shows the stress-strain response from a conventional triaxial extension (CTE) test. This test is performed at an initial confining pressure of 10 psi (68.9 kpa). In this test, the stress component \( \sigma_1 \) (vertical stress) is kept constant while the lateral stresses \( \sigma_2 \) and \( \sigma_3 \) are increased by the same amount.

In all the tests excepting the hydrostatic compression (HC) test, the three components of strains are plotted as a function of the octahedral shear stress, \( \tau_{\text{oct}} \). Thus, the plastic strains for each component
\[ \tau_{oct} = 2 \text{ gm/cc} \]
\[ \sigma_0 = 20 \text{ psi} \]
\[ 1 \text{ psi} = 6.89 \text{ kpa} \]

Figure 4.9. Stress-Strain Response Curves for Simple Shear Test
\[ \gamma_0 = 2 \text{ gm/cc} \]
\[ \sigma_0 = 10 \text{ psi} \]
\[ 1 \text{ psi} = 6.89 \text{ kpa} \]

Figure 4.10. Stress-Strain Response Curves for Conventional Triaxial Extension Test
can be calculated very easily. It is also evident from the Figs. 4.6 through 4.10 that at a density of 2 gm/cc this soil is highly compressible.

Limitations

**Test Apparatus.** The testing device used here is intended for testing samples under a three-dimensional state of stress such that the directions of principle stresses are always coincident with the directions of principal strains. Thus, the device is not suitable for studying the rotations of the principal directions of strains with respect to the principal directions of stresses. Such behavior may be important for studying induced anisotropy of materials.

In its present form, the device is capable of performing the tests only in the principal stress space. Thus, no shear stress can be applied directly to the sample.

Finally, since the device is a stress-controlled device, the softening behavior cannot be simulated.

**Sample Preparation.** It is intended to obtain an initially isotropic sample for the current test series. However, the sample preparation technique adopted here may cause anisotropy in the sample due to greater compaction in one direction. The degree of such anisotropy is dependent upon the intensity of compaction.
Procedure for Determining Material Constants

There are nine material constants associated with the proposed model as described in Chapter 3. These constants can be classified into three categories:

1) Elastic Constants
   \[ E, \nu \]

2) Constants for Ultimate Yielding
   \[ \alpha, \gamma, k \]

3) Constants for Hardening
   \[ \beta_a, \eta_1, \beta_b, \eta_2 \]

A number of tests will be required to obtain these material constants for a specific material. Detailed procedures for determining these constants are described below.

**Elastic Constants.** There are two elastic constants for an isotropic material, Young's modulus, \( E \) and Poisson's ratio, \( \nu \). It may be noted that bulk modulus, \( K \) and shear modulus, \( G \), may also be used. It will be assumed that unloading and reloading is elastic. Thus \( E \) and \( \nu \) can be found from the slopes of the unloading-reloading curves. Although any test can be used for this purpose, only hydrostatic compression test and conventional triaxial compression tests will be used to determine \( E \) and \( \nu \).

When the hydrostatic compression test data is plotted as mean pressure vs volumetric strain, the slope of the unloading-reloading curve gives the bulk modulus, \( K \), where \( K \) is related to \( E \) and \( \nu \) through the following equation:
To obtain $E$ and $\nu$ explicitly, a second equation is needed. A conventional triaxial compression (CTC) test data will be used for this purpose. In a CTC test, the cell pressure, $\sigma_2 (= \sigma_3)$, is kept constant while the axial stress, $\sigma_1$, is increased. The corresponding strains are axial strain, $\varepsilon_1$, and lateral strain, $\varepsilon_3 (= \varepsilon_2)$. When the CTC test results are plotted as $(\sigma_1 - \sigma_3)$ vs $2 (\varepsilon_1 - \varepsilon_3)$, the slope of the unloading-reloading curves gives the shearing modulus, $G$, where

$$G = \frac{E}{2(1 + \nu)}$$

(4.2)

Using Eqs. (4.1) and (4.2), $E$ and $\nu$ can be obtained explicitly.

An alternative to the above procedure is to obtain $E$ and $\nu$ directly from a CTC test. To do this, a CTC test data is plotted as $(\sigma_1 - \sigma_3)$ vs $\varepsilon_1$ and $(\sigma_1 - \sigma_3)$ vs $\varepsilon_3$. The slopes of the unloading-reloading curves of the first plot give the Young's modulus, $E$, while the slopes obtained from the second plot give $\frac{E}{\nu}$. Thus, dividing the slope of the first curve by the slope of the second curve, Poisson's ratio, $\nu$, may be obtained.

It may be noted that the Young's moduli of granular materials are frequently functions of the effective mean pressure. When more than one test is available, it is a good idea to obtain $E$ value from all the tests and then take a weighted average value. However, it is also possible to express the Young's modulus, $E$, as a function of the mean pressure.
Constants for Ultimate Yielding. These constants are \( \alpha, \gamma \) and \( k \). Physically, the constant \( k \) measures the cohesive strength of a material at zero normal stress. Thus, it is possible to obtain \( k \) prior to any major calculations. For example, for a cohesionless material, the constant \( k \) vanishes. However, in general, \( k \) can be determined along with \( \alpha \) and \( \gamma \), using any suitable regression technique.

As mentioned in Chapter 3, the state of ultimate yielding of a material point for \( F \) in Eq. 3.6 is realized when the growth function, \( \beta \), assumes a value of \( 3\alpha \). At that state, the yield condition takes the following form:

\[
F = J_{20} + \alpha \left[ J_1^2 - 3 J_1 J_3^{1/3} \right] - \gamma J_1 - k^2 = 0 \quad (4.3)
\]

There are only three material constants in Eq. (4.3). These constants may be obtained by evaluating Eq. (4.3) at ultimate condition for three different tests. Frequently, however, number of tests provided are more than the number of constants to be determined which makes the problem overdeterminate. A large list of optimization techniques (Fox, 1979) are available to solve for such system of equations. Although a least square technique may not be very efficient, it will be used to obtain the material constants because of its simplicity. The following steps are required to obtain \( \alpha, \gamma \) and \( k \):

1) Determination of the principal stresses, \( \sigma_1, \sigma_2, \sigma_3 \) at ultimate yield for all the tests available.
i1) Calculation of $J_1$, $J_{2D}$ and $J_3$ from $\sigma_1$, $\sigma_2$ and $\sigma_3$. This is done for every test.

i11) Equation (3.3) may be written as

\[
\begin{pmatrix}
-(J_1^2 - 3 J_1 J_3^{1/3}) & J_1 & 1
\end{pmatrix}
\begin{bmatrix}
\alpha \\
\gamma \\
k^2
\end{bmatrix} = J_{2D}
\]

(4.4)

where $\mathbf{J}$ represent a row matrix. Equation (4.4) is then evaluated at ultimate yield for every test. If $N$ number of tests is used, the final set of equations are

\[
\begin{bmatrix}
\alpha \\
\gamma \\
k^2
\end{bmatrix} = \{B\}
\]

(4.5)

where $[A] = \text{known coefficient matrix (NX3)}$

$\{B\} = \text{known vector (NX1)}$

In general, $N > 3$ in Eq. (4.5). To solve for $\alpha, \gamma, k^2$, any least square algorithm can be used.

**Constants for Hardening.** The growth function, $\beta$, (Chapter 3) is written as
\[ \beta = \beta_u \left( 1 - \frac{\beta_a}{\xi_1 (1 - \beta_b \cdot r_D^{n_2})} \right) \] (4.6)

where \( r_D = \xi_D / \xi \).

The constants for hardening are \( \beta_a, n_1, \beta_b \text{ and } n_2 \).

These four constants are determined in two steps. In the first step, \( \beta_a \) and \( n_1 \) are determined using the hydrostatic compression test data.

**Determination of \( \beta_a \) and \( n_1 \).** An isotropic material, when subjected to pure hydrostatic state of stress, experiences only volumetric deformation. Thus, \( \xi_D = 0 \) and \( \xi = \xi_V \). Consequently, Eq. (4.6) reduces to

\[ \beta = \beta_u \left( 1 - \frac{\beta_a}{n_1} \right) \] (4.7)

Rearranging,

\[ \frac{\beta_a}{n_1} = \left( 1 - \frac{\beta}{\beta_u} \right) \xi_V \] (4.8)

Taking natural log on both sides of Eq. (4.8), one obtains.

\[ \ln \left( \beta_a \right) - n_1 \ln \left( \xi_V \right) = \ln \left( 1 - \frac{\beta}{\beta_u} \right) \] (4.9)

Equation (4.9) represents a straight line when plotted in \( \ln \left( 1 - \frac{\beta}{\beta_u} \right) - \ln \left( \xi_V \right) \) space as shown in Fig. 4.11. Then \( \beta_a \) and \( n_1 \) are obtained from the intersection and the slope of the straight line, respectively. The following steps are required to obtain \( \beta_a \) and \( n_1 \) from the laboratory tests:
Figure 4.11. Typical Plot of $\ln (1 - \frac{\beta}{\beta_u})$ vs $-\ln (\xi_Y)$ for Hydrostatic Compression Test
i) Mean pressure, \( p \), vs volumetric strain, \( \varepsilon_V \), curve is drawn from an HC test (Fig. 4.12). A number of points, say \( N \), are chosen on the test curve. Using this unloading modulus, another curve, \( p-\xi_V \) (Fig. 4.13), is obtained such that

\[
\xi_V = \frac{1}{\sqrt{3}} \sum_{i=1}^{N} \Delta \varepsilon_V^p
\]  

(4.10)

where \( \Delta \varepsilon_V^p \) denotes the ith incremental plastic volumetric strain.

ii) For every point on the \( p-\xi_V \) curve, the mean pressure \( p \) (or \( J_1/3 \)) and the corresponding \( \xi_V \) is noted.

iii) The magnitude of the growth function for any state of stress can be obtained from the yield condition. For hydrostatic compression stress paths, \( J_{2D} = J_{3D} = 0 \), and the yield condition reduces to the form

\[
F = \alpha J_1^2 - \beta J_1^2 - \gamma J_1 - k^2 = 0
\]  

(4.11)

Since \( \alpha, \gamma \) and \( k \) are available, the growth function, \( \beta \), can calculated from the following relationship:

\[
\beta = 3 \left[ \alpha - \frac{\gamma}{J_1} - \left( \frac{k}{J_1} \right)^2 \right]
\]  

(4.12)

For \( N \) number of points selected on the \( p(J_1/3) \) vs \( \xi_V \) curve, the growth function, \( \beta \), is calculated.
Figure 4.12. A Typical Stress-Strain Response Curve for a Hydrostatic Compression Test Showing Unloading-Reloading Cycles

Figure 4.13. A Typical Plot of Mean Pressure vs Volumetric Plastic Strain Curve for Hydrostatic Compression Test
iv) For N number of points, $\ln \left(1 - \frac{\beta}{\beta_u}\right)$ and $\ln (\xi_V)$ are calculated. The corresponding points are noted on $\ln \left(1 - \frac{\beta}{\beta_u}\right)$ vs $\ln (\xi_V)$ space.

v) A straight line is drawn through these points in a least square sense as shown in Fig. 4.11. The slope of the straight line gives the constant $\eta_1$. The intercept with the $\ln \left(1 - \frac{\beta}{\beta_u}\right)$ axis represents $\ln (\beta_a)$, from which $\beta_a$ can be obtained.

Determination of $\beta_b$ and $\eta_2$. Since $\beta_a$ and $\eta_1$ are known, Eq. (4.6) may be written in the form

$$\beta_b \cdot r_D^{\eta_2} = \left[1 - \frac{\beta_a}{\xi \eta_1 \left(1 - \frac{\beta}{\beta_u}\right)}\right]$$

(4.13)

Taking natural log on both sides of Eq. (4.13)

$$\ln (\beta_b) + \eta_2 \ln (r_D) = \ln \left[1 - \frac{\beta_a}{\xi \eta_1 \left(1 - \frac{\beta}{\beta_u}\right)}\right]$$

(4.14)

Equation (4.14) can now be used to determine $\beta_b$ and $\eta_2$. The following steps are involved in the calculation of $\beta_b$ and $\eta_2$.

i) A number of tests are selected. Since conventional triaxial compression tests are commonly performed, a number of these
tests at various confining pressures may be used. However, the procedure is general and applicable for any stress path. Hence, although the following steps are explained with regard to a CTC test, they are also valid for any other path.

ii) A typical stress-strain curve for a CTC test is shown in Fig. 4.14. A number of points, say N, are selected on the loading portion of the stress-strain curve and the stresses, \( \sigma_i \) (i = 1, 2, 3) and the corresponding strains, \( \varepsilon_i \) (i = 1, 2, 3) at each point are noted. It is important here to note that the strains \( \varepsilon_i \) (or stress \( \sigma_3 \)) refer to the directions Z, X and Y, respectively (see Fig. 4.1).

iii) Using the slopes of the unloading-reloading curve, \( \varepsilon_i^p \) (i = 1, 2, 3) are obtained at each point where the superscript \( p \) denotes plastic. From \( \varepsilon_i^p \) (i = 1, 2, 3), the incremental plastic strain \( \Delta \varepsilon_i^p \) (i = 1, 2, 3) between any two adjacent points is obtained.

iv) At every point, \( \Delta \varepsilon \) and \( \Delta \varepsilon_D \) are then calculated using the incremental plastic strains, \( \Delta \varepsilon_i^p \) (i = 1, 2, 3), where

\[
\Delta \varepsilon = \left( (\Delta \varepsilon_1^p)^2 + (\Delta \varepsilon_2^p)^2 + (\Delta \varepsilon_3^p)^2 \right)^{1/2} \quad (4.15)
\]

and

\[
\Delta \varepsilon_D = \left( (\Delta e_1^p)^2 + (\Delta e_2^p)^2 + (\Delta e_3^p)^2 \right)^{1/2} \quad (4.16)
\]

where \( \Delta e_i^p = \Delta \varepsilon_i^p - \frac{1}{3} (\Delta \varepsilon_1^p + \Delta \varepsilon_2^p + \Delta \varepsilon_3^p) \).
v) Total $\xi$ and $\xi_D$ at j th point is then calculated according to the following equations:

$$\xi_j = \xi_0 + \sum_{i=1}^{j} \Delta \xi^i$$

(4.17)

and

$$\xi_D_j = \sum_{i=1}^{j} \Delta \xi_D^i$$

(4.18)

where $j$ refers to the point at which the quantities are calculated. $\Delta \xi^i$ and $\Delta \xi_D^i$ in Eq. (4.18) denote the ith increments of $\xi$ and $\xi_D$, respectively. $\xi_0$ is the value of $\xi$ corresponding to the initial confining pressure, $\sigma_0$, at which shearing started.

vi) The stress invariants $J_1$, $J_{2D}$ and $J_3$ are computed at each of N points from the known stresses, $\sigma_i$ ($i = 1, 2, 3$).

vii) The magnitude of $\beta$ is then calculated from the yield condition at each point using the equation

$$\beta = \frac{(J_{2D} + cJ_1^2 - \gamma J_1 - k^2)}{J_1 J_3^{1/3}}$$

(4.19)

viii) $\ln (r_D)$ and $\ln \left[ 1 - \frac{\beta a}{\xi} \right]$ are computed at each of the N points, where $r_D = \xi_D / \xi$. These are then
plotted in the $\ln (r_D) - \ln \left[ 1 - \frac{\beta_a}{\xi n_1 (1 - \frac{\beta_a}{\beta_u})} \right]$ space. A typical plot is shown in Fig. 4.15.

ix) A straight line is drawn through the points in a least square sense. The slope of the line gives $n_2$ while the intercept with $\ln \left[ 1 - \frac{\beta_a}{\xi n_1 (1 - \frac{\beta_a}{\beta_u})} \right]$ axis provides $\beta_b$.

The general procedure outlined above may be used for hand calculations. The process, however, is extremely tedious and time consuming. Moreover, chances of errors are also high if proper care is not taken during hand calculations. To avoid such difficulty, the whole procedure is computerized. The inputs to the computer code are the data for the discretized points on the stress-strain curves obtained from the laboratory tests. The code calculates the material constants for ultimate yielding as well as for hardening. The code has provisions for various plots when proper options are used. It may be noted that the code does not calculate the elastic parameters, $E$ and $\nu$. These parameters can be calculated from the slopes of the unloading-reloading curves of various tests. In this dissertation, the above mentioned code is used to obtain parameters for three different soils. These are explained in the following sections.
Figure 4.14. Typical Stress-Strain Response Curves for a Conventional Triaxial Compression Test

Figure 4.15. A Typical Plot to Determine the Hardening Constants $\beta_b$ and $n_2$
Material Constants for the Artificial Soil

Description of this soil is given in the section entitled "Laboratory Testing." The test results are presented in the Figs. 4.5 through 4.10. Since the properties of this soil are already described in a previous section, they will not be repeated here.

Elastic Constants \((E, \nu)\). Elastic constants are determined from the slopes of the unloading-reloading curves of HC and CTC tests. There are three CTC tests at three different confining pressures. It is observed that the \(E\) values obtained from all tests are different. This may be explained by the fact that the elastic moduli for granular materials are function of confining pressures. A number of \(E\) is obtained and then an average value is determined. The following values are found:

\[
E = 4000 \text{ psi (27560 kpa)}
\]

\[
\nu = 0.35
\]

Constants for Ultimate Yield \((\alpha, \gamma, k)\). It is mentioned earlier that the artificial soil is almost cohesionless. Thus, \(k = 0\) is assumed. \(\alpha\) and \(\gamma\) are determined using the computer procedure. Five tests are considered for this purpose. The values of the material constants are given below:

\[
\alpha = 0.162
\]

\[
\gamma = 1.542 \text{ psi (10.6 kpa)}
\]

\[
k = 0.0 \text{ psi (0.0 kpa)}
\]

Constants for Hardening \((\beta_a, \eta_1, \beta_b, \eta_2)\). Figure 4.16 shows the plot which provides the values of \(\beta_a\) and \(\eta_1\). The remaining constants,
Figure 4.16. Plot of $\ln (1 - \frac{\beta}{\beta_u})$ vs $-\ln (\xi_V)$ for Hydrostatic Compression Test of Artificial Soil
$\beta_b$ and $\eta_2$ are determined from Fig. 4.17. The values of the constants are

\begin{align*}
\beta_a &= 0.00217 \\
\eta_1 &= 1.376 \\
\beta_b &= 0.723 \\
\eta_2 &= 0.660
\end{align*}

It may be noted that the plots are obtained from the computer. Thus, no hand calculations are used to obtain the constants for hardening.

**Material Constants for a Silty Sand**

This soil is obtained from Urban Mass Transportation Administration (UMTA) test section at Pueblo, Colorado (Desai et al. 1982). The grain size distribution curve for this soil is shown in Fig. 4.18. The soil is well graded and has an optimum moisture content of 9% (Desai et al. 1982; Janardhanam, 1980).

Five tests, including the hydrostatic compression (HC) test, are used to obtain the material constants associated with the proposed model. Details of these tests are given by Desai et al. (1982), Janardhanam (1980), Siriwardane (1980) and Munster (1981). Table 4.1 shows the densities and moisture contents for these tests.

**Elastic Constants ($E$, $\nu$).** Elastic constants are determined from the unloading-reloading curves of the Hydrostatic Compression (HC) test and the Conventional Triaxial Compression (CTC) test. The values of the Young's Modulus, $E$ and the Poisson's Ratio, $\nu$, are
Figure 4.17. Plot of $\ln \left[ 1 - \frac{\beta_a}{\eta_1 (1 - \beta_u)} \right]$ vs $-\ln (r_D)$ for Artificial Soil
Figure 4.18. Grain Size Distribution Curve for the Silty Sand
Table 4.1 Densities, Moisture Contents and the Initial Confining Pressures for the Tests on the Silty Sand

<table>
<thead>
<tr>
<th>Test</th>
<th>Density (gm/cc)</th>
<th>Moisture Content (%)</th>
<th>Initial Confining Pressures (psi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>HC</td>
<td>1.86</td>
<td>9</td>
<td>-</td>
</tr>
<tr>
<td>CTC</td>
<td>1.92</td>
<td>9</td>
<td>10 (68.9 kpa)</td>
</tr>
<tr>
<td>SS</td>
<td>2.07</td>
<td>9</td>
<td>20 (137.8 kpa)</td>
</tr>
<tr>
<td>CTE</td>
<td>2.02</td>
<td>9</td>
<td>20 (137.8 kpa)</td>
</tr>
<tr>
<td>TC</td>
<td>2.03</td>
<td>9</td>
<td>25 (172.25 kpa)</td>
</tr>
</tbody>
</table>
E = 9386 psi (64670 kpa)
\( \nu = 0.36 \)

**Constants for Ultimate Yield (\( \alpha, \gamma, k \)).** These material constants are determined considering the stress state at ultimate yield for CTC, SS, CTE and TC tests. The computer code is used to obtain the constants \( \alpha, \gamma \) and \( k \). The procedure uses least square optimization technique to obtain an average value of all the constants. The values of \( \alpha, \gamma \) and \( k \) for the silty sand are as follows:

\[ \alpha = 0.156 \]
\[ \gamma = 3.007 \text{ psi (20.7 kpa)} \]
\[ k = 0.0 \text{ psi (0.0 kpa)} \]

**Constants for Hardening (\( \beta_a, \eta_1, \beta_b, \eta_2 \)).** As mentioned earlier, the constants \( \beta_a \) and \( \eta_1 \) are determined from the hydrostatic compression (HC) test. The remaining tests are used to obtain the average values of the material constants \( \beta_b \) and \( \eta_2 \). Although these constants can be determined by hand calculations, the computer procedure is used here to obtain from the computer code where the slope of the straight line is \( \eta_1 \) and the intercept with the ordinate is \( \ln (\beta_a) \) (see Fig. 4.19). The values obtained are

\[ \beta_a = 0.000617 \]
\[ \eta_1 = 1.0544 \]

The constants \( \beta_b \) and \( \eta_2 \) are also obtained using the computer procedure. Figure 4.20 shows the computer plot which is used to obtain \( \beta_b \) and \( \eta_2 \). The values of the constants are as follows:
Figure 4.19. Plot of $\ln \left( 1 - \frac{B}{B_u} \right)$ vs $-\ln (\xi_Y)$ for Hydrostatic Compression Test of the Silty Sand.
Figure 4.20. Plot of $-\ln \left( 1 - \frac{B_a}{\eta(1 - \frac{\partial}{\partial u})} \right)$ vs $-\ln (r_D)$ for the Silty Sand

CTC Test ($\sigma_0 = 10.0$ psi)
SS Test ($\sigma_0 = 20.0$ psi)
TC Test ($\sigma_0 = 25.0$ psi)
CTE Test ($\sigma_0 = 20.0$ psi)

1.0 psi = 6.89 kpa
\[ \beta_b = 0.854 \]
\[ n_2 = 0.802 \]

Material Constants for an Agricultural Soil

The agricultural soil is obtained from the grounds of Virginia Tech Agricultural Field Station. The soil is a well-graded orange sandy silty clay with occasional rock fragments. Optimum moisture is found to be about 20% (Markle, 1981). The specific gravity of the agricultural soil is approximately 2.65. The tests were carried out in the multiaxial cubical device described at the beginning of this chapter. A total of seven tests are used to determine the material constants. Densities and moisture contents for all the tests are presented in Table 4.2.

Elastic Constants \((E, \nu)\). The elastic constants are determined from the unloading-reloading curves of HC and the CTC tests. Average values of \(E\) and \(\nu\) are

\[ E = 4100 \text{ psi (28249 kpa)} \]
\[ \nu = 0.35 \]

Constants for Ultimate Yield \((\alpha, \gamma, k)\). The material has very little cohesion and, thus, \(k\) is assumed to be zero. \(\alpha\) and \(\gamma\) are determined
Table 4.2 Densities, Moisture Contents and the Initial Confining Pressures for the Tests on the Agricultural Soil

<table>
<thead>
<tr>
<th>Test</th>
<th>Density (gm/cc)</th>
<th>Moisture Content (%)</th>
<th>Initial Confining Pressures (psi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>HC</td>
<td>1.37</td>
<td>11.2</td>
<td>-</td>
</tr>
<tr>
<td>CTC</td>
<td>1.36</td>
<td>11.2</td>
<td>2.5 (17.23 kpa)</td>
</tr>
<tr>
<td>CTC</td>
<td>1.36</td>
<td>11.1</td>
<td>5.0 (34.45 kpa)</td>
</tr>
<tr>
<td>CTC</td>
<td>1.38</td>
<td>11.4</td>
<td>10.0 (68.9 kpa)</td>
</tr>
<tr>
<td>SS</td>
<td>1.37</td>
<td>10.7</td>
<td>5.0 (34.45 kpa)</td>
</tr>
<tr>
<td>TC</td>
<td>1.37</td>
<td>10.7</td>
<td>5.0 (34.45 kpa)</td>
</tr>
<tr>
<td>TE</td>
<td>1.39</td>
<td>10.9</td>
<td>5.0 (34.45 kpa)</td>
</tr>
</tbody>
</table>
from six tests using the computer procedure mentioned earlier. The following values are obtained:

\[
\alpha = 0.216 \\
\gamma = 1.425 \text{ psi (9.82 kpa)} \\
k = 0.0 \text{ psi (p.kpa)}
\]

**Constants for Hardening (\(\beta_a, \eta_1, \beta_b, \eta_2\)).** The constants \(\beta_a\) and \(\eta_1\) are obtained from the hydrostatic compression test. Figure 4.21 shows the plot which is used to obtain \(\beta_a\) and \(\eta_1\). The material constants \(\beta_b\) and \(\eta_2\) are obtained from all tests except HC test. Figure 4.22 shows an average curve which gives the values of \(\beta_b\) and \(\eta_2\). The values of the hardening constant are as follows:

\[
\beta_a = 0.00125 \\
\eta_1 = 1.173 \\
\beta_b = 0.808 \\
\eta_2 = 1.00
\]

**Comments on the Material Constants**

The elastic constants, \(E\) and \(\nu\), can be determined from the slopes of the unloading-reloading curves from a test. In this dissertation, only HC and CTC tests are considered for this purpose. As mentioned earlier, the Young's modulus of soil generally varies with confining pressure. Thus, a single number for \(E\) may not be appropriate; rather \(E\) should be expressed as a function of the confining pressure.
Figure 4.21. Plot of $\ln (1 - \frac{\beta}{\beta_u})$ vs $-\ln (\xi_V)$ for the Agricultural Soil
Figure 4.22. Plot of \( \ln \left[ 1 - \frac{\beta_a}{\xi_s(1 - \frac{D_0}{D_u})} \right] \) vs \(-\ln (r_0)\) for Agricultural Soil
The remaining constants for the proposed model are determined using a computer procedure. The inputs to the program are the discretized stress-strain data for the tests as well as the principal stresses at ultimate yielding. Ultimate yielding is defined by drawing an asymptotic line to the final range of the stress-strain response curve for any test. The advantage of using the computer procedure is twofold. Firstly, it eliminates the errors which may occur during hand calculations. Secondly, it calculates the material constants in a very short period of time. The computer procedure also gives detailed output of the computed plastic strains, \( \varepsilon_i^p \) (i = 1, 2, 3), values of the growth function, \( \beta \), values of \( \xi \), \( \xi_v \) and \( \xi_D \) at every point along with various plots. Thus, the user may use these outputs to calculate the material constants manually.

In the previous sections, the material constants for three different soils are obtained. A comparison of the values of the material constants for these soils is given in Table 4.3.

Finally, the accuracy of the material constants can be established by back-predicting the stress-strain responses from which they are determined. The following chapter is devoted to the verification of the proposed model.
Table 4.3 Comparison of the Material Constants Obtained for the Artificial Soil, the Silty Sand and the Agricultural Soil

<table>
<thead>
<tr>
<th>Material Constant</th>
<th>Artificial Soil</th>
<th>Silty Sand</th>
<th>Agricultural Soil</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Elastic Constants</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E$</td>
<td>4000 psi</td>
<td>9386 psi</td>
<td>4100 psi</td>
</tr>
<tr>
<td></td>
<td>(27560 kpa)</td>
<td>(64670 kpa)</td>
<td>(28249 kpa)</td>
</tr>
<tr>
<td>$v$</td>
<td>0.35</td>
<td>0.36</td>
<td>0.35</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.162</td>
<td>0.156</td>
<td>0.216</td>
</tr>
<tr>
<td><strong>Constants for Ultimate Yielding</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\gamma$</td>
<td>1.542 psi</td>
<td>3.007 psi</td>
<td>1.425 psi</td>
</tr>
<tr>
<td></td>
<td>(10.6 kpa)</td>
<td>(20.7 kpa)</td>
<td>(9.8 kpa)</td>
</tr>
<tr>
<td>$k$</td>
<td>0.0 psi</td>
<td>0.0 psi</td>
<td>0.0 psi</td>
</tr>
<tr>
<td></td>
<td>(0.0 kpa)</td>
<td>(0.0 kpa)</td>
<td>(0.0 kpa)</td>
</tr>
<tr>
<td>$\beta_a$</td>
<td>0.00217</td>
<td>0.000617</td>
<td>0.00125</td>
</tr>
<tr>
<td><strong>Constants for Hardening</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n_1$</td>
<td>1.376</td>
<td>1.0544</td>
<td>1.173</td>
</tr>
<tr>
<td>$\beta_b$</td>
<td>0.723</td>
<td>0.854</td>
<td>0.808</td>
</tr>
<tr>
<td>$n_2$</td>
<td>0.660</td>
<td>0.802</td>
<td>1.000</td>
</tr>
</tbody>
</table>
CHAPTER 5

VERIFICATION OF THE MODEL

General

This chapter is intended to verify the proposed constitutive model with respect to the cubical triaxial test data for a number of different soils. The parameters obtained in Chapter 4 will be used to back predict the stress-strain responses obtained from the laboratory mainly for a silty sand, an artificial soil and an agricultural soil. To do this, an explicit relation between the incremental stress and the incremental strain is obtained using the proposed constitutive law. In matrix notation, this can be written as

\[ \{ \text{d} \sigma \} = [C]^{\text{e-p}} \{ \text{d} \varepsilon \} \]  

(5.1)

The vectors, \( \{ \text{d} \sigma \} \) and \( \{ \text{d} \varepsilon \} \) have six independent components and the details are given in Appendix A. The coefficients of the 6 x 6 elasto-plastic constitutive matrix, \([C]^{\text{e-p}}\) are also given in Appendix A. Equation (5.1) describes a set of nonlinear differential equations involving the stress and the strain. To obtain the relations between the stress and the strain at any stage of loading, Eq. (5.1) is to be integrated; that is,

\[ \{ \sigma \} = \int \{ \text{d} \sigma \} = \int [C]^{\text{e-p}} \{ \text{d} \varepsilon \} \]  

(5.2)
In general, \([C]^{e-p}\) is a function of the previous state as well as the current state of the material. For simplicity, it is assumed that the elasto-plastic constitutive relation matrix, \([C]^{e-p}\), can be explicitly found from the previous state of the material; that is, the actual nonlinear stress-strain response can be approximated by a piecewise linear stress-strain curve as shown in Fig. 5.1. Thus, Eq. (5.2) can be approximated by the following summation:

\[
\{\sigma_i\} = \{\sigma_0\} + \sum_{i=1}^{N} \{\Delta\sigma_i\} = \{\sigma_0\} + \sum_{i=1}^{N} [C]^{e-p} \{d\varepsilon\}_i
\]

(5.3)

where \(\{\sigma_0\}\) is the vector of initial stresses, and \(N\) denotes the total number of linear segments used to approximate the actual nonlinear stress-strain behavior.

**Computer Procedure to Numerically Integrate the Incremental Stress-Strain Relationships**

A computer procedure is developed using Fortran IV language to perform the numerical integration in Eq. (5.3). Some of the important features of this integration routine are discussed below:

i) The increments of strain, \(\{d\varepsilon\}\), obtained from a numerical scheme such as the finite element method may not be small enough to represent the actual nonlinear stress-strain behavior with reasonable accuracy. Thus, the computed stresses using Eq. (5.3) may be far from being accurate. There are a number of techniques available to correct the stresses within an increment; for example, Newton-Raphson's
Figure 5.1. Schematic of the Simulated Stress-Strain Response Curve Using Incremental Method
method, the slope of the curve appears in the denominator. Thus, when the slopes are small or zero, this method does not work. Secant iterative method is an improved version of Newton-Raphson's method where the actual slope is substituted by its finite difference approximation. This technique is used by Sandler and Rubin (1979) for the Cap model. However, this technique may not converge, depending upon the type of nonlinearity and the value of the convergence tolerance. These difficulties can be avoided when subincrementation technique is used. In this technique, the total increment of strain is subdivided into a number of smaller increments called the subincrements. The elastoplastic constitutive relation matrix, \([C]^{e-p}\), is updated at every subincremental level. This updated \([C]^{e-p}\) matrix is used to compute the subincremental stress from the current subincrement of strain. Total stress is then computed by summing up all the subincrements of stresses. Thus, Eq. (5.3) can be rewritten in the following form:

\[
\{\sigma\} = \{\sigma_0\} + \sum_{i=1}^{N} \{\Delta\sigma\}_i = \{\sigma_0\} + \sum_{i=1}^{N} \sum_{j=1}^{N_s} [C]^{e-p}_{j-1} i\{\Delta(\Delta\sigma)\}_j \tag{5.4}
\]

where \(N_s\) denotes number of subincrements in a particular increment, \(i\{\Delta(\Delta\sigma)\}_j\) represents the jth subincrement of strain for the ith strain increment and \([C]^{e-p}_{j-1}\) represents the updated elastoplastic constitutive matrix at the end of the
(j-1) th subincrement of the ith strain increment.

Figure 5.2b shows a schematic of the subincrementation technique for one increment of strain.

The accuracy of this technique entirely depends upon the size of the subincrements. It is observed from a number of numerical experiments that a reasonable accuracy may be obtained when the following criterion is satisfied (see Fig. 5.2a):

\[
\text{Max } |\Delta(\Delta \varepsilon)| \leq 0.0005
\]  

(5.5)

This criteria is followed throughout the dissertation.

ii) It is mentioned in Chapter 3 that the hardening behavior is characterized by the trajectory of the plastic strain vector, \( \xi \), and a stress path factor, \( r_\varepsilon = \xi_D/\xi \), where \( \xi_D \) is the deviatoric part of \( \xi \) as defined by Eq. (3.11). For the first load increment, both \( \xi \) and \( \xi_D \) are zero since no deformation history prior to this load increment is known. The hardening parameters, \( \xi \) and \( \xi_D \) are assumed. However, these initial values of \( \xi \) and \( \xi_D \) can be quite important and can be selected by performing a parametric study. In the present study, the following initial values of \( \xi \) and \( \xi_D \) are used:

\[
\xi = e
\]  

(5.6)

\[
\xi_D = e/100
\]
\( \varepsilon_{\text{sub}} \) = Size of Subincremental Strain

1.0 Ksi = 1000.0 psi
1.0 psi = 6.89 Kpa

Uniaxial Test
\( \varepsilon_1 = 15.0\% \)
\( \varepsilon_2 = \varepsilon_3 = 5.3\% \)

*Figure 5.2a. Plot of the Calculated Stress as a Function of the Size of the Subincremental Strain, \( \varepsilon_{\text{sub}} \)
Figure 5.2b. Schematic of the Subincrementation Technique for Increment, $\Delta \varepsilon_1$
where \( e = 0.0001 \). It is important to note here that the initial values as defined by Eq. (5.6) may not be suitable for all geological materials. Thus, for every new material, a parametric study should be made to establish these initial values.

iii) Geological materials such as soils have very small tensile strength compared to the compressive strength. In fact, for all practical purposes, most soils are considered as no-tension materials. Thus, plasticity analyses for soils are often limited to compressive state of stress only. In an actual boundary value problem, some region of the soil domain may go into tension without failing the entire system. As a result, proper tension checks are important for a boundary value problem involving soil media. The following procedure is used when a material point goes into tension:

To check whether a material point is in a state of tension or not, the first invariant of the stress tensor, \( J_1 \), is calculated. If \( J_1 < 0 \), it is understood that the material point is in a state of tension. In general, it is possible to assign a finite value of tensile strength, \( t \), for soils. The first invariant of the stress tensor, \( J_1 \), is then compared with the tensile strength, \( t \). When
regular plasticity analysis is performed, and when

$$|J_1| < t$$  \hspace{1cm} (5.7)

it is assumed that the material is incapable of sustaining any shear stress. For such a condition, the calculated $J_1$ is replaced by $t$ and the deviatoric stress, $S_{ij}$, is set equal to zero.

It may be noted that for no-tension materials, $t = 0$. Thus, the stress should be zero at that material point. Under this condition, the material loses all its stiffnesses. In this program, this is accounted for by specifying a small value of the Young's modulus, $E$, in the form

$$E_{\text{small}} = \frac{E_{\text{actual}}}{500}$$  \hspace{1cm} (5.9)

where $E_{\text{actual}}$ is the initial value of the Young's modulus for the soil.

In general, the Poisson's ratio, $\nu$, should also be changed. In this study, however, the Poisson's ratio is
kept the same when tension cutoff criteria is satisfied. Thus, the shear modulus, $G$, is also reduced by a factor of 500.

v) Equation (5.4) shows how the stresses are computed when strains are given as input. For stress-controlled tests, it is more appropriate to compute the strains from stress inputs. For such case, Eq. (5.4) can be inverted to obtain

$$
\{\varepsilon\} = \sum_{i=1}^{N} \Delta \varepsilon = \sum_{i=1}^{N} \sum_{j=1}^{N_s} \{[D] e \cdot p \}^{i} \{\Delta (\Delta \sigma)\}^{j}
$$

(5.10)

where $N$ is the number of stress increments, $N_s$ is the number of subincrements in each stress increment, $\{[D] e \cdot p \}^{i}$ is the elasto-plastic compliance matrix for the $(j-1)$th subincrement and $\{\Delta (\Delta \sigma)\}^{j}$ is the jth subincrement of stress. The superscript $i$ in Eq. (5.13) denotes the ith stress increment. It may be noted that Eq. (5.13) is used to back-predict a number of triaxial tests for three different soils. These results are presented in the following sections.

**Verification With Respect to a Silty Sand**

The material constants for this soil are given in Chapter 4. The computer procedures discussed earlier are used to back-predict some of the stress-strain response curves obtained from the cubical triaxial tests.
The densities, $\gamma_0$, and the moisture contents for these tests are presented in Table 4.1. It is important to note that the material constants used here are the average values obtained from five different tests, as discussed in Chapter 4.

Comparison of Stress-Strain Responses

Figure 5.3 shows the comparison of the prediction and the observation for a hydrostatic compression (HC) test. It is seen from Fig. 5.3 that a good agreement is achieved between the prediction and the observation.

Figure 5.4 shows the stress-strain response curves for a conventional triaxial compression (CTC) test. The initial confining pressure, $\sigma_0$, for this test is 10 psi (68.9 kPa). The experimental results are also plotted in the same figure for comparison purpose. It is seen that the predictions compare well with the experimental results.

Figures 5.5 through 5.7 show the comparison of the stress-strain responses for a simple shear (SS) test, a triaxial compression (TC) test and a conventional triaxial extension (CTE) test. Initial confining pressures for the simple shear (SS) test and the conventional triaxial extension (CTE) test are 20 psi (137.8 kPa). For the triaxial compression (TC) test, the initial confining pressure is 25 psi (172.25 kPa). It is evident from Figs. 5.5 through 5.7 that the predictions from the model compare well with the experimental results.
Figure 5.3. Comparison of Stress-Strain Responses of Hydrostatic Compression Test for the Silty Sand

- Experimental
- Predicted

\[ \gamma_0 = 1.86 \text{ gm/cc} \]

\[ 1.0 \text{ psi} = 6.89 \text{ kpa} \]
\( \gamma_0 = 1.92 \text{ gm/cc} \)

1.0 psi = 6.89 kpa

\[ \text{Strain, } \varepsilon_2 = \varepsilon_3 (\%) \quad \text{Strain, } \varepsilon_1 (\%) \]

Figure 5.4. Comparison of Stress-Strain Responses of Conventional Triaxial Compression (CTC) Test for the Silty Sand (\( \sigma_0 = 10.0 \text{ psi} \))
Figure 5.5. Comparison of Stress-Strain Responses of Simple Shear (SS) Test for the Silty Sand ($\sigma_0 = 20.0$ psi)
Figure 5.6. Comparison of Stress-Strain Responses of Triaxial Compression (TC) Test for the Silty Sand ($\sigma_0 = 25.0$ psi)
Figure 5.7. Comparison of Stress-Strain Responses of Conventional Triaxial Extension (CTE) Test for the Silty Sand ($\sigma_0 = 20.0$ psi)
Comparison of Volumetric Responses

Conventionally, the volumetric strain, \( \varepsilon_V \), is plotted with respect to the axial strain, \( \varepsilon_1 \); in this dissertation, \( \varepsilon_1 \) is the strain in the vertical z-direction as shown in Fig. 4.1 in Chapter 4. For a general loading path, the use of \( \varepsilon_1 \) may not be meaningful. For example, in a conventional triaxial extension (CTE) test, \( \sigma_1 \) is kept constant and \( \sigma_2 \) and \( \sigma_3 \) are increased by the same amount. Thus, for a CTE test, \( \varepsilon_1 \) becomes tensile. On the other hand, for a conventional triaxial compression (CTC) test, \( \varepsilon_1 \) is compressive. These changes in the sign of \( \varepsilon_1 \) for different tests may cause difficulty in interpreting such plots. As alternatives to the axial strain, \( \varepsilon_1 \), the volumetric strain, \( \varepsilon_V \), may be plotted with respect to the trajectory of the total strain, \( \xi_T \), and the ratio \( \sqrt{\frac{J_2D}{J_1}} \) where

\[
\xi_T = \sqrt{\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2}
\]  

(5.11)

and \( J_1 \) is the first invariant of the stress tensor, \( \sigma_{ij} \), and \( J_{2D} \) is the second invariant of the deviatoric stress tensor, \( S_{ij} \) (see Chapter 2).

For purposes of comparison, the volumetric strain, \( \varepsilon_V \), is plotted with respect to all the above three alternatives.

Figures 5.8 through 5.11 show \( \varepsilon_V - \varepsilon_1 \) plots for CTC, SS, TC and CTE tests. Experimental results are also shown on the same plots. Figures 5.12 through 5.15 show the plots of volumetric strain with respect to the trajectory of the strain, \( \xi_T \), for the same tests. Alternatively, the \( \varepsilon_V - \sqrt{\frac{J_2D}{J_1}} \) plots are shown in Figs. 5.16 to 5.19.
Figure 5.8. Comparison of Volumetric Response of Conventional Triaxial Compression (CTC) Test for the Silty Sand ($\sigma_0 = 10.0$ psi)

$\gamma_0 = 1.92$ gm/cc

$1.0$ psi = 6.89 kpa
Figure 5.9. Comparison of Volumetric Response of Simple Shear (SS) Test for the Silty Sand ($\sigma_0 = 20.0$ psi)

- $\gamma_0 = 2.07$ gm/cc
- $1.0$ psi = $6.89$ kpa

End of Hydrostatic Compression
\( \gamma_0 = 2.03 \text{ gm/cc} \)

1.0 psi = 6.89 kpa

Figure 5.10. Comparison of Volumetric Response of Triaxial Compression (TC) Test for the Silty Sand \((\sigma_0 = 25.0 \text{ psi})\)
\[ \gamma_0 = 2.02 \text{ gm/cc} \]
\[ 1.0 \text{ psi} = 6.89 \text{ kpa} \]

Figure 5.11. Comparison of Volumetric Response of Conventional Triaxial Extension (CTE) Test for the Silty Sand (\( \sigma_0 = 20.0 \text{ psi} \))
\[ \gamma_0 = 1.92 \text{ gm/cc} \]

\[ 1.0 \text{ psi} = 6.89 \text{ kpa} \]

Figure 5.12. Plot of \( e_v - \xi_T \) of Conventional Triaxial Compression Test (CTC) Test for the Silty Sand (\( \sigma_0 = 10.0 \text{ psi} \))
\( \gamma_0 = 2.07 \text{ gm/cc} \)

1.0 psi = 6.89 kpa

Figure 5.13. Plot of \( \varepsilon_V - \xi_T \) of Simple Shear (SS) Test for the Silty Sand \((\sigma_0 = 20.0 \text{ psi})\)
Figure 5.14. Plot of $e_V$ - $\xi_T$ of Triaxial Compression (TC) Test for the Silty Sand ($\sigma_0 = 25.0$ psi)

$\gamma_0 = 2.03$ gm/cc

1.0 psi = 6.89 kpa

End of Hydrostatic Compression

Experimental

Predicted
\[ \gamma_0 = 2.02 \text{ gm/cc} \]
\[ 1.0 \text{ psi} = 6.89 \text{ kpa} \]

Figure 5.15. Plot of \( \varepsilon_V - \xi_T \) of Conventional Triaxial Extension (CTE) Test for the Silty Sand \((\sigma_0 = 20.0 \text{ psi})\)
Figure 5.16. Plot of $\varepsilon_V - \sqrt{J_2/D/J_1}$ of Conventional Triaxial Compression (CTC) Test for the Silty Sand ($\sigma_0 = 10.0$ psi)

$\gamma_0 = 1.92$ gm/cc
1.0 psi = 6.89 kpa
\( \gamma_0 = 2.07 \text{ gm/cc} \)

1.0 psi = 6.89 kpa

Figure 5.17. Plot of \( \varepsilon_V - \sqrt{J_{20}/J_1} \) of Simple Shear (SS) Test for the Silty Sand (\( \sigma_0 = 20.0 \text{ psi} \)).
\[ \gamma_0 = 2.03 \text{ gm/cc} \]

\[ 1.0 \text{ psi} = 6.89 \text{ kpa} \]

Figure 5.18. Plot of \( \varepsilon_V - \sqrt{\frac{J_2D}{J_1}} \) of Triaxial Compression (TC) Test for the Silty Sand (\( \sigma_0 = 25.0 \text{ psi} \))
$\gamma_0 = 2.02$ gm/cc
1.0 psi = 6.89 kpa

Figure 5.19. Plot of $\epsilon_V - \sqrt{J_2/J_1}$ of Conventional Triaxial Extension (CTE) Test for the Silty Sand ($\sigma_0 = 20.0$ psi)
It is evident from these plots that the predictions compare well with the experimental data.

Comparison of $r_V - r_D$ Plots

In general, the $r_V - r_D$ plots describe the effect of stress paths during plastic deformation. Details of this plot are given in Chapter 3 along with five such plots obtained for five different soils. In this section, such $r_V - r_D$ response curves are predicted for the four stress paths using the proposed constitutive model.

Figures 5.20 through 5.23 show the predictions of $r_V - r_D$ responses for a CTC test, an SS test, a TC test and a CTE test. The experimental results are also plotted in the same figures. It is observed that good agreement is achieved between the predictions and the experimental results.

Verification with Respect to An Artificial Soil

The material constants for this soil are given in Chapter 4. The description of the soil is also presented in the same chapter. A number of stress-strain response curves are predicted and compared with the experimental results.

Comparison of Stress-Strain Responses

Figure 5.24 shows the comparison between the predicted and observed stress-strain responses for a hydrostatic compression (HC) test. It is seen that the comparison is good.
Figure 5.20. $r_V - r_D$ Plot of Conventional Triaxial Compression (CTC) Test for the Silty Sand ($\sigma_0 = 10.0$ psi)

1.0 psi = 6.89 kpa
Figure 5.21: $r_V - r_D$ Plot of Simple Shear (SS) Test for the Silty Sand ($c_0 = 20.0$ psi)
Figure 5.22. $r_V - r_D$ Plot of Triaxial Compression (TC) Test for the Silty Sand ($\sigma_0 = 25.0$ psi)
Figure 5.23. $r_v - r_D$ Plot of Conventional Triaxial Extension (CTE) Test for the Silty Sand ($\sigma_0 = 20.0$ psi)

1.0 psi = 6.89 kpa
Figure 5.24. Comparison of Stress Strain Responses of Hydrostatic Compression (HC) Test for the Artificial Soil
Figures 5.25 through 5.27 show the stress-strain response curves for conventional triaxial compression (CTC) tests at initial confining pressures of 10 psi (68.9 kPa), 15 psi (103.35 kPa) and 20 psi (137.8 kPa), respectively. In all the figures, the ordinate represents octahedral shear stress, \( \tau_{\text{oct}} \), and the abscissa represents the principal strain components in percent. Experimental results are also plotted in the same figures for comparison purposes. Figure 5.28 shows comparison of the predicted and the observed stress-strain response curves for a simple shear (SS) test. Initial confining pressure for this test is 20 psi (137.8 kPa). The stress-strain response curves for a conventional triaxial extension (CTE) test is shown in Fig. 5.29. The initial confining pressure for this test is 10 psi (68.9 kPa). The experimental results are also plotted in the same figure.

It is evident from Figs. 5.24 through 5.29 that the predicted stress-strain responses are in agreement with the experimental results.

Comparison of Volumetric Responses

Figures 5.30 through 5.34 show plots of \( \varepsilon_V - \varepsilon_T \) for three conventional triaxial compression (CTC) tests, one simple shear (SS) test and one conventional triaxial extension (CTE) test. The specifications of these tests are given in Chapter 4 and will not be repeated here. It is seen from these figures that the model predicts greater dilatancy than observed experimentally for the artificial soil. In spite of this, the correlation between the model predictions and observations is reasonable.
\( \gamma_0 = 2.0 \text{ gm/cc} \)

1.0 psi = 6.89 kpa

\( \varepsilon_2 = \varepsilon_3 (\%) \)

\( \varepsilon_1 (\%) \)

Figure 5.25. Comparison of Stress-Strain Responses of Conventional Triaxial Compression (CTC) Test for the Artificial Soil (\( \sigma_0 = 10.0 \text{ psi} \))
Figure 5.26. Comparison of Stress-Strain Responses of Conventional Triaxial Compression (CTC) Test for the Artificial Soil ($\sigma_0 = 15.0 \text{ psi}$)

\[
y_0 = 2.0 \text{ gm/cc} \\
1.0 \text{ psi} = 6.89 \text{ kpa}
\]
Figure 5.27. Comparison of Stress-Strain Responses of Conventional Triaxial Compression (CTC) Test for the Artificial Soil ($\sigma_0 = 20.0$ psi)
Figure 5.28. Comparison of Stress-Strain Responses of Simple Shear (SS) Test for the Artificial Soil ($\sigma_0 = 20.0$ psi)
Figure 5.29. Comparison of Stress-Strain Responses of Conventional Triaxial Extension (CTE) Test for the Artificial Soil (σ₀ = 10.0 psi)
\[ Y_0 = 2.0 \text{ gm/cc} \]
\[ 1.0 \text{ psi} = 6.89 \text{ kpa} \]

Figure 5.30. Plot of \( \varepsilon_Y \) - \( \xi_T \) of Conventional Triaxial Compression (CTC) Test for the Artificial Soil \( (q_0 = 10.0 \text{ psi}) \)
\( \gamma_0 = 2.0 \text{ gm/cc} \)
1.0 psi = 6.89 kpa

Figure 5.31. Plot of \( \varepsilon_V - \xi_T \) of Conventional Triaxial Compression (CTC) Test for the Artificial Soil (\( \sigma_0 = 15.0 \text{ psi} \))
Figure 5.32. Plot of $\varepsilon_v - \xi_T$ of Conventional Triaxial Compression (CTC) Test for the Artificial Soil ($\sigma_0 = 20.0$ psi)

$\gamma_0 = 2.0$ gm/cc

1.0 psi = 6.89 kpa
Figure 5.33. Plot of $\varepsilon_V - \varepsilon_T$ of Simple Shear (SS) Test for the Artificial Soil ($\sigma_0 = 20.0$ psi)

\[ \gamma_0 = 2.0 \text{ gm/cc} \]
\[ 1.0 \text{ psi} = 6.89 \text{ kpa} \]
Figure 5.34. Plot of $\varepsilon_V - \xi_T$ of Conventional Triaxial Extension (CTE) Test for the Artificial Soil ($\sigma_0 = 10.0$ psi)

$\gamma_0 = 2.0 \text{ gm/cc}$

$1.0 \text{ psi} = 6.89 \text{ kpa}$
Figures 5.35 to 5.39 show the $\epsilon_V - \sqrt{J_2/D_{/J_1}}$ plots for the same tests discussed previously. For comparison purposes, experimental results are also plotted in the same figures. Again, the dilatant behavior predicted by the model is quite different from the observations.

Figures 5.40 through 5.44 show the $r_V - r_D$ plots for the above mentioned tests.

It may be important to note that the artificial soil is highly compressible and shows significant creep behavior when subjected to load. Thus, the total strain obtained from the tests can be expressed as

$$\epsilon = \epsilon^e + \epsilon^I$$  \hspace{1cm} (5.12)

where $\epsilon^I$ denotes the total inelastic strain which can be expressed as

$$\epsilon^I = \epsilon^P + \epsilon^C$$  \hspace{1cm} (5.13)

where $\epsilon^C$ is the creep strain and $\epsilon^P$ is the plastic strain. In general, separation of the inelastic strain, $\epsilon^I$, into plastic and creep parts requires knowledge of defining the creep behavior of the material. The creep behavior of materials is beyond the scope of the present study. Thus, the total inelastic strain, $\epsilon^I$, is considered to be $\epsilon^P$. The inaccuracy in the predictions may be partially attributed to this fact.

Verification with Respect to an Agricultural Soil

A total of seven tests are used to obtain the material constants for this soil. These tests are listed in Table 4.2. The moisture contents and the densities of these tests are also presented in the same
Figure 5.35. Plot of $\varepsilon_v - \sqrt{J_2}/J_1$ of Conventional Triaxial Compression (CTC) Test for the Artificial Soil ($\sigma_0 = 10.0$ psi)

$\gamma_0 = 2.0$ gm/cc
1.0 psi = 6.89 kpa
Figure 5.36. Plot of $\epsilon_v - \sqrt{J_{2D}/J_1}$ of Conventional Triaxial Compression (CTC) Test for the Artificial Soil ($\sigma_0 = 15.0$ psi)

$\gamma_0 = 2.0$ gm/cc

1.0 psi = 6.89 kpa
Figure 5.37. Plot of $\varepsilon_V - \sqrt{J_2/D}/J_1$ of Conventional Triaxial Compression (CTC) Test for the Artificial Soil ($\sigma_0 = 20.0$ psi)
Figure 5.38. Plot of $\varepsilon_v - \sqrt{J_{2D}/J_1}$ of Simple Shear (SS) Test for the Artificial Soil ($\sigma_0 = 20.0$ psi)

- $\gamma_0 = 2.0$ gm/cc
- 1.0 psi = 6.89 kpa
\[ \gamma_0 = 2.0 \text{ gm/cc} \]
\[ 1.0 \text{ psi} = 6.89 \text{ kpa} \]

Figure 5.39. Plot of \( \varepsilon_V - \sqrt{J_2/d} / J_1 \) of Conventional Triaxial Extension (CTE) Test for the Artificial Soil \( (\sigma_0 = 10.0 \text{ psi}) \)
Figure 5.40. $r_V - r_D$ Plot of Conventional Triaxial Compression (CTC) Test for the Artificial Soil ($\sigma_0 = 10.0$ psi)

1.0 psi = 6.89 kpa
Figure 5.41. $r_v - r_d$ Plot of Conventional Triaxial Compression (CTC) Test for the Artificial Soil ($\sigma_0 = 15.0$ psi)

- Experimental
- Predicted

1.0 psi = 6.89 kpa
Figure 5.42. $r_v - r_d$ Plot of Conventional Triaxial Compression (CTC) Test for the Artificial Soil ($\sigma_0 = 20.0$ psi)
Figure 5.43. \( r_v - r_D \) Plot of Simple Shear (SS) Test for the Artificial Soil \( (\sigma_0 = 20.0 \text{ psi}) \)
Figure 5.44. $r_V - r_D$ Plot of Conventional Triaxial Extension (CTE) Test for the Artificial Soil ($\sigma_0 = 10.0$ psi)

1.0 psi = 6.89 kpa
table. The material constants for this soil are given in Chapter 4. The following sections are intended to compare the predicted stress-strain responses with the experimental observations for a number of tests.

Comparison of Stress-Strain Responses

Figure 5.45 shows the comparison of the predicted and the observed stress-strain responses for a hydrostatic compression (HC) test. In Fig. 5.45 the ordinate represents mean pressure while the abscissa represents the volumetric strain in percent. The volumetric strain for this test is determined by summing the three principal strain components.

Figure 5.46 shows the stress-strain responses for a conventional triaxial compression (CTC) test. Initial confining pressure for this test is 5 psi (34.45 kPa). Experimental results are also plotted in the same figure. The ordinate in Fig. 5.46 represents octahedral shear stress and the abscissa represents the strain in percent. It may be noted that the strains are assumed to be positive when compressive.

Figure 5.47 shows the predicted stress-strain response curves for a simple shear test. Experimental results are also plotted in the same figure. Similar comparisons for triaxial compression (TC) test and the triaxial extension (TE) test are shown in Figs. 5.48 and 5.49, respectively. It is evident from the Figs. 5.45 through 5.49 that the model predictions are in close agreement with the observations.
Figure 5.45. Comparison of Stress-Strain Responses of Hydrostatic Compression (HC) Test for the Agricultural Soil

\[ \gamma_0 = 1.37 \text{ gm/cc} \]

\[ 1.0 \text{ psi} = 6.89 \text{ kpa} \]
Figure 5.46. Comparison of Stress-Strain Responses of Conventional Triaxial Compression (CTC) Test for the Agricultural Soil ($\sigma_0 = 5.0 \text{ psi}$)

$1.0 \text{ psi} = 6.89 \text{ kpa}$

$\gamma_0 = 1.36 \text{ gm/cc}$

- - - Experimental

- - - Predicted

Strain, $\varepsilon_2 = \varepsilon_3$ (%) Strain, $\varepsilon_1$ (%)
Figure 5.47. Comparison of Stress-Strain Responses of Simple Shear (SS) Test for the Agricultural Soil ($\sigma_0 = 5.0$ psi)

\[ \gamma_0 = 1.37 \text{ gm/cc} \]

1.0 psi = 6.89 kpa
Figure 5.48. Comparison of Stress-Strain Responses of Triaxial Compression (TC) Test for the Agricultural Soil ($\sigma_0 = 5.0$ psi)

$\gamma_0 = 1.37$ gm/cc

$1.0$ psi = $6.89$ kpa
Figure 5.49. Comparison of Stress-Strain Responses of Triaxial Extension (TE) Test for the Agricultural Soil ($\sigma_0 = 5.0$ psi)

\[
Y_0 = 1.39 \text{ gm/cc} \quad 1.0 \text{ psi} = 6.89 \text{ kpa}
\]
Comparison of Volumetric Responses

Figures 5.50 through 5.53 show the $\varepsilon_V - \varepsilon_T$ plots for a CTC, SS, TC and TE test, respectively. An alternative to this is $\varepsilon_V - \sqrt{J_{2D}/J_1}$ plot. Figures 5.54 to 5.57 show the $\varepsilon_V - \sqrt{J_{2D}/J_1}$ plots for the same tests. Experimental results are also plotted for comparison purposes. It is seen that a reasonable agreement is reached between the predictions and the observations.

Comments

Three different soils are considered for the verification of the proposed model. The material constants associated with the model are determined for each soil using a number of different tests for that soil. These material constants are used to back-predict the stress-strain responses of a selected number of tests for every soil. It is evident from these back-predictions that the model can represent the responses of geological materials with reasonable accuracy.
\[ \gamma_0 = 1.36 \text{ gm/cc} \]
\[ 1.0 \text{ psi} = 6.89 \text{ kpa} \]

Figure 5.50. Plot of \( \varepsilon_V - \xi_T \) of Conventional Triaxial Compression (CTC) Test for the Agricultural Soil \((\sigma_0 = 5.0 \text{ psi})\)
Figure 5.51. Plot of $\epsilon_V - \xi$ of Simple Shear (SS) Test for the Agricultural Soil ($\sigma_0 = 5.0$ psi)

$\gamma_0 = 1.37$ gm/cc

1.0 psi = 6.89 kpa
\( \gamma_0 = 1.37 \text{ gm/cc} \)

1.0 psi = 6.89 kpa

Figure 5.52. Plot of \( \varepsilon_V - \varepsilon_T \) of Triaxial Compression (TC) Test for the Agricultural Soil (\( \sigma_0 = 5.0 \text{ psi} \))
Figure 5.53. Plot of $\varepsilon_v - \xi_T$ of Triaxial Extension (TE) Test for the Agricultural Soil ($\sigma_0 = 5.0$ psi)

\[ \gamma_0 = 1.39 \text{ gm/cc} \]
\[ 1.0 \text{ psi} = 6.89 \text{ kpa} \]
Figure 5.54. Plot of $\varepsilon_V - \sqrt{J_{2D}/J_1}$ of Conventional Triaxial Compression (CTC) Test for the Agricultural Soil ($\sigma_0 = 5.0$ psi)

$\gamma_0 = 1.36$ gm/cc

1.0 psi = 6.89 kpa
\[ \gamma_0 = 1.37 \text{ gm/cc} \]
\[ 1.0 \text{ psi} = 6.89 \text{ kpa} \]

Figure 5.55. Plot of \( \varepsilon_V - \sqrt{J_2}/J_1 \) of Simple Shear (SS) Test for the Agricultural Soil \( (\sigma_0 = 5.0 \text{ psi}) \)
Figure 5.56. Plot of $\varepsilon_v - \sqrt{J_{2D}/J_1}$ of Triaxial Compression (TC) Test for the Agricultural Soil ($\sigma_0 = 5.0$ psi)

$\gamma_0 = 1.37$ gm/cc
1.0 psi = 6.89 kpa
Figure 5.57. Plot of $\varepsilon_Y - \sqrt{J_{20}/J_1}$ of Triaxial Extension (TE) Test for the Agricultural Soil ($\sigma_0 = 5.0$ psi)

$\gamma_0 = 1.39$ gm/cc

1.0 psi = 6.89 kpa
CHAPTER 6

FINITE ELEMENT METHOD

General

Most soil-structure interaction problems respond nonlinearly when subjected to external loads. In general, such nonlinearity can be classified into three groups: 1) geometric nonlinearity which arises from the nonlinear terms in the strain-displacement relationships, 2) material nonlinearity which is caused by the nonlinearity of the constitutive equations, and 3) boundary nonlinearity which arises because of the nonlinear interaction effects between two bodies in contact. In geomechanics such nonlinearity is also caused by the presence of joints, weak planes, faults, etc.

For most soil-structure interaction problems, closed-form solutions cannot be obtained. Thus, numerical techniques are often used. Among all available numerical techniques, the finite element method is one of the most efficient and versatile numerical methods. The versatility of the finite element method derives mainly from its capability of including nonlinearity in the solution of boundary value problems in a simple manner. Besides, the method is also capable of handling inhomogeneity and irregular geometry quite effectively. Furthermore, in geomechanics, the method allows one to include such factors as in situ stress, different types of loading, stress paths and interface conditions.

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Many boundary value problems in geomechanics are truly three-dimensional in nature. Thus, two-dimensional idealizations of this class of problems may not yield sufficiently accurate solutions when the finite element technique is used. Thus, three-dimensional finite element procedure is often required for some problems in geomechanics.

Three-dimensional finite element procedures allowing for material, geometric and boundary nonlinearities for problems in geomechanics have been developed and described by Phan (1979), Desai et al. (1982), Siriwardane (1980). The procedure and code used in this study are derived from these works with a number of modifications, particularly the implementation of the proposed constitutive model. The following sections are devoted to describing different aspects of the three-dimensional finite element procedure.

**Finite Element Formulation**

Nonlinear problems are often solved using an incremental method. In this method, the total load is discretized into a number of small increments, say N. The solution at every load increment is obtained by assuming the system to be linear within that load increment. The final solution is then obtained by summing up all the incremental solutions.

Most soil-structure interaction problems are stress path dependent; that is, the response of the system is dependent upon the past history of loading. Thus, the current solution of the problem depends on the known values of the static and the kinematic variables at the end of the previous load step when incremental method is used.
Let \( n \) and \( n+1 \) be two consecutive load steps such that all the static and kinematic variables are known at \( n \) (see Fig. 6.1). The objective is to obtain solutions for these variables at load step \( n+1 \). For quasi-static problems under isothermal condition, the static and kinematic variables are the stresses, \( \sigma_{ij} \), the strains, \( \varepsilon_{ij} \), and the displacements, \( u_i \). Thus, \( \sigma_{ij}^n, \varepsilon_{ij}^n, u_i^n \) are known and \( \sigma_{ij}^{n+1}, \varepsilon_{ij}^{n+1}, u_i^{n+1} \) are unknown where the superscripts \( n \) and \( n+1 \) denote the load steps at which they are calculated (see Fig. 6.1). Assuming the load increments to be small, \( \sigma_{ij}^{n+1}, \varepsilon_{ij}^{n+1}, u_i^{n+1} \) can be written in terms of \( \sigma_{ij}^n, \varepsilon_{ij}^n, u_i^n \) as follows:

\[
\begin{align*}
\sigma_{ij}^{n+1} &= \sigma_{ij}^n + \Delta \sigma_{ij} \\
\varepsilon_{ij}^{n+1} &= \varepsilon_{ij}^n + \Delta \varepsilon_{ij} \\
u_i^{n+1} &= u_i^n + \Delta u_i
\end{align*}
\] (6.1)

where \( \Delta \) denotes incremental quantity from step \( n \) to \( n+1 \). Thus, \( \Delta \sigma_{ij}, \Delta \varepsilon_{ij} \) and \( \Delta u_i \) are the basic unknowns between two consecutive load steps. Once these quantities are determined, Eqs. (6.1) can be used to obtain the solutions at step \( n+1 \).

It is important to note here that constitutive equations are required to obtain solution of any boundary value problem. For a nonlinear and inelastic system, the constitutive equations are also history dependent. Thus, for any load step \( n+1 \), the constitutive equations are also
Figure 6.1. Schematic of the Configurations of a Body at Load Steps n and n+1
unknown. Usually, this difficulty is avoided by evaluating the constitutive equations at the end of the previous load step, \( n \). Thus, the relation between \( \Delta \sigma_{ij} \) and \( \Delta \varepsilon_{ij} \) can be written as

\[
\Delta \sigma_{ij} = C_{ijkl}^n \Delta \varepsilon_{kl}^n
\]

(6.2)

where \( C_{ijkl}^n \) is the constitutive relation tensor at load step \( n \).

**Principle of Virtual Work**

Within the context of small deformation assumption, the principle of virtual work at step \( n+1 \) can be written as (Desai and Abel, 1972)

\[
\int_V C_{ij}^{n+1} \delta \varepsilon_{ij}^{n+1} dV = \int_V f_i^{n+1} \delta u_i^{n+1} dV + \int_S f_i^{n+1} \delta u_i^{n+1} dS
\]

(6.3)

where \( f_i \) denotes the prescribed body forces, \( \overline{T}_i \) denotes the prescribed surface tractions, \( V \) is the undeformed volume of the system, \( S \) is the undeformed surface area of the system and \( \delta \) denotes variation. Referring to Eqs. (6.1), \( \delta \varepsilon_{ij}^{n+1} \) and \( \delta u_i^{n+1} \) can be reduced to the forms

\[
\delta \varepsilon_{ij}^{n+1} = \delta \Delta \varepsilon_{ij}
\]

(6.4)

\[
\delta u_i^{n+1} = \delta \Delta u_i
\]
It should be mentioned here that the variations of \( \varepsilon_{ij}^n \) and \( u_i^n \) vanish because these are known quantities. Using Eqs. (6.1) and (6.4), the principle of virtual work can now be expressed in the form

\[
\int_V (\sigma_{ij}^n + \Delta \sigma_{ij}) \delta \Delta \varepsilon_{ij} \, dV = \int_V T_{i}^{n+1} \delta \Delta u_i \, dV + \int_S T_{i}^{n+1} \delta \Delta u_i \, dS
\]

(6.5)

Rearranging the terms, Eq. (6.5) can be written as

\[
\int_V \Delta \sigma_{ij} \delta \Delta \varepsilon_{ij} \, dV = \delta \Delta W - \int_V \sigma_{ij}^n \delta \Delta \varepsilon_{ij} \, dV
\]

(6.6)

where

\[
\delta \Delta W = \int_V T_{i}^{n+1} \delta \Delta u_i \, dV + \int_S T_{i}^{n+1} \delta \Delta u_i \, dS
\]

(6.7)

Using the constitutive equations given by Eq. (6.2), the principle of virtual work at step \( n+1 \) can be written as

\[
\int_V C_{ijkl} \Delta \varepsilon_{kl} \delta \Delta \varepsilon_{ij} \, dV = \delta \Delta W - \int_V \sigma_{ij}^n \delta \Delta \varepsilon_{ij} \, dV
\]

(6.8)

In matrix form,
\[
\int_\Omega \delta \{\Delta \varepsilon\}^T [C]^n \{\Delta \varepsilon\} \, dV = \int_\Omega \delta \{\Delta u\}^T \{\mathbf{T}\}^{n+1} \, dV + \\
\int_\partial \Omega \delta \{\Delta u\}^T \{\mathbf{T}\}^{n+1} \, dS - \int_\Omega \delta \{\Delta \varepsilon\}^T \{\sigma\}^n \, dV
\]

where \(\{\Delta \varepsilon\}\) is strain vector of order \(6 \times 1\), \([C]^n\) is the \(6 \times 6\) constitutive relation matrix at load step \(n\), \(\{\Delta u\}\) is the \(3 \times 1\) displacement vector, \(\{\mathbf{T}\}^{n+1}\) and \(\{\mathbf{T}\}^{n+1}\) denote the body force and surface traction vectors of order \(3 \times 1\), respectively, \(\{\sigma\}^n\) is the stress vector of order \(6 \times 1\) at step \(n\) and the superscript \(T\) denotes transpose.

Equation (6.9) represents the virtual work principle at load step \(n+1\). Finite element equations are now developed using this equation.

Finite Element Equations

Equation (6.9) is valid at any point in the continuum. In the finite element method, the domain is discretized into a number of finite elements as shown in Fig. 6.2 and the unknowns are evaluated at the nodal points of these elements. Thus, it is necessary to express Eq. (6.9) in terms of the nodal unknowns for each element. A brief discussion of how the finite element equations are formulated is given here. Details can be found in standard texts on the finite element method (Zienkiewicz, 1977; Gallagher, 1975; Desai and Abel, 1972; Bathe and Wilson, 1976).

Within an element, the incremental displacement, \(\Delta u_i\), is expressed in terms of the nodal displacements; that is,
Figure 6.2. Typical Discretization of a Domain Using Finite Elements
\[ \Delta u_i = N_m \Delta q_i^m \] (6.10)

where \( m \) is the nodal index and ranges from 1 to \( M \) which is equal to the total number of nodes in the element, \( N_m \) is the interpolation function at node \( m \) and \( \Delta q_i^m \) is the incremental displacement of node \( m \) in the direction \( i \). It may be noted that the repeated index, \( m \), in Eq. (6.10) represents summation over all the nodes (\( M \)) of the element. Thus, Eq. (6.10) can be written as

\[ \Delta u_i = N_1 \Delta q_i^1 + N_2 \Delta q_i^2 + \ldots + N_M \Delta q_i^M \] (6.11)

Equation (6.10) can also be written in matrix notation as

\[ \{\Delta u\} = [N] \{\Delta q\} \] (6.12)

where \( \{\Delta q\} \) is a vector of order \( 3M \times 1 \) and \( [N] \) is a matrix of order \( 3 \times 3M \).

Assuming small deformations, the incremental strains can be written as

\[ \Delta \varepsilon_{ij} = \frac{1}{2} \left[ \frac{\partial \Delta u_i}{\partial x_j} + \frac{\partial \Delta u_j}{\partial x_i} \right] \] (6.13)

where \( x_i \) refer to the coordinate axes (see Fig. 6.1).

Substituting Eq. (6.10) into Eq. (6.13), the incremental strain from step \( n \) to \( n+1 \) is expressed as
\[
\Delta e_{ij} = \frac{1}{2} \left[ \frac{\partial N_m}{\partial x_i} \Delta q_i^m + \frac{\partial N_m}{\partial x_j} \Delta q_j^m \right] \quad (6.14)
\]

In matrix notation

\[
\{\Delta \varepsilon\} = [B] \{\Delta \mathbf{q}\} \quad (6.15)
\]

where \([B]\) is the strain-displacement transformation matrix of order \(6 \times 3M\).

Using the Eqs. (6.12) and (6.15), the virtual work principle can be written as

\[
\delta \{\Delta \mathbf{q}\}^T \left( \int [B]^T [C]^n [B] \, dV \right) \{\Delta \mathbf{q}\} = \\
\delta \{\Delta \mathbf{q}\}^T \left( \int [N]^T \{f\}^{n+1} \, dV + \int [N]^T \{t\}^{n+1} \, dS \right) \\
-\delta \{\Delta \mathbf{q}\}^T \left( \int [B]^T \{\sigma\}^n \, dV \right) 
\]

Rearranging terms in Eq. (6.16)

\[
\delta \{\Delta \mathbf{q}\}^T ([k]^n \{\Delta \mathbf{q}\} = \delta \{\Delta \mathbf{q}\}^T ([q]^{n+1} - \{q_o\}^n) \quad (6.17)
\]

where
In Eq. 6.17, \([k]^n\) is the element stiffness matrix at step \(n\), \({Q}^{n+1}\) is external load vector at step \(n+1\) and \({Q}_o^n\) is the internal load vector at step \(n\). Since the virtual displacement is arbitrary, \(\delta \{\Delta q\}^T\) can be cancelled out from both sides of Eq. 6.17. The final equation can be written in the form

\[
[k]^n \{\Delta q\} = \{Q\}^{n+1} - \{Q}_o^n
\] (6.21)

Equation 6.21 shows the stiffness equations for a single finite element only. To solve any boundary value problem, the stiffnesses of all the elements are calculated and assembled to obtain global stiffness equations; that is,

\[
[K]^n \{\Delta r\} = \{R\}^{n+1} - \{R}_o^n
\] (6.22)

where \(\{\Delta r\}\) denotes the vector of all the nodal unknowns of the entire system. The load vectors \(\{R\}^{n+1}\) and \(\{R}_o^n\) are defined as
\[ \{R\}^{n+1} = \sum_{i=1}^{N_e} \{Q\}^{n+1}_i \]

\[ \{R_0\}^n = \sum_{i=1}^{N_e} \{Q_0\}^n_i \]

where \(N_e\) denotes the total number of elements of the system.

Equation (6.22) represents a system of simultaneous linear equations which can be solved by using any numerical scheme such as the Gaussian elimination. However, appropriate boundary conditions are to be introduced before solving them.

It is important to note that the derivation of the stiffness matrix, \([k]^n\) and the load vectors, \(\{Q\}^{n+1}\) and \(\{Q_0\}^n\) require integration over the domain of the element. Usually these integrations are difficult to perform by hand. Thus, for most problems, numerical integration is used. Details of numerical integration schemes are given by Zienkiewicz (1977) and Desai and Abel (1972).

Variable-Node Hexahedral Element

A variable-node isoparametric hexahedral element is used to discretize a three-dimensional continuum. The number of nodes can vary from 8 to 21 as shown in Fig. 6.3. Details of this element are given by Bathe and Wilson (1976).

Although the element can have any number of nodes between 8 to 21, in this dissertation only eight-noded brick elements are used.
Figure 6.3. 8 to 21 - Node Isoparametric Element
Nonlinear Numerical Techniques

There are basically two methods which are used in the nonlinear finite element analyses; namely, the incremental method and the iterative method. Brief descriptions of these methods are given below:

Incremental Method

In the incremental method, the load is divided into a number of increments, say N, as shown in Fig. 6.4. To compute the response of the system at a load step n+1, the stiffness matrix at step n is used. Thus, equilibrium is not satisfied exactly. However, accuracy of this technique depends on the size of the load increments and the type of nonlinearity. Figure 6.4 shows the schematic of incremental solution technique.

It is important to note that the incremental technique provides the whole history of deformation in contrast to iterative technique which yields only the final solution of the problem. The following section is devoted to describing the iterative technique.

Iterative Method

In the iterative method, the total load is applied at a time and iterations are performed to satisfy equilibrium. There are various iterative schemes that can be used to satisfy equilibrium. Of them, the Newton-Raphson method is commonly used. In this method, the stiffness is modified at the beginning of every iteration as shown in Fig. 6.5.
Figure 6.4. Symbolic of the Incremental Technique of Nonlinear Analysis
Figure 6.5. Schematic of the Newton-Raphson Iterative Technique

Figure 6.6. Schematic of the 'Initial Stress' Iterative Technique
As an alternative to this, Zienkiewicz et al. (1969) proposed an approach called the 'initial stress' method, also known as modified Newton-Raphson method. In this method, the stiffness matrix is kept constant while the load vector is modified at the beginning of every iteration. A schematic of 'initial stress' method is shown in Fig. 6.6. It may be noted that for the same convergence tolerance, number of iterations required for 'initial stress' method is higher than that required by Newton-Raphson method (Desai and Abel, 1972).

Although the iterative scheme satisfies equilibrium quite accurately, it does not yield the history of deformation. It only provides the final response of the system. Thus, often elasto-plastic analyses are performed using a combination of the incremental technique and the iterative technique. Figure 6.7 shows a schematic of the incremental-iterative scheme.

In the present study, an incremental-iterative scheme is adopted such that the stiffness matrix is modified at every equilibrium iteration.

**Interaction Effect**

In the finite element method, the body under consideration is assumed to be continuum. This is valid for domains which consist of only one type of material and do not contain any joints or discontinuities. Most problems of structural mechanics can be treated as continuum problems. Soil-structure interaction problems, on the other hand, often include such discontinuities in the form of joints or interfaces; that
Figure 6.7. Schematic of the Incremental-Iterative Technique

Load, $Q$

Displacement, $u$

$\Delta Q_i$

$\Delta Q_1$

Actual Load-Displacement Curve

$i \text{ goes from } 1 \text{ to } N$
is, contact surface between two dissimilar materials. Figure 6.8 shows some examples of interfaces and joints in soil-structure interaction problems.

In an actual deformation process, the interface region may experience relative slip or separation (Desai, 1981; Desai et al. 1983). Thus, the interface region needs special elements to model.

There are a number of interface elements available at this time to use in the finite element analysis (Goodman et al. 1968; Ghaboussi et al. 1973; Desai, 1974 and 1975; Herrmann, 1978; Katona, 1978; Heuze and Barbour, 1982; Desai, 1981; Desai et al. 1983; Faruque, 1980; Siriwardane, 1980; Lightner, 1981; Zaman, 1982).

In the present study, a three-dimensional 8-noded thin interface element is used. The thin-layer interface element is proposed by Desai (1981). Subsequently, this element has been used in a number of static and dynamic analysis (Desai et al. 1983; Siriwardane, 1980; Lightner, 1981; Zaman, 1982). Figure 6.9 shows an 8-noded three-dimensional interface element used in the present analysis.

The important features of this interface element are the following:

1) The thickness of the interface element is small compared to the other dimensions. However, a parametric study is required to select an appropriate thickness for the interface element. It is found that an aspect ratio between 0.01 and
Figure 6.8. Typical Soil-Structure Interaction Problems Showing Joints and Interfaces
Figure 6.9. A Typical 8-Noded Three-Dimensional Interface Element
Figure 6.10. Typical Shear Stress vs Relative Displacement Curve Obtained from a Direct Shear Test
0.001 yields reasonable solution. The aspect ratio is defined by $t_i/B$ where $t_i$ is the thickness and $B$ is the average width of the interface element.

2) In general, the stiffness properties of the interface element are quite different from the properties of the adjacent continuum elements. In this study it is assumed that the normal stiffness of the interface element is the same as other regular soil elements; however, the shearing stiffness, $K_G$, in the plane of the interface is different. In general, the shearing stiffness of the interface is determined by performing a series of direct shear tests at various normal loads. For each test, shear stress, $\tau$, is plotted as a function of the relative displacement, $u$, as shown in Fig. 6.10. The slope of the $\tau$-$u$ curve provides the shearing stiffness, $K_G$, of the interface. Assuming the thickness of the interface to be $t_i$, the shearing modulus, $G_{int}$, for the interface can be written as

$$G_{int} = K_G \cdot t_i$$  \hspace{1cm} (6.24)

3) To check the conditions of the interface element, the normal stress, $\sigma_n$, and the shear stress, $\tau$, on the plane of the interface element are calculated. With these values of $\sigma_n$ and $\tau$, three conditions are checked; that is,
\[ \sigma_n < 0 \] (6.25)

\[ \sigma_n > 0; \tau > C_a + \sigma_n \tan \phi \] (6.26)

\[ \sigma_n > 0; \tau < C_a + \sigma_n \tan \phi \] (6.27)

where \( C_a \) is the cohesion or adhesion and \( \phi \) is the friction angle for the interface material. In the case of interfaces between the dissimilar materials, \( \phi \) is the angle friction between these materials and can be obtained from direct shear tests.

The inequality \( \sigma_n < 0 \) (assuming compressive stress to be positive) signifies that the interface element is in separation mode. When this condition occurs, both the normal stiffnesses of the interface element are set equal to a small number. In this study, this is defined by dividing the original stiffness by 500. When \( \sigma_n > 0 \) and \( \tau > C_a + \sigma_n \tan \phi \), the interface element is in a state of slip; however, no separation has occurred. For this case, the shearing stiffnesses are reduced to small values; however, the normal stiffnesses are kept the same.

The inequalities, \( \sigma_n > 0 \) and \( \tau < C_a + \sigma_n \tan \phi \) signify that the interface element has suffered neither separation nor slip. In this case, no changes are made in the values of the stiffnesses of the interface element.
Implementation of the Proposed Constitutive Model

The proposed constitutive model is implemented into a three-dimensional finite element procedure developed by Phan (1979) and Desai et al. (1982). Although the procedure is capable of including both material and geometric nonlinearities, only material nonlinearity is considered for the present study. Besides the proposed model, the procedure also includes the following constitutive models:

1) Linear elastic model
2) Drucker-Prager model
3) Critical state model
4) Cap model (Sandler type)
5) Modified Cap model.

Descriptions of these models are given in Chapter 2.

This finite element procedure is used to solve a number of boundary value problems using various constitutive laws. The following chapter is devoted to describing the results of a number of boundary value problems.
CHAPTER 7

APPLICATIONS

General

As discussed previously, for many problems of Geomechanics closed form solutions cannot be obtained. Thus numerical techniques are often used for the analyses of this class of problems. Among all the Numerical methods, the finite element method is extensively used for the problems of Geomechanics. Main reason for this is that the finite element method can easily include such complexities as material nonlinearity, stress path dependency, inhomogeneity, anisotropy, irregular boundaries etc.

The finite element procedure described in Chapter 6 is used to obtain solutions for a number of boundary value problems in Geomechanics. This Chapter presents the results of the finite element analyses along with the corresponding experimental observations.

Simulation of the Triaxial Tests by the Finite Element Method

A number of triaxial tests on a Silty sand (as described in Chapter 4) are simulated by using the finite element procedure and the proposed constitutive model. The material constants for the Silty sand are also listed in Chapter 4. Since the state of stress and the strain
can be assumed to be homogeneous within the triaxial specimen (although local inhomogeneity may occur due to the directional nature of the plastic straining as described by Mroz, 1963), the sample can be represented by a single three-dimensional brick element. Because of the symmetry conditions about all three axes, only 1/8 th of the sample is considered as shown in Fig. 7.1. The boundary conditions are also shown in the same figure.

Hydrostatic Compression (HC) Test

This test is simulated up to a mean (confining) pressure of 80.0 psi (551.2 kpa) with increments of 5.0 psi (34.45 kpa). To satisfy equilibrium of the overall system only two iterations are performed at each increment level. Figure 7.2 shows the comparison of the prediction and the experimental result for the hydrostatic compression (HC) test. It is seen that a reasonable agreement is reached between the prediction and the observation.

Conventional Triaxial Compression (CTC) Test

The initial confining pressure for this test is 10 psi (68.9 kpa). To simulate this test, the sample is loaded hydrostatically up to 10 psi (68.9 kpa) and sheared following the conventional triaxial compression (CTC) test path. That is, the lateral stresses, $\sigma_2$ and $\sigma_3$, are kept constant while the vertical stress, $\sigma_1$, is increased. Two increments are used to simulate the 10 psi (68.9 kpa) initial confining pressure.
Figure 7.1. Finite Element Mesh to Model the Cubical Soil Specimen
Figure 7.2. Finite Element Simulation of Hydrostatic Compression (HC) Test for the Silty Sand
Subsequently, the vertical stress, $\sigma_1$, is increased with increments of 3.0 psi (20.67 kpa).

Figure 7.3 shows the comparison of the finite element results and the experimental results for the conventional triaxial compression (CTC) test. It is seen that the prediction is in agreement with the observation. It may be noted that for the CTC test the finite element prediction is less accurate than the prediction made in Chapter 5. This may be explained by the facts that in the finite element analysis the sizes of the stress increments are larger and only two iterations are performed to satisfy equilibrium at each increment level.

**Simple Shear (SS) Test**

This test is performed at an initial confining pressure of 20.0 psi (137.8 kpa). Thus the sample is loaded hydrostatically up to 20.0 psi (137.8 kpa) and then sheared following the simple shear (SS) test path. In the simple shear test, $\sigma_2$ is kept constant at the initial confining pressure. The vertical stress, $\sigma_1$, is increased while $\sigma_3$ is decreased by the same amount.

Hydrostatic phase of the test (up to 20.0 psi confining pressure) is simulated with increments of 5.0 psi (34.45 kpa). During Shearing the vertical stress, $\sigma_1$, is increased in increments of 2.0 psi (13.78 kpa) and $\sigma_3$ is decreased by the same amount.

To satisfy the overall equilibrium of the sample, four iterations are performed at each increment level. It is observed that for any
1.0 psi = 6.89 kpa

\[ \gamma_0 = 1.92 \text{ gm/cc} \]

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Experimental

---

Predicted

---

Figure 7.3. Finite Element Simulation of Conventional Triaxial Compression (CTC) Test for the Silty Sand (\( \sigma_0 = 10.0 \text{ psi} \))
test other than the hydrostatic compression test, equilibrium does not satisfy with reasonable accuracy if the number of iterations per load step is less than four. This observation, however, may not be valid for all soils. In general, this should be established when dealing with a new soil.

Figure 7.4. shows the stress-strain response curves for the simple shear (SS) test obtained from the finite element analysis. Experimental results are also plotted in the same figure for comparison.

Conventional Triaxial Extension (CTE) Test

This test is performed at an initial confining pressure of 20.0 psi (137.8 kpa). Four 5.0 psi (34.45 kpa) increments of stress are used to load the sample hydrostatically up to 20.0 psi (137.8 kpa). During shearing, the vertical stress, $\sigma_1$, is kept constant at 20.0 psi (137.8 kpa) while $\sigma_2$ and $\sigma_3$ are increased equally using 3.0 psi (20.67 kpa) stress increments.

Figure 7.5 shows the predicted stress-strain response curves for the conventional triaxial extension test obtained from the finite element analysis. Experimental results are also plotted in the same figure. It is seen that the comparison between the prediction and the observation is reasonably accurate.

Strip Footing Problem

In general, a strip footing problem can be considered as a two-dimensional plane strain problem. In this study, however, a three-
Figure 7.4. Finite Element Simulation of Simple Shear (SS) Test for the Silty Sand ($\sigma_0 = 20.0$ psi)

- Experimental
- Predicted

\[
\gamma_0 = 2.07 \text{ gm/cc}
\]
\[
1.0 \text{ psi} = 6.89 \text{ kpa}
\]
Figure 7.5. Finite Element Simulation of Conventional Triaxial Extension (CTE) Test for the Silty Sand ($\sigma_0 = 20.0$ psi)

\[ \gamma_0 = 2.02 \text{ gm/cc} \]
\[ 1.0 \text{ psi} = 6.89 \text{ kpa} \]
dimensional finite element procedure is used to analyze this strip footing problem. The plane strain condition is simulated in the 3-D finite element procedure by constraining the nodes in the direction normal to the plane of deformation.

Phan (1979) analyzed this strip footing problem using a number of different constitutive models and compared these results with the laboratory observations. The soil used for this study is the artificial soil as described in Chapter 4 of this dissertation. The density of the soil used for the laboratory testing is 2 gm/cc. Details of the Device and the testing program are given by Phan (1979) and Desai et al (1982) and will not be repeated here.

In this study the same problem is analyzed using the proposed constitutive model. The results are compared with the laboratory observation. To verify the effectiveness of the proposed model, the finite element results obtained by using the Cap model (DiMaggio and Sandler, 1971) is also compared with the current solution.

Figure 7.6a shows the dimensions of the rigid footing and the soil used for the laboratory testing. The width of the footing is 3.0 inches while the thickness is 0.75 inches.

The finite element mesh for this problem is shown in Fig. 7.6b. Since the footing is symmetric about the vertical axis passing through the center of the footing, only one half of the domain is discretized and used for the finite element analysis. A total of 21 eight-noded
Figure 7.6. Dimensions of the Test Model and the Finite Element Mesh for the Strip Footing Problem
brick elements are used to model the footing and the surrounding soil. The corresponding number of nodes is 64. In the present analysis a total of six load increments are used with two iterations per load step. This is consistent with the analyses performed by Phan (1979).

The material constants used for the present analysis are the following (also presented in Chapter 4):

\[ E = 4000 \text{ psi (27560 kpa)} \]
\[ \nu = 0.35 \]
\[ \alpha = 0.162 \]
\[ \gamma = 1.542 \text{ psi (10.6 kpa)} \]
\[ k = 0.0 \text{ psi (0.0 kpa)} \]
\[ \beta_a = 0.00217 \]
\[ \eta_1 = 1.376 \]
\[ \beta_b = 0.723 \]
\[ \eta_2 = 0.660 \]

Figure 7.7 shows the comparison of the load-displacement response obtained from the present analysis and the laboratory observation. The finite element analysis of the same problem using the Cap model is also plotted in the same figure for comparison (Phan, 1979). It is observed that the present analysis is in close agreement with the experimental results and the finite element results using Cap model.

Figures 7.8 and 7.9 show the displacement fields of the soil medium at 7.5 psi (51.68 kpa) and 15.0 psi (103.35 kpa), respectively.
Figure 7.7. Load-Displacement Curve for the Strip Footing
SCALES
Coordinate: 1 inch = 2.57 inches
Displacement: 1 inch = 0.40 inches

Figure 7.8. Displacement Field at $p = 7.5$ psi (51.68 kpa)
Figure 7.9. Displacement Field at $p = 15.0$ psi (103.35 kPa)
It is seen from these figures that the directions of the movement of the soil particles bend towards the right as the load is increased.

Three-Dimensional Soil-Tool Interaction

The behavior of a tool moving in soft soil is complex because of the three-dimensional interaction effects and the nonlinear and inelastic characteristics of the soil mass. There is no closed form solutions for such problems. Thus numerical methods such as the finite element method is often used to obtain solutions for these problems.

Prototype tests are conducted in a soil-bin facility to determine the draft force (force acting opposite to the direction of the movement of the tool) on the tool as a function of the resultant horizontal displacement (Desai et al, 1982). It may be noted that the artificial soil described in Chapter 4 is used for the testing purpose. The approximate density of the soil was about 2.0 gm/cc. Details of the tests and the testing facility are given by Durant (1979) and Durant et al (1979).

All tools used for the tests are made of cold-rolled steel 0.5 inches thick. The height and the width of the tools are varied for different tests.

In the present study, one particular test is simulated using the finite element procedure mentioned earlier. The proposed constitutive model is used to characterize the nonlinear and inelastic behavior of the artificial soil. The material constants for this soil are given in Chapter 4 and will not be repeated here. The tool is 2X4 inches in
dimensions as shown in Fig. 7.10. The finite element mesh used for the current study is also shown in Fig. 7.10. There are 300 nodes and 180 eight-noded isoparametric hexahedral elements including 7 thin-layer three-dimensional interface elements. The degrees-of-freedom in the x-direction (the direction of the movement of the tool) at the far ends of the mesh are constrained. However, it is found that the effect of fixing the degrees-of-freedom in this direction do not cause significant difference in the results in contrast to releasing these degrees-of-freedom. This may be due to the fact that the significant deformations occur only in the vicinity of the tool.

Figure 7.11 shows the comparison of the load-displacement response obtained from the present study with the experimental results. The load-displacement response obtained from finite element analysis using Cap Model is also plotted in the same figure for comparison (Phan, 1979). It is evident from Fig. 7.11 that the response obtained from the present analysis agrees with the Cap Model solutions. However, there is significant difference between the observed response and the present analysis. This may be due to the fact that only two equilibrium iterations are performed at each load level.

In order to understand the mechanism of soil-structure interaction, the displacements of the soil body and the tool are plotted at various sections for a load of 100 lbs. Figures 7.12 through 7.16 show the displacement fields at various sections.
Figure 7.10. Finite Element Mesh for the Soil-Tool Interaction Problem
Figure 7.11. Load vs Displacement Plot for the Tool. A Comparison with the Laboratory Observation
Figure 7.12. Displacement Field at Section $A_1 - A_2 - A_3 - A_4$
Figure 7.13. Displacement Field at Section $B_1 - B_2 - B_3 - B_4$

SCALES
Coordinate : 1 inch = 5.0 inches
Displacement : 1 inch = 1.0 inches
Figure 7.14. Displacement Field at Section $C_1 - C_2 - C_3 - C_4$

SCALES
Coordinate: 1 inch = 5.0 inches
Displacement: 1 inch = 1.0 inches
Figure 7.15. Displacement Field at Section $A_1 - A_2 - D_2 - D_1$

SCALES
Coordinate: 1 inch = 5.0 inches
Displacement: 1 inch = 1.0 inches
Figure 7.16. Displacement Field at Section $E_1 - E_2 - E_3 - E_4$

SCALES
Coordinate : 1 inch = 5.0 inches
Displacement : 1 inch = 1.0 inches
The general principles of mechanics are applicable to all materials, irrespective of their internal constitution. These principles alone do not provide sufficient number of equations to obtain solution of any boundary value problem. Additional equations are obtained by considering the internal constitution of the material.

There are many groups of constitutive theory, details of which are beyond the scope of the present study. Among them, the theory of plasticity describes the rate independent nonlinear and inelastic behavior of materials.

Geological materials often show nonlinear and inelastic response when subjected to external loads. Thus, the theory of plasticity can be effectively used to characterize the constitutive behavior of geological materials.

A large number of constitutive models are available at this time for geological materials. However, many of these models have inherent drawbacks. Thus, there is need for new developments in this area.

In the present study, a generalized approach is used to derive constitutive models for geological materials. The method, however, can be applied to other materials as well. Although any number of constitutive models can be established, in this study, only one form is fully investigated.
The proposed model has a number of material constants. Laboratory tests are required to determine these material constants. A series of triaxial tests are performed on an artificial soil using a cubical triaxial apparatus. The device is such that the samples can be subjected to three-dimensional state of stress by applying air pressure through the six open faces of the device. Besides the artificial soil, test data for a silty sand and an agricultural soil is also used (Desai et al. 1982, Markle, 1981). Details of the parameter determination are given in Chapter 4.

Using the test data for these three materials, the constants of the proposed model are determined. Table 4.3 shows a comparison of these material constants. These constants are then used to back-predict the stress-strain response from which they are calculated.

Finally, the proposed constitutive model is implemented in a three-dimensional finite element procedure. The finite element procedure is then used to obtain solutions of a number of boundary value problems.

Based on the investigation presented herein, the following conclusions can be made:

1) Appropriate constitutive laws are required to obtain reasonable solutions for many problems in geomechanics. Available constitutive models may not adequately characterize the behavior of many engineering materials. Thus, new and improved constitutive models are necessary.
In the current research, a generalized approach is used to develop constitutive models within the framework of plasticity theory. Although the present study is limited to geological materials only, the approach can also be used for developing constitutive models for other engineering materials such as concrete.

2) The proposed constitutive model plots continuous and convex in $\sqrt{J_{2D}} - J_1$ space. Thus, the normal to the yield surface at any point can be defined uniquely. Many available plasticity models such as critical state and Cap model represent the yielding process by two separate surfaces which intersect each other with slope discontinuity; that is, the normal at the point of intersection is nonunique. This arises difficulty when associated plasticity laws are used.

3) The proposed model is verified with respect to the triaxial tests on a number of different soils. Both stress-strain behavior and volumetric behavior are compared. It is observed that the model predicts accurately the behavior of the silty sand. However, for the artificial soil, the model seems to overpredict dilatancy during shearing. It may be mentioned that the artificial soil is highly compressible and shows strong creeping during testing. It is observed that the model predictions deviate from the observations when the strain in the axial direction is of the order of 9-10%.
The deviations of the model predictions from the experimental results may be attributed to these facts.

4) The proposed constitutive model is implemented into a three-dimensional finite element procedure. The finite element procedure is used to solve a number of boundary value problems. The numerical results are compared with the corresponding experimental observations. It is found that the proposed constitutive model can adequately characterize the nonlinear and inelastic behavior of a class of geological materials.

5) Besides the proposed constitutive model, a three-dimensional thin-layer interface element is also developed and implemented in the finite element procedure mentioned earlier. It is observed that for soil-structure interaction problems, the thin-layer interface element can adequately represent the interaction behavior.

6) Alternative forms of yield conditions are proposed (Chapter 3) which may be used instead of the proposed model. Further work is necessary to establish these alternatives and to explore their capabilities fully.
APPENDIX A

DERIVATION OF ELASTO-PLASTIC CONSTITUTIVE RELATIONS FOR THE PROPOSED MODEL

The yield function is written as

\[ F = J_{2D} + \alpha J_1^2 - \beta J_1 J_3^{1/3} - \gamma J_1 - k^2 = 0 \]  \hspace{1cm} (A.1)

where

\[ J_1 = \sigma_{ii} \]  \hspace{1cm} (A.2)

\[ J_{2D} = \frac{1}{2} S_{ij} S_{ji} \]  \hspace{1cm} (A.3)

\[ J_3 = \frac{1}{3} \sigma_{ij} \sigma_{jk} \sigma_{ki} \]  \hspace{1cm} (A.4)

\[ S_{ij} = \sigma_{ij} - \frac{1}{3} \sigma_{kk} \delta_{ij} \]  \hspace{1cm} (A.5)

\( \sigma_{ij} \) in Eq. (A.2) is the stress tensor.

The evolution function is expressed as

\[ \beta = \beta_u \left[ 1 - \frac{\beta_a}{n_1 (1 - \beta_b r_D n_2)} \right] \]  \hspace{1cm} (A.6)
where

\[ \xi = f(\varepsilon^P_{ij} \varepsilon^P_{ij})^{1/2} \quad (A.7) \]

\[ r_D = \xi_D / \xi \quad (A.8) \]

\[ \xi_D = f(\varepsilon^P_{ij} \varepsilon^P_{ij})^{1/2} \quad (A.9) \]

\[ \varepsilon^P_{ij} = \varepsilon_{ij}^P - \frac{1}{3} \varepsilon_{kk}^P \delta_{ij} \quad (A.10) \]

Consistency Condition

\[ dF = 0 \quad (A.11) \]

Using chain rule of differentiation,

\[ dF = \frac{\partial F}{\partial \varepsilon_{ij}} d\varepsilon_{ij} + \frac{\partial F}{\partial \varepsilon} d\varepsilon + \frac{\partial F}{\partial r_D} dr_D = 0 \quad (A.12) \]

where

\[ d\varepsilon = (\varepsilon^P_{ij} \varepsilon^P_{ij})^{1/2} \quad (A.13) \]

Using the definition of \( r_D \) given by Eq. (A.8),

\[ dr_D = \frac{\partial r_D}{\partial \varepsilon} d\varepsilon + \frac{\partial r_D}{\partial \xi_D} d\xi_D \quad (A.14) \]
Assuming associated plasticity, the incremental plastic strain, $\text{de}_{ij}^P$, can be expressed as

$$\text{de}_{ij}^P = \lambda \cdot \frac{\partial F}{\partial \sigma_{ij}}$$  \hspace{1cm} (A.15)

where $\lambda > 0$ governs the magnitude of the incremental plastic strain.

Using Eq. (A.15),

$$d\varepsilon = \lambda \cdot \gamma_F$$  \hspace{1cm} (A.16)

and

$$d\varepsilon_D = \lambda \cdot \gamma_{FD}$$  \hspace{1cm} (A.17)

where

$$\gamma_F = (\frac{\partial F}{\partial \sigma_{ij}} \cdot \frac{\partial F}{\partial \sigma_{ij}})^{1/2}$$  \hspace{1cm} (A.18)

$$\gamma_{FD} = [\left(\frac{\partial F}{\partial \sigma_{ij}} \cdot \frac{\partial F}{\partial \sigma_{ij}}\right)^2]^{1/2}$$  \hspace{1cm} (A.19)

The term $(\frac{\partial F}{\partial \sigma_{ij}})$ is the deviatoric part of $\frac{\partial F}{\partial \sigma_{ij}}$.

Using Eqs. (A.16) and (A.17) in Eq. (A.14), $d\varepsilon_D$ can be written as

$$d\varepsilon_D = \lambda \left[\frac{\partial r_D}{\partial \varepsilon} \cdot \gamma_F + \frac{\partial r_D}{\partial \varepsilon_D} \cdot \gamma_{FD}\right]$$  \hspace{1cm} (A.20)
Using the definition of $r_D$ given by Eq. (A.8), $dr_D$ can be written as

$$dr_D = \lambda \left[ - \frac{r_D}{\xi} \gamma_F + \frac{1}{\xi} \gamma_{FD} \right] \quad (A.21)$$

Using the expressions for $d\xi$ and $dr_D$, the consistency condition can now be written as

$$\frac{\partial F}{\partial \sigma_{ij}} \cdot d\sigma_{ij} + \lambda \left\{ \left( \frac{\partial F}{\partial \xi} \cdot \frac{r_D}{\xi} \cdot \frac{\partial F}{\partial r_D} \right) \gamma_F + \left( \frac{1}{\xi} \frac{\partial F}{\partial r_D} \right) \gamma_{FD} \right\} = 0 \quad (A.22)$$

Incremental stress, $d\sigma_{ij}$, is related to the incremental elastic strain, $d\varepsilon^e_{ij}$, through the generalized Hooke's Law; that is,

$$d\sigma_{ij} = C_{ijkl} \cdot d\varepsilon^e_{kl} \quad (A.23)$$

where $C_{ijkl}$ is the fourth order elastic constitutive relation tensor.

Assuming small deformation, the incremental strain, $d\varepsilon_{ij}$, can be written as the sum of incremental elastic strain, $d\varepsilon^e_{ij}$, and incremental plastic strain, $d\varepsilon^p_{ij}$; that is,

$$d\varepsilon_{ij} = d\varepsilon^e_{ij} + d\varepsilon^p_{ij} \quad (A.24)$$

Using Eqs. (A.15), (A.22), (A.23) and (A.24), the unknown $\lambda$ can be expressed as
where $H$ defines the hardening behavior and is given by

$$H = \left\{ (\frac{\partial F}{\partial \xi} - \frac{r_D}{\xi} \cdot \frac{\partial F}{\partial r_D}) \gamma_F + (\frac{1}{\xi} \cdot \frac{\partial F}{\partial r_D}) \gamma_{FD} \right\} \quad (A.26)$$

Finally, the elasto-plastic constitutive relation tensor, $C_{ijkl}^{e-p}$, can be written as

$$C_{ijkl}^{e-p} = C_{ijkl} - \frac{\partial F}{\partial \sigma_{pq}} C_{pqrs} \frac{\partial F}{\partial \sigma_{rs}} - H \quad (A.27)$$

Using chain rule of differentiation, the gradient tensor, $\frac{\partial F}{\partial \sigma_{ij}}$, can be written as

$$\frac{\partial F}{\partial \sigma_{ij}} = \frac{\partial F}{\partial J_1} \sigma_{ij} + \frac{\partial F}{\partial J_{2D}} S_{ij} + \frac{\partial F}{\partial J_{3D}} (S_{ik} S_{kj} - \frac{2}{3} J_{2D} \delta_{ij}) \quad (A.28)$$

Using the form of $F$ given by Eq. (A.1),

$$\frac{\partial F}{\partial J_1} = 2\alpha J_1 - \frac{\beta}{27 J_3^{2/3}} (27 J_3 + 6 J_1 J_{2D} + J_1^3) - \gamma \quad (A.29)$$

$$\frac{\partial F}{\partial J_{2D}} = 1 - \frac{2\beta J_1^2}{9 J_3^{2/3}} \quad (A.30)$$

$$\frac{\partial F}{\partial J_{3D}} = -\frac{\beta J_1}{3 J_3^{2/3}} \quad (A.31)$$
Evaluation of $H$ requires two derivatives, \( \frac{\partial F}{\partial \xi} \) and \( \frac{\partial F}{\partial r_D} \). Using the form of evolution function, $\beta$, given by Eq. (A.6), these can be expressed in the following form:

\[
\frac{\partial F}{\partial \xi} = - J_1 \frac{1}{3} \frac{\partial \beta}{\partial \xi} \quad (A.32)
\]

\[
\frac{\partial F}{\partial r_D} = - J_1 \frac{1}{3} \frac{\partial \beta}{\partial r_D} \quad (A.33)
\]

where

\[
\frac{\partial \beta}{\partial \xi} = \frac{\beta_u \cdot \beta_a \cdot n_1}{(n_1+1) \left( 1 - \beta_b \cdot r_D \right)^{n_2}} \quad (A.34)
\]

\[
\frac{\partial \beta}{\partial r_D} = - \frac{\beta_u \cdot \beta_a \cdot \beta_b \cdot n_2 \cdot r_D}{\xi^1 (1 - \beta_b \cdot r_D^2)^{n_2-1}} \quad (A.35)
\]

Thus, the incremental stress-strain relationships can be written as

\[
d\sigma_{ij} = C_{ijkl} e_{kl} e_{kl} \quad (A.36)
\]

In matrix form, Eq. (A.36) can be written as

\[
\{d\sigma\} = [C]e^{-p} \{d\varepsilon\} \quad (A.37)
\]
where

\[
\{d\sigma\}^T = [d\sigma_{11} \ d\sigma_{22} \ d\sigma_{33} \ d\sigma_{12} \ d\sigma_{23} \ d\sigma_{31}] \tag{A.38}
\]

\[
\{d\varepsilon\}^T + [d\varepsilon_{11} \ d\varepsilon_{22} \ d\varepsilon_{33} \ d\varepsilon_{12} \ d\varepsilon_{23} \ d\varepsilon_{31}] \tag{A.39}
\]

and

\[
[C]^{e-P} = [C] - \frac{[C] \{\frac{\partial F}{\partial \sigma}\} \{\frac{\partial F}{\partial \sigma}\}^T [C]}{\{\frac{\partial F}{\partial \sigma}\}^T [C] \{\frac{\partial F}{\partial \sigma}\} - H} \tag{A.40}
\]

The gradient factor, \(\{\frac{\partial F}{\partial \sigma}\}\), is

\[
\left\{\frac{\partial F}{\partial \sigma}\right\} = \begin{bmatrix}
\frac{\partial F}{\partial \sigma_{11}} \\
\frac{\partial F}{\partial \sigma_{22}} \\
\frac{\partial F}{\partial \sigma_{33}} \\
\frac{\partial F}{\partial \sigma_{12}} \\
\frac{\partial F}{\partial \sigma_{23}} \\
\frac{\partial F}{\partial \sigma_{31}}
\end{bmatrix} \tag{A.41}
\]
LIST OF REFERENCES


